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Christine Lescop

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| Christine Lescop. Invariants of links and 3-manifolds from graph configurations. 2020. hal-02465054

**HAL Id: hal-02465054**

**<https://hal.science/hal-02465054v1>**

Preprint submitted on 10 Oct 2024

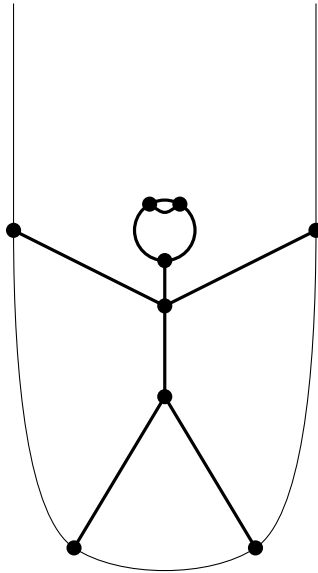
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# Invariants of links and 3-manifolds from graph configurations



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September 17, 2024

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**Keywords:** Knots, 3-manifolds, finite type invariants, homology 3-spheres, linking number, Theta invariant, Casson–Walker invariant, Jacobi diagrams, perturbative expansion of Chern–Simons theory, configuration space integrals, parallelizations of 3-manifolds, first Pontrjagin class

**MSC 2020:** 57K16 57K31 57K30 55R80 57R20 81Q30

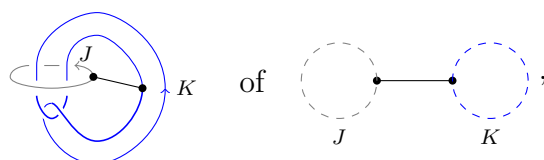


À Lucien Guillou,  
À nos 26 ans de bonheur ensemble,  
À nos deux enfants, Gaëlle et Ronan.

# Preface

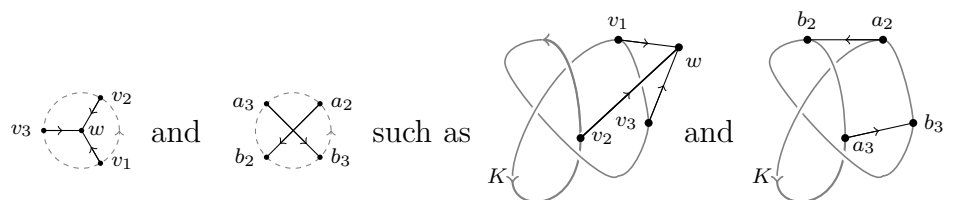
In this book, we explain how to count graph configurations to obtain invariants for 3-manifolds and knots in these 3-manifolds, and we investigate the properties of the obtained invariants.

The simplest of these invariants is the linking number of two disjoint knots in the ambient space  $\mathbb{R}^3$ . Gauss defined it in 1833 [Gau77]. As we review in Section 1.2, this linking number counts configurations



as the degree of an associated Gauss map.

Many mysterious knot invariants called “quantum invariants” were introduced in the mid-80s, starting with the Jones polynomial. Witten explained how to obtain many of them from the perturbative expansion of the Chern-Simons theory in a seminal article [Wit89]. This physicist viewpoint led Guadagnini, Martellini, Mintchev [GMM90] and Bar-Natan [BN95b] to show in what sense a coefficient  $w_2$  of the Jones polynomial counts configurations of the graphs



The theory of Vassiliev invariants reviewed in Chapter 6 associates a degree in  $(\mathbb{N} \cup \{\infty\})$  to a numerical knot invariant. The only knot invariants of degree 0 are the constant functions. The knot invariants of degree 2 are linear combinations of  $w_2$  and the constant function that maps every knot to 1. The Jones polynomial can be renormalized into a series whose coefficients are

finite-degree knot invariants. Altschüler and Freidel showed that every degree  $n$  real-valued knot invariant may be obtained by “counting” configurations of graphs with at most  $2n$  vertices as explained in this book [AF97]. The knot invariants counting graph configurations mentioned above are assembled in a *universal Vassiliev invariant*  $\mathcal{Z}(S^3, \cdot)$  valued in a product of vector spaces generated by some univalent graphs called *Jacobi diagrams*. Kontsevich had constructed another universal Vassiliev invariant with similar properties called *the Kontsevich integral* [Kon93, BN95a]. The Kontsevich integral  $Z_K$  may be defined combinatorially from planar knot diagrams. It has been extensively studied. To my knowledge, the coincidence of the spatial invariant  $\mathcal{Z}(S^3, \cdot)$  with  $Z_K$  is an open problem.

Developing the Witten approach further, Kontsevich outlined a way to count trivalent graphs in more general 3-manifolds and define a topological graded invariant  $\mathcal{Z}$  for them [Kon94]. These more general manifolds are the 3-dimensional  $\mathbb{Q}$ -spheres, simply called  *$\mathbb{Q}$ -spheres* in this book. They are the closed 3-manifolds with the same rational homology as the standard 3-dimensional sphere  $S^3$ . They include the 3-manifolds with the same  $\mathbb{Z}$ -homology as  $S^3$  called  *$\mathbb{Z}$ -spheres*. A  $\mathbb{Z}$ -sphere is a closed orientable three-manifold in which knots bound orientable compact surfaces. The degree one part  $\mathcal{Z}_1$  of the Kontsevich invariant  $\mathcal{Z}$  of  $\mathbb{Q}$ -spheres is determined by a real-valued invariant  $\Theta$  of  $\mathbb{Q}$ -spheres, which counts configurations of the graph  $\Theta$  in the manifold.

Ohtsuki, Habiro, Goussarov, and others developed theories of finite type invariants of  $\mathbb{Z}$ -spheres analogous to the Vassiliev theory for knots in  $\mathbb{R}^3$  [Oht96, GGP01]. Kuperberg and Thurston showed why the Kontsevich invariant  $\mathcal{Z}$  of  $\mathbb{Z}$ -spheres obtained by counting graphs in these manifolds is also *universal* with respect to the above theories [KT99]. This universality result implies that any real-valued invariant of  $\mathbb{Z}$ -spheres of degree  $2n$  (with respect to one of the equivalent developed theories) is a combination of invariants counting configurations of graphs with at most  $2n$  vertices.

In 1985, Casson had defined an invariant of  $\mathbb{Z}$ -spheres. The Casson invariant “counts” conjugacy classes of irreducible  $SU(2)$ -representations of the  $\mathbb{Z}$ -spheres fundamental groups. Their universality result allowed Kuperberg and Thurston to show that  $\Theta$  and the Casson invariant are proportional. In particular, the Casson invariant also “counts” configurations of the graph  $\Theta$ . For a  $\mathbb{Z}$ -sphere  $R$  equipped with a basepoint  $\infty$ , the configurations are counted in a suitable compactification  $C_2(R)$  of the space of ordered pairs of distinct points in the punctured manifold ( $\dot{R} = R \setminus \{\infty\}$ ). The set of counted configurations is the intersection in  $C_2(R)$  of three transverse codimension-2 submanifolds called *propagating chains*, and  $\Theta(R)$  is their algebraic intersection number. Dually, the invariant  $\Theta(R)$  is the integral over  $C_2(R)$  of the

cube of a *propagating* closed 2-form. Propagating chains and propagating forms both represent the linking form on  $R$ . We call them *propagators*. They are the main ingredient used to count graph configurations in this book. They are associated with the graph edges. They are precisely defined in Chapter 3. When  $R$  is  $\mathbb{Z}$ -sphere, results of Pontrjagin and Rohlin in the 1950s [Roh52] imply that the punctured  $\check{R}$  can be equipped with a preferred homotopy class of parallelizations. For a general  $\mathbb{Q}$ -sphere, the invariant  $\Theta$  is first introduced as an invariant of a pair  $(R, \tau)$ , where  $\tau$  is a parallelization of  $\check{R}$ . It is next corrected with the help of a relative first Pontrjagin class to become an invariant of  $R$ . Chapter 4 contains the complete construction of  $\Theta$ , and Chapter 5 establishes the needed properties of Pontrjagin classes.

Kuperberg and I associated explicit propagating chains to Morse functions and associated Morse flows. These propagators reviewed in Section 1.2.8 allowed me to express the Theta invariant in terms of Heegaard diagrams [Les15a]. With this type of propagator, the “counted” graph configurations either map an edge of the graph into a flow line, or map the edge ends into descending manifolds or ascending manifolds of critical points of the Morse function. Fukaya proposed such a way of counting graphs [Fuk96]. Many authors, including Watanabe and Shimizu, further studied it.

In the book’s second part, we define and study an invariant  $\mathcal{Z}(R, L)$  for a link  $L$  in a  $\mathbb{Q}$ -sphere  $R$ . This invariant generalizes both  $\mathcal{Z}(S^3, L)$  and  $\mathcal{Z}(R) = \mathcal{Z}(R, \emptyset)$ . Our definitions are more flexible than the original ones. We prove generalizations of the mentioned universality results in the book’s fourth part.

To get more properties of  $\mathcal{Z}$ , we cut our pairs  $(R, L)$  of links  $L$  in  $\mathbb{Q}$ -spheres  $R$  into pieces called *tangles in  $\mathbb{Q}$ -cylinders*. These pieces can be composed in various ways. In the book’s third part, we generalize  $\mathcal{Z}$  to a functorial invariant of framed tangles in  $\mathbb{Q}$ -cylinders, and we prove that it behaves well under the various allowed compositions.

Our first chapter is a more complete and much longer preface to this book. It contains several introductions. Section 1.1 is a short summary for experts. Other readers can start with Section 1.2, a slow informal introduction based on examples from which a broad audience can get the flavor of the studied topics and hopefully become interested. Section 1.3 is an independent, more formal, mathematical overview of the contents. It is accessible to beginners in topology after the warm-up of Section 1.2. Section 1.4 describes the book organization. Section 1.5 outlines why I wrote this book and what I consider original and new.

Apart from this preface and the first chapter, which has some parts written for experts and is sometimes imprecise, the rest of the book is precise, detailed, and mostly self-contained. The only prerequisites are basic notions

of algebraic topology and de Rham cohomology, surveyed in the appendices.

In 2018, Watanabe disproved a long-standing conjecture called the 4-dimensional Smale conjecture by constructing a topologically trivial  $S^4$ -bundle over  $S^2$ , which is not smoothly equivalent to the trivial bundle  $S^4 \times S^2$  [Wat18b]. He distinguished his exotic  $S^4 \times S^2$  from the standard  $S^4 \times S^2$  using characteristic classes introduced by Kontsevich [Kon94]. The involved Kontsevich–Watanabe characteristic class of a smooth topologically trivial  $S^4$ -bundle over  $S^2$  counts configurations of the complete graph  $\triangleleft$  with four vertices in the total space of the bundle. The ideas and techniques used by Watanabe are similar to those presented in this book. Even though we only count graph configurations in dimension 3, this book can also serve as an introduction to the work of Watanabe and other articles about invariants counting graph configurations in higher dimensions.

I thank the referees for their careful reading and their helpful comments.

# Part I

## Introduction



# Chapter 1

## Introductions

In this chapter, we propose several introductions to this book:

- a short abstract for experts in Section 1.1,
- a slow informal introduction based on examples in Section 1.2, from which a broad audience can get the flavor of the studied topics and hopefully become interested,
- an independent, more formal, mathematical overview of the contents, to which the experts can go directly, in Section 1.3, and
- a section on the genesis of this book in Section 1.5.

I apologize for some repetitions due to this structure. Section 1.4 describes the book organization. The chapter ends with a list of open problems in Section 1.6

Unlike this first chapter, which has some parts written for experts and is sometimes imprecise, the rest of the book is precise, detailed, and mostly self-contained. It relies only on the basic notions of algebraic topology and the basic notions of de Rham cohomology, surveyed in the appendices.

### 1.1 An abstract for experts

**Very first conventions.** Unless otherwise mentioned, manifolds are smooth, but they may have boundary and corners. Let  $\mathbb{K}$  be  $\mathbb{Z}$  or  $\mathbb{Q}$ . In this book, a  $\mathbb{K}$ -*sphere* is a compact oriented 3-dimensional manifold with the same homology with coefficients in  $\mathbb{K}$  as the standard unit sphere  $S^3$  of  $\mathbb{R}^4$ . The unit disk of  $\mathbb{C}$  is denoted by  $D_1$ . A  $\mathbb{K}$ -*ball* (resp. a  $\mathbb{K}$ -*cylinder*, a *genus  $g$   $\mathbb{K}$ -handlebody*) is a compact oriented 3-dimensional manifold  $A$  with the same



homology with coefficients in  $\mathbb{K}$  as the standard unit ball  $B^3$  of  $\mathbb{R}^3$ , (resp. the cylinder  $D_1 \times [0, 1]$ , the standard solid handlebody  $H_g$  of Figure 1.1), such that a neighborhood of the boundary of  $A$  (which is necessarily homeomorphic to the boundary of its model—ball, cylinder or handlebody) is identified with a neighborhood of the boundary of its model by a smooth diffeomorphism.

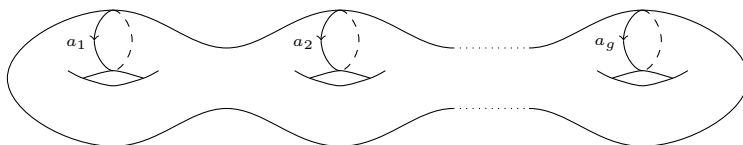


Figure 1.1: The genus  $g$  handlebody  $H_g$

In “ $\mathbb{Q}$ -spheres”,  $\mathbb{Q}$  is a shortcut for *rational homology*. So  $\mathbb{Q}$ -spheres are also called *rational homology spheres* or *rational homology 3-spheres*, while  $\mathbb{Z}$ -spheres are also called *integer homology 3-spheres*.

**Abstract.** In this book, following Edward Witten [Wit89], Maxim Kontsevich [Kon94], Greg Kuperberg and Dylan Thurston [KT99], we define an invariant  $\mathcal{Z}$  of  $n$ -component links  $L$  in rational homology 3-spheres  $R$ , and we study its properties. The invariant  $\mathcal{Z}$  is often called “the perturbative expansion of the Chern–Simons theory”. It is valued in a graded space  $\mathcal{A}(\sqcup_{j=1}^k S^1)$  generated by Jacobi diagrams  $\Gamma$  on  $\sqcup_{j=1}^k S^1$ . These diagrams are a special kind of Feynman diagrams. They are uni-trivalent. The invariant  $\mathcal{Z}(L)$  is a combination  $\mathcal{Z}(L) = \sum_{\Gamma} \mathcal{Z}_{\Gamma}(L) [\Gamma]$  for coefficients  $\mathcal{Z}_{\Gamma}(L)$  that “count” embeddings of  $\Gamma$  in  $R$  mapping the univalent vertices of  $\Gamma$  to  $L$ , in a sense explained in the book. The coefficients  $\mathcal{Z}_{\Gamma}(L)$  may be defined as integrals over configuration spaces or, in a dual way, as algebraic intersections in the same configuration spaces.

When  $R = S^3$ , the invariant  $\mathcal{Z}$  is a universal Vassiliev link invariant studied by many authors, including Enore Guadagnini, Maurizio Martellini, and Mihail Mintchev [GMM90], Dror Bar-Natan [BN95b], Raoul Bott and Clifford Taubes [BT94], Daniel Altschüler and Laurent Freidel [AF97], Dylan Thurston [Thu99], and Sylvain Poirier [Poi02]... This book contains a more flexible definition of this invariant.

*Rational LP-surgeries* in  $\mathbb{Q}$ -spheres are replacements of rational homology handlebodies by other rational homology handlebodies, in a way that does not change the linking number of curves outside the replaced handlebodies. We prove that the restriction of  $\mathcal{Z}$  to  $\mathbb{Q}$ -spheres (equipped with

empty links) is a universal finite type invariant with respect to rational LP-surgeries. Together with recent results of Gwénaél Massuyeau [Mas14] and Delphine Moussard [Mou12], this implies that the restriction of  $\mathcal{Z}$  to  $\mathbb{Q}$ -spheres contains the same information as the Lê–Murakami–Ohtsuki LMO invariant [LMO98] for these manifolds. This also implies that the degree one part of  $\mathcal{Z}$  is the Casson–Walker invariant.

We extend  $\mathcal{Z}$  to a functorial invariant of framed tangles in rational homology cylinders. We describe the behavior of this functor under various operations including some cabling operations. We also compute iterated derivatives of this extended invariant with respect to discrete derivatives associated to the main theories of finite type invariants.

## 1.2 A slow informal introduction for beginners

In this introduction, we describe the contents of this book to a broad audience, including graduate students. We start with examples to give a flavor of the topics studied in this book and to introduce ideas, conventions, and methods used later.

This book is about invariants of links and 3-manifolds that count graph configurations. The first example of such an invariant goes back to Carl Friedrich Gauss in 1833 [Gau77]. It is the linking number of two knots. We discuss it in Subsection 1.2.1.

At the end of the eighties, Edward Witten’s insight into the perturbative expansion of the Chern–Simons theory [Wit89] gave birth to trickier examples. Following Witten’s ideas, Enore Guadagnini, Maurizio Martellini, and Mihail Mintchev [GMM90], and Dror Bar-Natan [BN95b] defined and studied another knot invariant  $w_2$ , which counts configurations of some uni-trivalent graphs with 4 vertices. Maxim Kontsevich [Kon94], Raoul Bott and Clifford Taubes [BT94] [Bot96], Daniel Altschüler and Laurent Freidel [AF97], and others revisited and generalized the definition of  $w_2$ . In Subsection 1.2.5, we detail the example of  $w_2$ .

### 1.2.1 The linking number as a degree

Let  $S^1$  denote the unit circle of the complex plane  $\mathbb{C}$ . We use “:” as a symbol for “such that”. So, the circle  $S^1$  is the set  $\{z : z \in \mathbb{C}, |z| = 1\}$ . Consider a  $C^\infty$  embedding

$$J \sqcup K : S^1 \sqcup S^1 \hookrightarrow \mathbb{R}^3$$

of the disjoint union  $S^1 \sqcup S^1$  of two circles in the ambient space  $\mathbb{R}^3$ , as the example pictured in Figure 1.2. Such an embedding represents a *2-component link*.

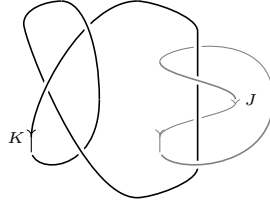


Figure 1.2: A 2-component link in  $\mathbb{R}^3$

It induces the *Gauss map*

$$\begin{array}{ccc}
 \begin{array}{c} \text{Diagram of } S^1 \times S^1 \\ \text{with axes 1 and 2} \end{array} & \xrightarrow{p_{JK}} & \begin{array}{c} \text{Diagram of } S^2 \\ \text{with axes 1 and 2} \end{array} \\
 p_{JK}: \quad S^1 \times S^1 & \rightarrow & S^2 \\
 (w, z) & \mapsto & \frac{1}{\|K(z) - J(w)\|} (K(z) - J(w)).
 \end{array}$$

**Definition 1.1.** The *Gauss linking number*  $lk_G(J, K)$  of the disjoint *knot embeddings*  $J$  and  $K$  is the degree of the Gauss map  $p_{JK}$ .

Below, we give our favorite definition of the degree for this book. To do so, we first agree on conventions used throughout the book.

### 1.2.2 On orientations of manifolds and degrees of maps

We work with smooth (i.e.,  $C^\infty$ ) manifolds locally diffeomorphic to open subspaces of  $[0, 1]^n$ . These manifolds are precisely described in Sections 2.1.1 and 2.1.4. They may have boundaries and ridges (or corners). The cube  $[0, 1]^3$  is an example of such a manifold. Its edges and its vertices are *ridges*.

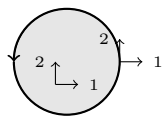
**Conventions 1.2.** Let  $M$  be such a manifold. The *interior* of  $M$  consists of the points with a neighborhood diffeomorphic to an open subspace of  $\mathbb{R}^n$ . The *boundary* of  $M$  is the complement of its interior in  $M$ .

An *orientation* of a real vector space  $V$  of positive dimension is a basis of  $V$  up to a change of basis with positive determinant. When  $V = \{0\}$ , an orientation of  $V$  is an element of  $\{-1, 1\}$ . An *orientation* of a smooth  $n$ -manifold is an orientation of its tangent space at each point of its interior,

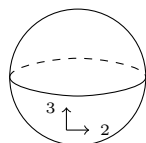
defined in a continuous way. (A local diffeomorphism  $h$  of  $\mathbb{R}^n$  is *orientation-preserving* at  $x$  if and only if the Jacobian determinant of its derivative  $T_x h$  is positive. If the transition maps  $\phi_j \circ \phi_i^{-1}$  of an *atlas*  $(\phi_i)_{i \in I}$  of a manifold  $M$  (as in Subsection 2.1.1) are orientation-preserving (at every point) for  $\{i, j\} \subseteq I$ , then the manifold  $M$  is *oriented* by this atlas.) Unless otherwise mentioned, manifolds are smooth, oriented, compact, and considered up to orientation-preserving diffeomorphisms, in this book. Products are oriented by the order of the factors. More generally, unless otherwise mentioned, the order of appearance of coordinates or parameters orients manifolds. When  $M$  is an oriented manifold,  $(-M)$  denotes the same manifold, equipped with the opposite orientation.

The boundary  $\partial M$  of an oriented manifold  $M$  is oriented by the *outward normal first* convention: If  $x \in \partial M$  is not in a ridge, then the outward normal to  $M$  at  $x$  followed by an oriented basis of  $T_x \partial M$  induce the orientation of  $M$ .

For example, the standard orientation of the disk in the plane induces the traditional counterclockwise orientation of the circle, as the following picture shows.



As another example, the sphere  $S^2$  is oriented as the boundary of the unit ball  $B^3$  of  $\mathbb{R}^3$ , which has the standard orientation induced by (thumb, index finger (2), middle finger (3)) of the right hand.



**Definitions 1.3.** Let  $M$  and  $N$  be smooth manifolds, and let  $p: M \rightarrow N$  be a smooth map from  $M$  to  $N$ . Assume that the boundary of  $M$  is empty. A point  $x$  of  $M$  is called a *regular point* of  $p$  if the tangent map

$$T_x p: T_x M \rightarrow T_y N$$

at  $x$  is surjective. A point  $y$  of  $N$  is called a *regular value* of  $p$  if  $p^{-1}(y)$  contains only regular points.

Our  $d$ -manifolds are covered by countably many open sets diffeomorphic to open subsets of  $[0, 1]^d$ . The following *Morse–Sard theorem* is proved in the book [Hir94] by Morris W. Hirsch. See [Hir94, Chapter 3, Theorem 1.3, p. 69].

**Theorem 1.4** (Morse–Sard theorem). *The set of regular values of a smooth map from a manifold to another such is dense.*<sup>1</sup>

If the source  $M$  of a smooth map  $p: M \rightarrow N$  is compact, then the set of its regular values is furthermore open. In general, the source may have boundary and ridges. Then a point  $y$  is a *regular value* of  $p$ , if  $y$  is a regular value of the restrictions of  $p$  to the interior of  $M$  and to all the open, smooth faces (or strata) of  $M$  (of any codimension). In particular, if the dimensions of  $M$  and  $N$  coincide, then a regular value is not in the image of the boundary  $\partial M$ .

**Definition 1.5.** Assume that  $M$  and  $N$  are oriented,  $M$  is compact, and the dimension of  $M$  coincides with the dimension of  $N$ . Then the (*differential*) *degree* of  $p$  at a regular value  $y$  of  $N$  is the (finite) sum running over the  $x \in p^{-1}(y)$  of the signs of the determinants of  $T_x p$  (in oriented charts). In this case, the differential degree of  $p$  extends as a continuous function  $\deg(p)$  from  $N \setminus p(\partial M)$  to  $\mathbb{Z}$ , as we prove in detail in Lemma 2.3. In particular, if the boundary of  $M$  is empty and if  $N$  is connected, then  $\deg(p)$  extends as a constant map from  $N$  to  $\mathbb{Z}$ , whose value is called the *degree* of  $p$ . See [Mil97, Chapter 5].

Figure 1.3 shows the values of  $\deg(p)$  for the pictured vertical projection  $p$  from the interval  $[0, 1]$  to  $\mathbb{R}$ .

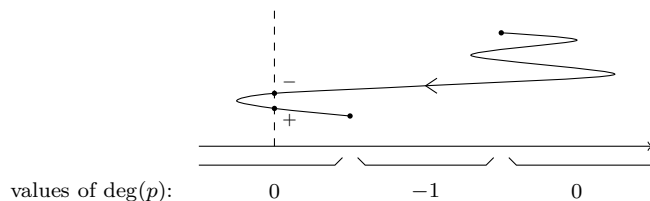


Figure 1.3: Degree of the vertical projection

Another easy example in higher dimensions is the case of an orientation-preserving embedding  $p$ . In this case,  $\deg(p)$  is 1 on the image of the interior of  $M$  and 0 outside the image of  $M$ .

<sup>1</sup>It is even *residual*, i.e., it contains the intersection of a countable family of dense open sets.

### 1.2.3 Back to the linking number

The Gauss linking number  $lk_G(J, K)$  can be computed from a link diagram as in Figure 1.2 as follows. It is the differential degree of  $p_{JK}$  at the vector  $Y$  that points towards us. The set  $p_{JK}^{-1}(Y)$  is the set of pairs  $(w, z)$  of points for which the projections of  $J(w)$  and  $K(z)$  coincide, and  $J(w)$  is under  $K(z)$ . They correspond to the *crossings*  $\begin{matrix} J \\ \nearrow \searrow \\ K \end{matrix}$  and  $\begin{matrix} K \\ \nearrow \searrow \\ J \end{matrix}$  of the diagram.

In a diagram, a crossing is *positive* if we turn counterclockwise from the arrow at the end of the upper strand towards the arrow at the end of the lower strand like  $\begin{matrix} \nearrow \\ \searrow \end{matrix}$ . Otherwise, it is *negative* like  $\begin{matrix} \searrow \\ \nearrow \end{matrix}$ .

Consider a positive crossing  $\begin{matrix} J \\ \nearrow \searrow \\ K \end{matrix}$ . Moving  $J(w)$  along  $J$  following the orientation of  $J$  moves  $p_{JK}(w, z)$  towards the southeast direction  $Tpdw$ . In the meantime, moving  $K(z)$  along  $K$  following the orientation of  $K$  moves  $p_{JK}(w, z)$  towards the northeast direction  $Tpdz$ . So the local orientation  $\begin{matrix} \nearrow^{Tpdz} \\ \searrow_{Tpdw} \end{matrix}$  induced by the image of  $p_{JK}$  around  $Y \in S^2$  is  $\begin{matrix} \nearrow \\ \searrow \\ 1 \end{matrix}$ . Therefore, the contribution of a positive crossing to the degree is 1. It is easy to deduce that the contribution of a negative crossing is  $(-1)$ . We denote the cardinality of a set  $A$  by  $|A|$ .

In particular,  $\left| \begin{matrix} J \\ \nearrow \searrow \\ K \end{matrix} \right|$  is the number of occurrences of  $\begin{matrix} J \\ \nearrow \searrow \\ K \end{matrix}$  in the diagram. We have thus proved that

$$\deg_Y(p_{JK}) = \left| \begin{matrix} J \\ \nearrow \searrow \\ K \end{matrix} \right| - \left| \begin{matrix} K \\ \nearrow \searrow \\ J \end{matrix} \right|.$$

So we have

$$lk_G(J, K) = \left| \begin{matrix} J \\ \nearrow \searrow \\ K \end{matrix} \right| - \left| \begin{matrix} K \\ \nearrow \searrow \\ J \end{matrix} \right|.$$

We similarly obtain  $\deg_{-Y}(p_{JK}) = \left| \begin{matrix} K \\ \nearrow \searrow \\ J \end{matrix} \right| - \left| \begin{matrix} J \\ \nearrow \searrow \\ K \end{matrix} \right|$ . This implies

$$\begin{aligned} lk_G(J, K) &= \left| \begin{matrix} K \\ \nearrow \searrow \\ J \end{matrix} \right| - \left| \begin{matrix} J \\ \nearrow \searrow \\ K \end{matrix} \right| \\ &= \frac{1}{2} \left( \left| \begin{matrix} J \\ \nearrow \searrow \\ K \end{matrix} \right| + \left| \begin{matrix} K \\ \nearrow \searrow \\ J \end{matrix} \right| - \left| \begin{matrix} K \\ \nearrow \searrow \\ J \end{matrix} \right| - \left| \begin{matrix} J \\ \nearrow \searrow \\ K \end{matrix} \right| \right) \end{aligned}$$

and  $lk_G(J, K) = lk_G(K, J)$ .

In the example of Figure 1.2, the Gauss linking number  $lk_G(J, K)$  is 2. For the *positive Hopf link* of Figure 1.4, we have  $lk_G(J, K) = 1$ . The Gauss linking number of the components of the *negative Hopf link* is equal to  $-1$ . It is zero for the *Whitehead link*.

Let  $\omega_S$  be a 2-form on  $S^2$ . The integral of the pull-back  $p_{JK}^*(\omega_S)$  of  $\omega_S$  over  $S^1 \times S^1$  is the integral over  $S^2$  of  $\deg(p_{JK})\omega_S$ . Since the differential degree  $\deg(p_{JK})$  of the Gauss map  $p_{JK}$  is constant and equal to  $lk_G(J, K)$

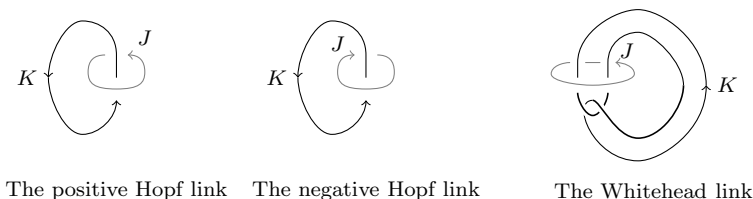


Figure 1.4: The Hopf links and the Whitehead link

on the set of regular values of  $p_{JK}$ , we have

$$\int_{S^1 \times S^1} p_{JK}^*(\omega_S) = \int_{S^2} \deg(p_{JK}) \omega_S = lk_G(J, K) \int_{S^2} \omega_S.$$

Denote the standard area form of  $S^2$  by  $4\pi\omega_{S^2}$ . So  $\omega_{S^2}$  is the homogeneous volume form of  $S^2$  such that  $\int_{S^2} \omega_{S^2} = 1$ . In 1833, Carl Friedrich Gauss defined the linking number of  $J$  and  $K$  as an integral [Gau77]. In modern notation, his definition may be written as

$$lk_G(J, K) = \int_{S^1 \times S^1} p_{JK}^*(\omega_{S^2}).$$

### 1.2.4 A first noninvariant count of graph configurations

The above Gauss linking number counts the *configurations* of the graph  $v_J \leftrightarrow v_K$ , for which  $v_J$  is on the knot  $J$  (or more precisely on the image  $J(S^1)$  of the knot embedding  $J$ ), the vertex  $v_K$  is on the knot  $K$ , and the edge from  $v_J$  to  $v_K$  is a straight segment with an arbitrary generic fixed direction  $X$  in  $S^2$ . Here, *generic* means that  $X$  is a regular value of  $p_{JK}$ .<sup>2</sup> A *configuration* of  $v_J \leftrightarrow v_K$  is an injection  $c: \{v_J, v_K\} \hookrightarrow (\mathbb{R}^3)^2$  such that  $c(v_J) = J(z_J)$  for some  $z_J \in S^1$  and  $c(v_K) = K(z_K)$  for some  $z_K \in S^1$ . The corresponding *configuration space* is parametrized by  $(z_J, z_K) \in S^1 \times S^1$ . It is diffeomorphic to  $S^1 \times S^1$ . The configurations such that the edge from  $v_J$  to  $v_K$  is a straight segment with direction  $X$  in  $S^2$  are in one-to-one correspondence with  $p_{JK}^{-1}(X)$ . The local degree of  $p_{JK}$  equips each of these configurations with a sign. The “count” of configurations with direction  $X$  is the sum of these signs, which is nothing but the degree of  $p_{JK}$  at  $X$ . So any choice of a generic  $X$  will give the same integral result, which will not be changed by a continuous deformation of our embedding among embeddings. The graph

<sup>2</sup>This is a generic condition thanks to the recalled Morse–Sard theorem 1.4.

$v_J \leftrightarrow v_K$  will also be denoted by  $J \overset{\circlearrowleft}{\leftrightarrow} \overset{\circlearrowright}{\circlearrowright} K$ . In this diagram, the dashed circles show to which component the vertices must map.

Let  $K: S^1 \hookrightarrow \mathbb{R}^3$  be a smooth embedding of the circle into  $\mathbb{R}^3$ . Such an embedding is called a *knot embedding*. An *isotopy* between two knot embeddings  $K$  and  $K_1$  is a smooth map  $\psi: [0, 1] \times S^1 \rightarrow \mathbb{R}^3$  such that the restriction  $\psi(t, \cdot)$  of  $\psi$  to  $\{t\} \times S^1$  is a knot embedding for any  $t \in [0, 1]$ ,  $\psi(0, \cdot) = K$ , and  $\psi(1, \cdot) = K_1$ . When there exists such an isotopy, the knot embeddings  $K$  and  $K_1$  are said to be *isotopic* or in the same *isotopy class*. A *knot* is an isotopy class of knot embeddings.

Let us now try to count the configurations

$$c: \{v_1, v_2\} \hookrightarrow (\mathbb{R}^3)^2$$

of the graph  $v_1 \leftrightarrow v_2$ , for which  $c(v_1)$  and  $c(v_2)$  are two distinct points on the (image of) the knot embedding  $K$ , and the edge from  $v_1$  to  $v_2$  is a straight segment with an arbitrary direction  $X$  in  $S^2$ . The graph  $v_1 \leftrightarrow v_2$  is also denoted by  $\overset{\circlearrowleft}{\leftrightarrow}$ . The associated *configuration space* is

$$\check{C}(K; \overset{\circlearrowleft}{\leftrightarrow}) = \left\{ \left( K(z), K(z \exp(2i\pi t)) \right) : z \in S^1, t \in ]0, 1[ \right\}.$$

(In this book, open and half-open intervals are denoted with square brackets. For example,  $]0, 1[$  denotes the open interval  $\{x : 0 < x < 1\}$ . Similarly,  $]0, 1]$  denotes the half-open interval  $\{x : 0 < x \leq 1\}$ .) The configuration space  $\check{C}(K; \overset{\circlearrowleft}{\leftrightarrow})$  is naturally identified with the open annulus  $S^1 \times ]0, 1[$ . We have a *Gauss direction map*

$$G_K: \begin{array}{ccc} \check{C}(K; \overset{\circlearrowleft}{\leftrightarrow}) & \rightarrow & S^2 \\ c & \mapsto & \frac{1}{\|K(z \exp(2i\pi t)) - K(z)\|} \left( K(z \exp(2i\pi t)) - K(z) \right) \end{array}$$

and the degree of  $G_K$  at an element  $X$  of  $S^2$  makes sense as soon as  $X$  is a regular value of  $G_K$  whose preimage is finite.

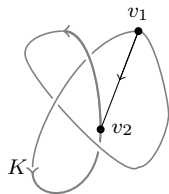


Figure 1.5: A configuration of a segment on  $K$

The annulus  $\check{C}(K; \overset{\circlearrowleft}{\leftrightarrow})$  can be compactified to the closed annulus  $C(K; \overset{\circlearrowleft}{\leftrightarrow}) = S^1 \times [0, 1]$ , to which  $G_K$  extends smoothly. The extended  $G_K$ , still denoted



by  $G_K$ , maps  $(z, 0) \in S^1 \times \{0\}$  to the direction of the tangent vector to  $K$  at  $z$ . It maps  $(z, 1) \in S^1 \times \{1\}$  to the opposite direction. The closed annulus  $C(K; \overleftrightarrow{\cdot})$  is an example of a smooth manifold whose boundary is

$$\partial C(K; \overleftrightarrow{\cdot}) = S^1 \times \{0\} \cup (-S^1 \times \{1\}).$$

The degree of  $G_K$  is a continuous map from  $S^2 \setminus G_K(\partial C(K; \overleftrightarrow{\cdot}))$  to  $\mathbb{Z}$ . Let us compute it for the following embeddings of the trivial knot.

Let  $O$  be an embedding of the circle to the horizontal plane.<sup>3</sup> The image under  $G_O$  of the whole annulus is in the horizontal great circle of  $S^2$ . The set of regular values of  $G_O$  is the complement of this circle, and hence the degree of  $G_O$  is zero on all this set.

Let  $K_1$  and  $K_{-1}$  be embeddings of  $S^1$  such that

- the images of  $K_1$  and  $K_{-1}$  project to the horizontal plane as in Figure 1.6,
- they lie in the horizontal plane everywhere except in a small ball around where they cross over, and
- they lie in the union of two orthogonal planes.

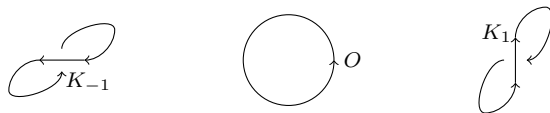
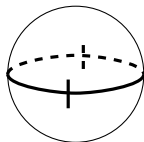


Figure 1.6: Diagrams of the trivial knot

The image of the boundary of  $C(K_{\pm 1}; \overleftrightarrow{\cdot}) = S^1 \times [0, 1]$  in  $S^2$  lies in the union of the great circles of the two planes. More precisely, it lies in the union of the horizontal plane and two vertical arcs, as in the following figure.



Therefore, when  $K = K_\varepsilon$ , for  $\varepsilon = \pm 1$ , the degree of  $G_{K_\varepsilon}$  (extends as a map, which) is constant on each side of our horizontal equator. Computing it at the North Pole  $\vec{N}$  as in Subsection 1.2.3, we find that the degree of  $G_{K_\varepsilon}$  is  $\varepsilon$  on the Northern Hemisphere. We similarly compute the degree of  $G_{K_\varepsilon}$  on the Southern Hemisphere. It is also  $\varepsilon$ .

<sup>3</sup>Here, “the” horizontal plane is the plane  $\mathbb{C} \times \{0\}$  of  $\mathbb{R}^3$  viewed as  $\mathbb{C} \times \mathbb{R}$ .

For any embedding  $K: S^1 \hookrightarrow \mathbb{R}^3$ , define

$$I_\theta(K) = \int_{\check{C}(K; \star)} G_K^*(\omega_{S^2}).$$

The real number  $I_\theta(K)$  is the integral of  $\deg(G_K)\omega_{S^2}$  over  $S^2$ . It can be seen as the *algebraic area* of  $G_K(C(K; \star))$ . The above degree evaluation allows us to compute the integrals  $I_\theta(O) = 0$ ,  $I_\theta(K_1) = 1$ , and  $I_\theta(K_{-1}) = -1$ .

More generally, say that a knot embedding  $K$  that lies in the union of the horizontal plane and a finite union of vertical planes so that the unit tangent vector to  $K$  is never vertical is *almost-horizontal*. The *writhe* of a generic almost-horizontal knot embedding is the number of positive crossings minus the number of negative crossings of its orthogonal projection onto the horizontal plane.<sup>4</sup> An almost-horizontal embedding  $K$  has a natural parallel  $K_\parallel$  (up to isotopy) obtained from  $K$  by (slightly) pushing it down.<sup>5</sup> For any almost-horizontal knot embedding  $K$ , the degree of  $G_K$  extends to a constant function of  $S^2$ . More precisely, we have the following lemma.

**Lemma 1.6.** *For any almost-horizontal knot embedding  $K$ , the degree of  $G_K$  at any regular value of  $G_K$  is the writhe of  $K$ , and we have*

$$I_\theta(K) = \int_{\check{C}(K; \star)} G_K^*(\omega_{S^2}) = lk(K, K_\parallel).$$

PROOF: As in the previous examples, for such a knot embedding  $K$ , the degree of  $G_K$  extends to a constant function on each hemisphere of  $S^2$ . It maps the regular values of these hemispheres to the writhe of  $K$ . In the complement of  $K$ , the parallel  $K_\parallel$  is isotopic to the parallel  $K_{\parallel, \ell}$  on the left-hand side of  $K$ . The formulas of Subsection 1.2.1 show that  $lk(K, K_{\parallel, \ell})$  is the writhe of  $K$ .  $\square$

A *knot invariant* is a function of embeddings that takes the same value on isotopic knots. Unlike the Gauss linking number, the integral  $I_\theta()$  is *not invariant under isotopy* since it takes distinct values on the isotopic knot embeddings  $K_{-1}$  and  $K_1$ .

In every isotopy class of embeddings of  $S^1$  into  $\mathbb{R}^3$ , we can construct an embedding  $K$  such that the degree of the direction map  $G_K$  extends as the constant map of  $S^2$  with value 0 (or any arbitrary integer) as follows:

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<sup>4</sup>The genericity of  $K$  implies that the orthogonal projection onto the horizontal plane of  $K$  is an immersion whose only multiple points are transverse double points.

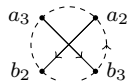
<sup>5</sup>A *parallel* of a knot embedding  $K$  in a 3-manifold  $M$  (such as  $\mathbb{R}^3$ ) is a knot embedding  $K_\parallel: S^1 \hookrightarrow M$  such that there exists an embedding  $f$  from  $[0, 1] \times S^1$  into  $M$  that restricts to  $\{0\} \times S^1$  as  $K$  and to  $\{1\} \times S^1$  as  $K_\parallel$ .

add kinks such as  $\curvearrowright$  or  $\curvearrowleft$  to a generic horizontal projection, and take a corresponding almost-horizontal embedding.

Since  $I_\theta$  varies continuously under an isotopy, it maps any isotopy class of embeddings of  $S^1$  into  $\mathbb{R}^3$  onto  $\mathbb{R}$ . In particular, there are embeddings  $K$  for which  $I_\theta(K)$  is not an integer. For such an embedding, the degree of the direction map  $G_K$  cannot be extended to a constant map on  $S^2$ .

### 1.2.5 A first knot invariant which counts graph configurations

Let us now try to count configurations  $c$  of the following graph:



A *configuration*  $c$  of that graph is an injection from the set  $\{b_3, a_2, a_3, b_2\}$  of vertices of  $\mathfrak{X}$  to  $\mathbb{R}^3$  such that the images  $c(b_3)$ ,  $c(a_2)$ ,  $c(a_3)$ , and  $c(b_2)$  of the vertices are on the knot  $K$ , and we successively meet  $c(b_3) = K(z)$ ,  $c(a_2) = K(z \exp(2i\pi\alpha_2))$ ,  $c(a_3) = K(z \exp(2i\pi\alpha_3))$ , and  $c(b_2) = K(z \exp(2i\pi\beta_2))$  along  $K$  following the orientation of  $K$ . The dashed circle shows the cyclic order of the four vertices. The associated *configuration space*  $\check{C}(K; \mathfrak{X})$  is

$$\left\{ \left( K(z), K(z \exp(2i\pi\alpha_2)), K(z \exp(2i\pi\alpha_3)), K(z \exp(2i\pi\beta_2)) \right) : \begin{array}{l} z \in S^1, (\alpha_2, \alpha_3, \beta_2) \in ]0, 1[^3, \alpha_2 < \alpha_3 < \beta_2 \end{array} \right\}.$$

For  $i \in \{2, 3\}$ , set  $e_i = (a_i, b_i)$ , and let  $G_{e_i}(c) = \frac{c(b_i) - c(a_i)}{\|c(b_i) - c(a_i)\|}$  denote the direction in  $S^2$  of the image under the configuration  $c$  of the edge  $e_i$ . The open configuration space  $\check{C}(K; \mathfrak{X})$  has the natural compactification

$$C(K; \mathfrak{X}) = S^1 \times \{(\alpha_2, \alpha_3, \beta_2) \in [0, 1]^3 : \alpha_2 \leq \alpha_3 \leq \beta_2\}.$$

The maps  $G_{e_2}$  and  $G_{e_3}$  smoothly extend to  $C(K; \mathfrak{X})$  as before, and

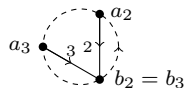
$$G_{\mathfrak{X}} = (G_{e_2}, G_{e_3}) : C(K; \mathfrak{X}) \rightarrow (S^2)^2$$

is a smooth map between two compact 4-manifolds.

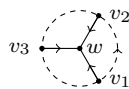
The *codimension-one faces*  $C(K; \mathfrak{X})$  are the four faces  $(\alpha_2 = 0)$ ,  $(\alpha_2 = \alpha_3)$ ,  $(\alpha_3 = \beta_2)$ , and  $(\beta_2 = 1)$ , on which  $c$  maps (at least) two consecutive vertices to the same point on  $K$ .<sup>6</sup> When  $G_{\mathfrak{X}}$  is locally an embedding near

<sup>6</sup>In this introduction, codimension-one faces are closed. Later, they will be open parts of the boundaries.

such a face, the degree of  $G_{\mathfrak{X}}$  changes by  $\pm 1$  when we cross the image of that face.<sup>7</sup> Thus, it suffices to determine the images of the interiors of these codimension-one faces and the local degree at one regular value to determine the degree of  $G_{\mathfrak{X}}$ , as a map from  $(S^2)^2 \setminus G_{\mathfrak{X}}(\partial C(K; \mathfrak{X}))$  to  $\mathbb{Z}$ . We associate the following figure to the face ( $\beta_2 = 1$ ) of  $C(K; \mathfrak{X})$ .



Let us now try to count configurations  $c$  of the following tripod  $\mathfrak{X}$ .



A *configuration*  $c$  of this tripod  $\mathfrak{X}$  is an injection from the set  $\{w, v_1, v_2, v_3\}$  of its vertices into  $\mathbb{R}^3$ , where the images  $c(v_1)$ ,  $c(v_2)$ , and  $c(v_3)$  of the vertices  $v_1$ ,  $v_2$ , and  $v_3$  are on the knot  $K$ , and we successively meet  $c(v_1)$ ,  $c(v_2)$ , and  $c(v_3)$  along  $K$  following the orientation of  $K$ .

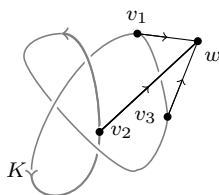


Figure 1.7: A configuration of the tripod on  $K$

Such a *configuration*  $c$  maps  $w$  to  $c(w) \in \mathbb{R}^3$ ,  $v_1$  to  $c(v_1) = K(z)$  for some  $z \in S^1$ ,  $v_2$  to  $c(v_2) = K(z \exp(2i\pi t_2))$ , and  $v_3$  to  $c(v_3) = K(z \exp(2i\pi t_3))$ . The set of these configurations is the *configuration space*  $\check{C}(K; \mathfrak{X})$ . It is an open 6-manifold parametrized by an open subspace of  $\mathbb{R}^3 \times S^1 \times \{(t_2, t_3) \in ]0, 1[^2 : t_2 < t_3\}$ . For  $i \in \underline{3} = \{1, 2, 3\}$ , set  $e_i = (v_i, w)$ , and let  $G_{e_i}(c) = \frac{c(w) - c(v_i)}{\|c(w) - c(v_i)\|}$  denote the direction of the image under  $c$  of the edge  $e_i$  in  $S^2$ . These edge directions together provide a map

$$\begin{aligned} \check{G}_{\mathfrak{X}}: \quad \check{C}(K; \mathfrak{X}) &\rightarrow (S^2)^3 \\ c &\mapsto (G_{e_1}(c), G_{e_2}(c), G_{e_3}(c)). \end{aligned}$$

from our open 6-manifold  $\check{C}(K; \mathfrak{X})$  to the 6-manifold  $(S^2)^3$ .

<sup>7</sup>See Lemma 2.3 for a precise statement.

For a regular value  $(X_1, X_2, X_3)$  of  $\check{G}_{\mathfrak{A}}$  whose preimage is finite, we can again count the configurations of the tripod such that the direction of the edge  $e_i$  is  $X_i$ , as the degree of  $\check{G}_{\mathfrak{A}}$  at  $(X_1, X_2, X_3)$ .

In Chapter 8, we construct a *compactification*  $C(K; \mathfrak{A})$  of  $\check{C}(K; \mathfrak{A})$  (and of many similar configuration spaces), using blow-up techniques, as William Fulton, Robert MacPherson [FM94], Scott Axelrod, Isadore Singer [AS92], and Maxim Kontsevich [Kon94] did. This compactification  $C(K; \mathfrak{A})$  is a smooth compact 6-manifold with boundary and ridges. Its interior is  $\check{C}(K; \mathfrak{A})$ . The map  $\check{G}_{\mathfrak{A}}$  extends to a smooth map  $G_{\mathfrak{A}}$  over  $C(K; \mathfrak{A})$ . *Regular values* of  $G_{\mathfrak{A}}$  are regular values of  $\check{G}_{\mathfrak{A}}$  in the complement of  $G_{\mathfrak{A}}(\partial C(K; \mathfrak{A}) = C(K; \mathfrak{A}) \setminus \check{C}(K; \mathfrak{A}))$ . They form an open dense subset  $\mathcal{O}$  of  $(S^2)^3$ , for which the differential degree of  $\check{G}_{\mathfrak{A}}$  makes sense. As mentioned above and proved in Lemma 2.3, this local integral degree extends to a continuous map on  $(S^2)^3 \setminus G_{\mathfrak{A}}(\partial C(K; \mathfrak{A}))$ . It restricts to a constant map on every connected component of  $(S^2)^3 \setminus G_{\mathfrak{A}}(\partial C(K; \mathfrak{A}))$ . In Lemma 7.12, we explicitly compute this local degree when  $K$  is the round circle  $O$  in a plane, as an example. Our computation shows that this degree cannot be extended to a constant map in this case.

Let  $p_i: (S^2)^3 \rightarrow S^2$  denote the projection to the  $i^{\text{th}}$  factor. Define the algebraic volume

$$I(K; \mathfrak{A}) = \int_{(S^2)^3} \deg(G_{\mathfrak{A}}) \wedge_{i=1}^3 p_i^*(\omega_{S^2}) = \int_{\check{C}(K; \mathfrak{A})} G_{\mathfrak{A}}^* (\wedge_{i=1}^3 p_i^*(\omega_{S^2}))$$

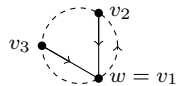
of the image of  $C(K; \mathfrak{A})$  under  $G_{\mathfrak{A}}$ . This algebraic volume  $I(K; \mathfrak{A})$  has no reason to be a knot isotopy invariant, and it is not.

To compute the map

$$\deg(G_{\mathfrak{A}}): (S^2)^3 \setminus G_{\mathfrak{A}}(\partial C(K; \mathfrak{A})) \rightarrow \mathbb{Z},$$

we look at  $G_{\mathfrak{A}}(\partial C(K; \mathfrak{A}))$ . Let us describe the compactification  $C(K; \mathfrak{A})$  on loci where  $c(w)$  approaches  $c(v_1)$  and  $c(v_1)$  is far from  $c(v_2)$  and  $c(v_3)$ . With our coordinates, this amounts to assuming  $c(w) = c(v_1) + \eta x$  for some  $\eta \in ]0, \varepsilon]$  for a small  $\varepsilon > 0$  and some  $x \in S^2$ , and  $(t_2, t_3) \in [\alpha, 1 - \alpha]^2$  for some  $\alpha \in ]0, 1/2[$ . This part of  $\check{C}(K; \mathfrak{A})$  is diffeomorphic to  $]0, \varepsilon] \times S^2 \times S^1 \times \{(t_2, t_3) \in [\alpha, 1 - \alpha]^2 : t_2 < t_3\}$ . Its closure in the compactification  $C(K; \mathfrak{A})$  is naturally diffeomorphic to  $[0, \varepsilon] \times S^2 \times S^1 \times \{(t_2, t_3) \in [\alpha, 1 - \alpha]^2 : t_2 \leq t_3\}$ . (We close  $]0, \varepsilon[$  at 0—we also relax the inequality  $t_2 < t_3$  but this is not important for us now.) In the compactification  $C(K; \mathfrak{A})$ , the image  $c(w)$  may coincide with  $c(v_1)$  (when  $\eta = 0$ ). The direction from  $c(v_1)$  to  $c(w)$  is still defined in this case. It is contained in the  $S^2$  factor. In particular,  $G_{\mathfrak{A}}$  extends to this part of  $C(K; \mathfrak{A})$ . The compactification creates the local

boundary  $\{0\} \times (-S^2) \times S^1 \times \{(t_2, t_3) \in ]\alpha, 1 - \alpha[^2 : t_2 \leq t_3\}$ . This local boundary is a 5-dimensional manifold. Its image under  $G_{\mathfrak{Z}}$  is the product by  $S^2$  of the image in  $(S^2)^2$  of the configuration space associated to the following graph by the natural Gauss map “direction of the edges”.

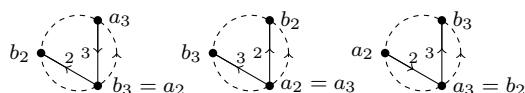


Again, the dashed circle represents the cyclic order of  $c(v_1)$ ,  $c(v_2)$ , and  $c(v_3)$  along  $K$ . The image of this local boundary creates a “wall” in  $(S^2)^3$  across which the local degree changes by  $\pm 1$ . (See Lemma 2.3 for a precise statement.)

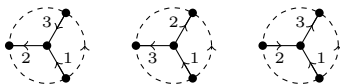
We recognize the picture associated to the face ( $\beta_2 = 1$ ) denoted by  $F_{\beta_2=1}$  of  $C(K; \mathfrak{Z})$ . We observe that the image of the corresponding face under  $G_{\mathfrak{Z}}$  coincides with the image of the face  $S^2 \times F_{\beta_2=1}$  of  $S^2 \times C(K; \mathfrak{Z})$  under

$$G'_{\mathfrak{Z}} = 1_{S^2} \times G_{\mathfrak{Z}}: S^2 \times C(K; \mathfrak{Z}) \rightarrow (S^2)^3.$$

So the combination  $(\deg(1_{S^2} \times G_{\mathfrak{Z}}) - \deg(G_{\mathfrak{Z}}))$  does not vary across the images of the corresponding faces (or at least not because of them). (There are some sign and orientation issues to check here, but we will carefully treat them in a broader generality in Section 7.1 and Lemma 9.14. Let the reader trust me that the signs are correct here.) We glued the images of  $G'_{\mathfrak{Z}}$  and of  $G_{\mathfrak{Z}}$  along  $G'_{\mathfrak{Z}}(S^2 \times F_{\beta_2=1})$  to make the union of these images behave as the image of a manifold without boundary, locally. Unfortunately, the three other faces of  $S^2 \times C(K; \mathfrak{Z})$  created other walls in  $(S^2)^3$  associated to the following figures:



To cancel these walls with the same type of faces of  $C(K; \mathfrak{Z})$  as before, we use Gauss maps associated to the following diagrams:



Let us describe these Gauss maps more precisely. For any subset  $I$  of  $\underline{3}$ , let  $\iota_I$  denote the diffeomorphism of  $(S^2)^3$  which maps  $(X_1, X_2, X_3)$  to  $(\varepsilon_1 X_1, \varepsilon_2 X_2, \varepsilon_3 X_3)$ , where  $\varepsilon_i = -1$  when  $i \in I$ , and  $\varepsilon_i = 1$  when  $i \notin I$ . For two elements  $i, j$  of  $\underline{3}$ , simply write  $\iota_i = \iota_{\{i\}}$  and  $\iota_{ij} = \iota_{\{i, j\}}$ . For a permutation  $\sigma$  of  $\underline{3}$ , let  $\sigma_*$  denote the diffeomorphism of  $(S^2)^3$  that maps  $(Y_1, Y_2, Y_3)$  to  $(Y_{\sigma^{-1}(1)}, Y_{\sigma^{-1}(2)}, Y_{\sigma^{-1}(3)})$ . (We have  $\sigma_*((X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)})) = (X_1, X_2, X_3)$ .)

The Gauss maps associated to the above diagrams are  $\iota_2 \circ (23)_* \circ G_{\mathfrak{S}_3}$ ,  $\iota_{23} \circ G_{\mathfrak{S}_3}$ , and  $\iota_3 \circ (23)_* \circ G_{\mathfrak{S}_3}$ , respectively. The combination

$$\begin{aligned} \deg(1_{S^2} \times G_{\mathfrak{S}_3}) - \deg(G_{\mathfrak{S}_3}) - \deg(\iota_{23} \circ G_{\mathfrak{S}_3}) \\ + \deg(\iota_2 \circ (23)_* \circ G_{\mathfrak{S}_3}) + \deg(\iota_3 \circ (23)_* \circ G_{\mathfrak{S}_3}) \end{aligned}$$

does not vary across the boundary of the image of  $1_{S^2} \times G_{\mathfrak{S}_3}$  as Figure 1.8 suggests.

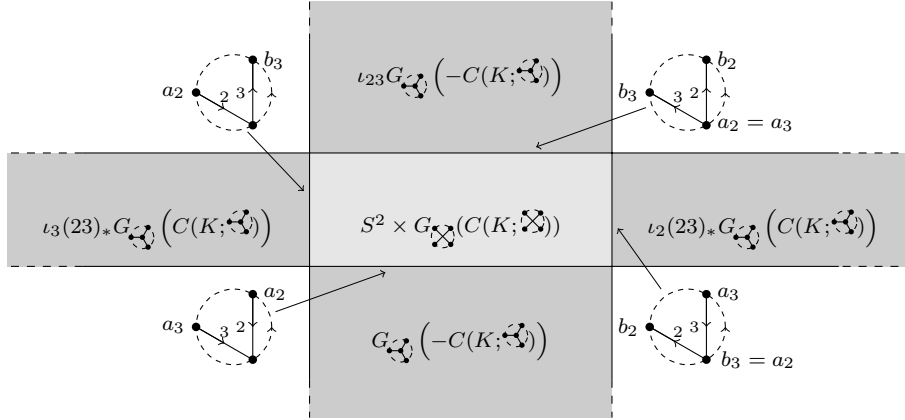


Figure 1.8: Variation of the image of  $1_{S^2} \times G$  across the boundary in  $(S^2)^3$  (pictured as 2-dimensional)

Unfortunately, this process introduces other walls, and we have not yet cancelled the walls due to the faces  $c(v_2) = c(w)$  and  $c(v_3) = c(w)$ . However, we have the following proposition, which is a corollary of Theorem 7.32, as it will be seen right after Theorem 7.32.

**Proposition 1.7.** *The map  $\tilde{w}_2$*

$$\begin{aligned} \frac{1}{24} \sum_{\sigma \in \mathfrak{S}_3} \deg\left(\sigma_*(1_{S^2} \times G_{\mathfrak{S}_3})\right) \\ - \frac{1}{48} \sum_{I \subset \underline{3}} (-1)^{|I|} \left( \deg(\iota_I \circ G_{\mathfrak{S}_3}) + \deg(\iota_I \circ (23)_* \circ G_{\mathfrak{S}_3}) \right), \end{aligned}$$

which is well-defined on an open dense subset of  $(S^2)^3$ , extends as a constant function of  $(S^2)^3$  whose value  $w_2(K)$  is in  $\frac{1}{48}\mathbb{Z}$ .

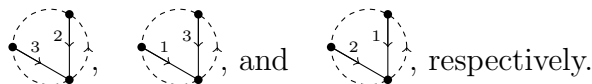
SKETCH OF PROOF: Let us show that the boundary

$$\sigma_*\left(S^2 \times G_{\mathfrak{S}_3}(\partial C(K; \mathfrak{S}_3))\right)$$

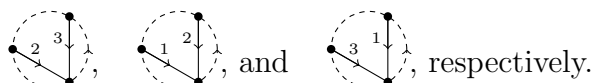
can be glued to the images of the faces  $c(v_i) = c(w)$  of  $C(K; \mathfrak{A})$  under the  $\iota_I \circ G_{\mathfrak{A}}$  or the  $\iota_I \circ (2\mathfrak{Z})_* \circ G_{\mathfrak{A}}$ , up to sign. For every permutation  $\sigma$  of  $\underline{3}$ , the boundary  $\sigma_*(S^2 \times G_{\mathfrak{A}}(\partial C(K; \mathfrak{A})))$  consists of the four faces



where the first one and the third one come with a coefficient  $-\frac{1}{24}$ , and the second one and the fourth one come with a coefficient  $\frac{1}{24}$ . The images of the (open) faces  $(c(v_1) = c(w))$ ,  $(c(v_2) = c(w))$ , and  $(c(v_3) = c(w))$  under  $G_{\mathfrak{A}}$  are



The images of the (open) faces  $(c(v_1) = c(w))$ ,  $(c(v_2) = c(w))$ , and  $(c(v_3) = c(w))$  under  $(2\mathfrak{Z})_* \circ G_{\mathfrak{A}}$  are



In order to obtain the images under the compositions of these maps by some  $\iota_I$ , we reverse the edge  $i$  when  $i \in I$ . Each of these faces appears twice (once for each orientation of the collapsed edge), with the same sign (since the antipodal map of  $S^2$  reverses the orientation) with a coefficient  $(-1)^{n(F)} \frac{1}{48}$ , where  $n(F)$  is the number of edges towards the bivalent vertex.

We leave the general discussion of signs to the reader, and we do not discuss all the faces of  $C(K; \mathfrak{A})$  in this sketch. Let us just mention that the image of the faces  $c(v_1) = c(v_2)$  is contained in the codimension-two subspace of  $(S^2)^3$ , for which at least two  $S^2$ -coordinates are equal or opposite. So these faces do not create walls and may be forgotten.  $\square$

The  $\sigma_*$  preserve the volume of  $(S^2)^3$ , and the  $\iota_I$  multiply the volume by  $(-1)^{|I|}$ . Therefore, the combination  $w_2(K)$  in Proposition 1.7 may also be written as

$$w_2(K) = \frac{1}{4} \int_{C(K; \mathfrak{A})} G_{\mathfrak{A}}^* (\wedge_{i=1}^2 p_i^* (\omega_{S^2})) - \frac{1}{3} \int_{C(K; \mathfrak{A})} G_{\mathfrak{A}}^* (\wedge_{i=1}^3 p_i^* (\omega_{S^2})).$$

Since  $w_2(\cdot)$  is valued in  $\frac{1}{48}\mathbb{Z}$ , and since  $w_2(K)$  varies continuously under an isotopy of  $K$ ,  $w_2(\cdot)$  is an isotopy invariant. Enore Guadagnini, Maurizio Martellini, and Mihail Mintchev [GMM90], and Dror Bar-Natan [BN95b] independently studied the invariant  $w_2(\cdot)$  under this integral form in 1990.



To prove its isotopy invariance, one can alternatively use Stokes' theorem to evaluate the variations of the integrals under a knot isotopy  $(t, z) \mapsto K_t(z)$ . In [BT94], Raoul Bott and Clifford Taubes used compactifications of the one-parameter configuration spaces  $\cup_{t \in [0,1]} C(K_t; \Gamma)$ , like those discussed above, to check invariance.

The above formulation of Proposition 1.7 presents the invariant  $w_2(K)$  as a discrete count of configurations of  $\bowtie$  and  $\bowtie$ , as Dylan Thurston [Thu99] and Sylvain Poirier [Poi02] first did independently.<sup>8</sup> For a generic triple  $(a, b, c)$  of  $(S^2)^3$ , we count the configurations of  $\bowtie$  for which the edge directions are a pair of noncolinear vectors in  $\{a, b, c, -a, -b, -c\}$ , and the configurations of  $\bowtie$  for which the edge directions are a triple of pairwise noncolinear vectors in  $\{a, b, c, -a, -b, -c\}$ , with some coefficients and some signs determined by the corresponding above local degrees.

Any knot of  $\mathbb{R}^3$  is obtained from the trivial knot by (isotopies and) a finite number of crossing changes  $\nearrow \rightarrow \searrow$ . So, in order to determine a real-valued knot invariant  $w$ , it suffices to know its value on the trivial knot and its variation  $w(\nearrow) - w(\searrow)$ , denoted by  $w(\bowtie)$ , under crossing change. (Here,  $\nearrow$  and  $\searrow$  represent knot diagrams that coincide outside a disk that they intersect as in the figure, and  $\bowtie$  represents a diagram that coincides with the former diagrams outside this disk.)

The variation  $w(\bowtie) = w(\nearrow) - w(\searrow)$  can be thought of as a discrete derivative of the invariant. The variation of this variation

$$\begin{aligned} w(\bowtie \bowtie) &= w(\nearrow \bowtie) - w(\searrow \bowtie) \\ &= w(\nearrow \nearrow) - w(\nearrow \searrow) - w(\searrow \nearrow) + w(\searrow \searrow) \end{aligned}$$

under a disjoint crossing change is thought of as a discrete second derivative of this invariant. In  $\mathbb{R}^3$ , a real-valued knot invariant  $w$  is actually determined by its value on the trivial knot and the discrete second derivative. (If the second derivative is zero, then the variation under a crossing change is independent of the knot, and it is the same as  $(w(\bowtie \bowtie) - w(\bowtie \searrow)) = 0$ .) In [GMM90], Enore Guadagnini, Maurizio Martellini, and Mihail Mintchev computed  $w_2(O) = -\frac{1}{24}$ . This is reproved in Examples 7.11 and Lemma 7.12. In [BN95b, §6.3], Dror Bar-Natan computed the above “discrete second derivative” for the invariant  $w_2$ , and he found:

$$w_2(\text{diagram with two crossings and dashed lines}) = 0 \quad \text{and} \quad w_2(\text{diagram with two crossings and dashed lines}) = 1,$$

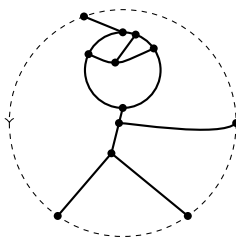
where the dashed lines indicate the connections inside  $K$  of the crossing strands. This allowed him to identify  $w_2$  with  $a_2 - \frac{1}{24}$ , where  $a_2(K) =$

<sup>8</sup>Michael Polyak and Oleg Viro obtained a similar result [PV01, Theorem 3.A, Section 3.5] in the setting of long knots of  $\mathbb{R}^3$ , with fewer involved gluings, and with induced combinatorial formulae in terms of knot diagrams.

$\frac{1}{2}\Delta''(K)(1)$  is half the second derivative of the Alexander polynomial of  $K$  at one, since  $a_2$  has the same discrete second derivative as  $w_2$  and  $a_2(O) = 0$ .

### 1.2.6 On other similar invariants of knots in $\mathbb{R}^3$

We can associate similar integrals over configuration space to every uni-trivalent graph  $\Gamma$  whose univalent vertices are ordered cyclically, as in the following figure:



We can also exhibit other similar combinations that provide isotopy invariants of knots in  $\mathbb{R}^3$ , as several authors, including Maxim Kontsevich [Kon94], Daniel Altschüler and Laurent Freidel [AF97], Dylan Thurston [Thu99], did. These invariants are all finite type invariants with respect to the following definition. An invariant valued in an abelian group is of *degree less than  $n$*  if all its discrete derivatives of order  $n$ , which generalize the previously studied discrete second derivative, vanish. A knot invariant is a *Vassiliev invariant* or a *finite type invariant* if it is of degree less than some integer  $n$ . (A precise definition is given in Section 6.1.) In [Kon93, BN95a], Dror Bar-Natan and Maxim Kontsevich proved the “fundamental theorem of Vassiliev invariants”, which determines the space of real-valued finite type invariants as the dual of an algebra  $\mathcal{A}$  generated by uni-trivalent graphs. Daniel Altschüler and Laurent Freidel [AF97]—and Dylan Thurston [Thu99] independently—constructed a “universal Vassiliev invariant”  $\mathcal{Z}$  of knots in  $\mathbb{R}^3$ , valued in  $\mathcal{A}$  such that, for any knot  $K$  of  $\mathbb{R}^3$ ,  $\mathcal{Z}(K)$  is a combination of classes of uni-trivalent graphs whose coefficients are integrals over corresponding configuration spaces.<sup>9</sup> Any real-valued finite type invariant may be expressed as  $\psi \circ \mathcal{Z}$  for some linear form  $\psi: \mathcal{A} \rightarrow \mathbb{R}$ . Therefore, every real-valued finite type knot invariant is a combination of integrals over configuration spaces associated to uni-trivalent graphs. It is still unknown whether finite type knot invariants distinguish all knots of  $\mathbb{R}^3$ , but many known polynomial invariants of knots of  $\mathbb{R}^3$ , such as the Alexander polynomial, the Jones polynomial, and the HOMFLYPT polynomial, factor through  $\mathcal{Z}$ .

<sup>9</sup>The original proof of the fundamental theorem of Vassiliev invariants relies on the construction of another—possibly equal—universal Vassiliev invariant called the *Kontsevich integral*.

### 1.2.7 On similar invariants of knots in other 3-manifolds

Recall that  $D_1 \times [0, 1]$  denotes the standard cylinder in  $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ , and that a *rational homology cylinder*  $\mathcal{C}$  is a compact oriented 3-manifold with the same boundary and the same rational homology as  $D_1 \times [0, 1]$  (i.e., the rational homology of a point).<sup>10</sup>

In this book, following ideas of Maxim Kontsevich [Kon94], Greg Kuperberg, and Dylan Thurston [KT99], we generalize  $\mathcal{Z}$  to links in more general 3-manifolds denoted by  $\check{R} = \check{R}(\mathcal{C})$ . These manifolds are constructed from  $\mathbb{R}^3$  by replacing  $D_1 \times [0, 1]$  by a rational homology cylinder  $\mathcal{C}$ . Such a more general 3-manifold  $\check{R}$  is called a *rational homology*  $\mathbb{R}^3$ . It looks like  $\mathbb{R}^3$  near  $\infty$ . The linking number can be defined as follows for a two-component link embedding  $J \sqcup K: S^1 \sqcup S^1 \rightarrow \check{R}$ .

Two submanifolds  $A$  and  $B$  in a manifold  $M$  are *transverse* if at each intersection point  $x \in A \cap B$ , we have  $T_x M = T_x A + T_x B$ . If two transverse oriented submanifolds  $A$  and  $B$  in an oriented manifold  $M$  are of *complementary dimensions* (i.e., if the sum of their dimensions is the dimension of  $M$ ), then the *sign of an intersection point* is  $+1$  if  $T_x M = T_x A \oplus T_x B$  as oriented vector spaces. Otherwise, the sign is  $-1$ . If  $A$  and  $B$  are compact, and if  $A$  and  $B$  are of complementary dimensions in  $M$ , then their *algebraic intersection* is the sum of the signs of the intersection points, it is denoted by  $\langle A, B \rangle_M$ .

When  $K$  bounds a compact oriented embedded surface  $\Sigma_K$  in  $\check{R}$  transverse to  $J$ , the *linking number* of  $J$  and  $K$  is the algebraic intersection number  $\langle J, \Sigma_K \rangle_{\check{R}}$  of  $J$  and  $\Sigma_K$  in  $\check{R}$ . In general, there is an oriented surface  $\Sigma_{nK}$  immersed in  $\check{R}$  whose (oriented!) boundary is a positive multiple  $nK$  of  $K$ , and  $lk(J, K) = \frac{1}{n} \langle J, \Sigma_{nK} \rangle_{\check{R}}$ .

For two-component links in  $\mathbb{R}^3$ , this definition coincides with Definition 1.1 of the Gauss linking number. See Proposition 2.9.

In the more general setting of a rational homology  $\mathbb{R}^3$ , instead of counting uni-trivalent graphs whose edge directions belong to a finite set of directions, we use the notion of *propagator* defined in Chapter 3. A propagator is a rational combination of oriented compact 4-manifolds in a suitable compactification  $C_2(R)$  (defined in Section 3.2) of the configuration space

$$\check{C}_2(R) = \{(x, y) \in \check{R}^2 : x \neq y\},$$

---

<sup>10</sup>This homological condition can be rephrased as “The (compact oriented) 3-manifold  $\mathcal{C}$  is connected, and every knot embedding  $K$  in  $\mathcal{C}$  bounds a rational chain in  $\mathcal{C}$ .” Let  $N(K)$  be a compact tubular neighborhood of  $K$  in  $\mathcal{C}$ , and let  $\mathring{N}(K)$  denote its interior. The latter condition is equivalent to the existence of a compact oriented surface in  $\mathcal{C} \setminus \mathring{N}(K)$  whose (oriented!) boundary is a disjoint union of curves in  $\partial N(K)$  that does not bound a compact oriented surface in  $N(K)$ .

which shares many properties with our *model propagator*

$$p_{S^2}^{-1}(X) = \overline{\{(x, x + tX) : x \in \mathbb{R}^3, t \in ]0, +\infty[ \}},$$

for  $X \in S^2$  (and  $\check{R} = \mathbb{R}^3$ ).

With these model propagators  $p_{S^2}^{-1}(X)$ , the direction of a configured edge  $(c(x), c(y))$  is  $X$  if and only if  $(c(x), c(y))$  belongs to the propagator  $p_{S^2}^{-1}(X)$ . More general propagators produce similar codimension-two constraints on configurations. They allow us to count uni-trivalent graphs whose configured edges belong to a finite set of propagators, with signs, as above.

In general, the boundary of a propagator of  $C_2(R)$  is in the boundary of  $C_2(R)$ , and, for any two-component link embedding  $J \sqcup K : S^1 \sqcup S^1 \rightarrow \check{R}$ , the algebraic intersection of  $J \times K \subset \check{C}_2(R)$  with a propagator in  $C_2(R)$  is the linking number of  $J$  and  $K$ . (Note that the linking number of two knots in  $\mathbb{R}^3$  is indeed this algebraic intersection with a model propagator.)

## 1.2.8 Morse propagators

Let us show examples of propagators in a general rational homology  $\mathbb{R}^3$ .

View a rational homology  $\mathbb{R}^3$  as the union of two genus  $g$  handlebodies as in Figure 1.9, where the two pieces are glued to each other by an a priori nontrivial diffeomorphism of  $\partial H_a$ .<sup>11</sup>

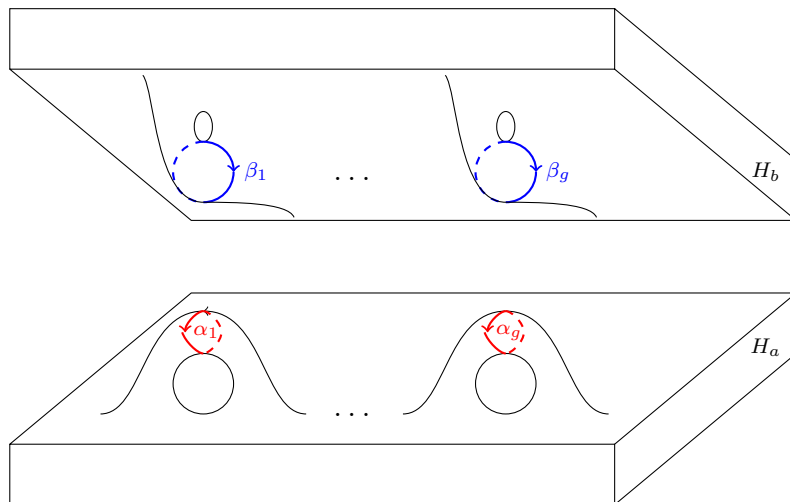


Figure 1.9:  $H_a$  and  $H_b$

<sup>11</sup>Unlike the handlebodies in the rest of this book, which are as in Section 1.1, the handlebodies of this section are not compact.

The handlebody  $H_a$  has  $g$  arbitrarily oriented meridian disks  $D(\alpha_i)$  centered at  $a_i$ , for  $i$  in  $\underline{g} = \{1, 2, \dots, g\}$ . The handlebody  $H_b$  has  $g$  arbitrarily oriented meridian disks  $D(\beta_j)$  centered at  $b_j$ , for  $j \in \underline{g}$ . The topology of  $\check{R}$  is determined by the curves  $\alpha_i = \partial D(\alpha_i)$  and  $\beta_j = \partial D(\beta_j)$  in the surface  $\partial H_a$ . The data  $(\partial H_a, (\alpha_i)_{i \in \underline{g}}, (\beta_j)_{j \in \underline{g}})$  is called a *Heegaard diagram*. From such data, we construct

- a *Morse function*  $f: \check{R} \rightarrow \mathbb{R}$  such that  $\partial H_a = f^{-1}(1/2)$  and the *critical points of  $f$*  (the points at which the derivative of  $f$  vanishes) are the  $a_i$ , which have index one and are in  $f^{-1}(1/3)$ , and the  $b_j$ , which have index two and are in  $f^{-1}(2/3)$ ,
- a gradient vector field  $\nabla: \check{R} \rightarrow T\check{R}$  associated to  $f$  and some metric  $\mathfrak{g}$  such that

$$T_x f(y \in T_x \check{R}) = \langle \nabla(x), y \rangle_{\mathfrak{g}},$$

and

- an associated *gradient flow*  $\phi: \mathbb{R} \times \check{R} \rightarrow \check{R}$  such that  $\phi(0, \cdot)$  is the Identity map and  $\frac{\partial}{\partial t} \phi(t, x)_{(u,x)} = \nabla(\phi(u, x))$ .<sup>12</sup>

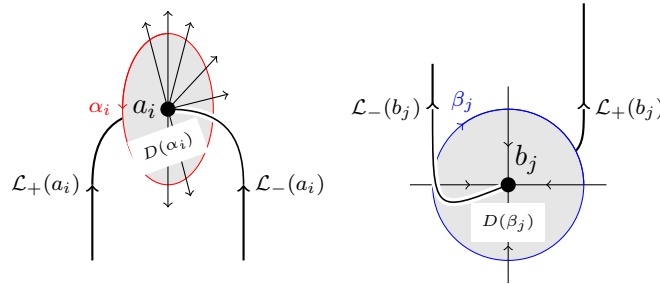
Let  $\mathcal{S}$  be the set of critical points of  $f$ . The *flow lines* of the flow  $\phi$  are the  $\phi(\mathbb{R} \times \{y\})$ , for the  $y$  in the complement  $\check{R} \setminus \mathcal{S}$  in  $\check{R}$  of  $\mathcal{S}$ . For the standard height function  $f_0$  and the standard metric of  $\mathbb{R}^3$ , the associated flow is

$$\left( (\phi_t = \phi(t, \cdot)): x \mapsto x + t\vec{N} \right),$$

where  $\vec{N} = (0, 0, 1)$ . Our rational homology  $\mathbb{R}^3$  (equipped with the metric  $\mathfrak{g}$ ) is assumed to coincide with  $\mathbb{R}^3$  (equipped with its standard metric) outside  $D_1 \times [0, 1]$ , and our Morse function (can be and) is assumed to coincide with  $f_0$  outside  $D_1 \times [0, 1]$ .

As a set, the complement  $\check{R} \setminus \mathcal{S}$  of the set of critical points is the disjoint union of the flow lines diffeomorphic to  $\mathbb{R}$ , which behave as follows. There are two flow lines  $\mathcal{L}_+(a_i)$  and  $\mathcal{L}_-(a_i)$  starting as vertical lines and ending at  $a_i$  that approach  $a_i$  at  $+\infty$ , as in Figure 1.10. (They start as vertical lines  $\{.\} \times ]-\infty, 0[.$ ) The closure in  $\check{R}$  of their union is a line  $\mathcal{L}(a_i)$ , called the *descending manifold* of  $a_i$ . It is oriented so that its algebraic intersection with  $D(\alpha_i)$  is 1. (This is not consistent with the bottom–top orientation of one of the flow lines. If the flow line with the orientation of the positive normal to  $D(\alpha_i)$  is  $\mathcal{L}_+(a_i)$ , then  $\mathcal{L}(a_i) = \overline{\mathcal{L}_+(a_i)} \cup (-\mathcal{L}_-(a_i))$ .) There are two flow lines  $\mathcal{L}_+(b_j)$  and  $\mathcal{L}_-(b_j)$  that approach  $b_j$  at  $-\infty$ . The closure of

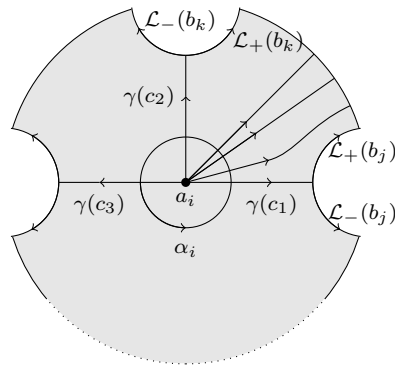
<sup>12</sup>The definition of this flow is justified in [Spi79, Chapter5], for example.

Figure 1.10:  $\mathcal{L}_+(a_i)$ ,  $\mathcal{L}_-(a_i)$ ,  $\mathcal{L}_+(b_j)$ ,  $\mathcal{L}_-(b_j)$ 

their union is a line  $\mathcal{L}(b_j)$ , which is called the *ascending manifold* of  $b_j$ . It is oriented so that its algebraic intersection with  $D(\beta_j)$  is 1.

The closure of the union of the flow lines that approach  $a_i$  at  $-\infty$  is called the *ascending manifold* of  $a_i$ . It is denoted by  $\mathcal{A}_i$ . Its intersection with  $H_a$  is  $D(\alpha_i)$ , and it is oriented like  $D(\alpha_i)$ . The closure of the union of the flow lines that approach  $b_j$  at  $+\infty$  is called the *descending manifold* of  $b_j$ . It is denoted by  $\mathcal{B}_j$ . Its intersection with  $H_b$  is  $D(\beta_j)$ , and it is oriented like  $D(\beta_j)$ . The ascending manifold  $\mathcal{A}_i$  is an immersion of Figure 1.11, which restricts to its interior as an embedding, where the flow lines  $\gamma(c_k)$  are flow lines through crossings  $c_k$  of  $\alpha_i \cap \beta_j$ , which approach  $a_i$  near  $-\infty$ , and  $b_j$  near  $+\infty$ . A figure for  $\mathcal{B}_j$  is obtained by reversing the arrows and changing  $a_i$  to  $b_j$ .

Except for the flow lines of the descending manifolds of the  $a_i$  and the flow lines of the ascending manifolds of the  $b_j$ , each flow line intersects  $\partial H_a$  once, transversally, with a positive sign.

Figure 1.11: The interior of  $\mathcal{A}_i$ .

The flow lines not in the above descending or ascending manifolds begin as vertical half-lines  $x \times ]-\infty, 0[$ . They end as vertical half-lines  $y \times ]1, \infty[$  for some  $x, y$  in  $\mathbb{R}^2$ . Except for the critical points, every point has a neighborhood

diffeomorphic to a cube  $]0, 1[^3$  such that, with the induced identification, the flow maps  $(t, x)$  to  $x + t\vec{N}$ , for any  $(t, x)$  in  $\mathbb{R} \times ]0, 1[^3$  such that  $x + t\vec{N} \in ]0, 1[^3$ .

In [Les15a, Theorem 4.2], Greg Kuperberg and I constructed a propagator from the gradient flow  $(\phi_t = \phi(t, \cdot))$  of a Morse function  $f$  without minima and maxima as above, as follows. Let  $P_\phi$  denote the closure in  $C_2(R)$  of  $\{(x, \phi_t(x)) : x \in \check{R} \setminus \mathcal{S}, t \in ]0, +\infty[ \}$ . Note that when  $(\phi_t : x \mapsto x + t\vec{N})$  is the flow associated to the standard height function  $f_0$  of  $\mathbb{R}^3$ ,  $P_\phi = p_{S^2}^{-1}(\vec{N})$  is one of our model propagators. Let

$$[\mathcal{J}_{ji}]_{(j,i) \in \{1, \dots, g\}^2} = \left[ \langle \alpha_i, \beta_j \rangle_{\partial H_a} \right]^{-1}$$

be the inverse matrix of the matrix of the algebraic intersection numbers  $\langle \alpha_i, \beta_j \rangle_{\partial H_a}$ . (This matrix is invertible because  $\check{R}$  is a rational homology  $\mathbb{R}^3$ .)

Let  $((\mathcal{B}_j \times \mathcal{A}_i) \cap C_2(R))$  denote the closure of  $((\mathcal{B}_j \times \mathcal{A}_i) \cap (\check{R}^2 \setminus \text{diagonal}))$  in  $C_2(R)$ , then

$$P(f, \mathfrak{g}) = P_\phi + \sum_{(i,j) \in \{1, \dots, g\}^2} \mathcal{J}_{ji} \left( (\mathcal{B}_j \times \mathcal{A}_i) \cap C_2(R) \right)$$

is an example of a propagator.

Pick four small generic perturbations  $P_1, P_2, P_3$ , and  $P_4$  of such a propagator (or of more general propagators as precisely defined in Section 3.3). The invariant  $w_2$  can be extended to knots  $K$  in  $\check{R}$  as follows. Let  $\Gamma$  be one of the graphs  $\mathfrak{X}, \mathfrak{Y}$ , or  $\mathfrak{Z}$ . Orient the edges of  $\Gamma$ . Number them by the data of an injection  $j_E$  from the set  $E(\Gamma)$  of the (plain) edges of  $\Gamma$  to  $\underline{4}$ . Thus, we can count the configurations of  $\Gamma$  such that the configured oriented edge numbered by  $i$  (viewed as the ordered pair of its ends) is in  $P_i$ , with signs precisely defined. Denote the average over the choices of such edge-orientations and numberings by  $I_a(K, \Gamma, (P_1, P_2, P_3, P_4))$ . (When averaging, we divide by the number  $2^{|E(\Gamma)|}$  of edge orientations and by the number  $\frac{4!}{(4-|E(\Gamma)|)!}$  of numberings.) Then

$$\begin{aligned} w_2(K) = I_a \left( K, \mathfrak{X}, (P_1, P_2, P_3, P_4) \right) &- I_a \left( K, \mathfrak{Y}, (P_1, P_2, P_3, P_4) \right) \\ &- 2I_a \left( K, \mathfrak{Z}, (P_1, P_2, P_3, P_4) \right) \end{aligned}$$

does not depend on the chosen propagators. Furthermore, if  $K$  is null-homologous, then  $w_2(K)$  is again  $\frac{1}{2}\Delta''(K)(1) - \frac{1}{24}$ , in this more general setting, as we prove in Theorem 18.43.<sup>13</sup> Note that when we compute  $w_2(K \subset \mathbb{R}^3)$  with model propagators,  $I_a(K, \mathfrak{Z}, (P_1, P_2, P_3, P_4))$  vanishes because the double edge gives contradictory constraints.

<sup>13</sup>In [Let22], David Leturcq completely expresses the Alexander polynomial of long null-homologous knots in  $\mathbb{Q}$ -spheres in terms of similar counts of configurations.

### 1.2.9 More about the contents of the book

The cited Altschüler–Freidel universal Vassiliev invariant of knots in  $\mathbb{R}^3$  also extends to knots in a rational homology  $\mathbb{R}^3$ . This book describes this extended invariant  $\mathcal{Z}$  using more general (and precisely defined) propagators. Kenji Fukaya proposed a way of counting configurations in a rational homology  $\mathbb{R}^3$  in [Fuk96]. Other authors, including Tadayuki Watanabe [Wat18a] further studied his approach and made it rigorous. In this book, we can view the resulting way of counting as a particular way of counting with the above “Morse propagators” associated to Heegaard diagrams. These Morse propagators are examples of the general “propagating chains” described in Chapter 3. Counts of configurations as above also yield invariants of the ambient rational homology  $\mathbb{R}^3$  and of its one-point compactification. This compactification is a *rational homology 3-sphere*, i.e., a connected oriented closed 3-manifold, where knots have a nontrivial multiple that bounds an immersed oriented compact surface.<sup>14</sup> In [Mou12], Delphine Moussard developed a theory of finite type invariants for rational homology 3-spheres. Her theory involves surgery operations called *rational Lagrangian-preserving surgeries* instead of crossing changes. We describe these rational Lagrangian-preserving surgeries, which replace a piece of a manifold by another such, in Subsection 1.3.2 below. The above counts of configurations yield a universal finite type invariant  $\mathcal{Z}$  of rational homology 3-sphere, with respect to Mousard’s theory. The universality of  $\mathcal{Z}$  follows from surgery formulae proved in [Les04b].

*Integer homology 3-spheres* are connected oriented closed 3-manifolds where knots bound an embedded oriented compact surface, as in the standard 3-sphere  $S^3$ . The invariant  $\mathcal{Z}$  also restricts to a universal finite type invariant for integer homology 3-spheres, with respect to the Ohtsuki theory of finite type invariants [Oht96] and other equivalent theories described in [GGP01], as first shown by Greg Kuperberg and Dylan Thurston in [KT99].

Let us say a little more about the contents of this book, which is mostly self-contained. We describe the background material in the appendices. Unlike in this informal introduction, we will give details and precise statements and carefully check all the assertions. Our general invariant  $Z$  is an infinite series of independent nontrivial invariants. To define it, we must equip any rational homology  $\mathbb{R}^3$  with a parallelization  $\tau: \check{R} \times \mathbb{R}^3 \rightarrow T\check{R}$ . We describe the space of suitable parallelizations up to homotopy and the associated Pontrjagin numbers in Chapter 5.

This book has four parts. In its present first part, we first state conven-

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<sup>14</sup>A manifold is said to be *closed* if it is compact and connected, and if its boundary is empty.



tions and known facts about 3-manifolds. Then we precisely define propagating chains associated with parallelizations and discuss the invariant  $\Theta$  of parallelized rational homology 3-spheres. This invariant  $\Theta$  is associated to the graph  $\Theta$ . It is the algebraic intersection of three transverse propagating chains. We explain how to get rid of the dependence of the parallelization of the rational homology 3-spheres with the help of relative Pontrjagin numbers to get an invariant of (unparallelized) rational homology 3-spheres in Section 4.3. This invariant is (six times) the Casson–Walker invariant of rational homology spheres. Andrew Casson and Kevin Walker originally defined it as a “count” of conjugacy classes of  $SU(2)$ -representations of the fundamental group of these manifolds [AM90, GM92, Mar88, Wal92].

Our general invariant  $\mathcal{Z}^f$  is an invariant of parallelized links in rational homology 3-spheres.<sup>15</sup> We define it in the second part of the book. It is valued in vector spaces generated by uni-trivalent graphs like  $\Theta$ ,  $\mathfrak{A}$ , and  $\mathfrak{B}$ . We describe these spaces of diagrams and their rich structures, which help formulate the properties of the invariants, in Chapter 6. We present the general definition of  $\mathcal{Z}^f$  for links in Chapter 7. We first give it in terms of integrals rather than in terms of discrete counts because the results are easier to write and prove in the world of differential forms, where no genericity hypotheses are required. We prove the consistency of our definition and the first properties of  $\mathcal{Z}^f$  in Chapters 9, 10, and 11, after the needed study of the compactifications of the involved configuration spaces in Chapter 8.

To compute and use an invariant of links or manifolds, it is interesting to cut links or manifolds into elementary pieces and understand how one can recover the invariant from the invariants of the pieces. In the third part of this book, we achieve this task with elementary pieces that are *tangles* in rational homology cylinders as in Figure 1.12. These tangles are cobordisms between planar configurations of points. We define them precisely in Section 13.1. They may be composed in many ways, horizontally, vertically, and by insertions in tubular neighborhoods of other tangle representatives. Such insertions are called *cablings*. We generalize  $\mathcal{Z}^f$  to tangles and describe the properties of our generalized  $\mathcal{Z}^f$  under the mentioned compositions in the third part of this book.

As already mentioned, a fundamental property of  $\mathcal{Z}^f$  is its universality among finite type invariants. For the restriction of  $\mathcal{Z}^f$  to links in  $\mathbb{R}^3$ , universality refers to the Vassiliev theory of finite type invariants based on crossing changes. For the restriction of  $\mathcal{Z}^f$  to rational homology spheres, universality refers to the Moussard theory based on rational Lagrangian-preserving surgeries. The proofs of universality involve computations of iterated discrete

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<sup>15</sup>*Framed links* or *parallelized links* are links equipped with a parallel (up to isotopy).

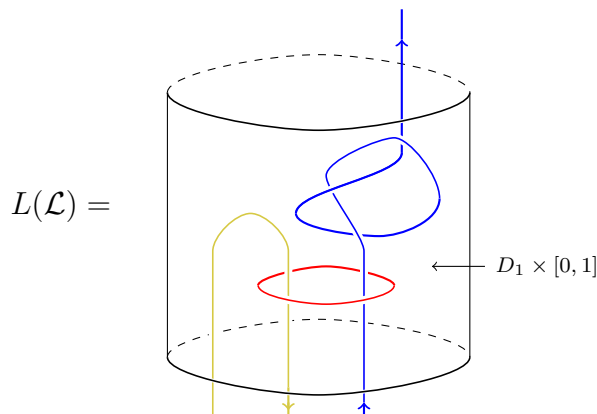


Figure 1.12: A tangle representative in  $D_1 \times [0, 1]$

derivatives of  $\mathcal{Z}^f$  in the same spirit as the Bar-Natan result recalled in the end of Subsection 1.2.5. The book’s fourth part presents these computations and some consequences for the general invariant  $\mathcal{Z}^f$  of framed tangles in rational homology cylinders.

## 1.3 A quicker introduction

This quicker introduction, for experienced topologists, is independent of the first one. Beginners can read it after the warm-up of the slower one. Here, we describe the invariant  $\mathcal{Z}$  of  $n$ -component links  $L$  in rational homology 3-spheres  $R$  studied in this book, more precisely. We also specify some notions vaguely introduced in the slow introduction of Section 1.2, and we say more on the mathematical landscape around  $\mathcal{Z}$ .

### 1.3.1 On the construction of $\mathcal{Z}$

The invariant  $\mathcal{Z}(L) = (\mathcal{Z}_k(L))_{k \in \mathbb{N}}$  of an  $n$ -component link  $L$  in a rational homology 3-sphere  $R$  is valued in a graded space generated by uni-trivalent graphs. Its degree  $k$  part is a sum

$$\mathcal{Z}_k(L) = \sum_{\Gamma} \mathcal{Z}_{\Gamma}(L) [\Gamma],$$

running over such graphs  $\Gamma$  with  $2k$  vertices. The coefficient  $\mathcal{Z}_{\Gamma}(L)$  “counts” embeddings of  $\Gamma$  in  $R$  mapping the univalent vertices of  $\Gamma$  to  $L$ , in a sense explained in the book. Let us slightly specify that sense.

For technical reasons, we remove a point  $\infty$  from our rational homology 3-spheres  $R$  to transform them into open manifolds  $\check{R}$ . When  $R$  is the standard sphere  $S^3$ , the punctured  $\mathbb{Q}$ -sphere  $\check{R}$  is  $\mathbb{R}^3$ .

Let  $\Delta(\check{R}^2)$  denote the diagonal of  $\check{R}^2$ . Following William Fulton, Robert MacPherson [FM94], Maxim Kontsevich [Kon94], Scott Axelrod, Isadore Singer [AS94, Section 5], and others, we will introduce a suitable smooth compactification  $C_2(R)$  (with boundary and ridges) of  $\check{R}^2 \setminus \Delta(\check{R}^2)$  such that the map

$$p_{S^2}: \begin{array}{ccc} (\mathbb{R}^3)^2 \setminus \Delta((\mathbb{R}^3)^2) & \rightarrow & S^2 \\ (x, y) & \mapsto & \frac{1}{\|y-x\|}(y-x) \end{array}$$

extends to  $C_2(S^3)$ . We will introduce a notion of *propagating chain* and the dual notion of *propagating form* for  $R$ . When  $R = S^3$ , for any  $X \in S^2$ , the submanifold  $p_{S^2}^{-1}(X)$  of  $C_2(S^3)$  is an example of a propagating chain. For any 2-form  $\omega_S$  on  $S^2$  such that  $\int_{S^2} \omega_S = 1$ , the form  $p_{S^2}^*(\omega_S)$  is an example of a propagating form. The propagating chain  $p_{S^2}^{-1}(X)$  is a *model propagating chain*. The propagating form  $p_{S^2}^*(\omega_S)$  is a *model propagating form*. In general, a propagating chain is a 4-dimensional rational chain (i.e., a finite rational combination of oriented compact 4-manifolds with possible ridges) of  $C_2(R)$ , while a propagating form is a closed 2-form on  $C_2(R)$ . Both have to satisfy some conditions on the boundary of  $C_2(R)$ , which make them share sufficiently many properties with our model propagating chains or forms.

In particular, for any propagating form  $\omega$ , for any propagating chain  $P$ , and for any two-component link  $(J, K): S^1 \sqcup S^1 \rightarrow \check{R}$ , we have

$$\int_{J \times K \subset C_2(R)} \omega = \langle J \times K, P \rangle_{C_2(R)} = lk(J, K),$$

where  $\langle \cdot, \cdot \rangle_{C_2(R)}$  stands for the algebraic intersection in  $C_2(R)$ , and  $lk$  is the linking number in  $R$ .

Propagating forms and propagating chains are both called *propagators* when their nature is clear from the context. The above equalities tell us in which way “propagators represent the linking form”.

A *Jacobi diagram*  $\Gamma$  on  $\sqcup_{i=1}^n S^1$  is a uni-trivalent graph  $\Gamma$  equipped with an isotopy class of injections from its set  $U(\Gamma)$  of univalent vertices into the domain  $\sqcup_{i=1}^n S^1$  of a link  $L$ . In the figures, we represent the domain of  $L$  by dashed circles and put the univalent vertices of  $\Gamma$  on their images under an injection of the given isotopy class as in Figure 1.13. Let  $V(\Gamma)$ ,  $T(\Gamma)$ , and  $E(\Gamma)$  respectively denote the set of vertices, trivalent vertices, and edges of  $\Gamma$ . The *configuration space*  $\check{C}(R, L; \Gamma)$  is the set of injections from  $V(\Gamma)$  to  $\check{R}$  that map the set  $U(\Gamma)$  of univalent vertices of  $\Gamma$  to  $L$  and induce the given

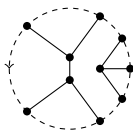


Figure 1.13: A (plain) Jacobi diagram on (the dashed)  $S^1$

isotopy class of injections. It is an open submanifold of  $\check{R}^{T(\Gamma)} \times L^{U(\Gamma)}$ . Assume that  $\Gamma$  has no *looped edge* like  $\rightarrow \bigcirc$  and that its edges are oriented. Then each edge  $e$  of  $\Gamma$  provides a natural restriction map  $p(\Gamma, e)$  from  $\check{C}(R, L; \Gamma)$  to  $\check{R}^2 \setminus \Delta(\check{R}^2)$ . When propagating forms  $\omega(e)$  are associated to the edges, this allows one to define a real number

$$I\left(R, L, \Gamma, (\omega(e))_e\right) = \int_{\check{C}(R, L; \Gamma)} \bigwedge_{e \in E(\Gamma)} p(\Gamma, e)^*(\omega(e)).$$

Similarly and dually, when propagating chains  $P(e)$  in general position are associated to the edges, one can define a rational number  $I(R, L, \Gamma, (P(e))_e)$  as the algebraic intersection of the codimension-two chains  $p(\Gamma, e)^{-1}(P(e))$  in  $\check{C}(R, L; \Gamma)$ .

For example, the Jacobi diagram  $\Gamma$  pictured as  $i \xrightarrow{\circ} \bullet \rightarrow \bullet \xrightarrow{\circ} j$ , is an oriented edge  $e$  from a univalent vertex that must go to the component  $K_i$  of  $L$  to another univalent vertex, which must go to another component  $K_j$  of  $L$ . The associated configuration space  $\check{C}(R, L; \Gamma)$  is  $K_i \times K_j$ . According to the given property of our propagators, we have

$$I\left(R, L, i \xrightarrow{\circ} \bullet \rightarrow \bullet \xrightarrow{\circ} j, \omega(e)\right) = \int_{K_i \times K_j \subset C_2(R)} \omega(e) = lk(K_i, K_j)$$

and

$$I\left(R, L, i \xrightarrow{\circ} \bullet \rightarrow \bullet \xrightarrow{\circ} j, P(e)\right) = \langle K_i \times K_j, P(e) \rangle_{C_2(R)} = lk(K_i, K_j)$$

for any propagating form  $\omega(e)$  and for any propagating chain  $P(e)$ .

As another example, consider the Jacobi diagram  $\Theta$  with two trivalent vertices and three edges  $e_1, e_2$ , and  $e_3$  from one vertex to the other. When  $R$  is a  $\mathbb{Z}$ -sphere, we will show how one can choose propagators  $P(e_i)$  and  $\omega(e_i)$  so that we have

$$I\left(R, \emptyset, \Theta, (\omega(e))_{e \in \{e_1, e_2, e_3\}}\right) = I\left(R, \emptyset, \Theta, (P(e))_{e \in \{e_1, e_2, e_3\}}\right) = 6\lambda_{CW}(R),$$

where  $\lambda_{CW}$  is the Casson invariant normalized as in [AM90, GM92, Mar88]. We can choose the above propagating forms so that  $\omega(e_1) = \omega(e_2) = \omega(e_3)$ .

In particular, the Casson invariant, which may be written as

$$\lambda_{CW}(R) = \frac{1}{6} \int_{C_2(R)} \omega(e_1)^3,$$

may be viewed as a “cube of the linking number”.

In general, when  $R$  is a  $\mathbb{Z}$ -sphere, one can choose a propagating form  $\omega$  for  $R$  and set  $\omega(e) = \omega$  for all edges  $e$  of Jacobi diagrams so that the real coefficient  $\mathcal{Z}_\Gamma(L)$  in “ $\mathcal{Z}(L) = \sum_\Gamma \mathcal{Z}_\Gamma(L) [\Gamma]$ ” is the product of  $I(R, L, \Gamma, (\omega)_e)$  and a constant which depends only on  $\Gamma$ . The coefficient  $\mathcal{Z}_\Gamma(L)$  can alternatively be obtained by averaging some  $I(R, L, \Gamma, (P(e))_e)$  over ways of equipping edges of  $\Gamma$  by propagating chains in a fixed set of generic propagating chains, and over ways of orienting the edges of  $\Gamma$ . The coefficients  $\mathcal{Z}_\Gamma(L)$  depend on propagator choices, but relations among Jacobi diagrams in the target space  $\mathcal{A}(\sqcup_{i=1}^n S^1)$  of  $\mathcal{Z}$ —generated by Jacobi diagrams—ensure that  $\mathcal{Z}(L)$  is an isotopy invariant. The invariant  $\mathcal{Z}$  may be thought of as a series of higher-order linking invariants.

The definition of  $\mathcal{Z}$  that is presented here is a generalization of the definition that I explained in detail for  $\mathbb{Q}$ -spheres in the unsubmitted preprint [Les04a], which was inspired by [KT99] and discussions with Dylan Thurston in Kyoto in 2001.

The present definition also includes links in  $\mathbb{Q}$ -spheres. Most of the additional arguments involved in the construction for links already appear in many places. We repeat them to make the book as self-contained as possible. We present many variants of the definitions and make them as flexible as possible because the flexibility has proved useful in many generalizations and applications of these constructions, such as equivariant constructions in [Les11, Les13], or the recent explicit computations of integrals over configuration spaces by David Leturcq [Let23, Let22], in addition to the applications presented in this book. David Leturcq obtained an expression of the Alexander polynomial of knots in  $\mathbb{Q}$ -spheres in terms of such integrals. His expression identifies some combinations of integrals over configuration spaces with coefficients of the Alexander polynomial.

With our flexible definition of a propagating chain, there is a natural propagating chain associated to a generic Morse function on a punctured  $\mathbb{Q}$ -sphere, and to a generic metric, as described in Subsection 1.2.8. The main part of such a propagator is the space of pairs of points on a gradient line such that the second point is after the first one. Greg Kuperberg and I constructed such a Morse propagator in [Les15a] for Morse functions without minima or maxima. Independent work of Tadayuki Watanabe [Wat18a] allows one to generalize these propagators to any Morse function. Thus, up to some corrections,  $\mathcal{Z}$  counts embeddings of graphs whose edges embed in gradient

lines of Morse functions as in a Fukaya article [Fuk96]. Tadayuki Watanabe has used similar constructions in his recent construction of exotic elements in the fundamental group of the group of diffeomorphisms of  $S^4$  [Wat18b]. This book contains a framework to study these questions precisely.<sup>16</sup>

We will also show how the construction of  $\mathcal{Z}$  extends to tangles in rational homology cylinders so that  $\mathcal{Z}$  extends to a functor  $\mathcal{Z}^f$  on a category of framed tangles with many important properties. These properties of the functor  $\mathcal{Z}^f$  are stated in Theorem 13.12, one of this book's main original theorems. They provide tools to reduce the computation of  $\mathcal{Z}^f$  to its evaluation at elementary pieces.

### 1.3.2 More mathematical context

**Finite type invariants.** The *finite type invariant* concept for knots was introduced in the 90's in order to classify knot invariants, with the work of Victor Vassiliev, Mikhail Goussarov, and Dror Bar-Natan, shortly after the birth of numerous quantum knot invariants, described by Vladimir Turaev in [Tur10]. Tomotada Ohtsuki extended this very useful concept to 3-manifold invariants [Oht96]. See also [Oht02]. Theories of finite type invariants in dimension 3 are defined from a set  $\mathcal{O}$  of operations on links or 3-manifolds. In the case of links in  $\mathbb{R}^3$ ,  $\mathcal{O}$  is the set  $\mathcal{O}_V$  of crossing changes  $\nearrow \leftrightarrow \searrow$ . The variation of an invariant  $\lambda$  under an operation of  $\mathcal{O}$  may be thought of as a discrete derivative. When  $k$  independent operations  $o_1, \dots, o_k$  on a pair  $(R, L)$  consisting of a link  $L$  in a  $\mathbb{Q}$ -sphere  $R$  are given, for a part  $I$  of  $\{1, \dots, k\}$  with cardinality  $|I|$ , the pair  $(R, L)((o_i)_{i \in I})$  is the pair obtained from  $(R, L)$  by applying the operations  $o_i$  for  $i \in I$ . Then the alternate sum

$$\sum_{I \subseteq \{1, \dots, k\}} (-1)^{|I|} \lambda((R, L)((o_i)_{i \in I}))$$

may be thought of as the  $k^{\text{th}}$  derivative of  $\lambda$  with respect to  $\{o_1, \dots, o_k\}$  at  $(R, L)$ . An *invariant of degree at most  $k$*  with respect to  $\mathcal{O}$  is an invariant whose degree  $k + 1$  derivatives vanish. A *finite type invariant* with respect to  $\mathcal{O}$  is an invariant that is of degree at most  $k$  for some positive integer  $k$ . Finite type invariants of links in  $\mathbb{R}^3$  with respect to the set of crossing changes are called *Vassiliev invariants*.

In this case of links in  $\mathbb{R}^3$ , Daniel Altschüler and Laurent Freidel [AF97] proved that the invariant  $\mathcal{Z}$  described in this book is a *universal Vassiliev invariant*, meaning that all real-valued Vassiliev invariants of links in  $\mathbb{R}^3$

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<sup>16</sup>We mainly develop the framework for tangles in dimension 3, but many arguments can be easily adapted in higher dimensions. See [Wat18b, Let21, Let23], for example.

factor through  $\mathcal{Z}$ . Since all the quantum invariants of [Tur10] can be viewed as sequences of finite type invariants,  $\mathcal{Z}$  also contains all these invariants such as the Jones polynomial, its colored versions, the HOMFLY polynomial... Dylan Thurston proved similar universality results in [Thu99] independently. He also showed that  $\mathcal{Z}$  is rational. Further substantial work of Sylvain Poirier in [Poi00] allowed me to identify the invariant  $\mathcal{Z}$  with the famous Kontsevich integral of links in  $\mathbb{R}^3$ —described in [BN95a], in [Oht02, Chapter 6], and in [CDM12] by Sergei Chmutov, Sergei Duzhin, and Jacob Mostovoy—up to a change of variables described in [Les02] in terms of an “anomaly”, which is sometimes called the *Bott–Taubes anomaly*.

Let us now describe operations on 3-manifolds. The boundary  $\partial A$  of a genus  $g$   $\mathbb{Q}$ -handlebody is a closed oriented genus  $g$  surface. The *Lagrangian*  $\mathcal{L}_A$  of a compact 3-manifold  $A$  is the kernel of the map induced by the inclusion from  $H_1(\partial A; \mathbb{Q})$  to  $H_1(A; \mathbb{Q})$ . (In Figure 1.1 of  $H_g$ , the Lagrangian of  $H_g$  is freely generated by the classes of the curves  $a_i$ .)

An *integral (resp. rational) Lagrangian-Preserving (or LP) surgery* ( $A'/A$ ) is the replacement of an integral (resp. rational) homology handlebody  $A$  embedded in the interior of a 3-manifold  $M$  by another such  $A'$  whose boundary  $\partial A'$  is identified with  $\partial A$  by an orientation-preserving diffeomorphism that sends  $\mathcal{L}_{A'}$  to  $\mathcal{L}_A$ .

Theories of finite type invariants of integer (resp. rational) homology 3-spheres  $R$  can be defined from the set  $\mathcal{O}_{\mathbb{Z}}^{\mathbb{Z}}$  (resp.  $\mathcal{O}_{\mathbb{Z}}^{\mathbb{Q}}$ ) of integral (resp. rational) LP-surgeries. For  $\mathbb{Z}$ -spheres, results of Kazuo Habiro [Hab00], Stavros Garoufalidis, Mikhail Goussarov, and Michael Polyak [GGP01], and Emmanuel Auclair and me [AL05] imply that the theory of real-valued finite type invariants with respect to  $\mathcal{O}_{\mathbb{Z}}^{\mathbb{Z}}$  is equivalent to the original theory defined by Tomotada Ohtsuki in [Oht96] using surgeries on algebraically split links. We will call this theory the *Ohtsuki–Goussarov–Habiro theory*.

Greg Kuperberg and Dylan Thurston first showed that the restriction of  $\mathcal{Z}$  to integer homology 3-spheres (equipped with empty links) is a *universal finite type invariant* of  $\mathbb{Z}$ -spheres with respect to  $\mathcal{O}_{\mathbb{Z}}^{\mathbb{Z}}$  in [KT99].

As in the case of links in  $\mathbb{R}^3$ , their proof of universality rests on a computation of the  $k^{\text{th}}$  derivatives of the degree  $k$  part  $\mathcal{Z}_k$  of the invariant  $\mathcal{Z} = (\mathcal{Z}_k)_{k \in \mathbb{Z}}$ , which proves that  $\mathcal{Z}_k$  is a degree  $k$  invariant whose  $k^{\text{th}}$  derivatives are universal in the following sense. All the  $k^{\text{th}}$  derivatives of degree  $k$  real-valued invariants factor through them.

The “Universality part” of this book will be devoted to a general computation of the  $k^{\text{th}}$  derivatives of the extension of  $\mathcal{Z}_k$  to tangles with respect both to  $\mathcal{O}_V$  and  $\mathcal{O}_{\mathbb{Z}}^{\mathbb{Q}}$  (which contains  $\mathcal{O}_{\mathbb{Z}}^{\mathbb{Z}}$ ). The resulting formulae stated in Theorems 17.30 and 18.6 are crucial properties of  $\mathcal{Z}$ . Theorem 18.6 is one of the main original results of this book.

I first proved the splitting formulae, which compute the  $k^{\text{th}}$  derivatives of the degree  $k$  part  $\mathcal{Z}_k$  with respect to  $\mathcal{O}_{\mathcal{L}}^{\mathbb{Q}}$ , in [Les04b] for the restriction of  $\mathcal{Z}_k$  to  $\mathbb{Q}$ -spheres. They allowed Delphine Moussard to classify finite type invariants of  $\mathbb{Q}$ -spheres with respect to  $\mathcal{O}_{\mathcal{L}}^{\mathbb{Q}}$  in [Mou12]. In particular, she proved that, when associated with the  $p$ -valuations of the cardinality  $|H_1|$  of the torsion first homology group,  $\mathcal{Z}$  is a *universal finite type invariant* of  $\mathbb{Q}$ -spheres with respect to  $\mathcal{O}_{\mathcal{L}}^{\mathbb{Q}}$ . Together with results of Gwénaél Mas-suyeau [Mas14] who proved that the LMO invariant  $Z_{LMO}$  of Thang Lê, Jun Murakami, and Tomotada Ohtsuki [LMO98] satisfies the same formulae, the Moussard classification implies that  $\mathcal{Z}$  and  $Z_{LMO}$  are equivalent in the sense that they distinguish the same  $\mathbb{Q}$ -spheres with identical  $|H_1|$ .

Thus, the invariant  $\mathcal{Z}$  is as powerful as the famous LMO invariant for  $\mathbb{Q}$ -spheres, and as the famous Kontsevich integral for links. The Kontsevich integral  $Z^K$  and its relations with the theory of quantum groups developed by Drinfeld and Jimbo have been extensively studied. See the books [Kas95, Oht02, CDM12], for instance. Thang Lê, Jun Murakami [LM96, Theorem 10], and Christian Kassel [Kas95, Theorem XX.8.3] independently showed how the Turaev quantum link invariants [Tur88], which include the HOMFLYPT and Kauffman polynomials, can be recovered from  $Z^K$ . See also [Oht02, Theorem 6.14]. An explicit way of recovering the Alexander polynomial from  $Z^K$  for knots in  $\mathbb{R}^3$  based on [BNG96] can be found in [CDM12, §11.2.4]. Tomotada Ohtsuki computed the two-loop part of the Kontsevich integral—which coincides with the two-loop part of  $\mathcal{Z}$  according to Corollary 12.29 and Note 10.14—for the genus one knots [Oht07]. Dror Bar-Natan, Thang Lê, and Dylan Thurston computed the Kontsevich integral of the trivial knot  $O$  [BNLT03].<sup>17</sup> Since  $\mathcal{Z}$  is obtained from the Kontsevich integral by a change of variables determined by the anomaly as in Corollary 12.29, these results have direct corollaries for  $\mathcal{Z}$ .

The LMO invariant and its generalizations for links in  $\mathbb{Q}$ -spheres [LMO98, Oht02, BNGRT02a, BNGRT02b, BNGRT04] are defined from the Kontsevich integral of links in  $\mathbb{R}^3$ , in a combinatorial way. Any compact oriented 3-manifold can be presented by a *framed* link of  $\mathbb{R}^3$ , which is a link equipped with a favorite parallel, according to a theorem proved by Raymond Lickorish and Andrew Wallace, independently and nicely reproved by Colin Rourke in [Rou85]. The *Kirby moves* are specific modifications of framed links that do not change the presented manifold. According to a theorem of Robion Kirby, two framed links present the same manifold if and only if they are related by a finite sequence of Kirby moves. The LMO invariant of a 3-manifold is

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<sup>17</sup>The computation for the torus knots follows as in [Les99, Proposition 3.16] or [CDM12, Section 9.3]. See also the computations of Julien Marché in [Mar04].



defined from the Kontsevich integral of a framed link that presents such a manifold. The proof of its invariance relies on the cited Kirby theorem. The combinatorial nature of  $Z_{LMO}$  allowed Dror Bar-Natan and Ruth Lawrence to compute  $Z_{LMO}$  for lens spaces and Seifert fibered spaces [BNL04].

When restricted to braids, the Kontsevich integral has a natural geometric meaning. It measures how the strands turn around each other (see [CDM12] or [Les99, Section 1]), and it defines morphisms from braid groups to algebras of horizontal chord diagrams. Dror Bar-Natan extended the Kontsevich integral to links [BN95a]. Thang Lê and Jun Murakami extended the Kontsevich integral to a functor from framed tangles to a category of Jacobi diagrams [LM96]. Next, Thang Lê, Jun Murakami, and Tomotada Ohtsuki defined the LMO invariant from the Le–Murakami–Kontsevich invariant of surgery presentations of the 3-manifolds using tricky algebraic manipulations of Jacobi diagrams with the help of Kirby calculus [LMO98, Oht02]. Though some of the physical meaning of the LMO invariant can be recovered from its universality properties, much of it gets lost in the manipulations.

The presented construction of  $\mathcal{Z}$  is much more physical, geometric, and natural—at least to me. It does not rely on the Kirby theorem and provides information about graph embeddings in  $\mathbb{Q}$ -spheres. On the other hand, some explicit computations or properties available for the LMO invariant have yet to be performed or proved for  $\mathcal{Z}$ .

## 1.4 Book organization

Chapter 2 completes our slow introduction to the linking number of Sections 1.2.1 and 1.2.3. It contains more conventions and arguments used throughout this book. Chapter 3 introduces our definitions of propagators. These propagators are the basic ingredients of all our constructions. Most of the time, they are associated with a parallelization of the 3-manifold. The Theta invariant is the simplest 3-manifold invariant that can be derived from the techniques described in this book. We present it in detail in Chapter 4. We first describe  $\Theta$  as an invariant of a parallelized punctured  $\mathbb{Q}$ -sphere  $(\check{R}, \tau)$ . The invariant  $\Theta(R, \tau)$  is the intersection of three propagating chains associated with the given parallelization  $\tau$  of  $\check{R}$  in the two-point configuration space  $C_2(R)$ . Equivalently, the invariant  $\Theta(R, \tau)$  is the integral over  $C_2(R)$  of the cube of a propagating form associated with  $\tau$ . Next, we transform  $\Theta$  to an invariant of  $\mathbb{Q}$ -spheres using relative Pontrjagin classes, also called Hirzebruch defects, as Greg Kuperberg and Dylan Thurston did in [KT99]. Like  $\Theta$ , the series  $\mathcal{Z}$  comes from a more natural invariant  $\mathcal{Z}^f$  of parallelized links  $(L, L_{\parallel})$  in parallelized punctured  $\mathbb{Q}$ -spheres, constructed with associ-

ated propagators. We transform  $\mathcal{Z}^f(R, L, L_{\parallel}, \tau)$  to an invariant  $\mathcal{Z}$  of links  $L$  in  $\mathbb{Q}$ -spheres using a function of linking numbers associated to the link parallelizations, Pontrjagin numbers associated to the manifold parallelizations, and constants  $\alpha$  and  $\beta$  called *anomalies*. Chapter 5 presents parallelizations of oriented 3-manifolds with boundaries and associated Pontrjagin numbers in detail. It closes this introductory part.

Thus, the book's first part describes the degree one part of the graded invariant  $\mathcal{Z}$  for links in  $\mathbb{Q}$ -spheres, which is determined by the linking numbers of the components and the  $\Theta$ -invariant of the ambient manifold. The book's second part is devoted to the general presentation of  $\mathcal{Z}$  for links in  $\mathbb{Q}$ -spheres. In this part, we first review various theories of finite type invariants for which specializations of  $\mathcal{Z}$  will be universal finite type invariants. This allows us to introduce the spaces of Jacobi diagrams in which  $\mathcal{Z}$  takes its values, in a natural way, in Chapter 6. The complete definitions of  $\mathcal{Z}$  for links in  $\mathbb{Q}$ -spheres are given in Chapter 7 without proofs of consistency. We show that these definitions make sense and do not depend on the involved choices of propagating forms in Chapters 9 and 10. Our proofs rely on the study of suitable compactifications of configuration spaces presented in Chapter 8 and on some standard arguments of the subject already appearing in many places starting with [Kon94], [BT94]... This second part of the book ends with discrete equivalent definitions of  $\mathcal{Z}$  in terms of propagating chains and algebraic intersections rather than propagating forms and integrals in Chapter 11. These definitions make clear that the invariant  $\mathcal{Z}$  is rational. The other main properties of  $\mathcal{Z}$  are precisely described in Sections 10.1, 10.6, 13.3, and Chapter 18. Some of them involve the extension of  $\mathcal{Z}$  to tangles, which can be found in Theorem 12.7.

The book's third part is devoted to this extension of  $\mathcal{Z}$  to tangles, still denoted by  $\mathcal{Z}$ . We introduce the framed version  $\mathcal{Z}^f$  of  $\mathcal{Z}$  for framed tangles in Definition 12.12. The spirit of the definition is the same. However, its justification is more difficult because the involved compactified configuration spaces are more complicated. They are no longer smooth manifolds with ridges and have additional types of faces. We first present the definition and properties of the extension without proofs in Chapters 12 and 13. Next, we justify them in Chapters 14 and 17, respectively. In particular, Chapter 17 contains the proofs of many properties of the link invariant  $\mathcal{Z}$ . These proofs involve easy-to-discretize variants of the functor  $\mathcal{Z}^f$ , which are interesting on their own and presented in Section 16.2. The third part of the book ends with the computation of the iterated derivatives of the generalized  $\mathcal{Z}$  with respect to crossing changes, in Section 17.6. This computation proves that the restriction of  $\mathcal{Z}$  to links in  $S^3$  is a universal Vassiliev invariant.

The fourth part focuses on the computation of the iterated derivatives

of the generalized  $\mathcal{Z}$  with respect to rational LP-surgeries. It begins with Chapter 18, stating the main results and their corollaries, and reducing their proofs to the proofs of two key propositions, presented in Chapter 20. The proofs of these propositions involve the introduction of a more flexible definition of  $\mathcal{Z}$  because the restriction of a parallelization of a  $\mathbb{Q}$ -sphere to the exterior of a  $\mathbb{Q}$ -handlebody does not necessarily extend to a  $\mathbb{Q}$ -handlebody that replaces the former one during a rational LP-surgery. Chapter 19 contains an extension of the notion of parallelization to a more flexible notion of *pseudo-parallelization* and a corresponding more flexible definition of  $\mathcal{Z}$ . Pseudo-parallelizations also have associated propagators and Pontrjagin numbers. They easily extend to arbitrary  $\mathbb{Q}$ -handlebodies. More flexible variants of the definition of  $\mathcal{Z}$  based on pseudo-parallelizations can be found in Chapter 21.

The book ends with two appendices. Appendix A lists the basic results and techniques of algebraic topology used in the book. Appendix B reviews the used properties of differential forms and de Rham cohomology.

Most chapters have their own detailed introduction. Many cross-references help the reader choose what she/he wants to read.

## 1.5 Book genesis

At first, this book aimed at presenting the results of two preprints [Les04a, Les04b] and lecture notes [Les15b]. It contains generalizations of the results of these preprints to wider settings. I have never submitted the preprints [Les04a, Les04b] for publication. Edward Witten's insight into the perturbative expansion of the Chern–Simons theory [Wit89] inspired the mathematical guidelines for constructing the invariant  $\mathcal{Z}$ . Maxim Kontsevich gave them in [Kon94, Section 2]. Greg Kuperberg and Dylan Thurston developed these guidelines in [KT99]. They defined  $\mathcal{Z}$  for  $\mathbb{Q}$ -spheres and sketched a proof that the restriction of  $\mathcal{Z}$  to  $\mathbb{Z}$ -spheres is a universal finite type invariant of  $\mathbb{Z}$ -spheres in the Ohtsuki–Goussarov–Habiro sense. This allowed them to identify the degree one part of  $\mathcal{Z}$  with the Casson invariant for  $\mathbb{Z}$ -spheres. I thank Dylan Thurston for explaining to me his joint work with Greg Kuperberg in Kyoto in 2001.

In [Les04b], I proved splitting formulae for  $\mathcal{Z}$ . These formulae compute derivatives of  $\mathcal{Z}$  with respect to rational LP-surgeries. They generalize similar Kuperberg–Thurston implicit formulae about Torelli surgeries. These formulae allowed me to identify the degree one part of  $\mathcal{Z}$  with the Walker generalization of the Casson invariant for  $\mathbb{Q}$ -spheres, in [Les04b, Section 6]. They also allowed Delphine Moussard to classify finite type invariants with

respect to these rational LP-surgeries and prove that all such real-valued finite type invariants factor through some “augmentation” of  $\mathcal{Z}$  by invariants derived from the order of the  $H_1(\cdot; \mathbb{Z})$ , in [Mou12]. In [Mas14], Gwénaél Massuyeau proved that the LMO invariant of Thang Lê, Jun Murakami, and Tomotada Ohtsuki [LMO98] satisfies the same splitting formulae as  $\mathcal{Z}$ . Thus, the Moussard classification implies that  $\mathcal{Z}$  and  $Z_{LMO}$  are equivalent in the sense that they distinguish the same  $\mathbb{Q}$ -spheres with identical  $|H_1(\cdot; \mathbb{Z})|$ . In order to write the proof of my splitting formulae, I needed to specify the definition of  $\mathcal{Z}$  and I described the Kontsevich–Kuperberg–Thurston construction in detail in [Les04a].

In [Les15b], mixing known constructions in the case of links in  $\mathbb{R}^3$  with the construction of  $\mathcal{Z}$  allowed me to define a natural extension of  $\mathcal{Z}$  as an invariant of links in  $\mathbb{Q}$ -spheres. This extension also generalizes invariants of links in  $\mathbb{R}^3$  defined by Enore Guadagnini, Maurizio Martellini, and Mihail Mintchev [GMM90], Dror Bar-Natan [BN95b], and by Raoul Bott and Clifford Taubes [BT94], which emerged after the Witten work [Wit89].<sup>18</sup> I also gave more flexible definitions of  $\mathcal{Z}$ .

In addition to the revisited contents of the preprints [Les04a, Les04b] and of the notes [Les15b], this book contains an extension of  $\mathcal{Z}$  as a functorial invariant of tangles in rational homology cylinders. It also contains the proofs of many properties of this extension, which imply simpler properties for  $\mathcal{Z}$ , such as the multiplicativity of  $\mathcal{Z}$  under connected sum, for example. This functorial extension, which generalizes the Poirier extension in [Poi00], and its properties are new. They appear only in this book (to my knowledge).

Most of the properties of  $\mathcal{Z}$  are very intuitive and rather easy to accept after some hand-waving. Writing complete proofs is often more complicated than one would expect. I hope I have succeeded in this task, which was sometimes much more difficult than I expected.

## 1.6 Some open questions

1. A Vassiliev invariant is *odd* if it distinguishes some knot from the same knot with the opposite orientation. Are there odd Vassiliev invariants?
2. More generally, do Vassiliev invariants distinguish knots in  $S^3$ ? In [Kup96], Greg Kuperberg proved that if they distinguish unoriented knots in  $S^3$ , then there exist odd Vassiliev invariants.

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<sup>18</sup>The relation between the perturbative expansion of the Chern–Simons theory of the Witten article and the configuration space integral viewpoint is explained by Michael Polyak in [Pol05] and by Justin Sawon in [Saw06]. See also [Oht02, Appendix F]

3. According to a theorem of Dror Bar-Natan and Ruth Lawrence [BNL04], the LMO invariant fails to distinguish rational homology spheres with isomorphic  $H_1$ . So, according to a Moussard theorem [Mou12], rational finite type invariants fail to distinguish  $\mathbb{Q}$ -spheres. Do finite type invariants distinguish  $\mathbb{Z}$ -spheres?
4. Compute the anomalies  $\alpha$  and  $\beta$  of Sections 10.3 and 10.2. For links in  $\mathbb{R}^3$ , I expressed the invariant  $\mathcal{Z}$  as a function of the Kontsevich integral  $Z^K$ , described in [BN95a, Oht02, CDM12], and of the Bott–Taubes anomaly  $\alpha$ , in [Les02]. The computation of  $\alpha$  would finish clarifying the relationship between  $\mathcal{Z}$  and  $Z^K$  for links in  $\mathbb{R}^3$ . See Note 10.14.
5. Find surgery formulae for  $\mathcal{Z}$ . Do the surgery formulae that define  $Z_{LMO}$  from  $Z^K$  define  $\mathcal{Z}$  from its restriction to links in  $\mathbb{R}^3$ ?
6. Compare  $\mathcal{Z}$  with the LMO invariant  $Z_{LMO}$  of Thang Lê, Jun Murakami, and Tomotada Ohtsuki described in [Oht02, Chapter 10].
7. Find relationships between  $\mathcal{Z}$  or other finite type invariants and Heegaard Floer homologies. Recall the propagators associated to Heegaard diagrams of [Les15a] from Subsection 1.2.8.
8. Andrew Kricker defined a lift  $\tilde{Z}^K$  of the Kontsevich integral  $Z^K$  (or the LMO invariant) for null-homologous knots in  $\mathbb{Q}$ -spheres [Kri00, GK04]. The Kricker lift is valued in a space  $\tilde{A}$  of trivalent diagrams whose edges are decorated by rational functions whose denominators divide the Alexander polynomial. Compare the Kricker lift  $\tilde{Z}^K$  with the equivariant configuration space invariant  $\tilde{Z}^c$  of [Les11, Les13] valued in the same diagram space  $\tilde{A}$ .
9. Does one obtain  $\mathcal{Z}$  from  $\tilde{Z}^c$  in the same way as one obtains  $Z^K$  from  $\tilde{Z}^K$ ?
10. Study extensions of  $\mathcal{Z}$  to manifolds with boundary. Dorin Cheptea, Kazuo Habiro, and Gwénaél Massuyeau introduced a functorial extension of the LMO invariant in [CHM08]. See also [HM21].

# Chapter 2

## More on manifolds and on the linking number

The first section of this chapter specifies some basic notions of differential topology, quickly and sometimes vaguely introduced in Subsection 1.2.2. It also contains some additional notation and conventions. The second section completes our discussion about the linking number of Sections 1.2.1 and 1.2.3.

### 2.1 More background material on manifolds

#### 2.1.1 Manifolds without boundary

This section presents a quick review of the notions of manifold and tangent bundle. We refer the reader to [Hir94, Chapter 1] by Morris Hirsch for a clean and complete introduction.

A *topological  $n$ -dimensional manifold  $M$  without boundary* is a Hausdorff topological space that is a union of open subsets  $U_i$  labeled in a countable set  $I$  ( $i \in I$ ), where every  $U_i$  is identified with an open subset  $V_i$  of  $\mathbb{R}^n$  by a homeomorphism  $\phi_i: U_i \rightarrow V_i$ , called a *chart*. Such a collection  $(\phi_i: U_i \rightarrow V_i)_{i \in I}$  of charts, for which  $\cup_{i \in I} U_i = M$ , is called an *atlas* of  $M$ . We consider manifolds up to homeomorphism. So homeomorphic manifolds are considered identical.

For  $r = 0, \dots, \infty$ , the topological manifold  $M$  has a  $C^r$ -structure (induced by the atlas  $(\phi_i)_{i \in I}$ ) or is a  $C^r$ -manifold, if, for each pair  $\{i, j\} \subset I$ , the *transition map*  $\phi_j \circ \phi_i^{-1}$  defined on  $\phi_i(U_i \cap U_j)$  is a  $C^r$ -diffeomorphism onto its image. The notion of  $C^s$ -maps,  $s \leq r$ , from such a manifold to another one can be induced naturally from the known case for which the manifolds are open subsets of some  $\mathbb{R}^n$ , thanks to the local identifications provided by

the charts. Manifolds of class  $C^r$  are considered up to  $C^r$ -diffeomorphism. They are called  $C^r$ -manifolds. *Smooth manifolds* are  $C^\infty$ -manifolds.

A  $C^r$  *embedding* from a  $C^r$  manifold  $A$  into a  $C^r$  manifold  $M$  is an injective  $C^r$  map  $j: A \hookrightarrow M$  such that, for any point  $a$  of  $A$ , there exist

- a  $C^r$  diffeomorphism  $\phi$  from an open neighborhood  $U$  of  $j(a)$  in  $M$  to an open subset  $V$  of  $\mathbb{R}^n$  and
- an open neighborhood  $U_A$  of  $a$  in  $A$  such that the restriction  $j|_{U_A}$  of  $j$  to  $U_A$  is a  $C^r$ -diffeomorphism onto its image, which may be written as  $j(U_A) = j(A) \cap U = \phi^{-1}(V \cap (\mathbb{R}^d \times \{(0, \dots, 0)\}))$ .

A *submanifold* of a manifold  $M$  is the image of such an embedding into  $M$ .

The *tangent space*  $T_x A$  to a  $C^r$  submanifold  $A$  of  $\mathbb{R}^n$  at a point  $x$  of  $A$ , for  $r \geq 1$  is the vector space of all tangent vectors to a curve (i.e., a 1-dimensional submanifold) of  $A$  at  $x$ . A well-known theorem [Hir94, Theorem 3.4, Chapter 1] asserts that any compact  $C^r$ -manifold, for  $r \geq 1$  may be embedded in some  $\mathbb{R}^d$ , and thus viewed as a submanifold of  $\mathbb{R}^d$ . The *tangent bundle*  $TA$  to  $A$  is the union over the elements  $x$  of  $A$  of the  $T_x A$ . Its *bundle projection*  $p: TA \rightarrow A$  maps an element  $v$  of  $T_x A$  to  $x$ . The tangent bundle to  $\mathbb{R}^n$  is canonically isomorphic to  $\mathbb{R}^n \times \mathbb{R}^n$ . A  $C^r$  diffeomorphism between two open sets of  $\mathbb{R}^n$ , together with its (first order) derivatives induces a canonical  $C^{r-1}$ -diffeomorphism between their tangent bundles. The notion of tangent bundle of any  $C^r$   $n$ -manifold, for  $r \geq 1$ , is naturally induced from the local identifications provided by the charts. A  $C^r$  map  $f$  from a  $C^r$ -manifold  $M$  to another one  $N$  has a well-defined *tangent map*, which is a map  $Tf: TM \rightarrow TN$  restricting as a linear map  $T_x f: T_x M \rightarrow T_{f(x)} N$  for any  $x$  of  $M$ .

## 2.1.2 More on low-dimensional manifolds

We now review classical results, which ensure that for  $n = 1, 2$  or  $3$ , any topological  $n$ -manifold may be equipped with a unique smooth (i.e.,  $C^\infty$ ) structure (up to diffeomorphism).

A topological manifold  $M$  as in the previous section *has a piecewise linear (or PL) structure* (induced by the atlas  $(\phi_i)_{i \in I}$ ) or *is a PL-manifold*, if, for each pair  $\{i, j\} \subset I$ , the transition map  $\phi_j \circ \phi_i^{-1}$  is a piecewise linear homeomorphism onto its image. *PL-manifolds* are considered up to *PL-homeomorphism*.

An  $n$ -dimensional *simplex* is the convex hull of  $(n+1)$  points that are not contained in an affine subspace of dimension  $(n-1)$  in some  $\mathbb{R}^k$ , with  $k \geq n$ .

For example, a 1-dimensional simplex is a closed interval, a 2-dimensional simplex is a solid triangle, and a 3-dimensional simplex is a solid tetrahedron. A topological space  $X$  has a *triangulation*, if it is a locally finite union of  $k$ -simplices (closed in  $X$ ), which are the simplices of the triangulation, such that

- the simplices are embedded in  $X$ ,
- every face of a simplex of the triangulation is a simplex of the triangulation,
- when two simplices of the triangulation are not disjoint, their intersection is a simplex of the triangulation.<sup>1</sup>

PL manifolds always have such triangulations.

When  $n \leq 3$ , the above notion of PL-manifold coincides with the notions of smooth and topological manifold, according to the following theorem. This is no longer true when  $n > 3$ . See [Kui99] by Nicolaas Kuiper.

**Theorem 2.1.** *Let  $n$  be a natural integer such that  $n \leq 3$ . Any topological  $n$ -manifold has a unique PL structure (up to PL homeomorphism). For any  $r \in (\mathbb{N} \setminus \{0\}) \cup \{\infty\}$ , any topological  $n$ -manifold has a unique  $C^r$ -structure (up to  $C^r$  diffeomorphism).*

This statement contains several theorems (see [Kui99]): The fact that any  $C^1$ -manifold in any dimension has a unique  $C^r$ -structure (up to  $C^r$ -diffeomorphism), for any  $r \in (\mathbb{N} \setminus \{0, 1\}) \cup \{\infty\}$ , follows from work of Hassler Whitney in 1936 [Whi36]. In 1934, Stewart Cairns [Cai35] equipped any  $C^1$ -manifold with a PL-structure. He proved that any PL manifold of dimension 3 arises as the image of a  $C^1$ -manifold under such a process [Cai40, Theorem III] in 1940. (This was already known in dimension less than 3.) Edwin Moise [Moi52] proved that any topological 3-manifold has a unique PL structure in 1952. (This was also already known in dimension less than 3.) In dimension 3, James Munkres [Mun60, Theorem 6.3] and Henry Whitehead [Whi61] completed this scheme in 1960 by their independent proofs of the uniqueness of a  $C^1$ -structure for any topological 3-manifold.

### 2.1.3 Connected sum

Let  $M_1$  and  $M_2$  be two smooth closed manifolds of dimension  $n$ . The *connected sum*  $M_1 \# M_2$  of  $M_1$  and  $M_2$  is defined as follows. For  $i \in \{1, 2\}$ , let

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<sup>1</sup>A collection of sets in  $X$  is *locally finite* if each point of  $X$  has a neighborhood in  $X$  that intersects finitely many sets of the collection.



$\phi_i: 2\mathring{B}^n \hookrightarrow M_i$  be a smooth embedding of the open ball  $2\mathring{B}^n$  of radius 2 (centered at the origin) of the Euclidean vector space  $\mathbb{R}^n$  into  $M_i$ , such that  $\phi_1$  is orientation-preserving and  $\phi_2$  is orientation-reversing. The elements of  $(2\mathring{B}^n \setminus \{0\})$  may be written as  $\lambda x$  for a unique pair  $(\lambda, x) \in ]0, 2[ \times S^{n-1}$ , where  $S^{n-1}$  is the unit sphere of  $\mathbb{R}^n$ . Let  $h: \phi_1(2\mathring{B}^n \setminus \{0\}) \rightarrow \phi_2(2\mathring{B}^n \setminus \{0\})$  be the diffeomorphism such that  $h(\phi_1(\lambda x)) = \phi_2((2 - \lambda)x)$  for any  $(\lambda, x) \in ]0, 2[ \times S^{n-1}$ .

Then

$$M_1 \# M_2 = (M_1 \setminus \{\phi_1(0)\}) \cup_h (M_2 \setminus \{\phi_2(0)\})$$

is the quotient space of  $(M_1 \setminus \{\phi_1(0)\}) \sqcup (M_2 \setminus \{\phi_2(0)\})$ , in which an element of  $\phi_1(2\mathring{B}^n \setminus \{0\})$  is identified with its image under  $h$ . As a topological manifold,  $M_1 \# M_2$  can be written as

$$M_1 \# M_2 = \left( M_1 \setminus \phi_1(\mathring{B}^n) \right) \cup_{\phi_1(S^{n-1}) \xrightarrow{h} \phi_2(S^{n-1})} \left( M_2 \setminus \phi_2(\mathring{B}^n) \right).$$

### 2.1.4 Manifolds with boundary and ridges

A *topological  $n$ -dimensional manifold  $M$  with possible boundary* is a Hausdorff topological space that is a union of open subsets  $U_i$  labeled in a set  $I$ , ( $i \in I$ ), where every  $U_i$  is identified with an open subset  $V_i$  of  $] -\infty, 0]^k \times \mathbb{R}^{n-k}$  by a chart  $\phi_i: U_i \rightarrow V_i$ . The *boundary* of  $] -\infty, 0]^k \times \mathbb{R}^{n-k}$  consists of the points  $(x_1, \dots, x_n)$  of  $] -\infty, 0]^k \times \mathbb{R}^{n-k}$  for which there exists  $i \leq k$  such that  $x_i = 0$ . The *boundary* of  $M$  consists of the points mapped to the boundary of  $] -\infty, 0]^k \times \mathbb{R}^{n-k}$  by a chart.

A map from an open subset  $O$  of  $] -\infty, 0]^k \times \mathbb{R}^{n-k}$  to an open subset of  $] -\infty, 0]^{k'} \times \mathbb{R}^{n'-k'}$  is smooth (resp.  $C^r$ ) at a point  $x \in O$  if it extends as a smooth (resp.  $C^r$ ) map from an open neighborhood of  $x$  in  $\mathbb{R}^n$  to  $\mathbb{R}^{n'}$ .

The topological manifold  $M$  is a *smooth manifold with ridges*, if, for each pair  $\{i, j\} \subset I$ , the map  $\phi_j \circ \phi_i^{-1}$  defined on  $\phi_i(U_i \cap U_j)$  is a smooth diffeomorphism onto its image.<sup>2</sup> When  $k \leq 1$  for any  $i$ , we simply say that  $M$  is a *smooth manifold with boundary*. The *codimension  $j$  boundary* of such a manifold  $M$ , which is denoted by  $\partial_j(M)$ , consists of the points that are mapped to points  $(x_1, \dots, x_n)$  of  $] -\infty, 0]^k \times \mathbb{R}^{n-k}$  for which there are at least  $j$  indices  $i \leq k$  such that  $x_i = 0$ . It is a closed subset of  $M$ . We have  $\partial M = \partial_1(M)$ .

The *codimension  $j$  faces* of such a smooth manifold  $M$  with ridges are the connected components of  $\partial_j(M) \setminus \partial_{j+1}(M)$ . They are smooth manifolds of dimension  $(n - j)$ . The *interior* of  $M$  is  $M \setminus \partial M$ .

For  $j \geq 2$ , the codimension  $j$  faces are called *ridges* of  $M$ .

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<sup>2</sup>Manifolds with ridges are often called manifolds with corners.

### 2.1.5 Algebraic intersections

We start this subsection with more orientation conventions and notation. Again, unless otherwise mentioned, manifolds are smooth, compact, and oriented. The *normal bundle* to a submanifold  $A$  in a manifold  $M$ , at a point  $x$ , is the quotient  $T_x M / T_x A$  of tangent bundles at  $x$ . We denote it by  $N_x A$  or  $N_x(A)$ . We orient  $N_x A$  so that (a lift of an oriented basis of)  $N_x A$  followed by (an oriented basis of)  $T_x A$  induce the orientation of  $T_x M$ . The orientation of  $N_x(A)$  is a *coorientation* of  $A$  at  $x$ . The regular preimage of a submanifold under a map  $f$  is oriented so that  $f$  preserves the coorientations.

Two submanifolds  $A$  and  $B$  in a manifold  $M$  are *transverse* if we have  $T_x M = T_x A + T_x B$  at each  $x \in A \cap B$ . As proved in [Hir94, Chapter 3 (Theorem 2.4 in particular)], transversality is a generic condition. The intersection  $A \cap B$  of two transverse submanifolds  $A$  and  $B$  in a manifold  $M$  is a manifold. We orient  $A \cap B$  so that the normal bundle to  $A \cap B$  is  $(N(A) \oplus N(B))$ , fiberwise. In order to give a meaning to the sum  $(N_x(A) \oplus N_x(B))$  at  $x \in A \cap B$ , pick a Riemannian metric on  $M$ . Such a metric identifies  $N_x(A)$  with  $T_x(A)^\perp$ ,  $N_x(B)$  with  $T_x(B)^\perp$ , and  $N_x(A \cap B)$  with  $T_x(A \cap B)^\perp = T_x(A)^\perp \oplus T_x(B)^\perp$ . Since the space of Riemannian metrics on  $M$  is convex, and therefore connected, the induced orientation of  $T_x(A \cap B)$  does not depend on the choice of a Riemannian metric.

Let  $A, B, C$  be three pairwise transverse submanifolds in a manifold  $M$  such that  $A \cap B$  is transverse to  $C$ . The oriented intersection  $(A \cap B) \cap C$  is a well-defined manifold. Our assumptions imply that at any  $x \in A \cap B \cap C$ , the sum  $(T_x A)^\perp + (T_x B)^\perp + (T_x C)^\perp$  is a direct sum  $(T_x A)^\perp \oplus (T_x B)^\perp \oplus (T_x C)^\perp$  for any Riemannian metric on  $M$ . So  $A$  is also transverse to  $B \cap C$ , and  $(A \cap B) \cap C = A \cap (B \cap C)$ . Thus, the intersection of transverse, oriented submanifolds is a well-defined associative operation, where *transverse submanifolds* are manifolds such that the elementary pairwise intermediate possible intersections are well-defined, as above. This intersection is also commutative when the codimensions of the submanifolds are even.

Recall from Subsection 1.2.7 that, for two transverse submanifolds  $A$  and  $B$  of complementary dimensions in a manifold  $M$ , the sign  $\pm 1$  of a point  $x \in A \cap B$  is  $+1$  if and only if  $T_x M = T_x A \oplus T_x B$  as oriented vector spaces. This is equivalent to the condition that the orientation of the normal bundle to  $x \in A \cap B$  coincides with the orientation of the ambient space  $M$ , that is that  $T_x M = N_x A \oplus N_x B$  (as oriented vector spaces again—exercise). If  $A$  and  $B$  are of complementary dimensions in  $M$ , and if  $A \cap B$  is finite, then the *algebraic intersection*  $\langle A, B \rangle_M$  of  $A$  and  $B$  is the sum of the signs of the intersection points.

In a manifold  $M$ , a  $k$ -dimensional *chain* (resp. a  $k$ -dimensional rational

*chain*) is a finite combination with coefficients in  $\mathbb{Z}$  (resp. in  $\mathbb{Q}$ ) of (smooth, compact, oriented)  $k$ -dimensional submanifolds  $C$  of  $M$  with boundary and ridges, up to the identification of  $(-1)C$  with  $(-C)$  and other natural identifications (e.g., a  $k$ -manifold  $A \cup B$  such that  $A \cap B$  is a  $(k-1)$ -submanifold of  $A \cup B$  is identified with the chain  $A + B$ ). The boundary  $\partial$  of chains is the linear map that maps a submanifold to its oriented boundary (with respect to the usual outward normal first convention). This boundary is the sum of the closures of the codimension-one faces when there are ridges. The canonical orientation of a point is the sign  $+1$ . So  $\partial[0, 1] = \{1\} - \{0\}$ . A  $k$ -dimensional chain whose boundary vanishes is a  $k$ -dimensional *cycle*, or a  $k$ -*cycle* for short.

If  $A_1, \dots, A_k$  are  $k$  transverse compact submanifolds of  $M$  whose codimension sum is the dimension of  $M$ , then their *algebraic intersection* is defined to be  $\langle A_1, \dots, A_k \rangle_M = \langle \cap_{i=1}^{k-1} A_i, A_k \rangle_M$ . If  $M$  is a connected manifold, which contains a point  $x$ , then the class of a 0-cycle in  $H_0(M; \mathbb{Q}) = \mathbb{Q}[x] = \mathbb{Q}$  is a well-defined number. The algebraic intersection  $\langle A_1, \dots, A_k \rangle_M$  can be equivalently defined to be the homology class of the (oriented) intersection  $\cap_{i=1}^k A_i$ . This algebraic intersection extends multilinearly to rational chains.

We will use the following lemma in Subsection 2.2.1.

**Lemma 2.2.** *Let  $A$  and  $B$  be two transverse submanifolds of a  $d$ -dimensional manifold  $M$  with disjoint boundaries. Let  $\beta$  denote the dimension of  $B$ . Then*

$$\partial(A \cap B) = (-1)^{d-\beta} \partial A \cap B + A \cap \partial B.$$

PROOF: Note that  $\partial(A \cap B) \subset \partial A \cup \partial B$ . At a point  $a \in \partial A$ ,  $T_a M$  is oriented by  $(N_a A, o, T_a \partial A)$ , where  $o$  is the outward normal to  $A$ . If  $a \in \partial A \cap B$ , then  $o$  is also an outward normal for  $A \cap B$ , and  $\partial(A \cap B)$  is cooriented by  $(N_a A, N_a B, o)$ , while  $\partial A \cap B$  is cooriented by  $(N_a A, o, N_a B)$ . At a point  $b \in A \cap \partial B$ , both  $\partial(A \cap B)$  and  $A \cap \partial B$  are cooriented by  $(N_b A, N_b B, o)$ .  $\square$

## 2.1.6 More on the degree

Here, we make the notion of *walls* in Subsection 1.2.5 more precise, by stating a lemma on the general behavior of the degree. We prove it with the Morse–Sard theorem 1.4. The formula of the lemma could also be justified with Stokes’ theorem.

**Lemma 2.3.** *Let  $n \in \mathbb{N}$ . Let  $M$  be a compact (oriented)  $n$ -manifold with possible boundary. Let  $N$  be a connected (oriented)  $n$ -manifold. Let  $f: M \rightarrow N$  be a smooth map. Let  $a$  and  $b$  be two distinct regular values of  $f$  in the interior  $\text{Int}(N)$  of  $N$ .*

Then there exists an embedding  $\gamma: [0, 1] \rightarrow \text{Int}(N)$  such that  $\gamma(0) = a$ ,  $\gamma(1) = b$ , and, for any  $x \in \partial M \cap f^{-1}(\gamma([0, 1]))$ ,

- $x$  is in an open face of  $\partial M$  (of codimension one in  $M$ ) and
- $T_{f(x)}N$  equals  $\eta(x)T_{f(x)}\gamma \oplus T_x f(T_x \partial M)$  as an oriented vector space for some  $\eta(x) = \pm 1$ .

For any such embedding  $\gamma$ , we have

$$\deg_a f - \deg_b f = \langle \gamma, f(\partial M) \rangle_N = \sum_{x \in f^{-1}(\gamma) \cap \partial M} \eta(x).$$

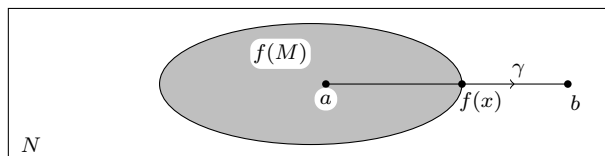


Figure 2.1: Lemma 2.3 for an embedding  $f$  from  $M$  to a rectangle  $N$

PROOF: Let  $B^{n-1}$  be the unit ball of  $\mathbb{R}^{n-1}$ . Since  $N$  is connected, there exists an (orientation-preserving) embedding  $\Psi: B^{n-1} \times [-1, 2] \rightarrow \text{Int}(N)$  such that  $\Psi(0, 0) = a$ ,  $\Psi(0, 1) = b$ , and  $\Psi(\varepsilon B^{n-1} \times ]-\varepsilon, \varepsilon[)$  and  $\Psi(\varepsilon B^{n-1} \times ]1 - \varepsilon, 1 + \varepsilon[)$  consist of regular values of  $f$  for some  $\varepsilon \in ]0, \frac{1}{100}[$ .

If all the elements of  $\Psi(\{0\} \times [0, 1])$  are regular values of  $f$ , then the preimage  $f^{-1}(\Psi(\{0\} \times [0, 1]))$  is a disjoint union of intervals between a point of  $f^{-1}(a)$  and a point of  $f^{-1}(b)$ , along which the sign of  $\det(T_x f)$  is constant, and  $\deg_a f$  is equal to  $\deg_b f$ . This proves that the degree is constant on any ball of regular values. Let us return to the general case and try to make a similar argument work. Set  $N_\varepsilon = \varepsilon \mathring{B}^{n-1} \times [0, 1]$ ,  $M_\varepsilon = f^{-1}(\Psi(N_\varepsilon))$ , and  $f_\varepsilon = \Psi^{-1} \circ f|_{M_\varepsilon}$ . For any  $c \in \Psi(\mathring{N}_\varepsilon)$ ,  $\deg_c f = \deg_{\Psi^{-1}(c)}(f_\varepsilon)$ .

Let  $p_B$  and  $p_I$  respectively denote the natural projections of  $\varepsilon \mathring{B}^{n-1} \times [0, 1]$  to its factors  $\varepsilon \mathring{B}^{n-1}$  and  $[0, 1]$ . The Morse–Sard theorem 1.4 guarantees the existence of a regular value  $y$  of  $p_B \circ f_\varepsilon$  in  $\varepsilon \mathring{B}^{n-1}$ . Let  $\gamma_y: [0, 1] \rightarrow N_\varepsilon$  map  $t$  to  $(y, t)$ . Then  $\gamma_y$  is cooriented by  $p_B$  (via  $T_w p_B: T_w N_\varepsilon \rightarrow \mathbb{R}^{n-1}$ ). So its preimage

$$f_\varepsilon^{-1}(\gamma_y) = f_\varepsilon^{-1}(\gamma_y([0, 1])) = (p_B \circ f_\varepsilon)^{-1}(y)$$

is a submanifold of dimension 1 of  $M$ , cooriented by  $p_B \circ f_\varepsilon$ . Let us show that the oriented boundary of  $f_\varepsilon^{-1}(\gamma_y)$  is

$$\partial f_\varepsilon^{-1}(\gamma_y) = -f_\varepsilon^{-1}((0, y)) + f_\varepsilon^{-1}((1, y)) + \sum_{x \in f_\varepsilon^{-1}(\gamma_y) \cap \partial M} \eta(x)x.$$

For a *regular point*  $x$  of  $f_\varepsilon$  (i.e., a point such that  $f_\varepsilon$  is a local diffeomorphism near  $x$ ), let  $\delta(x) = \pm 1$  denote the sign of the Jacobian determinant of  $f_\varepsilon$  at  $x$ . Near such a regular point  $x$  in  $(p_B \circ f_\varepsilon)^{-1}(y)$ , the curve  $\delta(x)f_\varepsilon^{-1}(\gamma_y)$  is oriented by  $p_I \circ f_\varepsilon$ . This shows that the above signs before  $f_\varepsilon^{-1}((0, y))$  and  $f_\varepsilon^{-1}((1, y))$  are correct.

Let us now check the sign associated to  $x \in f_\varepsilon^{-1}(\gamma_y) \cap \partial M$ . Let  $N_{x, \partial M}$  denote the outward normal to  $M$  at  $x$ . So we have  $T_x M = \mathbb{R}N_{x, \partial M} \oplus T_x \partial M$  and

$$T_{f_\varepsilon(x)} N_\varepsilon = \eta(x) T_{f_\varepsilon(x)} \gamma_y \oplus T_x f_\varepsilon (T_x \partial M) = \delta(x) (T_x f_\varepsilon (\mathbb{R}N_{x, \partial M}) \oplus T_x f_\varepsilon (T_x \partial M)).$$

Therefore, the outward normal  $N_{x, \partial M}$  to  $\partial M$  is oriented by  $\eta(x)\delta(x)p_I \circ f_\varepsilon$ , and the sign of the scalar product of  $T_x (f_\varepsilon^{-1}(\gamma_y))$  and  $N_{x, \partial M}$  is  $\eta(x)$ .

Writing that the homology class of  $\partial f_\varepsilon^{-1}(\gamma_y)$  is zero in  $H_0(M)$  shows

$$\deg_{(0, y)} f_\varepsilon - \deg_{(1, y)} f_\varepsilon = \sum_{x \in f_\varepsilon^{-1}(\gamma_y) \cap \partial M} \eta(x).$$

This proves that the lemma holds for  $\Psi(a_y = \gamma_y(0))$ ,  $\Psi(b_y = \gamma_y(1))$ , and  $\Psi \circ \gamma_y$ . Since  $\Psi(a_y)$  and  $a$  are in a common open ball  $B_a$  of regular values of  $f$  (in  $\text{Int}(N) \setminus f(\partial M)$ ), and since  $\Psi(b_y)$  and  $b$  are in another common ball of regular values of  $f$ , we may modify  $\Psi \circ \gamma_y$  to a smooth path  $\gamma$  from  $a$  to  $b$  with all the properties of the statement.

Let us finish by proving that for any smooth embedded path  $\gamma$  from  $a$  to  $b$  transverse to  $f(\partial M)$ , we have  $\deg_a f - \deg_b f = \langle \gamma, f(\partial M) \rangle_N$ . For any such path  $\gamma$ , we can construct a neighborhood embedding  $\Psi$  of  $\gamma$  as above, such that  $\Psi(0, t) = \gamma(t)$  for all  $t \in [0, 1]$ ,  $\Psi(y \times ]-\varepsilon, 1 + \varepsilon[)$  is transverse to  $f(\partial M)$  for any  $y \in \varepsilon \dot{B}^{n-1}$ , and  $\langle \gamma_y, f(\partial M) \rangle$  does not depend on  $y \in \varepsilon \dot{B}^{n-1}$ . So the previous study allows us to conclude.  $\square$

## 2.2 On the linking number, again

### 2.2.1 A general definition of the linking number

**Lemma 2.4.** *Let  $J$  and  $K$  be two rationally null-homologous disjoint cycles of respective dimensions  $j$  and  $k$  in a  $d$ -manifold  $M$ , where  $d = j + k + 1$ . There exists a rational  $(j + 1)$ -chain  $\Sigma_J$  bounded by  $J$  transverse to  $K$  and a rational  $(k + 1)$ -chain  $\Sigma_K$  bounded by  $K$  transverse to  $J$ . For any two such rational chains  $\Sigma_J$  and  $\Sigma_K$ , we have  $\langle J, \Sigma_K \rangle_M = (-1)^{j+1} \langle \Sigma_J, K \rangle_M$ . In particular,  $\langle J, \Sigma_K \rangle_M$  is a topological invariant of  $(J, K)$ . It is denoted by  $lk(J, K)$  and called the linking number of  $J$  and  $K$ . We have*

$$lk(J, K) = (-1)^{(j+1)(k+1)} lk(K, J).$$

PROOF: Since  $K$  is rationally null-homologous,  $K$  bounds a rational  $(k+1)$ -chain  $\Sigma_K$ . Without loss of generality,  $\Sigma_K$  is assumed to be transverse to  $\Sigma_J$ , so  $\Sigma_J \cap \Sigma_K$  is a rational 1-chain (which is a rational combination of circles and intervals). According to Lemma 2.2, we have

$$\partial(\Sigma_J \cap \Sigma_K) = (-1)^{d+k+1} J \cap \Sigma_K + \Sigma_J \cap K.$$

Furthermore, the sum of the coefficients of the points in the left-hand side must be zero since this sum vanishes for the boundary of an interval. This proves  $\langle J, \Sigma_K \rangle_M = (-1)^{d+k} \langle \Sigma_J, K \rangle_M$ . Therefore, this rational number is independent of the chosen  $\Sigma_J$  and  $\Sigma_K$ . Since we have

$$(-1)^{d+k} \langle \Sigma_J, K \rangle_M = (-1)^{j+1} (-1)^{k(j+1)} \langle K, \Sigma_J \rangle_M,$$

we get  $lk(J, K) = (-1)^{(j+1)(k+1)} lk(K, J)$ .  $\square$

**Remark 2.5.** Our sign convention for the linking number differs from that in [ST80, Section 77, page 288], where the linking number of cycles  $J$  and  $K$  as in the lemma is defined as  $\langle \Sigma_J, K \rangle_M$ , instead. The reason for our sign convention is justified in Remark 2.10.

In particular, the *linking number* of two rationally null-homologous disjoint links  $J$  and  $K$  in a 3-manifold  $M$  is the algebraic intersection of a rational chain bounded by one of the links and the other.

For  $\mathbb{K} = \mathbb{Z}$  or  $\mathbb{Q}$ , any knot is rationally null-homologous in a  $\mathbb{K}$ -sphere or in a  $\mathbb{K}$ -ball (defined in Section 1.1). So the linking number of two disjoint knots always makes sense in such a 3-manifold.

A *meridian*  $m_K$  of a knot  $K$  is the (oriented) boundary of a disk that intersects  $K$  once with a positive sign. See Figure 2.2. Note  $lk(K, m_K) = 1$ .

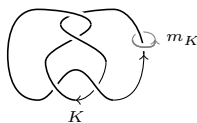


Figure 2.2: A meridian  $m_K$  of a knot  $K$

**Lemma 2.6.** *Let  $R$  be a  $\mathbb{Q}$ -sphere or a  $\mathbb{Q}$ -ball. Let  $K$  be a knot in  $R$ . Then  $H_1(R \setminus K; \mathbb{Q})$  is equal to  $\mathbb{Q}[m_K]$ . Let  $J$  be a knot of  $R$  disjoint from  $K$ . Then we have  $[J] = lk(J, K) [m_K]$  in  $H_1(R \setminus K; \mathbb{Q})$ . This equality provides an alternative definition for the linking number.*

PROOF: Exercise. Note that a chain bounded by  $J$  transverse to  $K$  in  $R$  provides a rational cobordism between  $J$  and a combination of meridians of  $K$ .  $\square$

The reader is also invited to check that the Gauss linking number  $lk_G$  of Subsection 1.2.1 coincides with the above linking number  $lk$  for two-component links of  $S^3$ , as an exercise. This is proved in the following subsection, see Proposition 2.9.

## 2.2.2 Generalizing the Gauss definition of the linking number and identifying the definitions

Let  $X$  and  $Y$  be two topological spaces. Recall that a *homotopy* from a continuous map  $f$  from  $X$  to  $Y$  to another such  $g$  is a continuous map  $H: [0, 1] \times X \rightarrow Y$  such that for any  $x \in X$ ,  $H(0, x) = f(x)$  and  $H(1, x) = g(x)$ . Two continuous maps  $f$  and  $g$  from  $X$  to  $Y$  are said to be *homotopic* if there exists a homotopy from  $f$  to  $g$ . A continuous map  $f$  from  $X$  to  $Y$  is a *homotopy equivalence* if there exists a continuous map  $g$  from  $Y$  to  $X$  such that  $g \circ f$  is homotopic to the identity map of  $X$  and  $f \circ g$  is homotopic to the identity map of  $Y$ . The topological spaces  $X$  and  $Y$  are said to be *homotopy equivalent*, or *of the same homotopy type* if there exists a homotopy equivalence from  $X$  to  $Y$ . Appendix A describes the homology  $H_*$  of topological spaces and continuous maps.

Let  $\Delta((\mathbb{R}^3)^2)$  denote the diagonal of  $(\mathbb{R}^3)^2$ .

**Lemma 2.7.** *The map*

$$\begin{aligned} p_{S^2}: (\mathbb{R}^3)^2 \setminus \Delta((\mathbb{R}^3)^2) &\rightarrow S^2 \\ (x, y) &\mapsto \frac{1}{\|y-x\|}(y-x) \end{aligned}$$

*is a homotopy equivalence. In particular*

$$H_i(p_{S^2}): H_i\left((\mathbb{R}^3)^2 \setminus \Delta((\mathbb{R}^3)^2); \mathbb{Z}\right) \rightarrow H_i(S^2; \mathbb{Z})$$

*is an isomorphism for all integer  $i$ , the space  $(\mathbb{R}^3)^2 \setminus \Delta((\mathbb{R}^3)^2)$  is a homology  $S^2$ , and  $[S] = H_2(p_{S^2})^{-1}[S^2]$  is a canonical generator of*

$$H_2\left((\mathbb{R}^3)^2 \setminus \Delta((\mathbb{R}^3)^2); \mathbb{Z}\right) = \mathbb{Z}[S].$$

PROOF: The configuration space  $((\mathbb{R}^3)^2 \setminus \Delta((\mathbb{R}^3)^2))$  is homeomorphic to  $\mathbb{R}^3 \times ]0, \infty[ \times S^2$  via the map

$$(x, y) \mapsto (x, \|y-x\|, p_{S^2}(x, y)).$$

□

As in Subsection 1.2.1, consider a two-component link  $J \sqcup K : S^1 \sqcup S^1 \hookrightarrow \mathbb{R}^3$ . This embedding induces an embedding

$$\begin{aligned} J \times K : S^1 \times S^1 &\hookrightarrow (\mathbb{R}^3)^2 \setminus \Delta((\mathbb{R}^3)^2) \\ (w, z) &\mapsto (J(w), K(z)). \end{aligned}$$

The map  $p_{JK}$  of Subsection 1.2.1 is the composition  $p_{S^2} \circ (J \times K)$ . We have

$$H_2(p_{JK}) [S^1 \times S^1] = \deg(p_{JK}) [S^2] = lk_G(J, K) [S^2]$$

in  $H_2(S^2; \mathbb{Z}) = \mathbb{Z} [S^2]$ . Thus we get

$$[(J \times K)(S^1 \times S^1)] = H_2(J \times K) [S^1 \times S^1] = lk_G(J, K) [S]$$

in  $H_2((\mathbb{R}^3)^2 \setminus \Delta((\mathbb{R}^3)^2); \mathbb{Z}) = \mathbb{Z} [S]$ . We will see that this definition of  $lk_G$  generalizes to links in rational homology spheres. Then we will prove that our generalized definition coincides with the general definition of linking numbers in this case.

For a manifold  $M$ , the normal bundle to the diagonal of  $M^2$  in  $M^2$  is identified with the tangent bundle to  $M$ , fiberwise, by the map

$$[(u, v)] \in \frac{(T_x M)^2}{\Delta((T_x M)^2)} \mapsto (v - u) \in T_x M.$$

A *parallelization*  $\tau$  of an oriented 3-manifold  $M$  is a (smooth) bundle isomorphism  $\tau : M \times \mathbb{R}^3 \rightarrow TM$  that restricts to  $x \times \mathbb{R}^3$  as an orientation-preserving linear isomorphism from  $x \times \mathbb{R}^3$  to  $T_x M$ , for any  $x \in M$ . It has long been known that any oriented 3-manifold is parallelizable (i.e., admits a parallelization). It is proved in Section 5.2. Therefore, a tubular neighborhood of the diagonal  $\Delta(M^2)$  in  $M^2$  is diffeomorphic to  $M \times \mathbb{R}^3$ .

**Lemma 2.8.** *Let  $R$  be a rational homology sphere and let  $\infty$  be a point of  $R$ . Set  $\check{R} = R \setminus \{\infty\}$ . Then  $\check{R}^2 \setminus \Delta(\check{R}^2)$  has the same rational homology as  $S^2$ . Let  $B$  be a ball in  $\check{R}$  and let  $x$  be a point inside  $B$ , then the class  $[S]$  of  $x \times \partial B$  is a canonical generator of  $H_2(\check{R}^2 \setminus \Delta(\check{R}^2); \mathbb{Q}) = \mathbb{Q} [S]$ .*

**PROOF:** In this proof, the homology coefficients are in  $\mathbb{Q}$ . We refer the reader to Section A.1. Since  $\check{R}$  has the homology of a point, the Künneth Formula (Theorem A.10) implies that  $\check{R}^2$  has the homology of a point. The excision axiom yields

$$\begin{aligned} H_*(\check{R}^2, \check{R}^2 \setminus \Delta(\check{R}^2)) &\cong H_*(\check{R} \times \mathbb{R}^3, \check{R} \times (\mathbb{R}^3 \setminus 0)) \\ &\cong H_*(\mathbb{R}^3, S^2) \cong \begin{cases} \mathbb{Q} & \text{if } * = 3, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$



Using the long exact sequence of the pair  $(\check{R}^2, \check{R}^2 \setminus \Delta(\check{R}^2))$ , we get  $H_*(\check{R}^2 \setminus \Delta(\check{R}^2)) \cong H_*(S^2)$ .  $\square$

Define the *Gauss linking number* of two disjoint links  $J$  and  $K$  in  $\check{R}$  so that

$$[(J \times K)(S^1 \times S^1)] = lk_G(J, K)[S]$$

in  $H_2(\check{R}^2 \setminus \Delta(\check{R}^2); \mathbb{Q})$ . Note that the two definitions of  $lk_G$  coincide when  $\check{R} = \mathbb{R}^3$ .

**Proposition 2.9.** *For two disjoint links  $J$  and  $K$  in  $\check{R}$ , we have*

$$lk_G(J, K) = lk(J, K)$$

PROOF: First recall that  $lk(J, K)$  is the algebraic intersection  $\langle J, \Sigma_K \rangle_R$  of  $J$  and a rational chain  $\Sigma_K$  bounded by  $K$ . Note that the definitions of  $lk(J, K)$  and  $lk_G(J, K)$  make sense when  $J$  and  $K$  are disjoint links. If  $J$  has several components  $J_i$ , for  $i = 1, \dots, n$ , then  $lk_G(\sqcup_{i=1}^n J_i, K) = \sum_{i=1}^n lk_G(J_i, K)$  and  $lk(\sqcup_{i=1}^n J_i, K) = \sum_{i=1}^n lk(J_i, K)$ . There is no loss of generality in assuming that  $J$  is a knot for the proof, which we do.

The chain  $\Sigma_K$  provides a rational cobordism  $C$  in  $\check{R} \setminus J$  between  $K$  and a combination of meridians of  $J$ . Thus, it provides the rational cobordism  $C \times J$  in  $\check{R}^2 \setminus \Delta(\check{R}^2)$ , which allows us to see that  $[J \times K] = lk(J, K)[J \times m_J]$  in  $H_2(\check{R}^2 \setminus \Delta(\check{R}^2); \mathbb{Q})$ . Similarly,  $\Sigma_J$  provides a rational cobordism between  $J$  and a meridian  $m_{m_J}$  of  $m_J$ . So  $[J \times m_J] = [m_{m_J} \times m_J]$  in  $H_2(\check{R}^2 \setminus \Delta(\check{R}^2); \mathbb{Q})$ . This shows

$$lk_G(J, K) = lk(J, K)lk_G(m_{m_J}, m_J).$$

Thus, it remains to prove that  $lk_G(m_{m_J}, m_J) = 1$  for a positive Hopf link  $(m_{m_J}, m_J)$ , as in Figure 1.4, in a standard ball embedded in  $\check{R}$ . Now, there is no loss of generality in assuming that our link is a Hopf link in  $\mathbb{R}^3$ , so the equality follows from that for the positive Hopf link in  $\mathbb{R}^3$ .  $\square$

**Remark 2.10.** Under the assumptions of Lemma 2.4, the reader can prove as an exercise that if  $M$  is connected and if  $B$  is a compact ball of  $M$  containing a point  $x$  in its interior, then  $J \times K$  is homologous to  $lk(J, K)(x \times \partial B)$  in  $M^2 \setminus \Delta(M^2)$ . In particular, Proposition 2.9 generalizes to all pairs  $(J, K)$  as in Lemma 2.4 naturally. This justifies our sign convention in Lemma 2.4.

# Chapter 3

## Propagators

For a two-component link  $(J, K)$  in  $\mathbb{R}^3$ , the definition of the linking number  $lk(J, K)$  can be rewritten as

$$lk(J, K) = \int_{J \times K} p_{S^2}^*(\omega) = \langle J \times K, p_{S^2}^{-1}(Y) \rangle_{(\mathbb{R}^3)^2 \setminus \Delta((\mathbb{R}^3)^2)}$$

for any 2-form  $\omega$  of  $S^2$  such that  $\int_{S^2} \omega = 1$ , and for any regular value  $Y$  of  $p_{JK}$ . Thus,  $lk(J, K)$  is the integral of a 2-form  $p_{S^2}^*(\omega)$  of  $(\mathbb{R}^3)^2 \setminus \Delta((\mathbb{R}^3)^2)$  along the 2-cycle  $[J \times K]$ , or it is the intersection of the 2-cycle  $[J \times K]$  with the 4-manifold  $p_{S^2}^{-1}(Y)$ . In order to adapt these definitions of the linking number to punctured rational homology 3-spheres  $\check{R} = R \setminus \{\infty\}$  and to build other invariants of links and rational homology spheres  $R$ , we compactify  $(\check{R})^2 \setminus \Delta((\check{R})^2)$  to a compact 6-manifold  $C_2(R)$ , in Section 3.2, using differential blow-ups described in Section 3.1. The above form  $p_{S^2}^*(\omega)$  extends to  $C_2(S^3 = \mathbb{R}^3 \cup \{\infty\})$  as a model *propagating form*. The closure of  $p_{S^2}^{-1}(Y)$  in  $C_2(S^3)$  is a model *propagating chain*. We define general propagating forms and propagating chains in  $C_2(R)$  in Section 3.3 by their behaviors on the created boundary of  $C_2(R)$ . The linking number in a rational homology sphere  $R$  is expressed in terms of these *propagators* as in the above equation, in Lemma 3.12. These propagators are the main ingredient in the definitions of the invariant  $\mathcal{Z}$  studied in this book.

### 3.1 Blowing up in real differential topology

For a vector space  $T$ ,  $S(T)$  denotes the quotient  $S(T) = (T \setminus \{0\})/\mathbb{R}^{+*}$ , where  $\mathbb{R}^{+*}$  acts by scalar multiplication. Recall that the *unit normal bundle* of a submanifold  $C$  in a smooth manifold is the fiber bundle whose fiber over  $x \in C$  is  $SN_x(C) = S(N_x(C))$ .

In this book, *blowing up* a submanifold  $C$  in a smooth manifold  $A$  is a canonical process, which transforms  $A$  into a smooth manifold  $\mathcal{B}\ell(A, C)$  by replacing  $C$  with the total space of its unit normal bundle. Unlike blow-ups in algebraic geometry, this blow-up of differential topology, which amounts to removing an open tubular neighborhood (thought of as infinitely small) of  $C$ , topologically, creates boundaries. Let us define it formally.

A smooth *submanifold transverse to the ridges* of a smooth manifold  $A$  is a subset  $C$  of  $A$  such that for any point  $x \in C$  there exists a smooth open embedding  $\phi$  from  $\mathbb{R}^c \times \mathbb{R}^e \times [0, 1]^d$  into  $A$  such that  $\phi(0) = x$  and the image of  $\phi$  intersects  $C$  exactly along  $\phi(0 \times \mathbb{R}^e \times [0, 1]^d)$ . Here  $c$  is the *codimension* of  $C$ ,  $d$  and  $e$  are integers, which depend on  $x$ .

**Definition 3.1.** Let  $C$  be a smooth submanifold transverse to the ridges of a smooth manifold  $A$ . The *blow-up*  $\mathcal{B}\ell(A, C)$  is the unique smooth manifold  $\mathcal{B}\ell(A, C)$  (with possible ridges) equipped with a canonical smooth projection

$$p_b: \mathcal{B}\ell(A, C) \rightarrow A$$

called the *blowdown map* such that

1. the restriction of  $p_b$  to  $p_b^{-1}(A \setminus C)$  is a canonical diffeomorphism onto  $A \setminus C$ , which identifies  $p_b^{-1}(A \setminus C)$  with  $A \setminus C$  (we will simply regard  $A \setminus C$  as a subset of  $\mathcal{B}\ell(A, C)$  via this identification),
2. there is a canonical identification of  $p_b^{-1}(C)$  with the total space  $SN(C)$  of the unit normal bundle to  $C$  in  $A$ ,
3. the restriction of  $p_b$  to  $p_b^{-1}(C) = SN(C)$  is the bundle projection from  $SN(C)$  to  $C$ ,
4. any smooth diffeomorphism  $\phi$  from  $\mathbb{R}^c \times \mathbb{R}^e \times [0, 1]^d$  onto an open subset  $\phi(\mathbb{R}^c \times \mathbb{R}^e \times [0, 1]^d)$  of  $A$  whose image intersects  $C$  exactly along  $\phi(0 \times \mathbb{R}^e \times [0, 1]^d)$ , for natural integers  $c, e, d$ , provides a smooth embedding

$$\begin{array}{ccc} [0, \infty[ \times S^{c-1} \times (\mathbb{R}^e \times [0, 1]^d) & \xrightarrow{\tilde{\phi}} & \mathcal{B}\ell(A, C) \\ (\lambda \in ]0, \infty[, v, x) & \mapsto & \phi(\lambda v, x) \\ (0, v, x) & \mapsto & T\phi(0, x)(v) \in SN(C) \end{array}$$

with open image in  $\mathcal{B}\ell(A, C)$ .

**PROOF THAT THE DEFINITION IS CONSISTENT:** Use local diffeomorphisms of the form  $\tilde{\phi}$  and charts on  $A \setminus C$  to construct an atlas for  $\mathcal{B}\ell(A, C)$ . These charts are obviously compatible over  $A \setminus C$ . We check compatibility for

charts  $\tilde{\phi}$  and  $\tilde{\psi}$  induced by embeddings  $\phi$  and  $\psi$  as in the statement. For those, transition maps may be written as

$$(\lambda, u, x) \mapsto \left( \tilde{\lambda} = \|p_1 \circ \psi^{-1} \circ \phi(\lambda u, x)\|, \tilde{u}, \tilde{x} = p_2 \circ \psi^{-1} \circ \phi(\lambda u, x) \right),$$

where  $p_1$  and  $p_2$  respectively denote the projections on the first and the second factors of  $\mathbb{R}^c \times (\mathbb{R}^e \times [0, 1]^d)$ , and

$$\tilde{u} = \begin{cases} \frac{p_1 \circ \psi^{-1} \circ \phi(\lambda u, x)}{\tilde{\lambda}} & \text{if } \lambda \neq 0 \\ \frac{T(p_1 \circ \psi^{-1} \circ \phi)(0, x)(u)}{\|T(p_1 \circ \psi^{-1} \circ \phi)(0, x)(u)\|} & \text{if } \lambda = 0 \end{cases}$$

In order to check that this is smooth, write

$$p_1 \circ \psi^{-1} \circ \phi(\lambda u, x) = \lambda \int_0^1 T(p_1 \circ \psi^{-1} \circ \phi)(t\lambda u, x)(u) dt,$$

and check that the integral does not vanish when  $\lambda$  is small enough. The restriction to  $S^{c-1}$  of  $T(p_1 \circ \psi^{-1} \circ \phi)(0, x)$  is an injection. So for any  $u_0 \in S^{c-1}$ , there exists a neighborhood of  $(0, u_0)$  in  $\mathbb{R} \times S^{c-1}$  such that for any  $(\lambda, u)$  in this neighborhood, we have the following condition about the scalar product

$$\left\langle T(p_1 \circ \psi^{-1} \circ \phi)(\lambda u, x)(u), T(p_1 \circ \psi^{-1} \circ \phi)(0, x)(u) \right\rangle > 0.$$

Therefore, there exists  $\varepsilon > 0$  such that for any  $\lambda \in ]-\varepsilon, \varepsilon[$  and for any  $u \in S^{c-1}$ , we have

$$\left\langle T(p_1 \circ \psi^{-1} \circ \phi)(\lambda u, x)(u), T(p_1 \circ \psi^{-1} \circ \phi)(0, x)(u) \right\rangle > 0.$$

Then

$$\tilde{\lambda} = \lambda \left\| \int_0^1 T(p_1 \circ \psi^{-1} \circ \phi)(t\lambda u, x)(u) dt \right\|$$

is a smooth function (defined even when  $\lambda \leq 0$ ) and

$$\tilde{u} = \frac{\int_0^1 T(p_1 \circ \psi^{-1} \circ \phi)(t\lambda u, x)(u) dt}{\left\| \int_0^1 T(p_1 \circ \psi^{-1} \circ \phi)(t\lambda u, x)(u) dt \right\|}$$

is smooth, too. Thus, our atlas is compatible. This defines  $\mathcal{B}(A, C)$  together with its smooth structure. The projection  $p_b$  maps  $\tilde{\phi}(\lambda, v, x)$  to  $\phi(\lambda v, x)$  in a chart as above. Thus, it is obviously smooth, and it has the desired properties.  $\square$

Note the following immediate proposition.

**Proposition 3.2.** *The blown-up manifold  $\mathcal{B}(A, C)$  is homeomorphic to the complement in  $A$  of an open tubular neighborhood of  $C$ . In particular,  $\mathcal{B}(A, C)$  is homotopy equivalent to  $A \setminus C$ . If  $C$  and  $A$  are compact, then  $\mathcal{B}(A, C)$  is compact and it is a smooth compactification of  $A \setminus C$ .*

□

Figure 3.1 first shows the result of blowing up  $(0, 0)$  in  $\mathbb{R}^2$ . The closures in  $\mathcal{B}(\mathbb{R}^2, (0, 0))$  of  $\{0\} \times \mathbb{R}^*$ ,  $\mathbb{R}^* \times \{0\}$ , and the diagonal of  $(\mathbb{R}^*)^2$  are next blown up in  $\mathcal{B}(\mathbb{R}^2, (0, 0))$ .

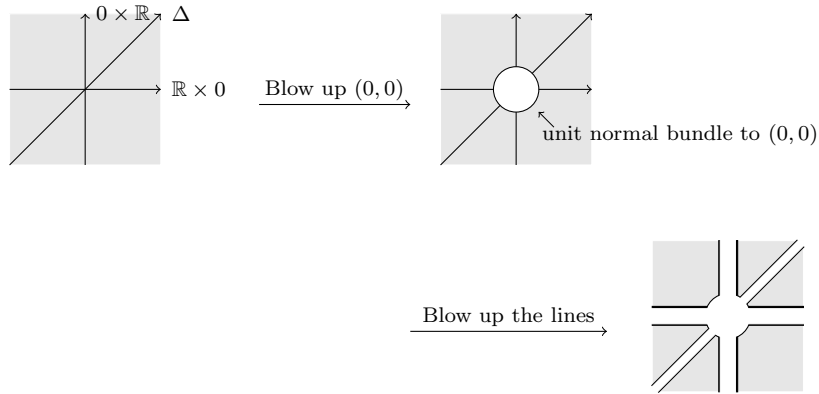


Figure 3.1: A composition of blow-ups

**Proposition 3.3.** *Let  $B$  and  $C$  be two smooth submanifolds transverse to the ridges of a  $C^\infty$  manifold  $A$ . Assume that  $C$  is a smooth submanifold of  $B$  transverse to the ridges of  $B$ .*

1. *The closure  $\overline{B \setminus C}$  of  $(B \setminus C)$  in  $\mathcal{B}(A, C)$  is a submanifold of  $\mathcal{B}(A, C)$ . It intersects*

$$SN(C) \subseteq \partial \mathcal{B}(A, C)$$

*as the unit normal bundle  $SN_B(C)$  to  $C$  in  $B$ . It is canonically diffeomorphic to  $\mathcal{B}(B, C)$ .*

2. *The blow-up  $\mathcal{B}(\mathcal{B}(A, C), \overline{B \setminus C})$  of  $\mathcal{B}(A, C)$  along  $\overline{B \setminus C}$  has a canonical differential structure of a manifold with ridges. The preimage of  $\overline{B \setminus C} \subset \mathcal{B}(A, C)$  in  $\mathcal{B}(\mathcal{B}(A, C), \overline{B \setminus C})$  under the canonical projection*

$$\mathcal{B}(\mathcal{B}(A, C), \overline{B \setminus C}) \longrightarrow \mathcal{B}(A, C)$$

*is the pull-back via the blowdown projection from  $\overline{B \setminus C}$  to  $B$  of the unit normal bundle to  $B$  in  $A$ .*

PROOF:

1. Let  $x \in C$ . It is always possible to choose an embedding  $\phi$  into  $A$  as in Definition 3.1 such that  $\phi(0) = x$  and  $\phi(\mathbb{R}^c \times \mathbb{R}^e \times [0, 1]^d)$  intersects  $C$  exactly along  $\phi(0 \times \mathbb{R}^e \times [0, 1]^d)$  and  $B$  exactly along  $\phi(0 \times \mathbb{R}^k \times [0, 1]^d)$ , with  $k > e$ . (First choose an embedding suitable for  $B$ , and then modify it to suit  $C$ .) Look at the induced chart  $\tilde{\phi}$  of  $\mathcal{B}\ell(A, C)$  near a point of  $\partial\mathcal{B}\ell(A, C)$ .

The intersection of  $(B \setminus C)$  with the image of  $\tilde{\phi}$  is

$$\tilde{\phi} \left( ]0, \infty[ \times (0 \times S^{k-e-1} \subset S^{c-1}) \times \mathbb{R}^e \times [0, 1]^d \right).$$

Thus, the closure of  $(B \setminus C)$  intersects the image of  $\tilde{\phi}$  as

$$\tilde{\phi} \left( [0, \infty[ \times (0 \times S^{k-e-1} \subset S^{c-1}) \times \mathbb{R}^e \times [0, 1]^d \right).$$

2. Together with the above given charts of  $\overline{B \setminus C}$ , the smooth injective map

$$\begin{aligned} \mathbb{R}^{c-k+e} \times S^{k-e-1} &\longrightarrow S^{c-1} \\ (u, y) &\longmapsto \frac{(u, y)}{\|(u, y)\|} \end{aligned}$$

identifies  $\mathbb{R}^{c-k+e}$  with the fibers of the normal bundle to  $\overline{B \setminus C}$  in  $\mathcal{B}\ell(A, C)$ . The blow-up process will therefore replace  $\overline{B \setminus C}$  with the quotient of the corresponding  $(\mathbb{R}^{c-k+e} \setminus \{0\})$ -bundle by  $]0, \infty[$ , which is of course the pull-back under the blowdown projection  $(\overline{B \setminus C} \longrightarrow B)$  of the unit normal bundle to  $B$  in  $A$ .

□

The fiber  $SN_c(C)$  is oriented as the boundary of a unit ball of  $N_c(C)$ .

### 3.2 The configuration space $C_2(R)$

Regard  $S^3$  as  $\mathbb{R}^3 \cup \{\infty\}$  or as two copies of  $\mathbb{R}^3$  identified along  $\mathbb{R}^3 \setminus \{0\}$  by the (exceptionally orientation-reversing) diffeomorphism  $x \mapsto x/\|x\|^2$ .

Let  $(-S_\infty^2)$  denote the unit normal bundle to  $\infty$  in  $S^3$ , so  $\mathcal{B}\ell(S^3, \infty) = \mathbb{R}^3 \cup S_\infty^2$  and  $\partial\mathcal{B}\ell(S^3, \infty) = S_\infty^2$ . There is a canonical orientation-preserving diffeomorphism  $p_\infty: S_\infty^2 \rightarrow S^2$ , such that  $x \in S_\infty^2$  is the limit in  $\mathcal{B}\ell(S^3, \infty)$  of a sequence of points of  $\mathbb{R}^3$  tending to  $\infty$  along any line of  $\mathbb{R}^3$  directed by  $p_\infty(x) \in S^2$ , in the direction of the line.

Let  $\mathring{B}_{1,\infty}$  (resp.  $B_{1,\infty}$ ) denote the complement of the closed (resp. open) ball of radius one of  $\mathbb{R}^3$  in  $S^3$ . Let  $\mathring{B}_{2,\infty}$  be the complement in  $S^3$  of the closed ball  $B(2)$  of radius 2 in  $\mathbb{R}^3$ .

Fix a rational homology sphere  $R$  and a point  $\infty$  of  $R$ . Set  $\check{R} = R \setminus \{\infty\}$ . Identify a neighborhood of  $\infty$  in  $R$  with  $\mathring{B}_{1,\infty}$ . The ball  $\mathring{B}_{2,\infty}$  is a smaller neighborhood of  $\infty$  in  $R$  via the understood identification. Then  $B_R = R \setminus \mathring{B}_{2,\infty}$  is a compact rational homology ball diffeomorphic to  $\mathcal{B}(R, \infty)$ .

Define the *configuration space*  $C_2(R)$  to be the compact 6-manifold with boundary and ridges obtained from  $R^2$  by blowing up first  $(\infty, \infty)$  in  $R^2$ , and next the closures of  $\{\infty\} \times \check{R}$ ,  $\check{R} \times \{\infty\}$ , and the diagonal of  $\check{R}^2$  in  $\mathcal{B}(R^2, (\infty, \infty))$ , as in Figure 3.1. Then  $\partial C_2(R)$  contains the unit normal bundle  $((T\check{R}^2/\Delta(T\check{R}^2)) \setminus \{0\})/\mathbb{R}^{+*}$  to the diagonal of  $\check{R}^2$ . This bundle is identified with the unit tangent bundle  $U\check{R}$  to  $\check{R}$  by the map  $([(x, y)] \mapsto [y - x])$ .

**Lemma 3.4.** *Let  $\check{C}_2(R) = \check{R}^2 \setminus \Delta(\check{R}^2)$ . The open manifold  $C_2(R) \setminus \partial C_2(R)$  is  $\check{C}_2(R)$ , and the inclusion  $\check{C}_2(R) \hookrightarrow C_2(R)$  is a homotopy equivalence. In particular,  $C_2(R)$  is a compactification of  $\check{C}_2(R)$  homotopy equivalent to  $\check{C}_2(R)$ . It has the same rational homology as the sphere  $S^2$ . The manifold  $C_2(R)$  is a smooth compact 6-dimensional manifold with boundary and ridges. Its boundary is*

$$\partial C_2(R) = p_{R^2}^{-1}(\infty, \infty) \cup (S_\infty^2 \times \check{R}) \cup (-\check{R} \times S_\infty^2) \cup U\check{R}.$$

Furthermore, there is a canonical smooth projection  $p_{R^2}: C_2(R) \rightarrow R^2$ .

PROOF: This lemma is a corollary of Propositions 3.2 and 3.3, and Lemma 2.8. We give a few additional arguments to check that we can perform the three blow-ups in  $\mathcal{B}(R^2, (\infty, \infty))$  simultaneously, and we take a closer look at the structure of  $p_{R^2}^{-1}(\infty, \infty)$  below.

Blowing up  $(\infty, \infty)$  in  $B_{1,\infty}^2$  transforms a neighborhood of  $(\infty, \infty)$  into the product  $[0, 1[ \times S^5$ . Explicitly, there is a map

$$\begin{aligned} \psi: [0, 1[ \times S^5 &\rightarrow \mathcal{B}(B_{1,\infty}^2, (\infty, \infty)) \\ (\lambda \in ]0, 1[, (x \neq 0, y \neq 0) \in S^5 \subset (\mathbb{R}^3)^2) &\mapsto \left( \frac{1}{\lambda\|x\|^2}x, \frac{1}{\lambda\|y\|^2}y \right) \\ (\lambda \in ]0, 1[, (0, y \neq 0) \in S^5 \subset (\mathbb{R}^3)^2) &\mapsto \left( \infty, \frac{1}{\lambda\|y\|^2}y \right) \\ (\lambda \in ]0, 1[, (x \neq 0, 0) \in S^5 \subset (\mathbb{R}^3)^2) &\mapsto \left( \frac{1}{\lambda\|x\|^2}x, \infty \right), \end{aligned}$$

which is a diffeomorphism onto its open image.

Here, the explicit image of  $(\lambda \in ]0, 1[, (x \neq 0, y \neq 0) \in S^5 \subset (\mathbb{R}^3)^2)$  is written in  $(\mathring{B}_{1,\infty} \setminus \{\infty\})^2 \subset \mathcal{B}(\mathring{B}_{1,\infty}^2, (\infty, \infty))$ , where  $\mathring{B}_{1,\infty} \setminus \{\infty\} \subset \mathbb{R}^3$ . The

image of  $\psi$  is a neighborhood of the preimage of  $(\infty, \infty)$  under the blowdown map

$$\mathcal{B}\ell(R^2, (\infty, \infty)) \xrightarrow{p_1} R^2.$$

This neighborhood respectively intersects  $\infty \times \check{R}$ ,  $\check{R} \times \infty$ , and  $\Delta(\check{R}^2)$  as  $\psi(]0, 1[ \times 0 \times S^2)$ ,  $\psi(]0, 1[ \times S^2 \times 0)$ , and  $\psi(]0, 1[ \times (S^5 \cap \Delta((\mathbb{R}^3)^2)))$ . In particular, the closures of  $\infty \times \check{R}$ ,  $\check{R} \times \infty$ , and  $\Delta(\check{R}^2)$  in  $\mathcal{B}\ell(R^2, (\infty, \infty))$  intersect the boundary  $\psi(0 \times S^5)$  of  $\mathcal{B}\ell(R^2, (\infty, \infty))$  as three disjoint spheres in  $S^5$ . They are respectively isomorphic to  $\infty \times \mathcal{B}\ell(R, \infty)$ ,  $\mathcal{B}\ell(R, \infty) \times \infty$ , and  $\Delta(\mathcal{B}\ell(R, \infty)^2)$ . Thus, the next steps will be three blow-ups along these three disjoint smooth manifolds.

These blow-ups will preserve the product structure  $\psi(]0, 1[ \times \cdot)$ . Thus,  $C_2(R)$  is a smooth compact 6-dimensional manifold with boundary, with three ridges  $S^2 \times S^2$  in  $p_{R^2}^{-1}(\infty, \infty)$ . A neighborhood of these ridges in  $C_2(R)$  is diffeomorphic to  $]0, 1[^2 \times S^2 \times S^2$ .  $\square$

Let  $\iota_{S^2}$  denote the *antipodal map* of  $S^2$ , which sends  $x$  to  $\iota_{S^2}(x) = -x$ .

**Lemma 3.5.** *The map  $p_{S^2}$  of Lemma 2.7 extends smoothly to  $C_2(S^3)$ . Its extension  $p_{S^2}$  satisfies*

$$p_{S^2} = \begin{cases} \iota_{S^2} \circ p_\infty \circ p_1 & \text{on } S_\infty^2 \times \mathbb{R}^3 \\ p_\infty \circ p_2 & \text{on } \mathbb{R}^3 \times S_\infty^2 \\ p_2 & \text{on } U\mathbb{R}^3 = \mathbb{R}^3 \times S^2, \end{cases}$$

where  $p_1$  and  $p_2$  denote the projections on the first and second factor with respect to the above expressions.

PROOF: Near the diagonal of  $\mathbb{R}^3$ , we have a chart of  $C_2(S^3)$

$$\psi_d: \mathbb{R}^3 \times ]0, \infty[ \times S^2 \longrightarrow C_2(S^3),$$

which maps  $(x \in \mathbb{R}^3, \lambda \in ]0, \infty[, y \in S^2)$  to  $(x, x + \lambda y) \in (\mathbb{R}^3)^2$ . Here,  $p_{S^2}$  extends as the projection onto the  $S^2$  factor.

Consider the orientation-reversing embedding  $\phi_\infty$

$$\begin{aligned} \phi_\infty: \mathbb{R}^3 &\longrightarrow (S^3 = \mathbb{R}^3 \cup \{\infty\}) \\ \mu(x \in S^2) &\mapsto \begin{cases} \infty & \text{if } \mu = 0 \\ \frac{1}{\mu}x & \text{otherwise.} \end{cases} \end{aligned}$$

Note that this chart induces the identification  $p_\infty$  of the unit normal bundle  $S_\infty^2$  to  $\{\infty\}$  in  $S^3$  with  $S^2$ . When  $\mu \neq 0$ ,

$$p_{S^2}(\phi_\infty(\mu x), y \in \mathbb{R}^3) = \frac{\mu y - x}{\|\mu y - x\|}.$$



Then  $p_{S^2}$  can be extended smoothly on  $S_\infty^2 \times \mathbb{R}^3$  (where  $\mu = 0$ ) by

$$p_{S^2}(x \in S_\infty^2, y \in \mathbb{R}^3) = -x.$$

Similarly, set  $p_{S^2}(x \in \mathbb{R}^3, y \in S_\infty^2) = y$ . Now, with the map  $\psi$  of the proof of Lemma 3.4, if  $x$  and  $y$  are not equal to zero and if they are distinct, then we have

$$p_{S^2} \circ \psi \left( (\lambda, (x, y)) \right) = \frac{\frac{y}{\|y\|^2} - \frac{x}{\|x\|^2}}{\left\| \frac{y}{\|y\|^2} - \frac{x}{\|x\|^2} \right\|} = \frac{\|x\|^2 y - \|y\|^2 x}{\| \|x\|^2 y - \|y\|^2 x \|}$$

when  $\lambda \neq 0$ . This map extends to  $\text{Bl}(R^2, (\infty, \infty))$  outside the boundaries of  $\infty \times \text{Bl}(R, \infty)$ ,  $\text{Bl}(R, \infty) \times \infty$  and  $\Delta(\text{Bl}(R, \infty)^2)$ , naturally, by keeping the same formula when  $\lambda = 0$ .

Let us check that the map  $p_{S^2}$  extends smoothly over the boundary of  $\Delta(\text{Bl}(R, \infty)^2)$ . There is a chart of  $C_2(R)$  near the preimage of this boundary in  $C_2(R)$

$$\psi_2: [0, \infty[ \times [0, \infty[ \times S^2 \times S^2 \longrightarrow C_2(S^3),$$

which maps  $(\lambda \in ]0, \infty[, \mu \in ]0, \infty[, x \in S^2, y \in S^2)$  to  $(\phi_\infty(\lambda x), \phi_\infty(\lambda(x + \mu y)))$ . With these coordinates, the map  $p_{S^2}$  may be written as

$$(\lambda, \mu, x, y) \mapsto \frac{y - 2\langle x, y \rangle x - \mu x}{\|y - 2\langle x, y \rangle x - \mu x\|}.$$

Therefore, it extends smoothly along  $\mu = 0$ . We check that  $p_{S^2}$  extends smoothly over the boundaries of  $\infty \times \text{Bl}(R, \infty)$  and  $\text{Bl}(R, \infty) \times \infty$  similarly.  $\square$

**Definition 3.6.** Let  $\tau_s$  denote the standard parallelization of  $\mathbb{R}^3$ . Say that a parallelization

$$\tau: \check{R} \times \mathbb{R}^3 \rightarrow T\check{R}$$

of  $\check{R}$  that coincides with  $\tau_s$  on  $\mathring{B}_{2, \infty} \setminus \{\infty\}$  is *asymptotically standard*. According to Proposition 5.5, such a parallelization exists. Such a parallelization identifies  $U\check{R}$  with  $\check{R} \times S^2$ .

**Proposition 3.7.** For any asymptotically standard parallelization  $\tau$  of  $\check{R}$ , there exists a smooth map  $p_\tau: \partial C_2(R) \rightarrow S^2$  such that

$$p_\tau = \begin{cases} p_{S^2} & \text{on } p_{R^2}^{-1}(\infty, \infty) \\ \iota_{S^2} \circ p_\infty \circ p_1 & \text{on } S_\infty^2 \times \check{R} \\ p_\infty \circ p_2 & \text{on } \check{R} \times S_\infty^2 \\ p_2 & \text{on } U\check{R} \stackrel{\tau}{=} \check{R} \times S^2, \end{cases}$$

where  $p_1$  and  $p_2$  denote the projections on the first and second factors with respect to the above expressions.

PROOF: This is a consequence of Lemma 3.5.  $\square$

**Definition 3.8.** Define an *asymptotic rational homology*  $\mathbb{R}^3$  to be a pair  $(\check{R}, \tau)$ , in which  $\check{R}$  is the union over  $]1, 2] \times S^2$  of a rational homology ball  $B_R$  and the complement  $\mathring{B}_{1,\infty} \setminus \{\infty\}$  of the unit ball in  $\mathbb{R}^3$ , and  $\tau$  is an asymptotically standard parallelization of  $\check{R}$ . Such a pair  $(\check{R}, \tau)$  defines the rational homology sphere  $R = \check{R} \cup \{\infty\}$  canonically. So “Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ ” is a shortcut for “Let  $R$  be a rational homology sphere, equipped with an embedding  $\phi_R: \mathring{B}_{1,\infty} \hookrightarrow R$  and an asymptotically standard parallelization  $\tau$  of the complement  $\check{R} = R \setminus \{\phi_R(\infty)\}$ , with respect to  $\phi_R$ ”. (The embedding  $\phi_R$  is understood. We will not mention it anymore. We will just view it as an inclusion  $\mathring{B}_{1,\infty} \hookrightarrow R$  and denote  $\phi_R(\infty)$  by  $\infty$ .)

### 3.3 On propagators

A *volume-one form of  $S^2$*  is a 2-form  $\omega_S$  of  $S^2$  such that  $\int_{S^2} \omega_S = 1$ . (See Appendix B for a short survey of differential forms and de Rham cohomology.)

Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . Recall the map

$$p_\tau: \partial C_2(R) \rightarrow S^2$$

of Proposition 3.7.

**Definition 3.9.** A *propagating chain* of  $(C_2(R), \tau)$  is a 4-chain  $P$  of  $C_2(R)$  such that  $\partial P = p_\tau^{-1}(a)$  for some  $a \in S^2$ . A *propagating form* of  $(C_2(R), \tau)$  is a closed 2-form  $\omega$  on  $C_2(R)$  whose restriction to  $\partial C_2(R)$  may be expressed as  $p_\tau^*(\omega_S)$  for some volume-one form  $\omega_S$  of  $S^2$ . We will call propagating chains and propagating forms *propagators* when their nature is clear from the context.

**Examples 3.10.** Recall the map  $p_{S^2}: C_2(S^3) \rightarrow S^2$  of Lemma 3.5. For any  $a \in S^2$ ,  $p_{S^2}^{-1}(a)$  is a propagating chain of  $(C_2(S^3), \tau_s)$ . For any 2-form  $\omega_S$  of  $S^2$  such that  $\int_{S^2} \omega_S = 1$ ,  $p_{S^2}^*(\omega_S)$  is a propagating form of  $(C_2(S^3), \tau_s)$ .

For our general  $\mathbb{Q}$ -sphere  $R$ , propagating chains of  $(C_2(R), \tau)$  exist because the 3-cycle  $p_\tau^{-1}(a)$  of  $\partial C_2(R)$  bounds in  $C_2(R)$  since  $H_3(C_2(R); \mathbb{Q}) = 0$ . Dually, propagating forms of  $(C_2(R), \tau)$  exist because the restriction induces a surjective map

$$H^2(C_2(R); \mathbb{R}) \rightarrow H^2(\partial C_2(R); \mathbb{R})$$

since  $H^3(C_2(R), \partial C_2(R); \mathbb{R})$  is trivial.

When  $R$  is a  $\mathbb{Z}$ -sphere, there exist propagating chains of  $(C_2(R), \tau)$  that are smooth 4-manifolds properly embedded in  $C_2(R)$ . See Corollary 11.10.

**Definition 3.11.** A *propagating form* of  $C_2(R)$  is a closed 2-form  $\omega_p$  on  $C_2(R)$  whose restriction to  $\partial C_2(R) \setminus UB_R$  is equal to  $p_\tau^*(\omega)$  for some volume-one form  $\omega_S$  of  $S^2$  and some asymptotically standard parallelization  $\tau$ . Similarly, a *propagating chain* of  $C_2(R)$  is a 4-chain  $P$  of  $C_2(R)$  such that  $\partial P \subset \partial C_2(R)$  and  $\partial P \cap (\partial C_2(R) \setminus UB_R)$  is equal to  $p_\tau^{-1}(a)$  for some  $a \in S^2$ . (These definitions do not depend on  $\tau$ .)

Explicit propagating chains associated to Heegaard splittings, constructed with Greg Kuperberg, are described in Subsection 1.2.8. They are integral chains multiplied by  $\frac{1}{|H_1(R; \mathbb{Z})|}$ , where  $|H_1(R; \mathbb{Z})|$  is the cardinality of  $H_1(R; \mathbb{Z})$ .

Since  $C_2(R)$  is homotopy equivalent to  $(\check{R}^2 \setminus \Delta(\check{R}^2))$ , Lemma 2.8 ensures that  $H_2(C_2(R); \mathbb{Q}) = \mathbb{Q}[S]$ , where the canonical generator  $[S]$  is the homology class of a fiber of  $U\check{R} \subset \partial C_2(R)$ . For a two-component link  $(J, K)$  of  $\check{R}$ , the homology class  $[J \times K]$  of  $J \times K$  in  $H_2(C_2(R); \mathbb{Q})$  is  $lk(J, K)[S]$ , according to Proposition 2.9.

**Lemma 3.12.** *Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . Let  $C$  be a two-cycle of  $C_2(R)$ . For any propagating chain  $P$  of  $C_2(R)$  transverse to  $C$  and for any propagating form  $\omega$  of  $C_2(R)$ , we have*

$$[C] = \left( \int_C \omega \right) [S] = \langle C, P \rangle_{C_2(R)} [S]$$

in  $H_2(C_2(R); \mathbb{Q}) = \mathbb{Q}[S]$ . In particular, we have

$$lk(J, K) = \int_{J \times K} \omega = \langle J \times K, P \rangle_{C_2(R)}.$$

for any two-component link  $(J, K)$  of  $\check{R}$ .

PROOF: Fix a propagating chain  $P$ , the algebraic intersection  $\langle C, P \rangle_{C_2(R)}$  depends only on the homology class  $[C]$  of  $C$  in  $C_2(R)$ . Similarly, since  $\omega$  is closed,  $\int_C \omega$  depends only on  $[C]$ . (Indeed, if  $C$  and  $C'$  cobound a chain  $D$  transverse to  $P$ , then  $C \cap P$  and  $C' \cap P$  cobound  $\pm(D \cap P)$ , and  $\int_{\partial D = C' - C} \omega = \int_D d\omega$  according to Stokes' theorem.) Furthermore, the dependence on  $[C]$  is linear. Therefore, it suffices to check the lemma for a chain that represents the canonical generator  $[S]$  of  $H_2(C_2(R); \mathbb{Q})$ . Any fiber of  $U\check{R}$  is such a chain.  $\square$

**Definition 3.13.** A propagating form  $\omega$  of  $C_2(R)$  is *homogeneous* if its restriction to  $\partial C_2(R) \setminus UB_R$  is  $p_\tau^*(\omega_{S^2})$  for the homogeneous volume form  $\omega_{S^2}$  of  $S^2$  of total volume 1.

**Definition 3.14.** Let  $\iota$  be the involution of  $C_2(R)$  that exchanges the two coordinates in  $\check{R}^2 \setminus \Delta(\check{R}^2)$ . An *antisymmetric* propagating form is a propagating form such that  $\iota_*(\omega) = -\omega$ .

**Example 3.15.** The propagating form  $p_{S^2}^*(\omega_{S^2})$  of  $(C_2(S^3), \tau)$  is antisymmetric.

The following easy lemma ensures the existence of antisymmetric propagating forms of  $(C_2(R), \tau)$ .

**Lemma 3.16.** *Let  $\omega_0$  be a propagating form of  $(C_2(R), \tau)$ . Then  $(-\iota^*(\omega_0))$  and  $\omega = \frac{1}{2}(\omega_0 - \iota^*(\omega_0))$  are propagating forms of  $(C_2(R), \tau)$ , and we have  $\iota^*(\omega) = -\omega$ . If  $\omega_0$  is homogeneous, then  $(-\iota^*(\omega_0))$  and  $\omega = \frac{1}{2}(\omega_0 - \iota^*(\omega_0))$  are also homogeneous.*

PROOF: Let  $\omega_S$  be the volume-one form of  $S^2$  such that  $\omega_0|_{\partial C_2(R)} = p_\tau^*(\omega_S)$ . Recall the antipodal map  $\iota_{S^2}$  of  $S^2$ . The form  $(-\iota_{S^2}^*(\omega_S))$  is a volume-one form of  $S^2$ , and we have  $(-\iota^*(\omega_0))|_{\partial C_2(R)} = p_\tau^*(-\iota_{S^2}^*(\omega_S))$ .  $\square$

Also note the following lemma.

**Lemma 3.17.** *Let  $\omega_a$  and  $\omega'_a$  be two propagating forms of  $(C_2(R), \tau)$ , respectively restricting to  $\partial C_2(R)$  as  $p_\tau^*(\omega_A)$  and  $p_\tau^*(\omega'_A)$ , for two volume-one forms  $\omega_A$  and  $\omega'_A$  of  $S^2$ . There exists a one-form  $\eta_A$  on  $S^2$  such that  $\omega'_A = \omega_A + d\eta_A$ . For any such  $\eta_A$ , there exists a one-form  $\eta$  on  $C_2(R)$  such that  $\omega'_a - \omega_a = d\eta$  and the restriction of  $\eta$  to  $\partial C_2(R)$  is  $p_\tau^*(\eta_A)$ .*

PROOF: Since  $\omega_a$  and  $\omega'_a$  are cohomologous, there exists a one-form  $\eta$  on  $C_2(R)$  such that  $\omega'_a = \omega_a + d\eta$ . Similarly, since  $\int_{S^2} \omega'_A = \int_{S^2} \omega_A$ , there exists a one-form  $\eta_A$  on  $S^2$  such that  $\omega'_A = \omega_A + d\eta_A$ . We have  $d(\eta - p_\tau^*(\eta_A)) = 0$  on  $\partial C_2(R)$ . The exact sequence with real coefficients

$$0 = H^1(C_2(R)) \longrightarrow H^1(\partial C_2(R)) \longrightarrow H^2(C_2(R), \partial C_2(R)) \cong H_4(C_2(R)) = 0,$$

implies  $H^1(\partial C_2(R); \mathbb{R}) = 0$ . Therefore, there exists a function  $f$  from  $\partial C_2(R)$  to  $\mathbb{R}$  such that

$$df = \eta - p_\tau^*(\eta_A)$$

on  $\partial C_2(R)$ . Extend  $f$  to a  $C^\infty$  map on  $C_2(R)$  and replace  $\eta$  with  $(\eta - df)$ .  $\square$



# Chapter 4

## The Theta invariant

Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . We are ready to define the topological invariant  $\Theta$  for  $(\check{R}, \tau)$  to be the algebraic triple intersection of three propagating chains of  $(C_2(R), \tau)$ , in Section 4.1. In Section 4.3, we use relative Pontrjagin classes introduced in Section 4.2 to turn  $\Theta$  into a topological invariant of rational homology 3-spheres.

### 4.1 The $\Theta$ -invariant of $(R, \tau)$

We defined the algebraic intersection  $\langle A, B, C \rangle_D$  of three transverse compact submanifolds  $A, B, C$  in a manifold  $D$  such that the sum of the codimensions of  $A, B$ , and  $C$  is the dimension of  $D$  in Subsection 2.1.5. It is the sum over the intersection points  $a$  of  $A \cap B \cap C$  of the associated signs, where the sign of  $a$  is positive if and only if the orientation of  $D$  is induced by the orientation of  $N_a A \oplus N_a B \oplus N_a C$ , where  $N_a A, N_a B$ , and  $N_a C$  are respectively identified with  $(T_a A)^\perp, (T_a B)^\perp$ , and  $(T_a C)^\perp$  with the help of a Riemannian metric.

**Theorem 4.1.** *Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . Let  $P_a, P_b$ , and  $P_c$  be three transverse propagators of  $(C_2(R), \tau)$  with respective boundaries  $p_\tau^{-1}(a), p_\tau^{-1}(b)$ , and  $p_\tau^{-1}(c)$  for three distinct points  $a, b$ , and  $c$  of  $S^2$ . Then*

$$\Theta(R, \tau) = \langle P_a, P_b, P_c \rangle_{C_2(R)}$$

*does not depend on the chosen propagators  $P_a, P_b$ , and  $P_c$ . It is a topological invariant of  $(R, \tau)$ . For any three propagating chains  $\omega_a, \omega_b$ , and  $\omega_c$  of  $(C_2(R), \tau)$ , we have*

$$\Theta(R, \tau) = \int_{C_2(R)} \omega_a \wedge \omega_b \wedge \omega_c.$$

PROOF: Since  $H_4(C_2(R); \mathbb{Q}) = 0$ , if the propagator  $P_a$  is replaced by a propagator  $P'_a$  with the same boundary, the 4-dimensional cycle  $(P'_a - P_a)$  bounds a 5-dimensional rational chain  $W$  transverse to  $P_b \cap P_c$ . The 1-dimensional chain  $W \cap P_b \cap P_c$  does not meet  $\partial C_2(R)$  since  $P_b \cap P_c$  does not meet  $\partial C_2(R)$ . Therefore, up to a well-determined sign, the boundary of  $W \cap P_b \cap P_c$  is  $P'_a \cap P_b \cap P_c - P_a \cap P_b \cap P_c$ , as in Figure 4.1. This proves

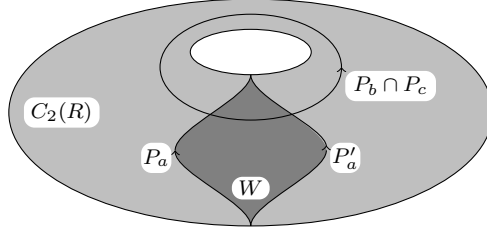


Figure 4.1: Proof of Theorem 4.1

that  $\langle P_a, P_b, P_c \rangle_{C_2(R)}$  is independent of  $P_a$  when  $a$  is fixed. Similarly, it is independent of  $P_b$  and  $P_c$  when  $b$  and  $c$  are fixed. Thus,  $\langle P_a, P_b, P_c \rangle_{C_2(R)}$  is a rational function on the connected set of triples  $(a, b, c)$  of distinct point of  $S^2$ . It is easy to see that this function is continuous. Therefore, it is constant.

Let us similarly prove that  $\int_{C_2(R)} \omega_a \wedge \omega_b \wedge \omega_c$  is independent of the propagating forms  $\omega_a$ ,  $\omega_b$ , and  $\omega_c$ . Let  $\omega'_a$  be a propagating form of  $(C_2(R), \tau)$ . Lemma 3.17 implies the existence of forms  $\eta$  and  $\eta_A$  such that  $\omega'_a - \omega_a = d\eta$  and the restriction of  $\eta$  to  $\partial C_2(R)$  is  $p_\tau^*(\eta_A)$ . So we have

$$\begin{aligned} \int_{C_2(R)} \omega'_a \wedge \omega_b \wedge \omega_c - \int_{C_2(R)} \omega_a \wedge \omega_b \wedge \omega_c &= \int_{C_2(R)} d(\eta \wedge \omega_b \wedge \omega_c) \\ &= \int_{\partial C_2(R)} \eta \wedge \omega_b \wedge \omega_c \\ &= \int_{\partial C_2(R)} p_\tau^*(\eta_A \wedge \omega_b \wedge \omega_c) = 0 \end{aligned}$$

since any 5-form on  $S^2$  vanishes. Thus,  $\int_{C_2(R)} \omega_a \wedge \omega_b \wedge \omega_c$  is independent of the propagating forms  $\omega_a$ ,  $\omega_b$ , and  $\omega_c$ . Now, we can choose the propagating forms  $\omega_a$ ,  $\omega_b$ , and  $\omega_c$  supported in tiny neighborhoods of  $P_a$ ,  $P_b$ , and  $P_c$  and respectively Poincaré dual to  $P_a$ ,  $P_b$ , and  $P_c$ . So the intersection of the three supports is a very small neighborhood of  $P_a \cap P_b \cap P_c$ , and we can easily see that  $\int_{C_2(R)} \omega_a \wedge \omega_b \wedge \omega_c = \langle P_a, P_b, P_c \rangle_{C_2(R)}$ . See Section 11.4, Section B.2, and Lemma B.4 in particular, for more details.  $\square$

In particular,  $\Theta(R, \tau)$  is equal to  $\int_{C_2(R)} \omega^3$  for any propagating chain  $\omega$  of  $(C_2(R), \tau)$ . Since such a propagating chain represents the linking number,  $\Theta(R, \tau)$  can be thought of as the *cube of the linking number with respect to  $\tau$* .

When  $\tau$  varies continuously,  $\Theta(R, \tau)$  varies continuously in  $\mathbb{Q}$ . So  $\Theta(R, \tau)$  is an invariant of the homotopy class of  $\tau$ .

**Remark 4.2.** Define a *combing* of  $\check{R}$  to be a section of  $U\check{R}$  that coincides with  $\tau_s(v)$  outside  $B_R$ , for some unit vector  $v$  of  $\mathbb{R}^3$ . Let  $X$  be a combing of  $\check{R}$ . Define a *propagating chain* of  $(C_2(R), X)$  to be a propagating chain of  $C_2(R)$  that intersects  $U\check{R}$  along the image of  $X$ . Define  $\tilde{\Theta}(R, X)$  to be the algebraic intersection of a propagating chain of  $(C_2(R), X)$ , a propagating chain of  $(C_2(R), -X)$ , and any other propagating chain of  $C_2(R)$ . It is easily proved in [Les15a, Theorem 2.1] that  $\tilde{\Theta}(R, X)$  depends only on  $R$  and on the homotopy class of  $X$  among combings. In particular,  $\Theta(R, \tau) = \tilde{\Theta}(R, \tau(v))$  depends only on the homotopy class of the combing  $\tau(v)$  of  $U\check{R}$ , for some unit vector  $v$  of  $\mathbb{R}^3$ . Further properties of the invariant  $\tilde{\Theta}(R, \cdot)$  of combings are studied in [Les15c]. I found an explicit formula [Les15a, Theorem 3.8] for the invariant  $\tilde{\Theta}(R, \cdot)$  from a Heegaard diagram of  $R$ . I computed it directly using the above definition of  $\tilde{\Theta}(R, \cdot)$  together with propagators associated to Morse functions constructed with Greg Kuperberg in [Les15a].

**Example 4.3.** Using (disjoint!) propagators  $p_{S^2}^{-1}(a)$ ,  $p_{S^2}^{-1}(b)$ ,  $p_{S^2}^{-1}(c)$  associated to three distinct points  $a$ ,  $b$ , and  $c$  of  $\mathbb{R}^3$ , as in Example 3.10, it is clear that

$$\Theta(S^3, \tau_s) = \langle p_{S^2}^{-1}(a), p_{S^2}^{-1}(b), p_{S^2}^{-1}(c) \rangle_{C_2(S^3)}$$

is zero.

## 4.2 Parallelizations of 3-manifolds and Pontrjagin classes

In this subsection,  $M$  denotes a smooth, compact oriented 3-manifold with possible boundary  $\partial M$ . Recall that a well-known theorem reproved in Section 5.2 ensures that such a 3-manifold is parallelizable.

Let  $GL^+(\mathbb{R}^3)$  denote the group of orientation-preserving linear isomorphisms of  $\mathbb{R}^3$ . Let  $[(M, \partial M), (GL^+(\mathbb{R}^3), 1)]_m$  denote the set of (continuous) maps

$$g: (M, \partial M) \longrightarrow (GL^+(\mathbb{R}^3), 1)$$

from  $M$  to  $GL^+(\mathbb{R}^3)$  that send  $\partial M$  to the identity element 1 of  $GL^+(\mathbb{R}^3)$ .

Let  $[(M, \partial M), (GL^+(\mathbb{R}^3), 1)]$  denote the group of homotopy classes of such maps, with the group structure induced by the multiplication of maps, using the multiplication in  $GL^+(\mathbb{R}^3)$ . For a map  $g$  in  $[(M, \partial M), (GL^+(\mathbb{R}^3), 1)]_m$ , define

$$\begin{aligned} \psi_{\mathbb{R}}(g): \quad M \times \mathbb{R}^3 &\longrightarrow M \times \mathbb{R}^3 \\ (x, y) &\mapsto (x, g(x)(y)). \end{aligned}$$



Let  $\tau_M: M \times \mathbb{R}^3 \rightarrow TM$  be a parallelization of  $M$ . Then any parallelization  $\tau$  of  $M$  that coincides with  $\tau_M$  on  $\partial M$  may be written as

$$\tau = \tau_M \circ \psi_{\mathbb{R}}(g)$$

for some  $g \in [(M, \partial M), (GL^+(\mathbb{R}^3), 1)]_m$ .

Thus, fixing  $\tau_M$  identifies the set of homotopy classes of parallelizations of  $M$  fixed on  $\partial M$  with the group  $[(M, \partial M), (GL^+(\mathbb{R}^3), 1)]$ . Since  $GL^+(\mathbb{R}^3)$  deformation retracts onto the group  $SO(3)$  of orientation-preserving linear isometries of  $\mathbb{R}^3$ , the group  $[(M, \partial M), (GL^+(\mathbb{R}^3), 1)]$  is isomorphic to  $[(M, \partial M), (SO(3), 1)]$ .

**Definition 4.4.** We regard  $S^3$  as  $B^3/\partial B^3$ , where  $B^3$  is the standard unit ball of  $\mathbb{R}^3$  viewed as the quotient of  $[0, 1] \times S^2$  where all points of  $\{0\} \times S^2$  are identified with each other. Let  $\chi_\pi: [0, 1] \rightarrow [0, 2\pi]$  be an increasing smooth bijection whose derivatives vanish at 0 and 1 such that  $\chi_\pi(1-\theta) = 2\pi - \chi_\pi(\theta)$  for any  $\theta \in [0, 1]$ . Let  $\rho: B^3 \rightarrow SO(3)$  be the map that sends  $(\theta \in [0, 1], v \in S^2)$  to the rotation  $\rho(\chi_\pi(\theta); v)$  with axis directed by  $v$  and with angle  $\chi_\pi(\theta)$ .

This map  $\rho$  induces the double covering map  $\tilde{\rho}: S^3 \rightarrow SO(3)$ , which orients  $SO(3)$  and which allows one to deduce the first three homotopy groups of  $SO(3)$  from those of  $S^3$ . The first three homotopy groups of  $SO(3)$  are  $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$ ,  $\pi_2(SO(3)) = 0$ , and  $\pi_3(SO(3)) = \mathbb{Z}[\tilde{\rho}]$ . For  $v \in S^2$ ,  $\pi_1(SO(3))$  is generated by the class of the loop that maps  $\exp(i\theta) \in S^1$  to the rotation  $\rho(\theta; v)$ . See Section A.2 and Theorem A.14 in particular.

Note that a map  $g$  from  $(M, \partial M)$  to  $(SO(3), 1)$  has a degree  $\deg(g)$ . The *degree*  $\deg(g)$  is the differential degree at a regular value (different from 1) of  $g$ . It can also be defined homologically, by  $H_3(g)[M, \partial M] = \deg(g)[SO(3), 1]$ .

We prove the following theorem, for which we claim no originality, in Chapter 5, as a direct consequence of Definition 5.13, Lemmas 5.2, 5.7, 5.8, and Propositions 5.10, 5.22, and 5.26.

**Theorem 4.5.** *For any compact connected oriented 3-manifold  $M$ , the group*

$$[(M, \partial M), (SO(3), 1)]$$

*is abelian and the degree*

$$\deg: [(M, \partial M), (SO(3), 1)] \longrightarrow \mathbb{Z}$$

*is a group homomorphism, which induces an isomorphism*

$$\deg: [(M, \partial M), (SO(3), 1)] \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}.$$

When  $\partial M$  is equal to  $\emptyset$  or  $S^2$ , there exists a canonical map  $p_1$  from the set of homotopy classes of parallelizations of  $M$ —that coincide with  $\tau_s$  near  $S^2$  if  $\partial M = S^2$ —to  $\mathbb{Z}$  such that

- we have  $p_1((\tau_s)|_{B^3}) = 0$  and
- for any map  $g$  in  $[(M, \partial M), (SO(3), 1)]_m$  and for any trivialization  $\tau$  of  $TM$ , we have

$$p_1(\tau \circ \psi_{\mathbb{R}}(g)) - p_1(\tau) = 2 \deg(g).$$

Definition 5.13 of the map  $p_1$  involves relative Pontrjagin classes. When  $\partial M = \emptyset$ , the map  $p_1$  coincides with the map  $h$  studied by Friedrich Hirzebruch in [Hir73, §3.1], and by Robion Kirby and Paul Melvin in [KM99] under the name of *Hirzebruch defect*.

Since  $[(M, \partial M), (SO(3), 1)]$  is abelian, the set of parallelizations of  $M$  that are fixed on  $\partial M$  is an affine space with translation group  $[(M, \partial M), (SO(3), 1)]$ .

Recall the map  $\rho: B^3 \rightarrow SO(3)$  from Definition 4.4. Let  $M$  be an oriented connected 3-manifold with possible boundary. For a ball  $B^3$  embedded in  $M$ , let  $\rho_M(B^3) \in [(M, \partial M), (SO(3), 1)]_m$  be a smooth map that coincides with  $\rho$  on  $B^3$  and that maps the complement of  $B^3$  to the identity element of  $SO(3)$ . The homotopy class of  $\rho_M(B^3)$  is well-defined.

**Lemma 4.6.** *We have  $\deg(\rho_M(B^3)) = 2$ .*

PROOF: Exercise. □

### 4.3 Defining a $\mathbb{Q}$ -sphere invariant from $\Theta$

Recall that an asymptotic rational homology  $\mathbb{R}^3$  is a pair  $(\check{R}, \tau)$ , where  $\check{R}$  is decomposed as the union over  $]1, 2] \times S^2$  of a rational homology ball  $B_R$  and the complement  $\check{B}_{1,\infty} \setminus \{\infty\}$  of the unit ball of  $\mathbb{R}^3$ , and  $\tau$  is an asymptotically standard parallelization of  $\check{R}$ . In this subsection, we prove the following proposition.

**Proposition 4.7.** *Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . For any map  $g$  in  $[(B_R, B_R \cap \check{B}_{1,\infty}), (SO(3), 1)]_m$  trivially extended to  $\check{R}$ , we have*

$$\Theta(R, \tau \circ \psi_{\mathbb{R}}(g)) - \Theta(R, \tau) = \frac{1}{2} \deg(g).$$

Theorem 4.5 allows us to derive the following corollary from Proposition 4.7.

**Corollary 4.8.** *Set  $\Theta(R) = \Theta(R, \tau) - \frac{1}{4}p_1(\tau)$ . Then  $\Theta(R)$  is an invariant of  $\mathbb{Q}$ -spheres.*

□

Since  $p_1(\tau_s) = 0$ , Example 4.3 shows that  $\Theta(S^3) = 0$ . We will prove  $\Theta(-R) = -\Theta(R)$  for any  $\mathbb{Q}$ -sphere  $R$  in Proposition 5.15.

More properties of  $\Theta$  will appear later in this book. We will first view this invariant as the degree one part of a much more general invariant  $\mathcal{Z}$  or  $\mathfrak{z}$  (introduced in Theorem 7.20 and in Corollary 10.9, respectively) in Corollary 10.11. The multiplicativity of  $\mathcal{Z}$  under connected sum stated in Theorem 10.26 implies that  $\Theta$  is additive under connected sum. The invariant  $\Theta$  will be identified with  $6\lambda_{CW}$ , where  $\lambda_{CW}$  is the Walker generalization of the Casson invariant to  $\mathbb{Q}$ -spheres, in Section 18.6. See Theorem 18.31. (The Casson–Walker invariant  $\lambda_{CW}$  is normalized as  $\frac{1}{2}\lambda_W$  for rational homology spheres, where  $\lambda_W$  is the Walker normalisation in [Wal92].) The equality  $\Theta = 6\lambda_{CW}$  will be obtained as a consequence of a *universality property* of  $\mathcal{Z}$  with respect to a theory of finite type invariants. We will also present a direct proof of a surgery formula satisfied by  $\Theta$  in Section 18.3.

Let us begin the proof of Proposition 4.7 with the following lemma.

**Lemma 4.9.** *The variation  $\Theta(R, \tau \circ \psi_{\mathbb{R}}(g)) - \Theta(R, \tau)$  in Proposition 4.7 is independent of  $\tau$ . Set  $\Theta'(g) = \Theta(R, \tau \circ \psi_{\mathbb{R}}(g)) - \Theta(R, \tau)$ . Then  $\Theta'$  is a homomorphism from  $\left[(B_R, B_R \cap \mathring{B}_{1,\infty}), (SO(3), 1)\right]$  to  $\mathbb{Q}$ .*

PROOF: For  $d = a, b$  or  $c$ , the propagator  $P_d$  of  $(C_2(R), \tau)$  of Theorem 4.1 can be assumed to be a product  $[-1, 0] \times p_{\tau}|_{UB_R}^{-1}(d)$  on a collar  $[-1, 0] \times UB_R$  of  $UB_R$  in  $C_2(R)$ . Since  $H_3([-1, 0] \times UB_R; \mathbb{Q}) = 0$ , the cycle

$$\left(\partial([-1, 0] \times p_{\tau}|_{UB_R}^{-1}(d)) \setminus (\{0\} \times p_{\tau}|_{UB_R}^{-1}(d))\right) \cup \left(\{0\} \times p_{\tau \circ \psi_{\mathbb{R}}(g)}|_{UB_R}^{-1}(d)\right)$$

bounds a chain  $G_d$ .

The chains  $G_a, G_b$ , and  $G_c$  can be assumed to be transverse. Construct the propagator  $P_d(g)$  of  $(C_2(R), \tau \circ \psi_{\mathbb{R}}(g))$  from  $P_d$  by replacing  $[-1, 0] \times p_{\tau}|_{UB_R}^{-1}(d)$  with  $G_d$  on  $[-1, 0] \times UB_R$ . Then we have

$$\Theta(R, \tau \circ \psi_{\mathbb{R}}(g)) - \Theta(R, \tau) = \langle G_a, G_b, G_c \rangle_{[-1, 0] \times UB_R}.$$

Using  $\tau$  to identify  $UB_R$  with  $B_R \times S^2$ , the cycle  $\partial G_d$  may be written as

$$\partial G_d = \left(\partial([-1, 0] \times B_R) \setminus (\{0\} \times B_R)\right) \times \{d\} \cup \left(\{0\} \times (\cup_{m \in B_R} (m, g(m)(d)))\right).$$

Therefore,  $\Theta(R, \tau \circ \psi_{\mathbb{R}}(g)) - \Theta(R, \tau)$  is independent of  $\tau$ . Then it is easy to see that  $\Theta'$  is a homomorphism from  $[(B_R, \partial B_R), (SO(3), 1)]$  to  $\mathbb{Q}$ .  $\square$

So, Theorem 4.5 and Lemma 4.6 reduce the proof of Proposition 4.7 to the proof that  $\Theta'(\rho_R(B^3)) = 1$ . It is easy to see that  $\Theta'(\rho_R(B^3)) = \Theta'(\rho)$ . Thus, we have reduced the proof of Proposition 4.7 to the proof of the following lemma.

**Lemma 4.10.** *We have  $\Theta'(\rho) = 1$ .*

To prove this lemma, we will compute the expression  $\langle G_a, G_{-a}, G_c \rangle_{[-1,0] \times UB^3}$  from the above proof of Lemma 4.9 with explicit  $G_a$  and  $G_{-a}$  in  $[-1, 0] \times UB^3$ . We will construct the chain  $G_a$  from a chain  $G(a)$  of  $B^3 \times S^2$  described in Lemma 4.12.

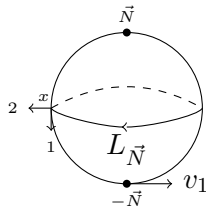
Again, we regard  $B^3$  as the quotient of  $[0, 1] \times S^2$  where all points of  $\{0\} \times S^2$  are identified with each other. We first prove the following lemma:

**Lemma 4.11.** *Recall the map  $\rho: B^3 \rightarrow SO(3)$  from Definition 4.4. Let  $a \in S^2$ . The point  $(-a)$  is a regular value of the map*

$$\begin{aligned} \rho_a: B^3 &\rightarrow S^2 \\ m &\mapsto \rho(m)(a) \end{aligned}$$

and its preimage (cooriented by  $S^2$  via  $\rho_a$ ) is the knot  $L_a = -\{\frac{1}{2}\} \times S_a$ , where  $S_a$  is the circle of  $S^2$  of vectors orthogonal to  $a$ , oriented as the boundary of the hemisphere that contains  $a$ .

**PROOF:** We prove the lemma when  $a$  is the North Pole  $\vec{N}$ . It is easy to see that  $\rho_{\vec{N}}^{-1}(-\vec{N}) = L_{\vec{N}}$  up to orientation.



Let  $(\frac{1}{2}, x) \in L_{\vec{N}}$ . Let  $C$  denote the great circle of  $S^2$  that contains  $\vec{N}$  and  $x$ . Orient  $C$  from  $x$  towards  $(-\vec{N})$ . When  $m$  moves along  $\{\frac{1}{2}\} \times C$  from  $(\frac{1}{2}, x)$  towards  $(\frac{1}{2}, -\vec{N})$ ,  $\rho(m)(\vec{N})$  moves from  $(-\vec{N})$  along  $C$  following the orientation of  $C$ , too. Let  $v_1$  denote the oriented unit tangent vector to  $C$  at  $(-\vec{N})$ . In our figure,  $x$  is on the left,  $C$  is oriented counterclockwise, and  $v_1$  points to the right. So  $S^2$  is oriented at  $(-\vec{N})$  by  $v_1$  and by the tangent vector  $v_2$  at  $(-\vec{N})$  towards us. In order to move  $\rho(\theta; v)(\vec{N})$  in the  $v_2$  direction, one increases  $\theta$ , so  $L_{\vec{N}}$  is cooriented and oriented as in the figure.  $\square$

**Lemma 4.12.** *Let  $a \in S^2$ . Let  $m \in B^3$ . Recall the notation from Lemma 4.11 above where  $\rho(m)(a) = \rho_a(m)$ . When  $m \notin L_a$ , let  $[a, \rho_a(m)]$  denote the unique geodesic arc of  $S^2$  with length  $\ell \in [0, \pi[$  from  $a$  to  $\rho_a(m)$ . For  $t \in [0, 1]$ , let  $X_t(m) \in [a, \rho_a(m)]$  be such that the length of  $[a, X_t(m)]$  is  $t\ell$ . Let  $G_h(a)$  be the closure of*

$$\left(\bigcup_{t \in [0, 1], m \in (B^3 \setminus L_a)} (m, X_t(m))\right)$$

in  $B^3 \times S^2$ . Let  $D_a$  be the disk image of  $[0, \frac{1}{2}] \times (-S_a)$  bounded by  $L_a$  in  $B^3$ . Set

$$G(a) = G_h(a) + D_a \times S^2.$$

Then  $G(a)$  is a chain of  $B^3 \times S^2$  such that

$$\partial G(a) = -(B^3 \times a) + \bigcup_{m \in B^3} (m, \rho_a(m)).$$

PROOF: The map  $X_t$  is well-defined on  $(B^3 \setminus L_a)$ . We have  $X_0(m) = a$  and  $X_1(m) = \rho_a(m)$ . Let us show how the definition of  $X_t$  extends smoothly to the manifold  $\mathcal{B}(B^3, L_a)$  obtained from  $B^3$  by blowing up  $L_a$ . The map  $\rho_a$  maps the normal bundle to  $L_a$  to a disk of  $S^2$  around  $(-a)$  by an orientation-preserving diffeomorphism on every fiber (near the origin). In particular,  $\rho_a$  induces a map from the unit normal bundle  $SN(L_a)$  to  $L_a$  in  $R$  to the unit normal bundle to  $(-a)$  in  $S^2$ . This map preserves the orientation of the fibers. Then for an element  $y$  of  $SN(L_a)$ , define  $X_t(y)$  as before on the half great circle  $[a, -a]_{\rho_a(-y)}$  from  $a$  to  $(-a)$  that is tangent to  $\rho_a(-y)$  at  $(-a)$ . (So  $\rho_a(-y)$  is an outward normal to  $[a, -a]_{\rho_a(-y)}$  at  $(-a)$ .) This continuously extends the definition of  $X_t$ . So we have

$$G_h(a) = \bigcup_{t \in [0, 1], m \in \mathcal{B}(B^3, L_a)} (p_{B^3}(m), X_t(m)),$$

and

$$\begin{aligned} \partial G_h(a) = & -(B^3 \times a) + \bigcup_{m \in B^3} (m, \rho_a(m)) \\ & + \bigcup_{t \in [0, 1], m \in -\partial \mathcal{B}(B^3, L_a)} (p_{B^3}(m), X_t(m)), \end{aligned}$$

where  $(-\partial \mathcal{B}(S^3, L_a))$  is oriented as the boundary of a tubular neighborhood of  $L_a$ . For any  $x \in L_a$ , the sphere  $S^2$  is covered with degree  $(-1)$  by the image of  $[0, 1] \times SN_x(L_a)$ , where the fiber  $SN_x(L_a)$  of  $SN(L_a)$  is oriented as the boundary of a disk in the fiber  $N_x(L_a)$  of the normal bundle. We may therefore write the last summand as  $(-L_a \times S^2)$ .  $\square$

PROOF OF LEMMA 4.10: We use the notation of the proof of Lemma 4.9 and construct an explicit  $G_a$  in  $[-1, 0] \times UB^3 \stackrel{\cong}{=} [-1, 0] \times B^3 \times S^2$ . Let  $\iota$  be

the endomorphism of  $UB^3$  over  $B^3$  that maps a unit vector to the opposite one. Recall the chain  $G(a)$  of Lemma 4.12. Set

$$G_a = \left( [-1, -2/3] \times B^3 \times a \right) + \left( \{-2/3\} \times G(a) \right) \\ + \left( [-2/3, 0] \times \cup_{m \in B^3} (m, \rho_a(m)) \right)$$

and

$$G_{-a} = \left( [-1, -1/3] \times B^3 \times (-a) \right) + \left( \{-1/3\} \times \iota(G(a)) \right) \\ + \left( [-1/3, 0] \times \cup_{m \in B^3} (m, \rho(m)(-a)) \right).$$

Then we have

$$G_a \cap G_{-a} = \left( [-2/3, -1/3] \times L_a \times (-a) \right) + \left( \{-2/3\} \times D_a \times (-a) \right) \\ - \left( \{-1/3\} \times \cup_{m \in D_a} (m, \rho_a(m)) \right).$$

Finally, according to the proof of Lemma 4.9,  $\Theta'(\rho)$  is the algebraic intersection of  $G_a \cap G_{-a}$  with  $P_c(\rho)$  in  $C_2(R)$ . This intersection coincides with the algebraic intersection of  $G_a \cap G_{-a}$  with any propagator of  $C_2(R)$ , according to Lemma 3.12. Therefore, we have

$$\Theta'(\rho) = \langle P_c, G_a \cap G_{-a} \rangle_{[-1,0] \times B^3 \times S^2} = -\deg_c(\rho_a: D_a \rightarrow S^2)$$

for any regular value  $c \neq -a$  of  $\rho_a|_{D_a}$ . Since the image of the quotient  $D_a$  of  $[0, \frac{1}{2}] \times (-S_a)$  under  $\rho_a$  covers the sphere with degree  $(-1)$ , we get  $\Theta'(\rho) = 1$ .  $\square$

So, Proposition 4.7 is proved.



# Chapter 5

## Parallelizations of 3-manifolds and Pontrjagin classes

In this chapter, we fix a smooth oriented connected 3-manifold  $M$  with possible boundary, and we study its parallelizations. In particular, we prove Theorem 4.5 in Sections 5.1, 5.3, and 5.7. We will use this theorem in our general constructions of link invariants in 3-manifolds as in the definition of  $\Theta$  in Section 4.3. This chapter also describes other properties of Pontrjagin classes, which will be used in the fourth part of this book in some universality proofs. Section 5.9 describes the structure of the space of parallelizations of oriented 3-manifolds, more precisely. It is not used in other parts of the book.

### 5.1 $[(M, \partial M), (SO(3), 1)]$ is an abelian group.

Again, we regard  $S^3$  as  $B^3/\partial B^3$ , and  $B^3$  as the quotient of  $[0, 1] \times S^2$  where all points of  $\{0\} \times S^2$  are identified with each other. Recall the map  $\rho: B^3 \rightarrow SO(3)$  of Definition 4.4, which maps  $(\theta \in [0, 1], v \in S^2)$  to the rotation  $\rho(\chi_\pi(\theta); v)$  with axis directed by  $v$  and with angle  $\chi_\pi(\theta)$ . Also recall that the group structure of  $[(M, \partial M), (SO(3), 1)]$  is induced by the multiplication of maps, using the multiplication of  $SO(3)$ .

Any  $g \in [(M, \partial M), (SO(3), 1)]_m$  induces a map

$$H_1(g; \mathbb{Z}): H_1(M, \partial M; \mathbb{Z}) \longrightarrow (H_1(SO(3), 1) = \mathbb{Z}/2\mathbb{Z}).$$

Since

$$\begin{aligned} H_1(M, \partial M; \mathbb{Z}/2\mathbb{Z}) &= H_1(M, \partial M; \mathbb{Z})/2H_1(M, \partial M; \mathbb{Z}) \\ &= H_1(M, \partial M; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}, \end{aligned}$$



we have

$$\begin{aligned} \text{Hom}(H_1(M, \partial M; \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) &= \text{Hom}(H_1(M, \partial M; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \\ &= H^1(M, \partial M; \mathbb{Z}/2\mathbb{Z}), \end{aligned}$$

and the image of  $H_1(g; \mathbb{Z})$  in  $H^1(M, \partial M; \mathbb{Z}/2\mathbb{Z})$  under the above isomorphisms is denoted by  $H^1(g; \mathbb{Z}/2\mathbb{Z})$ . (Equivalently,  $H^1(g; \mathbb{Z}/2\mathbb{Z})$  denotes the image of the generator of  $H^1(SO(3), 1; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$  under  $H^1(g; \mathbb{Z}/2\mathbb{Z})$  in  $H^1(M, \partial M; \mathbb{Z}/2\mathbb{Z})$ .)

For  $v \in S^2$ , let  $\gamma_v$  denote the loop that maps  $\exp(i\theta) \in S^1$  to the rotation  $\rho(\theta; v)$  with axis directed by  $v$  and with angle  $\theta$ . Let  $\mathbb{R}P_S^2$  denote the projective plane embedded in  $SO(3)$  consisting of the rotations of  $SO(3)$  of angle  $\pi$ . Note that  $\gamma_v(S^1) = \rho([0, 1] \times \{v\})$  and  $\mathbb{R}P_S^2 = \rho(\{1/2\} \times S^2)$  intersect once transversally. Thus, we have  $H_1(SO(3); \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} [\gamma_v]$  and  $H_2(SO(3); \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} [\mathbb{R}P_S^2]$ .

**Lemma 5.1.** *The map*

$$\begin{aligned} H^1(., \mathbb{Z}/2\mathbb{Z}): \quad & [(M, \partial M), (SO(3), 1)] \rightarrow H^1(M, \partial M; \mathbb{Z}/2\mathbb{Z}) \\ & [g] \quad \mapsto H^1(g; \mathbb{Z}/2\mathbb{Z}) \end{aligned}$$

*is a group homomorphism.*

PROOF: Let  $f$  and  $g$  be two elements of  $[(M, \partial M), (SO(3), 1)]_m$ . In order to prove  $H^1(fg; \mathbb{Z}/2\mathbb{Z}) = H^1(f; \mathbb{Z}/2\mathbb{Z}) + H^1(g; \mathbb{Z}/2\mathbb{Z})$ , it suffices to prove that, for any path  $\gamma: [0, 1] \rightarrow M$  whose image  $\gamma(\partial[0, 1])$  is in  $\partial M$ , we have

$$H_1(fg; \mathbb{Z}/2\mathbb{Z})([\gamma]) = H_1(f; \mathbb{Z}/2\mathbb{Z})([\gamma]) + H_1(g; \mathbb{Z}/2\mathbb{Z})([\gamma]),$$

where  $H_1(f; \mathbb{Z}/2\mathbb{Z})([\gamma]) = \langle f \circ \gamma, \mathbb{R}P_S^2 \rangle_{SO(3)} [\gamma_v]$  for the mod 2 algebraic intersection  $\langle f \circ \gamma, \mathbb{R}P_S^2 \rangle_{SO(3)}$ . Both sides of this equation depend only on the homotopy classes of  $f \circ \gamma$  and  $g \circ \gamma$ . Therefore, we may assume that the supports (closures of the preimages of  $SO(3) \setminus \{1\}$ ) of  $f \circ \gamma$  and  $g \circ \gamma$  are disjoint and easily conclude.  $\square$

**Lemma 5.2.** *Let  $M$  be an oriented connected 3-manifold with possible boundary. Recall that  $\rho_M(B^3) \in [(M, \partial M), (SO(3), 1)]_m$  is a map that coincides with the map  $\rho$  of Definition 4.4 on a ball  $B^3$  embedded in  $M$  and that maps the complement of  $B^3$  to the unit of  $SO(3)$ .*

1. *Any homotopy class of a map  $g$  from  $(M, \partial M)$  to  $(SO(3), 1)$ , such that  $H^1(g; \mathbb{Z}/2\mathbb{Z})$  is trivial, belongs to the subgroup*

$$\left\langle [\rho_M(B^3)] \right\rangle$$

*of  $[(M, \partial M), (SO(3), 1)]$  generated by  $[\rho_M(B^3)]$ .*

2. For any  $[g] \in [(M, \partial M), (SO(3), 1)]$ , we have  $[g]^2 \in \langle [\rho_M(B^3)] \rangle$ .
3. The group  $[(M, \partial M), (SO(3), 1)]$  is abelian.

PROOF: Let  $g \in [(M, \partial M), (SO(3), 1)]_m$ . Assume that  $H^1(g; \mathbb{Z}/2\mathbb{Z})$  is trivial. Choose a cell decomposition of  $M$  relative to its boundary, with only one three-cell, no zero-cell if  $\partial M \neq \emptyset$ , one zero-cell if  $\partial M = \emptyset$ , one-cells, and two-cells. See [Hir94, Chapter 6, Section 3]. Then after a homotopy relative to  $\partial M$ , we may assume that  $g$  maps the one-skeleton of  $M$  to 1. Next, since  $\pi_2(SO(3)) = 0$ , we may assume that  $g$  maps the two-skeleton of  $M$  to 1, and therefore that  $g$  maps the exterior of some 3-ball to 1. Now  $g$  becomes a map from  $B^3/\partial B^3 = S^3$  to  $SO(3)$ , and its homotopy class is  $k[\tilde{\rho}]$  in  $\pi_3(SO(3)) = \mathbb{Z}[\tilde{\rho}]$ . Therefore, the map  $g$  is homotopic to  $\rho_M(B^3)^k$ . This proves the first assertion.

Since  $H^1(g^2; \mathbb{Z}/2\mathbb{Z}) = 2H^1(g; \mathbb{Z}/2\mathbb{Z})$  is trivial, the second assertion follows.

For the third assertion, first note that  $[\rho_M(B^3)]$  belongs to the center of  $[(M, \partial M), (SO(3), 1)]$  because it can be supported in a small ball disjoint from the support (preimage of  $SO(3) \setminus \{1\}$ ) of a representative of any other element. Thus, according to the second assertion, any square is in the center. In particular, if  $f$  and  $g$  are elements of  $[(M, \partial M), (SO(3), 1)]$ , we have

$$(gf)^2 = (fg)^2 = (f^{-1}f^2g^2f)(f^{-1}g^{-1}fg),$$

where the first factor is equal to  $f^2g^2 = g^2f^2$ . Exchanging  $f$  and  $g$  yields  $f^{-1}g^{-1}fg = g^{-1}f^{-1}gf$ . Therefore, the commutator, which is a power of  $[\rho_M(B^3)]$ , thanks to Lemma 5.1 and to the first assertion, has a vanishing square, and thus a vanishing degree. So it is trivial.  $\square$

## 5.2 Any oriented 3-manifold is parallelizable.

In this subsection, we prove the following standard theorem. The spirit of our proof is the same as the Kirby proof in [Kir89, p.46]. But instead of assuming familiarity with the obstruction theory described by Norman Steenrod in [Ste51, Part III], we use this proof to introduce this theory.

**Theorem 5.3** (Stiefel). *Any orientable 3-manifold is parallelizable.*

**Lemma 5.4.** *Let  $M$  be an oriented 3-manifold. Let  $S$  be a closed surface, orientable or not, immersed in  $M$ . Then the restriction to  $S$  of the tangent bundle  $TM$  to  $M$  is trivializable.*

PROOF: Let us first prove that this bundle is independent of the immersion. It is the direct sum of the tangent bundle to the surface and of its normal one-dimensional bundle. This normal bundle is trivial when  $S$  is orientable. Otherwise, its unit bundle is the 2-fold orientation cover of the surface. (The orientation cover of  $S$  is its 2-fold orientable cover that is trivial over annuli embedded in the surface). Since any surface  $S$  can be immersed in  $\mathbb{R}^3$ , the restriction  $TM|_S$  is the pull-back of the trivial bundle of  $\mathbb{R}^3$  by such an immersion, and it is trivial.  $\square$

Using Stiefel-Whitney classes, the proof of Theorem 5.3 goes as follows. QUICK PROOF OF THEOREM 5.3: Let  $M$  be an orientable smooth 3-manifold, equipped with a smooth triangulation.<sup>1</sup> By definition, the *first Stiefel-Whitney class*  $w_1(TM) \in H^1(M; \mathbb{Z}/2\mathbb{Z} = \pi_0(GL(\mathbb{R}^3)))$  viewed as a map from  $\pi_1(M)$  to  $\mathbb{Z}/2\mathbb{Z}$  maps the class of a loop  $c$  embedded in  $M$  to 0 if  $TM|_c$  is orientable, and to 1 otherwise. It is the *obstruction* to the existence of a trivialization of  $TM$  over the one-skeleton of  $M$ . Since  $M$  is orientable, the first Stiefel-Whitney class  $w_1(TM)$  vanishes. So  $TM$  can be trivialized over the one-skeleton of  $M$ . The *second Stiefel-Whitney class*  $w_2(TM) \in H^2(M; \mathbb{Z}/2\mathbb{Z} = \pi_1(GL^+(\mathbb{R}^3)))$  viewed as a map from  $H_2(M; \mathbb{Z}/2\mathbb{Z})$  to  $\mathbb{Z}/2\mathbb{Z}$  maps the class of a connected closed surface  $S$  to 0 if  $TM|_S$  is trivializable, and to 1 otherwise. The second Stiefel-Whitney class  $w_2(TM)$  is the obstruction to the existence of a trivialization of  $TM$  over the two-skeleton of  $M$  when  $w_1(TM) = 0$ . According to the above lemma, we have  $w_2(TM) = 0$ , and  $TM$  can be trivialized over the two-skeleton of  $M$ . Then since  $\pi_2(GL^+(\mathbb{R}^3)) = 0$ , any parallelization over the two-skeleton of  $M$  can be extended as a parallelization of  $M$ .  $\square$

We detail the involved arguments without mentioning Stiefel-Whitney classes, (in fact, by almost defining  $w_2(TM)$ ) below. The elementary proof below can serve as an introduction to the obstruction theory used above.

ELEMENTARY PROOF OF THEOREM 5.3: Let  $M$  be an oriented 3-manifold. Choose a triangulation of  $M$ . For any cell  $c$  of the triangulation, define an arbitrary trivialization  $\tau_c: c \times \mathbb{R}^3 \rightarrow TM|_c$  such that  $\tau_c$  induces the orientation of  $M$ . This defines a trivialization  $\tau^{(0)}: M^{(0)} \times \mathbb{R}^3 \rightarrow TM|_{M^{(0)}}$  of  $M$  over the 0-skeleton  $M^{(0)}$  of  $M$ , which is the set of 0-dimensional cells of the triangulation. Let  $C_k(M)$  be the set of  $k$ -cells of the triangulation. Equip every cell with an arbitrary orientation. Let  $e \in C_1(M)$  be an edge of the triangulation. On  $\partial e$ , the trivialization  $\tau^{(0)}$  may be written as  $\tau^{(0)} = \tau_e \circ \psi_{\mathbb{R}}(g_e)$ , for a map  $g_e: \partial e \rightarrow GL^+(\mathbb{R}^3)$ . Since  $GL^+(\mathbb{R}^3)$  is connected, the map  $g_e$  extends to  $e$ , and  $\tau^{(1)} = \tau_e \circ \psi_{\mathbb{R}}(g_e)$  extends  $\tau^{(0)}$  to  $e$ . Doing so for all the edges extends

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<sup>1</sup>A theorem of Henry Whitehead proved in the Munkres book [Mun66] ensures the existence of such a triangulation.

$\tau^{(0)}$  to a trivialization  $\tau^{(1)}$  of the one-skeleton  $M^{(1)}$  of  $M$ , which is the union of the edges of the triangulation.

Let  $t$  be a triangle of the triangulation. There is a map  $g_t: \partial t \rightarrow GL^+(\mathbb{R}^3)$  such that  $\tau^{(1)} = \tau_t \circ \psi_{\mathbb{R}}(g_t)$  on  $\partial t$ . Let  $E(t, \tau^{(1)})$  be the homotopy class of  $g_t$  in  $(\pi_1(GL^+(\mathbb{R}^3)) = \pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z})$ . Then  $E(\cdot, \tau^{(1)}): C_2(M) \rightarrow \mathbb{Z}/2\mathbb{Z}$  extends linearly to a cochain, which is independent of the  $\tau_t$ . When  $E(\cdot, \tau^{(1)}) = 0$ ,  $\tau^{(1)}$  extends to a trivialization  $\tau^{(2)}$  over the two-skeleton of  $M$ , as before. Since  $\pi_2(GL^+(\mathbb{R}^3)) = 0$ , the trivialization  $\tau^{(2)}$  can next be extended over the three-skeleton of  $M$ . So it extends over  $M$ .

Let us now study the obstruction cochain  $E(\cdot, \tau^{(1)})$  whose vanishing guarantees the existence of a parallelization of  $M$ .

Choose a map  $d(e): (e, \partial e) \rightarrow (GL^+(\mathbb{R}^3), 1)$  for every edge  $e$  of the triangulation. Let  $\tau^{(1)'}$  be the trivialization of  $TM|_{M^{(1)}}$  obtained as above by changing the map  $g_e$  associated to  $e$  to  $d(e)g_e$ . Define the cochain

$$D(\tau^{(1)}, \tau^{(1)'}) : (\mathbb{Z}/2\mathbb{Z})^{C_1(M)} \rightarrow \mathbb{Z}/2\mathbb{Z}$$

that maps  $e$  to the homotopy class of  $d(e)$ . Then  $(E(\cdot, \tau^{(1)'}) - E(\cdot, \tau^{(1)}))$  is the coboundary of  $D(\tau^{(1)}, \tau^{(1)'})$ . (See Section A.1, before Theorem A.9.)

Let us prove that  $E(\cdot, \tau^{(1)})$  is a cocycle. Consider a 3-simplex  $T$ . The trivialization  $\tau^{(0)}$  extends to  $T$ . Assume that  $\tau_T$  is an extension of  $\tau^{(0)}$  to  $T$ , that  $\tau_t$  is the restriction of  $\tau_T$  to  $t$  for any face  $t$  of  $T$ , and that the above  $\tau^{(1)'}$  coincides with  $\tau_T$  on the edges of  $\partial T$ . Then we have  $E(\cdot, \tau^{(1)'})(\partial T) = 0$ . Since any coboundary maps  $\partial T$  to 0, we also have  $E(\cdot, \tau^{(1)})(\partial T) = 0$ .

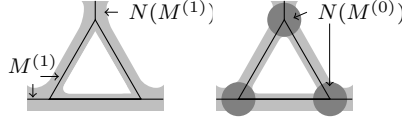
Now, it suffices to prove that the cohomology class of  $E(\cdot, \tau^{(1)})$  (which is equal to  $w_2(TM)$ ) vanishes in order to prove the existence of an extension  $\tau^{(1)'}$  of  $\tau^{(0)}$  on  $M^{(1)}$  that extends on  $M$ .

Since  $H^2(M; \mathbb{Z}/2\mathbb{Z}) = \text{Hom}(H_2(M; \mathbb{Z}/2\mathbb{Z}); \mathbb{Z}/2\mathbb{Z})$ , it suffices to prove that  $E(\cdot, \tau^{(1)})$  maps any 2-dimensional  $\mathbb{Z}/2\mathbb{Z}$ -cycle  $C$  to 0.

We represent the class of such a cycle  $C$  by a closed surface  $S$ , orientable or not, as follows.

Let  $N(M^{(0)})$  be a disjoint union of balls around the vertices and let  $N(M^{(1)})$  be a small regular neighborhood of  $M^{(1)}$  in  $M$  as in Figure 5.1. We assume that  $N(M^{(1)}) \cap (M \setminus N(M^{(0)}))$  is a disjoint union, over the edges  $e$ , of solid cylinders  $B_e$  identified with  $]0, 1[ \times D^2$ . The core  $]0, 1[ \times \{0\}$  of  $B_e = ]0, 1[ \times D^2$  is a connected part of the interior of the edge  $e$ . (As in Figure 5.1, the neighborhood  $N(M^{(1)})$  is thinner than  $N(M^{(0)})$ .)

Construct  $S$  in the complement of  $N(M^{(0)}) \cup N(M^{(1)})$  as the intersection of the support of  $C$  with this complement. Then the closure of  $S$  meets the part  $[0, 1] \times S^1$  of every  $\overline{B_e}$  as an even number of parallel intervals from  $\{0\} \times S^1$  to  $\{1\} \times S^1$ . Complete  $S$  in  $M \setminus N(M^{(0)})$  by connecting the intervals

Figure 5.1: The neighborhoods  $N(M^{(1)})$  and  $N(M^{(0)})$ 

pairwise in  $\overline{B_e}$  by disjoint bands. After this operation, the boundary of the closure of  $S$  is a disjoint union of circles in the boundary of  $N(M^{(0)})$ . Glue disjoint disks of  $N(M^{(0)})$  along these circles to finish the construction of  $S$ .

Extend  $\tau^{(0)}$  to  $N(M^{(0)})$ , assume that  $\tau^{(1)}$  coincides with this extension over  $M^{(1)} \cap N(M^{(0)})$ , and extend  $\tau^{(1)}$  to  $N(M^{(1)})$ . The bundle  $TM|_S$  is trivial, and we may choose a trivialization  $\tau_S$  of  $TM$  over  $S$  that coincides with our extension of  $\tau^{(0)}$  over  $N(M^{(0)})$ , over  $S \cap N(M^{(0)})$ . We have a cell decomposition of  $(S, S \cap N(M^{(0)}))$  with only 1-cells and 2-cells, for which the 2-cells of  $S$  are in one-to-one canonical correspondence with the 2-cells of  $C$ , and one-cells correspond bijectively to bands connecting two-cells in the cylinders  $B_e$ . These one-cells are equipped with the trivialization of  $TM$  induced by  $\tau^{(1)}$ . Then we can define 2-dimensional cochains  $E_S(\cdot, \tau^{(1)})$  and  $E_S(\cdot, \tau_S)$  as before, with respect to this cellular decomposition of  $S$ . The cochain  $(E_S(\cdot, \tau^{(1)}) - E_S(\cdot, \tau_S))$  is again a coboundary, and  $E_S(\cdot, \tau_S) = 0$ . So  $E_S(C, \tau^{(1)}) = 0$ , and we also have  $E(C, \tau^{(1)}) = 0$  since  $E(C, \tau^{(1)}) = E_S(C, \tau^{(1)})$ .  $\square$

Theorem 5.3 has the following immediate corollary.

**Proposition 5.5.** *Any punctured oriented 3-manifold  $\check{R}$  as in Definition 3.6 can be equipped with an asymptotically standard parallelization.*

PROOF: The oriented manifold  $R$  admits a parallelization  $\tau_0: R \times \mathbb{R}^3 \rightarrow TR$ . Over  $\mathring{B}_{1,\infty} \setminus \{\infty\}$ ,  $\tau_s = \tau_0 \circ \psi_{\mathbb{R}}(g)$  for a map  $g: \mathring{B}_{1,\infty} \setminus \{\infty\} \rightarrow GL^+(\mathbb{R}^3)$ . For  $r \in [1, 2]$ , let  $\mathring{B}_{r,\infty}$  (resp.  $B_{r,\infty}$ ) be the complement in  $S^3$  of the closed (resp. open) ball  $B(r)$  of radius  $r$  in  $\mathbb{R}^3$ . Since  $\pi_2(GL^+(\mathbb{R}^3)) = \{0\}$ , the restriction of  $g$  to  $B_{7/4,\infty} \setminus \mathring{B}_{2,\infty}$  extends to a map of  $\mathring{B}_{1,\infty} \setminus \mathring{B}_{2,\infty}$  that maps  $\mathring{B}_{1,\infty} \setminus \mathring{B}_{5/3,\infty}$  to 1. After smoothing, we get a smooth map  $\tilde{g}: \mathring{B}_{1,\infty} \setminus \{\infty\} \rightarrow GL^+(\mathbb{R}^3)$  that coincides with  $g$  on  $\mathring{B}_{2,\infty}$  and that maps  $\mathring{B}_{1,\infty} \setminus \mathring{B}_{3/2,\infty}$  to 1. Extend  $\tilde{g}$  to  $\check{R}$  so that it maps  $\check{R} \setminus \mathring{B}_{3/2,\infty}$  to 1. So  $\tau_0 \circ \psi_{\mathbb{R}}(\tilde{g})$  is an asymptotically standard parallelization as wanted.  $\square$

### 5.3 The homomorphism induced by the degree

In this section,  $M$  is a compact connected oriented 3-manifold, with or without boundary. Let  $S$  be a closed surface, orientable or not, embedded in the interior of our manifold  $M$ . Let  $\tau$  be a parallelization of our 3-manifold  $M$ . We define a twist  $g(S, \tau) \in [(M, \partial M), (SO(3), 1)]_m$  below.

The surface  $S$  has a tubular neighborhood  $N(S)$ , which is a  $[-1, 1]$ -bundle over  $S$ . This bundle admits (orientation-preserving) bundle charts  $\phi: [-1, 1] \times D \rightarrow N(S)$  for disks  $D$  of  $S$ , such that the changes of coordinates restrict to the fibers as  $\pm \mathbf{1}$ , where  $\mathbf{1}$  denotes the identity map. Let  $\phi(t, s)$  be in the image of such a chart. Let  $N_s = T_{\phi(0, s)}\phi([-1, 1] \times s)$  be the tangent vector to the fiber  $\phi([-1, 1] \times s)$  at  $\phi(0, s)$ . Define  $g(S, \tau)(\phi(t, s))$  to be the rotation with angle  $\pi(t + 1)$  and with axis

$$p_2\left(\tau^{-1}(N_s) = (\phi(0, s), p_2(\tau^{-1}(N_s)))\right).$$

Since this rotation coincides with the rotation with opposite axis and with opposite angle  $\pi(1 - t)$ , this provides a consistent definition of  $g(S, \tau)|_{N(S)}$ . Extend  $g(S, \tau)|_{N(S)}$  to  $M$  so that

$$g(S, \tau): (M, \partial M) \longrightarrow (SO(3), 1)$$

maps  $M \setminus N(S)$  to 1.

The homotopy class of  $g(S, \tau)$  depends only on the homotopy class of  $\tau$  and on the isotopy class of  $S$ . When  $M = B^3$ , when  $\tau$  is the standard parallelization of  $\mathbb{R}^3$ , and when  $\frac{1}{2}S^2$  denotes the sphere  $\frac{1}{2}\partial B^3$  inside  $B^3$ , the homotopy class of  $g(\frac{1}{2}S^2, \tau)$  coincides with the homotopy class of  $\rho$ .

We will see later (Proposition 5.33) that the homotopy class of  $g(S, \tau)$  depends only on the Euler characteristic  $\chi(S)$  of  $S$  and on the class of  $S$  in  $H_2(M; \mathbb{Z}/2\mathbb{Z})$ . Thus, we will simply denote  $g(S, \tau)$  by  $g(S)$ . We will also see (Corollary 5.34) that any element of  $[(M, \partial M), (SO(3), 1)]$  can be represented by some  $g(S)$ .

**Lemma 5.6.** *The morphism  $H^1(g(S, \tau); \mathbb{Z}/2\mathbb{Z})$  maps the generator of*

$$H^1(SO(3); \mathbb{Z}/2\mathbb{Z})$$

*to the mod 2 intersection with  $S$  in*

$$\text{Hom}(H_1(M, \partial M; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = H^1(M, \partial M; \mathbb{Z}/2\mathbb{Z}).$$

*Thus, the morphism  $H^1(\cdot; \mathbb{Z}/2\mathbb{Z}): [(M, \partial M), (SO(3), 1)] \rightarrow H^1(M, \partial M; \mathbb{Z}/2\mathbb{Z})$  is onto.*

PROOF: The first assertion is obvious.

The second one follows since  $H^1(M, \partial M; \mathbb{Z}/2\mathbb{Z})$  is the Poincaré dual of  $H_2(M; \mathbb{Z}/2\mathbb{Z})$  and since any element of  $H_2(M; \mathbb{Z}/2\mathbb{Z})$  is the class of a closed surface.  $\square$

**Lemma 5.7.** *The degree is a group homomorphism*

$$\deg: [(M, \partial M), (SO(3), 1)] \longrightarrow \mathbb{Z}$$

and  $\deg(\rho_M(B^3)^k) = 2k$ .

PROOF: It is easy to see that  $\deg(fg) = \deg(f) + \deg(g)$  when  $f$  or  $g$  is a power of  $[\rho_M(B^3)]$ .

Let us prove that  $\deg(f^2) = 2\deg(f)$  for any  $f$ . According to Lemma 5.6, there is an unoriented embedded surface  $S_f$  of the interior of  $C$  such that  $H^1(f; \mathbb{Z}/2\mathbb{Z}) = H^1(g(S_f, \tau); \mathbb{Z}/2\mathbb{Z})$  for some trivialization  $\tau$  of  $TM$ . According to Lemmas 5.1 and 5.2,  $fg(S_f, \tau)^{-1}$  is homotopic to some power of  $\rho_M(B^3)$ . So, it suffices to prove that the degree of  $g^2$  is  $2\deg(g)$  for  $g = g(S_f, \tau)$ . This can be done easily, by noticing that  $g^2$  is homotopic to  $g(S_f^{(2)}, \tau)$ , where  $S_f^{(2)}$  is the boundary of the tubular neighborhood of  $S_f$ . In general, we have

$$\deg(fg) = \frac{1}{2} \deg((fg)^2) = \frac{1}{2} \deg(f^2g^2) = \frac{1}{2} (\deg(f^2) + \deg(g^2)),$$

and the lemma is proved.  $\square$

Lemmas 5.2 and 5.7 imply the following lemma.

**Lemma 5.8.** *The degree induces an isomorphism*

$$\deg: [(M, \partial M), (SO(3), 1)] \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}.$$

Any group homomorphism  $\psi: [(M, \partial M), (SO(3), 1)] \longrightarrow \mathbb{Q}$  may be expressed as

$$\frac{1}{2} \psi(\rho_M(B^3)) \deg.$$

$\square$

Recall that  $M$  is a compact connected oriented 3-manifold in this section. So  $H^3(M, \partial M; \pi_3(SO(3)))$  is canonically isomorphic to  $\mathbb{Z}$ .

**Proposition 5.9.** *Let  $P: H^1(M, \partial M; \mathbb{Z}/2\mathbb{Z}) \rightarrow H_2(M; \mathbb{Z}/2\mathbb{Z})$  be the Poincaré duality isomorphism.*

Let  $i: H^3(M, \partial M; \pi_3(SO(3))) \rightarrow [(M, \partial M), (SO(3), 1)]$  be the group morphism that sends the canonical generator  $[M, \partial M]$  of  $H^3(M, \partial M; \pi_3(SO(3)))$  to  $[\rho_M(B^3)]$ . Then the sequence

$$0 \rightarrow H^3(M, \partial M; \pi_3(SO(3))) \xrightarrow{i} [(M, \partial M), (SO(3), 1)] \\ \xrightarrow{P \circ H^1([\cdot; \mathbb{Z}/2\mathbb{Z}])} H_2(M; \mathbb{Z}/2\mathbb{Z}) \rightarrow 0,$$

is a canonical exact sequence.

PROOF: The proposition is a consequence of Lemmas 5.2, 5.6, and 5.7.  $\square$

## 5.4 On the groups $SU(n)$

Let  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ . Let  $n \in \mathbb{N}$ . The stabilization maps induced by the inclusions

$$i: GL(\mathbb{K}^n) \longrightarrow GL(\mathbb{K} \oplus \mathbb{K}^n) \\ g \longmapsto i(g): (x, y) \mapsto (x, g(y))$$

are denoted by  $i$ . We represent elements of  $GL(\mathbb{K}^n)$  by matrices whose columns contain the coordinates of the images of the basis elements with respect to the standard basis of  $\mathbb{K}^n$ . View  $S^3$  as the unit sphere of  $\mathbb{C}^2$ . So, its elements are the pairs  $(z_1, z_2)$  of complex numbers such that  $|z_1|^2 + |z_2|^2 = 1$ . The group  $SU(2)$  may be identified with  $S^3$  by the homeomorphisms

$$m_r^{\mathbb{C}}: S^3 \longrightarrow SU(2) \quad \text{and} \quad \overline{m}_r^{\mathbb{C}}: S^3 \longrightarrow SU(2) \\ (z_1, z_2) \mapsto \begin{bmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{bmatrix} \quad (z_1, z_2) \mapsto \begin{bmatrix} \overline{z_1} & \overline{z_2} \\ -z_2 & z_1 \end{bmatrix}.$$

So the first nontrivial homotopy group of  $SU(2)$  is  $\pi_3(SU(2)) = \mathbb{Z} [\overline{m}_r^{\mathbb{C}}]$ , where  $[\overline{m}_r^{\mathbb{C}}] = -[m_r^{\mathbb{C}}]$  and  $\overline{m}_r^{\mathbb{C}}$  is a group homomorphism (it induces the group structure of  $S^3$ ). The long exact sequence associated to the fibration

$$SU(n-1) \xrightarrow{i} SU(n) \rightarrow S^{2n-1},$$

described in Theorem A.14, shows that  $i_*^n: \pi_j(SU(2)) \rightarrow \pi_j(SU(n+2))$  is an isomorphism for  $j \leq 4$  and  $n \geq 0$ . In particular, the group  $\pi_j(SU(4))$  is trivial for  $j \leq 2$  and we have

$$\pi_3(SU(4)) = \mathbb{Z} [i_*^2(\overline{m}_r^{\mathbb{C}})],$$



where  $i_*^2(\overline{m}_r^{\mathbb{C}})$  is the following map

$$i_*^2(\overline{m}_r^{\mathbb{C}}): (S^3 \subset \mathbb{C}^2) \longrightarrow SU(4)$$

$$(z_1, z_2) \longmapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \bar{z}_1 & \bar{z}_2 \\ 0 & 0 & -z_2 & z_1 \end{bmatrix}.$$

## 5.5 Definition of relative Pontrjagin numbers

Let  $M_0$  and  $M_1$  be two compact connected oriented 3-manifolds whose boundaries have collars identified by a diffeomorphism. Let  $\tau_0: M_0 \times \mathbb{R}^3 \rightarrow TM_0$  and  $\tau_1: M_1 \times \mathbb{R}^3 \rightarrow TM_1$  be two parallelizations (which respect the orientations) that agree on the collar neighborhoods of  $\partial M_0 = \partial M_1$ . Then the *relative Pontrjagin number*  $p_1(\tau_0, \tau_1)$  is the Pontrjagin obstruction to extending the trivialization of  $TW \otimes \mathbb{C}$  induced by  $\tau_0$  and  $\tau_1$  across the interior of a signature 0 cobordism  $W$  from  $M_0$  to  $M_1$ . Details follow.

Let  $M$  be a compact connected oriented 3-manifold. A *special complex trivialization* of  $TM$  is a trivialization of  $TM \otimes \mathbb{C}$  that is obtained from a trivialization  $\tau_M: M \times \mathbb{R}^3 \rightarrow TM$  by composing  $(\tau_M^{\mathbb{C}} = \tau_M \otimes_{\mathbb{R}} \mathbb{C}): M \times \mathbb{C}^3 \rightarrow TM \otimes \mathbb{C}$  by

$$\psi(G): M \times \mathbb{C}^3 \longrightarrow M \times \mathbb{C}^3$$

$$(x, y) \longmapsto (x, G(x)(y))$$

for a map  $G: (M, \partial M) \rightarrow (SL(3, \mathbb{C}), 1)$ , which is a map  $G: M \rightarrow SL(3, \mathbb{C})$  that maps  $\partial M$  to 1. The definition and properties of relative Pontrjagin numbers  $p_1(\tau_0, \tau_1)$ , given with more details below, are valid for pairs  $(\tau_0, \tau_1)$  of special complex trivializations.

The *signature* of a 4-manifold is the signature of the intersection form on its  $H_2(\cdot; \mathbb{R})$  (i.e., the number of positive entries minus the number of negative entries in a diagonalized version of this form). It is well known that any closed oriented three-manifold bounds a compact oriented 4-dimensional manifold. See [Rou85] for an elegant elementary proof. The signature of such a bounded 4-manifold may be changed arbitrarily by connected sums with copies of  $\mathbb{C}P^2$  or  $-\mathbb{C}P^2$ . A *cobordism from  $M_0$  to  $M_1$*  is a compact oriented 4-dimensional manifold  $W$  with ridges such that

$$\partial W = -M_0 \cup_{\partial M_0 \sim 0 \times \partial M_0} (-[0, 1] \times \partial M_0) \cup_{\partial M_1 \sim 1 \times \partial M_0} M_1,$$

where  $W$  is identified with an open subspace of one of the products  $[0, 1[ \times M_0$  or  $]0, 1] \times M_1$  near  $\partial W$ , as the following picture suggests.

$$\begin{array}{ccc}
\{0\} \times M_0 = M_0 & \boxed{W^4} & \{1\} \times M_1 = M_1 \\
& \nearrow \text{dashed line} & \\
& [0, 1] \times (-\partial M_0) & \xrightarrow{\vec{N}}
\end{array}$$

Let  $W = W^4$  be such a cobordism from  $M_0$  to  $M_1$ , with signature 0. Consider the complex 4-bundle  $TW \otimes \mathbb{C}$  over  $W$ . Let  $\vec{N}$  be the tangent vector to  $[0, 1] \times \{\text{pt}\}$  over  $\partial W$  (under the above identifications). Let  $\tau(\tau_0, \tau_1)$  denote the trivialization of  $TW \otimes \mathbb{C}$  over  $\partial W$  obtained by stabilizing either  $\tau_0$  or  $\tau_1$  into  $\vec{N} \oplus \tau_0$  or  $\vec{N} \oplus \tau_1$ . Then the obstruction to extending this trivialization to  $W$  is the relative first *Pontrjagin class*

$$p_1(W; \tau(\tau_0, \tau_1)) [W, \partial W] \in \left( H^4(W, \partial W; \mathbb{Z} = \pi_3(SU(4))) = \mathbb{Z} [W, \partial W] \right)$$

of the trivialization.

Now, we specify our sign conventions for this Pontrjagin class. They are the same as in [MS74]. In particular,  $p_1$  is the opposite of the complexified tangent bundle's second Chern class  $c_2$ . See [MS74, p. 174]. Let us describe these conventions. The determinant bundle of  $TW$  is trivial because  $W$  is oriented, and  $\det(TW \otimes \mathbb{C})$  is also trivial. Our parallelization  $\tau(\tau_0, \tau_1)$  over  $\partial W$  is special with respect to the trivialization of  $\det(TW \otimes \mathbb{C})$ . Equip  $M_0$  and  $M_1$  with Riemannian metrics that coincide near  $\partial M_0$ . Equip  $W$  with a Riemannian metric that coincides with the orthogonal product metric of one of the products  $[0, 1] \times M_0$  or  $[0, 1] \times M_1$  near  $\partial W$ . Equip  $TW \otimes \mathbb{C}$  with the associated Hermitian structure. Up to homotopy, assume that  $\tau(\tau_0, \tau_1)$  is unitary with respect to the Hermitian structure of  $TW \otimes \mathbb{C}$  and the standard Hermitian form of  $\mathbb{C}^4$ . Since  $\pi_i(SU(4)) = \{0\}$  when  $i < 3$ , the trivialization  $\tau(\tau_0, \tau_1)$  extends to a special unitary trivialization  $\tau$  outside the interior of a 4-ball  $B^4$  and defines

$$\tau: S^3 \times \mathbb{C}^4 \longrightarrow (TW \otimes \mathbb{C})|_{S^3}$$

over the boundary  $S^3 = \partial B^4$  of this 4-ball  $B^4$ . Over this 4-ball  $B^4$ , the bundle  $TW \otimes \mathbb{C}$  admits a special unitary trivialization

$$\tau_B: B^4 \times \mathbb{C}^4 \longrightarrow (TW \otimes \mathbb{C})|_{B^4}.$$

Then  $\tau_B^{-1} \circ \tau(v \in S^3, w \in \mathbb{C}^4) = (v, \phi(v)(w))$  for a map  $\phi: S^3 \longrightarrow SU(4)$  whose homotopy class may be written as

$$[\phi] = p_1(W; \tau(\tau_0, \tau_1)) [i_*^2(\overline{m}_r^{\mathbb{C}})] \in \pi_3(SU(4)),$$

where  $i_*^2(\overline{m}_r^{\mathbb{C}})$  was defined at the end of Section 5.4.

Define  $p_1(\tau_0, \tau_1) = p_1(W; \tau(\tau_0, \tau_1))$ .

**Proposition 5.10.** *Let  $M_0$  and  $M_1$  be two compact connected oriented 3-manifolds whose boundaries have collars identified by a diffeomorphism. Let  $\tau_0: M_0 \times \mathbb{C}^3 \rightarrow TM_0 \otimes \mathbb{C}$  and  $\tau_1: M_1 \times \mathbb{C}^3 \rightarrow TM_1 \otimes \mathbb{C}$  be two special complex trivializations (which respect the orientations) that agree on the collar neighborhoods of  $\partial M_0 = \partial M_1$ . The (first) Pontrjagin number  $p_1(\tau_0, \tau_1)$  is well defined by the above conditions.*

PROOF: According to the Novikov additivity theorem, if a closed 4-manifold  $Y$  is decomposed as  $Y = Y^+ \cup_X Y^-$ , where  $Y^+$  and  $Y^-$  are two 4-manifolds with boundary, embedded in  $Y$ , which intersect along a closed 3-manifold  $X$  (their common boundary, up to orientation), then we have

$$\text{signature}(Y) = \text{signature}(Y^+) + \text{signature}(Y^-).$$

According to a theorem of Vladimir Rohlin (see [Roh52] or [GM86, p. 18]), when  $Y$  is a compact oriented 4-manifold without boundary, we have

$$p_1(Y) = 3 \text{signature}(Y).$$

We only need to prove that  $p_1(\tau_0, \tau_1)$  is independent of the signature 0 cobordism  $W$ . Let  $W_E$  be a 4-manifold of signature 0 bounded by  $(-\partial W)$ . Then  $W \cup_{\partial W} W_E$  is a 4-dimensional manifold without boundary whose signature is

$$\text{signature}(W_E) + \text{signature}(W) = 0$$

by the Novikov additivity theorem. According to the Rohlin theorem, the first Pontrjagin class of  $W \cup_{\partial W} W_E$  is also zero. On the other hand, this first Pontrjagin class is the sum of the relative first Pontrjagin classes of  $W$  and  $W_E$  with respect to  $\tau(\tau_0, \tau_1)$ . These two relative Pontrjagin classes are opposite. Therefore, the relative first Pontrjagin class of  $W$  with respect to  $\tau(\tau_0, \tau_1)$  does not depend on  $W$ .  $\square$

Similarly, it is easy to prove the following proposition.

**Proposition 5.11.** *Under the above assumptions, except for the assumption on the signature of the cobordism  $W$ , we have*

$$p_1(\tau_0, \tau_1) = p_1(W; \tau(\tau_0, \tau_1)) - 3 \text{signature}(W).$$

$\square$

**Remark 5.12.** When  $\partial M_1 = \emptyset$  and  $M_0 = \emptyset$ , the map  $p_1(= p_1(\tau(\emptyset), \cdot))$  coincides with the map  $h$  studied by Friedrich Hirzebruch in [Hir73, §3.1], and by Robion Kirby and Paul Melvin in [KM99] under the name of *Hirzebruch defect*.

**Definition 5.13.** When  $(\check{R}, \tau)$  is an asymptotic rational homology  $\mathbb{R}^3$ , set

$$p_1(\tau) = p_1((\tau_s)|_{B^3}, \tau|_{B_R}),$$

with the notation of Proposition 5.10.

**Lemma 5.14.** *Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$  as in Definition 3.8. The parallelization  $\tau: \check{R} \times \mathbb{R}^3 \rightarrow T\check{R}$  induces the parallelization  $\bar{\tau}: (-\check{R}) \times \mathbb{R}^3 \rightarrow T(-\check{R})$  such that  $\bar{\tau}(x, v) = -\tau(x, v)$ .*

*Compose the orientation-preserving identification of a neighborhood of  $\infty$  in  $R$  with  $\mathring{B}_{1,\infty}$  by the (restriction of the) multiplication by  $(-1)$  in  $\mathbb{R}^3 \cup \{\infty\}$  in order to get an orientation-preserving identification of a neighborhood of  $\infty$  in  $(-R)$  with  $\mathring{B}_{1,\infty}$ . Then*

$$\left( -\check{R} = \overbrace{(-R)}^{\check{\phantom{R}}}, \bar{\tau} \right)$$

*is an asymptotic rational homology  $\mathbb{R}^3$ , and we have  $p_1(\bar{\tau}) = -p_1(\tau)$ .*

PROOF: Use a signature 0 cobordism  $W$  from  $\{0\} \times B^3$  to  $\{1\} \times B_R$  to compute  $p_1(\tau)$ . Extend the trivialization of  $TW \otimes \mathbb{C}$  on  $\partial W$ , which may be expressed as  $\vec{N} \oplus \tau_s$  on  $\{0\} \times B^3 \cup (-[0, 1] \times \partial B^3)$ , and  $\vec{N} \oplus \tau$  on  $\{1\} \times B_R$ , to a special trivialization on the complement of an open ball  $\mathring{B}^4$  in  $W$ . Let  $\overline{W}$  be the cobordism obtained from  $W$  by reversing the orientation of  $W$ . Equip  $\overline{W} \setminus \mathring{B}^4$  with the trivialization obtained from the above trivialization by a composition by  $\mathbf{1}_{\mathbb{R}} \times (-\mathbf{1})_{\mathbb{R}^3}$ . Then the changes of trivializations  $\phi: \partial B^4 \rightarrow SU(4)$  and  $\bar{\phi}: \partial \overline{B}^4 \rightarrow SU(4)$  are obtained from one another by the orientation-preserving conjugation by  $\mathbf{1}_{\mathbb{R}} \times (-\mathbf{1})_{\mathbb{R}^3}$ . Since  $\partial \overline{B}^4$  and  $\partial B^4$  have opposite orientations, we get the result.  $\square$

Back to the invariant  $\Theta$  defined in Corollary 4.8, we can now prove the following proposition.

**Proposition 5.15.** *For any  $\mathbb{Q}$ -sphere  $R$ , we have  $\Theta(-R) = -\Theta(R)$ .*

PROOF: Note that  $C_2(-R)$  is naturally identified with  $C_2(R)$ , with the same orientation. Recall the orientation-reversing diffeomorphism  $\iota$  of  $C_2(R)$  that exchanges the two coordinates in  $\check{R}^2 \setminus \Delta(\check{R}^2)$ . If  $\omega$  is a propagating form of  $(C_2(R), \tau)$ , then  $\iota^*(\omega)$  is a propagating form of  $(C_2(-R), \bar{\tau})$ . So we have  $\Theta(-R, \bar{\tau}) = \int_{C_2(-R)} \iota^*(\omega^3)$ . This proves  $\Theta(-R, \bar{\tau}) = -\Theta(R, \tau)$ . Corollary 4.8 and Lemma 5.14 yield the conclusion.  $\square$

## 5.6 On the groups $SO(3)$ and $SO(4)$

Let  $\mathbb{H}$  denote the vector space  $\mathbb{C} \oplus \mathbb{C}j$ . Set  $k = ij$ . The *conjugate* of an element  $(z_1 + z_2j)$  of  $\mathbb{H}$  is

$$\overline{z_1 + z_2j} = \overline{z_1} - z_2j.$$

**Lemma 5.16.** *The bilinear map that maps  $(z_1 + z_2j, z'_1 + z'_2j)$  to  $(z_1z'_1 - z_2\overline{z'_2}) + (z_2\overline{z'_1} + z_1z'_2)j$  maps  $(i, j)$  to  $k$ ,  $(j, k)$  to  $i$ ,  $(k, i)$  to  $j$ ,  $(j, i)$  to  $(-k)$ ,  $(k, j)$  to  $(-i)$ ,  $(i, k)$  to  $(-j)$ ,  $(i, i)$ ,  $(j, j)$ , and  $(k, k)$  to  $(-1)$ , and  $(z_1 + z_2j, \overline{z_1 + z_2j})$  to  $|z_1|^2 + |z_2|^2$ . It defines an associative product on  $\mathbb{H}$  such that this product and the addition make  $\mathbb{H}$  a field.*

PROOF: Exercise. □

The noncommutative field  $\mathbb{H}$ , which contains  $\mathbb{C}$  as a subfield, is the *field of quaternions*. It is equipped with the scalar product  $\langle \cdot, \cdot \rangle$  that makes  $(1, i, j, k)$  an orthonormal basis of  $\mathbb{H}$ . The associated norm maps  $(z_1 + z_2j)$  to

$$\sqrt{(z_1 + z_2j)\overline{z_1 + z_2j}}.$$

It is multiplicative. The unit sphere of  $\mathbb{H}$  is the sphere  $S^3$ . It is equipped with the group structure induced by the product of  $\mathbb{H}$ . The elements of  $\mathbb{H}$  are the *quaternions*. The *real part* of a quaternion  $(z_1 + z_2j)$  is the real part of  $z_1$ . The *pure quaternions* are the quaternions with zero real part.

For  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and  $n \in \mathbb{N}$ , the  $\mathbb{K}$  (Euclidean or Hermitian) oriented vector space with the direct orthonormal basis  $(v_1, \dots, v_n)$  is denoted by  $\mathbb{K}\langle v_1, \dots, v_n \rangle$ . The *cross product* or *vector product* of two elements  $v$  and  $w$  of  $\mathbb{R}^3 = \mathbb{R}\langle i, j, k \rangle$  is the element  $v \times w$  of  $\mathbb{R}^3$  such that for any  $x \in \mathbb{R}^3$ ,  $x \wedge v \wedge w = \langle x, v \times w \rangle i \wedge j \wedge k$  in  $\bigwedge^3 \mathbb{R}^3 = \mathbb{R}$ .

**Lemma 5.17.** *The product of two pure quaternions  $v$  and  $w$  is*

$$vw = -\langle v, w \rangle + v \times w.$$

*Every element of  $S^3$  may be expressed as  $\cos(\theta) + \sin(\theta)v$  for a unique  $\theta \in [0, \pi]$  and a pure quaternion  $v$  of norm 1, which is unique when  $\theta \notin \{0, \pi\}$ . For such an element, the restriction to  $\mathbb{R}\langle i, j, k \rangle$  of the conjugation*

$$R(\theta, v): w \mapsto (\cos(\theta) + \sin(\theta)v)w\overline{(\cos(\theta) + \sin(\theta)v)}$$

*is the rotation with axis directed by  $v$  and with angle  $2\theta$ .*

PROOF: It is easy to check the first assertion. The conjugation  $R(\theta, v)$  preserves the scalar product of  $\mathbb{H}$  and fixes  $\mathbb{R} \oplus \mathbb{R}v$ , pointwise. Therefore, it restricts to  $\mathbb{R}\langle i, j, k \rangle$  as an orthonormal transformation that fixes  $v$ . Let  $w$  be a pure quaternion orthogonal to  $v$ .

$$R(\theta, v)(w) = (\cos(\theta) + \sin(\theta)v)w(\cos(\theta) - \sin(\theta)v)$$

is equal to

$$\begin{aligned} R(\theta, v)(w) &= \cos^2(\theta)w - \sin^2(\theta)v w v + \cos(\theta)\sin(\theta)(v w - w v) \\ &= \cos(2\theta)w + \sin(2\theta)v \times w. \end{aligned}$$

□

**Lemma 5.18.** *The group morphism*

$$\begin{aligned} \tilde{\rho}: S^3 &\rightarrow SO(\mathbb{R}\langle i, j, k \rangle) = SO(3) \\ x &\mapsto (w \mapsto x.w.\bar{x}) \end{aligned}$$

*is surjective, and its kernel is  $\{-1, +1\}$ . The morphism  $\tilde{\rho}$  is a two-fold covering map, and this definition of  $\tilde{\rho}$  coincides with the previous one (after Definition 4.4), up to homotopy.*

PROOF: According to Lemma 5.17, the map  $\tilde{\rho}$  is surjective. Its kernel is the center of the group of unit quaternions, which is  $\{-1, +1\}$ . Thus, the map  $\tilde{\rho}$  is a two-fold covering map.

This two-fold covering map clearly coincides with the previous one, up to homotopy and orientation, since both classes generate  $\pi_3(SO(3)) = \mathbb{Z}$ . We take care of the orientation using the outward normal first convention to orient boundaries, as usual. Consider the diffeomorphism

$$\begin{aligned} \psi: ]0, \pi[ \times S^2 &\rightarrow S^3 \setminus \{-1, 1\} \\ (\theta, v) &\mapsto \cos(\theta) + \sin(\theta)v. \end{aligned}$$

We study  $\psi$  at  $(\pi/2, i)$ . At  $(\psi(\pi/2, i) = i)$ , the space  $\mathbb{R}\langle i, j, k \rangle$  is oriented by the outward normal  $o_i$  to  $S^2$  followed by the orientation of  $S^2$ . The field  $\mathbb{H}$  is oriented as  $\mathbb{R} \oplus \mathbb{R}\langle i, j, k \rangle$ . Since  $o_i$  coincides with the outward normal to  $S^3$  in  $\mathbb{H}$ , the sphere  $S^3$  is oriented by the opposite of the real part followed by the orientation of  $S^2$ . Since  $\cos$  is an orientation-reversing diffeomorphism at  $\pi/2$ ,  $\psi$  preserves the orientation near  $(\pi/2, i)$ . So the diffeomorphism  $\psi$  preserves the orientation everywhere, and the two maps  $\tilde{\rho}$  are homotopic. □

The following two group morphisms from  $S^3$  to  $SO(4)$  induced by the multiplication in  $\mathbb{H}$

$$\begin{aligned} m_\ell: S^3 &\rightarrow (SO(\mathbb{H}) = SO(4)) \\ x &\mapsto (m_\ell(x): v \mapsto x.v) \end{aligned}$$

$$\begin{aligned} \overline{m}_r: S^3 &\rightarrow SO(\mathbb{H}) \\ y &\mapsto (\overline{m}_r(y): v \mapsto v.\overline{y}) \end{aligned}$$

together induce the surjective group morphism

$$\begin{aligned} S^3 \times S^3 &\rightarrow SO(4) \\ (x, y) &\mapsto (v \mapsto x.v.\overline{y}). \end{aligned}$$

The kernel of this group morphism is  $\{(-1, -1), (1, 1)\}$ . So this morphism is a two-fold covering map. In particular, we have  $\pi_3(SO(4)) = \mathbb{Z}[m_\ell] \oplus \mathbb{Z}[\overline{m}_r]$ . Define

$$\begin{aligned} m_r: S^3 &\rightarrow (SO(\mathbb{H}) = SO(4)) \\ y &\mapsto (m_r(y): v \mapsto v.y). \end{aligned}$$

**Lemma 5.19.** *In  $\pi_3(SO(4)) = \mathbb{Z}[m_\ell] \oplus \mathbb{Z}[\overline{m}_r]$ , we have*

$$i_*([\tilde{\rho}]) = [m_\ell] + [\overline{m}_r] = [m_\ell] - [m_r].$$

PROOF: The  $\pi_3$ -product in  $\pi_3(SO(4))$  coincides with the product induced by the group structure of  $SO(4)$ .  $\square$

**Lemma 5.20.** *Recall that  $m_r$  denotes the map from the unit sphere  $S^3$  of  $\mathbb{H}$  to  $SO(\mathbb{H})$  induced by the right-multiplication. Denote the inclusions  $SO(n) \subset SU(n)$  by  $c$ . Then we have*

$$c_*([m_r]) = 2 [i_*^2(m_r^{\mathbb{C}})]$$

in  $\pi_3(SU(4))$ .

PROOF: Let  $\mathbb{H} + I\mathbb{H}$  denote the complexification of  $\mathbb{R}^4 = \mathbb{H} = \mathbb{R}\langle 1, i, j, k \rangle$ . Here, we have  $\mathbb{C} = \mathbb{R} \oplus I\mathbb{R}$  and  $I^2 = -1$ . When  $x \in \mathbb{H}$  and  $v \in S^3$ , we have  $c(m_r(v))(Ix) = Ixv$ . Let  $\varepsilon = \pm 1$ . Define

$$\mathbb{C}^2(\varepsilon) = \mathbb{C} \left\langle \frac{\sqrt{2}}{2}(1 + \varepsilon Ii), \frac{\sqrt{2}}{2}(j + \varepsilon Ik) \right\rangle.$$

Consider the quotient  $\mathbb{C}^4/\mathbb{C}^2(\varepsilon)$ . In this quotient, we have  $Ii = -\varepsilon 1$ ,  $Ik = -\varepsilon j$ , and, since  $I^2 = -1$ ,  $I1 = \varepsilon i$  and  $Ij = \varepsilon k$ . Therefore, this quotient is

isomorphic to  $\mathbb{H}$  as a real vector space with its complex structure  $I = \varepsilon i$ . Then it is easy to see that  $c(m_r(v))$  maps  $\mathbb{C}^2(\varepsilon)$  to 0 in this quotient, for any  $v \in S^3$ . We get  $c(m_r(v))(\mathbb{C}^2(\varepsilon)) = \mathbb{C}^2(\varepsilon)$ . Now, observe that  $\mathbb{H} + I\mathbb{H}$  is the orthogonal sum of  $\mathbb{C}^2(-1)$  and  $\mathbb{C}^2(1)$ . In particular,  $\mathbb{C}^2(\varepsilon)$  is isomorphic to the quotient  $\mathbb{C}^4/\mathbb{C}^2(-\varepsilon)$ , which is isomorphic to  $(\mathbb{H}; I = -\varepsilon i)$ , and  $c(m_r)$  acts on it by the right multiplication. Therefore, with respect to the orthonormal basis  $\frac{\sqrt{2}}{2}(1 - Ii, j - Ik, 1 + Ii, j + Ik)$ ,  $c(m_r(z_1 + z_2j = x_1 + y_1i + x_2j + y_2k))$  may be written as

$$c(m_r(x_1 + y_1i + x_2j + y_2k)) = \begin{bmatrix} x_1 + y_1I & -x_2 + y_2I & 0 & 0 \\ x_2 + y_2I & x_1 - y_1I & 0 & 0 \\ 0 & 0 & x_1 - y_1I & -x_2 - y_2I \\ 0 & 0 & x_2 - y_2I & x_1 + y_1I \end{bmatrix}.$$

Therefore, the homotopy class of  $c(m_r)$  is the sum of the homotopy classes of

$$(z_1 + z_2j) \mapsto \begin{bmatrix} m_r^{\mathbb{C}}(z_1, z_2) & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad (z_1 + z_2j) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & m_r^{\mathbb{C}} \circ \iota(z_1, z_2) \end{bmatrix},$$

where  $\iota(z_1, z_2) = (\overline{z_1}, \overline{z_2})$ . Since the first map is conjugate to  $i_*^2(m_r^{\mathbb{C}})$  by a fixed element of  $SU(4)$ , it is homotopic to  $i_*^2(m_r^{\mathbb{C}})$ . Since  $\iota$  induces the identity on  $\pi_3(S^3)$ , the second map is homotopic to  $i_*^2(m_r^{\mathbb{C}})$ , too.  $\square$

The following lemma finishes to determine the maps

$$c_*: \pi_3(SO(4)) \longrightarrow \pi_3(SU(4))$$

and  $c_*i_*: \pi_3(SO(3)) \longrightarrow \pi_3(SU(4))$ .

**Lemma 5.21.** *We have*

$$c_*([\overline{m}_r]) = c_*([m_\ell]) = -2 [i_*^2(m_r^{\mathbb{C}})] = 2 [i_*^2(\overline{m}_r^{\mathbb{C}})],$$

and

$$c_*\left(i_*([\tilde{\rho}])\right) = 4 [i_*^2(\overline{m}_r^{\mathbb{C}})].$$

PROOF: According to Lemma 5.19,  $i_*([\tilde{\rho}]) = [m_\ell] + [\overline{m}_r]$ . Using the conjugacy of quaternions, we have  $m_\ell(x)(v) = x.v = \overline{v}.\overline{x} = \overline{m}_r(x)(\overline{v})$ . Therefore,  $m_\ell$  is conjugated to  $\overline{m}_r$  via the conjugacy of quaternions, which lies in  $(O(4) \subset U(4))$ .

Since  $U(4)$  is connected, the conjugacy by an element of  $U(4)$  induces the identity on  $\pi_3(SU(4))$ . Thus, we get

$$c_*([m_\ell]) = c_*([\overline{m}_r]) = -c_*([m_r]) = -2 [i_*^2(m_r^{\mathbb{C}})] = 2 [i_*^2(\overline{m}_r^{\mathbb{C}})],$$

and  $c_*\left(i_*([\tilde{\rho}])\right) = c_*([m_\ell]) + c_*([\overline{m}_r]) = 4 [i_*^2(\overline{m}_r^{\mathbb{C}})]$ .  $\square$



## 5.7 Relating the Pontrjagin number to the degree

We finish proving Theorem 4.5 by proving the following proposition. See Lemmas 5.2, 5.7, and 5.8.

**Proposition 5.22.** *Let  $M_0$  and  $M$  be two compact connected oriented 3-manifolds whose boundaries have collars identified by a diffeomorphism. Let  $\tau_0: M_0 \times \mathbb{C}^3 \rightarrow TM_0 \otimes \mathbb{C}$  and  $\tau: M \times \mathbb{C}^3 \rightarrow TM \otimes \mathbb{C}$  be two special complex trivializations (which respect the orientations) that coincide on the collar neighborhoods of  $\partial M_0 = \partial M$ . Let  $[(M, \partial M), (SU(3), 1)]$  denote the group of homotopy classes of maps from  $M$  to  $SU(3)$  that map  $\partial M$  to 1. For any*

$$g: (M, \partial M) \longrightarrow (SU(3), 1),$$

define

$$\begin{aligned} \psi(g): \quad M \times \mathbb{C}^3 &\longrightarrow M \times \mathbb{C}^3 \\ (x, y) &\longmapsto (x, g(x)(y)). \end{aligned}$$

Then  $(p_1(\tau_0, \tau \circ \psi(g)) - p_1(\tau_0, \tau))$  is independent of  $\tau_0$  and  $\tau$ . Set

$$p'_1(g) = p_1(\tau_0, \tau \circ \psi(g)) - p_1(\tau_0, \tau).$$

The map  $p'_1$  induces an isomorphism from the group  $[(M, \partial M), (SU(3), 1)]$  to  $\mathbb{Z}$ , and, if  $g$  is valued in  $SO(3)$ , then we have

$$p'_1(g) = 2 \deg(g).$$

To prove Proposition 5.22, we first prove the following lemma.

**Lemma 5.23.** *Under the hypotheses of Proposition 5.22,*

$$p_1(\tau_0, \tau \circ \psi(g)) - p_1(\tau_0, \tau) = p_1(\tau, \tau \circ \psi(g)) = -p_1(\tau \circ \psi(g), \tau)$$

is independent of  $\tau_0$  and  $\tau$ .

**PROOF:** The equalities of the statement are easy to observe. Let us prove that  $p_1(\tau, \tau \circ \psi(g))$  is independent of  $\tau$ . Let  $\tau_W$  be the trivialization of the restriction to  $\partial([0, 1] \times M)$  of the complexified tangent bundle  $T[0, 1] \times M \otimes_{\mathbb{R}} \mathbb{C}$  to  $[0, 1] \times M$  given by  $T[0, 1] \oplus \tau$  on  $(\{0\} \times M) \cup ([0, 1] \times \partial M)$ , and  $T[0, 1] \oplus \tau \circ \psi(g)$  on  $\{1\} \times M$ . Let  $\tilde{g}: \partial([0, 1] \times M) \rightarrow SU(4)$  map  $(\{0\} \times M) \cup ([0, 1] \times \partial M)$  to 1 and coincide with  $i \circ g$  on  $\{1\} \times M$ . Then the obstruction  $p_1(\tau, \tau \circ \psi(g))$  to extending  $\tau_W$  to  $[0, 1] \times M$  is the obstruction to extending the map  $\tilde{g}$  to  $[0, 1] \times M$ . It lies in  $\pi_3(SU(4))$  since  $\pi_i(SU(4)) = 0$  for  $i < 3$ . It is independent of  $\tau$ .  $\square$

Lemma 5.23 guarantees that  $p'_1$  defines two group homomorphisms to  $\mathbb{Z}$  from  $[(M, \partial M), (SU(3), 1)]$  and from  $[(M, \partial M), (SO(3), 1)]$ .

**Lemma 5.24.** *Under the hypotheses of Proposition 5.22*

$$p'_1: [(M, \partial M), (SO(3), 1)] \rightarrow \mathbb{Z}$$

*is a group homomorphism and*

$$p'_1: [(M, \partial M), (SU(3), 1)] \rightarrow \mathbb{Z}$$

*is a group isomorphism.*

PROOF: Since  $\pi_i(SU(3))$  is trivial for  $i < 3$  and since  $\pi_3(SU(3)) = \mathbb{Z}$ , the group of homotopy classes  $[(M, \partial M), (SU(3), 1)]$  is generated by the class of a map that maps the complement of a 3-ball  $B$  to 1 and that factors through a map whose homotopy class generates  $\pi_3(SU(3))$  on  $B$ . By definition of the Pontrjagin classes,  $p'_1$  sends such a generator to  $\pm 1$ , and it induces an isomorphism from the group  $[(M, \partial M), (SU(3), 1)]$  to  $\mathbb{Z}$ .  $\square$

**Lemma 5.25.** *We have*

$$p'_1(\rho_M(B^3)) = 4.$$

PROOF: Let  $g = \rho_M(B^3)$ . Extend the map  $\tilde{g}$  of the proof of Lemma 5.23 by the constant map with value 1 outside  $[\varepsilon, 1] \times B^3 \cong B^4$ , for some  $\varepsilon \in ]0, 1[$ . We have

$$[\tilde{g}|_{\partial B^4}] = p_1(\tau, \tau \circ \psi(g)) [i_*^2(\overline{m}_r^{\mathbb{C}})]$$

in  $\pi_3(SU(4))$ . Since  $\tilde{g}|_{\partial B^4}$  is homotopic to  $c \circ i \circ \tilde{\rho}$ , Lemma 5.21 allows us to conclude.  $\square$

PROOF OF PROPOSITION 5.22: According to Lemmas 5.8 and 5.24, the restriction of  $p'_1$  to  $[(M, \partial M), (SO(3), 1)]$  is equal to  $p'_1(\rho_M(B^3)) \frac{\text{deg}}{2}$ . Conclude with Lemma 5.25.  $\square$

## 5.8 Properties of Pontrjagin numbers

**Proposition 5.26.** *Let  $M_0$  and  $M_1$  be two compact connected oriented 3-manifolds whose boundaries have collars identified by a diffeomorphism. Let  $\tau_0: M_0 \times \mathbb{C}^3 \rightarrow TM_0 \otimes \mathbb{C}$  and  $\tau_1: M_1 \times \mathbb{C}^3 \rightarrow TM_1 \otimes \mathbb{C}$  be two special complex trivializations (which respect the orientations) that agree on the collar neighborhoods of  $\partial M_0 = \partial M_1$ .*

*Then the first Pontrjagin number  $p_1(\tau_0, \tau_1)$  satisfies the following properties.*

1. Let  $M_2$  be a compact 3-manifold whose boundary has a collar neighborhood identified with a collar neighborhood of  $\partial M_0$ . Let  $\tau_2$  be a special complex trivialization of  $TM_2$  that agrees with  $\tau_0$  near  $\partial M_2$ . If two of the Lagrangians of  $M_0$ ,  $M_1$ , and  $M_2$  coincide in  $H_1(\partial M_0; \mathbb{Q})$ , then we have

$$p_1(\tau_0, \tau_2) = p_1(\tau_0, \tau_1) + p_1(\tau_1, \tau_2).$$

In particular, we also have  $p_1(\tau_1, \tau_0) = -p_1(\tau_0, \tau_1)$  since  $p_1(\tau_0, \tau_0) = 0$ .

2. Let  $D$  be a connected compact 3-manifold that contains  $M_0$  in its interior. Let  $\tau_D$  be a special complex trivialization of  $TD$  that restricts as the special complex trivialization  $\tau_0$  on  $TM_0$ . Let  $D_1$  be obtained from  $D$  by replacing  $M_0$  by  $M_1$ . Let  $\tau_{D_1}$  be the trivialization of  $TD_1$  that agrees with  $\tau_1$  on  $TM_1$ , and with  $\tau_D$  on  $T(D \setminus M_0)$ . If the Lagrangians

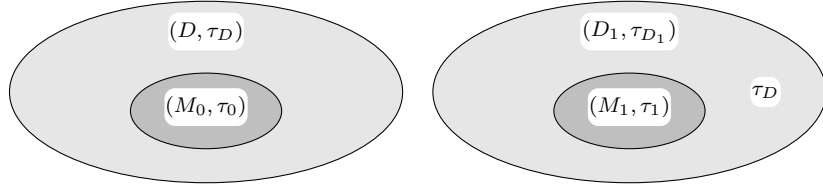


Figure 5.2: The manifolds  $D$  and  $D_1$  of Proposition 5.26

of  $M_0$  and  $M_1$  coincide, then we have

$$p_1(\tau_D, \tau_{D_1}) = p_1(\tau_0, \tau_1).$$

The proof uses a weak form of the Wall Non-Additivity theorem. We quote the weak form we need.

**Theorem 5.27** ([Wal69]). *Let  $Y$  be a compact oriented 4-manifold (with possible boundary). Let  $X$  be a 3-manifold properly embedded in  $Y$  that separates  $Y$  and that induces the splitting  $Y = Y^+ \cup_X Y^-$ , for two 4-manifolds  $Y^+$  and  $Y^-$  in  $Y$ , as in the following figure of  $Y$ :*

$$X^- \left( \begin{array}{c|c} Y^- & X Y^+ \end{array} \right) X^+$$

The manifold  $X$  is the intersection  $Y^+ \cap Y^-$ . It is oriented as a part of the boundary of  $Y^-$ . Set

$$X^+ = \overline{\partial Y^+ \setminus (-X)} \quad \text{and} \quad X^- = -\overline{\partial Y^- \setminus X}.$$

Let  $\mathcal{L}$ ,  $\mathcal{L}^-$ , and  $\mathcal{L}^+$  denote the Lagrangians of  $X$ ,  $X^-$ , and  $X^+$ , respectively. They are Lagrangian subspaces of  $H_1(\partial X, \mathbb{Q})$ . Then

$$(\text{signature}(Y) - \text{signature}(Y^+) - \text{signature}(Y^-))$$

is the signature of an explicit quadratic form on

$$\frac{\mathcal{L} \cap (\mathcal{L}^- + \mathcal{L}^+)}{(\mathcal{L} \cap \mathcal{L}^-) + (\mathcal{L} \cap \mathcal{L}^+)}.$$

Furthermore, this space is isomorphic to  $\frac{\mathcal{L}^+ \cap (\mathcal{L} + \mathcal{L}^-)}{(\mathcal{L}^+ \cap \mathcal{L}) + (\mathcal{L}^+ \cap \mathcal{L}^-)}$  and  $\frac{\mathcal{L}^- \cap (\mathcal{L} + \mathcal{L}^+)}{(\mathcal{L}^- \cap \mathcal{L}) + (\mathcal{L}^- \cap \mathcal{L}^+)}$ .

We do not describe the involved quadratic form because we use this theorem only when the above space is trivial.

**PROOF OF PROPOSITION 5.26:** Let us prove the first property. Let  $Y^- = W$  be a signature 0 cobordism from  $X^- = M_0$  to  $X = M_1$ . Let  $Y^+$  be a signature 0 cobordism from  $M_1$  to  $X^+ = M_2$ . Then it is enough to prove that the signature of  $Y = Y^+ \cup_X Y^-$  is zero. With the notation of Theorem 5.27, under our assumptions, the space  $\frac{\mathcal{L} \cap (\mathcal{L}^- + \mathcal{L}^+)}{(\mathcal{L} \cap \mathcal{L}^-) + (\mathcal{L} \cap \mathcal{L}^+)}$  is trivial. Therefore, according to the Wall theorem, the signature of  $Y$  is zero. The first property follows.

We now prove that under the assumptions of the second property, we have

$$p_1(\tau_D, \tau_{D_1}) = p_1(\tau_0, \tau_1).$$

Set  $Y^+ = ([0, 1] \times (D \setminus \overset{\circ}{M}_0))$  and  $X = -[0, 1] \times \partial M_0$ . Let  $Y^- = W$  be a signature 0 cobordism from  $M_0$  to  $M_1$ . Note that the signature of  $Y^+$  is zero. In order to prove the desired equality, it is enough to prove that the signature of  $Y = Y^+ \cup_X Y^-$  is zero. Here, we have  $H_1(\partial X; \mathbb{Q}) = H_1(\partial M_0) \oplus H_1(\partial M_0)$ . Let  $j: H_1(\partial M_0) \rightarrow H_1(D \setminus \overset{\circ}{M}_0)$  and  $j_{\partial D}: H_1(\partial D) \rightarrow H_1(D \setminus \overset{\circ}{M}_0)$  be the maps induced by inclusions. With the notation of Theorem 5.27, we have

$$\begin{aligned} \partial X &= -(\partial [0, 1]) \times \partial M_0, \\ X^- &= -\{1\} \times M_1 \cup (\{0\} \times M_0), \\ X^+ &= -[0, 1] \times \partial D \cup \left( (\partial [0, 1]) \times (D \setminus \overset{\circ}{M}_0) \right), \\ \mathcal{L} &= \{(x, -x) : x \in H_1(\partial M_0)\}, \\ \mathcal{L}^- &= \{(x, y) : x \in \mathcal{L}_{M_0}, y \in \mathcal{L}_{M_1}\}, \\ \mathcal{L}^+ &= \{(x, y) : (j(x), j(y)) = (j_{\partial D}(z \in H_1(\partial D)), -j_{\partial D}(z))\} \\ &= \{(y, -y) : j(y) \in \text{Im}(j_{\partial D})\} \oplus \{(x, 0) : j(x) = 0\}, \\ \mathcal{L} \cap \mathcal{L}^- &= \{(x, -x) : x \in (\mathcal{L}_{M_0} \cap \mathcal{L}_{M_1} = \mathcal{L}_{M_0})\}, \\ \mathcal{L} \cap \mathcal{L}^+ &= \{(x, -x) : j(x) \in \text{Im}(j_{\partial D})\}. \end{aligned}$$

Let us prove  $\mathcal{L} \cap (\mathcal{L}^- + \mathcal{L}^+) = (\mathcal{L} \cap \mathcal{L}^-) + (\mathcal{L} \cap \mathcal{L}^+)$ . For a subspace  $K$  of  $H_1(\partial M_0; \mathbb{Q})$ , let  $j_{MV}(K)$  be the subspace  $\{(x, -x) : x \in K\}$  of  $H_1(\partial X; \mathbb{Q})$ . Then we have  $\mathcal{L} = j_{MV}(H_1(\partial M_0))$ ,  $\mathcal{L} \cap \mathcal{L}^+ = j_{MV}(j^{-1}(\text{Im}(j_{\partial D})))$ , and

$$\mathcal{L} \cap (\mathcal{L}^- + \mathcal{L}^+) = \mathcal{L} \cap \mathcal{L}^+ + j_{MV}\left(\mathcal{L}_{M_1} \cap (\mathcal{L}_{M_0} + \text{Ker}(j))\right).$$

Since  $\mathcal{L}_{M_0} = \mathcal{L}_{M_1}$ , we have  $(\mathcal{L}_{M_1} \cap (\mathcal{L}_{M_0} + \text{Ker}(j))) = \mathcal{L}_{M_0}$ . So  $\mathcal{L} \cap (\mathcal{L}^- + \mathcal{L}^+) = (\mathcal{L} \cap \mathcal{L}^+) + j_{MV}(\mathcal{L}_{M_0})$ . Then the second property is proved because Wall's theorem guarantees the additivity of the signature in this case.  $\square$

**The parallelizations of  $S^3$ .** As a Lie group,  $S^3$  has two natural homotopy classes of parallelizations  $\tau_\ell$  and  $\tau_r$ , which we describe below. Identify the tangent space  $T_1 S^3$  to  $S^3$  at 1 with  $\mathbb{R}^3$  (arbitrarily, with respect to the orientation). For  $g \in S^3$ , the multiplication induces two diffeomorphisms  $m_\ell(g)$  and  $m_r(g)$  of  $S^3$ ,  $m_\ell(g)(h) = gh$  and  $m_r(g)(h) = hg$ . Let  $T(m_\ell(g))$  and  $T(m_r(g))$  denote their respective tangent maps at 1. Then we have

$$\tau_\ell\left(h \in S^3, v \in (\mathbb{R}^3 = T_1 S^3)\right) = \left(h, T_1(m_\ell(h))(v)\right)$$

and  $\tau_r(h, v) = (h, T_1(m_r(h))(v))$ .

**Proposition 5.28.** *We have  $p_1(\tau_\ell) = 2$  and  $p_1(\tau_r) = -2$ .*

PROOF: Regard  $S^3$  as the unit sphere of  $\mathbb{H}$ . So  $T_1 S^3 = \mathbb{R}\langle i, j, k \rangle$ . The unit ball  $B(\mathbb{H})$  of  $\mathbb{H}$  has the standard parallelization of a real vector space equipped with a basis. The trivialization  $\tau(\tau_\ell)$  induced by  $\tau_\ell$  on  $\partial B(\mathbb{H})$  is such that  $\tau_\ell(h \in S^3, v \in \mathbb{H}) = (h, hv) \in (S^3 \times \mathbb{H} = T\mathbb{H}|_{S^3})$ . So, we have  $c_*([m_\ell]) = p_1(\tau_\ell) [i_*^2(\overline{m}_r^C)]$  in  $\pi_3(SU(4))$  by Definition 5.13 of  $p_1$ . According to Lemma 5.21, we get  $p_1(\tau_\ell) = 2$ . We similarly get  $p_1(\tau_r) = -2$ .  $\square$

**On the image of  $p_1$ .** For  $n \geq 3$ , a *spin structure* of a smooth  $n$ -manifold is a homotopy class of parallelizations over a 2-skeleton of  $M$  (or, equivalently, over the complement of a point, if  $n = 3$  and if  $M$  is connected).

The class of the covering map  $\tilde{\rho}$  described after Definition 4.4 is the standard generator of  $\pi_3(SO(3)) = \mathbb{Z}[\tilde{\rho}]$ . Recall the map  $\rho_M(B^3)$  of Lemma 4.6. Set  $[\tilde{\rho}][\tau] = [\tau\psi_{\mathbb{R}}(\rho_M(B^3))]$ . The homotopy classes of parallelizations that induce a given spin structure constitute an affine space with translation group  $\pi_3(SO(3))$ . According to Theorem 4.5 and Lemma 4.6,  $p_1([\tilde{\rho}][\tau]) = p_1(\tau) + 4$ .

**Definition 5.29.** The *Rohlin invariant*  $\mu(M, \sigma)$  of a smooth closed 3-manifold  $M$ , equipped with a spin structure  $\sigma$ , is the mod 16 signature of a compact 4-manifold  $W$ , bounded by  $M$ , equipped with a spin structure that restricts to  $M$  as a stabilization of  $\sigma$ .

The *first Betti number* of  $M$  is the dimension of  $H_1(M; \mathbb{Q})$ . It is denoted by  $\beta_1(M)$ . Robion Kirby and Paul Melvin proved the following theorem [KM99, Theorem 2.6].

**Theorem 5.30.** *For any closed oriented 3-manifold  $M$ , for any parallelization  $\tau$  of  $M$ , we have*

$$\left( p_1(\tau) - \text{dimension}(H_1(M; \mathbb{Z}/2\mathbb{Z})) - \beta_1(M) \right) \in 2\mathbb{Z}.$$

Let  $M$  be a closed 3-manifold equipped with a given spin structure  $\sigma$ . Then  $p_1$  is a bijection from the set of homotopy classes of parallelizations of  $M$  that induce  $\sigma$  to

$$2\left( \text{dimension}(H_1(M; \mathbb{Z}/2\mathbb{Z})) + 1 \right) + \mu(M, \sigma) + 4\mathbb{Z}.$$

When  $M$  is a  $\mathbb{Z}$ -sphere,  $p_1$  is a bijection from the set of homotopy classes of parallelizations of  $M$  to  $(2 + 4\mathbb{Z})$ .

Thanks to Proposition 5.26(2), Theorem 5.30 implies Proposition 5.31 below.

**Proposition 5.31.** *Let  $M_0$  be the unit ball of  $\mathbb{R}^3$ . Let  $\tau_s$  be the standard parallelization of  $\mathbb{R}^3$ ,*

- *for any given  $\mathbb{Z}$ -ball  $M$ ,  $p_1(\cdot) = p_1((\tau_s)|_{B^3}, \cdot)$  defines a bijection from the set of homotopy classes of parallelizations of  $M$  that are standard near  $\partial M = S^2$  to  $4\mathbb{Z}$ .*
- *For any  $\mathbb{Q}$ -ball  $M$ , for any trivialization  $\tau$  of  $M$  that is standard near  $\partial M = S^2$ , we have*

$$\left( p_1(\tau) - \text{dimension}(H_1(M; \mathbb{Z}/2\mathbb{Z})) \right) \in 2\mathbb{Z}.$$

## 5.9 More on $[(M, \partial M), (SO(3), 1)]$

This section is a complement to the study of  $[(M, \partial M), (SO(3), 1)]$  started in Sections 5.1 and 5.3. It is not used later in this book. We show how to describe all the elements of  $[(M, \partial M), (SO(3), 1)]$  as twists across surfaces, and we describe the structure of  $[(M, \partial M), (SO(3), 1)]$  precisely, by proving the following theorem.

**Theorem 5.32.** *Let  $M$  be a compact oriented 3-manifold.*

*If all the closed surfaces embedded in  $M$  have an even Euler characteristic, then  $[(M, \partial M), (SO(3), 1)]$  is canonically isomorphic to  $H^3(M, \partial M; \mathbb{Z}) \oplus H_2(M; \mathbb{Z}/2\mathbb{Z})$ , and the degree maps  $[(M, \partial M), (SO(3), 1)]$  onto  $2\mathbb{Z}$ .*

*If  $H_1(M; \mathbb{Z})$  has no 2-torsion, then all the closed surfaces embedded in  $M$  have an even Euler characteristic.*

*If  $M$  is connected, and if there exists a closed surface  $S$  of  $M$  with odd Euler characteristic, then the degree maps  $[(M, \partial M), (SO(3), 1)]$  onto  $\mathbb{Z}$ , and  $[(M, \partial M), (SO(3), 1)]$  is isomorphic to  $\mathbb{Z} \oplus \text{Ker}(e\partial_B)$ , where*

$$e\partial_B: H_2(M; \mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z}$$

*maps the class of a surface to its Euler characteristic modulo 2, and the kernel of  $e\partial_B$  has a canonical image in  $[(M, \partial M), (SO(3), 1)]$ .*

**Representing the elements of  $[(M, \partial M), (SO(3), 1)]$  by surfaces.**

**Proposition 5.33.** *Let  $S$  be a surface, orientable or not, embedded in a 3-manifold  $M$  equipped with a parallelization  $\tau$ . Recall the map  $g(S, \tau)$  from the beginning of Section 5.3. If  $M$  is connected, then  $g(S, \tau)^2$  is homotopic to  $\rho_M(B^3)^{\chi(S)}$ . In particular, the homotopy class of  $g(S, \tau)$  depends only on  $\chi(S)$  and on the class of  $S$  in  $H_2(M; \mathbb{Z}/2\mathbb{Z})$ .*

**PROOF:** Assume that  $S$  is connected and oriented. Perform a homotopy of  $\tau$  so that  $p_2 \circ \tau^{-1}$  maps the positive normal  $N^+(S) = T_{(u,s)}([-1, 1] \times s)$  to  $u \times S$  to a fixed vector  $v$  of  $S^2$ , on  $[-1, 1] \times (S \setminus D)$ , for a disk  $D$  of  $S$ . Then there is a homotopy from  $[0, 1] \times [-1, 1] \times (S \setminus D)$  to  $SO(3)$ ,

- which factors through the projection onto  $[0, 1] \times [-1, 1]$ ,
- which maps  $(1, u, s)$  to the rotation  $g(S, \tau)^2(u, s)$  with axis  $v$  and angle  $2\pi(u + 1)$ , for any  $(u, s) \in [-1, 1] \times (S \setminus D)$ , and
- which maps  $(\partial([0, 1] \times [-1, 1]) \setminus (\{1\} \times [-1, 1])) \times (S \setminus D)$  to 1.

This homotopy extends to a homotopy  $h: [0, 1] \times [-1, 1] \times S \rightarrow SO(3)$  from  $h_0 = \rho_{[-1,1] \times S}(B^3)^k$  for some  $k \in \mathbb{Z}$ , to  $h_1 = g(S, \tau)^2|_{[-1,1] \times S}$ , such that  $h([0, 1] \times \{-1, 1\} \times D) = 1$ . Thus, the map  $g(S, \tau)^2$  is homotopic to  $\rho_M(B^3)^k$ .

Let us show that  $k$  depends only on the degree of the Gauss map  $G$  from  $(D, \partial D)$  to  $(S^2, v)$  that maps  $s \in (D = \{0\} \times D)$  to  $p_2(\tau^{-1}(N^+(s)))$ . Endow  $\mathbb{R}^3$  with an orthonormal basis beginning with  $v$ . Identify the tangent space to  $D$  with  $(D \times) \mathbb{R}^2$  and  $T([-1, 1] \times D)|_D$  with  $([-1, 1] \times D \times) (\mathbb{R}N^+(D) \oplus \mathbb{R}^2)$ . These identifications allow us to regard  $p_2 \circ \tau^{-1}$  as a map  $\Psi$  from  $[-1, 1] \times D$

to  $SO(3)$ . The map  $\Psi$  determines  $\tau$ . It sends  $[-1, 1] \times \partial D$  to  $(i(SO(2)) = SO(2))$ . Since  $\pi_2(SO(3))$  is trivial, the homotopy class of  $\Psi|_D$  is determined by its restriction  $\Psi|_{\partial D}$  to  $\partial D$ . The map from  $\pi_2(S^2)$  to  $\pi_1(SO(2) = S^1)$  in the long exact sequence associated to the fibration

$$SO(2) \xrightarrow{i} SO(3) \rightarrow S^2,$$

described in Theorem A.14 sends the class of  $G$  to the class of  $\Psi|_{\partial D}$ . So the degree of  $G$  determines the homotopy class of the restriction of  $\tau$  to  $D$  relative to  $\partial D$ . This homotopy class determines the homotopy class relative to  $[-1, 1] \times \partial D$  of the restriction of  $\tau$  to  $[-1, 1] \times D$ , which determines the homotopy class relative to  $\partial([-1, 1] \times D)$  of the restriction of  $g(S, \tau)^2$  to  $[-1, 1] \times D$ . Therefore, the degree  $d$  of  $G$  determines  $k$ . Cutting  $D$  into smaller disks shows that  $k$  depends linearly on  $d$ . Note that  $d$  is the degree of the Gauss map from  $S$  to  $S^2$  before the homotopy of  $\tau$ . For a standard sphere  $S^2$ , we have  $d = 1$  and  $g(S^2, \tau) = \rho_M(B^3)$ . So, we get  $k = 2d$ . It remains to see that the degree  $d$  of the Gauss map is  $\frac{\chi(S)}{2}$ . This is easy to observe for a standard embedding of  $S$  into  $\mathbb{R}^3$  equipped with its standard trivialization. Up to homotopy, the trivializations of  $TM|_S$  are obtained from the standard one by compositions by rotations with fixed axis  $v$  supported in neighborhoods of curves outside the preimage of  $v$ . So the degree (at  $v$ ) is independent of the trivialization. Thus, the proposition is proved when  $S$  is orientable and connected. When  $S$  is not orientable, the map  $g(S, \tau)^2$  is homotopic to  $\rho_M(B^3)^k$ , for some  $k$ , according to Lemma 5.2. Furthermore, it is homotopic to  $g(S^{(2)}, \tau)$ , where  $S^{(2)}$  is the orientable boundary of the tubular neighborhood of  $S$ . The Euler characteristic of  $S^{(2)}$  is  $2\chi(S)$ . So  $g(S, \tau)^4$  is homotopic to  $\rho_M(B^3)^{2k}$  and to  $\rho_M(B^3)^{2\chi(S)}$ . Since the arguments are local, they extend to the disconnected case and prove that  $g(S, \tau)^2$  is homotopic to  $\rho_M(B^3)^{\chi(S)}$  for any  $S$ . Then Proposition 5.9 and Lemma 5.6 allow us to conclude that the homotopy class of  $g(S, \tau)$  depends only on  $\chi(S)$  and on the class of  $S$  in  $H_2(M; \mathbb{Z}/2\mathbb{Z})$ .  $\square$

Hence,  $g(S, \tau)$  will be denoted by  $g(S)$ . Lemma 5.6 and Propositions 5.9 and 5.33 easily imply the following corollary.

**Corollary 5.34.** *All elements of  $[(M, \partial M), (SO(3), 1)]$  can be represented by  $g(S)$  for some embedded disjoint union of closed surfaces  $S$  of  $M$ .*

$\square$

**Structure of  $[(M, \partial M), (SO(3), 1)]$ .** Tensoring a free chain complex  $C_*(M; \mathbb{Z})$  whose homology is  $H_*(M; \mathbb{Z})$  by the short exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$



yields the associated long exact homology sequence

$$\dots \rightarrow H_*(M; \mathbb{Z}) \xrightarrow{\times 2} H_*(M; \mathbb{Z}) \rightarrow H_*(M; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\partial_B} H_{*-1}(M; \mathbb{Z}) \rightarrow \dots,$$

where  $\partial_B$  is the *Bockstein morphism*.

**Definition 5.35.** The *self-linking number* of a torsion element  $x$  of  $H_1(M; \mathbb{Z})$  is the linking number  $lk(c, c')$  of a curve  $c$  that represents  $x$  and a parallel  $c'$  of  $c$ , modulo  $\mathbb{Z}$ .<sup>2</sup> It belongs to  $\mathbb{Q}/\mathbb{Z}$ .

**Proposition 5.36.** *There is a canonical group homomorphism*

$$e\partial_B: H_2(M; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

that admits the following two equivalent definitions:

1. For any embedded surface  $S$ ,  $e\partial_B$  maps the class of  $S$  to the Euler characteristic of  $S$  modulo 2.
2. The map  $e\partial_B$  is the composition of the Bockstein morphism

$$\partial_B: H_2(M; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \text{2-torsion of } H_1(M; \mathbb{Z})$$

and the map

$$e: \text{2-torsion of } H_1(M; \mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

that maps the class of a curve  $x$  to 1 if the self-linking number of  $x$  is  $\frac{1}{2}$  modulo  $\mathbb{Z}$ , and to 0 otherwise.

**PROOF:** The map  $e\partial_B$  is well defined by the second definition. It is a group homomorphism. Let  $S$  be a connected closed surface. If  $S$  is orientable, then the Euler characteristic of  $S$  is even, and the long exact sequence shows that  $\partial_B([S]) = 0$ . Otherwise, there is a curve  $x$  (Poincaré dual to  $w_1(S)$ ) such that  $S \setminus x$  is orientable, and the boundary of the closure of the domain of the embedding  $S \setminus x$  maps to  $(\pm 2x)$ , so  $\partial_B([S]) = [x]$ , by definition. The characteristic curve  $x$  may be assumed to be connected. Then the tubular neighborhood of  $x$  in  $S$  is either a Möbius band or an annulus. In the first case, we have  $e([x]) = 1$  and  $\chi(S)$  is odd. Otherwise, we have  $e([x]) = 0$  and  $\chi(S)$  is even.  $\square$

**Proposition 5.37.** *Let  $M$  be an oriented connected 3-manifold. Then we have*

$$[(M, \partial M), (SO(3), 1)] \cong \mathbb{Z} \oplus \text{Ker}(e\partial_B),$$

and the degree maps  $[(M, \partial M), (SO(3), 1)]$  onto  $2\mathbb{Z}$  when  $e\partial_B = 0$ , and onto  $\mathbb{Z}$  otherwise.

<sup>2</sup>Equivalently, it is the linking number of two disjoint representatives of  $x$ , modulo  $\mathbb{Z}$ .

**PROOF OF PROPOSITION 5.37 AND THEOREM 5.32:** The class of a surface with even Euler characteristic in  $H_2(M; \mathbb{Z}/2\mathbb{Z})$  can be represented by a surface  $S$  with null Euler characteristic (a disjoint union with trivial bounding surfaces). According to Proposition 5.33, for such an  $S$ , the class of  $g(S)$  is a 2-torsion element of  $[(M, \partial M), (SO(3), 1)]$  called  $\sigma([S])$ . This defines a canonical partial section

$$\sigma: (\text{Ker}(e\partial_B) \subset H_2(M; \mathbb{Z}/2\mathbb{Z})) \rightarrow \text{Ker}(\text{deg}: [(M, \partial M), (SO(3), 1)] \rightarrow \mathbb{Z})$$

of the sequence of Proposition 5.9. Therefore, if  $e\partial_B = 0$ , we have

$$\left[ (M, \partial M), (SO(3), 1) \right] = \mathbb{Z} [\rho_M(B^3)] \oplus \sigma(\text{Ker}(e\partial_B) = H_2(M; \mathbb{Z}/2\mathbb{Z})).$$

If  $e\partial_B \neq 0$ , there exists a closed surface  $S_1$  with  $\chi(S_1) = 1$  in  $M$ . Since the degree is a group homomorphism from  $[(M, \partial M), (SO(3), 1)]$  to  $\mathbb{Z}$ , Proposition 5.33 implies  $\text{deg}(g(S_1)) = 1$  for such an  $S_1$ . Thus we get

$$\left[ (M, \partial M), (SO(3), 1) \right] = \mathbb{Z}[g(S_1)] \oplus \sigma(\text{Ker}(e\partial_B)).$$

□



## Part II

### The general invariants



# Chapter 6

## Introduction to finite type invariants and Jacobi diagrams

This chapter introduces the target space of the invariant  $\mathcal{Z}$  studied in this book. It is a space generated by uni-trivalent graphs called Jacobi diagrams. Dror Bar-Natan has studied this space in his foundational article [BN95a], where most of the results of this chapter come from. In this chapter, the field  $\mathbb{K}$  is  $\mathbb{Q}$  or  $\mathbb{R}$ .

### 6.1 Definition of finite type invariants

A  $\mathbb{K}$ -valued *invariant* of oriented 3-manifolds is a function from the set of 3-manifolds, considered up to orientation-preserving diffeomorphism, to  $\mathbb{K}$ . Let  $\sqcup_{i=1}^n S_i^1$  denote a disjoint union of  $n$  circles, where each  $S_i^1$  is a copy of  $S^1$ . Here, an  $n$ -component *link* in a 3-manifold  $R$  is an equivalence class of smooth embeddings  $L: \sqcup_{i=1}^n S_i^1 \hookrightarrow R$  under the equivalence relation that identifies two embeddings  $L$  and  $L'$  if and only if there is an orientation-preserving diffeomorphism  $h$  of  $R$  such that  $h(L) = L'$ .<sup>1</sup> A *knot* is a one-component link. A *link invariant* (resp. a *knot invariant*) is a function of links (resp. knots). For example,  $\Theta$  is an invariant of  $\mathbb{Q}$ -spheres, and the linking number is a rational invariant of two-component links in rational homology spheres.

In order to study a function, it is common to study its derivative, and the derivatives of its derivative. The derivative of a function is defined from its variations. For a function  $f$  from  $\mathbb{Z}^d = \bigoplus_{i=1}^d \mathbb{Z}e_i$  to  $\mathbb{K}$ , one can define its

---

<sup>1</sup>This relation is equivalent to the usual equivalence relation defined by isotopies when  $R$  is  $\mathbb{R}^3$  or  $S^3$ . In general 3-manifolds, two equivalent links are not necessarily isotopic, but the link invariants described in this book are invariant under the above equivalence relation.

first-order derivatives  $\frac{\partial f}{\partial e_i} : \mathbb{Z}^d \rightarrow \mathbb{K}$  by

$$\frac{\partial f}{\partial e_i}(z) = f(z + e_i) - f(z),$$

and check that all the first-order derivatives of  $f$  vanish if and only if  $f$  is constant. Inductively define an  $n$ -order derivative to be a first-order derivative of an  $(n - 1)$ -order derivative for a positive integer  $n$ . Then it can be checked that all the  $(n + 1)$ -order derivatives of a function  $f : \mathbb{Z}^d \rightarrow \mathbb{K}$  vanish if and only if  $f$  is a polynomial of degree not greater than  $n$  in the coordinates. In order to study topological invariants, we can similarly study their variations under *simple operations*.

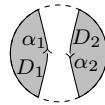
Below,  $X$  denotes one of the following sets



- $\mathbb{Z}^d$ ,
- the set  $\mathcal{K}$  of knots in  $\mathbb{R}^3$ , the set  $\mathcal{K}_k$  of  $k$ -component links in  $\mathbb{R}^3$ ,<sup>2</sup>
- the set  $\mathcal{M}$  of  $\mathbb{Z}$ -spheres, the set  $\mathcal{M}_{\mathbb{Q}}$  of  $\mathbb{Q}$ -spheres (up to orientation-preserving diffeomorphism).

and  $\mathcal{O}(X)$  denotes a set of *simple operations* acting on some elements of  $X$ .

For  $X = \mathbb{Z}^d$ ,  $\mathcal{O}(X)$  consists of the operations  $(z \rightarrow z \pm e_i)$ .

For knots or links in  $\mathbb{R}^3$ , the *simple operations* are *crossing changes*. A *crossing change ball* of a link  $L$  is a ball  $B$  of the ambient space, where  $L \cap B$  is a disjoint union of two arcs  $\alpha_1$  and  $\alpha_2$  properly embedded in  $B$ , and there exist two disjoint topological disks  $D_1$  and  $D_2$  embedded in  $B$ , such that, for  $i \in \{1, 2\}$ , the topological circle  $\partial D_i$  is the union of  $\alpha_i$  and an arc of  $\partial B$  as in the following picture:



After an isotopy, a projection of  $(B, \alpha_1, \alpha_2)$  looks like  or . A *crossing change* is a change that does not change  $L$  outside  $B$  and that modifies it inside  $B$  by a local move

$$\left( \text{crossing} \rightarrow \text{crossing} \right) \text{ or } \left( \text{crossing} \rightarrow \text{crossing} \right).$$

For the left move, the crossing change is *positive*. It is *negative* for the move of the right-hand side.

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<sup>2</sup>Recall that a knot is an isotopy class of knot embeddings.

For integer (resp. rational) homology 3-spheres, the simple operations are integral (resp. rational) *LP-surgeries*, defined in Subsection 1.3.2, and  $\mathcal{O}(\mathcal{M})$  (resp.  $\mathcal{O}(\mathcal{M}_{\mathbb{Q}})$ ) is denoted by  $\mathcal{O}_{\mathbb{Z}}^{\mathcal{L}}$  (resp.  $\mathcal{O}_{\mathbb{Q}}^{\mathcal{L}}$ ).

Say that crossing changes are *disjoint* if they sit inside disjoint 3-balls. Say that LP-surgeries  $(A'/A)$  and  $(B'/B)$  in a manifold  $R$  are *disjoint* if  $A$  and  $B$  are disjoint in  $R$ . Two operations on  $\mathbb{Z}^d$  are always *disjoint* (even if they look identical). In particular, disjoint operations *commute* (i.e., their result does not depend on which one is performed first). Set  $\underline{n} = \{1, 2, \dots, n\}$ . Consider the vector space  $\mathcal{F}_0(X) = \mathcal{F}_0(X; \mathbb{K})$  freely generated by  $X$  over  $\mathbb{K}$ . For an element  $x$  of  $X$  and  $n$  pairwise disjoint operations  $o_1, \dots, o_n$  acting on  $x$ , let  $x((o_i)_{i \in I})$  denote the element of  $X$  obtained by performing the operations  $o_i$  on  $x$  for  $i \in I$ . Define

$$[x; o_1, \dots, o_n] = \sum_{I \subseteq \underline{n}} (-1)^{|I|} x((o_i)_{i \in I}) \in \mathcal{F}_0(X).$$

Then define  $\mathcal{F}_n(X) = \mathcal{F}_n(X; \mathbb{K})$  as the  $\mathbb{K}$ -subspace of  $\mathcal{F}_0(X)$  generated by the  $[x; o_1, \dots, o_n]$ , for all  $x \in X$  equipped with  $n$  pairwise disjoint simple operations  $o_1, \dots, o_n$  acting on  $x$ . Since we have

$$[x; o_1, \dots, o_n, o_{n+1}] = [x; o_1, \dots, o_n] - [x(o_{n+1}); o_1, \dots, o_n],$$

we get  $\mathcal{F}_{n+1}(X) \subseteq \mathcal{F}_n(X)$  for all  $n \in \mathbb{N}$ .

**Definition 6.1.** A  $\mathbb{K}$ -valued function  $f$  on  $X$ , extends uniquely as a  $\mathbb{K}$ -linear map on

$$\mathcal{F}_0(X)^* = \text{Hom}(\mathcal{F}_0(X); \mathbb{K}),$$

which is still denoted by  $f$ . For an integer  $n \in \mathbb{N}$ , the invariant (or function)  $f$  is of *degree*  $\leq n$  if and only if  $f(\mathcal{F}_{n+1}(X)) = 0$ . The *degree* of such an invariant is the smallest integer  $n \in \mathbb{N}$  such that  $f(\mathcal{F}_{n+1}(X)) = 0$ . An invariant is of *finite type* if it is of degree  $n$  for some  $n \in \mathbb{N}$ . This definition depends on the chosen set of operations  $\mathcal{O}(X)$ . We fixed our choices for our sets  $X$ , but other choices could lead to different notions. See [GGP01] for  $X = \mathcal{M}$ .

Let  $\mathcal{I}_n(X) = (\mathcal{F}_0(X)/\mathcal{F}_{n+1}(X))^*$  be the space of invariants of degree at most  $n$ . Of course, we have  $\mathcal{I}_n(X) \subseteq \mathcal{I}_{n+1}(X)$  for all  $n \in \mathbb{N}$ .

**Exercise 6.2.** Prove that  $\mathcal{I}_n(\mathbb{Z}^d)$  is the space of polynomials of degree at most  $n$  on  $\mathbb{Z}^d$ .

**Lemma 6.3.** *If  $f \in \mathcal{I}_m(X)$  and  $g \in \mathcal{I}_n(X)$ , then  $fg \in \mathcal{I}_{m+n}(X)$ .*



PROOF: Let  $[x; (o_i)_{i \in \underline{m+n+1}}] \in \mathcal{F}_{m+n+1}(X)$ . The lemma is a direct consequence of the equality

$$fg\left([x; (o_i)_{i \in \underline{m+n+1}}]\right) = \sum_{J \subseteq \underline{m+n+1}} f\left([x; (o_j)_{j \in J}]\right)g\left([x((o_j)_{j \in J}); (o_i)_{i \in \underline{m+n+1} \setminus J}]\right),$$

which is proved as follows. The right-hand side is equal to

$$\begin{aligned} & \sum_{J \subseteq \underline{m+n+1}} (-1)^{|J|} \left( \sum_{K: K \subseteq J} (-1)^{|K|} f\left(x((o_i)_{i \in K})\right) \right) \left( \sum_{L: J \subseteq L} (-1)^{|L|} g\left(x((o_i)_{i \in L})\right) \right) \\ &= \sum_{(K,L): K \subseteq L \subseteq \underline{m+n+1}} (-1)^{|K|+|L|} f\left(x((o_i)_{i \in K})\right) g\left(x((o_i)_{i \in L})\right) \left( \sum_{J: K \subseteq J \subseteq L} (-1)^{|J|} \right), \end{aligned}$$

where  $\sum_{J: K \subseteq J \subseteq L} (-1)^{|J|} = \begin{cases} 0 & \text{if } K \subsetneq L \\ (-1)^{|K|} & \text{if } K = L. \end{cases} \quad \square$

**Lemma 6.4.** *Any  $n$ -component link in  $\mathbb{R}^3$  can be transformed to the trivial  $n$ -component link below by a finite number of disjoint crossing changes.*





PROOF: Let  $L$  be an (embedding representing an)  $n$ -component link in  $\mathbb{R}^3$ . Since  $\mathbb{R}^3$  is simply connected, there is a homotopy that carries  $L$  to the trivial link. Such a homotopy  $h: [0, 1] \times \sqcup_{i=1}^n S_i^1 \rightarrow \mathbb{R}^3$  can be chosen to be smooth and such that  $h(t, \cdot)$  is an embedding, except for finitely many times  $t_i$ ,  $0 < t_1 < \dots < t_i < t_{i+1} < \dots < 1$ , at which  $h(t_i, \cdot)$  is an immersion with one double point and no other multiple points, and the link  $h(t, \cdot)$  changes exactly by a crossing change when  $t$  crosses a  $t_i$ . (For an alternative elementary proof of this fact, see [Les05, Subsection 7.1] before Definition 7.5, for example).  $\square$

In particular, a degree 0 invariant of  $n$ -component links of  $\mathbb{R}^3$  must be constant since it does not vary under a crossing change.

- Exercise 6.5.**
1. Check that  $\mathcal{I}_1(\mathcal{K}) = \mathbb{K}c_0$ , where  $c_0$  is the constant map that maps any knot to 1.
  2. Check that the linking number is a degree 1 invariant of 2-component links of  $\mathbb{R}^3$ .
  3. Check that  $\mathcal{I}_1(\mathcal{K}_2) = \mathbb{K}c_0 \oplus \mathbb{K}lk$ , where  $c_0$  is the constant map that maps any two-component link to 1.

## 6.2 Introduction to chord diagrams

A *singular knot with  $n$  double points* in  $\mathbb{R}^3$  is an immersion of a circle with  $n$  transverse double points in  $\mathbb{R}^3$ . Such a double point  $\times$  can be *desingularized* in two ways, the *positive* one  $\nearrow$  and the *negative* one  $\nwarrow$ . For example, desingularizing the double points of the singular knot  in the positive way produces the knot . Note that the sign of the desingularization is defined from the orientations of the knot and of the ambient space.

Let  $o_i$  be disjoint negative crossing changes  $\nwarrow \rightarrow \nearrow$  to be performed on a knot  $K$ . We represent  $[K; o_1, \dots, o_n]$  as a singular knot with  $n$  double points such that  $K((o_i)_{i \in I})$  is obtained from the singular knot by desingularizing the crossings of  $I$  in the negative way, and the others in the positive way, for  $I \subseteq \underline{n}$ . Thus, singular knots represent elements of  $\mathcal{F}_0(\mathcal{K})$ . Three singular knots that coincide outside a ball, inside which they look as in the following *skein relation*

$$\times = \nearrow - \nwarrow,$$

satisfy this relation in  $\mathcal{F}_0(\mathcal{K})$ .

Define the *chord diagram*  $\Gamma_C([K; o_1, \dots, o_n])$  associated to  $[K; o_1, \dots, o_n]$  as follows. Draw the preimage of the associated singular knot with  $n$  double points as an oriented dashed circle equipped with the  $2n$  preimages of the double points, and join the pairs of preimages of a double point by a plain segment called a *chord*. For example, we have

$$\Gamma_C\left(\text{circle with two double points}\right) = \text{chord diagram with two chords}.$$

Formally, a *chord diagram* with  $n$  chords (on a circle) is a cyclic order of the  $2n$  ends of the  $n$  chords, up to a permutation of the chords and up to exchanging the two ends of a chord.

**Lemma 6.6.** *If  $f$  is an invariant of knots in  $\mathbb{R}^3$  of degree at most  $n$ , then  $f([K; o_1, \dots, o_n])$  depends only on  $\Gamma_C([K; o_1, \dots, o_n])$ .*

PROOF: Since  $f$  is of degree  $n$ ,  $f([K; o_1, \dots, o_n])$  is invariant under a crossing change outside the balls of the  $o_i$ , that is outside the double points of the associated singular knot. Therefore,  $f([K; o_1, \dots, o_n])$  depends only on the cyclic order of the  $2n$  arcs involved in the  $o_i$  on  $K$ . (A more detailed proof can be found in [Les05, Subsection 7.3].)  $\square$

Let  $\mathcal{D}_n$  be the  $\mathbb{K}$ -vector space freely generated by the  $n$ -chord diagrams

(i.e., the diagrams with  $n$  chords) on  $S^1$ . For example, we have

$$\mathcal{D}_0 = \mathbb{K} \circlearrowleft, \mathcal{D}_1 = \mathbb{K} \begin{array}{c} \bullet \\ | \\ \bullet \end{array}, \mathcal{D}_2 = \mathbb{K} \begin{array}{c} \bullet \quad \bullet \\ \curvearrowright \\ \bullet \quad \bullet \end{array} \oplus \mathbb{K} \begin{array}{c} \bullet \quad \bullet \\ \times \\ \bullet \quad \bullet \end{array}, \text{ and} \\ \mathcal{D}_3 = \mathbb{K} \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \curvearrowright \\ \bullet \quad \bullet \quad \bullet \end{array} \oplus \mathbb{K} \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \curvearrowright \\ \bullet \quad \bullet \quad \bullet \end{array} \oplus \mathbb{K} \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \times \\ \bullet \quad \bullet \quad \bullet \end{array} \oplus \mathbb{K} \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \times \\ \bullet \quad \bullet \quad \bullet \end{array} \oplus \mathbb{K} \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \times \\ \bullet \quad \bullet \quad \bullet \end{array}.$$

**Lemma 6.7.** *The map  $\phi_n$  from  $\mathcal{D}_n$  to  $\mathcal{F}_n(\mathcal{K})/\mathcal{F}_{n+1}(\mathcal{K})$  that maps an  $n$ -chord diagram  $\Gamma$  to some  $[K; o_1, \dots, o_n]$  whose diagram is  $\Gamma$  is well-defined and surjective.*

PROOF: Use the arguments of the proof of Lemma 6.6. □

For example, we have

$$\phi_3 \left( \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \times \\ \bullet \quad \bullet \quad \bullet \end{array} \right) = \left[ \begin{array}{c} \bullet \quad \bullet \\ \curvearrowright \\ \bullet \quad \bullet \end{array} \right].$$

Lemma 6.7 implies that

$$\phi_n^* : \left( \frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})} \right)^* \rightarrow \mathcal{D}_n^*$$

is injective. The kernel of the restriction below

$$\left( \mathcal{I}_n(\mathcal{K}) = \left( \frac{\mathcal{F}_0(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})} \right)^* \right) \rightarrow \left( \frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})} \right)^*$$

is  $\mathcal{I}_{n-1}(\mathcal{K})$ . Thus,  $\mathcal{I}_n(\mathcal{K})/\mathcal{I}_{n-1}(\mathcal{K})$  injects into  $\mathcal{D}_n^*$ , and the dimension of  $\mathcal{I}_n(\mathcal{K})$  is finite for all  $n$ . In particular,  $\mathcal{F}_0(\mathcal{K})/\mathcal{F}_{n+1}(\mathcal{K})$  is finite-dimensional, and the above restriction is surjective. Therefore, we have

$$\frac{\mathcal{I}_n(\mathcal{K})}{\mathcal{I}_{n-1}(\mathcal{K})} = \text{Hom} \left( \frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}; \mathbb{K} \right).$$

An *isolated chord* in a chord diagram is a chord between two points of  $S^1$  that are consecutive on the circle  $S^1$ .

**Lemma 6.8.** *Let  $D$  be a diagram on  $S^1$  that contains an isolated chord. Then  $\phi_n(D) = 0$ . Let  $D^1, D^2, D^3,$  and  $D^4$  be four  $n$ -chord diagrams identical outside three portions of circles, inside which they look like*

$$D^1 = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array}, D^2 = \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}, D^3 = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagdown \\ \bullet \quad \bullet \end{array}, \text{ and } D^4 = \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagup \\ \bullet \quad \bullet \end{array}.$$

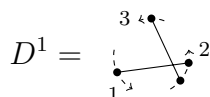
Then we have

$$\phi_n(-D^1 + D^2 + D^3 - D^4) = 0.$$

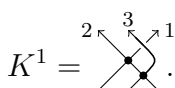
PROOF: For the first assertion, observe

$$\phi_n(\text{diagram}) = [\text{diagram}] - [\text{diagram}].$$

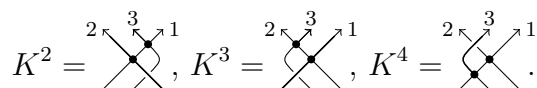
Let us prove the second one. We may represent



by a singular knot  $K^1$  with  $n$  double points, which intersects a ball as



Let  $K^2, K^3, K^4$  be the singular knots with  $n$  double points that coincide with  $K^1$  outside this ball, and that intersect this ball as shown in the pictures:



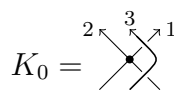
Then the chord diagram  $D(K^2)$  associated to  $K^2$  is  $D^2$ . Similarly, we have  $D(K^3) = D^3$  and  $D(K^4) = D^4$ . Therefore, we have  $\phi_n(-D^1 + D^2 + D^3 - D^4) = -[K^1] + [K^2] + [K^3] - [K^4]$ .

Thus, it is enough to prove that we have

$$-[K^1] + [K^2] + [K^3] - [K^4] = 0$$

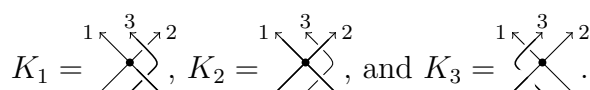
in  $\mathcal{F}_n(\mathcal{K})$ . Let us prove it.

Let  $K_0$  be the singular knot with  $(n - 1)$  double points that intersects our ball as



and that coincides with  $K^1$  outside this ball.

The strands 1 and 2 in the pictured double point are in the horizontal plane. They orient it. The strand 3 is vertical. It intersects the horizontal plane in a positive way between the tails of 1 and 2. Now, make 3 turn around the double point counterclockwise so that it successively becomes the knots with  $(n - 1)$  double points:



On its way, it goes successively through our four knots  $K^1$ ,  $K^2$ ,  $K^3$ , and  $K^4$  with  $n$  double points, which appear inside matching parentheses, in the following obvious identity in  $\mathcal{F}_{n-1}(\mathcal{K})$

$$([K_1] - [K_0]) + ([K_2] - [K_1]) + ([K_3] - [K_2]) + ([K_0] - [K_3]) = 0.$$

Now, we have  $[K^i] = \pm([K_i] - [K_{i-1}])$ , where the sign  $\pm$  is  $+$  when the vertical strand goes through an arrow from  $K_{i-1}$  to  $K_i$ , and minus when it goes through a tail. Therefore, the above equality can be written as

$$- [K^1] + [K^2] + [K^3] - [K^4] = 0.$$

□

Let  $\overline{\mathcal{A}}_n$  denote the quotient of  $\mathcal{D}_n$  by the *four-term relation* (4T). It is the quotient of  $\mathcal{D}_n$  by the vector space generated by the  $(-D^1 + D^2 + D^3 - D^4)$  for all the 4-tuples  $(D^1, D^2, D^3, D^4)$  as in Lemma 6.8. Call (1T) the relation that identifies a diagram with an isolated chord with 0. So  $\overline{\mathcal{A}}_n/(1T)$  is the quotient of  $\overline{\mathcal{A}}_n$  by the vector space generated by diagrams with an isolated chord.

According to Lemma 6.8 above, the map  $\phi_n$  induces a map

$$\overline{\phi}_n: \overline{\mathcal{A}}_n/(1T) \longrightarrow \frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}.$$

The fundamental theorem of *Vassiliev invariants* (which are finite type knot invariants) can now be stated.

**Theorem 6.9** (Bar-Natan, Kontsevich). *There exists a family*

$$\left( \overline{\mathcal{Z}}_n: \mathcal{F}_0(\mathcal{K}) \rightarrow \overline{\mathcal{A}}_n/(1T) \right)_{n \in \mathbb{N}}$$

of linear maps satisfying:

- $\overline{\mathcal{Z}}_n(\mathcal{F}_{n+1}(\mathcal{K})) = 0$ .
- Let  $\overline{\overline{\mathcal{Z}}}_n$  be the map induced by  $\overline{\mathcal{Z}}_n$  from  $\mathcal{F}_n(\mathcal{K})/\mathcal{F}_{n+1}(\mathcal{K})$  to  $\overline{\mathcal{A}}_n/(1T)$ . Then  $\overline{\overline{\mathcal{Z}}}_n \circ \overline{\phi}_n$  is the identity map of  $\overline{\mathcal{A}}_n/(1T)$ .

In particular, the inverse isomorphisms  $\overline{\overline{\mathcal{Z}}}_n$  and  $\overline{\phi}_n$  identify  $\mathcal{F}_n(\mathcal{K})/\mathcal{F}_{n+1}(\mathcal{K})$  with  $\overline{\mathcal{A}}_n/(1T)$ , and we have

$$\frac{\mathcal{I}_n(\mathcal{K})}{\mathcal{I}_{n-1}(\mathcal{K})} \cong (\overline{\mathcal{A}}_n/(1T))^*.$$

Maxim Kontsevich and Dror Bar-Natan proved this theorem by using the *Kontsevich integral*  $Z^K = (Z_n^K)_{n \in \mathbb{N}}$  [BN95a], for  $\mathbb{K} = \mathbb{R}$ . It is also true when  $\mathbb{K} = \mathbb{Q}$ . It is reproved in Section 17.6 using the invariant  $\mathcal{Z}$  studied in this book. An invariant  $\overline{\mathcal{Z}}$  as in the above statement has the following universality property.

**Corollary 6.10.** *For any real-valued finite type invariant  $f$  of knots in  $\mathbb{R}^3$  of degree at most  $n$ , there exist linear forms  $\psi_i: \overline{\mathcal{A}}_i/(1T) \rightarrow \mathbb{R}$ , for  $i = 0, \dots, n$ , such that*

$$f = \sum_{i=0}^n \psi_i \circ \overline{\mathcal{Z}}_i.$$

PROOF: Let  $\psi_n = f|_{\mathcal{F}_n} \circ \overline{\phi}_n$ , then  $(f - \psi_n \circ \overline{\mathcal{Z}}_n)$  is an invariant of degree at most  $n - 1$ . Conclude by induction.  $\square$

By projection (or up to (1T)), the invariant  $\overline{\mathcal{Z}}$  defines a universal Vassiliev knot invariant with respect to the following definition.

An invariant  $Y: \mathcal{F}_0(\mathcal{K}) \rightarrow \prod_{n \in \mathbb{N}} \overline{\mathcal{A}}_n/(1T)$  such that

- $Y_n(\mathcal{F}_{n+1}(\mathcal{K})) = 0$  and
- $Y_n$  induces a left inverse to  $\overline{\phi}_n$  from  $\mathcal{F}_n(\mathcal{K}; \mathbb{R})/\mathcal{F}_{n+1}(\mathcal{K}; \mathbb{R})$  to  $\overline{\mathcal{A}}_n/(1T)$

is called a *universal Vassiliev knot invariant*.

The terminology is justified because such an invariant contains all the real-valued Vassiliev knot invariants as in Corollary 6.10.

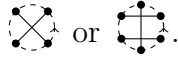
As Daniel Altschüler and Laurent Freidel proved in [AF97], the restriction of the invariant  $\mathcal{Z} = (\mathcal{Z}_n)_{n \in \mathbb{N}}$  to knots of  $\mathbb{R}^3$  also satisfies the properties of Theorem 6.9. So it is also a *universal Vassiliev knot invariant*. We will give alternative proofs of generalizations of this Altschüler–Freidel theorem in Section 17.6.

Similar characterizations of the spaces of finite type invariants of links in  $\mathbb{R}^3$ , integer homology 3-spheres, and rational homology 3-spheres will be presented in Section 17.6 and Chapter 18, respectively. For integer homology 3-spheres and rational homology 3-spheres, the most difficult parts of the proofs are consequences of the splitting formulae satisfied by  $\mathcal{Z}$ , stated in Theorem 18.6.

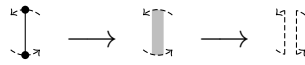
We end this subsection with an example of a nontrivial linear form on  $\overline{\mathcal{A}}_n/(1T)$ .

**Example 6.11.** We first define a function  $\check{w}_C$  of chord diagrams. Let  $\Gamma$  be a chord diagram. Immerse  $\Gamma$  in the unit disk  $D_1$  of the plane so that the

chords of  $\Gamma$  are embedded and attached to the left-hand side of the boundary  $S^1$  of  $D_1$ , as in our former pictures like



Attach disjoint oriented bands  $[0, 1]^2$ , one “around each chord”, so that  $(\partial [0, 1]) \times [0, 1]$  is a well-oriented neighborhood of the ends of the chord in the dashed circle  $S^1$ , as in the following figure. Perform the *surgery* on the dashed circle  $S^1$  that replaces  $(\partial [0, 1]) \times [0, 1]$  with  $[0, 1] \times \partial [0, 1]$  as in the following figure.



If the resulting naturally oriented one-manifold is connected, then  $\check{w}_C(\Gamma) = 1$ . Otherwise,  $\check{w}_C(\Gamma) = 0$ . For example, we have

$$\check{w}_C(\text{dashed circle}) = 1, \check{w}_C(\text{circle with vertical chord}) = 0, \text{ and } \check{w}_C(\text{circle with two crossing chords}) = 0.$$

Since  $\text{circle with two crossing chords}$  is connected, we have  $\check{w}_C(\text{circle with two crossing chords}) = 1$ . We also have

$$\check{w}_C(\text{circle with two parallel chords}) = \check{w}_C(\text{circle with two crossing chords}) = 0.$$

More generally, the reader can check that one of our surgeries changes the mod 2 congruence class of the number of connected components of the surgered manifold. So  $\check{w}_C(\Gamma) = 0$  for any chord diagram  $\Gamma$  with an odd number of chords. Extend  $\check{w}_C$  linearly over  $\mathcal{D}_n$ . For 4-tuples  $(D^1, D^2, D^3, D^4)$  as in Lemma 6.8, the reader can check that the extended  $\check{w}_C$  maps  $(-D^1 + D^2 + D^3 - D^4)$  to zero. It also maps diagrams with isolated chords to zero. Therefore,  $\check{w}_C$  induces a linear map  $w_C$  on  $\overline{\mathcal{A}}_n$  (and on  $\overline{\mathcal{A}}_n/(1T)$ ) for any  $n$ . For any chord diagram  $\Gamma$ , we have  $w_C([\Gamma]) = \check{w}_C(\Gamma)$ . Dror Bar-Natan and Stavros Garoufalidis introduced this linear map in [BNG96]. They called it the *Conway weight system*. It is zero when  $n$  is odd. It is not zero when  $n$  is even since we have

$$w_C([\text{circle with } n \text{ chords}]) \neq 0.$$

### 6.3 More spaces of diagrams

**Definition 6.12.** A *uni-trivalent graph*  $\Gamma$  is a 6-tuple

$$(H(\Gamma), E(\Gamma), U(\Gamma), T(\Gamma), p_E, p_V),$$

where

- $H(\Gamma)$ ,  $E(\Gamma)$ ,  $U(\Gamma)$ , and  $T(\Gamma)$  are finite sets, respectively called the *set of half-edges* of  $\Gamma$ , the *set of edges* of  $\Gamma$ , the *set of univalent vertices* of  $\Gamma$ , and the set of trivalent vertices of  $\Gamma$ ,
- $p_E: H(\Gamma) \rightarrow E(\Gamma)$  is a two-to-one map (every element of  $E(\Gamma)$  has two preimages under  $p_E$ ) and
- $p_V: H(\Gamma) \rightarrow U(\Gamma) \sqcup T(\Gamma)$  is a map such that every element of  $U(\Gamma)$  has one preimage under  $p_V$  and every element of  $T(\Gamma)$  has three preimages under  $p_V$ ,

up to isomorphism. In other words,  $\Gamma$  is a set  $H(\Gamma)$  equipped with two partitions, a partition into pairs (induced by  $p_E$ ), and a partition into singletons and triples (induced by  $p_V$ ), up to the bijections that preserve the partitions. These bijections are the *automorphisms* of  $\Gamma$ .

**Definition 6.13.** Let  $\mathcal{L}$  be a one-manifold, oriented or not. A *Jacobi diagram*  $\Gamma$  with support  $\mathcal{L}$ , also called *Jacobi diagram on  $\mathcal{L}$* , is a finite uni-trivalent graph  $\Gamma$  equipped with an isotopy class  $[i_\Gamma]$  of injections  $i_\Gamma$  from the set  $U(\Gamma)$  of univalent vertices of  $\Gamma$  into the interior of  $\mathcal{L}$ . For such a  $\Gamma$ , a  $\Gamma$ -compatible injection is an injection in the class  $[i_\Gamma]$ . An *orientation* of a trivalent vertex of  $\Gamma$  is a cyclic order on the set of the three half-edges that meet at this vertex. An *orientation* of a univalent vertex  $u$  of  $\Gamma$  is an orientation of the connected component  $\mathcal{L}(u)$  of  $i_\Gamma(u)$  in  $\mathcal{L}$ , for a choice of  $\Gamma$ -compatible  $i_\Gamma$ . This orientation associated to  $u$  is also called (and thought of as) a *local orientation of  $\mathcal{L}$  at  $u$* .<sup>3</sup> When  $\mathcal{L}$  is oriented, the orientation of  $\mathcal{L}$  orients the univalent vertices of  $\Gamma$  naturally.

A *vertex-orientation* of a Jacobi diagram  $\Gamma$  is an *orientation* of every vertex of  $\Gamma$ . A Jacobi diagram is *oriented* if it is equipped with a vertex-orientation.<sup>4</sup>

Unless otherwise mentioned, the supports of Jacobi diagrams are oriented, and we use the induced orientations of univalent vertices without mentioning them. Nevertheless, the above notion of local orientations will prove useful to state some properties of the invariant  $Z$  studied in this book, such as the behavior under cablings in Theorem 13.12.

We represent an oriented Jacobi diagram  $\Gamma$  by a planar immersion of  $\Gamma \cup \mathcal{L} = \Gamma \cup_{U(\Gamma)} \mathcal{L}$ , with the following conventions. We represent the (oriented)

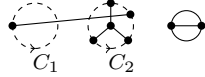
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<sup>3</sup>A *local orientation* of  $\mathcal{L}$  at  $u$  is simply an orientation of  $\mathcal{L}(u)$ . However, since different vertices are allowed to induce different orientations, we think of these orientations as being *local*, i.e., defined in a neighborhood of  $i_\Gamma(u)$  for a choice of  $\Gamma$ -compatible  $i_\Gamma$ .

<sup>4</sup>When  $\mathcal{L}$  is oriented, it suffices to specify the orientations of the trivalent vertices since  $\mathcal{L}$  orients the univalent vertices.



one-manifold  $\mathcal{L}$  by dashed lines, the edges of the diagram  $\Gamma$  by plain segments, and the vertices by big dots. The univalent vertices of  $U(\Gamma)$  are located at their images under a  $\Gamma$ -compatible injection  $i_\Gamma$ . We represent the orientation of a trivalent vertex by the counterclockwise order of the three half-edges that meet at the vertex. Here is an example of a picture of a Jacobi diagram  $\Gamma$  on the disjoint union  $\mathcal{L} = S^1 \sqcup S^1$  of two (oriented) circles:



The *degree* of such a diagram is half the number of all its vertices. Note that a chord diagram of  $\mathcal{D}_n$  is a degree  $n$  Jacobi diagram on  $S^1$  without trivalent vertices. For an (oriented) one-manifold  $\mathcal{L}$ ,  $\mathcal{D}_n(\mathcal{L})$  denotes the  $\mathbb{K}$ -vector space freely generated by the degree  $n$  oriented Jacobi diagrams on  $\mathcal{L}$ . For the (oriented) circle  $S^1$ , we have

$$\mathcal{D}_1(S^1) = \mathbb{K} \left( \text{circle with 1 vertex} \right) \oplus \mathbb{K} \left( \text{circle with 2 vertices} \right) \oplus \mathbb{K} \left( \text{circle with 3 vertices} \right) \oplus \mathbb{K} \left( \text{circle with 4 vertices} \right) \oplus \mathbb{K} \left( \text{circle with 5 vertices} \right).$$

For an (oriented) one-manifold  $\mathcal{L}$ ,  $\mathcal{A}_n(\mathcal{L})$  denotes the quotient of  $\mathcal{D}_n(\mathcal{L})$  by the following relations AS, Jacobi, and STU:

$$\begin{aligned} \text{AS (or antisymmetry): } & \quad \text{Y-shape} + \text{Y-shape} = 0 \\ \text{Jacobi: } & \quad \text{Y-shape} + \text{Y-shape} + \text{Y-shape} = 0 \\ \text{STU: } & \quad \text{Y-shape} = \text{Y-shape} - \text{Y-shape} + \text{Y-shape} \end{aligned}$$

Each of these relations relates oriented Jacobi diagrams which are identical outside the pictures. The quotient  $\mathcal{A}_n(\mathcal{L})$  is the largest quotient of  $\mathcal{D}_n(\mathcal{L})$  in which these relations hold. It is obtained by quotienting  $\mathcal{D}_n(\mathcal{L})$  by the  $\mathbb{K}$ -vector space generated by elements of  $\mathcal{D}_n(\mathcal{L})$  of the form

$$\left( \text{Y-shape} + \text{Y-shape} \right), \left( \text{Y-shape} + \text{Y-shape} + \text{Y-shape} \right), \text{ or } \left( \text{Y-shape} - \text{Y-shape} + \text{Y-shape} \right).$$

**Example 6.14.** We have

$$\mathcal{A}_1(S^1) = \mathbb{K} \left( \text{circle with 1 vertex} \right) \oplus \mathbb{K} \left( \text{circle with 3 vertices} \right).$$

**Note 6.15.** When  $\partial\mathcal{L} = \emptyset$ , any finite-dimensional Lie algebra equipped with a finite-dimensional representation and a nondegenerate bilinear symmetric invariant form provides a nontrivial linear map from  $\mathcal{A}_n(\mathcal{L})$  to  $\mathbb{K}$ . Such a map is called a *weight system*. See [BN95a], [CDM12, Chapter 6] or [Les05,

Section 6], for example. In the weight system constructions, the Jacobi relation for the Lie bracket ensures that the maps defined for oriented Jacobi diagrams factor through the Jacobi relation. In [Vog11], Pierre Vogel proved that the maps associated to Lie (super)algebras are sufficient to detect any nontrivial element of  $\mathcal{A}_n(\mathcal{L})$  until degree 15. He exhibited a nontrivial element of  $\mathcal{A}_{16}(\emptyset)$  that such maps cannot detect. Dror Bar-Natan originally called the Jacobi relation IHX in [BN95a] because we can write it as

$$\text{IHX} = \text{IHX}_1 - \text{IHX}_2,$$

up to AS. Note that the four entries in this IHX relation play the same role up to AS.

**Definition 6.16.** In figures, the orientation of a univalent vertex  $u$  of a Jacobi diagram on a nonoriented one-manifold  $\mathcal{L}$  is again represented by the counterclockwise cyclic order of the three half-edges that meet at  $u$  in a planar immersion of  $\Gamma \cup_{U(\Gamma)} \mathcal{L}$ , with respect to the following convention. The half-edge of  $u$  in  $\Gamma$  is attached to the left-hand side of  $\mathcal{L}$ , with respect to the local orientation of  $\mathcal{L}$  at  $u$ , as in the following picture:

$$\text{Diagram with upward arrow} \leftrightarrow \text{Diagram with downward arrow}$$

In other words, to represent the upward local orientation of  $\mathfrak{u}_C^i$  at  $u$ , we attach the half-edge of  $u$  in  $\Gamma$  as in the above figure, and to represent its downward orientation, we attach the half-edge of  $u$  as follows:

For a nonoriented one-manifold  $\mathcal{L}$ , the space  $\mathcal{D}_n(\mathcal{L})$  is the  $\mathbb{K}$ -vector space generated by the degree  $n$  oriented Jacobi diagrams on  $\mathcal{L}$ —where there are additional orientation choices for univalent vertices, and  $\mathcal{A}_n(\mathcal{L})$  is the quotient of  $\mathcal{D}_n(\mathcal{L})$  by the previous relations AS, Jacobi, and STU together with the additional antisymmetry relation

$$\text{Diagram 1} + \text{Diagram 2} = 0,$$

where the (unoriented) STU relation may be written as

$$\text{STU: Diagram 1} = \text{Diagram 2} - \text{Diagram 3}.$$

**Remark 6.17.** We can draw the unoriented STU relation above like the Jacobi relation up to AS:

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} = 0.$$

**Lemma 6.18.** *Let  $\mathcal{L}$  be an oriented one-manifold. Let  $\mathcal{L}^f$  denote the nonoriented manifold obtained from  $\mathcal{L}$  by forgetting the orientation. Let  $\Gamma^f$  be an oriented Jacobi diagram on  $\mathcal{L}^f$ . A univalent vertex  $u$  of  $\Gamma^f$  is  $\mathcal{L}$ -oriented if it induces the orientation of  $\mathcal{L}$ . Otherwise, it is  $(-\mathcal{L})$ -oriented. Let  $\Gamma(\Gamma^f, \mathcal{L})$  be the oriented Jacobi diagram on  $\mathcal{L}$  obtained from  $\Gamma^f$  by reversing the local orientation of the  $(-\mathcal{L})$ -oriented univalent vertices. Let  $k(\Gamma^f, \mathcal{L})$  be the number of  $(-\mathcal{L})$ -oriented univalent vertices of  $\Gamma^f$ . The linear map from  $\mathcal{D}_n(\mathcal{L}^f)$  into  $\mathcal{D}_n(\mathcal{L})$  that maps any oriented Jacobi diagram  $\Gamma^f$  to  $(-1)^{k(\Gamma^f, \mathcal{L})}\Gamma(\Gamma^f, \mathcal{L})$  and the linear canonical injection from  $\mathcal{D}_n(\mathcal{L})$  into  $\mathcal{D}_n(\mathcal{L}^f)$  induce canonical isomorphisms between  $\mathcal{A}_n(\mathcal{L})$  and  $\mathcal{A}_n(\mathcal{L}^f)$ , which are inverse to each other, for any integer  $n \in \mathbb{N}$ .*

PROOF: Exercise. □

The above lemma justifies using the same notation  $\mathcal{A}_n(\cdot)$  for oriented and unoriented supports. We draw Jacobi diagrams on oriented supports by attaching the half-edges of univalent vertices to the left-hand side of the support, to avoid confusion and get rid of the support orientation more easily.

**Notation 6.19.** When  $\mathcal{L} \neq \emptyset$ , let  $\check{\mathcal{A}}_n(\mathcal{L}) = \check{\mathcal{A}}_n(\mathcal{L}; \mathbb{K})$  denote the quotient of  $\mathcal{A}_n(\mathcal{L}) = \mathcal{A}_n(\mathcal{L}; \mathbb{K})$  by the vector space generated by the diagrams that have at least one connected component without univalent vertices.

So  $\check{\mathcal{A}}_n(\mathcal{L})$  is generated by the degree  $n$  oriented Jacobi diagrams whose (plain) connected components contain at least one univalent vertex.

**Lemma 6.20.** *The space  $\check{\mathcal{A}}_n(S^1)$  is the quotient by the relations AS and STU of the vector space generated by the degree  $n$  oriented Jacobi diagrams whose connected components contain at least one univalent vertex. In other words, the Jacobi relation is a consequence of the relations AS and STU in this vector space.*

PROOF: We want to prove that the Jacobi relation holds in the quotient by AS and STU of the space of uni-trivalent diagrams on  $S^1$  with at least one univalent vertex in each connected component. Consider three diagrams represented by three immersions which coincide outside a disk  $D$ , inside which they are as in the pictures involved in the Jacobi relation. Use STU as much as possible to remove all trivalent vertices that can be removed without changing the two vertices in  $D$ , on the three diagrams simultaneously. This transforms the Jacobi relation to be proved to a sum of similar relations, where one of the four entries of the disk is directly connected to  $S^1$ . Since the four entries play the same role in the Jacobi relation, we may assume that the Jacobi relation to be proved is

Using STU twice and AS transforms the summands of the left-hand side to diagrams that can be represented by three straight lines from the entries 1, 2, 3 to three fixed points of the horizontal line numbered from left to right. When the entry  $i \in \{1, 2, 3\}$  is connected to the point  $\sigma(i)$  of the horizontal dashed line, where  $\sigma$  is a permutation of  $\{1, 2, 3\}$ , the corresponding diagram is denoted by  $(\sigma(1)\sigma(2)\sigma(3))$ . Thus, the expansion of the left-hand side of the above equation is

$$\begin{aligned} & ((123) - (132) - (231) + (321)) \\ & - ((213) - (231) - (132) + (312)) , \\ & - ((123) - (213) - (312) + (321)) \end{aligned}$$

which vanishes, and the lemma is proved. □

**Proposition 6.21.** *The natural map from  $\mathcal{D}_n$  to  $\check{\mathcal{A}}_n(S^1)$  induces an isomorphism from the space  $\overline{\mathcal{A}}_n$  of chord diagrams to  $\check{\mathcal{A}}_n(S^1)$ .*

FIRST PART OF THE PROOF: According to STU, we have

in  $\check{\mathcal{A}}_n(S^1)$ . So the natural map from  $\mathcal{D}_n$  to  $\check{\mathcal{A}}_n(S^1)$  factors through  $4T$ . Since STU allows us to inductively write any oriented Jacobi diagram whose connected components contain at least one univalent vertex as a combination of chord diagrams, the induced map from  $\overline{\mathcal{A}}_n$  to  $\check{\mathcal{A}}_n(S^1)$  is surjective. We will prove injectivity in Section 6.4 by constructing an inverse map. □

## 6.4 Multiplying diagrams

Set  $\mathcal{A}(\mathcal{L}) = \prod_{n \in \mathbb{N}} \mathcal{A}_n(\mathcal{L})$ ,  $\check{\mathcal{A}}(\mathcal{L}) = \prod_{n \in \mathbb{N}} \check{\mathcal{A}}_n(\mathcal{L})$ , and  $\overline{\mathcal{A}} = \prod_{n \in \mathbb{N}} \overline{\mathcal{A}}_n$ .

Assume that a one-manifold  $\mathcal{L}$  is decomposed as a union of two one-manifolds  $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$  whose interiors in  $\mathcal{L}$  do not intersect. Let  $\Gamma_1$  be a Jacobi diagram with support  $\mathcal{L}_1$  and let  $\Gamma_2$  be a Jacobi diagram with support  $\mathcal{L}_2$ . Define the Jacobi diagram  $\Gamma_1 \sqcup \Gamma_2$  on  $\mathcal{L}$  to be the disjoint union of  $\Gamma_1$  and  $\Gamma_2$ . (Formally, a  $\Gamma_1 \sqcup \Gamma_2$ -compatible injection restricts to  $U(\Gamma_i)$  as a

$\Gamma_i$ -compatible injection, for  $i \in \{1, 2\}$ .) Define the *product associated to the decomposition*  $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ :

$$\mathcal{A}(\mathcal{L}_1) \times \mathcal{A}(\mathcal{L}_2) \longrightarrow \mathcal{A}(\mathcal{L})$$

to be the continuous bilinear map that maps  $([\Gamma_1], [\Gamma_2])$  to  $[\Gamma_1 \sqcup \Gamma_2]$ , for two diagrams  $\Gamma_1$  and  $\Gamma_2$  as above. In particular, the disjoint union of diagrams turns  $\mathcal{A}(\emptyset)$  into a commutative algebra graded by the degree, and it turns  $\mathcal{A}(\mathcal{L})$  into an  $\mathcal{A}(\emptyset)$ -module, for any 1-dimensional manifold  $\mathcal{L}$ .

An orientation-preserving diffeomorphism from a manifold  $\mathcal{L}$  to another one  $\mathcal{L}'$  induces a natural isomorphism from  $\mathcal{A}_n(\mathcal{L})$  to  $\mathcal{A}_n(\mathcal{L}')$  (and from  $\check{\mathcal{A}}_n(\mathcal{L})$  to  $\check{\mathcal{A}}_n(\mathcal{L}')$ ), for all  $n$ . Let  $I = [0, 1]$  be the compact oriented interval. If  $I = \mathcal{L}$ , and if we identify  $I$  with  $\mathcal{L}_1 = [0, 1/2]$  and with  $\mathcal{L}_2 = [1/2, 1]$  with respect to the orientation, then the above process turns  $\mathcal{A}(I)$  and  $\check{\mathcal{A}}(I)$  into algebras. The elements of  $\mathcal{A}(I)$  with nonzero degree zero part admit an inverse.

**Proposition 6.22.** *The algebra  $\mathcal{A}([0, 1])$  is commutative. The projection from  $[0, 1]$  to  $S^1 = [0, 1]/(0 \sim 1)$  induces an isomorphism from  $\mathcal{A}_n([0, 1])$  to  $\mathcal{A}_n(S^1)$  for all  $n$ . So  $\mathcal{A}(S^1)$  inherits a commutative algebra structure from this isomorphism. The choice of an oriented connected component  $\mathcal{L}_j$  of  $\mathcal{L}$  equips  $\mathcal{A}(\mathcal{L})$  with an  $\mathcal{A}([0, 1])$ -module structure  $\#_j$ , induced by the orientation-preserving inclusion from  $[0, 1]$  to a small part of  $\mathcal{L}_j$  outside the vertices, and the insertion of diagrams with support  $[0, 1]$  there.*

In order to prove this proposition, we present a useful trick in diagram spaces.

**Lemma 6.23.** *Let  $\mathcal{L}$  be a nonoriented one-manifold. Let  $\Gamma_1$  be an oriented Jacobi diagram (resp. a chord diagram) with support  $\mathcal{L}$  as in Definitions 6.13 and 6.16. Assume that  $\Gamma_1 \cup \mathcal{L}$  is immersed in the plane so that  $\Gamma_1 \cup \mathcal{L}$  meets an open annulus  $A$  embedded in the plane exactly along  $n + 1$  embedded arcs  $\alpha_1, \alpha_2, \dots, \alpha_n$ , and  $\beta$ , and one vertex  $v$  on  $\alpha_1$ , as in the examples below, so that*

1. *the  $\alpha_i$  are disjoint, they may be dashed or plain (they are dashed in the case of chord diagrams), they go from a boundary component of  $A$  to the other,*
2.  *$\beta$  is a plain arc going from the boundary of  $A$  to  $v$  on  $\alpha_1$ ,*
3. *the bounded component  $D$  of the complement of  $A$  does not contain a boundary point of  $\mathcal{L}$ ,*

4. the vertex-orientations are induced by the planar immersion by the local counterclockwise orders in the neighborhoods of vertices (as usual).

Let  $\Gamma_i$  be the diagram obtained from  $\Gamma_1$  by attaching the endpoint  $v$  of  $\beta$  to  $\alpha_i$  instead of  $\alpha_1$  on the same side, where the side of an arc is its side when going from the outside boundary component of  $A$  to the inside one  $\partial D$ , as in the examples below. Then we have

$$\sum_{i=1}^n \Gamma_i = 0$$

in  $\mathcal{A}(\mathcal{L})$  (resp. in the space  $\overline{\mathcal{A}}$  of Section 6.2).

**Examples 6.24.** We have

$$\Gamma_1 + \Gamma_2 = 0$$

and

$$\Gamma_1 + \Gamma_2 + \Gamma_3 = 0.$$

**Remarks 6.25.** The second example shows that the STU relation is equivalent to the relation of the statement when the bounded component  $D$  of  $\mathbb{R}^2 \setminus A$  intersects  $\Gamma_1$  in the neighborhood of a univalent vertex on  $\mathcal{L}$ . Similarly, the Jacobi relation is given by the statement's relation when  $D$  intersects  $\Gamma_1$  in the neighborhood of a trivalent vertex. Also note that AS corresponds to the case in which  $D$  intersects  $\Gamma_1$  along a dashed or plain arc.

**PROOF OF LEMMA 6.23:** Let us give the Bar-Natan [BN95a, Lemma 3.1] proof. See also [Vog11, Lemma 3.3]. Without loss of generality, assume that  $v$  is always attached on the left-hand side of the  $\alpha$ 's.

We first treat the case of trivalent diagrams. Add to the sum the contribution of the sum of the diagrams obtained from  $\Gamma_1$  by attaching  $v$  to each of the three (dashed or plain) half-edges of each vertex  $w$  of  $\Gamma_1 \cup \mathcal{L}$  in  $D$  on the right-hand side when the half-edges are oriented towards  $w$  (i.e., by attaching  $v$  to the hooks in  $\searrow_{\beta}^w$ ). The Jacobi and STU relations ensure that this contribution is trivial. Now, group the terms of the obtained sum

according to the edges of  $\Gamma_1 \cup \mathcal{L}$ , where  $v$  is attached. Observe that the sum is zero, edge by edge, by AS.

For chord diagrams, similarly add to the sum the contribution of the sum of the diagrams obtained from  $\Gamma_1$  by attaching  $v$  to each of the four (dashed) half-edges adjacent to each chord  $W$  of  $\Gamma_1 \cup \mathcal{L}$  in  $D$ , on the right-hand side when the half-edges are oriented towards  $W$  (i.e., by attaching  $v$  to the hooks in  $\begin{array}{c} \cdot \\ \vdots \\ \text{---} \\ \vdots \\ \cdot \end{array} \begin{array}{c} \cdot \\ \vdots \\ \text{---} \\ \vdots \\ \cdot \end{array}$ ). Thanks to the 4T relation, this contribution is trivial. Again, group the terms of the obtained sum according to the dashed edges of  $\Gamma_1 \cup \mathcal{L}$  where  $v$  is attached. Again observe that the sum is zero, edge by edge, by AS.  $\square$

END OF PROOF OF PROPOSITION 6.21:

As promised, we construct a map  $f$  from  $\check{\mathcal{A}}_n(S^1)$  to the space  $\overline{\mathcal{A}}_n$  of chord diagrams up to 4T and AS, and we prove that it is an inverse of the natural surjective map  $g$  from  $\overline{\mathcal{A}}_n$  to  $\check{\mathcal{A}}_n(S^1)$ . Let  $\mathcal{D}_{n,k}$  denote the vector space generated by the oriented uni-trivalent degree  $n$  diagrams on  $S^1$  that have at most  $k$  trivalent vertices and at least one univalent vertex per connected component. Then  $\mathcal{D}_{n,2n}$  is the vector space generated by the oriented uni-trivalent degree  $n$  diagrams on  $S^1$  with least one univalent vertex per connected component.

We will define linear maps  $\lambda_k$  from  $\mathcal{D}_{n,k}$  to  $\overline{\mathcal{A}}_n$  by induction on  $k$  so that

1.  $\lambda_0$  maps a chord diagram to its class in  $\overline{\mathcal{A}}_n$ ,
2. the restriction of  $\lambda_k$  to  $\mathcal{D}_{n,k-1}$  is  $\lambda_{k-1}$ , and
3.  $\lambda_k$  maps all the relations AS and STU that involve only elements of  $\mathcal{D}_{n,k}$  to zero.

When we have succeeded in such a task, the linear map that maps a diagram  $d$  of  $\mathcal{D}_{n,2n}$  with  $k$  trivalent vertices to  $\lambda_k(d)$  will factor through STU and AS. Then the induced map  $\bar{\lambda}$  will provide the desired inverse map and allow us to conclude the proof. Let us define our maps  $\lambda_k$  with the announced properties.

Let  $k \geq 1$ , assume that  $\lambda_{k-1}$  is defined on  $\mathcal{D}_{n,k-1}$  and that  $\lambda_{k-1}$  maps all the relations AS and STU that involve only elements of  $\mathcal{D}_{n,k-1}$  to zero. We want to extend  $\lambda_{k-1}$  on  $\mathcal{D}_{n,k}$  to a linear map  $\lambda_k$  that maps all the relations AS and STU that involve only elements of  $\mathcal{D}_{n,k}$  to zero.


Let  $d$  be a diagram with  $k$  trivalent vertices, and let  $e$  be an edge of  $d$  that contains one univalent vertex and one trivalent vertex. Set


$$\lambda \left( (d, e) = \begin{array}{c} \cdot \\ \vdots \\ \text{---} \\ \vdots \\ \cdot \end{array} \begin{array}{c} \cdot \\ \vdots \\ \text{---} \\ \vdots \\ \cdot \end{array} \right) = \lambda_{k-1} \left( \begin{array}{c} \cdot \\ \vdots \\ \text{---} \\ \vdots \\ \cdot \end{array} \begin{array}{c} \cdot \\ \vdots \\ \text{---} \\ \vdots \\ \cdot \end{array} - \begin{array}{c} \cdot \\ \vdots \\ \text{---} \\ \vdots \\ \cdot \end{array} \begin{array}{c} \cdot \\ \vdots \\ \text{---} \\ \vdots \\ \cdot \end{array} \right).$$

It suffices to prove that  $\lambda(d, e)$  is independent of our chosen edge  $e$  to conclude the proof by defining the linear map  $\lambda_k$ , which will obviously satisfy the desired properties, by


$$\lambda_k(d) = \lambda(d, e).$$

Assume that two different edges  $e$  and  $f$  of  $d$  connect a trivalent vertex to a univalent vertex. We prove that  $\lambda(d, e) = \lambda(d, f)$ . If  $e$  and  $f$  are disjoint, then the fact that  $\lambda_{k-1}$  satisfies STU allows us to express both  $\lambda(d, e)$  and  $\lambda(d, f)$  as the same combination of four diagrams with  $(k - 2)$  vertices and conclude. Thus, we assume that  $e$  and  $f$  are two different edges that share a trivalent vertex  $t$ . If there exists another trivalent vertex that is connected to  $S^1$  by an edge  $g$ , then  $\lambda(d, e) = \lambda(d, g) = \lambda(d, f)$ . Thus, we furthermore assume that  $t$  is the unique trivalent vertex connected to  $S^1$  by an edge. If  $t$  is the unique trivalent vertex, then its component is necessarily

like  and the fact that  $\lambda(d, e) = \lambda(d, f)$  is a consequence of (4T).

Otherwise, the component of  $t$  is of the form  where the dotted circle represents a dashed diagram with only one pictured entry. So we have

$$\lambda(d, e) = \lambda_{k-1} \left( \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array} \right).$$

Then  $\lambda(d, e)$  is zero because the expansion of  as a sum of chord diagrams commutes with any vertex in  $\overline{\mathcal{A}}_n$ , according to Lemma 6.23. We similarly have  $\lambda(d, f) = 0$ . So we get  $\lambda(d, e) = \lambda(d, f)$  in this last case.  $\square$

**Lemma 6.26.** *For any one-manifold  $\mathcal{L}$ , the class of a Jacobi diagram with one univalent vertex vanishes in  $\mathcal{A}_n(\mathcal{L})$ .*

PROOF: Exercise. (Use Lemma 6.23.)  $\square$

PROOF OF PROPOSITION 6.22: Let  $\Gamma'$  be a diagram with support  $\mathcal{L}$  and let  $\Gamma$  is a diagram with support  $I$ . Define  $[\Gamma] \#_j [\Gamma']$  to be the class in  $\mathcal{A}(\mathcal{L})$  of the diagram obtained by inserting  $\Gamma$  along  $\mathcal{L}_j$  outside the vertices of  $\Gamma$ , according to the given orientation. For example, we have

$$\left[ \begin{array}{c} \uparrow \\ \bullet \\ \bullet \\ \bullet \end{array} \right] \#_j \left[ \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array} \right] = \left[ \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array} \right] = \left[ \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array} \right].$$



As shown in the first example that illustrates Lemma 6.23, the independence of the choice of the insertion locus is a consequence of Lemma 6.23, where  $\Gamma_1$  is the disjoint union  $\Gamma \sqcup \Gamma'$ , and  $\Gamma_1$  intersects  $D$  along  $\Gamma \cup I$ . The continuous bilinear map

$$\begin{aligned} \mathcal{A}(I) \times \mathcal{A}(\mathcal{L}) &\longrightarrow \mathcal{A}(\mathcal{L}) \\ ([\Gamma], [\Gamma']) &\mapsto [\Gamma] \#_j [\Gamma'] \end{aligned}$$

endows  $\mathcal{A}(\mathcal{L})$  with the  $\mathcal{A}(I)$ -module structure  $\#_j$ .

Lemma 6.23 allows us to similarly prove that  $\mathcal{A}(I)$  is a commutative algebra. Now, it suffices to prove that the morphism from  $\mathcal{A}(I)$  to  $\mathcal{A}(S^1)$  induced by the identification of the two endpoints of  $I$  is an isomorphism. This is proved in the more general proposition below.  $\square$

**Proposition 6.27.** *Let  $n \in \mathbb{N}$ . Let  $\mathcal{L}$  be a disjoint union of circles. The projection from  $[0, 1]$  to  $S^1 = [0, 1]/(0 \sim 1)$  induces an isomorphism from  $\mathcal{A}_n([0, 1] \sqcup \mathcal{L})$  to  $\mathcal{A}_n(S^1 \sqcup \mathcal{L})$ .*

PROOF: The morphism from  $\mathcal{A}([0, 1] \sqcup \mathcal{L})$  to  $\mathcal{A}(S^1 \sqcup \mathcal{L})$  induced by the identification of the two endpoints of  $[0, 1]$  amounts to mod out  $\mathcal{A}([0, 1] \sqcup \mathcal{L})$  by the relation that identifies two diagrams obtained from one another by moving the nearest univalent vertex to an endpoint of  $[0, 1]$  near the other endpoint. Applying Lemma 6.23 (with  $\beta$  coming from the inside boundary of the annulus) shows that this relation is a consequence of the relations in  $\mathcal{A}([0, 1] \sqcup \mathcal{L})$ . So this morphism is an isomorphism from  $\mathcal{A}([0, 1] \sqcup \mathcal{L})$  to  $\mathcal{A}(S^1 \sqcup \mathcal{L})$ .  $\square$

As the following exercise shows, Proposition 6.27 would not be satisfied if  $\mathcal{L}$  were replaced by an interval.

**Exercise 6.28.** 1. Prove

$$\left[ \begin{array}{c} \uparrow \\ \bullet \\ \text{---} \\ \bullet \\ \downarrow \end{array} \right] \neq \left[ \begin{array}{c} \uparrow \\ \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \\ \downarrow \end{array} \right].$$

in  $\mathcal{A}_2(\downarrow \uparrow)$ .

2. Prove that  $\mathcal{A}_2(\downarrow \uparrow) \not\cong \mathcal{A}_2(\circlearrowleft \uparrow)$ .

**Lemma 6.29.** *Let  $\pi : \mathcal{L}' \rightarrow \mathcal{L}$  be a smooth map between two unoriented compact one-manifolds  $\mathcal{L}$  and  $\mathcal{L}'$ . Assume  $\pi(\partial \mathcal{L}') \subset \partial \mathcal{L}$ . Let  $\Gamma$  be an oriented Jacobi diagram on  $\mathcal{L}$  (as in Definitions 6.13 and 6.16) equipped with a  $\Gamma$ -compatible injection whose image avoids the critical values of  $\pi$ . Define  $\pi^*(\Gamma)$  to be the class in  $\mathcal{A}(\mathcal{L}')$  of the sum of all diagrams on  $\mathcal{L}'$  obtained from  $\Gamma$  by lifting each univalent vertex to one of its preimages under  $\pi$ . (These diagrams*

have the same vertices and edges as  $\Gamma$ , and the local orientations at univalent vertices are naturally induced by the local orientations of the corresponding univalent vertices of  $\Gamma$ .) Then  $\pi^*(\Gamma)$  depends only on the class of  $\Gamma$  in  $\mathcal{A}(\mathcal{L})$  and on the homotopy class of  $\pi$ . So one can unambiguously define the linear degree-preserving map

$$\pi^*: \mathcal{A}(\mathcal{L}) \longrightarrow \mathcal{A}(\mathcal{L}')$$

that maps the class of a diagram  $\Gamma$  as above to  $\pi^*(\Gamma)$ . Furthermore, the map  $\pi^*$  depends only on the homotopy class of  $\pi$  relative to the boundary of  $\mathcal{L}'$ .

PROOF: It suffices to see that this operation is compatible with STU and that nothing bad happens when a univalent vertex of  $\mathcal{L}$  moves across a critical value of  $\pi$ . □

**Remark 6.30.** Propositions 6.22 and 6.27 and Lemma 6.29 remain valid if  $\mathcal{A}$  is replaced by  $\check{\mathcal{A}}$ .

**Notation 6.31.** Let  $\mathcal{L}$  be a one-manifold. Let  $\mathcal{L}_0$  be a connected component of  $\mathcal{L}$ . Let

$$\mathcal{L}(r \times \mathcal{L}_0) = (\mathcal{L} \setminus \mathcal{L}_0) \sqcup \left( \sqcup_{i=1}^r \mathcal{L}_0^{(i)} \right)$$

be the manifold obtained from  $\mathcal{L}$  by *duplicating*  $\mathcal{L}_0$  ( $r-1$ ) times, that is by replacing  $\mathcal{L}_0$  by  $r$  copies of  $\mathcal{L}_0$ . Let  $\pi(r \times \mathcal{L}_0): \mathcal{L}(r \times \mathcal{L}_0) \longrightarrow \mathcal{L}$  be the associated map, which is the identity on  $(\mathcal{L} \setminus \mathcal{L}_0)$ , and the trivial  $r$ -fold covering from  $\sqcup_{i=1}^r \mathcal{L}_0^{(i)}$  to  $\mathcal{L}_0$ . The associated map is the *duplication map*:

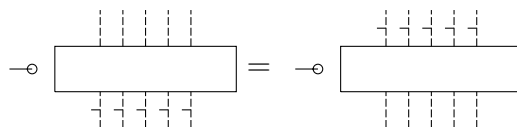
$$\pi(r \times \mathcal{L}_0)^*: \mathcal{A}(\mathcal{L}) \longrightarrow \mathcal{A}(\mathcal{L}(r \times \mathcal{L}_0)).$$

**Example 6.32.** We have

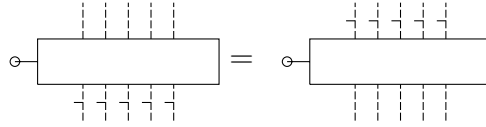
$$\pi(2 \times I)^* \left( \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \right) = \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \vdots \quad \vdots \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \vdots \quad \vdots \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}.$$

Note the following lemma.

**Lemma 6.33.** When  $\mathcal{L}$  is a disjoint union of  $r$  intervals, an  $r$ -duplicated vertex commutes with an element of  $\mathcal{A}(\mathcal{L})$ . This sentence is explained by the pictures below. In the first picture



there is a Jacobi diagram on  $\mathcal{L}$  inside the rectangle, and the picture represents the sum of the diagrams obtained by attaching the free end of an edge (the end with the empty circle) of some other part of a Jacobi diagram to each of the hooks attached to the vertical strands. The second picture



is similar, except that the edge with the free end is a part of a Jacobi diagram that is inside the box apart from this half-edge.

PROOF: This is a direct consequence of Lemma 6.23 as the pictures show. □

### 6.5 Coproduct on $\mathcal{A}(\mathcal{L})$

Recall that  $\mathbb{K}$  is  $\mathbb{Q}$  or  $\mathbb{R}$  in this chapter. All the results are valid over a commutative field  $\mathbb{K}$  of characteristic zero. Below, all tensor products are over the ground field  $\mathbb{K}$ . The canonical identification of  $V \otimes \mathbb{K}$  with  $V$  for a finite-dimensional vector space over  $\mathbb{K}$  will always be implicit. In this section,  $\mathcal{L}$  denotes a one-manifold.

For  $n \in \mathbb{N}$ , set

$$(\mathcal{A}(\mathcal{L}) \otimes \mathcal{A}(\mathcal{L}))_n = \bigoplus_{i=0}^n \mathcal{A}_i(\mathcal{L}) \otimes \mathcal{A}_{n-i}(\mathcal{L})$$

and

$$\mathcal{A}(\mathcal{L}) \hat{\otimes} \mathcal{A}(\mathcal{L}) = \prod_{n \in \mathbb{N}} (\mathcal{A}(\mathcal{L}) \otimes \mathcal{A}(\mathcal{L}))_n.$$

The topological product  $\mathcal{A}(\mathcal{L})$  is equipped with the following collection of linear maps

$$\Delta_n : \mathcal{A}_n(\mathcal{L}) \rightarrow (\mathcal{A}(\mathcal{L}) \otimes \mathcal{A}(\mathcal{L}))_n.$$

The image of the class of a Jacobi diagram  $\Gamma = \sqcup_{i \in I} \Gamma_i$  with  $|I|$  nonempty connected components  $\Gamma_i$ , numbered arbitrarily in a set  $I$ , is

$$\Delta_n([\Gamma]) = \sum_{J \subseteq I} [\sqcup_{i \in J} \Gamma_i] \otimes [\sqcup_{i \in (I \setminus J)} \Gamma_i].$$

Here, the connected components are not the connected components of  $\Gamma \cup_{U(\Gamma)} \mathcal{L}$ . They are the connected components of  $\Gamma$ , i.e., the connected components of the solid (i.e., nondashed) part of the figures. It is easy to check that  $\Delta_n$  is

well-defined. The family  $\Delta = (\Delta_n)_{n \in \mathbb{N}}$  defines a degree-preserving map from  $\mathcal{A}(\mathcal{L})$  to  $\mathcal{A}(\mathcal{L}) \hat{\otimes} \mathcal{A}(\mathcal{L})$ .

There is a well-defined continuous linear map  $\varepsilon: \mathcal{A}(\mathcal{L}) \rightarrow \mathbb{K}$  that maps the class of the empty diagram to 1 and  $\mathcal{A}_i(\mathcal{L})$  to 0 for any  $i > 0$ . The ground field  $\mathbb{K}$  is considered as a degree 0 vector space. So the map  $\varepsilon$  is a degree-preserving homomorphism. Recall that  $\mathbf{1}$  denotes the identity map.

In the following statement, we omit degrees. However, the following identities, which express the fact that  $\Delta$  is a graded *coproduct* with associated *counit*  $\varepsilon$ , are collections of identities between collections of degree-preserving linear maps between finite-dimensional vector spaces. For example, the *coassociativity* identity

$$(\Delta \otimes \mathbf{1}) \circ \Delta = (\mathbf{1} \otimes \Delta) \circ \Delta$$

means that

$$(\Delta \otimes \mathbf{1})_n \circ \Delta_n = (\mathbf{1} \otimes \Delta)_n \circ \Delta_n$$

for any  $n \in \mathbb{N}$ , where both maps are valued in

$$(\mathcal{A}(\mathcal{L}) \otimes \mathcal{A}(\mathcal{L}) \otimes \mathcal{A}(\mathcal{L}))_n = \bigoplus_{i,j,k: (i,j,k) \in \mathbb{N}^3, i+j+k=n} \mathcal{A}_i(\mathcal{L}) \otimes \mathcal{A}_j(\mathcal{L}) \otimes \mathcal{A}_k(\mathcal{L}).$$

**Lemma 6.34.** *We have  $(\varepsilon \otimes \mathbf{1}) \circ \Delta = (\mathbf{1} \otimes \varepsilon) \circ \Delta = \mathbf{1}$  and*

$$(\Delta \otimes \mathbf{1}) \circ \Delta = (\mathbf{1} \otimes \Delta) \circ \Delta.$$

PROOF: Exercise. □

Let

$$\begin{array}{ccc} \tau_n: & (\mathcal{A}(\mathcal{L}) \otimes \mathcal{A}(\mathcal{L}))_n & \rightarrow & (\mathcal{A}(\mathcal{L}) \otimes \mathcal{A}(\mathcal{L}))_n \\ & x \otimes y & \mapsto & y \otimes x. \end{array}$$

Then we also immediately have the identity

$$\tau \circ \Delta = \Delta,$$

which expresses the *cocommutativity* of  $\Delta$ .

## 6.6 Bialgebra structures

**Definition 6.35.** A *connected, finite type, commutative, cocommutative, graded bialgebra* over a field  $\mathbb{K}$  is the topological product  $\mathcal{H} = \prod_{n \in \mathbb{N}} \mathcal{H}_n$  of finite-dimensional vector spaces  $\mathcal{H}_n$  over  $\mathbb{K}$  equipped with

- a *multiplication*  $m = (m_n: (\mathcal{H} \otimes \mathcal{H})_n \rightarrow \mathcal{H}_n)_{n \in \mathbb{N}}$ , where  $(\mathcal{H} \otimes \mathcal{H})_n = \bigoplus_{i=0}^n \mathcal{H}_i \otimes \mathcal{H}_{n-i}$ ,

- a *coproduct*  $\Delta = (\Delta_n: \mathcal{H}_n \rightarrow (\mathcal{H} \otimes \mathcal{H})_n)_{n \in \mathbb{N}}$ ,
- a *unit*  $v: \mathbb{K} \rightarrow \mathcal{H}$ , which maps  $\mathbb{K}$  to  $\mathcal{H}_0$ , and which is an isomorphism from  $\mathbb{K}$  to  $\mathcal{H}_0$  (connectedness),
- a *counit*  $\varepsilon: \mathcal{H} \rightarrow \mathbb{K}$ , where  $\mathbb{K}$  is again assumed to be of degree 0,

where  $(m, \Delta, v, \varepsilon)$  are families of degree-preserving linear maps that satisfy

- the following identities, which express that  $(m, v)$  is an associative and commutative product with unit  $v(1)$ :

$$\begin{aligned} m \circ (m \otimes \mathbf{1}) &= m \circ (\mathbf{1} \otimes m) \\ m \circ (v \otimes \mathbf{1}) &= m \circ (\mathbf{1} \otimes v) = \mathbf{1} \\ m \circ \tau &= m, \end{aligned}$$

where  $\tau_n: (\mathcal{H} \otimes \mathcal{H})_n \rightarrow (\mathcal{H} \otimes \mathcal{H})_n$  maps  $x \otimes y$  to  $y \otimes x$ .

- the following identities, which express that  $(\Delta, \varepsilon)$  is a coassociative and cocommutative coproduct with counit  $\varepsilon$ :

$$\begin{aligned} (\Delta \otimes \mathbf{1}) \circ \Delta &= (\mathbf{1} \otimes \Delta) \circ \Delta \\ (\varepsilon \otimes \mathbf{1}) \circ \Delta &= (\mathbf{1} \otimes \varepsilon) \circ \Delta = \mathbf{1} \\ \tau \circ \Delta &= \Delta. \end{aligned}$$

- the following *compatibility identity*, which expresses the fact that  $\Delta$  is an algebra morphism and that  $m$  is a coalgebra morphism

$$\Delta \circ m = (m \otimes m) \circ (\mathbf{1} \otimes \tau \otimes \mathbf{1}) \circ (\Delta \otimes \Delta),$$

where the product on  $\mathcal{H} \hat{\otimes} \mathcal{H} = \prod_{n \in \mathbb{N}} (\mathcal{H} \otimes \mathcal{H})_n$  is defined from  $m$  so that it maps  $(a \otimes b) \otimes (a' \otimes b')$  to  $m(a \otimes a') \otimes m(b \otimes b')$ .

**Lemma 6.36.** *In a connected, finite type, commutative, cocommutative, graded bialgebra, we have  $\varepsilon \circ v = \mathbf{1}$  and  $\Delta(v(1)) = v(1) \otimes v(1)$ . The element  $v(1)$  is denoted by 1.*

PROOF: Since  $\Delta$  is degree-preserving, there exists  $k \in \mathbb{K}$  such that  $\Delta(v(1)) = kv(1) \otimes v(1)$ . Since  $(\varepsilon \otimes \mathbf{1}) \circ \Delta = \mathbf{1}$ , we have  $k\varepsilon(v(1)) = 1$ . Applying the compatibility identity to  $v(1) \otimes x$  yields  $\Delta(x) = k\Delta(x)$ . So we get  $k = 1 = \varepsilon(v(1))$ .  $\square$

In a connected, finite type, commutative, cocommutative, graded bialgebra, a *primitive* element is an element such that  $\Delta(x) = 1 \otimes x + x \otimes 1$ , and a *group-like* element is an element such that  $\Delta(x) = x \otimes x$  and  $\varepsilon(x) \neq 0$ .

The proof of the following lemma is straightforward and left to the reader.

**Lemma 6.37.** *Equipped with the product of Section 6.4, with the coproduct of Section 6.5, and with the counit that maps the class of the empty diagram to 1, the graded vector spaces  $\tilde{\mathcal{A}}(S^1)$ ,  $\mathcal{A}(S^1)$ , and  $\mathcal{A}(\emptyset)$  are connected, finite type, commutative, cocommutative, graded bialgebras.<sup>5</sup> The unit  $v(1)$  of these algebras is the class of the empty diagram. Furthermore, connected Jacobi diagrams are primitive elements in these algebras.*

□

Note the elementary lemma.

**Lemma 6.38.** *If  $y$  is a primitive element of a connected, finite type, commutative, cocommutative, graded bialgebra, then  $\exp(y)$  is group-like.*

PROOF: It suffices to prove that  $\Delta(y^n) = \sum_{k=0}^n \frac{n!}{k!(n-k)!} y^k \otimes y^{n-k}$ . When  $n = 0$ , this is Lemma 6.36. The compatibility identity implies  $\Delta(y^n) = \Delta(y^{n-1})(y \otimes v(1) + v(1) \otimes y)$ . □

We can now state a version of the Milnor-Moore theorem.

**Theorem 6.39.** *Let  $(\mathcal{H}; m, \Delta, v, \varepsilon)$  be a connected, finite type, commutative, cocommutative, graded bialgebra over a field  $\mathbb{K}$  of characteristic zero. Let  $\mathcal{P}_n$  denote the set of primitive elements of  $\mathcal{H}_n$ . It is a finite-dimensional vector space. Pick a basis  $b_n$  of each  $\mathcal{P}_n$ , for each  $n$ . For all  $n \in \mathbb{N}$ , the space  $\mathcal{H}_n$  is the vector space freely generated by the degree  $n$  monic monomials in the elements of  $b_{\leq n} = \cup_{k \in \mathbb{N}: k \leq n} b_k$ .<sup>6</sup>*

PROOF: Since  $\mathcal{P}_0 = \{0\}$  and  $\mathcal{H}_0$  is freely generated by the monomial  $v(1)$ , the theorem holds for  $n = 0$ . Let  $n \geq 1$ , let  $d_n$  be the set of degree  $n$  monic monomials in the elements of  $b_{\leq n-1}$ . We want to prove that  $\mathcal{H}_n$  is freely generated by  $d_n \sqcup b_n$ , by induction on  $n$ .

For  $x \in \mathcal{H}_n$ , set  $\Delta'(x) = \Delta(x) - x \otimes v(1) - v(1) \otimes x$ . According to Lemma 6.36, since  $(\varepsilon \otimes \mathbf{1}) \circ \Delta = (\mathbf{1} \otimes \varepsilon) \circ \Delta = \mathbf{1}$ , we have

$$\Delta'(x) \in \bigoplus_{i=1}^{n-1} \mathcal{H}_i \otimes \mathcal{H}_{n-i}$$

and  $\mathcal{P}_n$  is the kernel of  $\Delta'$ .

By induction,  $\mathcal{H}_i$  (resp.  $\mathcal{H}_{n-i}$ ) has a basis consisting of degree  $i$  (resp.  $(n-i)$ ) monic monomials in the elements of  $b_{\leq n-1}$ . Thus,  $\mathcal{H}_i \otimes \mathcal{H}_{n-i}$  has a

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<sup>5</sup>These spaces can be furthermore equipped with the linear *antipode* that maps a product  $\Pi$  of  $p$  primitive elements to  $(-1)^p \Pi$ . So they are *Hopf algebras*.

<sup>6</sup>By *monic monomials*, we mean monomials with coefficient one. So degree  $n$  monic monomials in the elements of  $b_{\leq n}$  are of the form  $\prod_{i \in I} p_i^{r(i)}$ , for elements  $p_i$  of  $b_{\leq n}$  of degree  $d(i)$ , and positive integers  $r(i)$  such that  $\sum_{i \in I} d(i)r(i) = n$ .

basis consisting of tensor products of these monomials. Multiplying two such monomials yields an element

$$\prod_{i \in I} p_i^{r(i)}$$

of  $d_n$ , where the  $p_i$  are distinct elements of  $b_{\leq n-1}$ , and the  $r(i)$  are positive integers. We have

$$\Delta \left( \prod_{i \in I} p_i^{r(i)} \right) = \sum_{k: I \rightarrow \mathbb{N}: 0 \leq k(i) \leq r(i), \forall i} \left( \prod_{i \in I} \frac{r(i)!}{k(i)!(r(i) - k(i))!} \right) \left( \prod_{i \in I} p_i^{k(i)} \right) \otimes \left( \prod_{i \in I} p_i^{r(i) - k(i)} \right).$$

This formula proves that  $\Delta'$  injects the vector space freely generated by the degree  $n$  monic monomials in the elements of  $b_{\leq n-1}$  into  $\bigoplus_{i=1}^{n-1} \mathcal{H}_i \otimes \mathcal{H}_{n-i}$ . Thus, the degree  $n$  monic monomials in the elements of  $b_{\leq n}$  form a free system of  $\mathcal{H}_n$ , and it suffices to prove that they generate  $\mathcal{H}_n$ . To do so, we only need to check that for every  $x \in \mathcal{H}_n$ , for every element  $d \in d_n$ , there exists a constant  $a(x, d)$  such that

$$\Delta'(x) = \sum_{d \in d_n} a(x, d) \Delta'(d).$$

Indeed, in this case,  $(x - \sum_{d \in d_n} a(x, d)d)$  is primitive. Fix  $x \in \mathcal{H}_n$ .

Let  $d = \prod_{i \in I} p_i^{r_d(i)} \in \mathcal{H}_n$ , where  $r_d(i) > 0$ , for any  $i \in I$ . Let  $E(d)$  be the set of maps  $k: I \rightarrow \mathbb{N}$  such that  $0 < \sum_{i \in I} k(i)$ ,  $k(i) \leq r_d(i)$ , and  $d(k) \stackrel{\text{def}}{=} \prod_{i \in I} p_i^{k(i)} \in \mathcal{H}_j$  for some  $j$  such that  $j < n$ . Then we have

$$\Delta'(d) = \sum_{k \in E(d)} c_{k, r_d} d(k) \otimes d(r_d - k)$$

with

$$c_{k, r_d} = \prod_{i \in I} \frac{r_d(i)!}{k(i)!(r_d(i) - k(i))!} \neq 0.$$

We also have

$$\Delta'(x) = \sum_{d \in d_n} \left( \sum_{k \in E(d)} c_k(x, d) d(k) \otimes d(r_d - k) \right).$$

Now, it suffices to prove that, for any  $d \in d_n$ , there exists  $a(x, d)$  such that  $c_k(x, d) = a(x, d)c_{k, r_d}$  for any  $k \in E(d)$ . Fix  $d \in d_n$  and set  $r = r_d$ .

Thanks to the coassociativity of  $\Delta$ , we have  $(\Delta \otimes \mathbf{1}) \circ \Delta(x) = (\mathbf{1} \otimes \Delta) \circ \Delta(x)$ . Therefore, if  $h(i) \leq k(i)$  for all  $i \in I$  and  $\sum_{i \in I} h(i) > 0$ , then the coefficient

$$c_{h,k}c_k(x, d) = c_h(x, d)c_{k-h, r-h}$$

of  $d(h) \otimes d(k-h) \otimes d(r-k)$  in  $(\Delta \otimes \mathbf{1}) \circ \Delta(x)$  determines both the coefficient  $c_h(x, d)$  of  $d(h) \otimes d(r-h)$  and the coefficient  $c_k(x, d)$  in  $\Delta'(x)$ . So, we have

$$c_h(x, d) = \frac{c_{h,k}}{c_{k-h, r-h}} c_k(x, d).$$

The coassociativity applied to  $d$  similarly implies

$$c_h(d, d) = \frac{c_{h,k}}{c_{k-h, r-h}} c_k(d, d),$$

where  $c_h(d, d) = c_{h,r}$ . So, we get

$$c_h(x, d) = \frac{c_{h,r}}{c_{k,r}} c_k(x, d).$$

Choose  $j \in I$ . Let  $\delta_j \in E(d)$  be such that  $\delta_j(j) = 1$  and  $\delta_j(i) = 0$  for all  $i \in I \setminus \{j\}$ . Set  $a(x, d) = c_{\delta_j}(x, d)/c_{\delta_j, r}$ . Then for all  $k \in E(d)$  such that  $k(j) \neq 0$ , we have  $c_k(x, d) = a(x, d)c_{k,r}$ . If  $\sum_{i \in I} r(i) > 2$ , then for any  $i \in I \setminus \{j\}$ , the map  $\delta_i + \delta_j$  is in  $E(d)$ . So we have  $c_{\delta_i}(x, d) = a(x, d)c_{\delta_i, r}$  for any  $i \in I$ , and therefore  $c_k(x, d) = a(x, d)c_{k,r}$  for any  $k \in E(d)$ . The only untreated case is  $I = \{i, j\}$  with  $r(i) = r(j) = 1$ . In this case, the cocommutativity of  $\Delta$  leads to the result.  $\square$

**Corollary 6.40.** *Under the assumptions of Theorem 6.39, there is a well-defined unique linear projection  $p^c$  from  $\mathcal{H}$  to  $\mathcal{P} = \prod_{n \in \mathbb{N}} \mathcal{P}_n$  that maps the products of two homogeneous elements of positive degree to 0, and that maps  $\mathcal{H}_0$  to 0.<sup>7</sup>*

$\square$

**Theorem 6.41.** *Let  $\mathcal{H}$  be a connected, finite type, commutative, cocommutative, graded bialgebra. Let  $\mathcal{P}$  be the space of its primitive elements, and let  $p^c: \mathcal{H} \rightarrow \mathcal{P}$  be the projection of Corollary 6.40. Any group-like element  $x$  of  $\mathcal{H}$  is the exponential of a unique primitive element of  $\mathcal{P}$ . This element is  $p^c(x)$ .*

<sup>7</sup>An element of  $\mathcal{H}$  is *homogeneous* if it belongs to some  $\mathcal{H}_j$  for some  $j$ .



PROOF: First note that  $p^c(\exp(y)) = y$  for any primitive element  $y$  of  $\mathcal{H}$ . Therefore, if  $\exp(y) = \exp(y')$  for two primitive elements  $y$  and  $y'$  of  $\mathcal{H}$ , then  $y = y'$ .

Set  $y_n = p^c(x)_n$ , and let us prove  $x = \exp(y)$ . Since  $\varepsilon(x) \neq 0$ , we have  $x_0 = k\nu(1)$  with  $k \neq 0$ . Since  $\Delta(x) = x \otimes x$ , we get  $k = 1$ . So the equality  $x = \exp(y)$  holds in degree 0. Assume that it holds until degree  $(n - 1)$ . We have  $\Delta_n(x_n) = \sum_{i=0}^n x_i \otimes x_{n-i}$ . According to Lemma 6.38, we also have  $\Delta_n(\exp(y)_n) = \sum_{i=0}^n \exp(y)_i \otimes \exp(y)_{n-i}$ . Our induction hypothesis ensures

$$\Delta_n(x_n - \exp(y)_n) = 1 \otimes (x_n - \exp(y)_n) + (x_n - \exp(y)_n) \otimes 1.$$

So  $(x_n - \exp(y)_n)$  is primitive, and we get  $(x_n - \exp(y)_n) = p^c(x_n - \exp(y)_n) = 0$ . □

# Chapter 7

## First definitions of $\mathcal{Z}$

In this chapter, we introduce the invariant  $\mathcal{Z}$  of links in  $\mathbb{Q}$ -spheres, which is the main object of this book. We illustrate the required definitions with many examples, skippable by a reader who only wants the definition.

### 7.1 Configuration spaces of Jacobi diagrams in 3-manifolds

Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ , as in Definition 3.8. Let  $\mathcal{L}$  be a disjoint union of  $k$  circles  $S_i^1$ ,  $i \in \underline{k}$ . Let

$$L: \mathcal{L} \longrightarrow \check{R}$$

denote a  $C^\infty$  embedding from  $\mathcal{L}$  to  $\check{R}$ . The embedding  $L$  is a *link embedding*. Let  $\Gamma$  be a Jacobi diagram with support  $\mathcal{L}$  as in Definition 6.13. Let  $U = U(\Gamma)$  denote the set of univalent vertices of  $\Gamma$ , and let  $T = T(\Gamma)$  denote the set of trivalent vertices of  $\Gamma$ . A *configuration* of  $\Gamma$  (with respect to  $L$ ) is an embedding

$$c: U \cup T \hookrightarrow \check{R}$$

whose restriction  $c|_U$  to  $U$  may be written as  $L \circ j$  for some  $\Gamma$ -compatible injection

$$j: U \hookrightarrow \mathcal{L}.$$

Denote the set of these configurations by  $\check{C}(R, L; \Gamma)$ , we have

$$\check{C}(R, L; \Gamma) = \left\{ c: U \cup T \hookrightarrow \check{R} : (\exists j \in [i_\Gamma] : c|_U = L \circ j) \right\}.$$

In  $\check{C}(R, L; \Gamma)$ , the univalent vertices move along  $L(\mathcal{L})$ , while the trivalent vertices move in the ambient space  $\check{R}$ . The configuration space  $\check{C}(R, L; \Gamma)$  is naturally an open submanifold of  $\mathcal{L}^U \times \check{R}^T$ .

An *orientation* of a set of cardinality at least 2 is a total order of its elements up to an even permutation. When  $L$  is oriented, such an orientation of the set  $V(\Gamma)$  of vertices of  $\Gamma$  orients  $\check{C}(R, L; \Gamma)$  naturally since it orders the oriented odd-dimensional factors of  $\mathcal{L}^U \times \check{R}^T$ . Below, we associate an orientation of  $\check{C}(R, L; \Gamma)$  to a vertex-orientation of  $\Gamma$  and an orientation of the set  $H(\Gamma)$  of half-edges of  $\Gamma$ .

Cut each edge of  $\Gamma$  into two half-edges. When an edge is oriented, define its *first* half-edge and its *second* one so that we meet the first half-edge first when following the orientation of the edge. When the edges of  $\Gamma$  are oriented, the orientations of the edges of  $\Gamma$  induce the following orientation of the set  $H(\Gamma)$ . Order the set  $E(\Gamma)$  of edges of  $\Gamma$  arbitrarily, and order  $H(\Gamma)$  as (First half-edge of the first edge, second half-edge of the first edge,  $\dots$ , second half-edge of the last edge). The induced orientation of  $H(\Gamma)$  is called the *edge-orientation* of  $H(\Gamma)$ . Note that it does not depend on the order of  $E(\Gamma)$ .

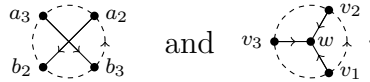
**Lemma 7.1.** *When  $\Gamma$  is equipped with a vertex-orientation, orientations of the manifold  $\check{C}(R, L; \Gamma)$  are in canonical one-to-one correspondence with orientations of the set  $H(\Gamma)$ .*

PROOF: Since  $\check{C}(R, L; \Gamma)$  is naturally an open submanifold of  $\mathcal{L}^U \times \check{R}^T$ , it inherits  $\mathbb{R}^{|U|+3|T|}$ -valued charts from  $\mathbb{R}$ -valued orientation-preserving charts of  $\mathcal{L}$ , with respect to the (possibly local as in Definition 6.13) orientation(s) of  $\mathcal{L}$ , and  $\mathbb{R}^3$ -valued orientation-preserving charts of  $\check{R}$ . To define the orientation of  $\mathbb{R}^{|U|+3|T|}$ , it suffices to identify its factors and order them (up to even permutation). Each factor is labeled by an element of  $H(\Gamma)$  as follows. The  $\mathbb{R}$ -valued local coordinate of an element of  $\mathcal{L}$  corresponding to the image under  $j$  of an element  $u$  of  $U$  sits in the factor labeled by the half-edge that contains  $u$ . The three  $\mathbb{R}$ -valued coordinates of the image under a configuration  $c$  of an element  $t$  of  $T$ , with respect to an arbitrary oriented local chart, belong to the factors labeled by the three half-edges containing  $t$  so that the cyclic order of the three half-edges induced by the vertex-orientation of  $\Gamma$  matches the order of the three factors.  $\square$

We will use Lemma 7.1 to orient  $\check{C}(R, L; \Gamma)$  as summarized in the following immediate corollary.

**Corollary 7.2.** *If  $\Gamma$  is equipped with a vertex-orientation  $o(\Gamma)$  and if the edges of  $\Gamma$  are oriented, then the induced edge-orientation of  $H(\Gamma)$  orients  $\check{C}(R, L; \Gamma)$ , via the canonical correspondence described in the proof of Lemma 7.1.*

**Examples 7.3.** In Subsection 1.2.5, the orientations of the configuration spaces  $\check{C}(K; \boxtimes)$  and  $\check{C}(K; \boxplus)$  were induced by the order of the given coordinates. We can check that these orientations are also induced by the edge-orientations and by the orientation of the vertex  $w$  in the following figures



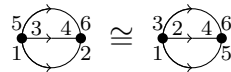
Recall that  $\check{C}(K; \boxplus)$ , which may be described as

$$\left\{ c: \{w, v_1, v_2, v_3\} \hookrightarrow \mathbb{R}^3 : \begin{array}{l} c(v_i) = K(z_i \in S^1), z_2 = \exp(2i\pi t_2)z_1, \\ z_3 = \exp(2i\pi t_3)z_1, 0 < t_2 < t_3 < 1 \end{array} \right\},$$

was regarded as an open submanifold of  $\mathbb{R}^3 \times (S^1)^3 = \{(X_1, X_2, X_3, z_1, z_2, z_3)\}$ , where  $(X_1, X_2, X_3) = c(w)$ .

Let us check that the above coordinates orient  $\check{C}(K; \boxplus)$  as the orientation of edges does. For  $i \in \underline{3}$ , let  $e(i)$  denote the edge from  $v_i$  to  $w$ . This orders the three factors of  $(S^1)^{E(\Gamma)}$ . Distribute the coordinates  $X_1, X_2, X_3$  so that  $X_i$  is on the second half-edge of  $e(i)$ . Then the vertex-orientation and the edge-orientation of  $\Gamma$  induce the orientation of  $\check{C}(K; \boxplus)$  represented by  $(z_1, X_1, z_2, X_2, z_3, X_3)$ , which is the same as the orientation represented by  $(X_1, X_2, X_3, z_1, z_2, z_3)$ . The case of  $\check{C}(K; \boxtimes)$  is left to the reader.

**Example 7.4.** Equip the diagram  $\ominus$  with its vertex-orientation induced by the picture. Orient its three edges so that they start from the same vertex. Then the orientation of  $\check{C}(R, L; \ominus)$  induced by this edge-orientation of  $\ominus$  matches the orientation of  $(\check{R} \times \check{R}) \setminus \Delta$  induced by the order of the two factors, where the first factor corresponds to the position of the vertex from which the three edges start, as shown in the following picture.



**Remark 7.5.** For a Jacobi diagram  $\Gamma$  equipped with a vertex-orientation  $o(\Gamma)$ , an orientation of  $V(\Gamma)$  induces the following orientation of  $H(\Gamma)$ . Fix a total order of  $V(\Gamma)$  that induces its given orientation. Then the corresponding orientation of  $H(\Gamma)$  is induced by a total order which starts with the half-edges adjacent to the first vertex, ordered with respect to  $o(\Gamma)$  if the vertex is trivalent, and continues with the half-edges adjacent to the second vertex, to the third one,  $\dots$  This orientation is called the *vertex-orientation* of  $H(\Gamma)$  associated to  $o(\Gamma)$  and to the orientation of  $V(\Gamma)$ . In particular, an orientation of  $H(\Gamma)$  (such as the edge-orientation of  $H(\Gamma)$  when the edges of  $\Gamma$  are oriented) and a vertex-orientation  $o(\Gamma)$  together induce an orientation of  $V(\Gamma)$ ,

namely the orientation of  $V(\Gamma)$  such that the induced vertex-orientation of  $H(\Gamma)$  matches the given orientation of  $H(\Gamma)$ .

The dimension of  $\check{C}(R, L; \Gamma)$  is

$$|U(\Gamma)| + 3|T(\Gamma)| = 2|E(\Gamma)|.$$

Since the degree of  $\Gamma$  is  $n = n(\Gamma) = \frac{1}{2}(|U(\Gamma)| + |T(\Gamma)|)$ , we have

$$|E(\Gamma)| = 3n - |U(\Gamma)|.$$

## 7.2 Configuration space integrals

**Definition 7.6.** Let  $A$  be a finite set. An  $A$ -numbered Jacobi diagram is a Jacobi diagram  $\Gamma$  whose edges are oriented, equipped with an injection  $j_E: E(\Gamma) \hookrightarrow A$ . Such an injection numbers the edges when  $A \subset \mathbb{N}$ . Let  $\mathcal{D}_n^e(\mathcal{L})$  denote the set of  $\underline{3n}$ -numbered degree  $n$  Jacobi diagrams with support  $\mathcal{L}$  without looped edges like  $\circlearrowleft$ .

Note that the injection  $j_E$  is a bijection for any diagram of  $\mathcal{D}_n^e(\mathcal{L})$  without univalent vertices.

**Examples 7.7.** We have

$$\begin{aligned} \mathcal{D}_1^e(\emptyset) &= \left\{ \begin{array}{c} \circlearrowleft \\ \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{2} \\ \xrightarrow{3} \end{array} \\ \circlearrowright \end{array} \right\}, \\ \mathcal{D}_1^e(S^1) &= \mathcal{D}_1^e(\emptyset) \sqcup \left\{ \begin{array}{c} \circlearrowleft \\ \xrightarrow{1} S^1 \\ \circlearrowright \end{array} \right\}, \text{ and} \\ \mathcal{D}_1^e(S_1^1 \sqcup S_2^1) &= \mathcal{D}_1^e(\emptyset) \sqcup (\mathcal{D}_1^e(S_1^1) \setminus \mathcal{D}_1^e(\emptyset)) \sqcup (\mathcal{D}_1^e(S_2^1) \setminus \mathcal{D}_1^e(\emptyset)) \\ &\quad \sqcup \left\{ \begin{array}{c} \circlearrowleft \\ \xrightarrow{1} \circlearrowleft \xrightarrow{2} \circlearrowright S_2^1 \\ \circlearrowright \end{array} \right\}, \\ &\quad \left\{ \begin{array}{c} \circlearrowleft \\ \xrightarrow{1} \circlearrowright \xrightarrow{2} \circlearrowleft S_2^1 \\ \circlearrowright \end{array} \right\}, \\ &\quad \left\{ \begin{array}{c} \circlearrowleft \\ \xrightarrow{1} \circlearrowright \xrightarrow{3} \circlearrowleft S_2^1 \\ \circlearrowright \end{array} \right\}, \\ &\quad \left\{ \begin{array}{c} \circlearrowleft \\ \xrightarrow{1} \circlearrowright \xrightarrow{2} \circlearrowleft S_2^1 \\ \circlearrowright \end{array} \right\}, \\ &\quad \left\{ \begin{array}{c} \circlearrowleft \\ \xrightarrow{1} \circlearrowright \xrightarrow{3} \circlearrowleft S_2^1 \\ \circlearrowright \end{array} \right\} \right\}. \end{aligned}$$

Let  $\Gamma$  be a numbered degree  $n$  Jacobi diagram with support  $\mathcal{L}$ . An edge  $e$  oriented from a vertex  $v_1$  to a vertex  $v_2$  of  $\Gamma$  induces the map

$$\begin{aligned} p_e: \check{C}(R, L; \Gamma) &\rightarrow C_2(R) \\ c &\mapsto (c(v_1), c(v_2)). \end{aligned}$$

Let  $o(\Gamma)$  be a vertex-orientation of  $\Gamma$ . For any  $i \in \underline{3n}$ , let  $\omega(i)$  be a propagating form of  $(C_2(R), \tau)$ . Let  $(\check{C}(R, L; \Gamma), o(\Gamma))$  denote the manifold  $\check{C}(R, L; \Gamma)$ , equipped with the orientation induced by  $o(\Gamma)$  and by the edge-orientation of  $\Gamma$ , as in Corollary 7.2. Define

$$I(R, L, \Gamma, o(\Gamma), (\omega(i))_{i \in \underline{3n}}) = \int_{(\check{C}(R, L; \Gamma), o(\Gamma))} \bigwedge_{e \in E(\Gamma)} p_e^* \left( \omega(j_E(e)) \right).$$

The convergence of this integral is a consequence of the following proposition, which will be proved in Chapter 8. (See the end of Section 8.2.)

**Proposition 7.8.** *There exists a smooth compactification of  $\check{C}(R, L; \Gamma)$ , which will be denoted by  $C(R, L; \Gamma)$ , to which the maps  $p_e$  extend smoothly.*

According to this proposition,  $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$  extends smoothly to  $C(R, L; \Gamma)$ , and we have

$$\int_{(\check{C}(R, L; \Gamma), o(\Gamma))} \bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e))) = \int_{(C(R, L; \Gamma), o(\Gamma))} \bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e))).$$

**Examples 7.9.** For any three propagating forms  $\omega(1)$ ,  $\omega(2)$ , and  $\omega(3)$  of  $(C_2(R), \tau)$ , we have

$$I\left(R, K_i \sqcup K_j: S_i^1 \sqcup S_j^1 \hookrightarrow \check{R}, \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} S_j^1, (\omega(i))_{i \in \underline{3}}\right) = lk(K_i, K_j)$$

and

$$I\left(R, \emptyset, \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array}, (\omega(i))_{i \in \underline{3}}\right) = \Theta(R, \tau)$$

for any numbering of the (nondashed) diagrams (exercise).

**Examples 7.10.** For any oriented trivalent numbered degree  $n$  Jacobi diagram  $(\Gamma, o(\Gamma))$ , we have

$$I(\Gamma, o(\Gamma)) = I(S^3, \emptyset, \Gamma, o(\Gamma), (p_{S^2}^*(\omega_{S^2}))_{i \in \underline{3n}}) = 0.$$

Indeed,  $I(\Gamma, o(\Gamma))$  is equal to

$$\int_{(\check{C}(S^3, \emptyset; \Gamma), o(\Gamma))} \left( \prod_{e \in E(\Gamma)} p_{S^2} \circ p_e \right)^* \left( \bigwedge_{e \in E(\Gamma)} \omega_{S^2} \right),$$

where

- $\bigwedge_{e \in E(\Gamma)} \omega_{S^2}$  is a product volume form of  $(S^2)^{E(\Gamma)}$  with total volume one,
- $\check{C}(S^3, \emptyset; \Gamma)$  is the space of injections of  $\underline{3n}$  into  $\mathbb{R}^3$ ,
- the degree of  $\bigwedge_{e \in E(\Gamma)} \omega_{S^2}$  is equal to the dimension of  $\check{C}(S^3, \emptyset; \Gamma)$ , and
- the map  $\prod_{e \in E(\Gamma)} p_{S^2} \circ p_e$  is never a local diffeomorphism since it is invariant under the action of global translations on  $\check{C}(S^3, \emptyset; \Gamma)$ .

**Examples 7.11.** Let  $O$  denote the representative of the unknot of  $S^3$ , that is the image of the embedding of the unit circle  $S^1$  of  $\mathbb{C}$  regarded as  $\mathbb{C} \times \{0\}$ , into  $\mathbb{R}^3$  regarded as  $\mathbb{C} \times \mathbb{R}$ . Let us compute

$$I(S^3, O, \Gamma, o(\Gamma), (p_{S^2}^*(\omega_{S^2}))_{i \in \underline{3n}}),$$

for the oriented Jacobi diagrams

$$\Gamma_1 = \text{⊖}, \Gamma_2 = \text{⊗}, \Gamma_3 = \text{⊙}, \text{ and } \Gamma_4 = \text{⊗}.$$

Since all edges are equipped with the same standard propagating form  $p_{S^2}^*(\omega_{S^2})$ , we do not number the edges. For  $i \in \underline{4}$ , set

$$I(\Gamma_i) = I(S^3, O, \Gamma_i, o(\Gamma_i), (p_{S^2}^*(\omega_{S^2}))).$$

We are about to prove  $I(\Gamma_1) = I(\Gamma_2) = I(\Gamma_3) = 0$  and  $I(\Gamma_4) = \frac{1}{8}$ . For  $i \in \underline{4}$ , set  $\Gamma = \Gamma_i$ . Recall that  $I(\Gamma)$  is equal to

$$\int_{(\check{C}(S^3, O; \Gamma), o(\Gamma))} \left( \prod_{e \in E(\Gamma)} p_{S^2} \circ p_e \right)^* \left( \bigwedge_{e \in E(\Gamma)} \omega_{S^2} \right).$$

When  $i \in \underline{2}$ , the image of  $\prod_{e \in E(\Gamma)} p_{S^2} \circ p_e$  lies in the subset of  $(S^2)^2$  consisting of the pair of horizontal vectors. Since the interior of this subset is empty, the form  $(\prod_{e \in E(\Gamma)} p_{S^2} \circ p_e)^*(\omega_{S^2}^{E(\Gamma)})$  vanishes identically. So we get  $I(\Gamma_i) = 0$ . When  $i = 3$ , the two edges with the same endpoints must have the same direction. So the image of  $\prod_{e \in E(\Gamma)} p_{S^2} \circ p_e$  lies in the subset of  $(S^2)^{E(\Gamma)}$ , where two  $S^2$ -coordinates are identical (namely those in the  $S^2$ -factors corresponding to the above pair of edges), and we obtain  $I(\Gamma_3) = 0$  as before.

**Lemma 7.12.** Let  $\Gamma = \text{⊗}$ . Then we have

$$I(S^3, O, \Gamma, o(\Gamma), (p_{S^2}^*(\omega_{S^2}))) = \frac{1}{8}.$$

**PROOF:** Let  $(S^1)_+^3$  be the subset of  $(S^1)^3$  consisting of triples  $(z_1, z_2, z_3)$  of pairwise distinct elements of  $S^1$  such that the orientation of  $S^1$  induces the cyclic order  $(z_1, z_2, z_3)$ . Then we write

$$\check{C} = \check{C}(S^3, O; \Gamma) = \left\{ (X, z_1, z_2, z_3) : X \in \mathbb{R}^3 \setminus \{z_1, z_2, z_3\}, (z_1, z_2, z_3) \in (S^1)_+^3 \right\},$$

where  $X = (X_1, X_2, X_3)$ . As explained in Example 7.3, these coordinates orient  $\check{C}$  as does the orientation of edges. Let  $\check{C}^+ = \{(X, z_1, z_2, z_3) \in \check{C} :$

$X_3 > 0\}$ . The reflection  $\sigma_h$  with respect to the horizontal plane acts on  $\check{C}$  by an orientation-reversing diffeomorphism, which changes  $X_3$  to  $(-X_3)$  and leaves the other coordinates unchanged. It also acts on  $S^2$  by an orientation-reversing diffeomorphism, which preserves the volume up to sign. Therefore, we have

$$I(\Gamma) = 2 \int_{\check{C}^+} \left( \prod_{e \in E(\Gamma)} p_{S^2} \circ p_e \right)^* \left( \bigwedge_{e \in E(\Gamma)} \omega_{S^2} \right).$$

Let  $S_+^2$  denote the set of elements of  $S^2$  with positive *height* (third coordinate), and let  $(S_+^2)^3$  be the set of elements of  $(S_+^2)^3$  which form a direct basis. Let us check that the volume of  $(S_+^2)^3$  is  $\frac{1}{16}$  (with respect to  $\bigwedge_{e \in E(\Gamma)} \omega_{S^2}$ ). The volume of  $(S_+^2)^3$  is  $\frac{1}{8}$ . This is also the volume of the subset of  $(S_+^2)^3$  consisting of triples of noncoplanar vectors. The involution that exchanges the last two vectors in the latter set sends the direct bases to the indirect ones, and it preserves the volume. Therefore, in order to prove  $I(\Gamma_4) = \frac{1}{8}$ , it suffices to prove that

$$\begin{aligned} \Psi: \quad \check{C}^+ &\rightarrow (S^2)^3 \\ c &\mapsto \left( \prod_{e \in E(\Gamma)} p_{S^2} \circ p_e \right) (c) \end{aligned}$$

is an orientation-preserving diffeomorphism onto  $(S_+^2)^3$ . Let  $c = (X, z_1, z_2, z_3)$  be a point of  $\check{C}^+$ . Let us first check that  $\Psi(c) \in (S_+^2)^3$ . For  $j \in \underline{3}$ , the vector  $\overrightarrow{z_j X}$  may be written as  $\lambda_j V_j$  for some  $\lambda_j \in ]0, +\infty[$  and for some  $V_j \in S_+^2$ . Since  $(V_1, \overrightarrow{z_1 z_2} = \lambda_1 V_1 - \lambda_2 V_2, \overrightarrow{z_1 z_3} = \lambda_1 V_1 - \lambda_3 V_3)$  is a direct basis of  $\mathbb{R}^3$ , so is  $(V_1, V_2, V_3)$ . Let us now compute the sign of the Jacobian of  $\Psi$  at  $c$ . Let  $T_c \Psi$  denote the tangent map to  $\Psi$  at  $c$ . For  $j \in \underline{3}$ , let  $Z_j$  denote the unit tangent vector to  $S^1$  at  $z_j$ , and let  $p_j: (\mathbb{R}^3)^3 \rightarrow \mathbb{R}^3/\mathbb{R}V_j$  be the projection onto the  $j^{\text{th}}$  factor composed by the projection onto the tangent space  $\mathbb{R}^3/\mathbb{R}V_j$  to  $S^2$  at  $V_j$ . Then  $p_j(T_c \Psi(Z_j)) = -Z_j$  in the tangent space to  $S^2$  at  $V_j$ , which is generated by (the projections onto  $\mathbb{R}^3/\mathbb{R}V_j$  of)  $-Z_j$  and any vector  $W_j$  such that  $\det(V_j, -Z_j, W_j) = 1$ . Reorder the oriented basis  $(-Z_1, W_1, -Z_2, W_2, -Z_3, W_3)$  of  $T(S^2)_+^3$  at  $(V_1, V_2, V_3)$  by a positive permutation of the coordinates as  $(W_1, W_2, W_3, -Z_1, -Z_2, -Z_3)$ . Writing the matrix of  $T_c \Psi$  with respect to this basis  $(W_1, W_2, W_3, -Z_1, -Z_2, -Z_3)$  of the target space and the basis  $(V_1, V_2, V_3)$  of  $T_X \mathbb{R}^3$  followed by  $(Z_1, Z_2, Z_3)$  for the domain  $T_c \check{C}^+$  produces a matrix whose last three columns contain 1 on the diagonals as only nonzero entries. In the quotient

$$T(S^2)_+^3 / (\mathbb{R}(-Z_1, 0, 0) \oplus \mathbb{R}(0, -Z_2, 0) \oplus \mathbb{R}(0, 0, -Z_3)),$$

$T_c \Psi(V_1)$  may be expressed as

$$T_c \Psi(V_1) = \det(V_2, -Z_2, V_1)W_2 + \det(V_3, -Z_3, V_1)W_3,$$



where  $\det(V_2, -Z_2, V_1) = \det(Z_2, V_2, V_1)$ . Let us prove that  $\det(Z_2, V_2, V_1) > 0$ . When  $z_2 = -1$ , projecting  $c$  on  $(\mathbb{R} = \frac{\mathbb{C}}{i\mathbb{R}}) \times \mathbb{R}$  produces a picture as in Figure 7.1 that makes this result clear. The general result follows easily.

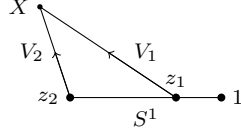


Figure 7.1: Partial projection of  $(X, z_1, z_2, z_3)$  on  $\frac{\mathbb{C}}{i\mathbb{R}} \times \mathbb{R}$  when  $z_2 = -1$

Finally, the Jacobian of  $\Psi$  at  $(X, z_1, z_2, z_3)$  is the determinant of

$$\begin{bmatrix} 0 & \det(Z_1, V_1, V_2) & \det(Z_1, V_1, V_3) \\ \det(Z_2, V_2, V_1) & 0 & \det(Z_2, V_2, V_3) \\ \det(Z_3, V_3, V_1) & \det(Z_3, V_3, V_2) & 0 \end{bmatrix},$$

which is

$$\begin{aligned} & \det(Z_1, V_1, V_2) \det(Z_2, V_2, V_3) \det(Z_3, V_3, V_1) \\ & + \det(Z_1, V_1, V_3) \det(Z_2, V_2, V_1) \det(Z_3, V_3, V_2). \end{aligned}$$

It is positive since all the involved terms are. Let us finally check that every element  $(V_1, V_2, V_3)$  of  $(S^2)_+^3$  has a unique element in its preimage. Construct the three lines of  $\mathbb{R}^3$  directed by  $V_1, V_2$ , and  $V_3$  through the origin of  $\mathbb{R}^3$ . The line directed by  $V_i$  intersects the horizontal plane at height  $(-1)$  at a point  $w_i$ . There is a unique circle in this horizontal plane that contains  $w_1, w_2$ , and  $w_3$ . Let  $\frac{1}{\lambda}$  be the radius of this circle, and let  $w_0$  be its center. Then the unique element of  $\Psi^{-1}(V_1, V_2, V_3)$  is  $(-\lambda w_0, \lambda w_1 - \lambda w_0, \lambda w_2 - \lambda w_0, \lambda w_3 - \lambda w_0)$ .  $\square$

### 7.3 Configuration space integrals associated to a chord

Let us now study the case of  $I(\bullet \xrightarrow{k} S_j^1, (\omega(i))_{i \in \mathbb{3}})$ . We will see that this integral depends on the chosen propagating forms and on the diagram numbering.

A *dilation* is a homothety with a positive ratio. Let  $U^+K_j$  denote the fiber space over  $K_j$  consisting of the tangent vectors to the knot  $K_j$  of  $\check{R}$  that orient  $K_j$  up to dilation. The fiber of  $U^+K_j$  consists of one point. So the total space of this *unit positive tangent bundle to  $K_j$*  is canonically diffeomorphic to  $K_j$ . Set  $U^-K_j = U^+(-K_j)$ .

For a knot  $K_j$  in  $\check{R}$ , we have

$$\check{C}(K_j; \overset{\leftarrow}{\curvearrowright} S_j^1) = \left\{ (K_j(z), K_j(z \exp(i\theta))) : (z, \theta) \in S^1 \times ]0, 2\pi[ \right\}.$$

Let  $C_j = C(K_j; \overset{\leftarrow}{\curvearrowright} S_j^1)$  be the closure of  $\check{C}(K_j; \overset{\leftarrow}{\curvearrowright} S_j^1)$  in  $C_2(R)$ . This closure is diffeomorphic to  $S^1 \times [0, 2\pi]$ , where  $S^1 \times \{0\}$  is identified with  $U^+K_j$ ,  $S^1 \times \{2\pi\}$  is identified with  $U^-K_j$ , and  $\partial C(K_j; \overset{\leftarrow}{\curvearrowright} S_j^1) = U^+K_j - U^-K_j$ .

**Lemma 7.13.** *For any  $i \in \underline{3}$ , let  $\omega(i)$  and  $\omega'(i)$  be propagating forms of  $(C_2(R), \tau)$ , which restrict to  $\partial C_2(R)$  as  $p_\tau^*(\omega(i)_{S^2})$  and  $p_\tau^*(\omega'(i)_{S^2})$ , respectively. Then there exists a one-form  $\eta(i)_{S^2}$  on  $S^2$  such that  $\omega'(i)_{S^2} = \omega(i)_{S^2} + d\eta(i)_{S^2}$ , and we have*

$$I\left(\overset{\leftarrow}{\curvearrowright} S_j^1, (\omega'(i))_{i \in \underline{3}}\right) - I\left(\overset{\leftarrow}{\curvearrowright} S_j^1, (\omega(i))_{i \in \underline{3}}\right) = \int_{U^+K_j} p_\tau^*(\eta(k)_{S^2}) - \int_{U^-K_j} p_\tau^*(\eta(k)_{S^2}).$$

PROOF: Such a form  $\eta(i)_{S^2}$  exists for any  $i$ . According to Lemma 3.17, there exists a one-form  $\eta(i)$  on  $C_2(R)$  such that  $\omega'(i) = \omega(i) + d\eta(i)$  and the restriction of  $\eta(i)$  to  $\partial C_2(R)$  is  $p_\tau^*(\eta(i)_{S^2})$ . Apply Stokes' theorem to  $\int_{C_j} (\omega'(k) - \omega(k)) = \int_{C_j} d\eta(k)$ .  $\square$

**Exercise 7.14.** Find a knot  $K_j$  of  $\mathbb{R}^3$  and a form  $\eta(k)$  of  $C_2(\mathbb{R}^3)$  such that the right-hand side of Lemma 7.13 does not vanish. (Use Lemma 3.17. Hints can be found in Section 7.5.)

Recall that a propagating form  $\omega$  of  $(C_2(R), \tau)$  is *homogeneous* if its restriction to  $\partial C_2(R)$  is  $p_\tau^*(\omega_{S^2})$  for the homogeneous volume form  $\omega_{S^2}$  of  $S^2$  of total volume 1.

**Lemma 7.15.** *For any  $i \in \underline{3}$ , let  $\omega(i)$  be a homogeneous propagating form of  $(C_2(R), \tau)$ . Then*

$$I\left(\overset{\leftarrow}{\curvearrowright} S_j^1, (\omega(i))_{i \in \underline{3}}\right)$$

*does not depend on the choices of the  $\omega(i)$ , it is denoted by  $I_\theta(K_j, \tau)$ .*

PROOF: It is a corollary of Lemma 7.13.  $\square$

## 7.4 First definition of $Z$

From now on, the coefficients of our spaces of Jacobi diagrams are in  $\mathbb{R}$ . ( $\mathbb{K} = \mathbb{R}$ .) Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . Let  $L: \mathcal{L} \hookrightarrow \check{R}$  be a link embedding.

Let  $\Gamma$  be a numbered Jacobi diagram  $\Gamma$  in the space  $\mathcal{D}_n^e(\mathcal{L})$  of Definition 7.6. Let  $[\Gamma, o(\Gamma)]$  denote the class in  $\mathcal{A}_n(\mathcal{L})$  of  $\Gamma$  equipped with a vertex-orientation  $o(\Gamma)$ . Then

$$I\left(R, L, \Gamma, o(\Gamma), (\omega(i))_{i \in \underline{3n}}\right) [\Gamma, o(\Gamma)] \in \mathcal{A}_n(\mathcal{L})$$

is independent of the vertex-orientation  $o(\Gamma)$  of  $\Gamma$ . It is simply denoted by

$$I\left(R, L, \Gamma, (\omega(i))_{i \in \underline{3n}}\right) [\Gamma].$$

**Notation 7.16.** For  $\Gamma \in \mathcal{D}_n^e(\mathcal{L})$ , set

$$\zeta_\Gamma = \frac{(3n - |E(\Gamma)|)!}{(3n)! 2^{|E(\Gamma)|}}.$$

Recall the definitions of propagating forms from Section 3.3. For any  $i \in \underline{3n}$ , let  $\omega(i)$  be a propagating form of  $C_2(R)$ . For  $n \in \mathbb{N}$ , set

$$Z_n\left(\check{R}, L, (\omega(i))_{i \in \underline{3n}}\right) = \sum_{\Gamma \in \mathcal{D}_n^e(\mathcal{L})} \zeta_\Gamma I\left(R, L, \Gamma, (\omega(i))_{i \in \underline{3n}}\right) [\Gamma] \in \mathcal{A}_n(\mathcal{L}).$$

This  $Z_n(\check{R}, L, (\omega(i))_{i \in \underline{3n}})$  is the hero of this book. Let us describe some of its variants.

Let  $\mathcal{A}_n^c(\emptyset)$  denote the subspace of  $\mathcal{A}_n(\emptyset)$  generated by connected trivalent Jacobi diagrams. Set  $\mathcal{A}^c(\emptyset) = \prod_{n \in \mathbb{N}} \mathcal{A}_n^c(\emptyset)$ , and let  $p^c: \mathcal{A}(\emptyset) \rightarrow \mathcal{A}^c(\emptyset)$  be the projection that maps the empty diagram and diagrams with several connected components to 0. Let  $\mathcal{D}_n^c$  denote the subset of  $\mathcal{D}_n^e(\emptyset)$  that contains the connected diagrams of  $\mathcal{D}_n^e(\emptyset)$ . For  $n \in \mathbb{N}$ , set

$$z_n\left(\check{R}, (\omega(i))_{i \in \underline{3n}}\right) = p^c\left(Z_n\left(\check{R}, \emptyset, (\omega(i))\right)\right).$$

$$z_n\left(\check{R}, (\omega(i))\right) = \sum_{\Gamma \in \mathcal{D}_n^c} \zeta_\Gamma I\left(R, \emptyset, \Gamma, (\omega(i))_{i \in \underline{3n}}\right) [\Gamma] \in \mathcal{A}_n^c(\emptyset).$$

When all the forms  $\omega(i)$  are equal to  $\omega(1)$ ,  $Z_n(\check{R}, L, \omega(1))$  and  $z_n(\check{R}, \omega(1))$  respectively denote  $Z_n(\check{R}, L, (\omega(i))_{i \in \underline{3n}})$  and  $z_n(\check{R}, (\omega(i))_{i \in \underline{3n}})$ .

We also use the projection  $\check{p}: \mathcal{A}(\mathcal{L}) \rightarrow \check{\mathcal{A}}(\mathcal{L})$  that maps the diagrams with connected components without univalent vertices to zero and that maps the other diagrams to themselves. Set  $\check{Z}_n = \check{p} \circ Z_n$ . For example, we have

$$\check{Z}_n(\check{R}, L, \omega(1)) = \check{p}\left(Z_n(\check{R}, L, \omega(1))\right).$$

We also remove the subscript  $n$  to denote the collection (or the sum) of the  $Z_n$  for  $n \in \mathbb{N}$ . For example, we write

$$\check{Z}(\check{R}, L, \omega(1)) = \left( \check{Z}_n(\check{R}, L, \omega(1)) \right)_{n \in \mathbb{N}} = \sum_{n \in \mathbb{N}} \check{Z}_n(\check{R}, L, \omega(1)) \in \check{\mathcal{A}}(\mathcal{L}).$$

As a first example, let us prove the following proposition.

**Proposition 7.17.** *Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ , then for any triple  $(\omega(i))_{i \in \underline{3}}$  of propagating forms of  $(C_2(R), \tau)$ , we have*

$$Z_1\left(\check{R}, \emptyset, (\omega(i))_{i \in \underline{3}}\right) = z_1\left(\check{R}, (\omega(i))\right) = \frac{\Theta(R, \tau)}{12} [\ominus]$$

in  $\mathcal{A}_1(\emptyset) = \mathcal{A}_1(\emptyset; \mathbb{R}) = \mathbb{R}[\ominus]$ .

**PROOF:** The diagram  $\ominus$  is the only trivalent diagram with 2 vertices without looped edges, and it is easy to check that  $\mathcal{A}_1(\emptyset) = \mathbb{R}[\ominus]$ . Each of the three edges goes from one vertex to the other. There are 4 elements in  $\mathcal{D}_1^e(\emptyset)$ , depending on whether the orientations of Edge 2 and Edge 3 coincide with the orientation of Edge 1. See Example 7.7. When the three edges start from the same vertex, call the corresponding element  $\theta_{++}$ , and order the vertices so that the vertex from which the edges start is first. Recall from Example 7.4 that the vertex-orientation  $o(\theta_{++})$  of  $\ominus$  induced by the picture  $\ominus$  and the edge-orientation of  $H(\theta_{++})$  orient  $C_2(R)$  as  $\check{R}^2 \setminus \Delta(\check{R}^2)$ , as in Corollary 7.2. Then, according to Theorem 4.1, for any triple  $(\omega(i))_{i \in \underline{3}}$  of propagating forms of  $(C_2(R), \tau)$ , we have

$$I\left(\theta_{++}, o(\theta_{++}), (\omega(i))_{i \in \underline{3}}\right) = \Theta(R, \tau).$$

Reversing the edge-orientation of Edge 2 transforms  $\theta_{++}$  to  $\theta_{-+}$  and  $\omega(2)$  to  $\iota^*(\omega(2))$ . It changes the edge-orientation of  $H(\theta_{\pm+})$ . According to Lemma 3.16, the form  $(-\iota^*(\omega(2)))$  is a propagating form of  $(C_2(R), \tau)$ . Therefore, we have

$$I\left(\theta_{++}, (\omega(i))_{i \in \underline{3}}\right) [\theta_{++}] = I\left(\theta_{-+}, (\omega(i))_{i \in \underline{3}}\right) [\theta_{-+}].$$

Similarly, the four graphs of  $\mathcal{D}_1^e(\emptyset)$  contribute in the same way to

$$Z_1\left(\check{R}, \emptyset, (\omega(i))_{i \in \underline{3}}\right) = \frac{4}{3!2^3} \Theta(R, \tau) [\ominus].$$

□

**Examples 7.18.** According to the computations in Example 7.10, we have

$$Z_n(\mathbb{R}^3, \emptyset, p_{S^2}^*(\omega_{S^2})) = z_n(\mathbb{R}^3, p_{S^2}^*(\omega_{S^2})) = 0$$

for  $n > 0$ , and  $Z_0(\mathbb{R}^3, \emptyset, p_{S^2}^*(\omega_{S^2})) = [\emptyset]$ , while  $z_0(\mathbb{R}^3, p_{S^2}^*(\omega_{S^2})) = 0$ .

For the embedding  $O$  of the trivial knot in  $\mathbb{R}^3$  of Example 7.11, we have

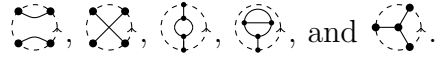
$$Z_0(\mathbb{R}^3, O, p_{S^2}^*(\omega_{S^2})) = 1 = [\emptyset].$$

Since  $I_\theta(O, \tau_s) = 0$ , we have  $Z_1(\mathbb{R}^3, O, p_{S^2}^*(\omega_{S^2})) = 0$ .

Let us now prove

$$Z_2(\mathbb{R}^3, O, p_{S^2}^*(\omega_{S^2})) = \frac{1}{24} \left[ \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right].$$

Note  $\iota_{S^2}^*(\omega_{S^2}) = -\omega_{S^2}$ , where  $\iota_{S^2}$  is the antipodal map of  $S^2$ . So, reversing the orientation of an edge does not change  $I(S^3, O, \Gamma, p_{S^2}^*(\omega_{S^2})) [\Gamma]$ , for a degree 2 numbered Jacobi diagram  $\Gamma$ , since it changes both the orientation of  $\check{C}(S^3, 0; \Gamma)$  and the sign of the form to be integrated. Thus,  $I(S^3, O, \Gamma, p_{S^2}^*(\omega_{S^2})) [\Gamma]$  depends only on the underlying Jacobi diagram. The degree 2 Jacobi diagrams all components of which have univalent vertices and without looped edges are



As proved in Example 7.11, the diagrams do not contribute to  $Z_2(\mathbb{R}^3, O, p_{S^2}^*(\omega_{S^2}))$ .

Since Lemma 6.26 implies  $[\text{circle with two vertices}] = 0$ , the diagram does not contribute either. Lemma 7.12 implies

$$I(S^3, O, \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}, (p_{S^2}^*(\omega_{S^2}))) \left[ \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] = \frac{1}{8} \left[ \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right].$$

When  $\Gamma = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}$ , we have  $\zeta_\Gamma = \frac{3!}{6!2^3}$ , and there are  $\frac{1}{3} \frac{6!2^3}{3!}$  numbered graphs of  $\mathcal{D}_2^\epsilon(S^1)$  that are isomorphic to  $\Gamma$  as a Jacobi diagram. This concludes the computation of  $Z_2(\mathbb{R}^3, O, p_{S^2}^*(\omega_{S^2}))$ .

See also Proposition 7.25 and Example 7.28. Enore Guadagnini, Maurizio Martellini, and Mihail Mintchev have performed alternative computations of similar quantities in [GMM90].

The following theorem is proved in Chapter 9. See Section 9.1, and Corollary 9.4 and Lemma 9.1, in particular.

**Theorem 7.19.** *Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . Let  $n \in \mathbb{N}$ . For any  $i \in \underline{6n}$ , let  $\omega(i)$  be a propagating form of  $C_2(R)$ . Then  $z_{2n}(\check{R}, (\omega(i))_{i \in \underline{6n}})$  is independent of the chosen  $\omega(i)$ , it depends only on the diffeomorphism class of  $R$ . It is denoted by  $z_{2n}(R)$ .*

For odd  $n$ ,  $z_n(\check{R}, (\omega(i))_{i \in \underline{3n}})$  depends on the chosen  $\omega(i)$ . Theorem 7.20 explains how to deal with this dependence when the  $\omega(i)$  are homogeneous propagating forms of  $(C_2(R), \tau)$ . Tadayuki Watanabe [Wat18a] and Tatsuro Shimizu [Shi16] have studied alternative compensations for this dependence.

We are going to prove the following theorem in the next chapters. The proof will be concluded in the end of Section 10.5.

**Theorem 7.20.** *Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . Let  $\mathcal{L} = \sqcup_{j=1}^k S_j^1$  be a disjoint union of  $k$  circles. Let  $L: \mathcal{L} \hookrightarrow \check{R}$  be an embedding. Let  $n \in \mathbb{N}$ . For any  $i \in \underline{3n}$ , let  $\omega(i)$  be a homogeneous propagating form of  $(C_2(R), \tau)$ .*

*Then  $Z_n(\check{R}, L, (\omega(i))_{i \in \underline{3n}})$  is independent of the chosen  $\omega(i)$ . It depends only on the diffeomorphism class of  $(R, L)$ , on  $p_1(\tau)$ , and on the  $I_\theta(K_j, \tau)$ , for the components  $K_j$  of  $L$ . It is denoted by  $Z_n(\check{R}, L, \tau)$ . More precisely, set*

$$Z(\check{R}, L, \tau) = (Z_n(\check{R}, L, \tau))_{n \in \mathbb{N}} \in \mathcal{A}(\sqcup_{j=1}^k S_j^1).$$

*There exist two constants  $\alpha \in \check{\mathcal{A}}(S^1; \mathbb{R})$  and  $\beta \in \mathcal{A}(\emptyset; \mathbb{R})$ , called anomalies,<sup>1</sup> such that  $\alpha_{2n} = 0$  and  $\beta_{2n} = 0$  for any  $n \in \mathbb{N}$ , and*

$$\exp\left(-\frac{1}{4}p_1(\tau)\beta\right) \prod_{j=1}^k \left(\exp(-I_\theta(K_j, \tau)\alpha) \#_j\right) Z(\check{R}, L, \tau) = \mathcal{Z}(R, L)$$

*depends only on the diffeomorphism class of  $(R, L)$ . Here  $\exp(-I_\theta(K_j, \tau)\alpha)$  acts on  $Z(\check{R}, L, \tau)$ , on the copy  $S_j^1$  of  $S^1$  as indicated by the subscript  $j$ , as in Proposition 6.22.*

Recall Notation 7.16. If  $\check{R} = \mathbb{R}^3$ , then the projection  $\check{\mathcal{Z}}(S^3, L)$  of  $Z(S^3, L)$  on  $\check{\mathcal{A}}(\sqcup_{j=1}^k S_j^1)$  is a *universal finite type invariant of links in  $\mathbb{R}^3$* , i.e., the projection of  $\check{\mathcal{Z}}_n$  onto  $\overline{\mathcal{A}}_n/(1T)$  satisfies the properties stated for  $\check{\mathcal{Z}}_n$  in Theorem 6.9. This result, due to Daniel Altschüler and Laurent Freidel [AF97], is proved in Section 17.6. See Theorem 17.30.

**Notation 7.21.** Let  $\check{\mathcal{Z}}(R, L)$  denote the projection of  $\mathcal{Z}(R, L)$  on  $\check{\mathcal{A}}(\sqcup_{j=1}^k S_j^1)$ , and set

$$\mathcal{Z}(R) = \mathcal{Z}(R, \emptyset).$$

---

<sup>1</sup>The anomaly  $\alpha$  is defined in Section 10.3, and  $\beta$  is defined in Definition 10.5.

The invariant  $\mathcal{Z}$  of rational homology 3-spheres is the Kontsevich configuration space invariant studied by Greg Kuperberg and Dylan Thurston in [KT99], and described in [Les04a] as  $Z_{KKT}$ .

We will see in Chapter 11 that the anomalies  $\alpha$  and  $\beta$  are *rational*, i.e., that  $\alpha \in \check{\mathcal{A}}(S^1; \mathbb{Q})$  (in Proposition 11.1) and  $\beta \in \mathcal{A}(\emptyset; \mathbb{Q})$  (in Theorem 11.8).

**Examples 7.22.** According to Example 7.18, we have  $Z(\mathbb{R}^3, \emptyset, \tau_s) = 1 = [\emptyset]$ . Since  $p_1(\tau_s) = 0$ , we also have  $\mathcal{Z}(S^3) = 1$ .

For the embedding  $O$  of the trivial knot in  $\mathbb{R}^3$  of Example 7.11, we have

$$I_\theta(O, \tau_s) = 0.$$

So, according to Example 7.18, we also have  $\mathcal{Z}_0(S^3, O) = 1$ ,  $\mathcal{Z}_1(S^3, O) = 0$ , and

$$\mathcal{Z}_2(S^3, O) = \frac{1}{24} \left[ \begin{array}{c} \bullet \\ \circlearrowleft \\ \bullet \end{array} \right].$$

**Definition 7.23.** Let  $\Gamma$  be a Jacobi diagram on an oriented one-manifold  $\mathcal{L}$  as in Definition 6.13. Let  $j: U(\Gamma) \hookrightarrow \mathcal{L}$  be a  $\Gamma$ -compatible injection from its set  $U(\Gamma)$  of univalent vertices to its support  $\mathcal{L}$ . An *automorphism* of  $\Gamma$  is a permutation of the set  $H(\Gamma)$  of half-edges of  $\Gamma$  that maps a pair of half-edges of an edge to another such and a triple of half-edges that contain a vertex to another such, and such that, for the induced bijection  $b$  of  $U(\Gamma)$ ,  $j \circ b$  is isotopic to  $j$ . (So the automorphisms preserve the components of the univalent vertices. They also preserve their linear order on intervals and their cyclic order on circles.) Let  $\text{Aut}(\Gamma)$  denote the set of automorphisms of  $\Gamma$ .

**Examples 7.24.** There are six automorphisms of  $\ominus$  that fix each vertex. They correspond to the permutations of the edges. The cardinality  $|\text{Aut}(\ominus)|$  of  $\text{Aut}(\ominus)$  is 12, and we have

$$\left| \text{Aut}(\hat{\mathfrak{G}}) \right| = 1 \text{ and } \left| \text{Aut}(\begin{array}{c} \bullet \\ \circlearrowleft \\ \bullet \end{array}) \right| = 3$$

Recall from Notation 7.16 that  $\zeta_\Gamma = \frac{(3n - |E(\Gamma)|)!}{(3n)! 2^{|E(\Gamma)|}}$ . Also recall that Lemma 3.16 ensures the existence of antisymmetric homogeneous propagating forms  $\omega$  of  $(C_2(R), \tau)$ .

**Proposition 7.25.** Let  $\mathcal{D}_n^u(\mathcal{L})$  denote the set of unnumbered, unoriented degree  $n$  Jacobi diagrams on  $\mathcal{L}$  without looped edges. Under the assumptions of Theorem 7.20, let  $\omega$  be an antisymmetric homogeneous propagating form of  $C_2(R)$ . Then we have

$$Z_n(\check{R}, L, \omega) = \sum_{\Gamma \in \mathcal{D}_n^u(\mathcal{L})} \frac{1}{|\text{Aut}(\Gamma)|} I(R, L, \Gamma, (\omega)_{i \in \underline{3n}}) [\Gamma].$$

PROOF: Set  $\omega(i) = \omega$  for any  $i$ . For a *numbered* graph  $\Gamma$  (i.e., a graph equipped with the structure described in Definition 7.6), there are  $\frac{1}{\zeta_\Gamma}$  ways of *renumbering* it (i.e., changing this structure), and  $|\text{Aut}(\Gamma)|$  of them will produce the same numbered graph. Therefore, we have

$$\sum_{\Gamma \in \mathcal{D}_n^e(\mathcal{L})} \zeta_\Gamma I\left(R, L, \Gamma, (\omega(i))_{i \in \underline{3n}}\right) [\Gamma] = \sum_{\Gamma \in \mathcal{D}_n^u(\mathcal{L})} \frac{1}{|\text{Aut}(\Gamma)|} I(R, L, \Gamma, (\omega)_{i \in \underline{3n}}) [\Gamma].$$

□

**Remark 7.26.** Let  $\omega$  be an antisymmetric homogeneous propagating form of  $(C_2(R), \tau)$ . The homogeneous definition of  $Z_n(\check{R}, L, \tau) = Z_n(\check{R}, L, \omega)$  above makes clear that  $Z_n(\check{R}, L, \tau)$  is a “measure” of graph configurations, where a graph configuration is an embedding of the set of vertices of a uni-trivalent graph into  $\check{R}$ , which maps univalent vertices to  $L(\mathcal{L})$ . The embedded vertices are connected by a set of abstract solid edges, which represent the measuring form. The factor  $\frac{1}{|\text{Aut}(\Gamma)|}$  ensures that every such configuration of an unnumbered, unoriented graph is “measured” once.

**Lemma 7.27.** *Recall Notation 7.21. Under the assumptions of Theorem 7.20, we have*

$$Z(\check{R}, L, \omega(1)) = Z(\check{R}, \emptyset, \omega(1)) \check{Z}(\check{R}, L, \omega(1)).$$

*In particular, Theorem 7.20 will imply*

$$\mathcal{Z}(R, L) = \mathcal{Z}(R) \check{\mathcal{Z}}(R, L).$$

PROOF: Remark 7.26 shows that the coefficients are correct. □

**Example 7.28.** For the embedding  $O$  of the trivial knot in  $\mathbb{R}^3$  of Examples 7.11, according to the computations performed in the series of examples 7.18, and with the notation of Theorem 7.20 and Notation 7.16, as in Example 7.22, we have

$$\begin{aligned} \mathcal{Z}_2(S^3, O) &= Z_2(\mathbb{R}^3, O, p_{S^2}^*(\omega_{S^2})) \\ &= \frac{1}{3} I\left(S^3, O, \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}, p_{S^2}^*(\omega_{S^2})\right) \left[ \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \\ &= \frac{1}{24} \left[ \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right]. \end{aligned}$$

We end this section by stating Theorems 7.30 and 7.32 about the numerical invariants obtained from  $\check{Z}_n$  by applying the Conway weight system  $w_C$  of Example 6.11. For  $n = 2$ , we get the invariant  $w_2$  discussed in Subsection 1.2.5.



Since  $w_C$  is multiplicative, and since  $w_C$  sends elements of odd degree to zero,  $w_C$  sends  $\exp(-I_\theta(K_j, \tau)\alpha)$  to the unit of  $\check{\mathcal{A}}(S^1)$ . So, with the notation of Theorem 7.20, we have  $w_C(\check{Z}_n(R, L)) = w_C(\check{Z}_n(\check{R}, L, \tau))$  for any  $n \in \mathbb{N}$ , and we can forget the anomalies for  $w_C \circ \check{Z}$ . Theorem 7.30 will tell us that we can furthermore omit the homogeneity assumptions on the forms for  $w_C \circ \check{Z}$  and reduce our averaging process. We will average only over some degree  $n$  graphs whose edges are numbered in  $\underline{3n-2}$ , or even in  $\underline{3}$  when  $n = 2$ , in some cases.

(Since  $\check{p}: \mathcal{A}(\mathcal{L}) \rightarrow \check{\mathcal{A}}(\mathcal{L})$  maps all graphs with less than 2 univalent vertices to zero (thanks to Lemma 6.26), the graphs that contribute to  $\check{Z}_n(\check{R}, L, \omega(1))$  have at most  $3n - 2$  edges. So it is natural to average only over these graphs.)

**Notation 7.29.** For a finite set  $A$ , let  $\mathcal{D}_{n,A}^e(\mathcal{L})$  denote the set of  $A$ -numbered degree  $n$  Jacobi diagrams with support  $\mathcal{L}$  without looped edges, as in Definition 7.6. (These diagrams have at most  $|A|$  edges.) The coefficient  $\zeta_\Gamma$  associated to a diagram  $\Gamma \in \mathcal{D}_{n,A}^e(\mathcal{L})$  is

$$\zeta_\Gamma = \frac{(|A| - |E(\Gamma)|)!}{|A|! 2^{|E(\Gamma)|}}.$$

For any  $i \in A$ , let  $\omega(i)$  be a propagating form of  $C_2(R)$ . Set

$$Z_{n,A}(\check{R}, L, (\omega(i))_{i \in A}) = \sum_{\Gamma \in \mathcal{D}_{n,A}^e(\mathcal{L})} \zeta_\Gamma I(\check{R}, L, \Gamma, (\omega(i))_{i \in A}) [\Gamma] \in \mathcal{A}_n(\mathcal{L}).$$

For  $m \in \mathbb{N}$ , set  $\mathcal{D}_{n,m}^e(\mathcal{L}) = \mathcal{D}_{n,\underline{m}}^e(\mathcal{L})$  and

$$Z_{n,m}(\check{R}, L, (\omega(i))_{i \in \underline{m}}) = Z_{n,\underline{m}}(\check{R}, L, (\omega(i))_{i \in \underline{m}}).$$

Note that  $\mathcal{D}_n^e = \mathcal{D}_{n,3n}^e$  and that  $Z_{n,3n} = Z_n$ . With the projection  $\check{p}: \mathcal{A}_n(S^1) \rightarrow \check{\mathcal{A}}_n(S^1)$  of Notation 7.16, set

$$\check{Z}_{n,m}(\check{R}, K, (\omega(i))_{i \in \underline{m}}) = \check{p}\left(Z_{n,m}(\check{R}, K, (\omega(i))_{i \in \underline{m}})\right).$$

**Theorem 7.30.** *Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . Let  $K: S^1 \hookrightarrow \check{R}$  be an embedding. Let  $n \in \mathbb{N}$ . For any  $i \in \underline{3n-2}$ , let  $\omega(i)$  be a propagating form of  $C_2(R)$ .*

*Let  $p^c$  be the projection given by Corollary 6.40 from  $\check{\mathcal{A}}(S^1)$  to the space  $\check{\mathcal{A}}^c(S^1)$  of its primitive elements. Recall the linear form  $w_C: (\check{\mathcal{A}}_n(S^1) = \overline{\mathcal{A}}_n) \rightarrow \mathbb{R}$  induced by the Conway weight system of Example 6.11.*

Then  $w_C(\check{Z}_{n,3n-2}(\check{R}, K, (\omega(i))_{i \in \underline{3n-2}}))$  and

$$w_C \circ p^c(\check{Z}_{n,3n-2}(\check{R}, K, (\omega(i))_{i \in \underline{3n-2}}))$$

are independent of the chosen  $\omega(i)$ . They depend only on the diffeomorphism class of  $(\check{R}, K)$ . They are respectively denoted by  $w_C \check{Z}_n(R, K)$  and

$$w_C p^c \check{Z}_n(R, K).$$

At the end of Subsection 9.1, the proof of Theorem 7.30 will be reduced to the proof of Proposition 9.7, which is proved in Subsection 9.3.

**Remark 7.31.** Assuming both Theorems 7.30 and 7.20, there is no notation conflict between them. Indeed, with the notation of Theorem 7.20, we have

$$\begin{aligned} w_C(\check{Z}_n(R, L)) &= w_C(\check{Z}_n(\check{R}, L, \tau)) = w_C(\check{Z}_n(\check{R}, L, \omega)) \\ &= \sum_{\Gamma \in \mathcal{D}_n^u(L)} \frac{1}{|\text{Aut}(\Gamma)|} I(R, L, \Gamma, (\omega)_{i \in \underline{3n}}) w_C([\Gamma]) \end{aligned}$$

for any homogeneous propagating form  $\omega$  of  $(C_2(R), \tau)$  and for any  $n \in \mathbb{N}$ , thanks to Proposition 7.25. When  $\omega(i) = \omega$  for all  $i$ , this is also the expression of

$$w_C(\check{Z}_{n,3n-2}(\check{R}, K, (\omega(i))_{i \in \underline{3n-2}})).$$

We prove the following theorem at the end of Section 9.3. It implies Proposition 1.7.

**Theorem 7.32.** *Let  $K: S^1 \hookrightarrow \mathbb{R}^3$  be a knot embedding. For  $i \in \underline{3}$ , let  $\omega_{S^2}(i)$  be a volume-one form of  $S^2$ , then  $w_C(\check{Z}_{2,3}(\mathbb{R}^3, K, (p_{S^2}^*(\omega_{S^2}(i)))_{i \in \underline{3}}))$  is independent of the chosen  $\omega_{S^2}(i)$ . It is an isotopy invariant of  $K$ , which coincides with  $w_C \check{Z}_2(S^3, K)$  and  $w_C p^c \check{Z}_2(S^3, K)$ .*

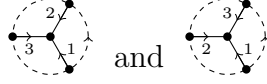
**PROOF OF PROPOSITION 1.7 ASSUMING THEOREM 7.32:** The only degree 2 Jacobi diagrams on  $S^1$  without looped edges, with at most three edges, and with no trivalent component are  $\mathfrak{X}$ ,  $\mathfrak{Y}$ , and  $\mathfrak{Z}$ . We have  $w_C([\mathfrak{X}]) = 1$ ,  $w_C([\mathfrak{Y}]) = 0$ ,  $w_C([\mathfrak{Z}]) = w_C([\mathfrak{Y}]) - w_C([\mathfrak{X}]) = -1$ . There are six elements of  $\mathcal{D}_{2,3}^e(S^1)$  isomorphic to  $\mathfrak{X}$ , one for each permutation  $\sigma$  of  $\underline{3}$ . They may be drawn as

$$\Gamma(\sigma) = \begin{array}{c} \text{---} \circ \text{---} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \circ \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \circ \text{---} \text{---} \end{array}$$

and we have  $\zeta_{\Gamma(\sigma)} = \frac{1}{3! \times 4}$ . For such a graph  $\Gamma(\sigma)$ , we also have

$$\begin{aligned} I\left(S^3, K, \Gamma(\sigma), \left(p_{S^2}^*(\omega_{S^2}(i))\right)_{i \in \underline{3}}\right) \\ = \int_{(S^2)^3} \deg(\sigma_* (1_{S^2} \times G_{\mathfrak{X}})) \wedge_{i=1}^3 p_i^*(\omega_{S^2}(i)). \end{aligned}$$

There are 16 elements of  $\mathcal{D}_{2,3}^e(S^1)$  isomorphic to  $\check{\mathcal{C}}$ . They are obtained from the two diagrams



by reversing the directions of some edges. For these diagrams  $\zeta_\Gamma = \frac{1}{3! \times 8}$ . Theorem 7.32 implies that for any embedding  $K: S^1 \hookrightarrow \mathbb{R}^3$ , we have

$$w_C \check{Z}_2(S^3, K) = \int_{(S^2)^3} \tilde{w}_2(K) \wedge_{i=1}^3 p_i^*(\omega_{S^2}(i))$$

for the locally constant degree map  $\tilde{w}_2(K)$  of Proposition 1.7 defined on an open dense subset of  $(S^2)^3$ . Any point  $(X_1, X_2, X_3)$  of this open dense subset of regular values of  $(S^2)^3$ , has an open connected neighborhood  $\prod_{i=1}^3 W_i$  of regular values, and there are volume-one 2-forms  $\omega_{S^2}(i)$  supported on  $W_i$ . For such forms,  $w_C \check{Z}_2(S^3, K) = \tilde{w}_2(K)(X_1, X_2, X_3)$ . So Theorem 7.32 implies that  $\tilde{w}_2(K)$  is constant.  $\square$

### 7.5 Straight links

A one-cycle  $c$  of  $S^2$  is *algebraically trivial* if, for any two points  $x$  and  $y$  outside its support, the algebraic intersection of an arc from  $x$  to  $y$  transverse to  $c$  with  $c$  is zero, or equivalently if the integral of any one-form of  $S^2$  along  $c$  is zero.

Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . Let  $K: S^1 \hookrightarrow \check{R}$  be a knot embedding. Recall the notation from Proposition 3.7 and Section 7.3. Set  $C_K = C(K; \check{\mathcal{C}} \rightarrow S^1)$ . We have  $C_K \subset C_2(R)$ .

**Lemma 7.33.** *If  $p_\tau(\partial C_K)$  is algebraically trivial, then for any propagating chain  $F$  of  $(C_2(R), \tau)$  transverse to  $C_K$  and for any propagating form  $\omega_p$  of  $(C_2(R), \tau)$ , we have*

$$\int_{C_K} \omega_p = \langle C_K, F \rangle_{C_2(R)} = I_\theta(K, \tau),$$

with respect to the definition of  $I_\theta(K, \tau)$  in Lemma 7.15. In particular, we have  $I_\theta(K, \tau) \in \mathbb{Q}$ , and  $I_\theta(K, \tau)$  is an integer when  $R$  is an integral homology sphere.

**PROOF:** According to Lemma 7.13, changing the propagating form  $\omega_p$  to  $\omega'_p$  adds some  $\int_{\partial C_K} p_\tau^*(\eta_S) = \int_{p_\tau(\partial C_K)} \eta_S$  for some one-form on  $S^2$ . Then by definition,  $\int_{C_K} \omega_p$  is independent of the propagating form  $\omega_p$  of  $(C_2(R), \tau)$ .

For a propagating chain  $F$  of  $(C_2(R), \tau)$  transverse to  $C_K$ , one can choose a propagating form dual to  $F$  and supported near  $F$  such that  $\int_{C_K} \omega_p = \langle C_K, F \rangle_{C_2(R)}$ . (See the end of Section B.2 and Lemma B.4 in particular, for details.) The rationality of  $I_\theta(K, \tau)$  follows from the rationality of  $F$ . Since  $F$  can be chosen to be an integral chain when  $R$  is a  $\mathbb{Z}$ -sphere, we get  $I_\theta(K, \tau) \in \mathbb{Z}$  in this case.  $\square$

**Remark 7.34.** One could have proved that  $\langle C_K, F \rangle_{C_2(R)}$  is independent of the chosen propagating chain  $F$  of  $(C_2(R), \tau)$  transverse to  $C_K$  directly, as follows. Let  $F$  and  $F'$  be two such propagating chains with respective boundaries  $p_\tau^{-1}(a)$  and  $p_\tau^{-1}(a')$ , where  $a$  and  $a'$  are in  $S^2$ . Let  $[a, a']$  be a path from  $a$  to  $a'$  in  $S^2$  transverse to  $p_\tau(\partial C_K)$ . There exists a rational chain  $W$  of  $C_2(R)$  transverse to  $C_K$  and to  $\partial C_2(R)$ , whose boundary is  $F' - F - p_\tau^{-1}([a, a'])$ . Then  $\partial C_K \cap W \subset C_K \cap \partial W$ , and  $\partial(C_K \cap W) = \pm C_K \cap \partial W$ . So we have

$$\langle C_K, \partial W \rangle_{C_2(R)} = \langle C_K, F' - F - p_\tau^{-1}([a, a']) \rangle_{C_2(R)} = 0.$$

So we get

$$\begin{aligned} \langle C_K, F' \rangle_{C_2(R)} - \langle C_K, F \rangle_{C_2(R)} &= \langle C_K, p_\tau^{-1}([a, a']) \rangle_{C_2(R)} \\ &= \langle \partial C_K, p_\tau^{-1}([a, a']) \rangle_{\partial C_2(R)} \\ &= \langle p_\tau(\partial C_K), [a, a'] \rangle_{S^2} = 0. \end{aligned}$$

**Lemma 7.35.** *Assume that  $p_\tau(\partial C_K)$  is algebraically trivial. Let  $Y \in S^2 \setminus p_\tau(\partial C_K)$ . Let  $Z: K \rightarrow S^2$  map  $k \in K$  to the vector  $Z(k)$  of  $S^2$  orthogonal to  $p_\tau(T_k K)$  in the half great circle of  $S^2$  that contains  $p_\tau(T_k K)$ ,  $Y$ , and  $p_\tau(-T_k K)$ . Define the parallel  $K_{\parallel, \tau, Y}$  by pushing a point  $k$  of  $K$  in the direction  $\tau(Z(k))$ . Then we have*

$$I_\theta(K, \tau) = lk(K, K_{\parallel, \tau, Y}).$$

PROOF: Thanks to Lemma 3.12,  $lk(K, K_{\parallel, \tau, Y})$  is the evaluation of any propagator of  $C_2(R)$  on  $K \times K_{\parallel, \tau, Y}$ . Let  $\omega_p$  be a propagating form of  $C_2(R)$  (as in Definition 3.11), which may be expressed as  $p_\tau^*(\omega_{-Y})$  on  $UN(K)$ , for a 2-form  $\omega_{-Y}$  of  $S^2$  supported in a geometric disk in  $S^2 \setminus p_\tau(\partial C_K)$  centered at  $(-Y)$ . Observe that the intersection of such a disk with the half great circle of  $S^2$  that contains  $p_\tau(T_k K)$ ,  $Y$ , and  $p_\tau(-T_k K)$  is empty for any  $k \in K$ .

Compute  $lk(K, K_{\parallel, \tau, Y})$  as the limit of  $I(K \overset{\curvearrowright}{\times} K_{\parallel, \tau, Y}, \omega_p)$ , when  $K_{\parallel, \tau, Y}$  tends to  $K$ . The configuration space  $K \times K_{\parallel, \tau, Y}$  is a torus of  $C_2(R)$ . When  $K_{\parallel, \tau, Y}$  tends to  $K$ , this torus tends to the union of the annulus  $C(K; \overset{\curvearrowright}{\times} K)$  and an annulus  $K \times_{\tau, Y} J$  contained in  $UR|_K$ , which fibers over  $K$  and whose

fiber over  $k \in K$  contains all the limit directions from  $k$  to a close point on  $K_{\parallel, \tau, Y}$ . This fiber is the half great circle of  $UR|_k$  that  $p_\tau$  maps to the half great circle of  $S^2$  that contains  $p_\tau(T_k K)$ ,  $Y$ , and  $p_\tau(-T_k K)$ . Thus  $K \times K_{\parallel, \tau, Y}$  is homologous to the torus

$$T = C_K \cup (K \times_{\tau, Y} J).$$

The integral of  $\omega_p$  on  $K \times_{\tau, Y} J$  is the integral of  $\omega_{-Y}$  along  $p_\tau(K \times_{\tau, Y} J)$ . It is zero since  $p_\tau(K \times_{\tau, Y} J)$  does not meet the support of  $\omega_{-Y}$ . Therefore, we have  $lk(K, K_{\parallel, \tau, Y}) = \int_{C_K} \omega_p = I_\theta(K, \tau)$ , thanks to Lemma 7.33.  $\square$

An isotopy class of parallels of a knot is called a *framing* (or a *parallelization*) of a knot.

**Corollary 7.36.** *A knot embedding  $K$  such that  $p_\tau(\partial C_K)$  is algebraically trivial, with respect to a parallelization  $\tau$ , has a canonical framing induced by  $\tau$ , which is the framing induced by a parallel  $K_{\parallel, \tau, Y}$  for an arbitrary  $Y \in S^2 \setminus p_\tau(\partial C_K)$ . For such a knot embedding  $K$ , for any propagating chain  $F$  of  $(C_2(R), \tau)$  transverse to  $C_K$ , and for any propagating form  $\omega_p$  of  $(C_2(R), \tau)$ , we have*

$$\int_{C_K} \omega_p = \langle C_K, F \rangle_{C_2(R)} = I_\theta(K, \tau) = lk(K, K_{\parallel, \tau, Y}).$$

PROOF: Since the linking number  $lk(K, K_{\parallel, \tau, Y})$  determines the framing, the corollary is a direct consequence of Lemmas 7.33 and 7.35.  $\square$

**Definition 7.37.** A knot embedding  $K: S^1 \hookrightarrow \check{R}$  is *straight* with respect to  $\tau$  if the curve  $p_\tau(U^+ K)$  of  $S^2$  is algebraically trivial (with the notation from Proposition 3.7 and Section 7.3). A link embedding is *straight* with respect to  $\tau$  if all its components are.

Straight knot embeddings and almost-horizontal knot embeddings in  $\mathbb{R}^3$  (defined before Lemma 1.6) are examples of knot embeddings  $K$  such that  $p_\tau(\partial C_K)$  is algebraically trivial. Therefore, Lemma 1.6 is a particular case of the above corollary. As a second corollary, we get the following lemma.

**Lemma 7.38.** *For any knot embedding  $K$  in  $\check{R}$ , there exists an asymptotically standard parallelization  $\tilde{\tau}$  homotopic to  $\tau$  such that the first vector  $\tilde{\tau}(\cdot; (1, 0, 0))$  of  $\tilde{\tau}$  is tangent to  $K$  and orients  $K$ . In this case, let  $K_{\tilde{\tau}}$  be the parallel of  $K$  obtained by pushing  $K$  in the direction of the second vector  $\tilde{\tau}(\cdot; (0, 1, 0))$  of  $\tilde{\tau}$ . Then we have*

$$I_\theta(K, \tilde{\tau}) = lk(K, K_{\tilde{\tau}}).$$

PROOF: In order to obtain  $\tilde{\tau}$ , it suffices to perform a homotopy of  $\tau$  around the image of  $K$  so that the first vector of  $\tilde{\tau}$  becomes tangent to  $K$  along  $K$ . Thus,  $K$  is straight with respect to  $(R, \tilde{\tau})$ . Apply Corollary 7.36.  $\square$

**Lemma 7.39.** *Let  $K_0: \{0\} \times S^1 \rightarrow \check{R}$  be a straight embedding with respect to  $\tau$ . Let  $K: [0, 1] \times S^1 \rightarrow \check{R}$  be an embedding such that its restriction  $K_1: \{1\} \times S^1 \rightarrow \check{R}$  is straight with respect to an asymptotically standard parallelization  $\tau_1$  homotopic to  $\tau$ . Then  $(I_\theta(K_1, \tau_1) - I_\theta(K_0, \tau))$  is an even integer. Furthermore, for any real number  $x$  congruent to  $I_\theta(K_0, \tau)$  modulo 2, there exists a straight embedding  $K_1$  isotopic to  $K_0$  such that  $I_\theta(K_1, \tau) = x$ .*

PROOF: Let  $H: t \mapsto \tau_t$  be a smooth homotopy from  $\tau = \tau_0$  to  $\tau_1$ . Let  $p_H: [0, 1] \times \partial C_2(R) \rightarrow S^2$  be the smooth map that restricts to  $\{t\} \times \partial C_2(R)$  as  $p_{\tau_t}$ . There is a closed 2-form  $\omega$  on  $[0, 1] \times C_2(R)$  that restricts to  $[0, 1] \times \partial C_2(R)$  as  $p_H^*(\omega_{S^2})$ . (Such a form may be obtained by modifying  $p_{C_2(R)}^* p_\tau^*(\omega_{S^2})$  in a collar neighborhood of  $U\check{R}$  using the homotopy  $H$ .) Then the integral of  $\omega$  over

$$\partial(\cup_{t \in [0, 1]} C(K; \overleftrightarrow{K}_t)) = C(K; \overleftrightarrow{K}_1) - C(K; \overleftrightarrow{K}_0) - \cup_{t \in [0, 1]} \partial C(K; \overleftrightarrow{K}_t)$$

vanishes. So we have

$$I_\theta(K_1, \tau_1) - I_\theta(K_0, \tau) = \int_{\cup_{t \in [0, 1]} \partial C(K; \overleftrightarrow{K}_t)} \omega.$$

This is the area in  $S^2$  of the integral cycle  $\cup_{t \in [0, 1]} p_{\tau_t}(\partial C(K; \overleftrightarrow{K}_t))$ . This cycle is the union of the two integral cycles

$$\cup_{t \in [0, 1]} p_{\tau_t}(U^+(K_t)) \text{ and } \cup_{t \in [0, 1]} p_{\tau_t}(-U^-(K_t))$$

which have the same integral area. So  $(I_\theta(K_1, \tau_1) - I_\theta(K_0, \tau))$  is an even integer.

Adding two small almost-horizontal kinks in a standard ball as in the end of Subsection 1.2.4 which turn in opposite direction and contribute with the same crossing sign like  $(\nearrow \text{ and } \searrow)$  or  $(\searrow \text{ and } \nearrow)$  preserves straightness and adds  $\pm 2$  to  $I_\theta$ .  $\square$

## 7.6 Second definition of $Z$

Let us state another version of Theorem 7.20 using straight links instead of homogeneous propagating forms. Recall  $\zeta_\Gamma = \frac{(3n - |E(\Gamma)|)!}{(3n)! 2^{|E(\Gamma)|}}$ .

**Theorem 7.40.** *Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . Let  $\mathcal{L} = \sqcup_{j=1}^k S_j^1$  be a disjoint union of  $k$  circles. Let  $L: \mathcal{L} \hookrightarrow \check{R}$  be a straight embedding with respect to  $\tau$ . For any  $i \in \underline{3n}$ , let  $\omega(i)$  be a propagating form of  $(C_2(R), \tau)$ . Then the element  $Z_n(\check{R}, L, (\omega(i))_{i \in \underline{3n}})$  of  $\mathcal{A}_n(\mathcal{L})$  defined in Notation 7.16 is independent of the chosen propagating forms  $\omega(i)$  of  $(C_2(R), \tau)$ . It is denoted by  $Z_n^s(\check{R}, L, \tau)$ . In particular, with the notation of Theorem 7.20, we have*

$$Z_n^s(\check{R}, L, \tau) = Z_n(\check{R}, L, \tau).$$

We will give the proof of this theorem in Section 10.4. Theorem 16.9 shows how  $Z_n(\check{R}, L, (\omega(i))_{i \in \underline{3n}})$  depends on the propagators without the straightness assumption.

Straight links  $L$  with respect to  $\tau$  are framed links  $(L, L_{\parallel, \tau})$  according to Corollary 7.36 and Lemma 7.35. So we can keep the information from the link framing and define the invariant of straight links

$$\mathcal{Z}^f(\check{R}, L, L_{\parallel, \tau}) = \exp\left(-\frac{1}{4}p_1(\tau)\beta\right)Z(\check{R}, L, \tau),$$

which depends only on  $(\check{R}, L)$  and on the  $lk(K_j, K_{j\parallel, \tau})$  for the components  $K_j$  of  $L$ , according to Theorem 7.20 and Corollary 7.36.

**Definition 7.41.** Recall the invariant  $\mathcal{Z}$  of Theorem 7.20. Define the framed link invariant  $\mathcal{Z}^f$  to satisfy

$$\mathcal{Z}^f\left(\check{R}, \sqcup_{j=1}^k K_j, \sqcup_{j=1}^k K_{j\parallel}\right) = \prod_{j=1}^k \left(\exp(lk(K_j, K_{j\parallel})\alpha) \#_j\right) \mathcal{Z}\left(R, \sqcup_{j=1}^k K_j\right),$$

for a link  $\sqcup_{j=1}^k K_j$  equipped with a parallel  $K_{j\parallel}$  for each component  $K_j$ .

Thanks to Theorem 7.20, Corollary 7.36, and Theorem 7.40, both definitions coincide for straight framed links  $(L, L_{\parallel, \tau})$ .

Again, we can reduce our averaging process when projecting to  $\check{\mathcal{A}}(\mathcal{L})$  and get the following theorem, also proved in Section 10.4.

**Theorem 7.42.** *Under the assumptions of Theorem 7.40, we have*

$$\check{p}\left(Z_n^s(\check{R}, L, \tau)\right) = \sum_{\Gamma \in \mathcal{D}_{n, 3n-2}^e(\mathcal{L})} \zeta_{\Gamma} I\left(R, L, \Gamma, (\omega(i))_{i \in \underline{3n-2}}\right) \check{p}([\Gamma]).$$

# Chapter 8

## Compactifications of configuration spaces

We use compactifications of our open configuration spaces  $\check{C}(R, L; \Gamma)$  associated to links  $L$  in a  $\mathbb{Q}$ -sphere  $R$  to study the behavior of our integrals and their dependence on the choice of propagating forms. More specifically, we prove the convergence of the integrals involved in the definitions of  $Z$  by finding a smooth compactification (with boundary and ridges), to which the integrated forms extend smoothly. The variation of an integral under the addition of an exact form  $d\eta$  is the integral of  $\eta$  on the codimension-one faces of the boundary, which need to be identified precisely. Therefore, the proofs of Theorems 7.40 and 7.20 (and their variants with reduced averages) require a deep knowledge of configuration spaces. We give all the useful statements in Sections 8.2 to 8.4. We prove all of them in Sections 8.5 to 8.8.

Before giving all the required general statements, we present the main features of the involved compactifications in some examples. William Fulton and Robert MacPherson introduced similar compactifications in [FM94]. Scott Axelrod and Isadore Singer [AS94, Section 5], Maxim Kontsevich [Kon94], Raoul Bott and Clifford Taubes [BT94], and Dev Sinha [Sin04] also investigated these compactifications.

### 8.1 An informal introduction

Our first example of a compactification  $C(R, L; \Gamma)$  of a configuration space  $\check{C}(R, L; \Gamma)$  is the closed annulus  $C(K; \overset{\circ}{\leftrightarrow}) = C(S^3, K; \overset{\circ}{\leftrightarrow})$  of Subsection 1.2.4. Our (more interesting) second example is the compactification  $C_2(R) = C(R, \emptyset; \Theta)$  studied in Section 3.2. Note that  $C(K; \overset{\circ}{\leftrightarrow})$  is the closure of  $\check{C}(K; \overset{\circ}{\leftrightarrow})$  in  $C_2(R)$ .



As in the example of  $C_2(R) = C(R, \emptyset; \Theta)$ , for a trivalent Jacobi diagram  $\Gamma$ , our general compactification  $C(R, \emptyset; \Gamma)$  of the space  $\check{C}(R, \emptyset; \Gamma) = \check{C}_{V(\Gamma)}(R)$  of injective maps from  $V(\Gamma)$  to  $\check{R}$  depends only on the finite set  $V = V(\Gamma)$  of vertices of  $\Gamma$ . It is denoted by  $C_V(R)$ . As in the example of  $C(S^3, K; \leftrightarrow)$ , for a Jacobi diagram  $\Gamma$  on the domain of a link  $L$  of  $\check{R}$ , we define the compactification  $C(R, L; \Gamma)$  of  $\check{C}(R, L; \Gamma)$  to be the closure of  $\check{C}(R, L; \Gamma)$  in  $C_{V(\Gamma)}(R)$  in Proposition 8.6. This is why we first study  $C_V(R)$  by generalizing the construction of  $C_2(R)$  performed in Section 3.2. In this general case, we start with  $R^V$  and blow up all the diagonals and all the loci that involve  $\infty$ , in the sense of Section 3.1, following a process precisely described in Theorem 8.4.

In this informal introduction, we forget about  $\infty$  and first discuss how we successively blow up the diagonals in the manifold  $(\mathbb{R}^3)^V$  of maps from  $V$  to  $\mathbb{R}^3$ . Thus, we get the preimage  $C_V[\mathbb{R}^3]$  of  $(\mathbb{R}^3)^V$  in  $C_{V(\Gamma)}(S^3)$  under the composition valued in  $(S^3)^V$  of the blowdown maps.

### 8.1.1 On the configuration space $C_V[\mathbb{R}^3]$ of four points in $\mathbb{R}^3$

The diagonal  $\Delta_V((\mathbb{R}^3)^V)$  is the set of constant maps from  $V$  to  $\mathbb{R}^3$ . The fiber of the normal bundle to the vector space  $\Delta_V((\mathbb{R}^3)^V)$  in  $(\mathbb{R}^3)^V$  is the vector space  $(\mathbb{R}^3)^V / \Delta_V((\mathbb{R}^3)^V)$ . It consists of maps  $c$  from  $V$  to  $\mathbb{R}^3$  up to global translation.<sup>1</sup> The fiber of the unit normal bundle to the diagonal  $\Delta_V((\mathbb{R}^3)^V)$  is the space  $\overline{\mathcal{S}}_V(\mathbb{R}^3)$  of nonconstant maps from  $V = \{v_1, v_2, \dots, v_{|V|}\}$  to  $\mathbb{R}^3$  up to (global) translation and up to dilation.<sup>2</sup> The space  $\overline{\mathcal{S}}_V(\mathbb{R}^3)$ , studied in Section 8.3, can be identified with the space of maps  $w: V \rightarrow \mathbb{R}^3$  that map  $v_1$  to 0 and such that  $\sum_{i=2}^{|V|} \|w(v_i)\|^2 = 1$ . It is diffeomorphic to a sphere of dimension  $(3|V| - 4)$ . Let  $B\mathcal{L}_1 = \mathcal{B}\ell((\mathbb{R}^3)^V, \Delta_V((\mathbb{R}^3)^V))$  be obtained from  $(\mathbb{R}^3)^V$  by blowing up  $\Delta_V((\mathbb{R}^3)^V)$ . We have a diffeomorphism

$$\psi: \mathbb{R}^3 \times [0, \infty[ \times \overline{\mathcal{S}}_V(\mathbb{R}^3) \rightarrow B\mathcal{L}_1,$$

which maps  $(u, \mu, w)$  to the map  $c: V \rightarrow \mathbb{R}^3$  such that  $c(v_i) = u + \mu w(v_i)$  for any  $i \in \underline{|V|}$ , with respect to the identification above. This map  $c$  is furthermore equipped with the data of the map  $w$  when  $c$  is constant, or equivalently when  $\mu = 0$ . In particular, this first blow-up equips each constant map  $c_0$  in the manifold  $B\mathcal{L}_1$  with the additional data of a nonconstant map  $w: V \rightarrow \mathbb{R}^3$  up to translation and dilation. Let  $c_0$  be the constant map which maps  $V$  to  $u$ , and let  $(c_0, w)$  denote  $\psi(u, 0, w)$ , then  $(c_0, w)$  is the limit in  $B\mathcal{L}_1$  of

<sup>1</sup>A map  $c$  is identified with the map  $(v \mapsto c(v) + W)$  for any  $W \in \mathbb{R}^3$ .

<sup>2</sup>In  $\overline{\mathcal{S}}_V(\mathbb{R}^3)$ , a map  $c$  is furthermore identified with the map  $(v \mapsto \lambda c(v))$  for any  $\lambda \in ]0, +\infty[$ .

$\psi(u, t, w)$ , when  $t > 0$  tends to 0. Therefore, we can think of the map  $w$  as an infinitesimal configuration. The first blow-up provides a magnifying glass, allowing us to see this infinitesimal configuration  $w$  of the vertices mapped to the same point  $u$  in  $\mathbb{R}^3$ .

When  $|V| = 2$ , we are done and  $B\ell_1$  is the preimage  $C_V[\mathbb{R}^3]$  of  $(\mathbb{R}^3)^2$  in  $C_2(S^3)$ . In general, we blow up the other *diagonals*  $\Delta_A((\mathbb{R}^3)^V)$  for all subsets  $A$  of  $V$ , where  $\Delta_A((\mathbb{R}^3)^V)$  is the subspace of  $(\mathbb{R}^3)^V$  consisting of the maps  $c$  that map  $A$  to a single element and such that  $c(V \setminus A) \subset \mathbb{R}^3 \setminus c(A)$ , as in Section 8.2.

Let us describe the process when  $V = V(\Gamma) = \{v_1, v_2, v_3, v_4\}$ . From now on, we restrict to this case in this subsection. The closure of the diagonal  $\Delta_A((\mathbb{R}^3)^V)$  in  $B\ell_1$  is  $\psi(\mathbb{R}^3 \times [0, \infty[ \times \Delta_A(\overline{\mathcal{S}}_V(\mathbb{R}^3)))$ , where  $\Delta_A(\overline{\mathcal{S}}_V(\mathbb{R}^3))$  is the subspace of  $\overline{\mathcal{S}}_V(\mathbb{R}^3)$  made of the classes of its maps  $c$  constant on  $A$  such that  $c(V \setminus A) \subset \mathbb{R}^3 \setminus c(A)$ . In particular, the closures of the diagonals  $\Delta_A((\mathbb{R}^3)^V)$  in  $B\ell_1$  for the subsets  $A$  of  $V$  of cardinality 3 are disjoint in  $B\ell_1$ . We blow up these closures in an arbitrary order. Since the closures are disjoint, the order of the blow-ups does not affect the result.

The fiber of the unit normal bundle of  $\Delta_A((\mathbb{R}^3)^V)$  in  $B\ell_1$  is the space  $\overline{\mathcal{S}}_A(\mathbb{R}^3)$  of nonconstant maps from  $A$  to  $\mathbb{R}^3$  up to (global) translation and dilation. Let  $B = A_{123} = \{v_1, v_2, v_3\}$ . View  $\overline{\mathcal{S}}_{123}(\mathbb{R}^3) = \overline{\mathcal{S}}_B(\mathbb{R}^3)$  as the space of maps  $w_{123}: V \rightarrow \mathbb{R}^3$  that map  $v_1$  and  $v_4$  to 0 and such that  $\|w_{123}(v_2)\|^2 + \|w_{123}(v_3)\|^2 = 1$ . We have a smooth embedding

$$\psi_2: \mathbb{R}^3 \times [0, \infty[^2 \times S^2 \times \overline{\mathcal{S}}_B(\mathbb{R}^3) \rightarrow \mathcal{B}\ell(B\ell_1, \overline{\Delta_B((\mathbb{R}^3)^V)})$$

that maps  $(u, \mu, \mu_{123}, W_4, w_{123})$  to the map  $c: V \rightarrow \mathbb{R}^3$  such that  $c(v_4) = u + \frac{\mu}{\sqrt{1+\mu_{123}^2}}W_4$  and  $c(v_i) = u + \frac{\mu}{\sqrt{1+\mu_{123}^2}}\mu_{123}w_{123}(v_i)$  for  $i \in \underline{3}$ . The preimage of  $\overline{\Delta_B((\mathbb{R}^3)^V)}$  under the blowdown map  $\mathcal{B}\ell(B\ell_1, \overline{\Delta_B((\mathbb{R}^3)^V)}) \rightarrow B\ell_1$  is  $\psi_2(\mathbb{R}^3 \times [0, \infty[ \times \{0\} \times S^2 \times \overline{\mathcal{S}}_B(\mathbb{R}^3))$ , where  $\mu_{123} = 0$ . The image of  $\psi_2$  contains a neighborhood of this preimage in  $\mathcal{B}\ell(B\ell_1, \overline{\Delta_B((\mathbb{R}^3)^V)})$ .

Here, this blow-up equips a map  $c$  of  $\Delta_B((\mathbb{R}^3)^V)$  with the additional data of the (infinitesimal, nonconstant) configuration  $w_{123}|_B: B \rightarrow \mathbb{R}^3$  up to translation and dilation. It equips a constant map  $c_0$  with value  $u$  in the closure of  $\Delta_B((\mathbb{R}^3)^V)$  in  $B\ell_1$  with such a configuration  $w_{123}|_B$  in addition to the former  $w$ , which maps  $B$  to 0 and  $v_4$  to some  $W_4 \in S^2$ . In this case, we denote the obtained configuration  $\psi_2(u, 0, 0, W_4, w_{123})$  by  $(c, w, w_{123}|_B)$ , and we have three observation scales. All the  $c(v_i)$  coincide at the first initial scale. At the second (infinitely smaller) scale  $w$  in  $B\ell_1$ , the  $w(v_i)$  coincide for  $i \in \underline{3}$ , but  $(w(v_4) - w(v_1))$  is not zero, and its direction is a vector  $W_4 \in S^2$ . At the third scale (infinitely smaller than the second one) in  $\mathcal{B}\ell(B\ell_1, \overline{\Delta_B((\mathbb{R}^3)^V)})$ , the configuration  $w_{123}|_B$  of  $\{v_1, v_2, v_3\}$  is visible up to

global translation and dilation. The first three pictures of Figure 8.1 show these three observation scales.

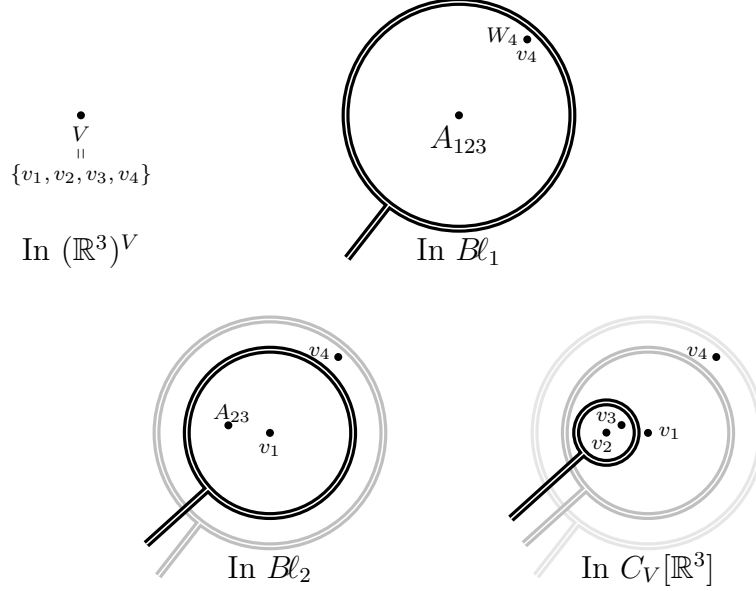


Figure 8.1: The magnifying glasses provided by the successive blow-ups from  $(\mathbb{R}^3)^V$  to  $C_V[\mathbb{R}^3]$ , for a configuration  $(c, w, w_{123}, w_{23})$

Let  $B\ell_2$  be the manifold obtained by blowing up the four closures of the diagonals  $\Delta_A((\mathbb{R}^3)^V)$  in  $B\ell_1$  for the subsets  $A$  of  $V$  of cardinality 3. We have local charts similar to  $\psi_2$  for  $B\ell_2$  in the neighborhoods of the disjoint blown-up loci.

Finally, we blow up (the preimages under the composition of the previous blowdown maps of) the closures of the diagonals  $\Delta_A((\mathbb{R}^3)^V)$  in  $B\ell_2$  for the subsets  $A$  of  $V$  of cardinality 2, in an arbitrary order, to get the manifold  $C_V[\mathbb{R}^3]$  of Section 8.6. Here the diagonals  $\Delta_{\{v_1, v_2\}}((\mathbb{R}^3)^V)$  and  $\Delta_{\{v_3, v_4\}}((\mathbb{R}^3)^V)$  are no longer disjoint. Nevertheless, the blow-up operations associated to  $A_{12} = \{v_1, v_2\}$  and  $A_{34} = \{v_3, v_4\}$ , which act on different coordinates, commute. In the neighborhood of the intersection of the corresponding blown-up loci in  $C_V[\mathbb{R}^3]$ , we have an embedding

$$\psi_3: \mathbb{R}^3 \times [0, \infty[ \times [0, 1/3]^2 \times (S^2)^3 \rightarrow C_V[\mathbb{R}^3],$$

which maps

$$(u, \mu', \mu_{12}, \mu_{34}, W_3, W_{12}, W_{34})$$

to the map  $c: V \rightarrow \mathbb{R}^3$  such that  $c(v_1) = u$ ,  $c(v_2) = u + \mu' \mu_{12} W_{12}$ ,  $c(v_3) = u + \mu' W_3$ ,  $c(v_4) = u + \mu'(W_3 + \mu_{34} W_{34})$  where  $W_3 \in S^2 = \overline{\mathcal{S}}_{\{A_{12}, A_{34}\}}(\mathbb{R}^3)$ ,

$W_{12} \in S^2 = \overline{\mathcal{S}}_{A_{12}}(\mathbb{R}^3)$ , and  $W_{34} \in S^2 = \overline{\mathcal{S}}_{A_{34}}(\mathbb{R}^3)$ . The configuration  $c$  is equipped with the map  $w$  up to translation and dilation when  $c$  is constant. (A representative  $w$  of this map sends the elements of  $V$  to  $w(v_1) = 0$ ,  $w(v_2) = \mu_{12}W_{12}$ ,  $w(v_3) = W_3$ , and  $w(v_4) = W_3 + \mu_{34}W_{34}$ .) The configuration  $c$  is equipped with  $W_{12}$  when the restriction of  $w$  (or  $c$ ) to  $A_{12}$  is constant. It is equipped with  $W_{34}$ , when the restriction of  $w$  (or  $c$ ) to  $A_{34}$  is constant. Figure 8.2 shows the three magnifying glasses that have popped up for a configuration  $(c, w, W_{12}, W_{34}) = \psi_3(u, 0, 0, 0, W_3, W_{12}, W_{34})$ .

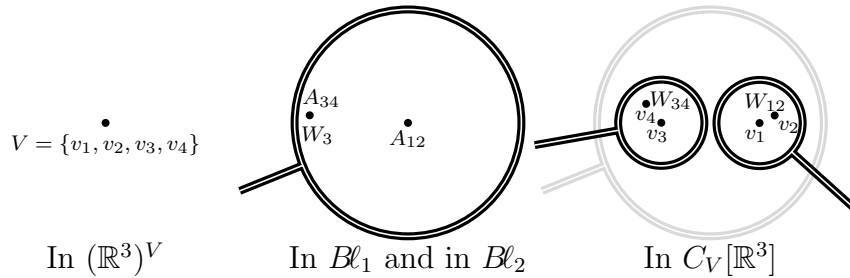


Figure 8.2: The magnifying glasses provided by the successive blow-ups from  $(\mathbb{R}^3)^V$  to  $C_V[\mathbb{R}^3]$ , for a configuration  $(c, w, W_{12}, W_{34})$

For a constant map  $c$ , whose associated infinitesimal  $w$  is constant on  $A_{123}$ , and whose next associated  $w_{123}$  is constant on  $A_{23} = \{v_2, v_3\}$ , the third blow-up family provides a fourth smaller observation scale. With this additional scale, we see a nonconstant map  $w_{23}: A_{23} \rightarrow \mathbb{R}^3$  up to global translation and dilation, as in Figure 8.1. The map  $w_{23}$  gives the direction from  $w_{23}(v_2)$  to  $w_{23}(v_3)$ .

### 8.1.2 More configuration spaces and their stratifications

In general, for a finite set  $V$  and for an integer  $d \in \mathbb{N} \setminus \{0\}$ , we transform  $(\mathbb{R}^d)^V$  to a manifold  $C_V[\mathbb{R}^d]$  by successively blowing up the closures of (the preimages under the composition of the previous blowdown maps of) the diagonals  $\Delta_A((\mathbb{R}^d)^V)$  associated to the subsets  $A$  of  $V$  of cardinality  $k$ , for  $k = |V|, |V| - 1, \dots, 2$  in this decreasing order, in Section 8.6. It will follow from Theorem 8.32 and Proposition 8.33 that there are natural smooth (quotients of) restriction maps from  $C_V[\mathbb{R}^d]$  to the space  $\overline{\mathcal{S}}_A(\mathbb{R}^d)$  of nonconstant maps from  $A$  to  $\mathbb{R}^d$  up to translation and dilation, for every subset  $A$  of  $V$  of cardinality at least 2. These restriction maps are smooth extensions of the natural (quotients of) restriction maps from the space  $\check{C}_V[\mathbb{R}^d]$  of injective maps from  $V$  to  $\mathbb{R}^d$  to the compact space  $\overline{\mathcal{S}}_A(\mathbb{R}^d)$ . We could define the

configuration space  $C_V[\mathbb{R}^d]$  to be the closure of the image of  $\check{C}_V[\mathbb{R}^d]$  in the product  $(\mathbb{R}^d)^V \times \prod_{A \subseteq V; |A| \geq 2} \overline{\mathcal{S}}_A(\mathbb{R}^d)$ , or in smaller spaces, as Dev Sinha did in [Sin04]. This would define its topology. However, the differential structures of our configuration spaces are essential for our purposes. This is why we study them in detail in this chapter.

**Definition 8.1.** The *partition associated to a map  $f$*  from a finite set  $V$  to some set  $X$  is the following set  $K(V; f)$  of subsets of  $V$ .

$$K(V; f) = \{f^{-1}(x); x \in f(V)\}.$$

In this book, a *partition* of a finite set  $V$  is a set of disjoint nonempty subsets of  $V$  whose union is  $V$ . The elements of a partition  $K(V)$  are called the *kids* of  $V$  (with respect to the partition). (We do not call them children because we use the initial of children in the notation of configuration spaces.) The *daughters* of  $V$  with respect to such a partition are its kids with cardinality at least 2, and its *sons* are the singletons of  $K(V)$ . (The daughters might bear smaller kids in other partitions.) We denote the set of daughters of  $V$  by  $D(V, K(V))$ , or by  $D(V)$  when  $K(V)$  is understood. We simply denote  $D(V, K(V; f))$  by  $D(V; f)$ .

**Definition 8.2.** A *parenthesization*  $\mathcal{P}$  of a finite set  $V$  is a set  $\mathcal{P} = \{A_i; i \in I\}$  of subsets of  $V$ , each of cardinality greater than one, such that, for any two distinct elements  $i, j$  of  $I$ , one of the following holds  $A_i \subset A_j$ ,  $A_j \subset A_i$  or  $A_i \cap A_j = \emptyset$ .

Every element  $x$  of the space  $C_V[\mathbb{R}^d]$  defines the following parenthesization  $\mathcal{P}(x)$  of  $V$ . The maximal elements (with respect to the inclusion) of  $\mathcal{P}(x)$  are the daughters of  $V$  with respect to  $(p_b(x) \in (\mathbb{R}^d)^V)$ . For any element  $A$  of  $\mathcal{P}(x)$ , the maximal strict subsets of  $A$  in  $\mathcal{P}(x)$  are the daughters of  $A$  with respect to the restriction of  $x$  to  $A$ . In our examples of Figures 8.1 and 8.2, the parenthesizations are  $\{V, A_{123}, A_{23}\}$  and  $\{V, A_{12}, A_{34}\}$ , respectively. They are in one-to-one correspondences with the magnifying glasses provided by the iterated blow-ups, or, equivalently, with the blow-ups that affected  $x$ . For a parenthesization  $\mathcal{P}$  of  $V$ , define the *stratum*  $C_{V, \mathcal{P}}[\mathbb{R}^d] = \{x \in C_V[\mathbb{R}^d] : \mathcal{P}(x) = \mathcal{P}\}$ . As we can see in the above examples, and as we will state in a larger generality in Proposition 8.34, such a stratum is a smooth manifold of codimension  $|\mathcal{P}|$  in  $C_V[\mathbb{R}^d]$ . As a set, the configuration space  $C_V[\mathbb{R}^d]$  is the disjoint union of the strata  $C_{V, \mathcal{P}}[\mathbb{R}^d]$ . So these strata define a *stratification* of  $C_V[\mathbb{R}^d]$ . The open codimension-one faces of  $C_V[\mathbb{R}^d]$  are the strata of codimension one. They are in one-to-one

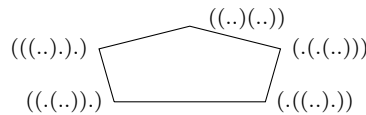
correspondence with the subsets  $A$  of  $V$  of cardinality greater than 1. The open face associated to  $A$  consists of the pairs  $(c \in \Delta_A((\mathbb{R}^d)^V), w_A \in \overline{\mathcal{S}}_A(\mathbb{R}^d))$  such that the restriction of  $c$  to  $(V \setminus A) \cup \{a \in A\}$  and the map  $w_A$  are injective.

The space  $\overline{\mathcal{S}}_V(\mathbb{R}^d)$  of nonconstant maps from  $V$  to  $\mathbb{R}^d$  up to translation and dilation is the preimage of a constant map under the blowdown map from  $\mathcal{B}((\mathbb{R}^d)^V, \Delta_V((\mathbb{R}^d)^V))$  to  $(\mathbb{R}^d)^V$ . In Theorem 8.11, we successively blow up the diagonals  $\Delta_A(\overline{\mathcal{S}}_V(\mathbb{R}^d))$  of this space, for strict subsets  $A$  of  $V$  of cardinality greater than 1, as above, to transform  $\overline{\mathcal{S}}_V(\mathbb{R}^d)$  to the preimage  $\mathcal{S}_V(\mathbb{R}^d)$  of a constant map under the composed blowdown map from  $C_V[\mathbb{R}^d]$  to  $(\mathbb{R}^d)^V$ . The space  $\mathcal{S}_V(\mathbb{R}^d)$  is presented in detail in Section 8.3. It is a compactification of the space  $\check{\mathcal{S}}_V(\mathbb{R}^d)$  of injective maps from  $V$  to  $\mathbb{R}^d$  up to translation and dilation.

The spaces  $\check{\mathcal{S}}_2(\mathbb{R}^d), \overline{\mathcal{S}}_2(\mathbb{R}^d), \mathcal{S}_2(\mathbb{R}^d)$  are identical, they consist of the classes of the maps  $w_X: \underline{2} \rightarrow \mathbb{R}^d$  such that  $w_X(1) = 0$  and  $w_X(2) = X$  for  $X \in S^{d-1}$ . So they are diffeomorphic to the unit sphere  $S^{d-1}$  of  $\mathbb{R}^d$ . In particular, the space  $\overline{\mathcal{S}}_2(\mathbb{R})$  has two elements, which are the classes of  $w^+$  and  $w^-$ , where  $w^+$  and  $w^-$  map 1 to 0,  $w^+(2) = 1$ , and  $w^-(2) = -1$ .

Let us discuss the spaces  $\check{\mathcal{S}}_V(\mathbb{R}), \overline{\mathcal{S}}_V(\mathbb{R}),$  and  $\mathcal{S}_V(\mathbb{R})$  in more detail.

**Example 8.3.** In general, for an integer  $k \geq 2$ ,  $\check{\mathcal{S}}_k(\mathbb{R})$  and its compactification  $\mathcal{S}_k(\mathbb{R})$  have  $k!$  connected components, which correspond to the orders of the  $c(i)$  in  $\mathbb{R}$ , for configurations  $c$  in  $\check{\mathcal{S}}_k(\mathbb{R})$ . Denote the connected component of  $\check{\mathcal{S}}_k(\mathbb{R})$  in which the configurations  $c$  satisfy  $c(1) < c(2) < \dots < c(k)$ , by  $\check{\mathcal{S}}_{<,k}(\mathbb{R})$ . Its respective closures in  $\mathcal{S}_k(\mathbb{R})$  and in  $\overline{\mathcal{S}}_k(\mathbb{R})$  are denoted by  $\mathcal{S}_{<,k}(\mathbb{R})$  and  $\overline{\mathcal{S}}_{<,k}(\mathbb{R})$ . Then we have  $\check{\mathcal{S}}_{<,3}(\mathbb{R}) = \{(0, t, 1) : t \in ]0, 1[ \}$ . The spaces  $\mathcal{S}_{<,3}(\mathbb{R})$  and  $\overline{\mathcal{S}}_{<,3}(\mathbb{R})$  coincide. They are the natural compactification  $[0, 1]$  of  $\check{\mathcal{S}}_{<,3}(\mathbb{R})$ . In this space, an element  $t$  of  $]0, 1[$  represents the injective configuration  $(0, t, 1)$ . In  $[0, 1]$ , the extremity 0 represents the limit configuration  $((..)) = \lim_{t \rightarrow 0}(0, t, 1)$  and 1 represents the limit configuration  $(.(..)) = \lim_{t \rightarrow 0}(0, 1 - t, 1)$ . The configuration space  $\mathcal{S}_{<,4}(\mathbb{R})$  is diffeomorphic to the following well-known pentagon:



The edge from  $((.(..)).)$  to  $((((..)).))$  is the preimage of the diagonal  $\Delta_3(\overline{\mathcal{S}}_{<,V}(\mathbb{R}))$  under the composed blowdown map from  $\mathcal{S}_{<,4}(\mathbb{R})$  to  $\overline{\mathcal{S}}_{<,4}(\mathbb{R})$ . It is naturally diffeomorphic to  $\mathcal{S}_{<,3}(\mathbb{R})$ . The edge from  $((((..)).))$  to  $((..)(..))$  is the closure of the preimage of the diagonal  $\Delta_2(\overline{\mathcal{S}}_{<,V}(\mathbb{R}))$ . Its interior is naturally diffeomorphic to  $\check{\mathcal{S}}_{<,\{2,3,4\}}(\mathbb{R})$ .

In general, for  $k \geq 3$ , the configuration space  $\mathcal{S}_{<,k}(\mathbb{R})$  is a *Stasheff polyhedron* [Sta63] of dimension  $(k - 2)$  whose *corners* (i.e., ridges of dimension 0) are labeled by *nonassociative words* in the letter  $\bullet$ , as in the above example. For any integer  $k \geq 2$ , a nonassociative word  $w$  with  $k$  letters represents a limit configuration  $w = \lim_{t \rightarrow 0} w(t)$ , where  $w(t) = (w_1(t) = 0, w_2(t), \dots, w_{k-1}(t), w_k(t) = 1)$  is an injective configuration inductively defined as follows for  $t \in ]0, \frac{1}{2}[$ . If  $w$  is the product  $uv$  of a nonassociative word  $u$  of length  $j \geq 1$  and a nonassociative word  $v$  of length  $(k - j) \geq 1$ , then  $w_i(t) = tu_i(t)$  when  $1 < i \leq j$ , and  $w_i(t) = 1 - t + tv_{i-j}(t)$  when  $j < i < k$ . For example, we have  $((\dots))_i(t) = (0, t^2, t, 1)$ . In a limit configuration associated to such a nonassociative word, points inside matching parentheses are thought of as infinitely closer to each other than they are to points outside these matching parentheses. The parenthesization associated as above to a nonassociative word is the set of strict subsets inside matching parentheses.<sup>3</sup>

## 8.2 General presentation of $C_V(R)$

Let  $V$  denote a finite set. We use this notation for a generic finite set since our sets will end up being sets of vertices of Jacobi diagrams. The space of maps from  $V$  to  $X$  is denoted by  $X^V$  as usual. For a subset  $A$  of  $V$  of cardinality at least 2, recall that the subspace of  $X^V$  consisting of maps  $c$  that map  $A$  to a single element and such that  $c(V \setminus A) \subset X \setminus c(A)$  is a diagonal denoted by  $\Delta_A(X^V)$ . In particular, if  $|V| \geq 2$ , the *small diagonal* consisting of constant maps is denoted by  $\Delta_V(X^V)$ .

As often in this book, we fix a rational homology sphere  $R$  and a point  $\infty$  of  $R$ . Recall that  $\check{C}_V(R)$  is the space of injective maps from  $V$  to  $(\check{R} = R \setminus \{\infty\})$ .

**Theorem 8.4.** *Define a compactification  $C_V(R)$  of  $\check{C}_V(R)$  as follows. For a nonempty  $A \subseteq V$ , let  $\Xi_A$  be the set of maps from  $V$  to  $R$  that map  $A$  to  $\infty$  and  $V \setminus A$  to  $R \setminus \{\infty\}$ . Start with  $R^V$ . Blow up  $\Xi_V$  (which is reduced to the point  $m = \infty^V$  such that  $m^{-1}(\infty) = V$ ). Set*

$$C_{V,|V|+1}(R) = \mathcal{B}(R^V, \Xi_V).$$

*For  $k = |V|, |V| - 1, \dots, 3, 2$ , let  $C_{V,k}(R)$  be obtained from  $C_{V,k+1}(R)$  by blowing up the closures of (the preimages under the composition of the previous blowdown maps of) the  $\Delta_A(\check{R}^V)$  such that  $|A| = k$  and the closures of (the preimages under the composition of the previous blowdown maps of)*

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<sup>3</sup>In Theorem 8.28, we will rather associate the set of all subsets inside matching parentheses.

the  $\Xi_J$  such that  $|J| = k - 1$ . At each step, the blown-up manifolds are smooth and transverse to the ridges, and  $C_{V,k}(R)$  is independent of the order of the blow-ups. The obtained manifold  $C_V(R) = C_{V,2}(R)$  is a smooth compact  $(3|V|)$ -manifold with ridges. The interior of  $C_V(R)$  is  $\check{C}_V(R)$ , and the composition of the blowdown maps gives rise to a canonical smooth blowdown projection  $p_b: C_V(R) \rightarrow R^V$ .

We prove the generalization Theorem 8.35 of Theorem 8.4 in Section 8.7 using the results of Sections 8.5 and 8.6. This generalization includes an alternative definition of  $C_V(R)$ .

Set  $C_n(R) = C_{\underline{n}}(R)$ .

As already announced, with the above definition, we have  $C_1(R) = \mathcal{B}(R, \infty)$ , and the configuration space  $C_2(R)$  is the compactification studied in Section 3.2. In particular, Theorem 8.4 is true when  $|V| \leq 2$ .

**Theorem 8.5.** *The configuration spaces  $\check{C}_V(R)$  have the following properties.*

1. *Under the assumptions of Theorem 8.4, for  $A \subset V$ , the restriction map*

$$p_A: \check{C}_V(R) \rightarrow \check{C}_A(R)$$

*extends to a smooth restriction map still denoted by  $p_A$  from  $C_V(R)$  to  $C_A(R)$  such that the following square commutes:*

$$\begin{array}{ccc} C_V(R) & \xrightarrow{p_A} & C_A(R) \\ p_b \downarrow & & \downarrow p_b \\ R^V & \xrightarrow{p_A} & R^A \end{array}$$

2. *For an open subset  $U$  of  $R$ , let  $C_V(U)$  denote  $p_b^{-1}(U^V)$ . If  $V = \sqcup_{i \in I} A_i$  and if  $(U_i)_{i \in I}$  is a family of disjoint open sets of  $R$ , then the product*

$$p_b^{-1} \left( \prod_{i \in I} U_i^{A_i} \right) \xrightarrow{\prod_{i \in I} p_{A_i}} \prod_{i \in I} C_{A_i}(U_i)$$

*of the above restriction maps is a diffeomorphism.*

The first part of this theorem will be a direct consequence of Proposition 8.33 and Theorem 8.35. Its second part comes from the locality of the blow-up operations. The spaces  $C_V(\check{R})$ , which involve only blow-ups along the diagonals, have been studied by Scott Axelrod and Isadore Singer [AS94, Section 5], and with more details by Dev Sinha [Sin04]. Their properties that are useful in this book are proved in Sections 8.5 and 8.6. William Fulton



and Robert MacPherson studied similar compactifications of configuration spaces in an algebraic geometry setting in [FM94].

Recall the configuration space

$$\check{C}(R, L; \Gamma) = \left\{ c: U \cup T \hookrightarrow \check{R} : (\exists j \in [i_\Gamma] : c|_U = L \circ j) \right\}$$

of Section 7.1.

**Proposition 8.6.** *The closure of  $\check{C}(R, L; \Gamma)$  in  $C_{V(\Gamma)}(R)$  is a smooth compact submanifold of  $C_{V(\Gamma)}(R)$  transverse to the ridges. It is denoted by  $C(R, L; \Gamma)$ .*

We will prove Proposition 8.6 in Section 8.8. Theorems 8.4 and 8.5, and Proposition 8.6 imply Proposition 7.8.

### 8.3 Configuration spaces associated to unit normal bundles to diagonals

For a vector space  $T$ , recall from Section 3.1 that  $S(T)$  denotes the quotient  $S(T) = (T \setminus \{0\})/\mathbb{R}^{+*}$  of  $T \setminus \{0\}$  by the dilations. If  $T$  is equipped with a Euclidean norm, then  $\mathbb{S}(T)$  denotes the unit sphere of  $T$ . In this case,  $S(T)$  and  $\mathbb{S}(T)$  are diffeomorphic.

**Definition 8.7.** Let  $V$  denote a finite set of cardinality at least 2. We use the notation  $T$  for a generic vector space since  $T$  will end up being a tangent space. Let

$$\overline{\mathcal{S}}_V(T) = S(T^V/\Delta_V(T^V)) = \frac{(T^V/\Delta_V(T^V)) \setminus \{0\}}{\mathbb{R}^{+*}}$$

be the space of nonconstant maps from  $V$  to  $T$  up to translation and dilation.

**Lemma 8.8.** *Let  $A$  be a subset of  $V$ . The fiber of the unit normal bundle to  $\Delta_A(R^V)$  in  $R^V$  over a configuration  $m$  is  $\overline{\mathcal{S}}_A(T_{m(A)}R)$ .*

PROOF: Exercise. □

As expected for the fiber of a unit normal bundle, we have the following lemma.

**Lemma 8.9.** *The topological space  $\overline{\mathcal{S}}_V(T)$  has a canonical smooth structure. When  $T$  is a vector space of dimension  $\delta$ , the space  $\overline{\mathcal{S}}_V(T)$  is diffeomorphic to a sphere of dimension  $((|V| - 1)\delta - 1)$ .*

PROOF: Choosing a basepoint  $b(V)$  for  $V$  and a basis for  $T$  identifies  $\overline{\mathcal{S}}_V(T)$  with the set  $\mathbb{S}(T^{V \setminus \{b(V)\}})$  of maps  $c: V \rightarrow T$  such that

- $c(b(V)) = 0$  and
- $\sum_{v \in V} \|c(v)\|^2 = 1,$

with respect to the Euclidean norm for which our basis is orthonormal. It is easy to see that changes of basepoints, or bases of  $T$  give rise to diffeomorphisms of spheres.  $\square$

**Convention 8.10.** In this chapter, we do not take care of orientations. Later, we will associate the following orientation of  $\overline{\mathcal{S}}_V(T)$  to an order of  $V$  and an orientation of  $T$ . Assume that  $V = \{v_1, \dots, v_n\}$  and that  $T$  is oriented. The order on  $V$  orients  $T^V$  and  $T^{V \setminus \{v_1\}}$ . Then the map from the boundary of the unit ball of  $T^{V \setminus \{v_1\}}$  to  $\overline{\mathcal{S}}_V(T)$  that maps an element  $(x_2, \dots, x_n)$  of  $\mathbb{S}(T^{V \setminus \{v_1\}})$  to the class of  $(0, x_2, \dots, x_n)$  is an orientation-preserving diffeomorphism.

When  $V = \underline{2}$ , fixing the basepoint  $b(\underline{2})$  of  $\underline{2}$  to be 1 identifies  $\overline{\mathcal{S}}_V(T)$  with the unit sphere  $\mathbb{S}(T)$  of  $T$ , if  $T$  is equipped with a Euclidean norm, as in the proof of Lemma 8.9.

For a strict subset  $A$  of  $V$  of cardinality at least 2, define the diagonal  $\Delta_A(\overline{\mathcal{S}}_V(T))$  to be the subset of  $\overline{\mathcal{S}}_V(T)$  consisting of classes of maps  $c$  from  $V$  to  $T$  that are constant on  $A$  and such that  $c(V \setminus A) \cap c(A) = \emptyset$ . Let  $\check{\mathcal{S}}_V(T)$  denote the subspace of  $\overline{\mathcal{S}}_V(T)$  consisting of *injective* maps from  $V$  to  $T$  up to translation and dilation. The following theorem defines a bigger compactification of  $\check{\mathcal{S}}_V(T)$ , which is also used in our study of the variations of  $Z$ , and especially in the definition of the anomalies. We will prove the following two theorems in Section 8.5. See Theorem 8.28 and Proposition 8.31.

**Theorem 8.11.** *Set  $\mathcal{S}_{V,|V|}(T) = \overline{\mathcal{S}}_V(T)$ . For  $k = |V| - 1, \dots, 3, 2$ , define  $\mathcal{S}_{V,k}(T)$  from  $\mathcal{S}_{V,k+1}(T)$  by blowing up the closures of (the preimages under the composition of the previous blowdown maps of) the  $\Delta_A(\overline{\mathcal{S}}_V(T))$  such that  $|A| = k$ . At each step, the blown-up manifolds are smooth and transverse to the ridges, and the resulting blown-up manifold is independent of the order choice.*

*Thus, this process gives rise to a canonical compact smooth manifold  $\mathcal{S}_V(T) = \mathcal{S}_{V,2}(T)$  with ridges. The interior of  $\mathcal{S}_V(T)$  is  $\check{\mathcal{S}}_V(T)$ .*

When  $V$  has two elements, we have  $\mathcal{S}_V(T) = \check{\mathcal{S}}_V(T) = \overline{\mathcal{S}}_V(T)$ . In general, the manifold  $\mathcal{S}_V(T)$  satisfies the following properties.

**Theorem 8.12.** *With the notation of Theorem 8.11, for any subset  $A$  of  $V$ , the restriction from  $\check{\mathcal{S}}_V(T)$  to  $\check{\mathcal{S}}_A(T)$  extends to a smooth map from  $\mathcal{S}_V(T)$  to  $\mathcal{S}_A(T)$ .*

The open codimension-one faces of  $\mathcal{S}_V(T)$  will be the loci for which only one blow-up along some  $\Delta_A(\overline{\mathcal{S}}_V(T))$  is involved, for a strict subset  $A$  of  $V$  such that  $|A| \geq 2$ . The blowdown map sends such a face  $f(A)(T)$  into  $\Delta_A(\overline{\mathcal{S}}_V(T))$ . As it will be seen in Lemma 8.22, the fiber of the unit normal bundle of  $\Delta_A(\overline{\mathcal{S}}_V(T))$  is  $\overline{\mathcal{S}}_A(T)$ . Thus, the following proposition will be clear in the end of Section 8.5, where a one-line proof is given.

**Proposition 8.13.** *The codimension-one faces of  $\mathcal{S}_V(T)$  are in one-to-one correspondence with the strict subsets  $A$  of  $V$  with cardinality at least 2. The (open) face  $f(A)(T)$  corresponding to such an  $A$  is*

$$f(A)(T) = \check{\mathcal{S}}_A(T) \times \check{\mathcal{S}}_{\{a\} \cup (V \setminus A)}(T)$$

for an element  $a$  of  $A$ . For a subset  $e$  of cardinality 2 of  $V$ , the restriction to  $f(A)(T)$  of the extended restriction

$$p_e: \mathcal{S}_V(T) \rightarrow \mathcal{S}_e(T)$$

may be described as follows:

- If  $e \subseteq A$ , then  $p_e$  is the composition of the natural projections

$$f(A)(T) \longrightarrow \check{\mathcal{S}}_A(T) \longrightarrow \mathcal{S}_e(T).$$

- If  $e \subseteq (V \setminus A) \cup \{a\}$ , then  $p_e$  is the composition of the natural projections

$$f(A)(T) \longrightarrow \check{\mathcal{S}}_{\{a\} \cup (V \setminus A)}(T) \longrightarrow \mathcal{S}_e(T).$$

- If  $e \cap A = \{a'\}$ , let  $\tilde{e}$  be obtained from  $e$  by replacing  $a'$  by  $a$ , then we have  $p_e = p_{\tilde{e}}$ .

The space  $\mathcal{S}_V(T_x \check{R})$  described in the above theorem is related to  $C_V(R)$  by the following proposition, which is a corollary of Theorem 8.32.

**Proposition 8.14.** *For  $x \in \check{R}$ , for  $p_b: C_V(R) \rightarrow R^V$ , we have*

$$p_b^{-1}(x^V) = \mathcal{S}_V(T_x \check{R}).$$

## 8.4 The codimension-one faces of $C(R, L; \Gamma)$

Recall that our terminology in Section 2.1.4 makes codimension-one faces open in the boundary of a manifold with ridges. Our codimension-one faces may be called *open codimension-one faces* in other references. We describe the codimension-one faces of the configuration space  $C(R, L; \Gamma)$  for a link  $L$  in a  $\mathbb{Q}$ -sphere  $R$  below with an outline of justification. Details are provided in Sections 8.5 to 8.8. Again, the codimension-one faces are the loci where only one blow-up has been performed.

In this book, the sign  $\subset$  stands for “is a strict subset of” or “ $\subseteq$  and  $\neq$ ”.

**Faces that involve blow-ups along diagonals  $\Delta_A$ .** Let  $A$  be a subset of the set  $V(\Gamma)$  of vertices of a Jacobi diagram  $\Gamma$  on the domain  $\mathcal{L}$  of a link  $L$ . Assume  $|A| \geq 2$ . Let us describe the (open) face  $F(A, L, \Gamma)$ , which comes from the blow-up along  $\Delta_A(\check{R}^{V(\Gamma)})$ . Such a face contains limit configurations that map  $A$  to a point of  $\check{R}$ . An element of such a face is described by its image  $m$  under the blowdown map

$$p_b: C_{V(\Gamma)}(R) \rightarrow R^{V(\Gamma)},$$

which maps  $F(A, L, \Gamma)$  to  $\Delta_A(\check{R}^{V(\Gamma)})$ , together with an element of the fiber

$$\overline{\mathcal{S}}_A(T_{m(A)}\check{R}).$$

Let  $a \in A$ . Let  $\check{\Delta}_A(\check{R}^{V(\Gamma)})$  denote the set of maps of  $\Delta_A(\check{R}^{V(\Gamma)})$  whose restriction to  $\{a\} \cup (V(\Gamma) \setminus A)$  is injective. Set

$$B(A, L, \Gamma) = \check{\Delta}_A(\check{R}^{V(\Gamma)}) \cap p_b(C(R, L; \Gamma)).$$

The face  $F(A, L, \Gamma)$  fibers over the subspace  $B(A, L, \Gamma)$ . When  $A$  contains no univalent vertex, the fiber over a point  $m$  is  $\check{\mathcal{S}}_A(T_{m(A)}\check{R})$ .

Let  $i_\Gamma$  be a  $\Gamma$ -compatible injection. Let  $\mathcal{L}_1$  be an oriented connected component of  $\mathcal{L}$ . Let  $U_1 = i_\Gamma^{-1}(\mathcal{L}_1)$ . The restriction to  $U_1$  of the injection  $i_\Gamma$  into  $\mathcal{L}_1$  induces a permutation  $\sigma$  of  $U_1$ , such that, travelling along  $\mathcal{L}_1$ , we meet  $i_\Gamma(v)$ ,  $i_\Gamma(\sigma(v))$ ,  $\dots$ ,  $i_\Gamma(\sigma^{|U_1|}(v) = v)$ , successively, for any element  $v$  of  $U_1$ . A *set of consecutive elements* of  $U_1$ , with respect to  $i_\Gamma$ , is a subset  $A_U$  of  $U_1$  that may be written as  $\{v, \sigma(v), \dots, \sigma^k(v)\}$  for some element  $v \in U_1$  and for  $k \leq |U_1| - 1$ . If  $A_U \neq U_1$ , then the first element  $v$  in such an  $A_U$  is unique, and  $\sigma$  induces the following unique linear order

$$v < \sigma(v) < \dots < \sigma^k(v)$$

on such a set  $A_U$  of consecutive elements  $U_1$ , which is said to be *compatible* with the isotopy class  $[i_\Gamma]$  of  $i_\Gamma$ . If  $A_U = U_1$ , every choice of an element  $v$  in  $A_U$  induces a *linear* (i.e., total) order

$$v < \sigma(v) < \cdots < \sigma^k(v),$$

which is said to be *compatible* with  $[i_\Gamma]$ .

Let  $A_U = A \cap U(\Gamma)$ . When  $A$  contains univalent vertices, if  $F(A, L, \Gamma)$  is nonempty, then  $A_U$  must be a set of consecutive vertices on a component  $\mathcal{L}_1$  of  $\mathcal{L}$  with respect to the given class  $[i_\Gamma]$  of injections of  $U(\Gamma)$  into  $\mathcal{L}_1$ . The fiber over a point  $m$  is the subset  $\check{\mathcal{S}}_A(T_{m(A)}\check{R}, L, \Gamma)$  of  $\check{\mathcal{S}}_A(T_{m(A)}\check{R})$  consisting of injections that map  $A_U$  on a line directed by  $T_{m(A)}L$ , so that the order induced by the line on  $A_U$  coincides with

- the linear order induced by  $[i_\Gamma]$ , if  $A_U$  is not the whole  $i_\Gamma^{-1}(\mathcal{L}_1)$ ,
- one of the  $|i_\Gamma^{-1}(\mathcal{L}_1)|$  linear orders compatible with the cyclic order induced by  $[i_\Gamma]$ , if  $A_U = i_\Gamma^{-1}(\mathcal{L}_1)$ .

In this latter case, neither the fiber nor  $F(A, L, \Gamma)$  is connected. Their connected components are in one-to-one correspondence with the compatible orders.

According to Theorem 8.5, for any pair  $e$  of  $V(\Gamma)$ , there exists a smooth restriction map from  $C_{V(\Gamma)}$  to  $C_e(R)$ . An order on  $e$  identifies  $C_e(R)$  with  $C_2(R)$ . We describe the natural restriction  $p_e$  to the (open) face  $F(A, L, \Gamma)$  below for a pair  $e$  of  $V(\Gamma)$ .

- If  $|e \cap A| \leq 1$ , then  $p_e$  is the composition of the natural projections

$$F(A, L, \Gamma) \longrightarrow \check{\Delta}_A(\check{R}^{V(\Gamma)}) \longrightarrow \check{C}_e(R).$$

- If  $e \subseteq A$ , then  $p_e$  maps an element of  $\check{\mathcal{S}}_A(T_{m(A)}\check{R}, L, \Gamma)$  to its projection in  $\check{\mathcal{S}}_e(T_{m(A)}\check{R}) \subset C_e(R)$ .

**Faces**  $F(V(\Gamma), L, \Gamma)$ . Let us study the following special case of the previous paragraph's faces, which will play a particular role. Consider a face  $F(V(\Gamma), L, \Gamma)$  where a connected graph  $\Gamma$  with at least one univalent vertex collapses and  $L$  is a knot embedding. Such a face has one connected component for each linear order of  $U(\Gamma)$  compatible with the cyclic order of  $U(\Gamma)$ .

**Notation 8.15.** A Jacobi diagram  $\check{\Gamma}$  on  $\mathbb{R}$  yields a diagram  $\text{cl}(\check{\Gamma})$  on  $S^1$ , where  $S^1$  is viewed as  $\mathbb{R} \cup \{\infty\}$  by adding  $\infty$  to  $\mathbb{R}$ . The natural orientation of  $\mathbb{R}$  orders  $U(\check{\Gamma})$ . For a Jacobi diagram  $\Gamma$  on  $S^1$ , represent a linear order of  $U(\Gamma)$  compatible with the cyclic order of  $U(\Gamma)$  by a Jacobi diagram  $\check{\Gamma}$  on  $\mathbb{R}$  such that  $\text{cl}(\check{\Gamma}) = \Gamma$  and the order of  $U(\Gamma)$  matches the order of  $U(\check{\Gamma})$ . Denote the corresponding connected component of  $F(V(\Gamma), L, \Gamma)$  by  $F(V(\Gamma), L, \check{\Gamma})$ .

Such a connected component fibers over the domain  $S^1$  of the knot. We denote the fiber over  $z \in S^1$  by  $\check{\mathcal{S}}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$ , where  $\vec{t}_{L(z)}$  is an oriented tangent vector to  $L$  at  $L(z)$ .

The space  $\check{\mathcal{S}}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$  is the space of injections of  $V(\check{\Gamma})$  into the vector space  $T_{L(z)}\check{R}$  that map  $U(\check{\Gamma})$  to the oriented line  $\mathbb{R}\vec{t}_{L(z)}$  directed by  $\vec{t}_{L(z)}$  with respect to the linear order of  $U(\check{\Gamma})$ , up to dilation and translation with respect to a vector in  $\mathbb{R}\vec{t}_{L(z)}$ . It is naturally a subspace of  $\check{\mathcal{S}}_{V(\Gamma)}(T_{L(z)}\check{R})$ . Indeed, modding out by all translations is equivalent to considering only configurations that map a univalent vertex to  $\mathbb{R}\vec{t}_{L(z)}$  and modding them out by translations along  $\mathbb{R}\vec{t}_{L(z)}$ .

**Lemma 8.16.** *The closure  $\overline{F}(V(\Gamma), L, \check{\Gamma})$  of  $F(V(\Gamma), L, \check{\Gamma})$  in  $C(R, L; \Gamma)$  is a manifold with ridges. The closure  $\mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$  of  $\check{\mathcal{S}}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$  in  $C(R, L; \Gamma)$  is canonically diffeomorphic to its closure in  $\mathcal{S}_{V(\Gamma)}(T_{L(z)}\check{R})$ . It is a smooth manifold with ridges.*

**PROOF:** The first assertion comes from Proposition 8.6 and from the fact that the closed faces of manifolds with ridges are manifolds with ridges (or from the proof of Lemma 8.17 at the end of Section 8.5). The space  $\mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$  is the closure of  $\check{\mathcal{S}}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$  in the fiber over  $L(z)^{V(\Gamma)}$  of  $C_{V(\Gamma)}(\check{R})$ , which is  $\mathcal{S}_{V(\Gamma)}(T_{L(z)}\check{R})$  according to Proposition 8.14.

Now  $\overline{F}(V(\Gamma), L, \check{\Gamma})$  fibers over  $S^1$ , and the fiber over  $S^1$  is also a manifold with ridges.  $\square$

Let  $A$  be a strict subset of  $V(\check{\Gamma})$  with cardinality at least 2 whose univalent vertices are consecutive on  $\mathbb{R}$ . Let  $a \in A$ . Let  $\check{\Delta}_A(\mathcal{S}_{V(\Gamma)}(T_{L(z)}\check{R}))$  denote the set of (classes of) maps of  $\mathcal{S}_{V(\Gamma)}(T_{L(z)}\check{R})$  whose restriction to  $A$  is constant and whose restriction to  $\{a\} \cup (V(\Gamma) \setminus A)$  is injective. Set

$$B(A, \vec{t}_{L(z)}; \check{\Gamma}) = \mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma}) \cap \check{\Delta}_A(\mathcal{S}_{V(\Gamma)}(T_{L(z)}\check{R})).$$

Define the (open) face  $f(A, \vec{t}_{L(z)}; \check{\Gamma})$  of  $\mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$  to be the space that fibers over the subspace  $B(A, \vec{t}_{L(z)}; \check{\Gamma})$ , whose fiber is the space of injections from  $A$  into  $T_{L(z)}\check{R}$  that map the univalent vertices of  $A$  to the oriented line  $\mathbb{R}\vec{t}_{L(z)}$  with respect to the linear order of  $U(\check{\Gamma}) \cap A$ , up to dilation and translation by a vector in  $\mathbb{R}\vec{t}_{L(z)}$ .

We will prove the following lemma at the end of Section 8.5.

**Lemma 8.17.** *The codimension-one faces of  $\mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$  are the faces  $f(A, \vec{t}_{L(z)}; \check{\Gamma})$  for the strict subsets  $A$  of  $V(\check{\Gamma})$  with cardinality at least 2 whose univalent vertices are consecutive on  $\mathbb{R}$ . The faces  $f(A, \vec{t}_{L(z)}; \check{\Gamma})$  are the intersections of  $\mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$  with the codimension-one faces  $f(A)(T_{L(z)}\check{R})$  of  $\mathcal{S}_{V(\check{\Gamma})}(T_{L(z)}\check{R})$  listed in Proposition 8.13. In particular, the restriction maps  $p_e$  from  $f(A)(T_{L(z)}\check{R})$  to  $\mathcal{S}_e(T_{L(z)}\check{R})$  of Proposition 8.13 restrict as restriction maps still denoted by  $p_e$  from  $f(A, \vec{t}_{L(z)}; \check{\Gamma})$  to  $\mathcal{S}_e(T_{L(z)}\check{R})$ .*

**Faces that involve  $\infty$ .** Let  $A$  be a nonempty subset of the set  $V(\Gamma)$  of vertices of a Jacobi diagram  $\Gamma$ . Let  $a \in A$ . Let us describe the (open) face  $F_\infty(A, L, \Gamma)$  that comes from the blow-up along  $\Xi_A$ . It contains limit configurations, which map  $A$  to  $\infty$ . If it is nonempty, then  $A$  contains no univalent vertices.

Let  $\check{\Xi}_A$  denote the set of maps of  $\Xi_A$  that restrict to injective maps on  $\{a\} \cup (V(\Gamma) \setminus A)$ , and set

$$B_\infty(A, L, \Gamma) = \check{\Xi}_A \cap p_b(C(R, L; \Gamma)).$$

The face  $F_\infty(A, L, \Gamma)$  fibers over the subspace  $B_\infty(A, L, \Gamma)$ .

**Notation 8.18.** Let  $\check{\mathcal{S}}(T_\infty R, A)$  denote the set of injective maps from  $A$  to  $(T_\infty R \setminus 0)$  up to dilation.

Note that  $\check{\mathcal{S}}(T_\infty R, A)$  is an open submanifold of the unit normal bundle of  $\Xi_A$ , which is nothing but  $S((T_\infty R)^A)$ . We have

$$F_\infty(A, L, \Gamma) = B_\infty(A, L, \Gamma) \times \check{\mathcal{S}}(T_\infty R, A).$$

For a pair  $e$  of  $V(\Gamma)$ , the natural restriction to  $F_\infty(A, L, \Gamma)$  of

$$p_e: C_{V(\Gamma)} \rightarrow C_e(R)$$

behaves in the following way.

- If  $e \subseteq V(\Gamma) \setminus A$ , then  $p_e$  is the composition of the natural maps

$$F_\infty(A, L, \Gamma) \rightarrow B_\infty(A, L, \Gamma) \rightarrow \check{C}_{V(\Gamma) \setminus A}(R) \rightarrow C_e(R).$$

- If  $e \subseteq A$ , then  $p_e$  is the composition of the natural maps

$$F_\infty(A, L, \Gamma) \rightarrow \check{\mathcal{S}}(T_\infty R, A) \rightarrow \check{\mathcal{S}}(T_\infty R, e) \hookrightarrow C_e(R).$$

- If  $e \cap A = \{a'\}$ , then  $p_e$  is the composition of the natural maps

$$\begin{aligned} F_\infty(A, L, \Gamma) &\rightarrow \check{C}_{e \setminus \{a'\}}(R) \times \check{\mathcal{S}}(T_\infty R, \{a'\}) \rightarrow \\ &\rightarrow \check{R}^{e \setminus \{a'\}} \times S(T_\infty R^{\{a'\}}) \hookrightarrow C_e(R). \end{aligned}$$

**Summary.** We have just outlined a proof of the following proposition.

**Proposition 8.19.** *The disjoint union of*

- *the faces  $F_\infty(A, L, \Gamma)$  for nonempty subsets  $A$  of  $T(\Gamma)$  and*
- *the faces  $F(A, L, \Gamma)$  for subsets  $A$  of  $V(\Gamma)$  with cardinality at least 2 such that  $A \cap U(\Gamma)$  is a (possibly empty) set of consecutive vertices on a connected component of  $\mathcal{L}$ ,*

*described above, embeds canonically in  $C(R, L; \Gamma)$ . Its image is the open codimension-one boundary  $\partial_1 C(R, L; \Gamma) \setminus \partial_2 C(R, L; \Gamma)$  of  $C(R, L; \Gamma)$ . Furthermore, for any ordered pair  $e$  of  $V(\Gamma)$ , the restriction to this codimension-one boundary of the map*

$$p_e: C(R, L; \Gamma) \rightarrow C_2(R)$$

*given by Theorem 8.4 is as described above.*

Proposition 8.19 is proved in Section 8.8. It will be used to prove that  $Z^s$  and  $Z$  are independent of the used propagating forms of Theorems 7.20 and 7.40, in Chapters 9 and 10.

All the results stated so far in this chapter are sufficient to understand the proofs of Theorems 7.20 and 7.40. We prove them in detail in the rest of this chapter, which can be skipped by a reader already convinced by the outline above.

## 8.5 Detailed study of $\mathcal{S}_V(T)$

In this section, we study the configuration space  $\mathcal{S}_V(T)$  presented in Section 8.3. We first prove Theorem 8.11. Let us first describe the transformations operated by the first blow-ups, locally.

Fix  $T$ , equip  $T$  and its powers with Euclidean norms. Fix  $\tilde{w}_0 \in \overline{\mathcal{S}}_V(T)$  and  $b(V) \in V$ . Identify  $\overline{\mathcal{S}}_V(T)$  with the unit sphere  $\mathbb{S}(T^V \setminus \{b(V)\})$  of  $T^V \setminus \{b(V)\}$ . Then  $\tilde{w}_0$  is viewed as a map from  $V$  to  $T$  such that  $\tilde{w}_0(b(V)) = 0$  and  $\sum_{v \in V} \|\tilde{w}_0(v)\|^2 = 1$ . Recall Definition 8.1 for a partition associated to a map and the associated notation.

**Definition 8.20.** A *based partition* of a finite set  $V$  equipped with a base-point  $b(V) \in V$  is a partition  $K(V)$  of  $V$  into nonempty subsets  $A$  equipped with basepoints  $b(A)$  such that

- $b(A) \in A$ , and



- if  $b(V) \in A$ , then  $b(A) = b(V)$ .

Fix  $\tilde{w}_0$ . Let  $K(V) = K(V; \tilde{w}_0)$  be the associated fixed partition. Fix associated basepoints so that  $K(V)$  becomes a based partition.

In general for a based subset  $A$  of  $V$  equipped with a based partition  $(K(A), b)$ , define the set  $O(A, K(A), b, T)$  of maps  $w : V \rightarrow T$  such that

- $\sum_{B \in K(A)} \|w(b(B))\|^2 = 1$ ,
- $w(b(A)) = 0$ ,  $w(V \setminus A) = \{0\}$ , and
- two elements of  $A$  that belong to different kids of  $A$  are mapped to different points of  $T$  by  $w$ .

There is a straightforward identification of  $O(V, K(V), b, T)$  with an open subset of  $\overline{\mathfrak{S}}_V(T)$ , which contains  $\tilde{w}_0$ . Let  $w_0 \in O(V, K(V), b, T)$  denote the element that corresponds to  $\tilde{w}_0$  under this identification. Set

$$W_V = O(V, K(V), b, T) \cap \left( \bigcap_{A \in D(V)} \Delta_A(\overline{\mathfrak{S}}_V(T)) \right).$$

Note that  $K(V)$  is a set that is naturally based by the element  $b(K(V))$  of  $K(V)$  that contains  $b(V)$ . Then  $W_V$  is an open subset of  $\mathfrak{S}(T^{K(V) \setminus \{b(K(V))\}})$ . It is the image  $\check{\mathfrak{S}}(T^{K(V) \setminus \{b(K(V))\}})$  of  $\check{\mathfrak{S}}_{K(V)}(T)$  under the canonical identification of  $\overline{\mathfrak{S}}_{K(V)}(T)$  with  $\mathfrak{S}(T^{K(V) \setminus \{b(K(V))\}})$ .

For  $A \in K(V)$ , view the elements of  $T^{A \setminus \{b(A)\}}$  as the maps from  $V$  to  $T$  that map  $(V \setminus A) \cup \{b(A)\}$  to 0, and let  $T_{< \varepsilon}^{A \setminus \{b(A)\}}$  denote the ball of its elements of norm smaller than  $\varepsilon$ . Note the easy lemma.

**Lemma 8.21.** *Let  $K(V)$  be a based partition of  $V$ . Let  $w_0 \in W_V$ . Then there exist an open neighborhood  $N(w_0)$  of  $w_0$  in  $W_V$  and an  $\varepsilon \in ]0, \infty[$  such that the map*

$$\begin{aligned} N(w_0) \times \prod_{A \in D(V)} T_{< \varepsilon}^{A \setminus \{b(A)\}} &\rightarrow \overline{\mathfrak{S}}_V(T) \\ (w, (\mu_A \tilde{w}_A)_{A \in D(V)}) &\mapsto w + \sum_{A \in D(V)} \mu_A \tilde{w}_A \end{aligned}$$

is an open embedding whose image does not meet diagonals that do not correspond to (nonstrict) subsets  $B$  of daughters of  $V$ .

□

In particular, the first blow-ups that will affect this neighborhood of  $\tilde{w}_0$  in  $\overline{\mathfrak{S}}_V(T)$  are blow-ups along diagonals corresponding to daughters of  $V$ . For any daughter  $A$  of  $V$ , the above identification identifies the normal bundle to  $\Delta_A(\overline{\mathfrak{S}}_V(T))$  with  $T^{A \setminus \{b(A)\}}$ , and the corresponding blow-up replaces the factor  $T_{< \varepsilon}^{A \setminus \{b(A)\}}$  with  $[0, \varepsilon[ \times \mathfrak{S}(T^{A \setminus \{b(A)\}})$ . Thus, it is clear that the blow-ups corresponding to different daughters of  $V$  commute. Note that our argument also proves the following lemma.

**Lemma 8.22.** *Let  $A$  be a subset of  $V$ . The fiber of the unit normal bundle to  $\Delta_A(\overline{\mathcal{S}}_V(T))$  is  $\overline{\mathcal{S}}_A(T)$ .*

□

When performing the blow-ups along the diagonals corresponding to the daughters of  $V$ , we replace  $\mu_A \tilde{w}_A \in T^{A \setminus \{b(A)\}}$ , for  $\mu_A \in [0, \varepsilon[$  and  $\tilde{w}_A \in \mathbb{S}(T^{A \setminus \{b(A)\}})$ , with  $(\mu_A, \tilde{w}_A)$ . Thus, we replace  $0 \in \prod_{A \in D(V)} T^{A \setminus \{b(A)\}}_{< \varepsilon}$  with the set of normal vectors  $\tilde{w}_A$  that pop up during the blow-ups.

**Lemma 8.23.** *In particular, with the notation of Lemma 8.21, we get a chart of*

$$\mathcal{B} \left( O(V, K(V), b, T), (\Delta_A(\overline{\mathcal{S}}_V(T)) \cap O(V, K(V), b, T))_{A \in D(V)} \right),$$

which maps

$$(w, (\mu_A, \tilde{w}_A)_{A \in D(V)}) \in N(w_0) \times \prod_{A \in D(V)} ([0, \varepsilon[ \times \mathbb{S}(T^{A \setminus \{b(A)\}}))$$

to the element

$$\left( w + \sum_{A \in D(V)} \mu_A \tilde{w}_A, (\tilde{w}_A)_{A \in D(V): \mu_A=0} \right)$$

of

$$\mathcal{B} \left( O(V, K(V), b, T), (\Delta_A(\overline{\mathcal{S}}_V(T)) \cap O(V, K(V), b, T))_{A \in D(V)} \right),$$

where the  $\tilde{w}_A$  are the normal vectors viewed in  $\overline{\mathcal{S}}_A(T)$  that popped up during the blow-ups.

We can construct an atlas of  $\mathcal{B}(O(V, K(V), b, T), (\Delta_A(\overline{\mathcal{S}}_V(T)))_{A \in D(V)})$  with charts of this form.

□

In order to conclude and get charts of manifolds with ridges, we blow up the

$$\overline{\mathcal{S}}_A(T) \cong \mathbb{S}(T^{A \setminus \{b(A)\}})$$

for the daughters  $A$  of  $V$ , and we iterate. Such an iteration produces a parenthesization of  $V$  as in Definition 8.2.

**Definition 8.24.** A  $\Delta$ -parenthesization of  $V$  is a parenthesization  $\mathcal{P}$  of  $V$  such that  $V \in \mathcal{P}$ . The *daughters* of an element  $A$  of a parenthesization  $\mathcal{P}$  (with respect to  $\mathcal{P}$ ) are the maximal elements of  $\mathcal{P}$  strictly included in  $A$ .

The *mother* of an element  $A$  in  $\mathcal{P}$  is the smallest element of  $\mathcal{P}$  that strictly contains  $A$  (if there is one). A  $\Delta$ -parenthesization  $\mathcal{P}$  is organized as a tree, in which the vertices are the elements of  $\mathcal{P}$  and the edges are in one-to-one correspondence with the pairs (daughter, mother) of  $\mathcal{P}^2$ . We orient its edges from the daughter to her mother, as in Figure 8.3. The *sons* of an element  $A$  of  $\mathcal{P}$  are the singletons consisting of elements of  $A$  that do not belong to a daughter of  $A$  (with respect to  $\mathcal{P}$ ). Any element  $A$  of  $\mathcal{P}$  is equipped with the set  $K(A, \mathcal{P})$  ( $= K(A)$  when  $\mathcal{P}$  is fixed) of the *kids* of  $A$ , which are its daughters and its sons.

**Example 8.25.** The trees associated to the parenthesizations  $\{V, A_{123}, A_{23}\}$  and  $\{V, A_{12}, A_{34}\}$ , which correspond to Figures 8.1 and 8.2 in Section 8.1, are pictured in Figure 8.3. With respect to the parenthesization  $\{V, A_{123}, A_{23}\}$ , the daughter  $A_{123}$  of  $V$  has two kids, its son  $\{1\}$  and its daughter  $A_{23}$ .

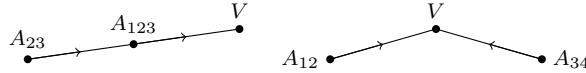


Figure 8.3: Trees associated to  $\Delta$ -parenthesizations

**Definition 8.26.** Let  $\mathcal{T}(V)$  (resp.  $\mathcal{T}_\Delta(V)$ ) denote the set of parenthesizations (resp.  $\Delta$ -parenthesizations) of  $V$ . Fix  $\mathcal{P} \in \mathcal{T}_\Delta(V)$ . Choose a basepoint  $b(A) = b(A; \mathcal{P})$  for any  $A \in \mathcal{P}$  so that if  $A$  and  $B$  are in  $\mathcal{P}$ , if  $B \subset A$ , and if  $b(A) \in B$ , then  $b(B) = b(A)$ . When  $A \in \mathcal{P}$ ,  $D(A)$  denotes the set of daughters of  $A$ . The basepoint  $b(B)$  of a son  $B$  of  $A \in \mathcal{P}$  is its unique point. A  $\Delta$ -parenthesization equipped with basepoints as above is called a *based  $\Delta$ -parenthesization*.

Recall the canonical identification of the set  $O(A, K(A), b, T)$  of Definition 8.20 with an open subset of  $\overline{\mathcal{S}}_A(T)$ . Set

$$W_A = O(A, K(A), b, T) \cap \left( \bigcap_{B \in D(A)} \Delta_B(\overline{\mathcal{S}}_A(T)) \right).$$

Note that  $W_A$  may be identified canonically with an open subset of the sphere  $\overline{\mathcal{S}}_{K(A)}(T)$ .

For  $((\mu_A)_{A \in \mathcal{P} \setminus \{V\}}, (w_A)_{A \in \mathcal{P}}) \in (\mathbb{R}^+)^{\mathcal{P} \setminus \{V\}} \times \prod_{A \in \mathcal{P}} W_A$ , and for  $B \in \mathcal{P}$ , define  $v_B((\mu_A)_{A \in \mathcal{P} \setminus \{V\}}, (w_A)_{A \in \mathcal{P}})$  as the following map from  $B$  to  $T$ :

$$\begin{aligned} v_B((\mu_A), (w_A)) &= \sum_{C \in \mathcal{P}: C \subseteq B} \left( \prod_{D \in \mathcal{P}: C \subseteq D \subseteq B} \mu_D \right) w_C \\ &= w_B + \sum_{C \in D(B)} \mu_C \left( w_C + \sum_{D \in D(C)} \mu_D (w_D + \dots) \right). \end{aligned}$$

The construction of  $v_B$  is illustrated in Figure 8.4. We can see it on the subtree whose vertices are the subsets of  $B$ . The map  $v_B$  is the sum over the vertices  $C$  of this subtree of the maps  $w_C$  associated to its vertices, multiplied by the products of the coefficients  $\mu_D$  associated to the edges of the path from  $C$  to  $B$ .

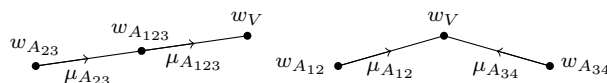


Figure 8.4: About the construction of  $v_B$

Note that  $v_B((\mu_A), (w_A)) \in O(B, K(B), b, T)$  when the  $\mu_A$  are small enough. Also note the following easy lemma.

**Lemma 8.27.** *For any  $(w_A^0)_{A \in \mathcal{P}} \in \prod_{A \in \mathcal{P}} W_A$ , there exists a neighborhood  $N((w_A^0))$  of  $0 \times (w_A^0)$  in  $(\mathbb{R}^+)^{\mathcal{P} \setminus \{V\}} \times \prod_{A \in \mathcal{P}} W_A$  such that, for any  $((\mu_A)_{A \in \mathcal{P} \setminus \{V\}}, (w_A)_{A \in \mathcal{P}}) \in N((w_A^0))$ ,*

- *if  $v_V((\mu_A), (w_A))$  is constant on  $B$  for a subset  $B$  of  $V$ , then  $B$  is included in (or equal to) a daughter of  $V$ ,*
- *if  $\mu_A \neq 0$  for any  $A \in \mathcal{P}$ , then  $v_V((\mu_A), (w_A))$  is an injective map from  $V$  to  $T$ .*

□

When the construction of Theorem 8.11 makes sense, we denote a point of our configuration space  $\mathcal{S}_V(T)$  as a tuple

$$\left( v_V((\mu_A), (w_A)), \left( v_B((\mu_A), (w_A)) \right)_{B: \mu_B=0} \right),$$

which contains its blowdown projection  $v_V((\mu_A), (w_A))$  in  $\overline{\mathcal{S}}_V(T)$  followed by the normal vectors  $v_B((\mu_A), (w_A))$  that have popped up during the blow-ups. Lemma 8.22 ensures that the normal vectors are elements of  $\overline{\mathcal{S}}_B(T)$  for some  $B$ . These normal vectors are nonconstant maps from  $B$  to  $T$  up to translation and dilation.

**Theorem 8.28.** *Theorem 8.11 is correct and defines  $\mathcal{S}_V(T)$ . Let  $\mathcal{P}$  be a  $\Delta$ -parenthesization of  $V$ . Let  $\mathcal{S}_{V, \mathcal{P}}(T)$  be the space of the elements that have been transformed by the blow-ups along the closures of the  $(\Delta_A(T))_{A \in \mathcal{P} \setminus \{V\}}$  and that have not been transformed by other blow-ups. Then  $\mathcal{S}_{V, \mathcal{P}}(T)$  is canonically diffeomorphic to*

$$\prod_{A \in \mathcal{P}} \check{\mathcal{S}}_{K(A)}(T).$$

The composed blowdown map sends  $((w_A)_{A \in \mathcal{P}}) \in \prod_{A \in \mathcal{P}} \check{\mathcal{S}}_{K(A)}(T)$  to the map that sends an element of a kid  $B$  of  $V$  to  $w_V(B)$ . For  $A \in \mathcal{P} \setminus \{V\}$ , the  $w_A$  are (similarly identified with) the unit normal vectors that have appeared during the blow-ups.

As a set,  $\mathcal{S}_V(T)$  is the disjoint union over the  $\Delta$ -parenthesizations  $\mathcal{P}$  of  $V$  of the  $\mathcal{S}_{V, \mathcal{P}}(T)$ .

Any based  $\Delta$ -parenthesization  $\mathcal{P}$  of  $V$  and any  $(w_A^0)_{A \in \mathcal{P}} \in \prod_{A \in \mathcal{P}} W_A$  provide the following smooth open embedding  $\psi(\mathcal{P}, (w_A^0)_{A \in \mathcal{P}})$  from a neighborhood  $N((w_A^0))$  as in Lemma 8.27 to  $\mathcal{S}_V(T)$ :

$$\begin{aligned} N((w_A^0)) &\hookrightarrow \mathcal{S}_V(T) \\ ((\mu_A)_{A \in \mathcal{P} \setminus \{V\}}, (w_A)_{A \in \mathcal{P}}) &\mapsto \left( \begin{array}{c} v_V((\mu_A), (w_A)), \\ (v_B((\mu_A), (w_A)))_{\{B \in \mathcal{P} \setminus \{V\} : \mu_B = 0\}} \end{array} \right). \end{aligned}$$

This embedding restricts to  $N((w_A^0)) \cap ((\mathbb{R}^{+*})^{\mathcal{P} \setminus \{V\}} \times \prod_{A \in \mathcal{P}} W_A)$  as a diffeomorphism onto an open subset of  $\check{\mathcal{S}}_V(T)$ . Furthermore, the open images of the embeddings  $\psi(\mathcal{P}, (w_A^0)_{A \in \mathcal{P}})$  for different  $(\mathcal{P}, (w_A^0)_{A \in \mathcal{P}})$  cover  $\mathcal{S}_V(T)$ , and the codimension of  $\mathcal{S}_{V, \mathcal{P}}(T)$  in  $\mathcal{S}_V(T)$  equals the cardinality of  $\mathcal{P} \setminus \{V\}$ .

PROOF: Note that the images of the embeddings corresponding to  $\mathcal{P} = \{V\}$  cover  $\check{\mathcal{S}}_V(T)$  trivially. The theorem is obviously true when  $|V| = 2$ . We proceed by induction on  $|V|$ . By induction, for each daughter  $A$  of  $V$ , the space  $\mathcal{S}_A(T)$  is covered by images of smooth open embeddings  $\psi(\mathcal{P}_A, (w_B^0)_{B \in \mathcal{P}_A})$  from  $N((w_B^0)_{B \in \mathcal{P}_A})$  to some open subset  $U_A$  of  $\mathcal{S}_A(T)$  associated to parenthesizations  $\mathcal{P}_A$  of  $A$ , as in the statement. So Lemma 8.23 leaves us with the proof that  $\psi(\mathcal{P}, (w_A^0)_{A \in \mathcal{P}})$  is smooth and open for any  $(w_A^0)_{A \in \mathcal{P}} \in \prod_{A \in \mathcal{P}} W_A$ , for a small neighborhood  $N((w_A^0))$  satisfying the conditions of Lemma 8.27. It suffices to check that the change of coordinates replacing the coordinates  $(\tilde{\mu}_A, \tilde{w}_A)$  written as  $(\mu_A, \tilde{w}_A)$  in Lemma 8.23 by the coordinates  $(\mu_A, v_A)$  compatible with  $\psi(\mathcal{P}, (w_A^0)_{A \in \mathcal{P}})$  is smooth and open for each daughter  $A$  of  $V$ . We have  $\tilde{\mu}_A \tilde{w}_A = \mu_A v_A$ ,  $\sum_{a \in A} \|\tilde{w}_A(a)\|^2 = 1$ , and  $\sum_{B \in D(A)} \|v_A(b(B))\|^2 = 1$ . We get

$$\begin{aligned} \tilde{w}_A &= \left( \sum_{a \in A} \|v_A(a)\|^2 \right)^{-1/2} v_A, \quad v_A = \left( \sum_{B \in D(A)} \|\tilde{w}_A(b(B))\|^2 \right)^{-1/2} \tilde{w}_A, \\ \tilde{\mu}_A &= \left( \sum_{a \in A} \|v_A(a)\|^2 \right)^{1/2} \mu_A, \quad \text{and} \quad \mu_A = \left( \sum_{B \in D(A)} \|\tilde{w}_A(b(B))\|^2 \right)^{1/2} \tilde{\mu}_A, \end{aligned}$$

where  $\sum_{a \in A} \|v_A(a)\|^2 \geq 1$ , and  $\sum_{B \in D(A)} \|\tilde{w}_A(b(B))\|^2 > 0$ .  $\square$

The above proof also proves the following two propositions, with the notation  $\partial_r$  introduced in the beginning of Section 2.1.4.

**Proposition 8.29.** *Let  $\mathcal{T}_{r,\Delta}(V)$  be the subset of  $\mathcal{T}_\Delta(V)$  consisting of the  $\Delta$ -parenthesizations of  $V$  of cardinality  $r$ . Then*

$$\partial_{r-1}(\mathcal{S}_V(T)) \setminus \partial_r(\mathcal{S}_V(T)) = \sqcup_{\mathcal{P} \in \mathcal{T}_{r,\Delta}(V)} \mathcal{S}_{V,\mathcal{P}}(T).$$

For two  $\Delta$ -parenthesizations  $\mathcal{P}$  and  $\mathcal{P}'$ , if  $\mathcal{P} \subset \mathcal{P}'$ , then  $\mathcal{S}_{V,\mathcal{P}'}(T) \subset \overline{\mathcal{S}_{V,\mathcal{P}}(T)}$ .

PROOF: We can deduce the first assertion from the charts of Theorem 8.28. Let  $c_0 = (w_A)_{A \in \mathcal{P}'} \in \mathcal{S}_{V,\mathcal{P}'}(T)$  and let  $\Psi = \psi(\mathcal{P}', (w_A)_{A \in \mathcal{P}'}): N((w_A)) \rightarrow \mathcal{S}_V(T)$  be a smooth open embedding as in Theorem 8.28. Let  $\varepsilon \in ]0, \infty[$  be such that  $[0, \varepsilon]^{\mathcal{P}' \setminus \{V\}} \times \{(w_A)_{A \in \mathcal{P}'}\} \subset N((w_A)_{A \in \mathcal{P}'})$ . For  $t \in [0, 1]$ , set

$$\mu_A(t) = \begin{cases} \varepsilon t & \text{if } A \in \mathcal{P}' \setminus \mathcal{P} \\ 0 & \text{if } A \in \mathcal{P} \setminus \{V\} \end{cases}$$

and let  $c(t) = \Psi((\mu_A(t))_{A \in \mathcal{P}' \setminus \{V\}}, (w_A)_{A \in \mathcal{P}'})$ . Then  $c(t) \in \mathcal{S}_{V,\mathcal{P}}(T)$  for any  $t \in ]0, 1]$ , and  $\lim_{t \rightarrow 0} c(t) = c_0$ .  $\square$

**Proposition 8.30.** *Any injective linear map  $\phi$  from a vector space  $T$  to another such  $T'$  induces a canonical embedding  $\phi_*: \mathcal{S}_V(T) \rightarrow \mathcal{S}_V(T')$ . This embedding maps an element  $((w_A)_{A \in \mathcal{P}})$  of  $\mathcal{S}_{V,\mathcal{P}}(T)$  to the element  $((\phi \circ w_A)_{A \in \mathcal{P}})$  of  $\mathcal{S}_{V,\mathcal{P}}(T')$ . If  $\psi$  is another injective linear map from a vector space  $T'$  to a third vector space  $T''$ , then we have  $(\psi \circ \phi)_* = \psi_* \circ \phi_*$ .*

$\square$

Finally, let us check the following proposition, which implies Theorem 8.12.

**Proposition 8.31.** *Let  $A$  be a finite subset of cardinality at least 2 of a finite set  $V$ . For  $\mathcal{P} \in \mathcal{T}_\Delta(V)$ , define*

$$\mathcal{P}_A = \{B \cap A : B \in \mathcal{P}, |B \cap A| \geq 2\}$$

in  $\mathcal{T}_\Delta(A)$ . For  $C \in \mathcal{P}_A$ , let  $\hat{C}$  be the smallest element of  $\mathcal{P}$  that contains  $C$  or is equal to  $C$ . For

$$w = (w_B \in \check{\mathcal{S}}_{K(B)}(T))_{B \in \mathcal{P}} \in \mathcal{S}_{V,\mathcal{P}}(T),$$

and for  $C \in \mathcal{P}_A$ , define  $w'_C$  to be the natural restriction of  $w_{\hat{C}}$  to  $K(C, \mathcal{P}_A)$ . Then set

$$p_A(w) = ((w'_C)_{C \in \mathcal{P}_A}) \in \mathcal{S}_{A,\mathcal{P}_A}(T).$$

This consistently defines a smooth map

$$p_A: \mathcal{S}_V(T) \rightarrow \mathcal{S}_A(T).$$

The map  $p_A$  is the unique continuous extension from  $\mathcal{S}_V(T)$  to  $\mathcal{S}_A(T)$  of the restriction map from  $\check{\mathcal{S}}_V(T)$  to  $\check{\mathcal{S}}_A(T)$ .

PROOF: It is easy to see that this restriction map is well-defined. In order to prove that it is smooth, use the charts of Theorem 8.28. Fix a  $\Delta$ -parenthesization  $\mathcal{P}$  of  $V$ . Let  $\mathcal{P}_A$  be the induced  $\Delta$ -parenthesization of  $A$ . Fix basepoints  $b_A$  for  $\mathcal{P}_A$  according to the rule in Definition 8.26. Fix basepoints for the elements  $B$  of  $\mathcal{P}$  so that  $b(B) = b_A(A \cap B)$  when  $B \cap A \in \mathcal{P}_A$ , and  $b(B) \in B \cap A$  when  $B \cap A \neq \emptyset$ . According to Theorem 8.28, it suffices to prove smoothness in charts involving the based  $\Delta$ -parenthesizations  $\mathcal{P}$  and  $\mathcal{P}_A$ . So it suffices to prove that the projections on the factors that contain the restrictions maps  $w'_C$  are smooth for  $C \in \mathcal{P}_A$ , and that the projections on the factors that contain the dilation factors  $\mu_C$  are smooth for  $C \in \mathcal{P}_A \setminus \{A\}$ .

With our charts and our conditions on the basepoints, for any  $C \in \mathcal{P}_A$ , we have

$$w'_C = \frac{1}{g(C, w_{\hat{C}})} w_{\hat{C}}|_{K(C, \mathcal{P}_A)},$$

where

$$g(C, w_{\hat{C}}) = \sqrt{\sum_{D \in K(C, \mathcal{P}_A)} \|w_{\hat{C}}(D)\|^2}$$

is not zero since  $w_{\hat{C}}$  is nonconstant on  $K(C, \mathcal{P}_A)$ . So, the projection on the factor of  $w'_C$  is smooth.

Let  $C \in \mathcal{P}_A \setminus \{A\}$ , and let  $m(C)$  denote the *mother* of  $C$  in  $\mathcal{P}_A$ . Let  $E$  be the set of basepoints of the kids of  $m(C)$  distinct from  $C$  with respect to  $\mathcal{P}_A$ . Consider the elements  $B_i \in \mathcal{P}$ , for  $i = 1, 2, \dots, k(B)$ , such that  $C = B_i \cap A$  for any  $i \leq k(C)$ , where  $(\hat{C} = B_1) \subset B_2 \cdots \subset B_{k(C)}$ . We have  $b(B_i) = b_A(C)$  for any  $i \in k(C)$  and  $b(\widehat{m(C)}) = b_A(m(C))$ .

Then the restrictions of the configurations  $w'_{m(C)} + \mu_C w'_C$  and

$$w_{\widehat{m(C)}} + \left( \prod_{i=1}^{k(C)} \mu_{B_i} \right) w_{\hat{C}}$$

to  $C \cup E$  coincide up to dilation. So  $g(m(C), w_{\widehat{m(C)}}) \mu_C w'_C$  coincides with

$$\left( \prod_{i=1}^{k(C)} \mu_{B_i} \right) g(C, w_{\hat{C}}) w'_C$$

on  $C$ , and we have

$$\mu_C = \left( \prod_{i=1}^{k(C)} \mu_{B_i} \right) \frac{g(C, w_C)}{g(m(C), \widehat{w_{m(C)}})}.$$

Thus  $\mu_C$  is smooth (it is defined even when the  $\mu_{B_i}$  are negative).  $\square$

Proposition 8.13 follows from Propositions 8.29 and 8.31.  $\square$

**PROOF OF LEMMA 8.17:** The structure of a smooth manifold with ridges of  $\mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$  in Lemma 8.16 can be alternatively deduced from the charts of Theorem 8.28. These charts also show that  $\mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$  is a submanifold transverse to the ridges of  $\mathcal{S}_{V(\check{\Gamma})}(T_{L(z)}\check{R})$ . So its codimension-one faces are the intersections of  $\mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$  with the codimension-one faces of  $\mathcal{S}_{V(\check{\Gamma})}(T_{L(z)}\check{R})$ . Then Lemma 8.17 follows from Proposition 8.13.  $\square$

## 8.6 Blowing up diagonals

In the rest of this chapter,  $M$  is a smooth manifold without boundary of dimension  $\delta > 0$ . It is not necessarily oriented. The set of injective maps from  $V$  to  $M$  is denoted by  $\check{C}_V[M]$  with brackets instead of parentheses (we have  $\check{C}_V[\check{R}] = \check{C}_V(\check{R})$ , but  $\check{C}_V[R] \neq \check{C}_V(R)$ ).

**Theorem 8.32.** *Let  $M$  be a manifold. Let  $V$  be a finite set.*

*Set  $C_{V,|V|+1}[M] = M^V$ . For  $k = |V|, \dots, 3, 2$ , define  $C_{V,k}[M]$  from  $C_{V,k+1}[M]$  by blowing up the closures of (the preimages under the composition of the previous blowdown maps of) the  $\Delta_A(M^V)$  such that  $|A| = k$ . At each step, the blown-up manifolds are smooth and transverse to the ridges, and the resulting blown-up manifold is independent of the order choice. Thus, this process gives rise to a canonical smooth manifold  $C_V[M] = C_{V,2}[M]$  with ridges equipped with its composed blowdown projection*

$$p_b: C_V[M] \rightarrow M^V.$$

- *Let  $f \in M^V$  be a map from  $V$  to  $M$ . Then  $p_b^{-1}(f)$  is canonically diffeomorphic to  $\prod_{A \in D(V;f)} \mathcal{S}_A(T_{f(A)}M)$ . We will denote an element  $x \in p_b^{-1}(f)$  of  $C_V[M]$  by  $(f \in M^V, (w_A \in \mathcal{S}_A(T_{f(A)}M))_{A \in D(V;f)})$ , with the notation of Definition 8.1.*
- *For any open subset  $U$  of  $M$ ,  $C_V[U] = p_b^{-1}(U^V)$ .*
- *The space  $\check{C}_V[M]$  is dense in  $C_V[M]$ .*



- If  $M$  is compact, then  $C_V[M]$  is compact, too.
- Any choice of a basepoint  $b = b(V)$  of  $V$  and of an open embedding

$$\phi: \mathbb{R}^\delta \rightarrow M$$

induces the diffeomorphism

$$\psi(\phi, b): \mathbb{R}^\delta \times \mathbb{R}^+ \times \mathcal{S}_V(\mathbb{R}^\delta) \rightarrow C_V[\phi(\mathbb{R}^\delta)]$$

described below.

Let  $u \in \mathbb{R}^\delta$ ,  $\mu \in \mathbb{R}^+$ , and  $n \in \mathcal{S}_V(\mathbb{R}^\delta)$ . Then  $\psi(\phi, b)$  satisfies

$$p_b(\psi(\phi, b)(u, \mu, n)) = \phi \circ (u + \mu p_b(n)),$$

where  $p_b(n)$  is viewed as a map from  $V$  to  $\mathbb{R}^\delta$  such that  $p_b(n)(b) = 0$  and  $\sum_{v \in V} \|p_b(n)(v)\|^2 = 1$ , and  $(u + \mu p_b(n))$  denotes the map from  $V$  to  $\mathbb{R}^\delta$  obtained from  $p_b(n)$  by composition by the homothety with ratio  $\mu$ , followed by the translation of vector  $u$ . For  $u \in \mathbb{R}^\delta$  and  $n \in \check{\mathcal{S}}_V(\mathbb{R}^\delta)$ , the map  $p_b(\psi(\phi, b)(u, 0, n))$  is constant with value  $\phi(u)$  and we have

$$\psi(\phi, b)(u, 0, n) = (\phi(u)^V, (T_u \phi)_*(n)).$$

The restriction of  $\psi(\phi, b)$  to  $\mathbb{R}^\delta \times \mathbb{R}^{+*} \times \check{\mathcal{S}}_V(\mathbb{R}^\delta)$  is a diffeomorphism onto  $\check{C}_V[\phi(\mathbb{R}^\delta)]$ .

- An embedding  $\phi$  of a manifold  $M_1$  into another such  $M_2$  induces the canonical embedding  $\phi_*$  such that

$$\phi_* \left( f \in M_1^V, (w_A \in \mathcal{S}_A(T_{f(A)} M_1))_{A \in D(V; f)} \right) = (\phi \circ f, (T_{f(A)} \phi)_*(w_A))$$

from  $C_V[M_1]$  to  $C_V[M_2]$ . If  $\psi$  is an embedding from  $M_2$  to another manifold  $M_3$ , then we have  $(\psi \circ \phi)_* = \psi_* \circ \phi_*$ .

PROOF: Start with  $M = \mathbb{R}^\delta$  equipped with its usual Euclidean norm. Fix  $b(V)$ . Any map  $f$  from  $V$  to  $\mathbb{R}^\delta$  may be written as  $f(b(V)) + y$ , for a unique element  $y \in (\mathbb{R}^\delta)^{V \setminus \{b(V)\}}$ . Then blowing up  $(\mathbb{R}^\delta)^V$  along the diagonal  $\Delta_V((\mathbb{R}^\delta)^V)$ , described by the equation  $y = 0$ , replaces  $(\mathbb{R}^\delta)^{V \setminus \{b(V)\}}$  with  $\mathbb{R}^+ \times \mathcal{S}((\mathbb{R}^\delta)^{V \setminus \{b(V)\}})$  and provides a diffeomorphism from  $\mathbb{R}^\delta \times \mathbb{R}^+ \times \mathcal{S}((\mathbb{R}^\delta)^{V \setminus \{b(V)\}})$  to  $\mathcal{B}((\mathbb{R}^\delta)^V, \Delta_V((\mathbb{R}^\delta)^V))$ .

The diagonals corresponding to strict subsets of  $V$  are products by  $\mathbb{R}^\delta \times \mathbb{R}^+$  of the diagonals corresponding to the same subsets for the manifold  $\mathcal{S}((\mathbb{R}^\delta)^{V \setminus \{b(V)\}}) \cong \check{\mathcal{S}}_V(\mathbb{R}^\delta)$ , which was studied in the previous subsection.

Thus,  $C_V[\mathbb{R}^\delta]$  is well-defined, and our diffeomorphism lifts to a diffeomorphism from  $\mathbb{R}^\delta \times \mathbb{R}^+ \times \mathcal{S}_V(\mathbb{R}^\delta)$  to  $C_V[\mathbb{R}^\delta]$ . So the composition of this diffeomorphism with the product of the charts obtained in Theorem 8.28 by the identity map yields an atlas of  $C_V[\mathbb{R}^\delta]$ .

For a diffeomorphism  $\phi$  from  $\mathbb{R}^\delta$  to an open subspace  $U$  of a manifold, the diffeomorphism

$$\phi^V: (\mathbb{R}^\delta)^V \rightarrow U^V$$

preserves diagonals. So  $C_V[U]$  is well-defined for any open subset  $U$  diffeomorphic to  $\mathbb{R}^\delta$ , in a manifold  $M$ . Furthermore,  $\phi^V$  lifts as a natural diffeomorphism  $\phi_*: C_V[\mathbb{R}^\delta] \rightarrow C_V[U]$ .

Note that the elements of  $C_V[\mathbb{R}^\delta]$  have the prescribed form. Since the normal bundle to a diagonal  $\Delta_A(\check{C}_V[U])$  is  $((TU^A/\Delta_A(TU)) \setminus \{0\})/\mathbb{R}^{+*}$ , the diffeomorphism  $\phi_*$  from  $C_V[\mathbb{R}^\delta]$  to  $C_V[U]$  maps

$$x = \left( f \in (\mathbb{R}^\delta)^V, (w_A \in \mathcal{S}_A(\mathbb{R}^\delta))_{A \in D(V;f)} \right)$$

to

$$\phi_*(x) = \left( \phi \circ f, ((T_{f(A)}\phi)_*(w_A))_{A \in D(V;f)} \right).$$

Then the elements of  $C_V[U]$  have the prescribed form, too.

In order to prove that  $C_V[M]$  is well-defined for a manifold  $M$ , it suffices to see that it is well-defined over an open neighborhood of any point  $c$  of  $M^V$ . Such a map  $c$  defines the partition  $K(V; c)$ . There exist pairwise disjoint open neighborhoods  $U_A$  diffeomorphic to  $\mathbb{R}^\delta$  of the  $c(A)$ , for  $A \in K(V) = K(V; c)$ . It is easy to see that  $C_V[M]$  is well-defined over  $\prod_{A \in K(V)} U_A^A$  and that it is canonically isomorphic to  $\prod_{A \in K(V)} C_A[U_A]$ , there. Thus,  $C_V[M]$  is well-defined, and its elements have the prescribed form.

If  $\phi$  is a diffeomorphism from a manifold  $M_1$  to another such  $M_2$ , then the diffeomorphism

$$\phi^V: M_1^V \rightarrow M_2^V$$

preserves diagonals. So it lifts as a natural diffeomorphism  $\phi_*: C_V[M_1] \rightarrow C_V[M_2]$ , which behaves as stated.

Then the study of the map induced by an embedding from a manifold  $M_1$  into another such  $M_2$  can be easily reduced to the case of a linear embedding from  $\mathbb{R}^k$  to  $\mathbb{R}^\delta$ . For such an embedding, use the identification of  $C_V[\mathbb{R}^\delta]$  with  $\mathbb{R}^\delta \times \mathbb{R}^+ \times \mathcal{S}_V(\mathbb{R}^\delta)$ .

The other statements are easy to check. □

**Proposition 8.33.** *Let  $B$  be a subset of  $V$ , then the restriction from  $\check{C}_V[M]$  to  $\check{C}_B[M]$  extends uniquely to a smooth map  $p_B$  from  $C_V[M]$  to  $C_B[M]$ . Let*

$f \in M^V$ . The elements  $A$  of  $D(B; f|_B)$  are of the form  $B \cap \hat{A}$  for a unique  $\hat{A}$  in  $D(V; f)$ . Then we have

$$p_B \left( f, (w_C \in \mathcal{S}_C(T_{f(C)}M))_{C \in D(V; f)} \right) = \left( f|_B, (w_{\hat{A}|_A})_{A \in D(B; f|_B)} \right).$$

PROOF: It is obvious when  $|B| = 1$ . Assume that  $|B| \geq 2$ . In order to check that this restriction map is smooth, it is enough to prove that it is smooth when  $M = \mathbb{R}^\delta$ . Assume that  $b(V) \in B$ . Use the diffeomorphism of Theorem 8.32 to write an element of  $C_V[\mathbb{R}^\delta]$  as  $(u \in \mathbb{R}^\delta, \mu \in \mathbb{R}^+, n \in \mathcal{S}_V(\mathbb{R}^\delta))$ . Then the restriction maps  $(u, \mu, n)$  to  $(u, \|p_b(n)|_B\| \mu, n|_B)$ . It is smooth according to Proposition 8.31.  $\square$

An element  $x = (f \in M^V, (w_A \in \mathcal{S}_A(T_{f(A)}M))_{A \in D(V; f)})$  of  $C_V[M]$  induces the parenthesization  $\mathcal{P}(x)$  of  $V$  that is the union over the elements  $A$  of  $D(V; f)$  of the  $\mathcal{P}_A(x)$  such that  $w_A \in \mathcal{S}_{A, \mathcal{P}_A(x)}(T_{f(A)}M)$ . (See Theorem 8.28.) Let  $\mathcal{P}$  be a parenthesization of  $V$ . Set  $C_{V, \mathcal{P}}[M] = \{x \in C_V[M] : \mathcal{P}(x) = \mathcal{P}\}$ .

The following proposition is easy to observe. (See Proposition 8.29.)

**Proposition 8.34.** *Let  $\mathcal{T}_r(V)$  be the subset of  $\mathcal{T}(V)$  consisting of the parenthesizations of  $V$  of cardinality  $r$ . Then we have*

$$\partial_r(C_V[M]) \setminus \partial_{r+1}(C_V[M]) = \sqcup_{\mathcal{P} \in \mathcal{T}_r(V)} C_{V, \mathcal{P}}[M].$$

Furthermore, for any two parenthesizations  $\mathcal{P}$  and  $\mathcal{P}'$  such that  $\mathcal{P} \subset \mathcal{P}'$ , we have  $C_{V, \mathcal{P}'}[M] \subset C_{V, \mathcal{P}}[M]$ .  $\square$

## 8.7 Blowing up $\infty$

For a finite set  $V$ , let  $V^+$  be obtained from  $V$  by adding a special element  $v_\infty$  to  $V$ . We have  $V^+ = V \sqcup \{v_\infty\}$ .

We state and prove the following generalization of Theorem 8.4, where  $C_V(R) = C_V[R, \infty]$ .

**Theorem 8.35.** *Let  $V$  be a finite set. Let  $M$  be a manifold without boundary of dimension  $\delta$ , and let  $\infty \in M$ . Set  $\check{M} = M \setminus \{\infty\}$ . Recall the manifold  $C_{V^+}[M]$  of Theorem 8.32. Let  $p_b|_{\{v_\infty\}}: C_{V^+}[M] \rightarrow M$  map a configuration to its value at  $v_\infty$ . Define  $C_V^+[M, \infty]$  to be the preimage of  $\infty$  under this map. Then  $C_V^+[M, \infty]$  is a smooth submanifold of  $C_{V^+}[M]$  transverse to the ridges equipped with charts induced by the local models of Theorems 8.32 and 8.28.*

It is the closure in  $C_{V^+}[M]$  of  $\check{C}_V[\check{M}] \times \{\infty\}^{\{v_\infty\}}$ , where  $\check{C}_V[\check{M}] \subset \check{M}^V$  is the space of injective maps from  $V$  to  $\check{M}$ .

Furthermore, the manifold  $C_V^+[M, \infty]$  is canonically diffeomorphic to the manifold  $C_V[M, \infty]$  obtained from  $M^V$  by the following process.

For a nonempty  $A \subseteq V$ , let  $\Xi_A$  be the set of maps from  $V$  to  $M$  that map  $A$  to  $\infty$ , and  $V \setminus A$  to  $\check{M}$ .

Start with  $M^V$ . Blow up  $\Xi_V$ , which is reduced to the point  $m = \infty^V$  such that  $m^{-1}(\infty) = V$ . Set

$$C_{V,|V|+1}[M, \infty] = \mathcal{B}(M^V, \infty^V).$$

Then for  $k = |V|, |V| - 1, \dots, 3, 2$ , define  $C_{V,k}[M, \infty]$  from  $C_{V,k+1}[M, \infty]$  by blowing up the closures of (the preimages under the composition of the previous blowdown maps of) the  $\Delta_A(\check{M}^V)$  such that  $|A| = k$  and the closures of (the preimages under the composition of the previous blowdown maps of) the  $\Xi_J$  such that  $|J| = k - 1$ . At each step, the blown-up manifolds are smooth and transverse to the ridges, and  $C_{V,k}[M, \infty]$  is independent of the possible order choice of the blow-ups. The obtained manifold  $C_V[M, \infty] = C_{V,2}[M, \infty]$  is a smooth manifold of dimension  $\delta|V|$ , with ridges. It is compact if  $M$  is compact. The interior of  $C_V[M, \infty]$  is  $\check{C}_V[\check{M}]$ , and the composition of the blowdown maps gives rise to a canonical smooth blowdown projection  $p_b: C_V[M, \infty] \rightarrow M^V$ .

PROOF: Let  $U_\infty$  be a small open neighborhood of  $\infty$  in  $M$ . Then the local models given by a composition of the local models of Theorems 8.32 and 8.28 make clear that  $C_V^+[U_\infty, \infty]$  has a canonical smooth structure and that  $C_V^+[U_\infty, \infty]$  is the closure in  $C_{V^+}[U_\infty]$  of  $\check{C}_V[U_\infty \setminus \{\infty\}] \times \{\infty\}^{\{v_\infty\}}$ .

Let  $U$  be an open subset of  $M$  disjoint from  $U_\infty$ . Let  $A$  be a subset of  $V$ . Consider the map  $p_b|_V: C_V^+[M, \infty] \rightarrow M^V$ . Observe

$$(p_b|_V)^{-1}(U_\infty^{A^+} \times U^{V \setminus A}) = C_A^+[U_\infty, \infty] \times C_{V \setminus A}[U].$$

The preimage of  $(U_\infty^A \times U^{V \setminus A})$  under  $p_b: C_V[M, \infty] \rightarrow M^V$  is similarly canonically diffeomorphic to  $C_A[U_\infty, \infty] \times C_{V \setminus A}[U]$ .

So we are left with the identification of a small open neighborhood of  $(p_b|_V)^{-1}(\infty^V)$  in  $C_V^+[U_\infty, \infty]$  and a small open neighborhood of  $p_b^{-1}(\infty^V)$  in  $C_V[U_\infty, \infty]$  for an arbitrarily small neighborhood  $U_\infty$  of  $\infty$  and a finite set  $V$ . Assume without loss that  $U_\infty$  is identified with  $\mathbb{R}^\delta$  by a diffeomorphism  $\varphi: \mathbb{R}^\delta \rightarrow U_\infty$  such that  $\varphi(0) = \infty$ . The naturality of the constructions leaves us with the case  $(U_\infty, \infty) = (\mathbb{R}^\delta, 0)$ .

Equip  $V^+$  with the basepoint  $v_\infty$ . The first blow-up along the small diagonal of  $V^+$  transforms  $(\mathbb{R}^\delta)^{V^+}$  to

$$C_{V^+,|V^+|}[\mathbb{R}^\delta] \cong (\mathbb{R}^\delta)^{\{v_\infty\}} \times \mathbb{R}^+ \times \overline{\mathcal{S}}_{V^+}(\mathbb{R}^\delta),$$

where the space of  $\overline{\mathcal{S}}_{V^+}(\mathbb{R}^\delta)$  of nonconstant maps from  $V^+$  to  $\mathbb{R}^\delta$  up to translation and dilation is naturally identified with the space  $S((\mathbb{R}^\delta)^V)$  of nonzero maps from  $V$  to  $\mathbb{R}^\delta$  up to dilation. The first factor  $(\mathbb{R}^\delta)^{\{v_\infty\}}$  contains the value of the configuration at  $v_\infty$ . It is zero on  $p_b(C_V^+[\mathbb{R}^\delta, 0])$ . Since  $S((\mathbb{R}^\delta)^V)$  is the unit normal bundle of  $\{0\}^V$  in  $M^V$ , we get a canonical diffeomorphism from  $C_{V^+,|V^+|}[\mathbb{R}^\delta] \cap (p_b|_{\{v_\infty\}})^{-1}(0)$  to  $C_{V,|V|+1}[\mathbb{R}^\delta, 0]$ .

Proceed by induction to get a canonical diffeomorphism from  $C_{V^+,k}[\mathbb{R}^\delta] \cap (p_b|_{\{v_\infty\}})^{-1}(0)$  to  $C_{V,k}[\mathbb{R}^\delta, 0]$  for all  $k$ .  $\square$

Theorem 8.5 is a direct consequence of Proposition 8.33 and Theorem 8.35. (Recall  $C_V(R) = C_V[R, \infty]$ .)  $\square$

An element  $x$  of  $C_V^+[M, \infty]$  induces a parenthesization  $\mathcal{P}^+(x) = \mathcal{P}(x \in C_{V^+}[M])$  of  $V^+$  as before Proposition 8.34. By Theorems 8.28 and 8.32 the element  $x$  can be written as

$$\left( p_b(x)|_V, (w_B \in \check{\mathcal{S}}_{K(B)}(T_{x(B)}M))_{B \in \mathcal{P}^+(x)} \right).$$

Let  $\mathcal{P}_s^+(x)$  denote the set of elements of  $\mathcal{P}^+(x)$  containing  $v_\infty$ . This set is totally ordered by the inclusion. So is

$$\mathcal{P}_s(x) = \{A \setminus \{v_\infty\} : A \in \mathcal{P}_s^+(x)\} = \{V(1), V(2), \dots, V(\sigma)\}$$

with  $V(i+1) \subset V(i)$ . We have  $x(V(i)) = \infty$  for all  $i$ . Let  $K_d^s(V(i))$  denote the set of kids of  $V(i)^+$  that do not contain  $v_\infty$ .

Recall from Notation 8.18 that  $\check{\mathcal{S}}(T_\infty M, A)$  denotes the set of injective maps from  $A$  to  $(T_\infty M \setminus 0)$  up to dilation. The natural basepoint choice of  $v_\infty$  identifies the set  $\check{\mathcal{S}}_{K(V(i)^+)}(T_\infty M)$  of injective maps from  $K(V(i)^+)$  to  $T_\infty M$  up to dilation and translation with  $\check{\mathcal{S}}(T_\infty M, K_d^s(V(i)))$ .

Set  $\mathcal{P}_d(x) = \mathcal{P}^+(x) \setminus \mathcal{P}_s^+(x)$ . When  $x$  is seen as an element of  $C_V[M, \infty]$ , the set  $\mathcal{P}_s(x)$  is the set of subsets  $B$  of  $V$  such that  $x$  has been transformed by the blow-up along (the closure of the preimage of)  $\Xi_B$  and  $\mathcal{P}_d(x)$  is the set of subsets  $B$  of  $V$  such that  $x$  has been transformed by the blow-up along (the closure of the preimage of)  $\Delta_B(M^V)$ .

The proof of Theorem 8.35 also proves the following proposition.

**Proposition 8.36.** *Let  $\mathcal{P}^+$  be a parenthesization of  $V^+ = V \sqcup \{v_\infty\}$ . Set*

$$C_{V,\mathcal{P}^+}[M, \infty] = \{x \in C_V^+[M, \infty] : \mathcal{P}^+(x) = \mathcal{P}^+\}.$$

Use the canonical identification of  $C_V^+[M, \infty]$  with  $C_V[M, \infty]$ . Then the stratum  $C_{V, \mathcal{P}^+}[M, \infty]$  is an open part of

$$\partial_{|\mathcal{P}^+|}(C_V[M, \infty]) \setminus \partial_{|\mathcal{P}^+|+1}(C_V[M, \infty]).$$

Let  $\mathcal{P}_s^+$  be the set of elements of  $\mathcal{P}^+$  containing  $v_\infty$ . Set  $\mathcal{P}_d = \mathcal{P}^+ \setminus \mathcal{P}_s^+$  and  $\mathcal{P}_s = \{A \setminus \{v_\infty\} : A \in \mathcal{P}_s^+\}$ . For an element  $B$  of  $\mathcal{P}_s \cup \{V\}$ , let  $K_d^s(B)$  denote the set of kids of  $B^+$  that do not contain  $v_\infty$ . The stratum  $C_{V, \mathcal{P}^+}[M, \infty]$  fibers over the space  $\check{C}_{K_d^s(V)}[\check{M}]$  of injective maps  $c$  from  $K_d^s(V)$  to  $\check{M}$ . Its fiber over such an injective map  $c$  is

$$\left( \prod_{B \in \mathcal{P}_s} \check{S}(T_\infty M, K_d^s(B)) \right) \times \left( \prod_{A \in \mathcal{P}_d} \check{S}_{K(A)}(T_{c(A)} M) \right) = \prod_{A \in \mathcal{P}^+} \check{S}_{K(A)}(T_{c(A)} M).$$

**Corollary 8.37.** *An element of  $C_V[M, \infty]$  is a map  $c$  from  $V$  to  $M$ , equipped with*

- a parenthesization  $\mathcal{P}^+$  of  $V^+ = V \sqcup \{v_\infty\}$  and induced parenthesizations  $\mathcal{P}_s$  and  $\mathcal{P}_d$  of  $V$ , as in Proposition 8.36,
- an element  $f_B$  of  $\check{S}(T_\infty M, K_d^s(B))$  for any element  $B$  of  $\mathcal{P}_s$ ,
- an element  $w_A$  of  $\check{S}_{K(A)}(T_{c(A)} M)$  for each  $A \in \mathcal{P}_d$ .

## 8.8 Finishing the proofs of the statements of Sections 8.2 and 8.4

**PROOF OF PROPOSITION 8.6:** To study the closure of  $\check{C}(R, L; \Gamma)$  in  $C_{V(\Gamma)}(R)$ , we study its intersection with some  $p_b^{-1}(\prod_{i \in I} U_i^{V_i})$  for disjoint small compact  $U_i$ . We assume that at most one  $U_i$  contains  $\infty$  and that this  $U_i$  does not meet the link. So the corresponding  $V_i$  does not contain univalent vertices, and Theorem 8.4 gives the structure of the corresponding factor.

Thus, it is enough to study  $p_A(\overline{\check{C}(R, L; \Gamma)}) \cap C_A(\phi(\mathbb{R}^3))$  when

- $\phi$  is an embedding from  $\mathbb{R}^3$  to  $\check{R}$  that maps the vertical line  $\mathbb{R}\vec{v}$  through the origin oriented from bottom to top onto  $\phi(\mathbb{R}^3) \cap L$ , so that  $\phi$  identifies  $(\mathbb{R}^3, \mathbb{R}\vec{v})$  with  $(\phi(\mathbb{R}^3), \phi(\mathbb{R}^3) \cap L)$ ,<sup>4</sup>
- the univalent vertices of  $A$  form a nonempty subset  $A_U = U(\Gamma) \cap A$  of consecutive vertices on the component  $L_i$  of  $L$  such that  $\phi(\mathbb{R}^3) \cap L = \phi(\mathbb{R}^3) \cap L_i$ .

<sup>4</sup>Here,  $L$  also denotes the image  $L(\mathcal{L})$  of  $L$ .

Let  $\mathcal{O}(A_U)$  denote the set of the linear orders  $<$  on  $A_U$  compatible with  $[i_\Gamma]$ . (This is a singleton unless  $A_U$  contains all the univalent vertices of  $L_i$ .) Via the natural maps induced by  $\phi$ , the space  $\check{C}_A(\phi(\mathbb{R}^3))$  is identified with the set of injections from  $A$  to  $\mathbb{R}^3$ , and  $p_A(\check{C}(R, L; \Gamma)) \cap \check{C}_A(\phi(\mathbb{R}^3))$  is identified with the disjoint union over  $\mathcal{O}(A_U)$  of the subsets  $\check{C}_A(\mathbb{R}^3, A_U, < \in \mathcal{O}(A_U))$  of injections that map  $A_U$  to  $\mathbb{R}\vec{v}$  so that the order induced by  $\mathbb{R}\vec{v}$  coincides with  $<$ . Fix  $< \in \mathcal{O}(A_U)$ , and write  $A_U = \{v_1, \dots, v_k\}$  so that  $v_1 < v_2 \cdots < v_k$ .

Fix  $b(A) = v_1$ , and study the closure of  $\check{C}_A(\mathbb{R}^3, A_U, <)$  in  $\mathbb{R}^3 \times \mathbb{R}^+ \times \mathcal{S}_A(\mathbb{R}^3)$ , using the diffeomorphism  $\psi(\phi, v_1)$  of Theorem 8.32. This closure is the closure of  $\mathbb{R}\vec{v} \times \mathbb{R}^+ \times \check{\mathcal{S}}_A(\mathbb{R}^3, A_U, <)$ , where  $\check{\mathcal{S}}_A(\mathbb{R}^3, A_U, <)$  is the quotient of  $\check{C}_A(\mathbb{R}^3, A_U, <)$  by translations by vectors of  $\mathbb{R}\vec{v}$  and dilations. Then the charts of Theorem 8.28 (used with basepoints that are as much as possible in  $A_U$ ) make clear that the closure of  $\check{\mathcal{S}}_A(\mathbb{R}^3, A_U, <)$  in  $\mathcal{S}_A(\mathbb{R}^3)$  consists of the limit configurations  $c$  such that  $(c(v_j) - c(v_i))$  is nonnegatively colinear with  $\vec{v}$  at any scale (i.e., in any infinitesimal configuration  $w$  that has popped up during the blow-ups) for any  $i$  and  $j$  in  $\{1, \dots, k\}$  such that  $i < j$ , and that this closure is a smooth submanifold of  $\mathcal{S}_A(\mathbb{R}^3)$  transverse to the ridges.  $\square$

This proof also proves the following lemma.

**Lemma 8.38.** *The codimension-one faces of  $C(R, L; \Gamma)$  are the intersections of  $C(R, L; \Gamma)$  with the codimension-one faces of  $C_{V(\Gamma)}(R)$ .*

$\square$

Proposition 8.19 then follows, with the help of Propositions 8.33 and 8.36.

$\square$

## 8.9 Alternative descriptions of configuration spaces

Apart from Lemma 8.39, this section will not be used in this book. It mentions other presentations of the configuration spaces studied in Sections 8.5 to 8.7 without proofs. Most of the proofs are left to the reader as exercises.

Let  $V$  be a finite set of cardinality at least 2. Let  $\mathcal{P}_{\geq 2} = \mathcal{P}_{\geq 2}(V)$  be the set of its (nonstrict) subsets of cardinality at least 2. The smooth blowdown projection from  $\mathcal{S}_A(T)$  to  $\overline{\mathcal{S}}_A(T)$  for an  $A \in \mathcal{P}_{\geq 2}(V)$  may be composed with the smooth restriction map from  $\mathcal{S}_V(T)$  to  $\mathcal{S}_A(T)$  to produce a smooth map  $\pi_A$  from  $\mathcal{S}_V(T)$  to  $\overline{\mathcal{S}}_A(T)$ .

**Lemma 8.39.** *The product over the subsets  $A$  of  $V$  with cardinality at least 2 of the  $\pi_A$  is a smooth embedding*

$$\prod_{A \in \mathcal{P}_{\geq 2}} \pi_A: \mathcal{S}_V(T) \hookrightarrow \prod_{A \in \mathcal{P}_{\geq 2}} \overline{\mathcal{S}}_A(T)$$

and the image of  $\mathcal{S}_V(T)$  is the closure of the image of the restriction of  $\prod_{A \in \mathcal{P}_{\geq 2}} \pi_A$  to  $\tilde{\mathcal{S}}_V(T)$ .

PROOF: The injectivity of  $\prod_{A \in \mathcal{P}_{\geq 2}} \pi_A$  can be seen from the description of  $\mathcal{S}_V(T)$  as a set, which is given in Theorem 8.28.  $\square$

**Proposition 8.40.** *The image  $(\prod_{A \in \mathcal{P}_{\geq 2}} \pi_A)(\mathcal{S}_V(T))$  is the subset of*

$$\prod_{A \in \mathcal{P}_{\geq 2}} \overline{\mathcal{S}}_A(T)$$

consisting of the elements  $((c_A)_{A \in \mathcal{P}_{\geq 2}})$  such that for any two elements  $A$  and  $B$  of  $\mathcal{P}_{\geq 2}$  such that  $B \subset A$ , the restriction of  $c_A$  to  $B$  coincides with  $c_B$  if it is not constant.

PROOF: Exercise.  $\square$

Thus,  $\mathcal{S}_V(T)$  can be defined as its image described in the above proposition. Similar definitions involving only cardinality 2 or 3 subsets of  $V$  can be found in [Sin04].

For a smooth manifold  $M$  without boundary, we have similar smooth maps  $\pi_A$  from  $C_V[M]$  to  $\mathcal{B}(M^A, \Delta_A(M^A))$ . They also define a smooth map

$$\prod_{A \in \mathcal{P}_{\geq 2}} \pi_A: C_V[M] \hookrightarrow \prod_{A \in \mathcal{P}_{\geq 2}} \mathcal{B}(M^A, \Delta_A(M^A)).$$

The elements of  $\mathcal{B}(M^A, \Delta_A(M^A))$  are maps  $c$  from  $A$  to  $M$  that are equipped with an element  $w \in \overline{\mathcal{S}}_A(T_{c(A)}M)$  when they are constant.

**Proposition 8.41.** *The map*

$$\prod_{A \in \mathcal{P}_{\geq 2}} \pi_A: C_V[M] \hookrightarrow \prod_{A \in \mathcal{P}_{\geq 2}} \mathcal{B}(M^A, \Delta_A(M^A))$$

is an embedding. Its image is the subset of  $\prod_{A \in \mathcal{P}_{\geq 2}} \mathcal{B}(M^A, \Delta_A(M^A))$  consisting of the elements  $((c_A)_{A \in \mathcal{P}_{\geq 2}})$  such that for any two elements  $A$  and  $B$  of  $\mathcal{P}_{\geq 2}$  such that  $B \subset A$ ,



- the restriction to  $B$  of the map  $p_b(c_A): A \rightarrow M$  coincides with  $p_b(c_B)$ , and,
- if  $p_b(c_A)$  is constant, then the restriction to  $B$  of  $(w_A(c_A) \in \overline{\mathcal{S}}_A(T_{c_A(A)}M))$  is  $w_B(c_B)$  if this restriction is not constant.

PROOF: Exercise. □

Again,  $C_V[M]$  can be defined as its image described in the above proposition, and similar definitions involving only cardinality 2 and 3 subsets of  $V$  can be found in [Sin04].

We may obtain similar statements for  $C_V[M, \infty]$ , where  $\infty \in M$ , from the definition of  $C_V[M, \infty]$  as  $C_V^+[M, \infty]$  in Theorem 8.35.

More information about the homotopy groups and the homology of the configuration spaces  $\check{C}_V[\mathbb{R}^d]$  and  $\check{C}_V[S^d]$  can be found in the book [FH01] by Edward Fadell and Sufian Husseini. See also the book [CILW24] by Ricardo Campos, Najib Idrissi, Pascal Lambrechts, and Thomas Willwacher.

# Chapter 9

## Dependence on the propagating forms

This chapter shows how our combinations of integrals over configuration spaces depend on the chosen propagating forms.

### 9.1 Introduction

In this section, we give a first general description of the variation of  $Z$  when propagating forms change in Proposition 9.2. Then we show how this proposition and a preliminary lemma (9.1) apply to prove Theorem 7.19 and two other lemmas (9.5 and 9.6), about independence of chosen propagating forms as in Definition 3.11. The typical proof of Proposition 9.2 will occupy the next sections.

Again, any closed 2-form on  $\partial C_2(R)$  extends to  $C_2(R)$  because the restriction induces a surjective map  $H^2(C_2(R); \mathbb{R}) \rightarrow H^2(\partial C_2(R); \mathbb{R})$  since we have

$$H^3(C_2(R), \partial C_2(R); \mathbb{R}) = 0.$$

**Lemma 9.1.** *Let  $(\check{R}, \tau_0)$  be an asymptotic rational homology  $\mathbb{R}^3$  as in Definition 3.8. Let  $\tau: [0, 1] \times \check{R} \times \mathbb{R}^3 \rightarrow T\check{R}$  be a smooth map whose restriction to  $\{t\} \times \check{R} \times \mathbb{R}^3$  is an asymptotically standard parallelization  $\tau_t$  of  $\check{R}$  for any  $t \in [0, 1]$ . Define  $p_\tau: [0, 1] \times \partial C_2(R) \rightarrow [0, 1] \times S^2$  by  $p_\tau(t, x) = (t, p_{\tau_t}(x))$ .*

*Let  $\omega_0$  and  $\omega_1$  be two propagating forms of  $C_2(R)$  that restrict to  $\partial C_2(R) \setminus UB_R$  as  $p_{\tau_0}^*(\omega_{0,S^2})$  and as  $p_{\tau_1}^*(\omega_{1,S^2})$ , respectively, for two volume-one forms  $\omega_{0,S^2}$  and  $\omega_{1,S^2}$  of  $S^2$ .*

*Then there exist*

- *a closed 2-form  $\tilde{\omega}_{S^2}$  on  $[0, 1] \times S^2$  whose restriction to  $\{t\} \times S^2$  is  $\omega_{t,S^2}$  for  $t \in \{0, 1\}$ ,*

- for any such  $\tilde{\omega}_{S^2}$ , a closed 2-form  $\omega^\partial$  on  $[0, 1] \times \partial C_2(R)$  whose restriction to  $\{t\} \times \partial C_2(R)$  is  $\omega_t|_{\partial C_2(R)}$  for  $t \in \{0, 1\}$ , and whose restriction to  $[0, 1] \times (\partial C_2(R) \setminus U(B_R))$  is  $p_\tau^*(\tilde{\omega}_{S^2})$ , and,
- for any such compatible  $\tilde{\omega}_{S^2}$  and  $\omega^\partial$ , a closed 2-form  $\omega$  on  $[0, 1] \times C_2(R)$  whose restriction to  $\{t\} \times C_2(R)$  is  $\omega_t$  for  $t \in \{0, 1\}$ , and whose restriction to  $[0, 1] \times \partial C_2(R)$  is  $\omega^\partial$ .

If  $\omega_0$  and  $\omega_1$  are propagating forms of  $(C_2(R), \tau_0)$  and  $(C_2(R), \tau_1)$ , then we may choose  $\omega^\partial = p_\tau^*(\tilde{\omega}_{S^2})$  on  $[0, 1] \times \partial C_2(R)$ .

PROOF: As in Lemma 3.17, there exists a one-form  $\eta_{S^2}$  on  $S^2$  such that  $d\eta_{S^2} = \omega_{1,S^2} - \omega_{0,S^2}$ . Define the closed 2-form  $\tilde{\omega}_{S^2}$  on  $[0, 1] \times S^2$  by

$$\tilde{\omega}_{S^2} = p_{S^2}^*(\omega_{0,S^2}) + d(tp_{S^2}^*(\eta_{S^2})),$$

where  $t$  is the coordinate on  $[0, 1]$ .

Now, the form  $\omega^\partial$  is defined on the boundary of  $[0, 1] \times U(B_R)$ , and it extends as a closed 2-form  $\omega^\partial$  as desired there because the restriction induces a surjective map  $H^2([0, 1] \times U(B_R); \mathbb{R}) \rightarrow H^2(\partial([0, 1] \times U(B_R)); \mathbb{R})$  since we have

$$H^3([0, 1] \times U(B_R), \partial([0, 1] \times U(B_R)); \mathbb{R}) = 0.$$

Finally, the desired form  $\omega$  is defined on the boundary of  $[0, 1] \times C_2(R)$ . It similarly extends as a closed 2-form to  $[0, 1] \times C_2(R)$ .  $\square$

When  $A$  is a subset of the set of vertices  $V(\Gamma)$  of a numbered Jacobi diagram  $\Gamma$  with support a one-manifold  $\mathcal{L}$ , the set of edges of  $\Gamma$  between two elements of  $A$  is denoted by  $E(\Gamma_A)$  (edges of  $\Gamma$  are plain), and  $\Gamma_A$  is the subgraph of  $\Gamma$  consisting of the vertices of  $A$  and the edges of  $E(\Gamma_A)$  together with the natural restriction to  $U(\Gamma) \cap A$  of the isotopy class of injections from  $U(\Gamma)$  to  $\mathcal{L}$  associated to  $\Gamma$ .

The following proposition, whose proof occupies most of this chapter, is crucial in the study of the variations of  $Z$ .

**Proposition 9.2.** *Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . Let  $L = \sqcup_{j=1}^k K_j$  be an embedding of  $\mathcal{L} = \sqcup_{j=1}^k S_j^1$  into  $\check{R}$ . Let  $\tau: [0, 1] \times \check{R} \times \mathbb{R}^3 \rightarrow T\check{R}$  be a smooth map whose restriction to  $\{t\} \times \check{R} \times \mathbb{R}^3$  is an asymptotically standard parallelization  $\tau_t$  of  $\check{R}$  for any  $t \in [0, 1]$ .<sup>1</sup> Define  $p_\tau: [0, 1] \times \partial C_2(R) \rightarrow [0, 1] \times S^2$  by  $p_\tau(t, x) = (t, p_{\tau_t}(x))$ .*

*Let  $n$  and  $m$  be positive integers. For  $i \in \underline{m}$ , let  $\tilde{\omega}(i)$  be a closed 2-form on  $[0, 1] \times C_2(R)$  whose restriction to  $\{t\} \times C_2(R)$  is denoted by  $\tilde{\omega}(i, t)$ , for any  $t \in [0, 1]$ .*

<sup>1</sup>This homotopy  $\tau$  is not useful for this statement, but we will use this notation later.

Assume that  $\tilde{\omega}(i)$  restricts to  $[0, 1] \times (\partial C_2(R) \setminus UB_R)$  as  $p_r^*(\tilde{\omega}_{S^2}(i))$ , for some closed two-form  $\tilde{\omega}_{S^2}(i)$  on  $[0, 1] \times S^2$  such that  $\int_{\{t\} \times S^2} \tilde{\omega}_{S^2}(i) = 1$  for  $t \in [0, 1]$ . Recall Notation 7.29 and set

$$Z_{n,m}(t) = \sum_{\Gamma \in \mathcal{D}_{n,m}^e(\mathcal{L})} \zeta_\Gamma I\left(R, L, \Gamma, (\tilde{\omega}(i, t))_{i \in \underline{m}}\right) [\Gamma] \in \mathcal{A}_n(\sqcup_{j=1}^k S_j^1)$$

and  $Z_n(t) = Z_{n,3n}(t)$ . Let  $\mathcal{D}_{n,m}^{e,F}(\mathcal{L})$  denote the set of pairs  $(\Gamma, A)$  such that

- $\Gamma \in \mathcal{D}_{n,m}^e(\mathcal{L})$ ,
- $A \subseteq V(\Gamma)$ ,  $|A| \geq 2$ ,
- $\Gamma_A$  is a connected component of  $\Gamma$ ,
- $|A| \equiv 2 \pmod{4}$  if  $A \cap U(\Gamma) = \emptyset$ , and
- $A \cap U(\Gamma)$  is a set of consecutive vertices on a component  $\mathcal{L}_A$  of  $\mathcal{L}$  if  $A \cap U(\Gamma) \neq \emptyset$ .

For  $(\Gamma, A) \in \mathcal{D}_{n,m}^{e,F}(\mathcal{L})$ , set

$$I(\Gamma, A) = \int_{[0,1] \times F(A,L,\Gamma)} \bigwedge_{e \in E(\Gamma)} p_e^* \left( \tilde{\omega}(j_E(e)) \right) [\Gamma],$$

where  $p_e: [0, 1] \times C(R, L; \Gamma) \rightarrow [0, 1] \times C_2(R)$  is the product by the identity map  $\mathbf{1}_{[0,1]}$  of  $[0, 1]$  of the previous  $p_e$ ,  $j_E$  is the edge-numbering map of Definition 7.6, and the face  $F(A, L, \Gamma)$  of  $C(R, L; \Gamma)$  is described in Section 8.4. Set  $\mathcal{D}_n^{e,F}(\mathcal{L}) = \mathcal{D}_{n,3n}^{e,F}(\mathcal{L})$ . Then we have

$$Z_n(1) - Z_n(0) = \sum_{(\Gamma, A) \in \mathcal{D}_n^{e,F}(\mathcal{L})} \zeta_\Gamma I(\Gamma, A)$$

and

$$Z_{n,3n-2}(1) - Z_{n,3n-2}(0) = \sum_{(\Gamma, A) \in \mathcal{D}_{n,3n-2}^{e,F}(\mathcal{L})} \zeta_\Gamma I(\Gamma, A).$$

This statement simplifies as in Corollary 9.4 when  $L = \emptyset$  using the projection  $p^c: \mathcal{A}(\emptyset) \rightarrow \mathcal{A}^c(\emptyset)$ , which maps diagrams with several connected components to 0. Recall that  $\mathcal{D}_n^c$  is the subset of  $\mathcal{D}_n^e(\emptyset)$  whose elements are the numbered diagrams of  $\mathcal{D}_n^e(\emptyset)$  with one connected component.

For an oriented connected trivalent diagram  $\Gamma$ , the face  $F(V(\Gamma), \emptyset, \Gamma)$  fibers over  $\check{R}$ , and the fiber over  $x \in \check{R}$  is the space  $\check{\mathcal{S}}_{V(\Gamma)}(T_x \check{R})$  of injections from  $V(\Gamma)$  to  $T_x \check{R}$ , up to translation and dilation. See Section 8.3. We also denote this face by  $\check{\mathcal{S}}_{V(\Gamma)}(T \check{R})$ .

**Lemma 9.3.** *Let  $\Gamma \in \mathcal{D}_n^c$  be equipped with a vertex-orientation, which induces an orientation of  $C(R, \emptyset; \Gamma)$  as in Corollary 7.2. These orientations induce the orientation of  $V(\Gamma)$  described in Remark 7.5. The orientation of  $F(V(\Gamma), \emptyset, \Gamma)$  as part of the boundary of  $C(R, \emptyset; \Gamma)$  can be alternatively described as follows. The face  $F(V(\Gamma), \emptyset, \Gamma)$  is oriented as the local product  $\check{R} \times \text{fiber}$ , where the fiber is oriented as in Convention 8.10, using the above orientation of  $V(\Gamma)$ .*

PROOF: The dilation factor for the quotient  $\check{\mathcal{S}}_{V(\Gamma)}(T_x \check{R})$  plays the role of an inward normal for  $C(R, \emptyset; \Gamma)$ . The orientation of  $C(R, \emptyset; \Gamma)$  near the boundary is given by the orientation of  $\check{R}$ , followed by this inward normal, followed by the fiber orientation.  $\square$

For any pair  $e$  of  $V(\Gamma)$ , we have a natural restriction map

$$p_e: \check{\mathcal{S}}_{V(\Gamma)}(T\check{R}) \rightarrow \check{\mathcal{S}}_e(T\check{R}) \cong U\check{R},$$

which provides natural restriction maps

$$p_e: [0, 1] \times \check{\mathcal{S}}_{V(\Gamma)}(T\check{R}) \rightarrow [0, 1] \times \check{\mathcal{S}}_e(T\check{R})$$

by multiplication by  $\mathbf{1}_{[0,1]}$ .

Proposition 9.2 has the following corollary.

**Corollary 9.4.** *Assume  $L = \emptyset$ . Recall Notation 7.16. Under the assumptions of Proposition 9.2, set  $z_n(t) = p^c(Z_n(t))$  so that we have*

$$z_n(t) = z_n \left( R, (\tilde{\omega}(i, t))_{i \in \underline{3n}} \right).$$

*Then  $z_n(1) = z_n(0)$  if  $n$  is even, and  $(z_n(1) - z_n(0))$  depends only on  $(\tilde{\omega}(i)|_{[0,1] \times UB_R})_{i \in \underline{3n}}$  if  $n$  is odd. Define*

$$z_n \left( [0, 1] \times UB_R; (\tilde{\omega}(i))_{i \in \underline{3n}} \right) = \sum_{\Gamma \in \mathcal{D}_n^c} \zeta_\Gamma I(\Gamma, V(\Gamma)),$$

where

$$I(\Gamma, V(\Gamma)) = \int_{[0,1] \times \check{\mathcal{S}}_{V(\Gamma)}(TB_R)} \bigwedge_{e \in E(\Gamma)} p_e^* \left( \tilde{\omega}(j_E(e)) \right) [\Gamma].$$

Then we have

$$z_n(1) - z_n(0) = z_n \left( [0, 1] \times UB_R; (\tilde{\omega}(i))_{i \in \underline{3n}} \right)$$

for any odd integer  $n$ .

PROOF: We have

$$\int_{[0,1] \times \check{\mathcal{S}}_{V(\Gamma)}(T(\check{R} \setminus B_R))} \bigwedge_{e \in E(\Gamma)} p_e^* \left( \tilde{\omega}(j_E(e)) \right) = 0$$

because the integrated form factors through  $[0, 1] \times \check{\mathcal{S}}_{V(\Gamma)}(\mathbb{R}^3)$  via a map induced by  $p_\tau$ , which is fixed and independent of  $\tau$ , there. In particular, the  $I(\Gamma, V(\Gamma))$  depend only on  $(\tilde{\omega}(i)|_{[0,1] \times U_{B_R}})_{i \in \underline{3n}}$ .  $\square$

PROOF OF THEOREM 7.19 ASSUMING PROPOSITION 9.2: Changing propagating forms  $\omega(i)_0$  of  $C_2(R)$  to other ones  $\omega(i)_1$  provides forms  $\tilde{\omega}(i)$  on  $[0, 1] \times C_2(R)$  as in Lemma 9.1. Then Corollary 9.4 guarantees that  $z_{2n}(\check{R}, \emptyset, (\omega(i)))$  does not depend on the used propagating forms (which are not normalized on  $U(B_R)$  and hence do not depend on parallelizations).  $\square$

**Lemma 9.5.** *Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . Let  $L: \mathcal{L} \hookrightarrow \check{R}$  be a link embedding. For any  $i \in \underline{3n}$ , let  $\omega(i)$  be a homogeneous propagating form of  $(C_2(R), \tau)$ . Then, as stated in Theorem 7.20,  $Z_n(\check{R}, L, (\omega(i)))$  is independent of the chosen  $\omega(i)$ .<sup>2</sup> Denote it by  $Z_n(\check{R}, L, \tau)$ .*

PROOF ASSUMING PROPOSITION 9.2: To prove this lemma, it suffices to prove that if some homogeneous  $\omega(i) = \tilde{\omega}(i, 0)$  is changed to another homogeneous propagating form  $\tilde{\omega}(i, 1)$ , then  $Z_n(\check{R}, L, (\omega(i)))$  is unchanged. According to Lemma 3.17, under these assumptions, there exists a one-form  $\eta$  on  $C_2(R)$  such that

- $\tilde{\omega}(i, 1) = \tilde{\omega}(i, 0) + d\eta$  and
- $\eta|_{\partial C_2(R)} = 0$ .

Let  $p_{C_2}: [0, 1] \times C_2(R) \rightarrow C_2(R)$  denote the projection on the second factor. Define closed 2-forms  $\tilde{\omega}(j)$  on  $[0, 1] \times C_2(R)$  by

- $\tilde{\omega}(j) = p_{C_2}^*(\omega(j))$  if  $j \neq i$ , and
- $\tilde{\omega}(i) = p_{C_2}^*(\tilde{\omega}(i, 0)) + d(tp_{C_2}^*(\eta))$ .

Then the variation of  $Z_n(\check{R}, L, (\omega(i)))$  is  $Z_n(1) - Z_n(0)$ , with the notation of Proposition 9.2, where all the forms involved in some  $I(\Gamma, A)$ , except  $p_{e(i)}^*(\tilde{\omega}(i))$ , for the possible edge  $e(i)$  such that  $j_E(e(i)) = i$ , factor through  $p_{C_2}^*$ . Thus, if  $i \notin \text{Im}(j_E)$ , then all the forms factor through  $p_{C_2}^*$ , and  $I(\Gamma, A)$  vanishes.

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<sup>2</sup>The proof of Theorem 7.20 will be concluded in the end of Section 10.5.

Locally,  $F(A, L, \Gamma)$  is diffeomorphic to the product of  $F(A, L, \Gamma_A)$  by  $C(R, L; \Gamma \setminus \Gamma_A)$ . If  $e(i)$  is not an edge of  $\Gamma_A$ , then the form  $\bigwedge_{e \in E(\Gamma_A)} p_e^*(\tilde{\omega}(j_E(e)))$  factors through  $F(A, L, \Gamma_A)$  whose dimension is  $2|E(\Gamma_A)| - 1$ , and the form vanishes. If  $e(i)$  is an edge of  $\Gamma_A$ , then the part  $p_{e(i)}^*(d(tp_{C_2}^*(\eta)))$  vanishes since  $\eta$  vanishes on  $\partial C_2(R)$ . Thus  $\bigwedge_{e \in E(\Gamma_A)} p_e^*(\omega(j_E(e)))$  still factors through  $F(A, L, \Gamma_A)$ . So, when  $(R, L, \tau)$  is fixed,  $Z_n(\check{R}, L, (\omega(i)))$  is independent of the chosen homogeneous  $\omega(i)$ .  $\square$

**Lemma 9.6.** *Let  $(\check{R}, \tau_0)$  be an asymptotic rational homology  $\mathbb{R}^3$ . Let  $n \in \mathbb{N}$ . For any  $i \in \underline{3n}$ , let  $\omega(i)$  be a propagating form of  $(C_2(R), \tau_0)$ . Then  $Z_n(\check{R}, \emptyset, (\omega(i))) = Z_n(\check{R}, \emptyset, \tau_0)$  with the notation of Lemma 9.5. Furthermore,  $Z_n(\check{R}, \emptyset, \tau_0)$  depends only on the homotopy class of  $\tau_0$ .*

**PROOF ASSUMING PROPOSITION 9.2:** Let  $\tau: [0, 1] \times \check{R} \times \mathbb{R}^3 \rightarrow T\check{R}$  be a smooth map whose restriction to  $\{t\} \times \check{R} \times \mathbb{R}^3$  is an asymptotically standard parallelization  $\tau_t$  of  $\check{R}$  for any  $t \in [0, 1]$ . Define  $p_\tau: [0, 1] \times \partial C_2(R) \rightarrow [0, 1] \times S^2$  by  $p_\tau(t, x) = (t, p_{\tau_t}(x))$ . For any  $i \in \underline{3n}$ , let  $\omega(i)_0$  be a (non-necessarily homogeneous) propagating form of  $(C_2(R), \tau_0)$ , and let  $\omega(i)_1$  be a propagating form of  $(C_2(R), \tau_1)$ . It suffices to prove that

$$Z_n(\check{R}, \emptyset, (\omega(i)_1)) - Z_n(\check{R}, \emptyset, (\omega(i)_0)) = 0.$$

Use forms  $\omega(i)$  on  $[0, 1] \times C_2(R)$  provided by Lemma 9.1, which restrict to  $[0, 1] \times \partial C_2(R)$  as  $p_\tau^*(\tilde{\omega}_{S^2})$ , to express this variation as in Proposition 9.2. Here, a face  $F(A, \emptyset, \Gamma)$  is an open dense subset of the product of  $F(A, \emptyset, \Gamma_A)$  by  $\check{C}(R, \emptyset; \Gamma \setminus \Gamma_A)$ , and  $\tau$  identifies  $[0, 1] \times F(A, \emptyset, \Gamma_A)$  with  $[0, 1] \times \check{R} \times \check{\mathcal{S}}_A(\mathbb{R}^3)$ . The form  $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$  pulls back through  $[0, 1] \times \check{\mathcal{S}}_A(\mathbb{R}^3) \times \check{C}(R, \emptyset; \Gamma \setminus \Gamma_A)$ , and it vanishes.  $\square$

The following variant of Proposition 9.2 implies Theorem 7.30. We prove it in Section 9.3.

**Proposition 9.7.** *Under the assumptions of Proposition 9.2, the following statement is also true. For  $\Gamma \in \mathcal{D}_{n,m}^e(\mathcal{L})$  and for a connected component  $\Gamma_A$  of  $\Gamma$  (with no univalent vertices or) whose univalent vertices are consecutive on one component  $\mathcal{L}_A$  of  $\mathcal{L}$ , let  $\Gamma^{\text{rev}}(A)$  denote the graph obtained from  $\Gamma$  by reversing the order on the univalent vertices of  $\Gamma_A$  induced by  $i_\Gamma$ .<sup>3</sup> Set*

$$I'(\Gamma, A) = \frac{1}{2} \int_{[0,1] \times F(A,L,\Gamma)} \bigwedge_{e \in E(\Gamma)} p_e^*(\tilde{\omega}(j_E(e))) \left( [\Gamma] - (-1)^{|E(\Gamma_A)|} [\Gamma^{\text{rev}}(A)] \right).$$

<sup>3</sup>When  $\mathcal{L}_A$  is oriented, this order is a linear order if  $U(\Gamma)$  has vertices of  $V(\Gamma) \setminus A$  on  $\mathcal{L}_A$ ; it is cyclic, otherwise. When  $\mathcal{L}_A$  is not oriented, the order is not defined, but reversing the order is well-defined in any case. When  $\mathcal{L}_A$  is oriented, we have  $[\Gamma] = [\Gamma_A \#_{\mathcal{L}_A} \Gamma_{V(\Gamma) \setminus A}]$  and  $[\Gamma^{\text{rev}}(A)] = [\Gamma_A^{\text{rev}}(A) \#_{\mathcal{L}_A} \Gamma_{V \setminus A}]$ .

Then we have

$$Z_n(1) - Z_n(0) = \sum_{(\Gamma, A) \in \mathcal{D}_n^{e, F}(\mathcal{L})} \zeta_\Gamma I'(\Gamma, A)$$

and

$$Z_{n, 3n-2}(1) - Z_{n, 3n-2}(0) = \sum_{(\Gamma, A) \in \mathcal{D}_{n, 3n-2}^{e, F}(\mathcal{L})} \zeta_\Gamma I'(\Gamma, A).$$

**PROOF OF THEOREM 7.30 ASSUMING PROPOSITION 9.7:** There is a map  $\text{rev}$  from  $\mathcal{A}(S^1)$  to itself that sends the class of a diagram  $\Gamma$  to the class of the diagram obtained from  $\Gamma$  by reversing the order of the univalent vertices on  $S^1$  and by multiplying the class by  $(-1)^{|T(\Gamma)|}$ . The composition  $w_C \circ \text{rev}$  equals  $w_C$ . Furthermore,  $w_C$  sends odd-degree diagrams to zero, and  $w_C$  is multiplicative with respect to the multiplication of  $\mathcal{A}(S^1)$ . So for any  $(\Gamma, A) \in \mathcal{D}_{n, m}^{e, F}(S^1)$  as in the above statement such that  $\Gamma$  has no component without univalent vertices,  $(w_C([\Gamma]) - (-1)^{|E(\Gamma_A)|} w_C([\Gamma^{\text{rev}}(A)]))$  is equal to

$$w_C\left([\Gamma_{V(\Gamma) \setminus A}]\right) \left(w_C([\Gamma_A]) - (-1)^{|E(\Gamma_A)| + |T(\Gamma_A)|} w_C(\text{rev}([\Gamma_A]))\right),$$

which is zero since  $|E(\Gamma_A)| + |T(\Gamma_A)|$  is even when the degree of  $\Gamma_A$  is even. So  $w_C(I'(\Gamma, A))$  is always zero, and  $w_C(\check{Z}_{n, 3n-2}(\check{R}, K, (\omega(i))_{i \in \underline{3n-2}}))$  is independent of the chosen  $\omega(i)$ . The variation of  $p^c(\check{Z}_{n, 3n-2}(\check{R}, K, (\omega(i))_{i \in \underline{3n-2}}))$  is

$$\sum_{(\Gamma, A) \in \mathcal{D}_{n, 3n-2}^{e, F}(S^1)} \zeta_\Gamma p^c \check{p}(I'(\Gamma, A)),$$

where  $w_C$  sends  $p^c \check{p}(I'(\Gamma, A))$  to zero, for the same reasons as above.  $\square$

## 9.2 Sketch of proof of Proposition 9.2

According to Stokes' theorem, for any  $\Gamma \in \mathcal{D}_{n, m}^e(\mathcal{L})$ , where  $m = 3n$  or  $m = 3n - 2$ , we have

$$\begin{aligned} I(R, L, \Gamma, (\tilde{\omega}(i, 1))_{i \in \underline{m}}) &= I(R, L, \Gamma, (\tilde{\omega}(i, 0))_{i \in \underline{m}}) \\ &\quad + \sum_F \int_{[0, 1] \times F} \bigwedge_{e \in E(\Gamma)} p_e^*(\tilde{\omega}(j_E(e))), \end{aligned}$$

where the sum runs over the codimension-one faces  $F$  of  $C(R, L; \Gamma)$ , which are described in Proposition 8.19. Let  $\tilde{p}_e: C(R, L; \Gamma) \rightarrow C_e(R)$  be the natural restriction and set

$$p(\Gamma) = \mathbf{1}_{[0, 1]} \times \prod_{e \in E(\Gamma)} \tilde{p}_e: [0, 1] \times C(R, L; \Gamma) \rightarrow [0, 1] \times \prod_{e \in E(\Gamma)} C_e(R).$$



For an edge  $e_0$  of  $E(\Gamma)$ , let  $p_{j_E(e_0)}: [0, 1] \times \prod_{e \in E(\Gamma)} C_e(R) \rightarrow [0, 1] \times C_2(R)$  be the composition of the natural projection onto  $[0, 1] \times C_{e_0}(R)$  and the natural identification of  $[0, 1] \times C_{e_0}(R)$  with  $[0, 1] \times C_2(R)$ . Define the form

$$\Omega_{E(\Gamma)} = \bigwedge_{e \in E(\Gamma)} p_{j_E(e)}^* \left( \tilde{\omega}(j_E(e)) \right)$$

on  $[0, 1] \times \prod_{e \in E(\Gamma)} C_e(R)$ . Set

$$I(\Gamma, A) = \int_{[0,1] \times F(A,L,\Gamma)} p(\Gamma)^*(\Omega_{E(\Gamma)}) [\Gamma]$$

for any subset  $A$  of  $V(\Gamma)$  of cardinality at least 2, where  $F(A, L, \Gamma)$  is empty (and hence  $I(\Gamma, A) = 0$ ) if  $A \cap U(\Gamma)$  is not a set of consecutive vertices on one component of  $\mathcal{L}$ . Set

$$I(\Gamma, A, \infty) = \int_{[0,1] \times F_\infty(A,L,\Gamma)} p(\Gamma)^*(\Omega_{E(\Gamma)}) [\Gamma]$$

for any subset  $A$  of  $V(\Gamma)$  of cardinality at least 1, where  $F_\infty(A, L, \Gamma)$  is empty (and hence  $I(\Gamma, A, \infty) = 0$ ) if  $A \cap U(\Gamma)$  is not empty. We have

$$Z_{n,m}(1) - Z_{n,m}(0) = \sum_{\Gamma \in \mathcal{D}_{n,m}^e(\mathcal{L})} \left( \sum_{A \in \mathcal{P}_{\geq 2}(V(\Gamma))} \zeta_\Gamma I(\Gamma, A) + \sum_{A \in \mathcal{P}_{\geq 1}(V(\Gamma))} \zeta_\Gamma I(\Gamma, A, \infty) \right),$$

where  $\mathcal{P}_{\geq 1}(V(\Gamma))$  denotes the set of the (nonstrict) nonempty subsets of  $V(\Gamma)$ . In order to prove Proposition 9.2, it suffices to prove that the codimension-one faces  $F$  of the  $C(R, L; \Gamma)$  that do not appear in the statement of Proposition 9.2 do not contribute.

This is the consequence of Lemmas 9.8 to 9.14, together with the analysis before Lemma 9.13.

**Lemma 9.8.** *For any  $\Gamma \in \mathcal{D}_{n,m}^e(\mathcal{L})$ , For any nonempty subset  $A$  of  $V(\Gamma)$ , we have  $I(\Gamma, A, \infty) = 0$ .*

PROOF: Recall  $F_\infty(A, L, \Gamma) = B_\infty(A, L, \Gamma) \times \check{\mathcal{S}}(T_\infty R, A)$  from Section 8.4. Let  $E_C$  be the set of the edges of  $\Gamma$  that contain an element of  $V(\Gamma) \setminus A$  and an element of  $A$ . Let  $p_2$  denote the projection of  $F_\infty(A, L, \Gamma)$  onto  $\check{\mathcal{S}}(T_\infty R, A)$ . For  $e \in E_A \cup E_C$ , the map  $P_e: (S^2)^{E_A \cup E_C} \rightarrow S^2$  is the projection onto the factor indexed by  $e$ . We prove that there exists a smooth map

$$g: \check{\mathcal{S}}(T_\infty R, A) \rightarrow (S^2)^{E_A \cup E_C}$$

such that  $\bigwedge_{e \in E_A \cup E_C} p_e^* \left( \tilde{\omega}(j_E(e)) \right)$  is equal to

$$\left( \mathbf{1}_{[0,1]} \times (g \circ p_2) \right)^* \left( \bigwedge_{e \in E_A \cup E_C} \left( \mathbf{1}_{[0,1]} \times P_e \right)^* \left( \tilde{\omega}_{S^2}(j_E(e)) \right) \right).$$

If  $e \in E_A \cup E_C$ , then  $p_e(F_\infty(A, L, \Gamma)) \subset \partial C_2(R) \setminus U(\check{R})$ , and we have

$$\left( \mathbf{1}_{[0,1]} \times p_e \right)^* \left( \tilde{\omega}(j_E(e)) \right) = \left( \mathbf{1}_{[0,1]} \times (p_\tau \circ p_e) \right)^* \left( \tilde{\omega}_{S^2}(j_E(e)) \right).$$

If  $e \in E_C$ , then  $p_\tau \circ p_e$  depends only on the projection on  $S(T_\infty R)$  of the vertex at  $\infty$  (of  $A$ ). If  $e \in E_A$ , then  $p_\tau \circ p_e$  factors through  $\check{\mathcal{S}}(T_\infty R, e)$ . So in both cases, the map  $p_\tau \circ p_e$  factors through  $\check{\mathcal{S}}(T_\infty R, A)$ . Thus it may be expressed as  $((P_e \circ g) \circ p_2)$ . Therefore, if the degree of the form  $\bigwedge_{e \in E_A \cup E_C} p_e^* \left( \tilde{\omega}_{S^2}(j_E(e)) \right)$  is bigger than the dimension  $3|A|$  of  $[0, 1] \times \check{\mathcal{S}}(T_\infty R, A)$ , this form vanishes on  $[0, 1] \times F_\infty(A, L, \Gamma)$ . The degree of the form is  $(2|E_A| + 2|E_C|)$ , and we have

$$3|A| = 2|E_A| + |E_C|.$$

So the integral vanishes unless  $E_C$  is empty. In this case, all the  $p_\tau \circ p_e$ , for  $e \in E_A$  factor through the conjugates under the inversion  $(x \mapsto x/\|x\|^2)$  of the translations that make sense, and the form  $\bigwedge_{e \in E_A} p_e^* \left( \tilde{\omega}_{S^2}(j_E(e)) \right)$  factors through the product by  $[0, 1]$  of the quotient of  $\check{\mathcal{S}}(T_\infty R, A)$  by these translation conjugates. So it vanishes, too.  $\square$

If there exists a smooth map from  $[0, 1] \times F(A, L, \Gamma)$  to a manifold of strictly smaller dimension that factorizes the restriction of

$$p(\Gamma) = \left( \mathbf{1}_{[0,1]} \times \prod_{e \in E(\Gamma)} \tilde{p}_e \right): [0, 1] \times C(R, L; \Gamma) \rightarrow [0, 1] \times \prod_{e \in E(\Gamma)} C_e(R)$$

to  $[0, 1] \times F(A, L, \Gamma)$ , then we have  $I(\Gamma, A) = 0$ . We use this principle to get rid of some faces.

**Lemma 9.9.** *Let  $\Gamma \in \mathcal{D}_{n,m}^e(\mathcal{L})$ . For any subset  $A$  of  $V(\Gamma)$  such that the graph  $\Gamma_A$  defined before Proposition 9.2 is not connected and  $\Gamma_A$  is not a pair of univalent vertices, we have  $I(\Gamma, A) = 0$ .*

PROOF: In the fiber  $\check{\mathcal{S}}_A(T_{m(A)}\check{R}, L, \Gamma)$  of  $F(A, L, \Gamma)$  we may translate one connected component of  $\Gamma_A$  whose set of vertices is  $C$ , independently, without changing the restriction of  $p(\Gamma)$  to  $F(A, L, \Gamma)$ . The translation vector is in  $T_{m(A)}L$  when  $C$  contains univalent vertices. Unless  $C$  and  $A \setminus C$  are reduced

to a univalent vertex, the quotient of  $\check{\mathcal{S}}_A(T_{m(A)}\check{R}, L, \Gamma)$  by these translations has a smaller dimension than  $\check{\mathcal{S}}_A(T_{m(A)}\check{R}, L, \Gamma)$ , and  $p(\Gamma)$  factors through the corresponding quotient of  $[0, 1] \times F(A, L, \Gamma)$ .  $\square$

**Lemma 9.10.** *Let  $\Gamma \in \mathcal{D}_{n,m}^e(\mathcal{L})$ . Let  $A$  be a subset of  $V(\Gamma)$  such that  $|A| \geq 3$ . If some trivalent vertex of  $A$  belongs to exactly one edge of  $\Gamma_A$ , then  $I(\Gamma, A) = 0$ .*

PROOF: Let  $b$  be the mentioned trivalent vertex. Let  $e$  be its edge in  $\Gamma_A$ , and let  $d \in A$  be the other element of  $e$ . The group  $]0, \infty[$  acts on the map  $t$  from  $A$  to  $T_{c(b)}R$  by moving  $t(b)$  on the half-line from  $t(d)$  through  $t(b)$ , by multiplying  $(t(b) - t(d))$  by a scalar. When  $|A| \geq 3$ , this action defined on an open dense subset of  $\check{\mathcal{S}}_A(T_{m(A)}\check{R}, L, \Gamma)$  is not trivial, and  $p(\Gamma)$  factors through the corresponding quotient of an open dense subset of  $[0, 1] \times F(A, L, \Gamma)$ , which is of smaller dimension.  $\square$

### 9.3 Cancellations of nondegenerate faces

From now on, we will study cancellations that are no longer individual, and orientations must be seriously considered. Recall that the codimension-one faces are oriented as parts of the boundary of  $C(R, L; \Gamma)$ , with the outward normal first convention, where  $C(R, L; \Gamma)$  is oriented by an orientation of  $L$  and an order on  $V(\Gamma)$ . The relations between an orientation of  $V(\Gamma)$ , which orients  $C(R, L; \Gamma)$ , a vertex-orientation of  $\Gamma$ , and an edge-orientation of the set  $H(\Gamma)$  of half-edges of  $\Gamma$  are explained in Lemma 7.1, Corollary 7.2, and Remark 7.5. Fortunately, we do not have to fix everything to compare similar orientations.

**Lemma 9.11.** *Let  $\Gamma \in \mathcal{D}_{n,m}^e(\mathcal{L})$ . Let  $A$  be a subset of  $V(\Gamma)$  such that at least one element of  $A$  belongs to exactly two edges of  $\Gamma_A$ . Let  $\mathcal{E}(\Gamma, A)$  denote the set of graphs of  $\mathcal{D}_{n,m}^e(\mathcal{L})$  that are isomorphic to  $\Gamma$  by an isomorphism that is only allowed to change the labels and the orientations of the edges of  $\Gamma_A$ . Such an isomorphism preserves  $A$ , and we have*

$$\sum_{\tilde{\Gamma}: \tilde{\Gamma} \in \mathcal{E}(\Gamma, A)} \zeta_{\tilde{\Gamma}} I(\tilde{\Gamma}, A) = 0.$$

PROOF: Let us first check that the isomorphisms of the statement preserve  $A$ . The vertices of the elements of  $\mathcal{D}_{n,m}^e(\mathcal{L})$  are not numbered. A vertex is characterized by the half-edges that contain it. Therefore, the isomorphisms of the statement preserve the vertices of  $V(\Gamma) \setminus A$ . So they preserve  $A$

setwise. These isomorphisms also preserve the vertices that have adjacent edges outside  $E(\Gamma_A)$  pointwise. The isomorphisms described below actually induce the identity map on  $V(\Gamma)$ .

Among the vertices of  $A$  that belong to exactly two edges of  $\Gamma_A$  and one edge  $j_E^{-1}(k)$  of  $\Gamma$  outside  $\Gamma_A$ , choose the vertices such that  $k$  is minimal. If there is one such vertex, then call this vertex  $v_m$ . Otherwise, there are two choices, and  $v_m$  is chosen to be the vertex that belongs to the first half-edge of  $j_E^{-1}(k)$ .

We first describe an orientation-reversing diffeomorphism of the complement of a codimension-three submanifold of  $F(A, L, \Gamma)$ . Let  $v_j$  and  $v_k$  denote the (possibly equal) two other vertices of the two edges of  $\Gamma_A$  that contain  $v_m$ . Consider the transformation  $S$  of the space  $\overline{\mathcal{S}}_A(T_{c(A)}R)$  of nonconstant maps  $f$  from  $A$  to  $T_{c(A)}R$  up to translations and dilations that maps  $f$  to  $S(f)$ , where

$$\begin{aligned} S(f)(v_\ell) &= f(v_\ell) \text{ if } v_\ell \neq v_m, \text{ and} \\ S(f)(v_m) &= f(v_j) + f(v_k) - f(v_m). \end{aligned}$$

This is an orientation-reversing involution of  $\overline{\mathcal{S}}_A(T_{c(A)}R)$ . The set of elements of  $\check{\mathcal{S}}_A(T_{c(A)}R)$  whose image under  $S$  is not in  $\check{\mathcal{S}}_A(T_{c(A)}R)$  is a codimension-three submanifold of  $\check{\mathcal{S}}_A(T_{c(A)}R)$ . The fibered product of  $S$  by the identity of the base  $B(A, L, \Gamma)$  is an orientation-reversing smooth involution outside a codimension-three submanifold  $F_S$  of  $F(A, L, \Gamma)$ . It is still denoted by  $S$ , as its product by  $\mathbf{1}_{[0,1]}$  is, too.

Now, let  $\sigma(A; \Gamma)(\tilde{\Gamma})$  be obtained from  $(\tilde{\Gamma} \in \mathcal{E}(\Gamma, A))$  by exchanging the labels of the two edges of  $\Gamma_A$  that contain  $v_m$  and by reversing their orientations if (and only if) they both start or end at  $v_m$ . Then, as Figure 9.1 shows, we have

$$p(\tilde{\Gamma}) \circ S = p(\sigma(A; \Gamma)(\tilde{\Gamma})).$$

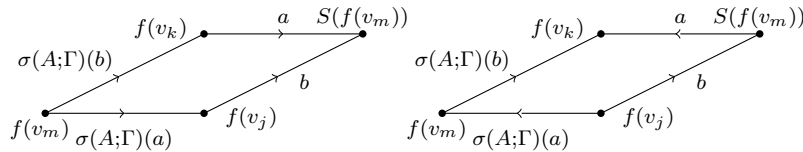


Figure 9.1: The parallelogram transformation  $S$

Recall

$$I(\Gamma, A) = \int_{[0,1] \times F(A, L, \Gamma)} p(\Gamma)^*(\Omega_{E(\Gamma)}) [\Gamma],$$

with the map

$$p(\Gamma) = \left( \mathbf{1}_{[0,1]} \times \prod_{e \in E(\Gamma)} \tilde{p}_e \right) : [0, 1] \times C(R, L; \Gamma) \rightarrow [0, 1] \times \prod_{e \in E(\Gamma)} C_e(R)$$

and  $\Omega_{E(\Gamma)} = \bigwedge_{e \in E(\Gamma)} p_{j_E(e)}^*(\tilde{\omega}(j_E(e)))$ . We have

$$\begin{aligned} I(\tilde{\Gamma}, A) &= \int_{[0,1] \times (F(A, L, \tilde{\Gamma}) \setminus F_S)} p(\tilde{\Gamma})^*(\Omega_{E(\Gamma)})[\tilde{\Gamma}] \\ &= - \int_{[0,1] \times (F(A, L, \tilde{\Gamma}) \setminus F_S)} S^* \left( p(\tilde{\Gamma})^*(\Omega_{E(\Gamma)}) \right) [\tilde{\Gamma}] \\ &= - \int_{[0,1] \times (F(A, L, \tilde{\Gamma}) \setminus F_S)} (p(\tilde{\Gamma}) \circ S)^*(\Omega_{E(\Gamma)})[\tilde{\Gamma}] \\ &= - \int_{[0,1] \times (F(A, L, \tilde{\Gamma}) \setminus F_S)} p(\sigma(A; \Gamma)(\tilde{\Gamma}))^*(\Omega_{E(\Gamma)})[\tilde{\Gamma}] \\ &= -I(\sigma(A; \Gamma)(\tilde{\Gamma}), A) \end{aligned}$$

since we have  $[\tilde{\Gamma}] = [\sigma(A; \Gamma)(\tilde{\Gamma})]$ . Now,  $\sigma(A; \Gamma)$  defines an involution of  $\mathcal{E}(\Gamma, A)$ , and we get

$$\sum_{\tilde{\Gamma} \in \mathcal{E}(\Gamma, A)} I(\tilde{\Gamma}, A) = \sum_{\tilde{\Gamma} \in \mathcal{E}(\Gamma, A)} I(\sigma(A; \Gamma)(\tilde{\Gamma}), A) = - \sum_{\tilde{\Gamma} \in \mathcal{E}(\Gamma, A)} I(\tilde{\Gamma}, A) = 0.$$

□

Maxim Kontsevich observed the symmetry of the above proof in [Kon94].

**Lemma 9.12.** *Let  $\Gamma \in \mathcal{D}_n^e(\mathcal{L})$ . Let  $A$  be a subset of  $V(\Gamma)$  such that  $\Gamma_A$  is a connected component of  $\Gamma$ . Let  $\Gamma^{\text{eo}}(A)$  denote the graph obtained from  $\Gamma$  by reversing all the orientations of the edges of  $\Gamma_A$ . Recall the notation of Proposition 9.2. If  $\Gamma_A$  is a diagram of even degree without univalent vertices, then we have*

$$I(\Gamma, A) + I(\Gamma^{\text{eo}}(A), A) = 0.$$

*If  $(\Gamma, A)$  is an element of the set  $\mathcal{D}_{n,m}^{e,F}(\mathcal{L})$  defined in Proposition 9.2 and if  $\Gamma_A$  has univalent vertices, let  $\Gamma^{\text{eo,rev}}(A)$  denote the graph obtained from  $\Gamma^{\text{eo}}(A)$  by reversing the order of the univalent vertices of  $\Gamma_A$  on  $\mathcal{L}_A$ . Recall*

$$I'(\Gamma, A) = \frac{1}{2} \int_{[0,1] \times F(A, L, \Gamma)} \bigwedge_{e \in E(\Gamma)} p_e^*(\tilde{\omega}(j_E(e))) \left( [\Gamma] - (-1)^{|E(\Gamma_A)|} [\Gamma^{\text{rev}}(A)] \right)$$

*from Proposition 9.7. Then we have*

$$I(\Gamma, A) + I(\Gamma^{\text{eo,rev}}(A), A) = I'(\Gamma, A) + I'(\Gamma^{\text{eo,rev}}(A), A).$$

PROOF: Set  $\bar{\Gamma}(A) = \Gamma^{\text{eo,rev}}(A)$  in both cases. The opposite of the identity map of  $T_{c(A)}\check{R}$  induces a diffeomorphism from the fiber of  $F(A, L, \Gamma)$  to the fiber of  $F(A, L, \bar{\Gamma}(A))$ , which induces a diffeomorphism from  $F(A, L, \Gamma)$  to  $F(A, L, \bar{\Gamma}(A))$  over the identity map of the base. Denote by  $\mathcal{S}$  the product of this diffeomorphism by  $\mathbf{1}_{[0,1]}$ . Let us carefully discuss orientations to determine when this diffeomorphism preserves the orientation.

Order and orient the vertices of  $\Gamma$  so that the corresponding orientation of  $H(\Gamma)$ , as in Remark 7.5, is induced by the edge-orientation of  $\Gamma$ . There is a natural bijection from  $V(\Gamma)$  to  $V(\bar{\Gamma}(A))$ . This bijection is the identity on the set  $V(\Gamma) \setminus A$  of vertices of  $\Gamma \setminus \Gamma_A$ , unaffected by the modifications. When  $A$  is not a pair of vertices in a  $\theta$ -component, a vertex of  $\Gamma_A$  is characterized by the labels of the edges that contain it. Such a vertex is sent to the vertex of  $\bar{\Gamma}(A)$  with the same set of labels of adjacent edges. When  $A$  is a pair of vertices in a  $\theta$ -component, the vertex at which an edge of  $\Gamma_A$  labeled by  $i$  begins is sent to the vertex of  $\bar{\Gamma}(A)$  at which the edge of  $\bar{\Gamma}(A)$  labeled by  $i$  ends.

Order the vertices of  $\bar{\Gamma}(A)$  like the vertices of  $\Gamma$  if  $|E(\Gamma_A)|$  is even, and permute two vertices if  $|E(\Gamma_A)|$  is odd. Orient  $C(R, L; \Gamma)$  and  $C(R, L; \bar{\Gamma}(A))$  with respect to the above orders of  $V(\Gamma)$  and  $V(\bar{\Gamma}(A))$ , using the orientations of  $R$  and  $L$ . Then  $\mathcal{S}$  reverses the orientation if and only if  $|E(\Gamma_A)|$  is even since  $|V(\Gamma_A)|$  is even.

Orient the vertices of  $\bar{\Gamma}(A)$  like the vertices of  $\Gamma$ . So the orientation of  $H(\bar{\Gamma}(A))$  associated to that vertex-orientation and to the above order of vertices is induced by the edge-orientation of  $\bar{\Gamma}(A)$ . When orientations are fixed as above, set  $I_0(\Gamma, A) = \int_{[0,1] \times F(A, L, \Gamma)} p(\Gamma)^*(\Omega_{E(\Gamma)})$  so that  $I(\Gamma, A) = I_0(\Gamma, A) [\Gamma]$ . Let  $e$  be an edge of  $\Gamma$ . Up to its edge-orientation, the edge  $e$  is also an edge of  $\bar{\Gamma}(A)$ , and  $p_{j_E(e)} \circ p(\bar{\Gamma}(A))$  restricts to  $[0, 1] \times F(A, L, \bar{\Gamma}(A))$  as  $p_{j_E(e)} \circ p(\Gamma) \circ \mathcal{S}^{-1}$ . We have

$$\begin{aligned} I_0(\bar{\Gamma}(A), A) &= \int_{[0,1] \times F(A, L, \bar{\Gamma}(A))} p(\bar{\Gamma}(A))^*(\Omega_{E(\bar{\Gamma}(A))}) \\ &= \int_{[0,1] \times F(A, L, \bar{\Gamma}(A))} \bigwedge_{e \in E(\Gamma)} \left( p_{j_E(e)} \circ p(\bar{\Gamma}(A)) \right)^* \left( \tilde{\omega}(j_E(e)) \right) \\ &= \int_{[0,1] \times F(A, L, \bar{\Gamma}(A))} (\mathcal{S}^{-1})^* \left( \bigwedge_{e \in E(\Gamma)} \left( p_{j_E(e)} \circ p(\Gamma) \right)^* \left( \tilde{\omega}(j_E(e)) \right) \right) \\ &= (-1)^{|E(\Gamma_A)|+1} \int_{[0,1] \times F(A, L, \Gamma)} \bigwedge_{e \in E(\Gamma)} \left( p_{j_E(e)} \circ p(\Gamma) \right)^* \left( \tilde{\omega}(j_E(e)) \right) \\ &= (-1)^{|E(\Gamma_A)|+1} I_0(\Gamma, A). \end{aligned}$$

In particular, if  $\Gamma_A$  is a connected diagram of even degree without univalent vertices, then  $|E(\Gamma_A)|$  is even, and  $[\Gamma]$  is equal to  $[\bar{\Gamma}(A)]$  and  $[\Gamma^{\text{eo}}(A)]$ . So we

have  $I(\Gamma, A) = -I(\Gamma^{\text{eo}}(A), A)$ . Otherwise, we have

$$\begin{aligned} I(\Gamma, A) + I(\overline{\Gamma}(A), A) &= \frac{1}{2} (I_0(\Gamma, A) - (-1)^{|E(\Gamma_A)|} I_0(\overline{\Gamma}(A), A)) [\Gamma] \\ &\quad + \frac{1}{2} (I_0(\overline{\Gamma}(A), A) - (-1)^{|E(\Gamma_A)|} I_0(\Gamma, A)) [\Gamma^{\text{rev}}(A)] \\ &= I'(\Gamma, A) + I'(\overline{\Gamma}(A), A). \end{aligned}$$

□

Lemmas 9.9, 9.10, and 9.11 allow us to get rid of the pairs  $(\Gamma, A)$  with  $|A| \geq 3$  such that

- at least one element of  $A$  does not have all its adjacent edges in  $E(\Gamma_A)$ , or
- $\Gamma_A$  is disconnected.

Lemma 9.12 rules out the pairs  $(\Gamma, A)$  such that  $\Gamma_A$  is an even degree connected component of  $\Gamma$ , without univalent vertices (where  $|A| \equiv 0 \pmod{4}$ ). Therefore, according to Lemma 9.8, we are left with

- the pairs  $(\Gamma, A)$  of the statement of Proposition 9.2, for which  $\Gamma_A$  is a connected component of  $\Gamma$  (which may be an edge between two univalent vertices) and
- the following pairs, for which  $|A| = 2$  and
  - $\Gamma_A$  is an edge between two trivalent vertices,
  - $\Gamma_A$  is an edge between a trivalent vertex and a univalent one, or
  - $\Gamma_A$  is a pair of isolated consecutive univalent vertices,

since Lemma 9.9 rules out the disconnected  $\Gamma_A$  with a trivalent vertex, and Lemma 9.11 rules out  $\Gamma_A = \diamond$ .

The following lemma allows us to eliminate the pairs  $(\Gamma, A)$  such that  $\Gamma_A$  is an edge between two trivalent vertices using the Jacobi relation.

**Lemma 9.13.** *The contributions to  $(Z_{n,m}(1) - Z_{n,m}(0))$  of the faces  $F(A, L, \Gamma)$  for which  $\Gamma_A$  is an edge between two trivalent vertices cancel. More precisely, let  $\Gamma \in \mathcal{D}_{n,m}^e(\mathcal{L})$ . Let  $A$  be a subset of  $V(\Gamma)$  such that  $\Gamma_A$  is an edge  $e(\ell)$  with label  $\ell$ . Let  $\Gamma/\Gamma_A$  be the labeled edge-oriented graph obtained from  $\Gamma$  by contracting  $\Gamma_A$  to one point, as in Figure 9.2. (The labels of the edges of  $\Gamma/\Gamma_A$  belong to  $\underline{m} \setminus \{\ell\}$ . The graph  $\Gamma/\Gamma_A$  has one four-valent vertex, and its other vertices are univalent or trivalent.) Let  $\mathcal{E}(\Gamma; A)$  be the subset of  $\mathcal{D}_{n,m}^e(\mathcal{L})$  that*

contains the graphs  $\tilde{\Gamma}$  equipped with a pair  $A$  of vertices joined by an edge  $e(\ell)$  with label  $\ell$  such that  $\tilde{\Gamma}/\tilde{\Gamma}_A$  is equal to  $\Gamma/\Gamma_A$ . Then we have

$$\sum_{\tilde{\Gamma} : \tilde{\Gamma} \in \mathcal{E}(\Gamma; A)} \zeta_{\tilde{\Gamma}} I(\tilde{\Gamma}, A) = 0.$$

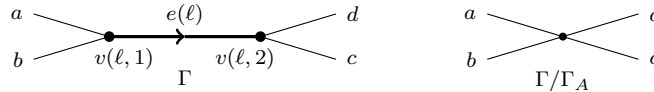


Figure 9.2: The graph  $\Gamma$ , its bold subgraph  $\Gamma_A$ , and  $\Gamma/\Gamma_A$

PROOF: Let us prove that there are 6 graphs in  $\mathcal{E}(\Gamma; A)$ . Let  $a, b, c, d$  be the four half-edges of  $\Gamma/\Gamma_A$  that contain its four-valent vertex. In  $\tilde{\Gamma}$ , the edge  $e(\ell)$  goes from a vertex  $v(\ell, 1)$  to a vertex  $v(\ell, 2)$ . The vertex  $v(\ell, 1)$  is adjacent to the first half-edge of  $e(\ell)$  and to two half-edges of  $\{a, b, c, d\}$ . The unordered pair of  $\{a, b, c, d\}$  adjacent to  $v(\ell, 1)$  determines  $\tilde{\Gamma}$  as an element of  $\mathcal{D}_{n,m}^e(\mathcal{L})$  and there are 6 graphs in  $\mathcal{E}(\Gamma; A)$  labeled by the pairs of elements of  $\{a, b, c, d\}$ . They are  $\Gamma = \Gamma_{ab}, \Gamma_{ac}, \Gamma_{ad}, \Gamma_{bc}, \Gamma_{bd}$ , and  $\Gamma_{cd}$ .

The face  $F(A, L, \Gamma)$  is fibered over  $B(A, L, \Gamma)$  with fiber  $\tilde{\mathcal{S}}_A(T_{c(v(\ell, 1))}R) =_{\tau} S^2$ , which contains the direction of the vector from  $c(v(\ell, 1))$  to  $c(v(\ell, 2))$ . Consistently order the vertices of the  $\Gamma_{..}$  starting with  $v(\ell, 1), v(\ell, 2)$  (the other vertices are in natural correspondences for different  $\Gamma_{..}$ ). Use these orders to orient the configuration spaces  $C(R, L; \Gamma_{..})$ .

The oriented face  $F(A, L, \Gamma_{..})$  and the map

$$p(\Gamma_{..}) : [0, 1] \times (F(A, L, \Gamma_{..}) \subset C(R, L; \Gamma_{..})) \longrightarrow [0, 1] \times \prod_{e \in E(\Gamma_{..})} C_2(R)^e$$

are the same for all the elements  $\Gamma_{..}$  of  $\mathcal{E}(\Gamma; A)$ . Therefore, the

$$I_0(\Gamma_{..}, A) = \int_{[0, 1] \times F(A, L, \Gamma_{..})} p(\Gamma_{..})^*(\Omega_{E(\Gamma_{..})})$$

are the same for all the elements  $\Gamma_{..}$  of  $\mathcal{E}(\Gamma; A)$  (for our consistent orders of the vertices), and the sum of the statement is

$$\sum_{\tilde{\Gamma} : \tilde{\Gamma} \in \mathcal{E}(\Gamma; A)} \zeta_{\tilde{\Gamma}} I_0(\tilde{\Gamma}, A)[\tilde{\Gamma}] = \zeta_{\Gamma} I_0(\Gamma, A) \sum_{\tilde{\Gamma} : \tilde{\Gamma} \in \mathcal{E}(\Gamma; A)} [\tilde{\Gamma}].$$

Let  $e_1$  be the first half-edge of  $e(\ell)$ , and let  $e_2$  be the other half-edge of  $e(\ell)$ . Equip  $\Gamma = \Gamma_{ab}$  with a vertex-orientation, represented by  $(a, b, e_1)$  at  $v(\ell, 1)$ ,

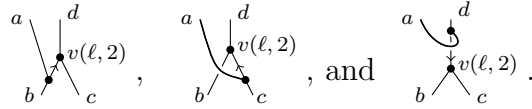


and  $(c, d, e_2)$  at  $v(\ell, 2)$ , which is consistent with its given edge-orientation (i.e., such that the edge-orientation of  $H(\Gamma)$  is equivalent to its vertex-orientation, with respect to the above order of vertices). The orientation of  $H(\Gamma)$  is represented by  $(a, b, e_1, c, d, e_2, \dots)$ . It induces the edge-orientation of  $H(\Gamma)$ , which is the same for all the elements of  $\mathcal{E}(\Gamma; A)$ .

Thus, permuting the letters  $b, c, d$  cyclically gives rise to two other graphs  $(\Gamma_{ac}$  and  $\Gamma_{ad})$  in  $\mathcal{E}(\Gamma; A)$  equipped with suitable vertex-orientations, respectively represented by

$$(a, c, e_1) \text{ at } v(\ell, 1), \text{ and } (d, b, e_2) \text{ at } v(\ell, 2), \text{ or} \\ (a, d, e_1) \text{ at } v(\ell, 1), \text{ and } (b, c, e_2) \text{ at } v(\ell, 2).$$

The three other elements of  $\mathcal{E}(\Gamma; A)$  with their suitable vertex-orientation are obtained from the three previous ones by exchanging the ordered pair before  $e_1$  with the ordered pair before  $e_2$ . This does not change the unlabeled vertex-oriented graph. The first three graphs can be represented by the following three graphs identical outside the pictured disk:



Then the sum  $\sum_{\tilde{\Gamma}: \tilde{\Gamma} \in \mathcal{E}(\Gamma; A)} [\tilde{\Gamma}]$  is zero thanks to the Jacobi relation. □

Now, we get rid of the remaining faces with the help of the STU relation.

**Lemma 9.14.** *The contributions to  $(Z_n(1) - Z_n(0))$  or to  $(Z_{n,3n-2}(1) - Z_{n,3n-2}(0))$  of the faces  $F(A, L, \Gamma)$  such that*

- $\Gamma_A$  is an edge between a trivalent vertex and a univalent vertex or
- $A$  is a pair of consecutive univalent vertices and  $\Gamma_A$  is not an edge of  $\Gamma$

cancel. More precisely, let  $\Gamma \in \mathcal{D}_{n,m}^e(\mathcal{L})$ , let  $A$  be a pair of consecutive univalent vertices of  $\Gamma$  on a component of  $\mathcal{L}$ , and assume that  $\Gamma_A$  is not an edge of  $\Gamma$ . Let  $\Gamma/\Gamma_A$  be the labeled edge-oriented graph obtained from  $\Gamma$  by contracting  $\Gamma_A$  to one point. (The labels of the edges of  $\Gamma/\Gamma_A$  belong to  $\underline{m}$ ,  $\Gamma/\Gamma_A$  has one bivalent vertex injected on  $\mathcal{L}$ .) Let  $\mathcal{E}(\Gamma/\Gamma_A)$  be the subset of  $\mathcal{D}_{n,m}^e(\mathcal{L})$  that contains the graphs  $\tilde{\Gamma}$  equipped with a pair  $A$  of vertices that are either

- two consecutive univalent vertices or
- a univalent vertex and a trivalent vertex connected by an edge,

such that  $\Gamma/\Gamma_A$  is equal to  $\tilde{\Gamma}/\tilde{\Gamma}_A$ . If  $m = 3n$  or if  $m = 3n - 2$ , then

$$\sum_{\tilde{\Gamma} : \tilde{\Gamma} \in \mathcal{E}(\Gamma; A)} \zeta_{\tilde{\Gamma}} I(\tilde{\Gamma}, A) = 0.$$

PROOF: Note that the face  $F(A, L, \Gamma)$  has two connected components if the only univalent vertices of  $\Gamma$  on the component of  $\mathcal{L}$  of the univalent vertices of  $A$  are the two vertices of  $A$ . The two connected components correspond to the two possible linear orders of  $A$  at the collapse.

Below, we consider these connected components as two different faces, and a face corresponds to a subset  $A$  equipped with a linear order compatible with  $i_\Gamma$ . In particular, the graph and its face are determined by the labeled edge-oriented graph  $\Gamma/\Gamma_A$  obtained from  $\Gamma$  by contracting  $A$  to one point, together with a linear order of the two half-edges of the bivalent vertex. Let  $k \in \underline{m} \setminus j_E(E(\Gamma))$ . Define  $\Gamma_k^+$  (resp.  $\Gamma_k^-$ ) to be the graph in  $\mathcal{D}_{n,m}^e(\mathcal{L})$  with an edge  $e(k)$  such that  $j_E(e(k)) = k$ , which goes from a univalent vertex  $u$  to a trivalent vertex  $t$  (resp. from a trivalent vertex  $t$  to a univalent vertex  $u$ ) forming a pair  $A = \{u, t\}$  such that  $\Gamma_k^+ / (\Gamma_k^+)_A$  (resp.  $\Gamma_k^- / (\Gamma_k^-)_A$ ) coincides with  $\Gamma/\Gamma_A$ .

Order the sets of vertices of the  $\Gamma_k^\pm$  by putting the vertices of  $A$  first, with respect to the order induced by the edge orientation (source first), and so that the order of the remaining vertices is the same for all  $\Gamma_k^\pm$ . Let  $o(V(\Gamma) \setminus A)$  denote this order of  $V(\Gamma) \setminus A$ . For  $(\tilde{\Gamma}, A) \in \mathcal{E}(\Gamma/\Gamma_A)$  such that  $A$  is an ordered pair of univalent vertices of  $\tilde{\Gamma}$ , order  $V(\tilde{\Gamma})$  by putting the vertices of  $A$  first with respect to the linear order induced by the collapse, and next the others with the same order  $o(V(\Gamma) \setminus A)$  as for the  $\Gamma_k^\pm$ .

Let  $\phi: \mathbb{R}^3 \rightarrow \check{R}$  be an orientation-preserving diffeomorphism onto a neighborhood of the image of  $A$  in a configuration of  $F(A, L, \Gamma)$ . There exists  $\rho_\phi: \mathbb{R}^3 \rightarrow GL^+(\mathbb{R}^3)$  such that  $T_x \phi(\rho_\phi(x)(\vec{v})) = \tau(\phi(x), \vec{v})$ , for any  $\vec{v} \in (\mathbb{R}^3 = T_x \mathbb{R}^3)$ . Then for  $\Gamma_k^+$ , the configuration space is locally diffeomorphic to  $\mathcal{L} \times \check{R} \times \dots$ , where  $\mathcal{L}$  contains the position  $c(u) = \phi(x)$  of the univalent vertex  $u$  of  $A$ , and  $\check{R}$  contains the position  $c(t)$  of the trivalent vertex  $t$  of  $A$ . This position  $c(t) = \phi(x + \lambda \rho_\phi(x)(\vec{v}))$  is described by a small positive  $\lambda$ , which plays the role of an inward normal near the collapse, and a vector  $\vec{v}$  of  $S^2$ , which is equal to  $p_\tau \circ p_{e(k)}(c)$  when  $\lambda$  reaches 0. The face is diffeomorphic to  $S^2 \times \mathcal{L} \times \dots$ , where the projection onto the factor  $S^2$  is  $p_\tau \circ p_{e(k)}$ , and the dots contain the coordinates of the remaining vertices, which are the same for all the considered diagrams.

For  $\Gamma_k^-$ , the configuration space is locally diffeomorphic to  $\check{R} \times \mathcal{L} \times \dots$ , where  $\mathcal{L}$  contains the position  $c(u) = \phi(x)$  of the univalent vertex  $u$  of  $A$ , and  $\check{R}$  contains the position  $c(t)$  of the trivalent vertex  $t$  of  $A$ . This position

$c(t) = \phi(x - \lambda \rho_\phi(x)(\vec{v}))$  is described by a small positive  $\lambda$ , which still plays the role of an inward normal near the collapse, and a vector  $\vec{v}$  of  $S^2$ , which is  $p_\tau \circ p_{e(k)}(c)$  when  $\lambda$  reaches 0. The face is again diffeomorphic to  $S^2 \times \mathcal{L} \times \dots$ , where the projection onto the factor  $S^2$  is  $p_\tau \circ p_{e(k)}$ , and the dots contain the coordinates of the remaining vertices, which are the same as for the  $\Gamma_k^+$ .

For  $(\tilde{\Gamma}, A) \in \mathcal{E}(\Gamma/\Gamma_A)$  such that  $A$  is an ordered pair of univalent vertices of  $\tilde{\Gamma}$ , the configuration space is locally diffeomorphic to  $\mathcal{L} \times \mathcal{L} \times \dots$ , where the first  $\mathcal{L}$  contains the position  $c(u_1) = \phi(x)$  of the first univalent vertex  $u_1$  of  $A$ , and the second  $\mathcal{L}$  contains the position  $c(u_2)$  of the vertex  $u_2$  that follows  $u_1$  along  $\mathcal{L}$ . The position  $c(u_2) = \phi(x + \lambda \rho_\phi(x)(\vec{t}))$  is again described by a small positive  $\lambda$ , which plays the role of an inward normal near the collapse, where  $p_\tau$  maps the oriented unit tangent vector to  $L$  at  $c(u_1)$  to  $\vec{t} \in S^2$  when  $\lambda$  reaches 0. The face is diffeomorphic to  $\mathcal{L} \times \dots$  with the same notation as before. So the previous faces are the products by  $S^2$  of this one, and  $p_\tau \circ p_{e(k)}$  is the projection to the factor  $S^2$ . Since the other  $p_e$  do not depend on this factor  $S^2$ ,  $\int_{[0,1] \times F(A,L,\Gamma_k^+)} p(\Gamma_k^+)^*(\Omega_{E(\Gamma_k^+)})$  is equal to

$$\int_{(t,c) \in [0,1] \times F(A,L,\Gamma)} \left( \int_{\{t\} \times \check{S}_{e(k)}(T_{c(A)}\check{R})} \tilde{\omega}(k) \right) p(\Gamma)^*(\Omega_{E(\Gamma)}),$$

where  $\{t\} \times \check{S}_{e(k)}(T_{c(A)}\check{R})$  is the factor  $S^2$  above. Furthermore, we have

$$\int_{\{t\} \times \check{S}_{e(k)}(T_{c(A)}\check{R})} \tilde{\omega}(k) = 1$$

since the integral of the closed form  $\tilde{\omega}(k)$  over any representative of the homology class of the fiber of the unit tangent bundle of  $\check{R}$  in  $[0,1] \times \partial C_2(R)$  is 1.

This argument, which also works for  $\Gamma_k^-$ , implies that all the integrals  $I_0(\tilde{\Gamma}, A) = \int_{[0,1] \times F(A,L,\tilde{\Gamma})} p(\tilde{\Gamma})^*(\Omega_{E(\tilde{\Gamma})})$  coincide for all the  $(\tilde{\Gamma}, A) \in \mathcal{E}(\Gamma/\Gamma_A)$  equipped with orders of their vertices as above. So it suffices to prove

$$\zeta_\Gamma([\Gamma] + [\Gamma']) + \sum_{k \in \underline{m} \setminus j_E(E(\Gamma))} \zeta_{\Gamma_k^+}([\Gamma_k^+] + [\Gamma_k^-]) = 0,$$

where  $\Gamma'$  is the graph obtained from  $\Gamma$  by permuting the order of the two univalent vertices on  $\mathcal{L}$ , and all the graphs  $\tilde{\Gamma}$  are vertex-oriented so that the vertex-orientation of  $H(\tilde{\Gamma})$  induced by the fixed order of the vertices coincides with the edge-orientation of  $H(\tilde{\Gamma})$  (as in Remark 7.5), for  $m = 3n - 2$  and for  $m = 3n$ .

Let  $a$  and  $b$  denote the half-edges of  $\Gamma$  that contain the vertices of  $A$ . Without loss of generality, assume that the vertex of  $b$  follows the vertex

of  $a$  on  $\mathcal{L}$  for  $\Gamma$  (near the connected face). Let  $o_V(H(\Gamma) \setminus \{a, b\})$  be an order of  $H(\Gamma) \setminus \{a, b\}$ , such that the order  $(a, b, o_V(H(\Gamma) \setminus \{a, b\}))$  (i.e.,  $(a, b)$  followed by the elements  $H(\Gamma) \setminus \{a, b\}$  ordered by  $o_V(H(\Gamma) \setminus \{a, b\})$ ) induces the edge-orientation of  $H(\Gamma)$ .

Orient the vertices of  $V(\Gamma) \setminus A$  in  $\Gamma$  so that  $o_V(H(\Gamma) \setminus \{a, b\})$  is induced by the order  $o(V(\Gamma) \setminus A)$  and this vertex-orientation. Let  $f$  (resp.  $s$ ) denote the first (resp. second) half-edge of  $e(k)$  in  $\Gamma_k^\pm$ . Then  $(f, s, a, b, o_V(H(\Gamma) \setminus \{a, b\}))$  induces the edge-orientation of  $H(\Gamma_k^\pm)$ . Equip the trivalent vertex of  $A$  in  $\Gamma_k^\pm$  with the vertex-orientation  $((f \text{ or } s), a, b)$ , which corresponds to the picture

$$\begin{array}{c} b \swarrow a \\ \bullet \rightarrow \\ \bullet \rightarrow \end{array}, \tag{9.1}$$

and equip the other vertices of  $\Gamma_k^\pm$  with the same vertex-orientation as their vertex-orientation in  $\Gamma$ . Then the vertex-orientation of  $H(\Gamma_k^+)$  is induced by  $(f, s, a, b, o_V(H(\Gamma) \setminus \{a, b\}))$  and coincides with its edge-orientation. Similarly, the vertex-orientation of  $H(\Gamma_k^-)$  is induced by  $(f, a, b, s, o_V(H(\Gamma) \setminus \{a, b\}))$  and coincides with its edge-orientation. Thus, we have  $[\Gamma_k^+] = [\Gamma_k^-]$  for any  $k$ , and  $[\Gamma_k^+]$  is independent of  $k$ .

Note that  $[\Gamma]$  looks like  $\begin{array}{c} b \swarrow a \\ \bullet \rightarrow \\ \bullet \rightarrow \end{array}$  locally. It coincides with  $[\Gamma_k^+]$  outside the pictured part. But  $\Gamma'$  must be equipped with the opposite vertex-orientation, and  $(-[\Gamma'])$  looks like  $\begin{array}{c} b \downarrow a \\ \bullet \rightarrow \\ \bullet \rightarrow \end{array}$ .

Thus, it suffices to prove

$$\zeta_\Gamma \left( \left[ \begin{array}{c} b \swarrow a \\ \bullet \rightarrow \\ \bullet \rightarrow \end{array} \right] - \left[ \begin{array}{c} b \downarrow a \\ \bullet \rightarrow \\ \bullet \rightarrow \end{array} \right] \right) + 2(m - |E(\Gamma)|)\zeta_{\Gamma_k^+} \left[ \begin{array}{c} b \swarrow a \\ \bullet \rightarrow \\ \bullet \rightarrow \end{array} \right] = 0$$

if  $m \in \{3n - 2, 3n\}$ . With the expression of the  $\zeta_\Gamma$  in Notation 7.16 and Notation 7.29, this equality is equivalent to the STU relation when  $m > |E(\Gamma)|$ . In particular, it is equivalent to the STU relation when  $m = 3n$ . When  $m = 3n - 2$ , if  $m = |E(\Gamma)|$ , then  $\Gamma$  has exactly 2 univalent vertices, so we have  $([\Gamma] + [\Gamma']) = 0$ , and the equality is still true.  $\square$

Proposition 9.2 is now proved. Proposition 9.7 follows from Proposition 9.2 and Lemma 9.12.  $\square$

**PROOF OF THEOREM 7.32:** We follow the face cancellations in the proof of Proposition 9.2, to study the effect of changing  $\omega_{S^2}(i) = \tilde{\omega}_{S^2}|_{\{0\} \times S^2}$  to  $\omega'_{S^2}(i) = \tilde{\omega}_{S^2}|_{\{1\} \times S^2}$ , for a closed two-form  $\tilde{\omega}_{S^2}$  on  $[0, 1] \times S^2$  as in Lemma 9.1, for some  $i \in \underline{3}$ . We use the form  $\omega(i) = p_{[0,1] \times S^2}^*(\tilde{\omega}_{S^2})$  on  $[0, 1] \times C_2(S^3)$ . Here, the involved graphs have no looped edges, 4 vertices, at most 3 edges, and hence at most one trivalent vertex. They are  $\otimes$ ,  $\Downarrow$ , and  $\Leftarrow$ . The only cancellation that requires an additional argument is the cancellation of the

faces, for which  $\Gamma$  is isomorphic to  $\textcircled{3}$  and  $A$  is a pair of univalent vertices of  $\Gamma$ . (The cancellation of Lemma 9.14 would involve  $\textcircled{3}$ .) In this case, the open face  $F(A, L, \Gamma)$  is the configuration space of 2 vertices on the knot (one of them stands for the two vertices of  $A$ ) and a trivalent vertex in  $\mathbb{R}^3$ . The integral  $\int_{[0,1] \times F(A,L,\Gamma)} \bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$  is the pull-back of a 6-form on  $[0,1] \times (S^2)^3$ , by a map whose image is in the codimension 2 subspace of  $[0,1] \times (S^2)^3$  in which two  $S^2$ -coordinates coincide. Indeed, the two edges that contain the vertices of  $A$  have the same direction. Therefore, we have  $I(\Gamma, A) = 0$  for these faces, and  $w_C(\check{Z}_{2,3}(\mathbb{R}^3, K, (p_{S^2}^*(\omega_{S^2}(i)))_{i \in \underline{3}}))$  is independent of the chosen  $\omega_{S^2}(i)$ . Conclude with the arguments of Remark 7.31.  $\square$

# Chapter 10

## First properties of $Z$ and anomalies

### 10.1 Some properties of $Z(\check{R}, L, \tau)$

Lemma 9.5 allows us to set

$$Z_n(\check{R}, L, \tau) = Z_n(\check{R}, L, (\omega(i)))$$

for any collection  $(\omega(i))$  of homogeneous propagating forms of  $(C_2(R), \tau)$ , under the assumptions of Theorem 7.20. We still have to study how  $Z_n(\check{R}, L, \tau)$  varies when  $\tau$  varies inside its homotopy class when  $L \neq \emptyset$ , but the naturality of the construction of  $Z_n$  already implies the following proposition.

**Proposition 10.1.** *Let  $R$  be the  $\mathbb{Q}$ -sphere equipped with its neighborhood  $\mathring{B}_{1,\infty}$  of  $\infty$  of the beginning of Section 3.2. Let  $\psi$  be an orientation-preserving diffeomorphism from  $R$  to  $\psi(R)$ . Use the restriction of  $\psi$  to the ball  $\mathring{B}_{1,\infty}$  as an identification of  $\mathring{B}_{1,\infty}$  with a neighborhood of  $\psi(\infty)$  in  $\psi(R)$ . Define  $\psi_*(\tau) = T\psi \circ \tau \circ (\psi^{-1} \times \mathbf{1}_{\mathbb{R}^3})$ . Then we have*

$$Z_n(\psi(\check{R}), \psi(L), \psi_*(\tau)) = Z_n(\check{R}, L, \tau)$$

for all  $n \in \mathbb{N}$ , where  $p_1(\psi_*(\tau)) = p_1(\tau)$ .

PROOF: The diffeomorphism  $\psi$  induces natural diffeomorphisms  $\psi_*$  from  $C_2(R)$  to  $C_2(\psi(R))$ , and from the  $\check{C}(R, L; \Gamma)$  to the  $\check{C}(\psi(R), \psi(L); \Gamma)$ . If  $\omega$  is a homogeneous propagating form of  $(C_2(R), \tau)$ , then  $(\psi_*^{-1})^*(\omega)$  is a homogeneous propagating form of  $(C_2(\psi(R)), \psi_*(\tau))$  since the restriction of  $(\psi_*^{-1})^*(\omega)$  to  $U\psi(\check{R})$  is  $(T\psi^{-1})^*(p_\tau^*(\omega_{S^2})) = (p_\tau \circ T\psi^{-1})^*(\omega_{S^2}) = p_{\psi_*(\tau)}^*(\omega_{S^2})$ .

For any Jacobi diagram  $\Gamma$  on the domain  $\mathcal{L}$  of  $L$ , we have

$$\begin{aligned} I(\psi(R), \psi(L), \Gamma, (\psi_*^{-1})^*(\omega)) &= \int_{\check{C}(\psi(R), \psi(L); \Gamma)} \bigwedge_{e \in E(\Gamma)} p_e^*((\psi_*^{-1})^*(\omega)) \\ &= \int_{\check{C}(\psi(R), \psi(L); \Gamma)} (\psi_*^{-1})^* \left( \bigwedge_{e \in E(\Gamma)} p_e^*(\omega) \right) \\ &= I(R, L, \Gamma, \omega), \end{aligned}$$

where  $\Gamma$  is equipped with an implicit orientation  $o(\Gamma)$ . Therefore, we have

$$Z_n(\psi(\check{R}), \psi(L), \psi_*(\tau)) = Z_n(\check{R}, L, \tau)$$

for all  $n \in \mathbb{N}$ . □

We study some other properties of  $Z_n(\check{R}, L, \tau)$ .

Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . Thanks to Lemma 9.6,  $Z_n(\check{R}, \emptyset, \tau)$  depends only on the homotopy class of  $\tau$  for any integer  $n$ . Set  $Z_n(R, \tau) = Z_n(\check{R}, \emptyset, \tau)$  and  $Z(R, \tau) = (Z_n(R, \tau))_{n \in \mathbb{N}}$ . Using Notation 7.16, let  $z_n(R, \tau) = p^c(Z_n(R, \tau))$  be the connected part of  $Z_n(R, \tau)$ , and set  $z(R, \tau) = (z_n(R, \tau))_{n \in \mathbb{N}}$ .

We give a direct elementary proof of the following proposition, which could also be proved in the same way as Corollary 10.4 below.

**Proposition 10.2.** *For any propagating form  $\omega$  of  $C_2(R)$ , we have*

$$Z(R, \omega) = \exp(z(R, \omega)).$$

*In particular, for any asymptotically standard parallelization  $\tau$  of  $R$ , we have*

$$Z(R, \tau) = \exp(z(R, \tau)).$$

**PROOF:** Let  $\Gamma$  be a trivalent Jacobi diagram whose components are isomorphic to some  $\Gamma_i$  for  $i \in \underline{r}$  and such that  $\Gamma$  has  $k_i$  connected components isomorphic to  $\Gamma_i$ . Then we have

$$I(R, \emptyset, \Gamma, (\omega)_{i \in \underline{3n}}) [\Gamma] = \prod_{i=1}^r \left( I(R, \emptyset, \Gamma_i, (\omega)_{i \in \underline{3 \deg(\Gamma_i)}})^{k_i} [\Gamma_i]^{k_i} \right)$$

and  $|\text{Aut}(\Gamma)| = \prod_{i=1}^r (k_i! (\text{Aut}(\Gamma_i))^{k_i})$ . Conclude with Proposition 7.25. □

Recall the coproduct maps  $\Delta_n$  defined in Section 6.5.

**Proposition 10.3.** *Under the assumptions of Theorem 7.20, we have*

$$\Delta_n(Z_n(\check{R}, L, \tau)) = \sum_{i=0}^n Z_i(\check{R}, L, \tau) \otimes Z_{n-i}(\check{R}, L, \tau).$$

PROOF: Let  $T_i = Z_i(R, L, \tau) \otimes Z_{n-i}(R, L, \tau)$ . We have

$$T_i = \sum \frac{1}{|\text{Aut}(\Gamma')|} I(\Gamma', (\omega)_{j \in \underline{3i}}) \frac{1}{|\text{Aut}(\Gamma'')|} I(\Gamma'', (\omega)_{j \in \underline{3(n-i)}}) [\Gamma'] \otimes [\Gamma''],$$

where the sum runs over the pairs  $(\Gamma', \Gamma'')$  in  $\mathcal{D}_i^u(\mathcal{L}(L)) \times \mathcal{D}_{n-i}^u(\mathcal{L}(L))$ . Use Remark 7.26 to view the summands as a measure of configurations of graphs  $\Gamma' \sqcup \Gamma''$  (which may correspond to several elements of  $\mathcal{D}_n^u(\mathcal{L}(L))$ ) together with a choice of an embedded subgraph  $\Gamma'$ .  $\square$

**Corollary 10.4.** *Recall the projection  $p^c$  of Corollary 6.40 from  $\check{\mathcal{A}}(S^1)$  to the space  $\check{\mathcal{A}}^c(S^1)$  of its primitive elements. If  $L$  has one component, set*

$$\check{z}(R, L, \tau) = p^c(\check{Z}(\check{R}, L, \tau)).$$

Then we have

$$\check{Z}(\check{R}, L, \tau) = \exp(\check{z}(R, L, \tau)).$$

PROOF: This is a direct consequence of Lemma 6.37, Proposition 10.3 and Theorem 6.41.  $\square$

## 10.2 On the anomaly $\beta$

We now study how  $z_n(R, \tau)$ , which is defined before Proposition 10.2, depends on  $\tau$ .

**Definition 10.5.** Let  $\rho: (B^3, \partial B^3) \rightarrow (SO(3), 1)$  be the map of Definition 4.4. Extend it to  $\mathbb{R}^3$  by considering  $B^3$  as the unit ball of  $\mathbb{R}^3$  and by letting  $\rho$  map  $(\mathbb{R}^3 \setminus B^3)$  to 1. Consider the parallelization  $\tau_s \circ \psi_{\mathbb{R}}(\rho)$ , where  $\psi_{\mathbb{R}}(\rho)(x, v) = (x, \rho(x)(v))$ . Set

$$\beta_n = z_n(S^3, \tau_s \circ \psi_{\mathbb{R}}(\rho)).$$

**Proposition 10.6.** *Let  $(\check{R}, \tau_0)$  be an asymptotic rational homology  $\mathbb{R}^3$ , and let  $\tau_1$  be a parallelization of  $\check{R}$  that coincides with  $\tau_0$  outside  $B_R$ . Then we have*

$$z_n(R, \tau_1) - z_n(R, \tau_0) = \frac{p_1(\tau_1) - p_1(\tau_0)}{4} \beta_n$$

for any integer  $n$ .



Proposition 10.6 is an easy consequence of Proposition 10.7 below. The latter looks more complicated, but it is very useful since it offers more practical definitions of the *anomaly*

$$\beta = (\beta_n)_{n \in \mathbb{N}}$$

when applied to  $(\check{R}, \tau_0, \tau_1) = (\mathbb{R}^3, \tau_s, \tau_s \circ \psi_{\mathbb{R}}(\rho))$  (and to the case in which  $\tilde{\omega}_{S^2}(i)$  is the pull-back of  $\omega_{0,S^2}(i)$  under the natural projection from  $[0, 1] \times S^2$  to  $S^2$ ).

**Proposition 10.7.** *Let  $(\check{R}, \tau_0)$  be an asymptotic rational homology  $\mathbb{R}^3$ , and let  $\tau_1$  be a parallelization of  $\check{R}$  that coincides with  $\tau_0$  outside  $B_R$ . For  $i \in \underline{3n}$ , let  $\omega_{0,S^2}(i)$  and  $\omega_{1,S^2}(i)$  be two volume-one forms on  $S^2$ . Then there exists a closed two-form  $\tilde{\omega}_{S^2}(i)$  on  $[0, 1] \times S^2$  such that the restriction of  $\tilde{\omega}_{S^2}(i)$  to  $\{t\} \times S^2$  is  $\omega_{t,S^2}$  for  $t \in \{0, 1\}$ . For any such forms  $\tilde{\omega}_{S^2}(i)$ , there exist closed 2-forms  $\tilde{\omega}(i)$  on  $[0, 1] \times U\check{R}$  such that*

- the restriction of  $\tilde{\omega}(i)$  to  $\{t\} \times U\check{R}$  is  $p_{\tau_t}^*(\omega_{t,S^2}(i))$  for  $t \in \{0, 1\}$ ,
- the restriction of  $\tilde{\omega}(i)$  to  $[0, 1] \times (U(\check{R} \setminus B_R))$  is  $(\mathbf{1}_{[0,1]} \times p_{\tau_0})^*(\tilde{\omega}_{S^2}(i))$ .

Then we have

$$\begin{aligned} z_n(R, \tau_1) - z_n(R, \tau_0) &= z_n\left([0, 1] \times UB_R; (\tilde{\omega}(i))_{i \in \underline{3n}}\right) \\ &= \sum_{\Gamma \in \mathcal{D}_n^c} \zeta_{\Gamma} \int_{[0,1] \times \check{S}_{V(\Gamma)}(TB_R)} \bigwedge_{e \in E(\Gamma)} p_e^*(\tilde{\omega}(j_E(e))) [\Gamma] \\ &= \frac{p_1(\tau_1) - p_1(\tau_0)}{4} \beta_n, \end{aligned}$$

and  $\beta_n = 0$  if  $n$  is even. (Recall that the orientation of  $\check{S}_{V(\Gamma)}(T\check{R})$  is defined in Lemma 9.3.)

PROOF: The existence of  $\tilde{\omega}_{S^2}(i)$  comes from Lemma 9.1. In order to prove the existence of  $\tilde{\omega}(i)$ , which is defined on  $\partial([0, 1] \times S^2 \times B_R)$  by the conditions, we need to extend it to  $[0, 1] \times S^2 \times B_R$ . The obstruction belongs to

$$H^3([0, 1] \times S^2 \times B_R, \partial([0, 1] \times S^2 \times B_R)) \cong H_3([0, 1] \times S^2 \times B_R),$$

which is trivial. So  $\tilde{\omega}(i)$  extends as desired. In order to prove that the first equality is a consequence of Corollary 9.4, extend the forms  $\tilde{\omega}(i)$  of the statement to  $[0, 1] \times C_2(R)$  as forms that satisfy the conditions in Proposition 9.2. First extend the  $\tilde{\omega}(i)$  to  $[0, 1] \times (\partial C_2(R) \setminus UB_R)$  as  $(\mathbf{1}_{[0,1]} \times p_{\tau_0})^*(\tilde{\omega}_{S^2}(i))$ . Next extend the restriction of  $\tilde{\omega}(i)$  to  $\{0\} \times \partial C_2(R)$  (resp. to  $\{1\} \times \partial C_2(R)$ ) on

$\{0\} \times C_2(R)$  (resp. on  $\{1\} \times C_2(R)$ ) as a propagating form of  $(C_2(R), \tau_0)$  (resp. of  $(C_2(R), \tau_1)$ ) as in Section 3.3. Thus,  $\tilde{\omega}(i)$  is defined consistently on  $\partial([0, 1] \times C_2(R))$ , and it extends as a closed form that satisfies the assumptions in Proposition 9.2 as in Lemma 9.1. Corollary 9.4, Lemma 9.6, and Lemma 9.12 yield

$$z_n(R, \tau_1) - z_n(R, \tau_0) = \sum_{\Gamma \in \mathcal{D}_c^n} \frac{1}{(3n)!2^{3n}} \int_{[0,1] \times \check{S}_{V(\Gamma)}(TB_R)} \bigwedge_{e \in E(\Gamma)} p_e^* \left( \tilde{\omega}(j_E(e)) \right) [\Gamma],$$

which is zero if  $n$  is even. So everything is proved when  $n$  is even. Assume that  $n$  is odd.

There exists a map  $g: (\check{R}, \check{R} \setminus B_R) \rightarrow (SO(3), 1)$  such that  $\tau_1 = \tau_0 \circ \psi_{\mathbb{R}}(g)$ . Using  $\tau_0$  to identify  $\check{S}_{V(\Gamma)}(T\check{R})$  with  $\check{R} \times \check{S}_{V(\Gamma)}(\mathbb{R}^3)$  makes clear that  $(z_n(R, \tau_0 \circ \psi_{\mathbb{R}}(g)) - z_n(R, \tau_0))$  does not depend on  $\tau_0$ . For any  $g: (\check{R}, \check{R} \setminus B_R) \rightarrow (SO(3), 1)$ , set  $z'_n(g) = z_n(R, \tau_0 \circ \psi_{\mathbb{R}}(g)) - z_n(R, \tau_0)$ . Then  $z'_n$  is a homomorphism from  $[(B_R, \partial B_R), (SO(3), 1)]$  to the vector space  $\mathcal{A}_n^c(\emptyset)$  over  $\mathbb{R}$ . Theorem 4.5 and Lemma 4.6 yield

$$z'_n(g) = \frac{\deg(g)}{2} z'_n(\rho_{B_R}(B^3)).$$

It is easy to see that  $z'_n(\rho_{B_R}(B^3))$  is independent of  $\check{R}$ . Since Example 7.18 shows  $z_n(S^3, \tau_s) = 0$ , we have  $z'_n(\rho_{B_R}(B^3)) = \beta_n$  by Definition 10.5. Recall  $p_1(\tau_0 \circ \psi_{\mathbb{R}}(g)) - p_1(\tau_0) = 2 \deg(g)$  from Theorem 4.5.  $\square$

**Remark 10.8.** The anomaly  $\beta$  is the opposite of the constant  $\xi$  defined in [Les04a, Section 1.6].

**Corollary 10.9.** *Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ , then*

$$z_n(R, \tau) - \frac{p_1(\tau)}{4} \beta_n$$

*is independent of  $\tau$ . Set  $\mathfrak{z}_n(R) = z_n(R, \tau) - \frac{p_1(\tau)}{4} \beta_n$ ,  $\mathfrak{z}(R) = (\mathfrak{z}_n(R))_{n \in \mathbb{N}}$ , and*

$$\mathcal{Z}(R) = \exp(\mathfrak{z}(R)).$$

*Then*

$$\mathcal{Z}(R) = Z(R, \tau) \exp\left(-\frac{p_1(\tau)}{4} \beta\right)$$

*is the invariant  $\mathcal{Z}(R, \emptyset)$  that was announced in Theorem 7.20.*

**PROOF:** See Proposition 10.2.  $\square$

**Proposition 10.10.** *The degree-one part of the anomaly  $\beta$  is  $\beta_1 = \frac{1}{12} [\Theta]$ .*

PROOF: According to Proposition 7.17, we have  $z_1(R, \tau) = \frac{\Theta(R, \tau)}{12} [\Theta]$ . Proposition 10.6 implies  $z_1(R, \tau_1) - z_1(R, \tau_0) = \frac{p_1(\tau_1) - p_1(\tau_0)}{4} \beta_1$ , while Corollary 4.8 implies

$$\Theta(R, \tau_1) - \Theta(R, \tau_0) = \frac{1}{4} (p_1(\tau_1) - p_1(\tau_0)).$$

□

**Corollary 10.11.** *Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ , then we have  $Z_1(R, \tau) = z_1(R, \tau) = \frac{\Theta(R, \tau)}{12} [\Theta]$  and*

$$\mathcal{Z}_1(R) = \mathfrak{z}_1(R) = \frac{\Theta(R)}{12} [\Theta]$$

in  $\mathcal{A}_1(\emptyset) = \mathbb{R} [\Theta]$ .

PROOF: The first equality is a direct consequence of Proposition 7.17. The second one follows from Corollary 4.8, Corollary 10.9, and Proposition 10.10. □

**Remark 10.12.** According to Proposition 10.7, the even-degree part of the anomaly  $\beta$  vanishes. The values of  $\beta_{2n+1}$  are unknown when  $n \geq 1$ . We may hope them to be zero, but I do not know any conjecture about them.

### 10.3 On the anomaly $\alpha$

We define the *anomaly*

$$\alpha = (\alpha_n)_{n \in \mathbb{N}},$$

which is sometimes called the *Bott–Taubes anomaly*, below. Let  $v \in S^2$ . Let  $D_v$  denote the linear map

$$\begin{aligned} D_v: \mathbb{R} &\longrightarrow \mathbb{R}^3 \\ 1 &\mapsto v. \end{aligned}$$

Let  $\mathcal{D}_n^c(\mathbb{R})$  denote the set of degree  $n$ , connected,  $\underline{3n-2}$ -numbered diagrams on  $\mathbb{R}$  with at least one univalent vertex, without looped edges. As in Definition 7.6, a degree  $n$  diagram  $\check{\Gamma}$  is *numbered* if the edges of  $\check{\Gamma}$  are oriented and if  $E(\check{\Gamma})$  is equipped with an injection  $j_E: E(\check{\Gamma}) \hookrightarrow \underline{3n-2}$ , which numbers its edges. Let  $\check{\Gamma} \in \mathcal{D}_n^c(\mathbb{R})$ . Define  $\check{C}(D_v; \check{\Gamma})$  as in Section 7.1, where the

line  $D_v$  of  $\mathbb{R}^3$  replaces the link  $L$  of  $\check{R}$ , and  $\mathbb{R}$  replaces the domain  $\mathcal{L}$ . Let  $\check{Q}(v; \check{\Gamma})$  be the quotient of  $\check{C}(D_v; \check{\Gamma})$  by the translations parallel to  $D_v$  and by the dilations. Then  $\check{Q}(v; \check{\Gamma})$  is the space denoted by  $\check{\mathcal{S}}(\mathbb{R}^3, v; \check{\Gamma})$  before Lemma 8.16. Let  $Q(v; \check{\Gamma})$  denote the closure of  $\check{Q}(v; \check{\Gamma})$  in  $\mathcal{S}_{V(\check{\Gamma})}(\mathbb{R}^3)$ . According to Lemma 8.16, the space  $Q(v; \check{\Gamma})$ , which coincides with  $\mathcal{S}(\mathbb{R}^3, v; \check{\Gamma})$ , is a compact smooth manifold with ridges.

To each edge  $e$  of  $\check{\Gamma}$ , associate a map  $p_{e, S^2}$ , which maps a configuration of  $\check{Q}(v; \check{\Gamma})$  to the direction of the vector from the origin of  $e$  to its end in  $S^2$ . This map extends to  $Q(v; \check{\Gamma})$  according to Theorem 8.12.

Now, define  $\check{Q}(\check{\Gamma})$  (resp.  $Q(\check{\Gamma})$ ) as the total space of the fibration over  $S^2$  whose fiber over  $v$  is  $\check{Q}(v; \check{\Gamma})$  (resp.  $Q(v; \check{\Gamma})$ ). The configuration space  $\check{Q}(\check{\Gamma})$  and its compactification  $Q(\check{\Gamma})$  carry natural smooth structures. The configuration space  $\check{Q}(\check{\Gamma})$  is oriented as follows, when a vertex-orientation  $o(\check{\Gamma})$  is given. Equip  $\check{C}(D_v; \check{\Gamma})$  with its orientation induced by Corollary 7.2, as before. Orient  $\check{Q}(v; \check{\Gamma})$  so that  $\check{C}(D_v; \check{\Gamma})$  is locally homeomorphic to the oriented product (translation vector  $z$  in  $\mathbb{R}v$ , ratio of homothety  $\lambda \in ]0, \infty[$ )  $\times \check{Q}(v; \check{\Gamma})$ . Next orient  $\check{Q}(\check{\Gamma})$  with the (base(=  $S^2$ )  $\oplus$  fiber) convention. (We can summarize this by saying that the  $S^2$ -coordinates replace  $(z, \lambda)$ .)

**Proposition 10.13.** *For  $i \in \underline{3n-2}$ , let  $\omega(i, S^2)$  be a volume-one form of  $S^2$ . Define*

$$I(\check{\Gamma}, o(\check{\Gamma}), \omega(i, S^2)) = \int_{\check{Q}(\check{\Gamma})} \bigwedge_{e \in E(\check{\Gamma})} p_{e, S^2}^*(\omega(j_E(e), S^2)).$$

Set

$$\alpha_n = \frac{1}{2} \sum_{\check{\Gamma} \in \mathcal{D}_n^c(\mathbb{R})} \zeta_{\check{\Gamma}} I(\check{\Gamma}, o(\check{\Gamma}), \omega(i, S^2)) [\check{\Gamma}, o(\check{\Gamma})] \in \mathcal{A}(\mathbb{R}),$$

where  $\zeta_{\check{\Gamma}} = \frac{(3n-2-|E(\check{\Gamma})|)!}{(3n-2)!2^{|E(\check{\Gamma})|}}$ . Then  $\alpha_n$  does not depend on the chosen  $\omega(i, S^2)$ , we have  $\alpha_1 = \frac{1}{2} [\check{\zeta}]$ , and  $\alpha_{2n} = 0$  for all  $n$ .

PROOF: Let us first prove that  $\alpha_n$  does not depend on the chosen  $\omega(i, S^2)$ , by proving that its variation vanishes when  $\omega(i, S^2)$  is changed to some  $\tilde{\omega}(i, 1, S^2)$ . According to Lemma 9.1, there exists a closed 2-form  $\tilde{\omega}(i, S^2)$  on  $[0, 1] \times S^2$  whose restriction to  $\{0\} \times S^2$  is  $\omega(i, S^2) = \tilde{\omega}(i, 0, S^2)$  and whose restriction to  $\{1\} \times S^2$  is  $\tilde{\omega}(i, 1, S^2)$ . According to Stokes' theorem, for any  $\check{\Gamma} \in \mathcal{D}_n^c(\mathbb{R})$ , we have

$$\begin{aligned} I(\check{\Gamma}, o(\check{\Gamma}), (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3n-2}}) - I(\check{\Gamma}, o(\check{\Gamma}), (\tilde{\omega}(i, 0, S^2))_{i \in \underline{3n-2}}) \\ = \sum_F \int_{[0, 1] \times F} \bigwedge_{e \in E(\check{\Gamma})} p_{e, S^2}^*(\tilde{\omega}(j_E(e), S^2)), \end{aligned}$$

where  $p_{e,S^2}: [0, 1] \times \check{Q}(\check{\Gamma}) \rightarrow [0, 1] \times S^2$  denotes the product by  $\mathbf{1}_{[0,1]}$  of  $p_{e,S^2}$ , and the sum runs over the codimension-one faces  $F$  of  $Q(\check{\Gamma})$ . These faces fiber over  $S^2$ , and the fibers over  $v \in S^2$  are the codimension-one faces  $f(A, v; \check{\Gamma})$  of  $Q(v, \check{\Gamma}) = \mathcal{S}(\mathbb{R}^3, v; \check{\Gamma})$  for the strict subsets  $A$  of  $V(\check{\Gamma})$  with cardinality at least 2 whose univalent vertices are consecutive on  $\mathbb{R}$ , as in Lemma 8.17. Let  $F(A, \check{\Gamma})$  denote the face with fiber  $f(A, v; \check{\Gamma})$ . Now, it suffices to prove that the contributions of all the  $F(A, \check{\Gamma})$  vanish.

When the product of all the  $p_{e,S^2}$  factors through a quotient of  $[0, 1] \times F(A, \check{\Gamma})$  of smaller dimension, the face  $F(A, \check{\Gamma})$  does not contribute. This allows us to get rid of

- the faces  $F(A, \check{\Gamma})$  for which  $\check{\Gamma}_A$  is not connected and  $A$  is not a pair of univalent vertices of  $\check{\Gamma}$ , as in Lemma 9.9, and
- the faces  $F(A, \check{\Gamma})$  for which  $|A| \geq 3$  and  $\check{\Gamma}_A$  has a univalent vertex that was trivalent in  $\check{\Gamma}$ , as in Lemma 9.10.

We also have faces that cancel each other for graphs that are identical outside their  $\check{\Gamma}_A$  part.

- The faces  $F(A, \check{\Gamma})$  (which are not already listed) such that  $\check{\Gamma}_A$  has at least one bivalent vertex cancel (by pairs) by the parallelogram identification as in Lemma 9.11.
- The faces  $F(A, \check{\Gamma})$  for which  $\check{\Gamma}_A$  is an edge between two trivalent vertices, cancel by triples, thanks to the Jacobi relation as in Lemma 9.13.
- Similarly, two faces for which  $A$  is a pair of (necessarily consecutive in  $\mathbb{R}$ ) univalent vertices of  $\check{\Gamma}$ , cancel  $(3n - 2 - |E(\Gamma)|)$  faces  $F(\check{\Gamma}', A')$  for which  $\check{\Gamma}'_{A'}$  is an edge between a univalent vertex of  $\check{\Gamma}$  and a trivalent vertex of  $\check{\Gamma}$ , thanks to the STU relation (and to Lemma 6.26) as in Lemma 9.14.

Here, there are no faces left, and  $\alpha_n$  does not depend on the chosen  $\omega(i, S^2)$ .

The computation of  $\alpha_1$  is straightforward.

Let us prove that  $\alpha_n = 0$  for any even  $n$ . Let  $\check{\Gamma}$  be a numbered graph, and let  $\check{\Gamma}^{eo}$  be obtained from  $\check{\Gamma}$  by reversing the orientations of the  $|E|$  edges of  $\check{\Gamma}$ . Consider the map  $r$  from  $\check{Q}(\check{\Gamma}^{eo})$  to  $\check{Q}(\check{\Gamma})$  that composes a configuration by the multiplication by  $(-1)$  in  $\mathbb{R}^3$ . It sends a configuration over  $v \in S^2$  to a configuration over  $(-v)$ . It is a fibered space map over the orientation-reversing antipode of  $S^2$ . Equip  $\check{\Gamma}$  and  $\check{\Gamma}^{eo}$  with the same vertex-orientation and with the same orders on their vertex sets. Then our map  $r$  is orientation-preserving if and only if  $|T(\check{\Gamma})| + 1$  is even. The vertex-orientations of  $H(\check{\Gamma})$

and  $H(\check{\Gamma}^{eo})$  can be consistent with both of the edge-orientations of  $H(\check{\Gamma})$  and  $H(\check{\Gamma}^{eo})$  if and only if  $|E(\check{\Gamma})|$  is even. We have  $p_{e,S^2,\check{\Gamma}^{eo}} = p_{e,S^2,\check{\Gamma}} \circ r$  for all the edges  $e$  of  $\check{\Gamma}^{eo}$ . Since  $|E(\check{\Gamma})| = n + |T(\check{\Gamma})|$ , we get

$$I(\check{\Gamma}^{eo}, \omega(i, S^2)) [\check{\Gamma}^{eo}] = (-1)^{n+1} I(\check{\Gamma}, \omega(i, S^2)) [\check{\Gamma}].$$

□

**Note 10.14.** It is known that  $\alpha_3 = 0$  [Poi02, Proposition 1.4]. Sylvain Poirier also found that  $\alpha_5 = 0$  with the help of a Maple program. Furthermore, according to [Les02, Corollary 1.4],  $\alpha_{2n+1}$  is a combination of diagrams with two univalent vertices, and  $\mathcal{Z}(S^3, L) = \check{\mathcal{Z}}(S^3, L)$  is obtained from the Kontsevich integral  $Z^K$  by inserting  $d$  times the plain part of  $2\alpha$  on each degree  $d$  connected component of a diagram. A similar statement is valid for the functorial extension of  $\mathcal{Z}$  to tangles described in the book’s third part. See Corollary 12.29 and Lemma 13.19, which provides an alternative definition of  $\alpha$ . It has been conjectured that  $\alpha_n = 0$  for  $n > 1$  and incorrect proofs have circulated.

## 10.4 Dependence on the forms for straight links

In this section, we prove Theorems 7.40 and 7.42. In order to do it, we will prove the following lemma.

**Lemma 10.15.** *Under the assumptions of Theorem 7.40, the element*

$$Z_{n,3n-2}(\check{R}, L, (\omega(i))_{i \in \underline{3n-2}}) = \sum_{\Gamma \in \mathcal{D}_{n,3n-2}^e(\mathcal{L})} \zeta_{\Gamma} I(R, L, \Gamma, (\omega(i))_{i \in \underline{3n-2}}) [\Gamma]$$

*of  $\mathcal{A}_n(\mathcal{L})$  is independent of the chosen propagating forms  $\omega(i)$  of  $(C_2(R), \tau)$ .*

**Lemma 10.16.** *Lemma 10.15 implies Theorem 7.40 and Theorem 7.42.*

**PROOF:** Let us prove that Lemma 10.15 implies Theorem 7.40.  $\mathcal{A}_n(\mathcal{L}) = \mathcal{A}_n(\emptyset) \oplus \mathcal{A}'_n(\mathcal{L})$ , where  $\mathcal{A}'_n(\mathcal{L})$  is the subspace of  $\mathcal{A}_n(\mathcal{L})$  generated by the Jacobi diagrams with at least one univalent vertex. Since we know from Lemma 9.6 that  $Z_n(\check{R}, \emptyset, (\omega(i))_{i \in \underline{3n}})$  is independent of the chosen propagating forms  $\omega(i)$  of  $(C_2(R), \tau)$ , we focus on the projection  $Z'_n(\check{R}, L, (\omega(i))_{i \in \underline{3n}})$  of

$$Z_n(\check{R}, L, (\omega(i))_{i \in \underline{3n}}) = \sum_{\Gamma \in \mathcal{D}_n^e(\mathcal{L})} \zeta_{\Gamma} I(R, L, \Gamma, (\omega(i))_{i \in \underline{3n}}) [\Gamma] \in \mathcal{A}_n(\mathcal{L})$$

onto  $\mathcal{A}'_n(\mathcal{L})$ . This sum is a sum over diagrams with at least two univalent vertices, according to Lemma 6.26. Recall Notation 7.29. Lemma 10.15 guarantees that

$$Z_{n,I}(\check{R}, L, (\omega(i))_{i \in I}) = \sum_{\Gamma \in \mathcal{D}_{n,I}^e(\mathcal{L})} \zeta_\Gamma I(R, L, \Gamma, (\omega(i))_{i \in I}) [\Gamma] \in \mathcal{A}_n(\mathcal{L})$$

is independent of the chosen  $\omega(i)$  for any subset  $I$  of  $\underline{3n}$  with cardinality  $(3n-2)$ . Observe

$$Z'_n(\check{R}, L, (\omega(i))_{i \in \underline{3n}}) = \frac{2((3n-2)!)}{(3n)!} \sum_{I \subset \underline{3n}: |I|=3n-2} Z_{n,I}(\check{R}, L, (\omega(i))_{i \in I})$$

since for a numbered graph  $\Gamma$  of  $\mathcal{D}_n^e(\mathcal{L})$ , the coefficient of  $I(R, L, \Gamma, (\omega(i))_{i \in \underline{3n}})$  is  $\frac{(3n-|E(\Gamma)|)!}{(3n)!2^{|E(\Gamma)|}} [\Gamma]$  in the left-hand side, and

$$\begin{aligned} & \frac{2((3n-2)!)}{(3n)!} \sum_{I \subset \underline{3n}: |I|=3n-2, j_E(E(\Gamma)) \subseteq I} \frac{(3n-2-|E(\Gamma)|)!}{(3n-2)!2^{|E(\Gamma)|}} [\Gamma] \\ &= \frac{2((3n-2)!)}{(3n)!} \frac{(3n-|E(\Gamma)|)!}{2((3n-2-|E(\Gamma)|)!)} \frac{(3n-2-|E(\Gamma)|)!}{(3n-2)!2^{|E(\Gamma)|}} [\Gamma] \end{aligned}$$

in the right-hand-side. Thus, Lemma 10.15 and Lemma 9.5 imply Theorem 7.40.

Lemma 10.15 also directly implies that

$$\sum_{\Gamma \in \mathcal{D}_{n,3n-2}^e(\mathcal{L})} \zeta_\Gamma I(R, L, \Gamma, (\omega(i))_{i \in \underline{3n-2}}) \check{p}([\Gamma])$$

is independent of the chosen  $\omega(i)$ , so it implies Theorem 7.42 as above.  $\square$

**PROOF OF LEMMA 10.15:** Let  $L: \mathcal{L} \hookrightarrow \check{R}$  be a straight embedding with respect to  $\tau$ . It suffices to prove that  $Z_{n,3n-2}(\check{R}, L, (\omega(i))_{i \in \underline{3n-2}})$  does not change when some  $\omega(i)$  is changed to  $\omega(i) + d\eta$  for some one-form  $\eta$  on  $C_2(R)$ , which restricts to  $\partial C_2(R)$  as  $p_\tau^*(\eta_{S^2})$  for some one-form  $\eta_{S^2}$  on  $S^2$ , as in Lemma 3.17. Assume that the forms  $\omega(j)$  restrict to  $\partial C_2(R)$  as  $p_\tau^*(\omega_{S^2}(j))$ . Set  $\tilde{\omega}(i, 0) = \omega(i)$ . Let  $p_{C_2}: [0, 1] \times C_2(R) \rightarrow C_2(R)$  and  $p_{S^2}: [0, 1] \times S^2 \rightarrow S^2$  denote the projections onto the second factor. Define the closed 2-form  $\tilde{\omega}_{S^2}(i)$  on  $[0, 1] \times S^2$  by

$$\tilde{\omega}_{S^2}(i) = p_{S^2}^*(\omega_{S^2}(i)) + d(tp_{S^2}^*(\eta_{S^2})),$$

where  $t$  is the coordinate on  $[0, 1]$ . Define the closed 2-form  $\tilde{\omega}(i)$  on  $[0, 1] \times C_2(R)$  to be

$$\tilde{\omega}(i) = p_{C_2}^*(\omega(i)) + d(tp_{C_2}^*(\eta)).$$

For  $j \in \underline{3n} \setminus \{i\}$ , define  $\tilde{\omega}_{S^2}(j) = p_{S^2}^*(\omega_{S^2}(j))$  and  $\tilde{\omega}(j) = p_{C_2}^*(\omega(j))$ . For any  $j \in \underline{3n}$ , let  $\tilde{\omega}(j, t)$  denote the restriction of  $\tilde{\omega}(j)$  to  $\{t\} \times C_2(R)$ . Thus, it suffices to prove that  $Z_{n,3n-2}(1) = Z_{n,3n-2}(0)$ , with

$$Z_{n,3n-2}(t) = \sum_{\Gamma \in \mathcal{D}_{n,3n-2}^e(\mathcal{L})} \zeta_\Gamma I\left(R, L, \Gamma, (\tilde{\omega}(i, t))_{i \in \underline{3n-2}}\right) [\Gamma] \in \mathcal{A}_n(\mathcal{L}).$$

Proposition 9.2 expresses  $(Z_{n,3n-2}(1) - Z_{n,3n-2}(0))$  as a sum over numbered graphs  $\Gamma$  equipped with a connected component  $\Gamma_A$ , with no univalent vertex, or whose univalent vertices form a nonempty set of consecutive vertices in  $\Gamma$  on some component  $S_j^1$  of  $\mathcal{L}$ . The faces for which  $A$  has no univalent vertex do not contribute, as in the proof of Lemma 9.6. So we focus on the remaining faces.

Such a  $F(A, L, \Gamma)$  may split according to the possible compatible linear orders of the univalent vertices of  $\Gamma_A$ , represented by lifts  $\check{\Gamma}_A$  of  $\Gamma_A$  on  $\mathbb{R}$  as in Notation 8.15. We view  $Z_{n,3n-2}(1) - Z_{n,3n-2}(0)$  as a sum over pairs  $(\Gamma, \check{\Gamma}_A)$  of terms

$$\zeta_\Gamma I(\Gamma, \check{\Gamma}_A) = \zeta_\Gamma \int_{[0,1] \times F(\check{\Gamma}_A, L, \Gamma)} \bigwedge_{e \in E(\Gamma)} p_e^*(\tilde{\omega}(j_E(e))) [\Gamma]$$

associated to the corresponding face components denoted by  $F(\check{\Gamma}_A, L, \Gamma)$ .

For a fixed numbered graph  $\check{\Gamma}_A$  on  $\mathbb{R}$  as above, we study the sum of the contributions  $\zeta_\Gamma I(\check{\Gamma}, \check{\Gamma}_A)$  running over the graphs  $\check{\Gamma}$  such that

- the graph  $\check{\Gamma}_A$  is a subgraph of  $\check{\Gamma}$  when  $\mathbb{R}$  is identified with a part of  $S_j^1$ ,
- the univalent vertices of  $\check{\Gamma}_A$  are consecutive on  $S_j^1$  in  $\check{\Gamma}$ , with respect to their linear order, and
- $\check{\Gamma} \setminus \check{\Gamma}_A$  is equal to a fixed  $\Gamma \setminus \check{\Gamma}_A$ , as above

Recall from Proposition 6.22 that  $[\check{\Gamma}] = [\check{\Gamma}_A] \#_j [\Gamma \setminus \check{\Gamma}_A]$  in  $\check{\mathcal{A}}(\mathcal{L})$  for any such pair  $(\check{\Gamma}, \check{\Gamma}_A)$ , provided that all the vertices of the graph  $\check{\Gamma}$  inherit their orientations from a fixed vertex-orientation of  $\Gamma$ .

If  $i \notin j_E(E(\check{\Gamma}_A))$ , then the  $I(\check{\Gamma}, \check{\Gamma}_A)$  vanish because all the  $p_e^*(\tilde{\omega}(j_E(e)))$  for  $e \in E(\check{\Gamma}_A)$  factor through the projection onto  $F(\check{\Gamma}_A, L, \Gamma)$  whose dimension is  $(2|E(\check{\Gamma}_A)| - 1)$ .



If  $i \in j_E(E(\check{\Gamma}_A))$ , let  $e(i)$  be the edge such that  $j_E(e(i)) = i$ . The sum of the contributions  $\zeta.I(\cdot, \check{\Gamma}_A)$  involving  $\check{\Gamma}_A$  factors through

$$I = \int_{[0,1] \times \cup_{c(v) \in K_j} \check{\mathcal{S}}(T_{c(v)}\check{R}, \vec{t}_{c(v)}; \check{\Gamma}_A)} p_{e(i)}^* \left( d(tp_{C_2}^*(\eta)) \right) \bigwedge_{e \in E(\check{\Gamma}_A) \setminus e(i)} p_e^* \left( \tilde{\omega}(j_E(e)) \right),$$

where  $\vec{t}_{c(v)}$  denotes the unit tangent vector to  $K_j$  at  $c(v)$ .

Recall that  $\check{Q}(\check{\Gamma}_A)$  was defined in Section 10.3, together with natural maps

$$p_{e,S^2}: \check{Q}(\check{\Gamma}_A) \rightarrow S^2.$$

Let  $p_{e,S^2}$  also denote  $\mathbf{1}_{[0,1]} \times p_{e,S^2}: [0,1] \times \check{Q}(\check{\Gamma}_A) \rightarrow [0,1] \times S^2$ . The form

$$p_{e(i)}^* \left( d(tp_{C_2}^*(\eta)) \right) \bigwedge_{e \in E(\check{\Gamma}_A) \setminus e(i)} p_e^* \left( \tilde{\omega}(j_E(e)) \right)$$

is the pull-back of the closed form

$$\Omega = p_{e(i),S^2}^* \left( d(t\eta_{S^2}) \right) \bigwedge_{e \in E(\check{\Gamma}_A) \setminus e(i)} p_{e,S^2}^* \left( \tilde{\omega}_{S^2}(j_E(e)) \right)$$

on  $[0,1] \times \check{Q}(\check{\Gamma}_A)$  under the projection

$$[0,1] \times \cup_{c(v) \in K_j} \check{\mathcal{S}}(T_{c(v)}\check{R}, \vec{t}_{c(v)}; \check{\Gamma}_A) \rightarrow [0,1] \times \check{Q}(\check{\Gamma}_A).$$

The image of this projection is the product by  $[0,1]$  of the restriction of the bundle  $\check{Q}(\check{\Gamma}_A)$  over  $p_\tau(U^+K_j)$ , and  $I$  is the integral of  $\Omega$  along this image. Compute the integral by integrating first along the fibers of  $\check{Q}(\check{\Gamma}_A)$ , next along  $[0,1]$ . Afterwards, the integral  $I$  becomes the integral of a one-form along  $p_\tau(U^+K_j) \subset S^2$ . So it vanishes because  $K_j$  is straight.  $\square$

## 10.5 The general variation for homogeneous propagating forms

Set  $\mathcal{D}^c(\mathbb{R}) = \cup_{n \in \mathbb{N}} \mathcal{D}_n^c(\mathbb{R})$ , where  $\mathcal{D}_n^c(\mathbb{R})$  is the set of degree  $n$  connected  $(3n - 2)$ -numbered Jacobi diagrams on  $\mathbb{R}$  introduced in the beginning of Section 10.3. In this section, we write various sums over numbered diagrams, but all the edges of a diagram are equipped with the same propagating forms. So neither the set in which the edges are numbered nor its cardinality matters, provided that the cardinality is greater than the possible number of edges for a given degree. (See Proposition 7.25 and Remark 7.31.)

**Proposition 10.17.** *Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . Let  $L = \sqcup_{j=1}^k K_j$  be an embedding of  $\mathcal{L} = \sqcup_{j=1}^k S_j^1$  into  $\check{R}$ . Let  $\tilde{\omega}(0)$  and  $\tilde{\omega}(1)$  be two homogeneous propagating forms of  $C_2(\check{R})$ . Let  $\tilde{\omega}$  be a closed 2-form on  $[0, 1] \times \partial C_2(\check{R})$  whose restriction  $\tilde{\omega}(t)$  to  $\{t\} \times (\partial C_2(\check{R}) \setminus UB_R)$  is  $p_\tau^*(\omega_{S^2})$  for any  $t \in [0, 1]$ , and whose restriction  $\tilde{\omega}(t)$  to  $\{t\} \times \partial C_2(\check{R})$  coincides with the restriction to  $\partial C_2(\check{R})$  of the given  $\tilde{\omega}(t)$  for  $t \in \{0, 1\}$ . For any  $j \in \underline{k}$ , define  $I_j = \sum_{\Gamma_B \in \mathcal{D}^c(\mathbb{R})} \zeta_{\Gamma_B} I(\Gamma_B, K_j, \tilde{\omega})$ , where*

$$I(\Gamma_B, K_j, \tilde{\omega}) = \int_{u \in [0,1]} \int_{w \in K_j} \int_{\check{S}(T_w \check{R}, \vec{t}_w; \Gamma_B)} \bigwedge_{e \in E(\Gamma_B)} p_e^*(\tilde{\omega}(u)) [\Gamma_B],$$

and  $\vec{t}_w$  denotes the unit tangent vector to  $K_j$  at  $w$ . Define

$$z(\tilde{\omega}) = \sum_{n \in \mathbb{N}} z_n([0, 1] \times UB_R; \tilde{\omega})$$

as in Corollary 9.4. Then we have

$$Z(\check{R}, L, \tilde{\omega}(1)) = \left( \prod_{j=1}^k \exp(I_j) \#_j \right) Z(\check{R}, L, \tilde{\omega}(0)) \exp(z(\tilde{\omega})),$$

where  $\#_j$  stands for the insertion on a diagram on  $\mathbb{R}$  on the component  $S_j^1$  of  $\mathcal{L}$ .

We will actually prove the following lemma.

**Lemma 10.18.** *Recall Notation 7.16. Under the assumptions of Proposition 10.17, we have*

$$\check{Z}(\check{R}, L, \tilde{\omega}(1)) = \left( \prod_{j=1}^k \exp(I_j) \#_j \right) \check{Z}(\check{R}, L, \tilde{\omega}(0)).$$

**Lemma 10.19.** *Lemma 10.18 implies Proposition 10.17.*

PROOF: When  $L = \emptyset$ , Proposition 10.17 follows from Corollary 9.4, Proposition 10.2, and Lemma 9.1, which ensures that there exists a closed 2-form  $\tilde{\omega}$  on  $[0, 1] \times C_2(\check{R})$  that extends the 2-form  $\tilde{\omega}$  of the statement. Conclude with Lemma 7.27.  $\square$

Let us begin the proof of Lemma 10.18 with the proof of the following corollary of Proposition 9.2.

**Lemma 10.20.** *Under the assumptions of Proposition 10.17, let  $\tilde{\omega}$  be a closed 2-form on  $[0, 1] \times C_2(R)$  which extends the 2-form  $\tilde{\omega}$  of Proposition 10.17. For any  $t \in [0, 1]$ , let  $\tilde{\omega}(t)$  denote the restriction to  $\{t\} \times C_2(R)$  of  $\tilde{\omega}$ . Set*

$$\check{Z}(t) = \left( \check{Z}_n(R, L, \tilde{\omega}(t)) \right)_{n \in \mathbb{N}}.$$

For  $\Gamma_B \in \mathcal{D}^c(\mathbb{R})$ ,  $u \in [0, 1]$ , and  $j \in \underline{k}$ , set

$$\eta(R, L, \Gamma_B, K_j, \tilde{\omega})(u) = \int_{w \in K_j} \int_{\check{S}(T_w \check{R}, \vec{t}_w; \Gamma_B)} \bigwedge_{e \in E(\Gamma_B)} p_e^*(\tilde{\omega}(u)) [\Gamma_B],$$

where  $\vec{t}_w$  denotes the unit tangent vector to  $K_j$  at  $w$ , and set

$$\gamma_j(u) = \sum_{\Gamma_B \in \mathcal{D}^c(\mathbb{R})} \zeta_{\Gamma_B} \eta(R, L, \Gamma_B, K_j, \tilde{\omega})(u).$$

Then  $\check{Z}(t)$  is differentiable, and we have

$$\check{Z}'(t)dt = \left( \sum_{j=1}^k \gamma_j(t) \#_j \right) \check{Z}(t).$$

**PROOF OF LEMMA 10.20:** The variations of  $\check{Z}_n(t)$  are given by Proposition 9.2, by sending the diagrams with components without univalent vertices to 0. They involve only faces  $F(A, L, \Gamma)$  for which  $\Gamma_A$  is a connected component of  $\Gamma$ , with univalent vertices on one component of  $\mathcal{L}$ . Again, such a face may split according to the possible compatible linear orders of the univalent vertices of  $\Gamma_A$ , represented by lifts  $\check{\Gamma}_A$  of  $\Gamma_A$  on  $\mathbb{R}$ , as in Notation 8.15. The corresponding face component is denoted by  $F(\check{\Gamma}_A, L, \Gamma)$ , and the corresponding integral is

$$I(\Gamma, \check{\Gamma}_A) = \int_{[0,t] \times F(\check{\Gamma}_A, L, \Gamma)} \bigwedge_{e \in E(\Gamma)} p_e^*(\tilde{\omega}) [\Gamma].$$

Proposition 9.2 implies

$$\check{Z}_n(t) - \check{Z}_n(0) = \sum_{\substack{(\Gamma, A) : \Gamma \in \mathcal{D}_n^e(\mathcal{L}), A \subseteq V(\Gamma), |A| \geq 2, \\ \text{every component of } \Gamma \text{ has univalent vertices,} \\ \Gamma_A \text{ is a connected component of } \Gamma, \\ \text{the univalent vertices of } \Gamma_A \text{ are consecutive} \\ \text{on one component of } \Gamma, \text{ and} \\ \check{\Gamma}_A \text{ is a compatible lift of } \Gamma_A \text{ on } \mathbb{R}.}} \zeta_{\Gamma} I(\Gamma, \check{\Gamma}_A),$$

where the set of univalent vertices of  $\check{\Gamma}_A$  is equipped with the unique linear order induced by  $\Gamma$  if there are univalent vertices of  $\Gamma \setminus \Gamma_A$  on the component of  $\Gamma_A$ , and with one of the linear orders compatible with  $\Gamma$  otherwise.

This expression implies that  $\check{Z}_n$  (valued in a finite-dimensional vector space) is differentiable. (For any smooth compact  $d$ -dimensional manifold  $C$  and for any smooth  $(d+1)$ -form  $\omega$  on  $[0, 1] \times C$ , the function  $(t \mapsto \int_{[0,t] \times C} \omega)$  is differentiable.) Assume that the vertices of  $\Gamma_A$  are on a component  $K_j$  of  $L(\sqcup_{j=1}^k S_j^1)$ . The forms associated to edges of  $\Gamma_A$  are integrated along  $[0, 1] \times (\cup_{c(v) \in K_j} \check{\mathcal{S}}(T_{c(v)} \check{R}, \vec{t}_{c(v)}; \check{\Gamma}_A))$ , where  $\vec{t}_{c(v)}$  denotes the unit tangent vector to  $K_j$  at  $c(v)$ . So they do not depend on the configuration of  $(V(\Gamma) \setminus A)$ . The other forms are integrated along  $\check{C}(R, L; \Gamma \setminus \Gamma_A)$  at  $u \in [0, 1]$ .

Group the contributions of the pairs  $(\Gamma, \check{\Gamma}_A)$  with common  $(\Gamma \setminus \Gamma_A, \check{\Gamma}_A)$  to view the global variation  $(\check{Z}(t) - \check{Z}(0))$  as

$$\sum_{j=1}^k \int_0^t \left( \sum_{\check{\Gamma}_A \in \mathcal{D}^c(\mathbb{R})} \zeta_{\check{\Gamma}_A} \eta(R, L, \check{\Gamma}_A, K_j, \tilde{\omega})(u) \#_j \right) \check{Z}(u).$$

Use Proposition 7.25 and Remark 7.26 to check that the coefficients are correct. So we get

$$\check{Z}(t) - \check{Z}(0) = \int_0^t \left( \sum_{j=1}^k \gamma_j(u) \#_j \right) \check{Z}(u).$$

□

PROOF OF LEMMA 10.18: Set  $I_j(t) = \int_0^t \gamma_j(u)$ . We have  $\check{Z}_0(t) = 1$ . The equation

$$\check{Z}'(t) dt = \left( \sum_{j=1}^k \gamma_j(t) \#_j \right) \check{Z}(t)$$

of Lemma 10.20 determines  $\check{Z}(t)$  as a function of  $\check{Z}(0)$  by induction on the degree, and we have  $\check{Z}(t) = \prod_{j=1}^k \exp(I_j(t)) \#_j \check{Z}(0)$ . □

Let us now apply Lemma 10.20 to study the variation of the quantity  $\check{z}(R, L, \tau)$  of Corollary 10.4 when  $\tau$  varies smoothly.

**Lemma 10.21.** *Let  $(\tau(t))_{t \in [0,1]}$  define a smooth homotopy of asymptotically standard parallelizations of  $\check{R}$ . We have*

$$\frac{\partial}{\partial t} \check{Z}(\check{R}, L, \tau(t)) = \left( \sum_{j=1}^k \frac{\partial}{\partial t} \left( 2 \int_{[0,t] \times U+K_j} p_{\tau(\cdot)}^*(\omega_{S^2}) \right) \alpha \#_j \right) \check{Z}(\check{R}, L, \tau(t)).$$

PROOF: Fix a homogeneous propagating form  $\omega$  of  $(C_2(R), \tau(0))$  and a form  $\tilde{\omega}$  on  $[0, 1] \times C_2(R)$  such that  $\tilde{\omega}(t)$  is a homogeneous propagating form  $\omega$  of  $(C_2(R), \tau(t))$  for all  $t \in [0, 1]$  as in Lemma 9.1. Lemma 10.20 ensures

$$\frac{\partial}{\partial t} \check{Z}(\check{R}, L, \tau(t)) dt = \left( \sum_{j=1}^k \gamma_j(t) \#_j \right) \check{Z}(\check{R}, L, \tau(t)),$$

with

$$\gamma_j(u) = \sum_{\Gamma_B \in \mathcal{D}^c(\mathbb{R})} \zeta_{\Gamma_B} \eta(R, L, \Gamma_B, K_j, \tilde{\omega})(u),$$

and

$$\eta(R, L, \Gamma_B, K_j, \tilde{\omega})(u) = \int_{w \in K_j} \int_{\check{\mathcal{S}}(T_w \check{R}, \vec{t}_w; \Gamma_B)} \bigwedge_{e \in E(\Gamma_B)} p_e^*(p_{\tau(u)}^*(\omega_{S^2})) [\Gamma_B].$$

The restriction of  $p_{\tau(\cdot)}$  from  $[0, 1] \times U^+ K_j$  to  $S^2$  induces a map

$$p_{a, \tau, \Gamma_B}: [0, 1] \times \cup_{w \in K_j} \check{\mathcal{S}}(T_w \check{R}, \vec{t}_w; \Gamma_B) \rightarrow \check{Q}(\Gamma_B)$$

over  $(p_{\tau(\cdot)}: [0, 1] \times U^+ K_j \rightarrow S^2)$ , which restricts to the fibers as the identity map, for any  $\Gamma_B \in \mathcal{D}^c(\mathbb{R})$ . (Recall the definition of  $\check{Q}(\Gamma_B)$  from the beginning of Section 10.3.) We have

$$\int_0^1 \eta(R, L, \Gamma_B, K_j, \tilde{\omega})(u) = \int_{\text{Im}(p_{a, \tau, \Gamma_B})} \left( \bigwedge_{e \in E(\Gamma_B)} p_{e, S^2}^*(\omega_{S^2}) \right) [\Gamma_B].$$

Integrating  $(\bigwedge_{e \in E(\Gamma_B)} p_{e, S^2}^*(\omega_{S^2})) [\Gamma_B]$  along the fiber in  $\check{Q}(\Gamma_B)$  yields a two-form on  $S^2$ . This two-form is homogeneous because everything is. Thus, it may be expressed as  $2\alpha(\Gamma_B)\omega_{S^2} [\Gamma_B]$ , where  $\alpha(\Gamma_B) \in \mathbb{R}$  and

$$\sum_{\Gamma_B \in \mathcal{D}^c(\mathbb{R})} \zeta_{\Gamma_B} \alpha(\Gamma_B) [\Gamma_B] = \alpha.$$

So we get

$$\int_0^t \eta(R, L, \Gamma_B, K_j, \tilde{\omega})(u) = 2\alpha(\Gamma_B) \int_{[0, t] \times U^+ K_j} p_{\tau(\cdot)}^*(\omega_{S^2}) [\Gamma_B],$$

and  $\gamma_j(t) = 2 \frac{\partial}{\partial t} \left( \int_{[0, t] \times U^+ K_j} p_{\tau(\cdot)}^*(\omega_{S^2}) \right) \alpha dt.$  □

**Corollary 10.22.** *The product*

$$\prod_{j=1}^k \exp\left(-I_\theta(K_j, \tau(t))\alpha\right) \#_j \check{Z}(\check{R}, L, \tau(t))$$

*does not change when  $\tau$  varies by a smooth homotopy.*

PROOF: With the notation of Lemma 7.15, we have

$$\check{Z}_1(\check{R}, L, \tau(t)) = \frac{1}{2} \sum_{j=1}^k I_\theta(K_j, \tau(t)) [\overset{\leftarrow}{\circlearrowleft} S^j].$$

Therefore, Lemma 10.21 and Proposition 10.13 imply

$$\frac{\partial}{\partial t} I_\theta(K_j, \tau(t)) = 2 \frac{\partial}{\partial t} \int_{[0,t] \times U + K_j} p_{\tau(\cdot)}^*(\omega_{S^2}).$$

(The reader can also check it directly as an exercise.) So Lemma 10.21 implies

$$\frac{\partial}{\partial t} \check{Z}(\check{R}, L, \tau(t)) = \sum_{j=1}^k \left( \frac{\partial}{\partial t} I_\theta(K_j, \tau(t)) \alpha \#_j \right) \check{Z}(\check{R}, L, \tau(t)).$$

Therefore, the derivative of

$$\prod_{j=1}^k \exp\left(-I_\theta(K_j, \tau(t))\alpha\right) \#_j \check{Z}(\check{R}, L, \tau(t))$$

with respect to  $t$  vanishes.  $\square$

PROOF OF THEOREM 7.20: According to the naturality of Proposition 10.1, Lemma 9.5, Proposition 10.7, Corollary 10.9, and Proposition 10.13, it suffices to prove that

$$\prod_{j=1}^k \left( \exp(-I_\theta(K_j, \tau)\alpha) \#_j \right) \check{Z}(\check{R}, L, \tau) \in \check{\mathcal{A}}(\mathcal{L})$$

is independent of the homotopy class of parallelization  $\tau$ .

When  $\tau$  changes in a ball that does not meet the link, the forms can be changed only in the neighborhoods of the unit tangent bundle to this ball. Apply Proposition 10.17 again to  $\check{Z}$ , where the  $p_e^*(\tilde{\omega}(u))$  are independent of  $u$  over  $K_j$ . So we get that

$$\prod_{j=1}^k \left( \exp(-I_\theta(K_j, \tau)\alpha) \#_j \right) \check{Z}(\check{R}, L, \tau) \in \check{\mathcal{A}}(\mathcal{L})$$

is invariant under the natural action of  $\pi_3(SO(3))$  on the homotopy classes of parallelizations, thanks to Corollary 10.22.

Now examine the effect of the twist of the parallelization by a map  $g: (B_R, 1) \rightarrow (SO(3), 1)$ . Without loss of generality, assume that  $p_\tau(U^+K_j) = v$  for some  $v \in S^2$  and  $g$  maps  $K_j$  to rotations with axis  $v$  for any  $j \in \underline{k}$ . We want to compare  $\check{Z}(\check{R}, L, \tau \circ \psi_{\mathbb{R}}(g))$  with  $\check{Z}(\check{R}, L, \tau)$ . There exists a closed form  $\omega$  on  $[0, 1] \times UB_R$  equal to  $p_\tau^*(\omega_{S^2})$  on  $\partial([0, 1] \times UB_R) \setminus (\{1\} \times UB_R)$  and equal to  $p_{\tau \circ \psi_{\mathbb{R}}(g)}^*(\omega_{S^2})$  on  $\{1\} \times UB_R$ . Extend this form to a closed form  $\Omega$  on  $[0, 1] \times C_2(R)$ , which restricts to  $[0, 1] \times (\partial C_2(R) \setminus UB_R)$  as  $p_\tau^*(\omega_{S^2})$ , as in Lemma 9.1. Let  $\Omega(t)$  denote the restriction of  $\Omega$  to  $\{t\} \times C_2(R)$ . According to Proposition 10.17, we have

$$\check{Z}(\check{R}, L, \tau \circ \psi_{\mathbb{R}}(g)) = \prod_{j=1}^k (\exp(I_j) \#_j) \check{Z}(\check{R}, L, \tau),$$

where  $I_j = \int_0^1 \gamma_j(u)$ , with  $\gamma_j(t) = \sum_{\Gamma_B \in \mathcal{D}^c(\mathbb{R})} \zeta_{\Gamma_B} \eta(R, L, \Gamma_B, K_j, \Omega)(t)$  and

$$\eta(R, L, \Gamma_B, K_j, \Omega)(t) = \int_{w \in K_j} \int_{\check{S}(T_w \check{R}, \vec{t}_w; \Gamma_B)} \left( \bigwedge_{e \in E(\Gamma_B)} p_e^*(\Omega(t)) \right) [\Gamma_B].$$

It suffices to prove  $I_j = (I_\theta(K_j, \tau \circ \psi_{\mathbb{R}}(g)) - I_\theta(K_j, \tau))\alpha$ . Proposition 10.17 implies that the degree one part  $I_{1,j}$  of  $I_j$  is

$$\begin{aligned} I_{1,j} &= \check{Z}_1(\check{R}, K_j, \tau \circ \psi_{\mathbb{R}}(g)) - \check{Z}_1(\check{R}, K_j, \tau) \\ &= \frac{1}{2} (I_\theta(K_j, \tau \circ \psi_{\mathbb{R}}(g)) - I_\theta(K_j, \tau)) [\hat{\mathfrak{C}}]. \end{aligned}$$

Let  ${}^\tau\psi(g^{-1}): UB_R \rightarrow UB_R$  denote the map induced by  $\tau \circ \psi_{\mathbb{R}}(g^{-1}) \circ \tau^{-1}$ . Recall  $p_\tau = p_{S^2} \circ \tau^{-1}$ . So we have

$$p_{\tau \circ \psi_{\mathbb{R}}(g)} = p_{S^2} \circ \psi_{\mathbb{R}}(g^{-1}) \circ \tau^{-1} = p_\tau \circ {}^\tau\psi(g^{-1}).$$

Let  $(. - 1): [1, 2] \rightarrow [0, 1]$  map  $x$  to  $x - 1$ . Set  ${}_{-1}{}^\tau\psi(g^{-1}) = ((. - 1) \times {}^\tau\psi(g^{-1}))$ . Extend  $\Omega$  over  $[0, 2] \times C_2(R)$  so that  $\Omega$  restricts to  $[1, 2] \times UB_R$  as  ${}_{-1}{}^\tau\psi(g^{-1})^*(\Omega)$ . For any  $\Gamma_B$ , the map  ${}_{-1}{}^\tau\psi(g^{-1})$  induces an orientation-preserving diffeomorphism

$${}_{-1}{}^\tau\psi(g^{-1})_*: [1, 2] \times \cup_{w \in K_j} \check{S}(T_w \check{R}, \vec{t}_w; \Gamma_B) \rightarrow [0, 1] \times \cup_{w \in K_j} \check{S}(T_w \check{R}, \vec{t}_w; \Gamma_B)$$

such that  $p_e \circ {}_{-1}{}^\tau\psi(g^{-1})_* = {}_{-1}{}^\tau\psi(g^{-1}) \circ p_e$  for any edge  $e$  of  $\Gamma_B$ . Using these diffeomorphisms  ${}_{-1}{}^\tau\psi(g^{-1})_*$  to pull back  $(\bigwedge_{e \in E(\Gamma_B)} p_e^*(\Omega(t)))$  proves  $\gamma_j(t+1) = \gamma_j(t)$ . In particular, we get

$$I_j(2) = \int_0^2 \gamma_j(u) = 2I_j.$$

Set  $\check{Z}(2) = \check{Z}(R, L, \tau \circ \psi_{\mathbb{R}}(g)^2)$ . We have

$$\check{Z}(2) = \prod_{j=1}^k \exp\left(\left(I_{\theta}(K_j, \tau \circ \psi_{\mathbb{R}}(g)^2) - I_{\theta}(K_j, \tau)\right)\alpha\right) \#_j \check{Z}(\check{R}, L, \tau),$$

since  $g^2$  is homotopic to the trivial map outside a ball (see Lemma 5.2, 2). Proposition 10.17 implies

$$\check{Z}(2) = \prod_{j=1}^k (\exp(2I_j) \#_j) \check{Z}(\check{R}, L, \tau).$$

By induction on the degree, we get

$$2I_j = (I_{\theta}(K_j, \tau \circ \psi_{\mathbb{R}}(g)^2) - I_{\theta}(K_j, \tau))\alpha.$$

The degree one part of this equality implies

$$2(I_{\theta}(K_j, \tau \circ \psi_{\mathbb{R}}(g)) - I_{\theta}(K_j, \tau)) = I_{\theta}(K_j, \tau \circ \psi_{\mathbb{R}}(g)^2) - I_{\theta}(K_j, \tau).$$

We get  $I_j = (I_{\theta}(K_j, \tau \circ \psi_{\mathbb{R}}(g)) - I_{\theta}(K_j, \tau))\alpha$ , as desired. □

## 10.6 Some more properties of $Z$

When  $\check{R} = \mathbb{R}^3$ , then  $\mathcal{Z}(S^3, L) = \check{\mathcal{Z}}(S^3, L)$  is the configuration space invariant studied by Daniel Altschüler, Laurent Freidel [AF97], Dylan Thurston [Thu99], Sylvain Poirier [Poi02], and others, after the work of many people including Edward Witten [Wit89], Enore Guadagnini, Maurizio Martellini, Mihail Mintchev [GMM90], Maxim Kontsevich [Kon94, Kon93], Raoul Bott and Clifford Taubes [BT94], Dror Bar-Natan [BN95b]...

**Reversing a link component orientation.** The following proposition is obvious from the definition of  $Z$ .

**Proposition 10.23.** *Let  $L: \sqcup_{j=1}^k S_j^1 \rightarrow R$  be a link in a  $\mathbb{Q}$ -sphere  $R$ . For a Jacobi diagram  $\Gamma$  on  $\sqcup_{j=1}^k S_j^1$ , let  $U_j(\Gamma)$  denote the set of univalent vertices of  $\Gamma$  mapped to  $S_j^1$ . This set is cyclically ordered by  $S_j^1$ . When the orientation of the component  $L(S_j^1)$  is reversed,  $\mathcal{Z}(L)$  is modified by reversing the circle  $S_j^1$  (that is reversing the cyclic order of  $U_j(\Gamma)$ ) in classes  $[\Gamma]$  of diagrams  $\Gamma$  on  $\sqcup_{j=1}^k S_j^1$  and multiplying them by  $(-1)^{|U_j(\Gamma)|}$  in  $\mathcal{A}(\sqcup_{j=1}^k S_j^1)$ .*

*In other words, we can forget the orientation of the link  $L$  and view  $\mathcal{Z}(L)$  as valued in  $\mathcal{A}(\sqcup_{j=1}^k S_j^1)$ , where the  $S_j^1$  are not oriented, as in Definitions 6.13 and 6.16.*



**Remark 10.24.** The orientation of a component  $L(S_j^1)$  is used in two ways. It defines a cyclic order on  $U_j(\Gamma)$ , and it defines the orientation of the vertices of  $U_j(\Gamma)$  as in Definition 6.13. The local orientation of  $S_j^1$  near the image of a vertex orients the corresponding local factor of the configuration space. The cyclic order is encoded in the isotopy class of the injection of  $U_j$  into the domain  $S_j^1$ .

**Numbering of the link components.** The following proposition is obvious from the definition of  $\mathcal{Z}$ .

**Proposition 10.25.** *When the numbering of the components of  $L$  is changed,  $\mathcal{Z}(L)$  is modified by the corresponding change of numbering of the circles  $S_j^1$  in diagram classes of  $\mathcal{A}(\sqcup_{j=1}^k S_j^1)$ .*

For a link  $L: \mathcal{L}(L) \rightarrow R$  in a  $\mathbb{Q}$ -sphere  $R$ ,  $\mathcal{Z}(R, L)$  is valued in  $\mathcal{A}(\mathcal{L}(L))$ . The one-manifold  $\mathcal{L}(L)$  is a disjoint union of oriented circles, which have been numbered so far. However, the numbers may be changed to any decoration that marks the component.

This gives sense to the statement of the following theorem.

**Connected sums.**

**Theorem 10.26.** *For any two links  $L_1$  and  $L_2$  in rational homology spheres  $R_1$  and  $R_2$ , we have*

$$\mathcal{Z}(R_1 \# R_2, L_1 \sqcup L_2) = \mathcal{Z}(R_1, L_1) \mathcal{Z}(R_2, L_2).$$

We prove a generalization of Theorem 10.26 in Section 17.2. See Theorem 17.9 in particular. See also Section 13.3. The proof given in Section 17.2 can be read without reading the intermediate chapters. Theorem 10.26 and Corollary 10.11 yield the following corollary.

**Corollary 10.27.** *For any two rational homology spheres  $R_1$  and  $R_2$ , we have*

$$\Theta(R_1 \# R_2) = \Theta(R_1) + \Theta(R_2).$$

**Reversing the orientation of the ambient space.**

**Lemma 10.28.** *Under the assumptions of Lemma 5.14, we have*

$$Z_n(-\check{R}, L, \bar{\tau}) = (-1)^n Z_n(\check{R}, L, \tau).$$

PROOF: If  $\omega$  is a homogeneous propagating form of  $(C_2(R), \tau)$ , then  $(-\omega)$  is a homogeneous propagating form of  $(C_2(-R), \bar{\tau})$ . Let  $\Gamma$  be a degree  $n$  numbered Jacobi diagram. When the orientation of  $R$  is reversed, the orientation of  $\check{C}(R, L; \Gamma)$  is reversed if and only if  $|T(\Gamma)|$  is odd. Thus, the integrals will be multiplied by  $(-1)^{|E(\Gamma)|+|T(\Gamma)|}$ . We have  $2|E(\Gamma)| = 3|T(\Gamma)| + |U(\Gamma)|$ , and hence  $2n = 2(|E(\Gamma)| - |T(\Gamma)|)$ .  $\square$

**Theorem 10.29.** *For any link  $L$  in a rational homology sphere  $R$ , we have*

$$\mathcal{Z}_n(-R, L) = (-1)^n \mathcal{Z}_n(R, L).$$

PROOF: Theorem 7.20 implies

$$\mathcal{Z}(-R, L) = \exp\left(-\frac{1}{4}p_1(\bar{\tau})\beta\right) \prod_{j=1}^k \left(\exp(-I_\theta(K_j, \bar{\tau})\alpha) \#_j\right) \mathcal{Z}(-\check{R}, L, \bar{\tau}).$$

Lemma 10.28 implies  $I_\theta(K_j, \bar{\tau}) = -I_\theta(K_j, \tau)$ , and Lemma 5.14 yields  $p_1(\bar{\tau}) = -p_1(\tau)$ . So Lemma 10.28 implies the result since  $\alpha$  and  $\beta$  vanish in even degrees.  $\square$



# Chapter 11

## Rationality

In this chapter, we give equivalent definitions of  $\mathcal{Z}$  based on algebraic intersections of propagating chains, and we prove that  $\mathcal{Z}$  and the anomalies  $\alpha$  and  $\beta$  are rational.

### 11.1 From integrals to algebraic intersections

In order to warm up, we first prove the following rationality result, which is due to Sylvain Poirier [Poi02] and Dylan Thurston [Thu99], independently.

**Proposition 11.1.** *The anomaly  $\alpha$  of Section 10.3 is rational, i.e., it belongs to  $\check{\mathcal{A}}(\mathbb{R}; \mathbb{Q})$ . For any link  $L: \mathcal{L} \rightarrow \mathbb{R}^3$ , the value  $\mathcal{Z}(S^3, L)$  is also rational. It belongs to  $\mathcal{A}(\mathcal{L}; \mathbb{Q})$ .*

PROOF: Let us fix  $n$  and prove that  $\alpha_n$  is in  $\check{\mathcal{A}}_n(\mathbb{R}; \mathbb{Q})$ . For any degree  $n$  numbered Jacobi diagram  $\check{\Gamma}$  on  $\mathbb{R}$ , define the smooth map

$$g(\check{\Gamma}): Q(\check{\Gamma}) \times (S^2)^{\underline{3n-2} \setminus j_E(E(\check{\Gamma}))} \rightarrow (S^2)^{3n-2}$$

as the product  $(\prod_{e \in E(\check{\Gamma})} p_{e, S^2}) \times \mathbf{1}((S^2)^{\underline{3n-2} \setminus j_E(E(\check{\Gamma}))})$ . Note that a regular value of  $g(\check{\Gamma})$  is not in the image of  $\partial Q(\check{\Gamma}) \times (S^2)^{\underline{3n-2} \setminus j_E(E(\check{\Gamma}))}$ . According to the Morse–Sard theorem 1.4, the set of regular values of  $g(\check{\Gamma})$  is dense. Since  $Q(\check{\Gamma})$  is compact, so are  $\partial Q(\check{\Gamma})$ , the boundary of the domain of  $g(\check{\Gamma})$ , and the subset of the domain of  $g(\check{\Gamma})$  consisting of the points at which the derivative of  $g(\check{\Gamma})$  is not surjective. Therefore, the set of regular values of  $g(\check{\Gamma})$  is open. Thus, the finite intersection over all the  $\check{\Gamma} \in \mathcal{D}_n^c(\mathbb{R})$  of the sets of regular values of the  $g(\check{\Gamma})$  is also open and dense. Let  $\prod_{i=1}^{3n-2} B(x_i)$  be a product of open balls of  $S^2$  that is in this intersection. Then for any  $\check{\Gamma} \in \mathcal{D}_n^c(\mathbb{R})$ , the local degree of  $g(\check{\Gamma})$  (which is an integer) is constant over

$\prod_{i=1}^{3n-2} B(X_i)$ . In particular, if  $\omega(i, S^2)$  is a volume-one form of  $S^2$  that is supported on  $B(X_i)$  for each  $i \in \underline{3n-2}$ , then  $I(\check{\Gamma}, o(\check{\Gamma}), \omega(i, S^2))$ , which is nothing but this integral local degree, is an integer for any  $\check{\Gamma}$  in  $\mathcal{D}_n^e(\mathbb{R})$ . Thus  $\alpha_n$ , which is defined in Proposition 10.13, is in  $\check{\mathcal{A}}_n(\mathbb{R}; \mathbb{Q})$ .

For a fixed  $n$  and a given  $k$ -component link  $L$  of  $S^3$ , there exists a similar product  $\prod_{i=1}^{3n} B_L(Y_i)$  of open balls of  $S^2$  consisting of points of  $(S^2)^{3n}$  that are regular values for all maps

$$\left( \prod_{e \in E(\Gamma)} p_{e, S^2} \right) \times \mathbf{1} \left( (S^2)^{\underline{3n} \setminus j_E(E(\Gamma))} \right) : C(S^3, L; \Gamma) \times (S^2)^{\underline{3n} \setminus j_E(E(\Gamma))} \rightarrow (S^2)^{3n}$$

associated to Jacobi diagrams  $\Gamma$  of  $\mathcal{D}_n^e(\mathcal{L})$ . Then if  $\omega(i, S^2)$  is a volume-one form of  $S^2$  that is supported on  $B_L(Y_i)$  for each  $i \in \underline{3n}$ ,

$$I \left( S^3, L, \Gamma, \left( p_{S^2}^*(\omega(i, S^2)) \right) \right)$$

is an integer for every  $\Gamma$  of  $\mathcal{D}_n^e(\mathcal{L})$ .

If the link is straight, then Theorem 7.40 implies that  $Z_n(\mathbb{R}^3, L, \tau_s)$  is rational. Thus,  $Z(\mathbb{R}^3, L, \tau_s)$  is rational for any straight link  $L$  of  $\mathbb{R}^3$ . In particular  $I_\theta(K, \tau_s)$  is rational for any component  $K$  of a straight link  $L$ , and Theorem 7.20 together with the rationality of  $\alpha$  implies that  $\mathcal{Z}(S^3, L)$  is rational.  $\square$

With the notation of the above proof, the  $P(i) = p_{S^2}^{-1}(y_i) \subset C_2(S^3)$  for  $y_i \in B_L(Y_i)$  are propagating chains such that, for any  $\Gamma$  of  $\mathcal{D}_n^e(\mathcal{L})$ , the intersection over  $E(\Gamma)$  of the  $p_e^{-1}(P(j_E(e)))$  in  $C(S^3, L; \Gamma)$  is transverse. The integral  $I(S^3, L, \Gamma, (p_{S^2}^*(\omega(i, S^2))))$  is nothing but their algebraic intersection.

We will use Version 7.40 of Theorem 7.20 to replace the configuration space integrals with algebraic intersections in configuration spaces and thus prove the rationality of  $Z^s$  for straight links in any rational homology sphere as follows.

**Definition 11.2.** A smooth map  $f: B \rightarrow A$  is *transverse* to a submanifold  $C$  of  $A$  along a subset  $K$  of  $B$  if

$$T_{f(x)}A = T_x f(T_x B) + T_{f(x)}C$$

for any point  $x$  of  $K \cap f^{-1}(C)$ . When  $A$  or  $B$  have ridges, we furthermore require this equality to hold when  $A$  or  $B$  are replaced by all their open faces (of any dimension).

A smooth map  $f: B \rightarrow A$  is *transverse* to a submanifold  $C$  of  $A$  if it is transverse to  $C$  along  $B$ . Say that a smooth map  $f: B \rightarrow A$  is *transverse* to

a rational chain  $C$  of  $A$ , which is a multiple of a union of compact smooth embedded submanifolds with boundaries and ridges  $\cup_{k \in J} C_k$ , if  $f$  is transverse to  $C_k$  for any  $k \in J$ .

A *rational simplicial chain*, which is a rational combination of simplices in a triangulated smooth manifold, is an example of what we call a rational chain. Rational multiples of compact immersion images provide other examples of chains. An immersion image will be represented as a union of embedded manifolds by decomposing the domain as a union of compact manifolds with boundaries and ridges glued along their boundaries.

Recall from Notation 7.29 that  $\mathcal{D}_{k,3n}^e(\mathcal{L})$  denotes the set of  $3n$ -numbered degree  $k$  Jacobi diagrams with support  $\mathcal{L}$  without looped edges. Let  $\mathcal{D}_{3n}^e(\mathcal{L}) = \cup_{k \in \mathbb{N}} \mathcal{D}_{k,3n}^e(\mathcal{L})$ . Note that  $\mathcal{D}_n^e(\mathcal{L}) = \mathcal{D}_{n,3n}^e(\mathcal{L})$  but  $3n$ -numbered Jacobi diagrams may have a degree different from  $n$ .

**Definition 11.3.** Say that a family  $(P(i))_{i \in 3n}$  of propagating chains of  $(C_2(R), \tau)$  is in *general  $3n$  position* with respect to a link  $L: \mathcal{L} \rightarrow \check{R}$  if for any  $\Gamma \in \mathcal{D}_{3n}^e(\mathcal{L})$  and for any subset  $E$  of  $E(\Gamma)$ , the map

$$p(\Gamma, E) = \prod_{e \in E} p_e: C(R, L; \Gamma) \rightarrow (C_2(R))^{j_E(E)}$$

is transverse to  $\prod_{e \in E} P(j_E(e))$ .

For such a family  $(P(i))_{i \in 3n}$  in general  $3n$  position, the intersection

$$\cap_{e \in E(\Gamma)} p_e^{-1} \left( P(j_E(e)) \right)$$

consists of a finite number of points  $x$ , which sit in the interior of  $C(R, L; \Gamma)$ , and, for each such  $x$ , the following conditions are satisfied.

- For every edge  $e \in E(\Gamma)$ ,  $p_e(x)$  meets the union of smooth embedded 4-manifolds with boundaries that constitute  $P(j_E(e))$  in the interior of finitely many of these manifolds. The family  $(\Delta_{j_E(e), i})_{i \in J(e, x)}$  of met manifolds is indexed by a finite set  $J(e, x)$ .<sup>1</sup>
- For every map  $i: E(\Gamma) \rightarrow \cup_{e \in E(\Gamma)} J(e, x)$  such that  $i(e) \in J(e, x)$ , the local maps, from small open neighborhoods of  $x$  in  $C(R, L; \Gamma)$  to the product over  $E(\Gamma)$  of the fibers of the locally trivialized normal bundles to the  $\Delta_{j_E(e), i(e)}$ , are local diffeomorphisms.

We will prove the following lemma in Section 11.3.

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<sup>1</sup>The  $\Delta_{j_E(e), i}$  are smooth embedded 4-simplices when  $P(j_E(e))$  is a simplicial chain.

**Lemma 11.4.** *Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . For any link  $L: \mathcal{L} \hookrightarrow \check{R}$ , for any integer  $n$ , there exists a family  $(P(i))_{i \in \underline{3n}}$  of propagating chains of  $(C_2(R), \tau)$  in general  $3n$ -position with respect to  $L$ .*

Let  $(P(i))_{i \in \underline{3n}}$  be such a family. The codimension-two rational chains  $p_e^{-1}(P(j_E(e)))$  are cooriented by the coorientation of  $P(j_E(e))$  in  $C_2(R)$ . Define  $I(\Gamma, o(\Gamma), (P(i))_{i \in \underline{3n}})$  to be the algebraic intersection in  $(C(R, L; \Gamma), o(\Gamma))$  of the chains  $p_e^{-1}(P(j_E(e)))$  over the edges  $e$  of  $E(\Gamma)$ .

For any finite set  $V$ , equip  $C_V(R)$  with a Riemannian metric that is symmetric with respect to permutations of elements of  $V$ . Let  $d$  denote the associated distance. Our choice of distance will not matter thanks to the following easy lemma.

**Lemma 11.5.** *All the distances associated to Riemannian metrics are equivalent on a compact smooth manifold.*

PROOF: Let  $g_1$  and  $g_2$  be two Riemannian metrics on the compact manifold  $M$ , let  $\|\cdot\|_1$  and  $\|\cdot\|_2$  be the two associated norms on tangent vectors, and let  $d_1$  and  $d_2$  be the two associated distances. View the unit tangent bundle  $UM$  of  $M$  as the set of unit tangent vectors to  $M$  with respect to  $\|\cdot\|_1$ . Then the image of  $UM$  under the continuous map  $\|\cdot\|_2$  is a compact interval  $[a, b]$  with  $a > 0$ , and we have

$$a \leq \frac{\|x\|_2}{\|x\|_1} \leq b$$

for any nonzero tangent vector  $x$  of  $M$ . Let  $p$  and  $q$  be two distinct points of  $M$ . For any smooth path  $\gamma: [0, 1] \rightarrow M$  such that  $\gamma(0) = p$  and  $\gamma(1) = q$ , we have

$$d_2(p, q) \leq \int_0^1 \|\gamma'(t)\|_2 dt \leq b \int_0^1 \|\gamma'(t)\|_1 dt.$$

Therefore, we get  $d_2(p, q) \leq b d_1(p, q)$ . Similarly, we have  $d_1(p, q) \leq \frac{d_2(p, q)}{a}$ .  $\square$

**Definition 11.6.** For a subset  $X$  of  $C_V(R)$  and for  $\varepsilon > 0$ , set

$$N_\varepsilon(X) = \{x \in C_V(R) : d(x, X) < \varepsilon\}.$$

For a small positive number  $\eta$ , a closed 2-form  $\omega(i)$  on  $C_2(R)$  is said to be  $\eta$ -dual to  $P(i)$ , if it is supported in  $N_\eta(P(i))$  and if  $\int_D \omega(i) = \langle D, P(i) \rangle_{C_2(R)}$  for any 2-dimensional disk  $D$  embedded in  $C_2(R)$  transverse to  $P(i)$  whose boundary sits outside  $N_\eta(P(i))$ .

We will prove the following lemma in Section 11.4.

**Lemma 11.7.** *Assume Lemma 11.4. Under its hypotheses, for any  $\eta > 0$ , there exist propagating forms  $\omega(i)$  of  $(C_2(R), \tau)$   $\eta$ -dual to the  $P(i)$  of Lemma 11.4. If  $\eta$  is small enough, then we have*

$$I\left(\Gamma, o(\Gamma), (P(i))_{i \in \underline{3n}}\right) = I\left(\Gamma, o(\Gamma), (\omega(i))_{i \in \underline{3n}}\right)$$

for any  $\Gamma \in \mathcal{D}_{k, \underline{3n}}^e(\mathcal{L})$ , where  $k \leq n$ .

Thus,  $I(\Gamma, o(\Gamma), (\omega(i))_{i \in \underline{3n}})$  is rational in this case, and we get the following theorem.

**Theorem 11.8.** *The anomaly  $\beta$  is rational, i.e., it belongs to  $\mathcal{A}(\emptyset; \mathbb{Q})$ . Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . For any link  $L: \mathcal{L} \rightarrow \check{R}$ , the value  $\mathcal{Z}(R, L)$  belongs to  $\mathcal{A}(\mathcal{L}; \mathbb{Q})$ .*

**PROOF OF THE THEOREM ASSUMING LEMMAS 11.4 AND 11.7:** Theorems 7.40 and 7.20, Lemmas 11.4 and 11.7 imply that  $Z(S^3, \tau) = Z(\mathbb{R}^3, \emptyset, \tau)$  is rational for any  $\tau$ . So  $z(S^3, \tau) = p^c(Z(S^3, \tau))$  is rational for any  $\tau$ , too, and, by Definition 10.5, the anomaly  $\beta$  is rational. Therefore, Theorem 7.40, Lemmas 11.4, and 11.7 also imply that  $Z(\check{R}, L, \tau)$  is rational for any pair  $((\check{R}, \tau), L)$  such that  $(\check{R}, \tau)$  is an asymptotic rational homology  $\mathbb{R}^3$  and  $L$  is a straight link with respect to  $\tau$ . In particular, the integral  $I_\theta(K, \tau)$  is rational for any component  $K$  of a straight link  $L$ . Since  $p_1(\tau)$  and the anomalies  $\alpha$  and  $\beta$  are rational, Theorem 7.20 now implies that  $\mathcal{Z}(R, L)$  is rational.  $\square$

## 11.2 More on general propagating chains

By a theorem of René Thom [Tho54, Théorème II.27, p. 55], any integral codimension 2 homology class in a manifold can be represented as the class of an embedded closed (oriented) submanifold. We prove a relative version of this result below, following Thom's original proof in this particular case of his theorem.

**Theorem 11.9.** *Let  $A$  be a compact smooth (oriented) manifold with boundary. Let  $C$  be a smooth codimension 2 closed (oriented) submanifold of  $\partial A$  such that the homology class of  $C$  vanishes in  $H_{\dim(A)-3}(A; \mathbb{Z})$ . Then there exists a compact smooth codimension 2 submanifold  $B$  of  $A$  transverse to  $\partial A$  whose boundary is  $C$ .*



PROOF: Let us first sketch Thom’s proof with his notation. The normal bundle to  $C$  in  $\partial A$  is an oriented disk bundle. It is the pull-back of a universal disk bundle  $A_{SO(2)}$  over a compact classifying space  $B_{SO(2)}$  via a map  $f_C$  from  $C$  to  $B_{SO(2)}$ . See [MS74, p. 145]. Like René Thom [Tho54, p. 28, 29], define the Thom space  $M(SO(2))$  of  $SO(2)$  to be the space obtained from the total space  $A_{SO(2)}$  by identifying its subspace  $E_{SO(2)}$  consisting of the points in the boundaries of the fibers  $D^2$  of  $A_{SO(2)}$  with a single point  $a$ . Regard  $B_{SO(2)}$  as the zero section of  $A_{SO(2)}$ . So  $B_{SO(2)}$  sits inside  $M(SO(2))$ .

The map  $f_C$  extends canonically to  $\partial A$ . Its extension  $f_{\partial A}$  injects the fibers of an open tubular neighborhood of  $C$  in  $\partial A$  to fibers of  $A_{SO(2)}$  and maps the complement of such a neighborhood to  $a$ . Thus  $C$  is the preimage of  $B_{SO(2)}$ . In order to prove the theorem, it suffices to extend the map  $f_{\partial A}$  to a map  $f_A$  from  $A$  to  $M(SO(2))$  so that, in a neighborhood of any point of  $f_A^{-1}(B_{SO(2)})$ , the differential of a local projection to the fiber of the normal bundle to  $B_{SO(2)}$  composed with  $f_A$  is well-defined and surjective. (The fiber of the normal bundle to  $B_{SO(2)}$  is isomorphic to the tangent space to a fiber of  $A_{SO(2)}$ .) Indeed, the compact submanifold  $B = f_A^{-1}(B_{SO(2)})$  of  $A$ , with respect to such an extension, has the desired properties.

The map  $f_{\partial A}$  can be extended as a continuous map, using the fact that  $M(SO(2))$  is a  $K(\mathbb{Z}; 2)$  [Tho54, ii), p. 50]. In other words, the only nontrivial homotopy group of  $M(SO(2))$  is its  $\pi_2$ , which is isomorphic to  $\mathbb{Z}$ .

Let us now give some details about the above sketch and show how the pair  $(B_{SO(2)}, M(SO(2)))$  can be replaced by  $(\mathbb{C}P^N, \mathbb{C}P^{N+1})$  for some large integer  $N$ , following [Tho54, ii), p. 50]. View the fiber of a disk bundle as the unit disk of  $\mathbb{C}$ . The corresponding complex line bundle over  $C$  injects into a trivial complex bundle  $\mathbb{C}^{N+1} \times C$  as in [MS74, Lemma 5.3, p. 61] for some integer  $N$ , by some map  $(f_{1,C}, \mathbf{1}(C))$ . Therefore, it is the pull-back of the tautological complex line bundle  $\gamma_N^1$  over  $B'_{SO(2)} = \mathbb{C}P^N$  by the map  $f'_C: C \rightarrow \mathbb{C}P^N$  that sends a point  $x$  of  $C$  to the image of the fiber over  $x$  under  $f_{1,C}$ . The disk bundle  $A'_{SO(2)}$  associated to  $\gamma_N^1$  is diffeomorphic to the normal bundle to  $\mathbb{C}P^N$  in  $\mathbb{C}P^{N+1}$  by the inverse of the following (orientation-reversing) map:

$$\begin{aligned} \mathbb{C}P^{N+1} \setminus \{(0, \dots, 0, 1)\} &\rightarrow \gamma_N^1 \\ [z_1, \dots, z_{N+1}, z] &\mapsto \left( \frac{\bar{z}}{\sum_{i=1}^{N+1} |z_i|^2} (z_1, \dots, z_{N+1}), [z_1, \dots, z_{N+1}] \right). \end{aligned}$$

This map also shows that the space  $M'(SO(2))$ , obtained from  $A'_{SO(2)}$  by identifying  $E'_{SO(2)} = \partial A'_{SO(2)}$  to a point, is homeomorphic to the whole  $\mathbb{C}P^{N+1}$ .

The long exact sequence associated to the fibration  $S^1 \hookrightarrow S^{2N+3} \rightarrow \mathbb{C}P^{N+1}$  implies that  $\pi_2(\mathbb{C}P^{N+1}) = \mathbb{Z}[\mathbb{C}P^1]$  and that  $\pi_i(\mathbb{C}P^{N+1})$  is trivial

for any  $i \in \underline{2N+2} \setminus \{2\}$ . See Theorem A.14. Assume that  $2N+2$  is bigger than the dimension of  $A$  without loss of generality.

It is not hard to see that  $\pi_2(M'(SO(2))) = H_2(M'(SO(2)); \mathbb{Z})$  is freely generated by the class of the image  $[F]$  of a fiber under the identification of  $E'_{SO(2)}$  with the point  $a$ . (Indeed, since  $B'_{SO(2)}$  is connected, the homology class of  $[F]$  is well-defined. Since  $M'(SO(2)) \setminus B'_{SO(2)}$  is contractible and since any 2-cycle is homologous to a 2-cycle that is transverse to  $B'_{SO(2)}$ , any homology class of degree 2 is a multiple of  $[F]$ , which therefore generates  $H_2(M'(SO(2)); \mathbb{Z})$ .)

Extend  $f'_C$  to a map  $f_{\partial A}$  valued in  $M'(SO(2))$  as before so that  $C = f_{\partial A}^{-1}(B'_{SO(2)})$ . Recall that any smooth manifold is triangulable [Cai35], [Whi40]. Fix a triangulation for  $(A, \partial A)$  transverse to  $C$ . In particular,  $C = f_{\partial A}^{-1}(B'_{SO(2)})$  avoids the 1-skeleton. Extend  $f_{\partial A}$  skeleton by skeleton starting with the zero and one-skeleta, for which there is no obstruction to extending  $f_{\partial A}$  to a map valued in  $M'(SO(2)) \setminus B'_{SO(2)}$ , which is connected. There is no obstruction to extending  $f_{\partial A}$ , as a map valued in the contractible  $M'(SO(2)) \setminus B'_{SO(2)}$ , to the two-skeleton of  $(A, \partial A)$ , but such a map would not necessarily extend to the three-skeleton. Let  $f_A^{(2)}$  be an arbitrary generic extension valued in  $M'(SO(2))$  of  $f_{\partial A}$  to the two-skeleton of  $(A, \partial A)$ . Define the 2-cochain  $c(f_A^{(2)})$  with  $\mathbb{Z}$ -coefficients such that  $c(f_A^{(2)})(D)$  is the algebraic intersection of  $B'_{SO(2)}$  and  $f_A^{(2)}(D)$  in  $M'(SO(2))$  for any 2-cell  $D$  of  $A$ . Then  $f_A^{(2)}$  extends to the 3-skeleton if and only if this cochain (which is fixed on  $\partial A$  and Poincaré dual to  $C$  on  $\partial A$ ) is a cocycle. Thus, to prove that  $f_{\partial A}$  extends to the 3-skeleton, it suffices to prove that the class of  $c(f_A^{(2)})|_{\partial A}$  in  $H^2(\partial A; \mathbb{Z})$  is in the natural image of  $H^2(A; \mathbb{Z})$ , or, equivalently, that its image in  $H^3(A, \partial A; \mathbb{Z})$ , by the boundary map of the long cohomology exact sequence of  $(A, \partial A)$ , vanishes. This image is represented by a cochain that maps a 3-cell  $\mathcal{B}$  of  $(A, \partial A)$  to the algebraic intersection of  $\partial \mathcal{B}$  and  $C$ , which is, up to a fixed sign, the algebraic intersection of  $\mathcal{B}$  and  $C$  (pushed inside  $A$ ). Therefore, the class in  $H^3(A, \partial A; \mathbb{Z})$  of this relative cocycle is Poincaré dual to the class of  $C$  in  $H_{\dim(A)-3}(A)$ , which vanishes. So  $f_{\partial A}$  can be extended to the 3-skeleton. Since the next homotopy groups  $\pi_i(M'(SO(2)))$ , for  $3 \leq i < \dim(A)$ , vanish, there is no obstruction to extending  $f_{\partial A}$  to the manifold  $A$ .

Finally, make  $f_A$  smooth, using an approximation theorem [Hir94, Chapter 2, Theorem 2.6] of continuous maps by smooth maps, and make  $f_A$  transverse to  $B'_{SO(2)}$ , with the help of a transversality theorem [Hir94, Chapter 3, Theorem 2.1].  $\square$

**Corollary 11.10.** *If  $R$  is a  $\mathbb{Z}$ -sphere, for any asymptotically standard parallelization  $\tau$  of  $\tilde{R}$ , for any  $X \in S^2$ , there exists a 4-dimensional submanifold*

of  $C_2(R)$  transverse to the ridges whose boundary is  $p_\tau^{-1}(X)$ .

PROOF: First extend  $p_\tau$  as a regular map from a regular neighborhood  $N(\partial C_2(R))$  of  $\partial C_2(R)$ , where  $N(\partial C_2(R))$  is a smooth cobordism with ridges embedded in  $C_2(R)$  from a smooth manifold  $\partial C'_2(R)$  without ridges to  $\partial C_2(R)$ , and  $N(\partial C_2(R))$  is homeomorphic to the product  $[0, 1] \times \partial C_2(R)$ . Then apply Theorem 11.9 to  $C'_2(R) = C_2(R) \setminus \text{Int}(N(\partial C_2(R)))$  and to  $p_\tau|_{\partial C'_2(R)}^{-1}(X)$ .  $\square$

When  $R$  is a  $\mathbb{Q}$ -sphere, perform the same first step as in the above proof. Take a collar neighborhood of  $\partial C'_2(R)$  in  $N(\partial C_2(R))$ , which is (diffeomorphic to and) identified with  $[0, 8] \times \partial C'_2(R)$  so that  $\partial C'_2(R) = \{0\} \times \partial C'_2(R)$ . Assume that  $p_\tau$  factors through the projection to  $\partial C'_2(R)$  on  $[0, 8] \times \partial C'_2(R)$ . There exists a positive integer  $k$  such that  $kp_\tau|_{\partial C'_2(R)}^{-1}(X)$  is null-homologous in  $C'_2(R)$ . Let  $p_k: S^2 \rightarrow S^2$  be a degree  $k$  map that does not fix  $X$  and such that  $X$  is a regular value of  $p_k$  with  $k$  preimages. Then  $(p_k \circ p_\tau|_{\partial C'_2(R)})^{-1}(X)$  bounds a 4-manifold  $P'$  properly embedded in  $C'_2(R)$  according to Theorem 11.9. For  $j \in \underline{k}$ , let  $\{\gamma_j: [0, 4] \rightarrow S^2\}_{j \in \underline{k}}$  be a collection of smooth injective paths ending at  $X = \gamma_j(4)$  whose images do not meet outside  $X$  and such that  $p_k^{-1}(X) = \{\gamma_j(0) : j \in \underline{k}\}$ . Also assume that all the derivatives of  $\gamma_j$  vanish at 0 and 4. Consider

$$\begin{aligned} p_{[0,8]} \times p_\tau: [0, 8] \times \partial C'_2(R) &\rightarrow [0, 8] \times S^2 \\ (t, x) &\mapsto (t, p_\tau(x)). \end{aligned}$$

Then

$$\begin{aligned} P = p_\tau^{-1}|_{N(\partial C_2(R)) \setminus ([0,4] \times \partial C'_2(R))}(X) &+ \frac{1}{k} P' \\ &+ \frac{1}{k} (p_{[0,8]} \times p_\tau)|_{[0,4] \times \partial C'_2(R)}^{-1} \left( \left\{ (t, \gamma_j(t)) : j \in \underline{k}, t \in [0, 4] \right\} \right) \end{aligned}$$

is a propagating chain of  $(\check{R}, \tau)$ . See Figure 11.1.

### 11.3 Existence of transverse propagating chains

In this section, we prove Lemma 11.4.

In order to warm up, we first prove a weak version of this lemma. The proof is a straightforward adaptation of a proof of Thom [Tho54, p. 23, 24, Lemma I.4].

Assume that  $R$  is an integer homology 3-sphere. Let  $(P(i))_{i \in \underline{3n}}$  be a family of propagating chains of  $(C_2(R), \tau)$  for an asymptotically standard parallelization  $\tau$  of  $\check{R}$ . Assume that these chains are submanifolds of  $C_2(R)$  transverse to  $\partial C_2(R)$  as in Corollary 11.10.

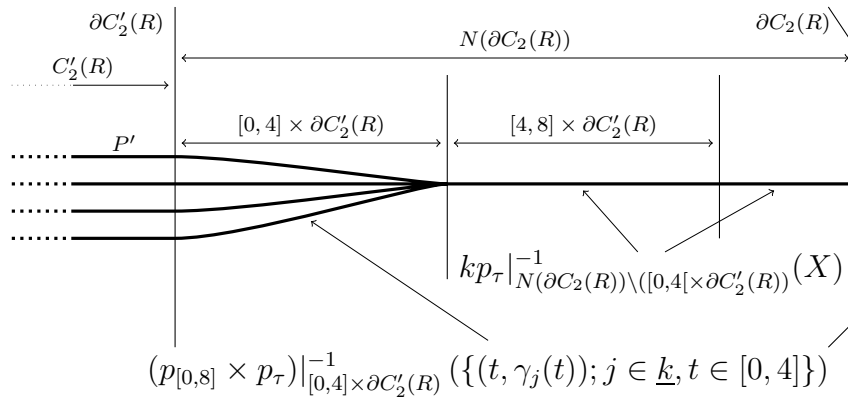


Figure 11.1: A multiple of a propagating chain of  $(\check{R}, \tau)$

Let  $N(P(i))$  denote the normal bundle to  $P(i)$  embedded in  $C_2(R)$  as a tubular neighborhood whose fibers  $N_x(P(i))$  over a point  $x \in P(i)$  are disks embedded in  $C_2(R)$ . Let  $(K_{i,j})_{j \in J(i)}$  be a finite cover of  $P(i)$  by compact subspaces  $K_{i,j}$  embedded in open subspaces  $\mathcal{O}_{i,j}$  of  $P(i)$  equipped with bundle charts  $(\psi_{i,j}: N(P(i))|_{\mathcal{O}_{i,j}} \rightarrow A_{i,j} \times D^2)_{j \in J}$ . We assume that the above  $A_{i,j}$  are  $\mathbb{R}^4$ ,  $\mathbb{R}^+ \times \mathbb{R}^3$  or  $(\mathbb{R}^+)^2 \times \mathbb{R}^2$ , and that the  $\psi_{i,j}$  are bundle charts over diffeomorphisms  $\phi_{i,j}: \mathcal{O}_{i,j} \rightarrow A_{i,j}$ . We also assume that for any  $\{j, k\} \subset J(i)$ , for any  $x \in \mathcal{O}_{i,j} \cap \mathcal{O}_{i,k}$ , the map

$$v \mapsto p_{D^2} \circ \psi_{i,k} \circ \psi_{i,j}^{-1}(\phi_{i,j}(x), v)$$

is a linear map of  $SO(2)$ , where  $p_{D^2}$  denotes the natural projection onto  $D^2$ . Consider the space  $\mathcal{H}_i$  of smooth diffeomorphisms of  $N(P(i))$  that are isotopic to the identity map, that fix a neighborhood of  $\partial N(P(i))$  pointwise, and that map any fiber of  $N(P(i))$  to itself. Equip this space  $\mathcal{H}_i$  with the following distance  $d$ .<sup>2</sup> Each fiber is equipped with the distance  $d_P$  induced by the norm of  $\mathbb{R}^2$ . This allows us to define a  $C^0$  distance  $d_0$  between two elements  $h$  and  $k$  of  $\mathcal{H}_i$  by

$$d_0(h, k) = \sup_{x \in N(P(i))} d_P(h(x), k(x)).$$

Since  $A_{i,j} \times D^2$  is a subset of  $\mathbb{R}^6$ , the differential of a map  $\psi_{i,j} \circ h \circ \psi_{i,j}^{-1}$  for  $h \in \mathcal{H}_i$  maps every element  $x$  of  $A_{i,j} \times D^2$  to a linear map of  $\mathbb{R}^6$ . The norm  $\|L\|$  of a linear map  $L$  of  $\mathbb{R}^6$  is defined to be  $\|L\| = \sup_{x \in S^5} \|L(x)\|$ . For  $h$

<sup>2</sup>This distance induces the strong (or weak, which is the same since  $N(P(i))$  is compact)  $C^1$ -topology. See [Hir94, Chapter 2, p.35].

and  $k$  in  $\mathcal{H}_i$ , set

$$d_{(1)}(h, k) = \sup_{j \in J(i), x \in \phi_{i,j}(K_{i,j}) \times D^2} (\|T_x(\psi_{i,j} \circ h \circ \psi_{i,j}^{-1}) - T_x(\psi_{i,j} \circ k \circ \psi_{i,j}^{-1})\|)$$

and

$$d(h, k) = \sup (d_0(h, k), d_{(1)}(h, k)).$$

**Lemma 11.11.** *Under the above hypotheses, there is a dense open subset of  $\prod_{i=1}^{3n} \mathcal{H}_i$  such that for any  $(h_i)$  in this subset, the chains obtained from the  $P(i)$  by replacing  $P(i)$  with  $h_i^{-1}(P(i))$  are in general  $3n$  position with respect to  $L$  in the sense of Definition 11.3.*

PROOF: We will first list finitely many sufficient conditions on the  $(h_i)$ , which guarantee the conclusion “For any graph  $\Gamma$  of  $\mathcal{D}_{3n}^e(\mathcal{L})$ , and for any subset  $E$  of  $E(\Gamma)$ , the map

$$p(\Gamma, E) = \prod_{e \in E} p_e : C(R, L; \Gamma) \rightarrow (C_2(R))^{j_E(E)}$$

is transverse to  $\prod_{e \in E} h_{j_E(e)}^{-1} P(j_E(e))$ .” Next we will prove that each of these conditions is realized in an open dense subset of  $\prod_{i=1}^{3n} \mathcal{H}_i$ .

Extend the elements of  $\mathcal{H}_i$  to diffeomorphisms of  $C_2(R)$ , by the identity map of  $C_2(R) \setminus \dot{N}(P(i))$ . The propagating chains obtained from the  $P(i)$  by replacing  $P(i)$  with  $h_i^{-1}(P(i))$  are in general  $3n$  position with respect to  $L$  if and only if the following condition  $(*) (\Gamma, E, \ell)$  holds for any triple  $(\Gamma, E, \ell)$ , where  $\Gamma \in \mathcal{D}_{3n}^e(\mathcal{L})$ ,  $E \subseteq E(\Gamma)$ , and  $\ell$  is a map  $\ell : E \rightarrow \cup_{i \in \underline{3n}} J(i)$  such that  $\ell(e) \in J(j_E(e))$ .

$(*) (\Gamma, E, \ell)$ : The map  $p(\Gamma, E)$  is transverse to  $\prod_{e \in E} h_{j_E(e)}^{-1} (P(j_E(e)))$  along  $p(\Gamma, E)^{-1} (\prod_{e \in E} h_{j_E(e)}^{-1} (K_{j_E(e), \ell(e)}))$  (as in Definition 11.2).

In order to prove our lemma, it suffices to prove that, for any of the finitely many  $(\Gamma, E, \ell)$  as above, the set  $\mathcal{H}(\Gamma, E, \ell)$  in which the condition  $(*) (\Gamma, E, \ell)$  is realized is a dense open subset of  $\prod_{i=1}^{3n} \mathcal{H}_i$ . This condition is equivalent to

$(*) (\Gamma, E, \ell)$ : the map  $(\prod_{e \in E} h_{j_E(e)}) \circ p(\Gamma, E)$  is transverse to  $\prod_{e \in E} P(j_E(e))$  along  $p(\Gamma, E)^{-1} (\prod_{e \in E} h_{j_E(e)}^{-1} (K_{j_E(e), \ell(e)}))$ .

Set

$$C_{E, \ell} = C(R, L; \Gamma) \cap p(\Gamma, E)^{-1} \left( \prod_{e \in E} \psi_{j_E(e), \ell(e)}^{-1} (\phi_{j_E(e), \ell(e)} (K_{j_E(e), \ell(e)}) \times D^2) \right).$$

The condition  $(*) (\Gamma, E, \ell)$  can equivalently be written as “ $(0)_{e \in E}$  is a regular value of the map

$$\prod_{e \in E} (p_{D^2} \circ \psi_{j_E(e), \ell(e)} \circ h_{j_E(e)} \circ p_e)$$

on  $C_{E,\ell}$ .”

Note that the set of regular values of this map on the compact domain  $C_{E,\ell}$  is open. Therefore, if  $(h_i)_{i \in 3n} \in \mathcal{H}(\Gamma, E, \ell)$  and if the  $d_0(h_i, h'_i)$  are small enough, the preimage of  $(0)_{e \in E}$  under the restriction of  $\prod_{e \in E} p_{D^2} \circ \psi_{j_E(e), \ell(e)} \circ h'_{j_E(e)} \circ p(\Gamma, E)$  to  $C_{E,\ell}$  consists of regular points of  $\prod_{e \in E} p_{D^2} \circ \psi_{j_E(e), \ell(e)} \circ h_{j_E(e)} \circ p(\Gamma, E)$ . These points are regular for  $\prod_{e \in E} p_{D^2} \circ \psi_{j_E(e), \ell(e)} \circ h'_{j_E(e)} \circ p(\Gamma, E)$  provided that the  $d(h_i, h'_i)$  are small enough. Therefore, the set  $\mathcal{H}(\Gamma, E, \ell)$  is open.

To prove density, we use explicit deformations of the  $h_i \in \mathcal{H}_i$ , for a given  $(h_i)_{i \in 3n} \in \prod_{i=1}^{3n} \mathcal{H}_i$ . Fix a smooth map  $\chi: D^2 \rightarrow [0, 1]$ , which maps the disk of radius  $\frac{1}{2}$  to 1 and the complement of the disk of radius  $\frac{3}{4}$  to 0. For each compact  $K_{i,j}$ , such that  $j \in J(i)$ , fix a smooth map  $\chi_{i,j}: A_{i,j} \rightarrow [0, 1]$  that maps  $\phi_{i,j}(K_{i,j})$  to 1, and that vanishes outside a compact of  $A_{i,j}$ . For  $w \in D^2$ , define

$$h_{i,j,w}: \begin{array}{ccc} A_{i,j} \times D^2 & \rightarrow & A_{i,j} \times D^2 \\ (x, v) & \mapsto & (x, v + \chi(v)\chi_{i,j}(x)w). \end{array}$$

Note that  $h_{i,j,w}$  is a diffeomorphism as soon as  $\|w\|$  is smaller than a fixed positive number  $\eta < \frac{1}{2}$ . Extend  $\psi_{i,j}^{-1} \circ h_{i,j,w} \circ \psi_{i,j}$  by the identity map outside  $N(P(i))|_{\mathcal{O}_{i,j}}$ . Note that there exists a constant  $C$  such that  $d(\psi_{i,j}^{-1} \circ h_{i,j,w} \circ \psi_{i,j} \circ h_i, h_i) \leq C\|w\|$ .

Thus, it suffices to prove that, for any  $\varepsilon$  such that  $0 < \varepsilon < \eta$ , there exists  $(w_e)_{e \in E}$  with  $\|w_e\| < \varepsilon$  such that the restriction of

$$\prod_{e \in E} (\psi_{j_E(e), \ell(e)}^{-1} \circ h_{j_E(e), \ell(e), w_e} \circ \psi_{j_E(e), \ell(e)} \circ h_{j_E(e)}) \circ p(\Gamma, E)$$

to  $(C_{E,\ell})$  is transverse to  $\prod_{e \in E} P(j_E(e))$  along  $C_{E,\ell}$ . Since this happens when  $(-w_e)_{e \in E}$  is a regular value of the restriction of

$$\prod_{e \in E} (p_{D^2} \circ \psi_{j_E(e), \ell(e)} \circ h_{j_E(e)}) \circ p(\Gamma, E)$$

to  $C_{E,\ell}$ , and since such regular values form a dense set according to the Morse–Sard theorem, the lemma is proved.  $\square$

The magic in the Thom proof above is that it proves the density of manifolds in general  $3n$  position without bothering to construct a single one.

Lemma 11.11 does not quite prove Lemma 11.4 for two reasons. First, the  $h_i$  do not fix the boundary of  $\partial C_2(R)$  pointwise, so the perturbations  $h_i^{-1}(P(i))$  are no longer propagating chains of  $(C_2(R), \tau)$ . Second, we have to deal with immersed manifolds (multiplied by an element of  $\mathbb{Q}$ ) rather than embedded ones when  $R$  is not an integer homology sphere.

To deal with this latter issue, we start with immersions  $f_i$  of manifolds  $\tilde{P}(i)$  to  $C_2(R)$  whose images  $f_i(\tilde{P}(i))$  represent chains  $kP$  as in the end of Section 11.2, and (extended) immersions  $f_i$  of the pull-backs  $N(\tilde{P}(i))$  of the normal bundles to their images. Our immersions  $f_i$  have the following properties. The restriction to  $f_i^{-1}(C'_2(R))$  of each immersion  $f_i$  is an embedding. The preimage  $f_i^{-1}(N(\partial C_2(R)))$  has  $k$  connected components  $C_{j,i}$  ( $j \in \underline{k}$ ) in  $N(\tilde{P}(i))$ , and  $f_i$  embeds each of these  $k$  connected components into  $N(\partial C_2(R))$ . We will think of the intersection with a preimage of  $f_i(\tilde{P}(i)) \cap N(\partial C_2(R))$  as the sum of the intersections with the preimages of the  $f_i(C_{i,j})$ , and argue with covers of  $\tilde{P}(i)$  rather than covers of its image. So this latter issue is not a big one—if we do not require the boundary conditions. We keep this in mind, and we no longer discuss this issue.

The first issue is more serious. We want the boundaries of our propagating chains to be equal to  $p_\tau|_{\partial C_2(R)}^{-1}(X_i)$  for some  $X_i \in S^2$ . Recall that  $p_\tau$  also denotes a regular extension of  $p_\tau$  on  $N(\partial C_2(R))$ , that a collar  $[0, 8] \times \partial C'_2(R)$  of  $\partial C'_2(R)$  in  $N(\partial C_2(R))$  has been fixed, and that  $p_\tau$  factors through the natural projection onto  $\partial C'_2(R)$  in this collar. For an interval  $I$  included in  $[0, 8]$ , set

$$N_I = I \times \partial C'_2(R).$$

For  $a \in [1, 8]$ , set

$$N_{[a,9]} = N(\partial C_2(R)) \setminus [0, a[ \times \partial C'_2(R).$$

We will actually impose that our propagating chains intersect  $N_{[7,9]}$  as  $p_\tau|_{N_{[7,9]}}^{-1}(X_i)$ , by modifying our immersions  $f_i$  provided by the construction of the end of Section 11.2, only on  $N_{[4,7[}$ .

We first describe appropriate choices for the  $X_i$ , to allow transversality near the boundaries.

Let  $\Gamma$  of  $\mathcal{D}_{\underline{3n}}^e(\mathcal{L})$ , let  $E$  be a subset of  $E(\Gamma)$ . A condition on  $(X_i)_{i \in \underline{3n}}$  is that  $(X_i)_{i \in j_E(E)}$  is a regular (for the restriction to any stratum of  $C(R, L; \Gamma)$ ) value of the map

$$\prod_{e \in E} p_\tau \circ p_e$$

from

$$C(\Gamma, E) = C(R, L; \Gamma) \cap \bigcap_{e \in E} p_e^{-1}(N(\partial C_2(R)))$$

to  $(S^2)^{j_E(E)}$ . According to the Morse–Sard theorem 1.4, this condition holds when  $(X_i)_{i \in \underline{3n}}$  is in a dense subset of  $(S^2)^{3n}$ , which is furthermore open since  $C(\Gamma, E)$  is compact. Thus, this condition holds for any of the finitely many

pairs  $(\Gamma, E)$  as above, if  $(X_i)_{i \in \underline{3n}}$  belongs to the intersection of the corresponding open dense subsets of  $(S^2)^{3n}$ , which is still open and dense.

Fix  $(X_i)_{i \in \underline{3n}}$  in this open dense subset of  $(S^2)^{3n}$ . Now, we refer to the proof of Lemma 11.11 and adapt it to produce the desired family of propagating chains of Lemma 11.4. We fix a finite cover  $(K_{i,j})_{j \in J(i)}$  of  $\tilde{P}(i) \cap f_i^{-1}(C'_2(R) \cup N_{[0,7]})$ . We assume that this cover contains a special element  $K_{i,0} = \tilde{P}(i) \cap f_i^{-1}(N_{[5,7]})$  and that  $K_{i,j}$  is a compact subset of  $\tilde{P}(i) \cap f_i^{-1}(C'_2(R) \cup N_{[0,5]})$  for any  $j \in J'(i) = J(i) \setminus \{0\}$ .

When  $j \in J'(i)$ ,  $K_{i,j}$  is embedded in an open subspace  $\mathcal{O}_{i,j}$  of  $\tilde{P}(i) \cap f_i^{-1}(C'_2(R) \cup N_{[0,6]})$ . These  $\mathcal{O}_{i,j}$  are diffeomorphic to  $\mathbb{R}^4$  via diffeomorphisms  $\phi_{i,j}: \mathcal{O}_{i,j} \rightarrow \mathbb{R}^4$ , and we have bundle charts  $(\psi_{i,j}: N(\tilde{P}(i))|_{\mathcal{O}_{i,j}} \rightarrow \mathbb{R}^4 \times D^2)$ , for  $j \in J'(i)$ , as in the proof of Lemma 11.11.

The bundle  $N(K_{i,0})$  is trivialized by  $p_\tau$  in the following way. Fix a small neighborhood  $D_i$  of  $X_i$  in  $S^2$  diffeomorphic to the standard disk  $D^2$  and a diffeomorphism  $\psi_{D,i}$  from  $D_i$  to  $D^2$ . Without loss of generality, assume that  $N(K_{i,0}) = f_i^{-1}(p_\tau^{-1}(D_i) \cap N_{[5,7]})$ , and identify  $N(K_{i,0})$  with  $K_{i,0} \times D^2$  so that the projection onto  $D^2$  may be expressed as  $p_{D^2} = \psi_{D,i} \circ p_\tau \circ f_i$ .

The space  $\mathcal{H}_i$  is now the space of smooth diffeomorphisms of  $N(\tilde{P}(i))$  that are isotopic to the identity map, that fix a neighborhood of  $\partial N(\tilde{P}(i))$  and a neighborhood of  $f_i^{-1}(f_i(N(\tilde{P}(i))) \cap (N_{[7,9]}))$  pointwise, and that map any fiber of  $N(\tilde{P}(i))$  to itself. The space  $\mathcal{H}_i$  is equipped with a distance similar to that described before Lemma 11.11.

We want to prove that the subspace of  $\prod_{i \in \underline{3n}} \mathcal{H}_i$  consisting of the  $(h_i)_{i \in \underline{3n}}$  such that the  $f_i(h_i^{-1}(\tilde{P}(i)))$  are in general  $3n$  position with respect to  $L$ , in the sense of Definition 11.3, is open and dense.

It is open as in the proof of Lemma 11.11.<sup>3</sup>

Moreover, for any  $\Gamma$  of  $\mathcal{D}_{\underline{3n}}^e(\mathcal{L})$ , for any triple  $(E_X, E_N, E_C)$  of pairwise disjoint subsets of  $E(\Gamma)$ , the subset  $\mathcal{H}(\Gamma, E_X, E_N, E_C)$  of  $\prod_{i \in \underline{3n}} \mathcal{H}_i$  such that the restriction of  $p(\Gamma, E_N \cup E_C)$  to  $C(R, L; \Gamma) \cap p(\Gamma, E_X)^{-1} \prod_{e \in E_X} (N_{[5,9]} \cap p_\tau^{-1}(X_{j_E(e)}))$  is transverse to

$$\prod_{e \in E_N \cup E_C} \left( f_{j_E(e)} \left( h_{j_E(e)}^{-1} \left( \tilde{P}(j_E(e)) \right) \right) \right)$$

---

<sup>3</sup>If we ask only for transversality of the

$$p(\Gamma, E) = \prod_{e \in E} p_e: C(R, L; \Gamma) \rightarrow (C_2(R))^{j_E(E)}$$

to  $\prod_{e \in E} f_{j_E(e)}(h_{j_E(e)}^{-1}(\tilde{P}(j_E(e))))$  along  $\prod_{e \in E} p_e^{-1}(C'_2(R) \cup N_{[0,5]})$ , then density could also be proved as in Lemma 11.11.



along

$$p(\Gamma, E_X)^{-1} \prod_{e \in E_X} (N_{[5,9]} \cap p_\tau^{-1}(X_{j_E(e)})) \cap p(\Gamma, E_N)^{-1} \prod_{e \in E_N} (N_{[5,9]}) \\ \cap p(\Gamma, E_C)^{-1} \prod_{e \in E_C} (C'_2(R) \cup N_{[0,5]})$$

is open.<sup>4</sup> The  $\mathcal{H}(\Gamma, E_X, \emptyset, E_C)$  are furthermore dense as in the proof of Lemma 11.11.

In order to prove Lemma 11.4, it suffices to prove that for any  $\Gamma$  of  $\mathcal{D}_{3n}^e(\mathcal{L})$ , for any pair  $(E_N, E_C)$  of disjoint subsets of  $E(\Gamma)$ , the subset  $\mathcal{H}(\Gamma, \emptyset, E_N, E_C)$  of  $\prod_{i \in \underline{3n}} \mathcal{H}_i$  is dense. To do that, we fix  $(h_i)_{i \in \underline{3n}} \in \prod_{i \in \underline{3n}} \mathcal{H}_i$  and  $\varepsilon \in ]0, 1[$ , and we prove that there exists  $(h'_i)_{i \in \underline{3n}} \in \mathcal{H}(\Gamma, \emptyset, E_N, E_C)$  such that  $\max_{i \in \underline{3n}} (d(h_i, h'_i)) < \varepsilon$ . There exists  $\eta \in ]0, 1[$  such that the restriction of  $h_i$  to

$$f_i^{-1} \left( f_i(N(\tilde{P}(i))) \cap N_{[7-2\eta, 9]} \right)$$

is the identity map for any  $i \in j_E(E_N)$ .

For  $i \in j_E(E_N)$ , our  $h'_i$  will be constructed as some  $h_{i, \eta, w} \circ h_i$ . Let  $\chi_\eta$  be a smooth map from  $[4, 9]$  to  $[0, 1]$  that maps  $[5, 7 - 2\eta]$  to 1 and that maps the complement of  $[5 - \eta, 7 - \eta]$  to 0. Recall our smooth map  $\chi: D^2 \rightarrow [0, 1]$ , which maps the disk of radius  $\frac{1}{2}$  to 1 and the complement of the disk of radius  $\frac{3}{4}$  to 0. For  $w \in D^2$  define

$$h_{\eta, w}: \begin{array}{l} [4, 9] \times D^2 \rightarrow D^2 \\ (t, v) \mapsto v + \chi(v)\chi_\eta(t)w. \end{array}$$

Define  $h_{i, \eta, w} \in \mathcal{H}_i$ , for  $w$  sufficiently small, to coincide with the identity map outside

$$f_i^{-1} \left( f_i(N(\tilde{P}(i))) \cap N_{[5-\eta, 7-\eta]} \right),$$

and with the map that sends  $(p, v) \in \left( \tilde{P}(i) \cap f_i^{-1}(N_{\{t\}}) \right) \times D^2$

$$\left( \subset \left( \left( \tilde{P}(i) \cap f_i^{-1}(N_{[5-\eta, 7-\eta]}) \right) \times D^2 = N(\tilde{P}(i)) \cap f_i^{-1}(N_{[5-\eta, 7-\eta]}) \right) \right)$$

to  $(p, h_{\eta, w}(t, v))$ , for  $t \in [5 - \eta, 7 - \eta]$ .<sup>5</sup> There exists  $u \in ]0, 1[$  such that, as soon as  $\|w\| < u$ ,  $h_{i, \eta, w}$  is indeed a diffeomorphism and  $d(h_{i, \eta, w} \circ h_i, h_i) < \varepsilon$ .

<sup>4</sup>Our hypotheses on  $(X_i)_{i \in \underline{3n}}$  guarantee that  $C(R, L; \Gamma) \cap p(\Gamma, E_X)^{-1} \prod_{e \in E_X} (N_{[5,9]} \cap p_\tau^{-1}(X_{j_E(e)}))$  is a manifold.

<sup>5</sup>For any  $t \in [0, 9]$ , we assume  $f_i \left( \left( \tilde{P}(i) \cap f_i^{-1}(N_{\{t\}}) \right) \times D^2 \right) \subset N_{\{t\}}$ , without loss of generality.

Fix  $(h'_i)_{i \in J_E(E_C)}$  such that  $d(h_i, h'_i) < \varepsilon$  and  $(h'_i)_{i \in J_E(E_C)} \times (h_i)_{i \notin J_E(E_C)}$  is in the dense open set  $\cap_{E_x \subseteq E_N} \mathcal{H}(\Gamma, E_x, \emptyset, E_C)$  (this does not impose anything on  $(h_i)_{i \notin J_E(E_C)}$ ).

After reducing  $u$ , we may now assume that as soon as  $\|w\| < u$ , for any  $E_x \subseteq E_N$ ,  $p(\Gamma, E_C \cup E_x)$  is transverse to

$$\prod_{e \in E_C} f_{j_E(e)} \left( \left( h'_{j_E(e)} \right)^{-1} \left( \tilde{P}(j_E(e)) \right) \right) \\ \times \prod_{e \in E_x} f_{j_E(e)} \left( \left( h_{j_E(e), \eta, w_{j_E(e)}} \circ h_{j_E(e)} \right)^{-1} \left( \tilde{P}(j_E(e)) \right) \right)$$

along

$$p(\Gamma, E_C)^{-1} \left( \prod_{e \in E_C} (C'_2(R) \cup N_{[0,5]}) \right) \cap p(\Gamma, E_x)^{-1} \left( \prod_{e \in E_x} (N_{[7-2\eta,9]}) \right)$$

since  $h_{j_E(e)}^{-1}(\tilde{P}(j_E(e))) = p_\tau^{-1}(X_{j_E(e)})$  on  $N_{[7-2\eta,9]}$ . Furthermore,  $p(\Gamma, E_C \cup E_N)$  is transverse to

$$M \left( (h'_i)_{i \in J_E(E_C)}, (h_{i, \eta, w_i})_{i \in J_E(E_N)} \right) = \prod_{e \in E_C} \left( f_{j_E(e)} \left( \left( h'_{j_E(e)} \right)^{-1} \left( \tilde{P}(j_E(e)) \right) \right) \right) \\ \times \prod_{e \in E_N} f_{j_E(e)} \left( \left( h_{j_E(e), \eta, w_{j_E(e)}} \circ h_{j_E(e)} \right)^{-1} \left( \tilde{P}(j_E(e)) \right) \right)$$

along

$$p(\Gamma, E_C)^{-1} \left( \prod_{e \in E_C} (C'_2(R) \cup N_{[0,5]}) \right) \cap p(\Gamma, E_N)^{-1} \left( \prod_{e \in E_N} N_{[5,9]} \right),$$

if and only if, for any subset  $E_x$  of  $E_N$ , the following condition  $(*) (E_x)$  holds.  
 $(*) (E_x) : p(\Gamma, E_C \cup E_N)$  is transverse to  $M \left( (h'_i)_{i \in J_E(E_C)}, (h_{i, \eta, w_i})_{i \in J_E(E_N)} \right)$   
 along

$$p(\Gamma, E_C)^{-1} \left( \prod_{e \in E_C} (C'_2(R) \cup N_{[0,5]}) \right) \cap p(\Gamma, E_x)^{-1} \left( \prod_{e \in E_x} N_{[7-2\eta,9]} \right) \\ \cap p(\Gamma, E_N \setminus E_x)^{-1} \left( \prod_{e \in E_N \setminus E_x} N_{[5,7-2\eta]} \right).$$

Let  $\mathring{D}_u$  denote the open disk of  $\mathbb{R}^2$  centered at 0 of radius  $u$ . Our former hypotheses guarantee transversality of  $p(\Gamma, E_C \cup E_x)$  as soon as the  $\|w_i\|$  are smaller than  $u$  for  $i \in E_x$ . So the condition  $(*)(E_x)$  is realized as soon as  $(w_i)_{i \in j_E(E_N)}$  is in an open dense subset  $\mathcal{D}(E_x)$  of  $\mathring{D}_u^{j_E(E_N)}$ . Thus, we have the desired transversality when  $(w_i)_{i \in j_E(E_N)}$  is in the intersection of the open dense subsets  $\mathcal{D}(E_x)$  over the subsets  $E_x$  of  $E_N$ .  $\square$

## 11.4 More on forms dual to transverse propagating chains

Though Lemma 11.7 is not surprising, we prove it and refine it in this section. We use its refinement in Chapter 17. Recall the notation of Definition 11.6, and let  $D_\varepsilon$  ( resp.  $\mathring{D}_\varepsilon$ ) denote the closed ( resp. open) disk of  $\mathbb{C}$  centered at 0 with radius  $\varepsilon$ .

**Lemma 11.12.** *Recall that our configuration spaces are equipped with Riemannian metrics. Let  $(\check{R}, \tau)$  be an asymptotic rational homology  $\mathbb{R}^3$ . Let  $L: \mathcal{L} \rightarrow \check{R}$  be a link in  $\check{R}$ . Let  $n \in \mathbb{N}$ , and let  $(P(i))_{i \in \underline{3n}}$  be a family of propagating chains of  $(C_2(R), \tau)$  in general  $3n$  position with respect to  $L$ . For any  $\varepsilon > 0$ , there exists  $\eta > 0$  such that for any  $i \in \underline{3n}$ , for any  $\Gamma \in \mathcal{D}_{\underline{3n}}^e(\mathcal{L})$ , and for any  $e \in E(\Gamma)$  with associated restriction map  $p_e: C(R, L; \Gamma) \rightarrow C_2(R)$ , we have*

$$p_e^{-1} \left( N_\eta \left( P(j_E(e)) \right) \right) \subset N_\varepsilon \left( p_e^{-1} \left( P(j_E(e)) \right) \right).$$

**PROOF:** Of course, it is enough to prove the lemma for a fixed  $(\Gamma, e)$ . Set  $i = j_E(e)$ . The compact  $p_e(C(R, L; \Gamma) \setminus N_\varepsilon(p_e^{-1}(P(i))))$  does not meet  $P(i)$ . So there exists  $\eta > 0$  such that this compact does not meet  $N_\eta(P(i))$  either. This implies  $p_e^{-1}(N_\eta(P(i))) \subset N_\varepsilon(p_e^{-1}(P(i)))$ .  $\square$

**Lemma 11.13.** *Let  $\Gamma \in \mathcal{D}_{\underline{3n}}^e(\mathcal{L})$ . Assume that the hypotheses of Lemma 11.12 are satisfied. Then the intersection in  $C(R, L; \Gamma)$  over the edges  $e$  of  $E(\Gamma)$  of the codimension 2 rational chains  $p_e^{-1}(P(j_E(e)))$  is a finite set  $I_S(\Gamma, (P(i))_{i \in \underline{3n}})$ . Furthermore, for any  $\varepsilon > 0$ , there exists  $\eta > 0$  such that we have*

$$\bigcap_{e \in E(\Gamma)} p_e^{-1} \left( N_\eta \left( P(j_E(e)) \right) \right) \subset N_\varepsilon \left( I_S \left( \Gamma, (P(i))_{i \in \underline{3n}} \right) \right)$$

for any  $\Gamma \in \mathcal{D}_{\underline{3n}}^e(\mathcal{L})$ . So, for any family  $(\omega(i))_{i \in \underline{3n}}$  of propagating forms of  $(C_2(R), \tau)$   $\eta$ -dual to the  $P(i)$ , the form  $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$  is supported in  $N_\varepsilon(I_S(\Gamma, (P(i))_{i \in \underline{3n}}))$ .

Moreover, if  $N_\varepsilon(I_S(\Gamma, (P(i))))$  is a disjoint union over the points  $x$  of  $I_S(\Gamma, (P(i)))$  of the  $N_\varepsilon(x)$ , then the integral over  $N_\varepsilon(x)$  of  $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$  is the rational intersection number of the rational chains  $p_e^{-1}(P(j_E(e)))$  at  $x$ . When all the  $P(j_E(e))$  are embedded manifolds with coefficient 1 near  $p_e(x)$ , this intersection number is the sign of  $x$  with respect to an orientation  $o(\Gamma)$  of  $\Gamma$ .

PROOF: Again, it suffices to prove the lemma for a fixed  $\Gamma \in \mathcal{D}_{3n}^e(\mathcal{L})$ . We refer to the description of the image under  $p(\Gamma) = \prod_{e \in E(\Gamma)} p_e$  of an intersection point  $x$  after Definition 11.3.

Fix such an  $x$ . For each edge  $e$ ,  $p_e(x)$  sits inside a nonsingular open 4-dimensional smooth ball  $\delta_e$  of a smooth piece  $\Delta_{j_E(e),k}$  of  $P(j_E(e))$ . Consider a tubular neighborhood  $N_u(\delta_e)$  whose fibers are disks  $D_\theta$  orthogonal to  $\delta_e$  of radius  $\theta$ . The bundle  $N_u(\delta_e)$  is isomorphic to  $\delta_e \times D_\theta$ , with respect to a trivialization of  $N_u(\delta_e)$ . Another trivialization would compose the diffeomorphism from  $N_u(\delta_e)$  to  $\delta_e \times D_\theta$  by a map  $(x, v) \mapsto (v, \phi(x)(v))$  for some  $\phi: \delta_e \rightarrow SO(2)$ .

The projection  $p_e(x)$  may sit simultaneously in different nonsingular 4-dimensional smooth parts  $\Delta_{j_E(e),k}$  of  $P(j_E(e))$ . Let  $K(e, x)$  be the finite set of components  $\Delta_{j_E(e),k}$  of  $P(j_E(e))$  such that  $p_e(x) \in \Delta_{j_E(e),k}$ . We first focus on one element of  $K(e, x)$  for each  $e$ , and next take the sum over all the choices in  $\prod_{e \in E(\Gamma)} K(e, x)$  multiplied by the products of the coefficients of the elements of  $K(e, x)$  in the rational chains  $P(j_E(e))$ . Similarly, our forms  $\eta$ -dual to the  $P(i)$  are thought of and constructed as linear combinations of forms  $\eta$ -dual to the elements of  $K(e, x)$ .

Without loss of generality, assume that  $\varepsilon$  is small enough so that we have

$$p_e(N_\varepsilon(x)) \subset \delta_e \times D_\theta$$

for any edge  $e$  of  $\Gamma$  and for any  $\delta_e = \delta_{e,k}$  associated to an element  $\Delta_{j_E(e),k}$  of  $K(e, x)$ , and so that  $p_e(N_\varepsilon(x))$  does not meet the components  $\Delta_{j_E(e),k}$  of  $P(j_E(e))$  that are not in  $K(e, x)$ . Reduce  $\varepsilon$  and choose  $\eta < \theta$  small enough so that  $p_e(N_\varepsilon(x))$  does not meet the neighborhoods  $N_\eta(\Delta_{j_E(e),k})$  of these components, either.

Let  $p_{D_\theta}: \delta_e \times D_\theta \rightarrow D_\theta$  denote the natural projection. Let  $\omega_\eta$  be a volume-one form supported on  $\mathring{D}_\eta$ . Forms  $\eta$ -dual to  $P(j_E(e))$  can be constructed by patching forms  $(p_{D_\theta})^*(\omega_\eta)$  (multiplied by the coefficients of the  $\Delta_{j_E(e),k}$ ) together, as in Lemma B.4. Conversely, for any form  $\omega(j_E(e))$   $\eta$ -dual to a piece  $\Delta_{j_E(e),k}$  of  $P(j_E(e))$  that contains  $\delta_e$ , there exists a one-form  $\alpha_e$  on  $\delta_e \times D_\theta$ , such that  $\omega(j_E(e)) = p_{D_\theta}^*(\omega_\eta) + d\alpha_e$  on  $\delta_e \times D_\theta$ . Then we have

$$\int_{\{x \in \delta_e\} \times \partial D_\eta} \alpha_e = \int_{\{x \in \delta_e\} \times D_\eta} \omega(j_E(e)) - p_{D_\theta}^*(\omega_\eta) = 0.$$

So  $\alpha_e$  is exact on  $\delta_e \times (D_\theta \setminus D_\eta)$ , and  $\alpha_e$  can and will be assumed to be supported on  $\delta_e \times D_\eta$ .

In the neighborhood  $N_\varepsilon(x)$  of  $x$ ,  $\prod_{e \in E(\Gamma)} p_{D_\theta} \circ p_e$  is a local diffeomorphism around  $x$ . Without loss of generality, assume that  $\eta$  and  $\varepsilon$  are small enough so that

$$\Pi_p = \prod_{e \in E(\Gamma)} p_{D_\theta} \circ p_e : N_\varepsilon(x) \rightarrow D_\theta^{E(\Gamma)}$$

restricts to a diffeomorphism from  $\Pi_p^{-1}(D_{2\eta}^{E(\Gamma)})$  to  $D_{2\eta}^{E(\Gamma)}$ , for each  $x$  (and for each choice in  $\prod_{e \in E(\Gamma)} K(e, x)$ ). If  $\prod_{e \in E(\Gamma)} K(e, x)$  has one element, and if the coefficient of the element of  $K(e, x)$  in  $P(j_E(e))$  is 1 for any edge  $e$ , then we have

$$\int_{N_\varepsilon(x)} \bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e))) = \int_{N_\varepsilon(x)} \bigwedge_{e \in E(\Gamma)} p_e^*(p_{D_\theta}^*(\omega_\eta)).$$

Indeed, changing one  $\omega(j_E(e))$  to  $(p_{D_\theta})^*(\omega_\eta)$  amounts to add the integral obtained by replacing  $\omega(j_E(e))$  by  $d\alpha_e$ . Since all the forms are closed, this latter integral is the integral over  $\Pi_p^{-1}(\partial(D_{2\eta}^{E(\Gamma)}))$  of the form obtained by replacing  $d\alpha_e$  by  $\alpha_e$ , which is zero since the whole form is supported in  $\Pi_p^{-1}(D_\eta^{E(\Gamma)})$ . Therefore, the integral is the sign of the intersection point  $x$  with respect to the given orientation and coorientations.

The open neighborhoods  $N_\varepsilon(x)$  may be assumed to be disjoint from each other for distinct  $x$ . Consequently, since  $C(R, L; \Gamma)$  is compact, the set of intersection points  $x$  is finite. Consider the complement  $C^c(\eta_0)$  in  $C(R, L; \Gamma)$  of the union over the intersection points  $x$  of the  $N_\varepsilon(x)$ . Since  $p_{e_1}^{-1}(P(j_E(e_1)))$  does not meet  $\bigcap_{e \in E(\Gamma) \setminus \{e_1\}} p_e^{-1}(P(j_E(e)))$  in  $C^c(\eta_0)$ , there is an  $\varepsilon_1 > 0$  such that  $\overline{N_{\varepsilon_1}(p_{e_1}^{-1}(P(j_E(e_1))))}$  does not meet  $\bigcap_{e \in E(\Gamma) \setminus \{e_1\}} p_e^{-1}(P(j_E(e)))$  either in  $C^c(\eta_0)$ . Iterating, we find  $\varepsilon_2 > 0$  such that

$$C^c(\eta_0) \cap \bigcap_{e \in E(\Gamma)} N_{\varepsilon_2} \left( p_e^{-1} \left( P(j_E(e)) \right) \right) = \emptyset.$$

According to Lemma 11.12,  $\eta$  can be reduced so that  $p_e^{-1}(N_\eta(P(i))) \subset N_{\varepsilon_2}(p_e^{-1}(P(i)))$  for any  $i$ . Then  $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$  is supported where we want it to be. □

Lemma 11.7 follows. □

Theorem 11.8 is now proved. □

## 11.5 A discrete definition of the anomaly $\beta$

In this section, we give a discrete definition of the anomaly  $\beta$  and mention a few recent results of Kévin Corbineau on  $\beta_3$ .

**Lemma 11.14.** *Let  $n \in \mathbb{N}$ . Let  $\Gamma \in \mathcal{D}_n^c$ . Recall the compactification  $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$  of  $\check{\mathcal{S}}_{V(\Gamma)}(\mathbb{R}^3)$  from Theorem 8.11. For any edge  $e = j_E^{-1}(i)$  of  $\Gamma$ , we have a canonical projection*

$$p_e: B^3 \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3) \rightarrow B^3 \times S^2.$$

Let  $i \in \underline{3n}$ . When  $\Gamma$  is fixed, set  $p_i = p_{j_E^{-1}(i)}$ . For any  $a_i \in S^2$ , define the following cooriented chains of  $B^3 \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$ :

$$A(\Gamma, i, a_i) = p_i^{-1}(B^3 \times \{a_i\}),$$

$$B(\Gamma, i, a_i) = p_i^{-1}\left(\left\{\cup_{m \in B^3} (m, \rho(m)(a_i))\right\}\right),$$

and

$$H(\Gamma, i, a_i) = p_i^{-1}(G(a_i)),$$

where  $\rho$  is introduced in Definition 4.4, and the chain  $G(a_i)$  of  $B^3 \times S^2$  is introduced in Lemma 4.12. The codimension of  $A(\Gamma, i, a_i)$  and  $B(\Gamma, i, a_i)$  is 2, while the codimension of  $H(\Gamma, i, a_i)$  is 1. An element  $(a_1, \dots, a_{3n})$  of  $(S^2)^{3n}$  is  $\beta_n$ -admissible if for  $h \in \underline{3n}$  and for any  $\Gamma \in \mathcal{D}_n^c$ , the intersection of the  $A(\Gamma, i, a_i)$  for  $i \in \underline{h-1}$ , the  $B(\Gamma, i, a_i)$  for  $i \in \underline{3n} \setminus \underline{h}$ , and  $H(\Gamma, h, a_h)$  is transverse. Then the sets of elements of  $(S^2)^{3n}$  that are  $\beta_n$ -admissible is an open dense subset of  $(S^2)^{3n}$ .

PROOF: The principle of the proof is the same as the proof of Proposition 11.1. See also Section 11.3. This lemma is proved in detail in [Cor16].  $\square$

**Proposition 11.15.** *Recall the orientation of  $\check{\mathcal{S}}_{V(\Gamma)}(\mathbb{R}^3)$  of Lemma 9.3. For any  $\beta_n$ -admissible element  $(a_1, \dots, a_{3n})$  of  $(S^2)^{3n}$ , we have*

$$\beta_n = \sum_{h=1}^{3n} \sum_{\Gamma \in \mathcal{D}_n^c} \frac{1}{(3n)!2^{3n}} I(\Gamma, h) [\Gamma],$$

with

$$I(\Gamma, h) [\Gamma] = \left\langle \cap_{i=1}^{h-1} A(\Gamma, i, a_i), H(\Gamma, h, a_h), \cap_{i=h+1}^{3n} B(\Gamma, i, a_i) \right\rangle_{B^3 \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)} [\Gamma].$$

PROOF: For  $a \in S^2$  and  $t \in [0, 1]$ , define the following chain

$$G(a, t) = [0, t] \times B^3 \times \{a\} + (\{t\} \times G(a)) + \left\{ (u, m, \rho(m)(a)) : u \in [t, 1], m \in B^3 \right\}$$

of  $[0, 1] \times B^3 \times S^2$ . Let  $(t_i)_{i \in \underline{3n}}$  be a strictly decreasing sequence of  $]0, 1[$ . Let  $\Gamma \in \mathcal{D}_n^c$ . For  $i \in \underline{3n}$ , let  $p_i$  also denote the canonical projection associated to  $e = j_E^{-1}(i)$  from  $[0, 1] \times B^3 \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$  to  $[0, 1] \times B^3 \times S^2$ .

If  $(a_1, \dots, a_{3n})$  is  $\beta_n$ -admissible, then for any  $\Gamma \in \mathcal{D}_n^c$ , the intersection of the  $p_i^{-1}(G(a_i, t_i))$  is transverse and equal to

$$\sqcup_{h=1}^{3n} \{t_h\} \times \left( \bigcap_{i=1}^{h-1} A(\Gamma, i, a_i) \cap H(\Gamma, h, a_h) \cap \left( \bigcap_{i=h+1}^{3n} B(\Gamma, i, a_i) \right) \right).$$

Indeed, it is clear that the intersection may be expressed as above at the times  $t \in \{t_h : h \in \underline{3n}\}$ . Since  $(a_1, \dots, a_{3n})$  is  $\beta_n$ -admissible, this intersection is transverse at these times. So it does not intersect the boundaries of the  $H(\Gamma, h, a_h)$ . Therefore, there is no intersection in  $([0, 1] \setminus \{t_h : h \in \underline{3n}\}) \times B^3 \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$ .

Then for any  $\alpha > 0$ , there exist closed 2-forms  $\tilde{\omega}(i)$  on  $[0, 1] \times \mathbb{R}^3 \times S^2$ , as in Proposition 10.7, applied when  $\tau_0 = \tau_s$  and  $\tau_1 = \tau_0 \circ \psi_{\mathbb{R}}(\rho)$  on  $UB^3$ , such that  $\tilde{\omega}(i)$  is  $\alpha$ -dual to  $G(a_i, t_i)$ , for any  $i$ .

Theorem 4.5 yields  $p_1(\tau_0 \circ \psi_{\mathbb{R}}(\rho)) - p_1(\tau_0) = 2 \deg(\rho) = 4$ . Therefore, Proposition 10.7 implies

$$\beta_n = \sum_{\Gamma \in \mathcal{D}_n^c} \frac{1}{(3n)! 2^{3n}} \int_{[0,1] \times \check{\mathcal{S}}_{V(\Gamma)}(TB^3)} \bigwedge_{e \in E(\Gamma)} p_e^* \left( \tilde{\omega}(j_E(e)) \right) [\Gamma],$$

where  $\bigwedge_{e \in E(\Gamma)} p_e^* \left( \tilde{\omega}(j_E(e)) \right) = \bigwedge_{i=1}^{3n} p_i^* \left( \tilde{\omega}(i) \right)$ .

As in Section 11.4, for  $\alpha$  small enough,  $\int_{[0,1] \times \check{\mathcal{S}}_{V(\Gamma)}(TB^3)} \bigwedge_{i=1}^{3n} p_i^* \left( \tilde{\omega}(i) \right)$  is the algebraic intersection of the  $p_i^{-1}(G(a_i, t_i))$ .

For the signs, note that the coorientation of  $\{t_h\} \times G(a_h)$  in  $[0, 1] \times B^3 \times S^2$  is represented by the orientation of  $[0, 1]$ , followed by the coorientation of  $G(a_h)$  in  $B^3 \times S^2$ .  $\square$

In his Ph. D. thesis [Cor16, Théorème 2.15], Kévin Corbineau obtained the following simplified expression for  $\beta_3$ .

**Theorem 11.16.** *For  $j \in \underline{n}$ , set*

$$H_h(\Gamma, j, a_j) = p_j^{-1}(G_h(a_j)),$$

where the chain  $G_h(a_j)$  of  $B^3 \times S^2$  is introduced in Lemma 4.12. Let  $\mathcal{D}_3^c(T)$  be the set of numbered graphs in  $\mathcal{D}_3^c$  isomorphic to



For any element  $\beta_3$ -admissible  $(a_1, \dots, a_9)$  of  $(S^2)^9$ , we have

$$\beta_3 = \sum_{j=2}^8 \sum_{\Gamma \in \mathcal{D}_3^c(T)} \frac{1}{(9)!2^9} I_h(\Gamma, j) [\Gamma],$$

with

$$I_h(\Gamma, j) [\Gamma] = \left\langle \cap_{i=1}^{j-1} A(\Gamma, i, a_i), H_h(\Gamma, j, a_j), \cap_{i=j+1}^9 B(\Gamma, i, a_i) \right\rangle_{B^3 \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)} [\Gamma].$$

The Ph. D. thesis of Kévin Corbineau also contains an algorithm to compute  $\beta_3$ .





**Part III**  
**Functoriality**



Recall that  $D_1$  denotes the closed disk of  $\mathbb{C}$  centered at 0 with radius 1. In this book, a *rational homology cylinder* (or  $\mathbb{Q}$ -cylinder) is a compact oriented 3-manifold, whose boundary neighborhood is identified with a boundary neighborhood  $N(\partial(D_1 \times [0, 1]))$  of  $D_1 \times [0, 1]$ , and which has the same rational homology as a point.

Roughly speaking,  $q$ -tangles are parallelized cobordisms between limit planar configurations of points up to dilation and translation in rational homology cylinders. We describe the category of  $q$ -tangles and its structures precisely in Section 13.1. Framed links in rational homology spheres are particular  $q$ -tangles. They are cobordisms between empty configurations.

In this third part of the book, we define a functorial extension to  $q$ -tangles of the invariant  $\mathcal{Z}^f$  of framed links in  $\mathbb{Q}$ -spheres defined in Section 7.6, and we prove that it has a lot of useful properties. These properties are listed in Theorem 13.12. They ensure that  $\mathcal{Z}^f$  is a functor, which behaves naturally with respect to other structures of the category of  $q$ -tangles, such as cabling or duplication. They allow one to reduce the computation of  $\mathcal{Z}^f$  for links to its computation for elementary pieces of the links.

Section 12.1 introduces particular  $q$ -tangles, for which the involved planar configurations are injective. Section 12.3 introduces other particular  $q$ -tangles, for which the involved planar configurations are corners of the Stasheff polyhedra of Example 8.3. In Chapter 12, we define  $\mathcal{Z}^f$  for these particular  $q$ -tangles without proofs. We also state a functoriality result, a monoidality result, and a duplication property, under simple hypotheses, to introduce the involved structures and motivate their introduction. These results are just particular cases of Theorem 13.12.

In Chapter 13, we state our general Theorem 13.12, and we describe our strategy towards a consistent definition of  $\mathcal{Z}^f$  for general  $q$ -tangles in Section 13.2. Our proofs involve convergence results, which rely on intricate compactifications of configuration spaces described in Chapter 14. In Chapter 15, we study  $\mathcal{Z}^f$  as a holonomy for the  $q$ -tangles that are paths in spaces of planar configurations. In Chapter 16, we introduce discretizable versions of  $\mathcal{Z}^f$ . We use these discretizable versions in the proofs of some important properties of  $\mathcal{Z}^f$  given in Chapter 17. The consistency of our strategy for the definition of  $\mathcal{Z}^f$  is shown in Chapters 14, 15, and 16. The proof of Theorem 13.12 will be finished in Chapter 17.

This functoriality part contains a generalization of results of Sylvain Poirier [Poi00], who constructed the functor  $\mathcal{Z}^f$  and proved Theorem 13.12 for combinatorial  $q$ -tangles of  $\mathbb{R}^3$ . We recall his results in Section 12.4.



# Chapter 12

## A first introduction to the functor $\mathcal{Z}^f$

In Section 12.1, we extend the definition of the invariant  $\mathcal{Z}$  of Theorem 7.20 to long tangle representatives as in Figure 1.12. Then we define the framed version  $\mathcal{Z}^f$  of  $\mathcal{Z}$  and state that it is multiplicative under the allowed vertical compositions in Section 12.2.

In Section 12.3, we state that  $\mathcal{Z}^f$  reaches a limit with nice cabling properties when some vertical infinite strands of the long tangle representatives approach each other. We thus define the restriction of  $\mathcal{Z}^f$  to *combinatorial  $q$ -tangles*, which are parallelized cobordisms between limit configurations on the real line in rational homology cylinders. This definition is due to Sylvain Poirier [Poi00] when the involved rational homology cylinder is the standard one  $D_1 \times [0, 1]$ . In Section 12.4, we list sufficiently many properties of the Poirier restriction of  $\mathcal{Z}^f$  to characterize the restriction of  $\mathcal{Z}^f$  to combinatorial  $q$ -tangles, in terms of the anomaly  $\alpha$ .

### 12.1 Extension of $\mathcal{Z}$ to long tangles

View  $\mathbb{R}^3$  as  $\mathbb{C} \times \mathbb{R}$ , where  $\mathbb{C}$  is horizontal and  $\mathbb{R}$  is vertical, oriented upwards. For a rational homology cylinder  $\mathcal{C}$ ,  $\check{R}(\mathcal{C})$  denotes the asymptotically standard  $\mathbb{Q}$ -homology  $\mathbb{R}^3$  obtained by replacing the standard cylinder  $\mathcal{C}_0 = D_1 \times [0, 1]$  in  $\mathbb{R}^3$  by  $\mathcal{C}$ .

**Definition 12.1.** A *long tangle representative* (or LTR for short) in  $\check{R}(\mathcal{C})$  is an embedding  $L: \mathcal{L} \hookrightarrow \check{R}(\mathcal{C})$  of a one-manifold  $\mathcal{L}$ , as in Figure 1.12, such that

- $L(\mathcal{L})$  intersects the closure  $\check{\mathcal{C}}_0^c$  of the complement of  $\mathcal{C}_0$  in  $\mathbb{R}^3$  as

$$L(\mathcal{L}) \cap \check{\mathcal{C}}_0^c = (c^-(B^-) \times ]-\infty, 0]) \cup (c^+(B^+) \times [1, \infty[)$$

for two finite sets  $B^-$  and  $B^+$  and two injective maps  $c^-: B^- \hookrightarrow \text{Int}(D_1)$  and  $c^+: B^+ \hookrightarrow \text{Int}(D_1)$ , which are called the *bottom configuration* and the *top configuration* of  $L$ , respectively, and

- $L(\mathcal{L}) \cap \mathcal{C}$  is a compact one-manifold whose unoriented boundary is  $(c^-(B^-) \times \{0\}) \cup (c^+(B^+) \times \{1\})$ .

For a  $\underline{3n}$ -numbered degree  $n$  Jacobi diagram with support  $\mathcal{L}$  without looped edges, let  $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$  be its configuration space defined as in Section 7.1.<sup>1</sup> The univalent vertices on a *strand*, which is the image under  $L$  of an open connected component of  $\mathcal{L}$  (diffeomorphic to  $\mathbb{R}$ ), move along this whole long component, as in Figure 12.1.

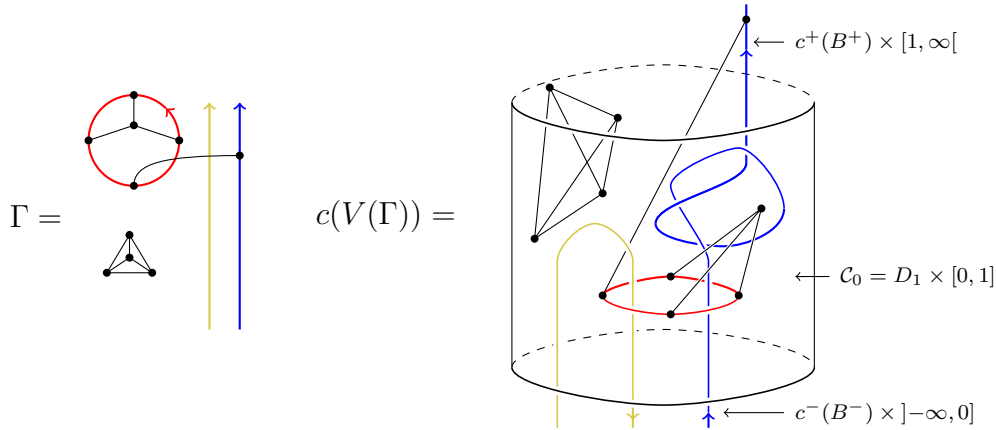


Figure 12.1: A (black) Jacobi diagram  $\Gamma$  on the domain of an LTR  $L$  and a configuration  $c$  of  $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$

For any  $i \in \underline{3n}$ , let  $\omega(i)$  be a propagating form of  $(C_2(R(\mathcal{C})), \tau)$ . Let  $o(\Gamma)$  be a vertex-orientation of  $\Gamma$ . As in Section 7.2, define

$$I(\mathcal{C}, L, \Gamma, o(\Gamma), (\omega(i))_{i \in \underline{3n}}) = \int_{(\check{C}(\check{R}(\mathcal{C}), L; \Gamma), o(\Gamma))} \bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e))),$$

where  $(\check{C}(\check{R}(\mathcal{C}), L; \Gamma), o(\Gamma))$  denotes the manifold  $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$ , equipped with the orientation induced by the vertex-orientation  $o(\Gamma)$  and by the edge-orientation of  $\Gamma$ , as in Corollary 7.2, and

$$I(\mathcal{C}, L, \Gamma, (\omega(i))_{i \in \underline{3n}}) [\Gamma] = I(\mathcal{C}, L, \Gamma, o(\Gamma), (\omega(i))_{i \in \underline{3n}}) [\Gamma, o(\Gamma)].$$

<sup>1</sup>The only differences are that  $\mathcal{L}$  is not necessarily a disjoint union of circles and that  $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$  was denoted by  $\check{C}(R(\mathcal{C}), L; \Gamma)$ .

**Theorem 12.2.** *The above integral is convergent.*

We prove this theorem in Section 14.2. See Lemma 14.24. Again, its proof involves appropriate compactifications  $C(\check{R}(\mathcal{C}), L; \Gamma)$  of the configuration spaces  $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$ . The compactifications are more complicated in this case. We study them in Chapter 14.

As an example, let us compute  $I(\mathcal{C}, L, \Gamma, o(\Gamma), (\omega(i))_{i \in \underline{3n}})$  when

- $\mathcal{C} = \mathcal{C}_0 = D_1 \times [0, 1]$ ,
- $L$  is an LTR  $\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}$  whose bottom and top configurations coincide and map  $B^- = B^+$  to  $\{-\frac{1}{2}, \frac{1}{2}\}$ ,
- $L(\mathcal{L}) \cap \mathcal{C}_0$  projects to  $\mathbb{R}^2$  as  $\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}$ ,
- $(\Gamma, o(\Gamma))$  is the vertex-oriented diagram  $\begin{array}{c} \uparrow \uparrow \\ \bullet \bullet \\ \downarrow \downarrow \end{array}$  whose chord is oriented and numbered, and
- the propagating forms  $\omega(i)$  pull back through  $p_{S^2}: C_2(S^3) \rightarrow S^2$ .

**Lemma 12.3.** *We have*

$$I\left(\mathcal{C}_0, \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}, \begin{array}{c} \uparrow \uparrow \\ \bullet \bullet \\ \downarrow \downarrow \end{array}, (p_{S^2}^*(\omega_{i,S}))_{i \in \underline{3}}\right) = I\left(\mathcal{C}_0, \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}, \begin{array}{c} \uparrow \uparrow \\ \bullet \bullet \\ \downarrow \downarrow \end{array}, (p_{S^2}^*(\omega_{i,S}))_{i \in \underline{3}}\right) = 1$$

for any arbitrary numbering of the edge of the involved Jacobi diagram and any choice of volume-one forms  $\omega_{i,S}$  of  $S^2$ .

PROOF: Let us compute

$$I\left(\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}, \begin{array}{c} \uparrow \uparrow \\ \bullet \bullet \\ \downarrow \downarrow \end{array}, (p_{S^2}^*(\omega_{i,S}))_{i \in \underline{3}}\right).$$

The configuration space

$$\check{C} = \left(\check{C}(\mathbb{R}^3, \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}; \Gamma), o(\Gamma)\right)$$

is naturally diffeomorphic to  $] -\infty, \infty[ \times ] -\infty, \infty[$ , where the first factor parametrizes the height of the vertex on the left strand oriented from bottom to top and the second one parametrizes the height of the vertex on the right strand.



The map  $p_{S^2}$  maps  $] -\infty, 0]^2$  and  $] 1, \infty[^2$  to the vertical circle through the horizontal real direction. Therefore, the integral of  $p_{S^2}^*(\omega_{i,S})$  vanishes there, and the integral of  $p_{S^2}^*(\omega_{i,S})$  over  $\check{C}$  is the integral of  $p_{S^2}^*(\omega_{i,S})$  over  $] -\infty, \infty[^2 \setminus (] -\infty, 0[^2 \cup ] 1, \infty[^2)$  or over  $] -\infty, \infty[^2 \setminus ([ -\infty, 0]^2 \cup ] 1, \infty]^2)$ , to which  $p_{S^2}^*(\omega_{i,S})$  extends naturally. The boundary of this domain, which is drawn in Figure 12.2, is mapped to the vertical half circle between the two vertical directions north  $\vec{N}$  and south  $(-\vec{N})$  through the horizontal east direction  $\vec{E}$  towards the right.

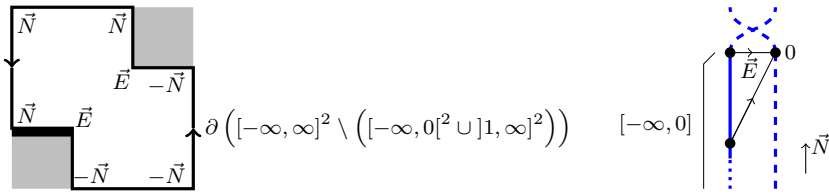


Figure 12.2: Images of boundary points of  $] -\infty, \infty[^2 \setminus (] -\infty, 0[^2 \cup ] 1, \infty[^2)$  under  $p_{S^2}$ . The right-hand side shows the computation for the very thick part  $] -\infty, 0] \times \{0\}$ .

Thus,  $p_{S^2}(\partial C(\mathbb{R}^3, \bigcirc; \Gamma))$  is algebraically trivial (as in the beginning of Section 7.5), and the differential degree of  $p_{S^2}$  is constant on the set of regular values of  $p_{S^2}$ , according to Lemma 2.3. It can be computed as in Subsection 1.2.3 at the vector that points towards the reader. It is equal to one. Thus we have

$$I \left( \bigcirc, \hat{\uparrow}, \hat{\uparrow}, (p_{S^2}^*(\omega_{i,S}))_{i \in \underline{3}} \right) = 1$$

for any arbitrary numbering of the edge of  $\Gamma = \hat{\uparrow}$  and for any choice of volume-one forms  $\omega_{i,S}$  of  $S^2$ . Similarly, for the opposite orientation of the edge of  $\Gamma$ , we have

$$I \left( \bigcirc, \hat{\downarrow}, \hat{\downarrow}, (p_{S^2}^*(\omega_{i,S}))_{i \in \underline{3}} \right) = 1.$$

□

**Definition 12.4.** A *parallelization* of  $\mathcal{C}$  is a parallelization of  $\check{R}(\mathcal{C})$  that agrees with the standard parallelization of  $\mathbb{R}^3$  outside  $\mathcal{C}$ . A *parallelized rational homology cylinder*  $(\mathcal{C}, \tau)$  is a rational homology cylinder equipped with such a parallelization.

The following lemma shows other important examples of computations.

**Lemma 12.5.** *Let  $K: \mathbb{R} \hookrightarrow \check{R}(\mathcal{C})$  be a component of  $L$ . Let  $\tau$  be a parallelization of  $\mathcal{C}$  (which is standard near  $\partial\mathcal{C}$  by Definition 12.4). For any  $i \in \underline{3}$ , let  $\omega(i)$  and  $\omega'(i)$  be propagating forms of  $(C_2(R(\mathcal{C})), \tau)$ , which restrict to  $\partial C_2(R(\mathcal{C}))$  as  $p_\tau^*(\omega(i)_{S^2})$  and  $p_\tau^*(\omega'(i)_{S^2})$ , respectively. Let  $\eta(i)_{S^2}$  be a one-form on  $S^2$  such that  $\omega'(i)_{S^2} = \omega(i)_{S^2} + d\eta(i)_{S^2}$ . Then when  $K$  goes from bottom to top or from top to bottom, we have*

$$\begin{aligned} I\left({}_k\hat{\mathcal{K}}^K, (\omega'(i))_{i \in \underline{3}}\right) - I\left({}_k\hat{\mathcal{K}}^K, (\omega(i))_{i \in \underline{3}}\right) &= \int_{U+K} p_\tau^*(\eta(k)_{S^2}) \\ &= \int_{p_\tau(U+K)} \eta(k)_{S^2}. \end{aligned}$$

When  $K$  goes from bottom to bottom (resp. from top to top), let  $S(K)$  be the half-circle from  $-\vec{N}$  to  $\vec{N}$  (resp. from  $\vec{N}$  to  $-\vec{N}$ ) through the horizontal direction from the initial vertical half-line of  $K$  (the first encountered one) to the final one, then we have

$$I\left({}_k\hat{\mathcal{K}}^K, (\omega'(i))_{i \in \underline{3}}\right) - I\left({}_k\hat{\mathcal{K}}^K, (\omega(i))_{i \in \underline{3}}\right) = \int_{p_\tau(U+K) \cup S(K)} \eta(k)_{S^2}.$$

In particular,  $I({}_k\hat{\mathcal{K}}^K, (\omega(i))_{i \in \underline{3}})$  depends only on  $\omega(k)_{S^2}$ . It is also denoted by  $I(\hat{\mathcal{K}}^K, \omega(k)_{S^2})$ .

PROOF: In any case, the configuration space  $\check{C}(\check{R}(\mathcal{C}), L; \hat{\mathcal{K}}^K)$  is identified naturally with the interior of a triangle, as in the left part of Figure 12.3. When  $K$  goes from bottom to top,  $p_\tau$  extends smoothly to the triangle, as a map that sends the horizontal side and the vertical side to  $\vec{N}$ . Conclude as in Lemmas 7.13 and 7.15. The case in which  $K$  goes from top to bottom is similar.

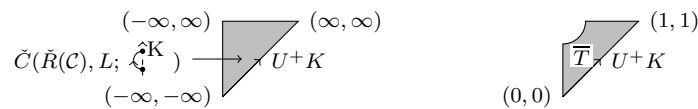


Figure 12.3: Compactifications of configuration spaces for the proof of Lemma 12.5

Let us study the case in which  $K$  goes from top to top. Let  $d_1 = -\{z_1\} \times [1, \infty]$  and  $d_2 = \{z_2\} \times [1, \infty]$  denote the vertical half-lines of  $K$  above  $\mathcal{C}$ , where  $d_1$  is before  $d_2$ . View  $K$  as a path composition  $d_1(K \cap (D_1 \times [0, 1]))d_2$  and parametrize

$$K = d_1 \downarrow \uparrow d_2 \text{ by } m: \begin{aligned} ]0, 1[ &\rightarrow K \\ t \in ]0, 1/3] &\mapsto (z_1, 1/(3t)) \\ t \in [2/3, 1[ &\mapsto (z_2, 1/(3(1-t))). \end{aligned}$$

Let  $T_0 = \{(t_1, t_2) \in ]0, 1[^2 : t_1 < t_2\}$ . We study the integral of  $\omega'(k) - \omega(k) = d\eta(k)$  over  $\check{C}(\check{R}(\mathcal{C}), L; \hat{\zeta}^K) = m^2(T_0)$ . View  $T_0$  as the set

$$T = \left\{ (t_1, t_2, \alpha) \in ]0, 1[^2 \times \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[ : t_1 < t_2, \tan(\alpha) = \frac{1/(3(1-t_2)) - 1/(3t_1)}{|z_2 - z_1|} \right\}$$

denoted by  $T$ . Note that when  $(t_1, t_2) \in ]0, 1/3[\times]2/3, 1[$ , and when  $(t_1, t_2, \alpha) \in T$ , we have

$$p_\tau(m(t_1), m(t_2)) = \cos \alpha \frac{z_2 - z_1}{|z_2 - z_1|} + \sin \alpha \vec{N} \in S(K).$$

Let  $\overline{T}$  be the closure of  $T$  in  $[0, 1]^2 \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ . This closure, drawn in the right part of Figure 12.3, is a smooth blow-up of  $\{(t_1, t_2) \in [0, 1]^2 : t_1 \leq t_2\}$  at  $(0, 1)$  (with corners), where  $(0, 1)$  lifts as  $(0, 1) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$  in  $\overline{T}$ . The map  $m^2$  extends as a smooth map valued in  $C_2(R(\mathcal{C}))$  on  $\overline{T}$ . Its composition with  $p_\tau$  sends the vertical side of  $\overline{T}$  to  $-\vec{N}$ , its horizontal side to  $\vec{N}$ , and the blown-up upper-left corner to  $S(K)$ . The integral of  $d\eta(k)$  over  $\check{C}(\check{R}(\mathcal{C}), L; \hat{\zeta}^K)$  is the integral of  $d\eta(k)$  over  $m^2(\overline{T})$ . So it is the integral of  $\eta(k)$  over  $m^2(\partial\overline{T})$ , where  $\eta(k)$  can be assumed to be equal to  $p_\tau^*(\eta(k)_{S^2})$ , as in Lemma 3.17. Furthermore,  $p_\tau \circ m^2$  restricts to  $\partial\overline{T}$  as a degree one map onto  $p_\tau(U^+K) \cup S(K)$ . So the stated conclusion follows. The case in which  $K$  goes from bottom to bottom can be treated similarly.  $\square$

**Definition 12.6.** Recall the notation of Lemma 12.5. For a long component  $K: \mathbb{R} \hookrightarrow \check{R}(\mathcal{C})$  of a tangle in a parallelized  $\mathbb{Q}$ -cylinder  $(\mathcal{C}, \tau)$ , define

$$I_\theta(K, \tau) = 2I(\hat{\zeta}^K, \omega_{S^2}).$$

Recall the definition of  $I_\theta(K, \tau)$  for a closed component  $K$  of  $\check{R}(\mathcal{C})$  from Lemma 7.15.

The factor 2 in the definition of  $I_\theta$  for long components in Definition 12.6 may seem unnatural. It allows to get homogeneous formulas in Theorem 12.7 below. Theorem 12.7 generalizes Theorem 7.20 to long tangle representatives. It will be proved in Section 14.3.

**Theorem 12.7.** *Let  $\mathcal{C}$  be a rational homology cylinder equipped with a parallelization  $\tau$  (standard near  $\partial\mathcal{C}$ ). Let  $L: \mathcal{L} \hookrightarrow \check{R}(\mathcal{C})$  be a long tangle representative in  $\check{R}(\mathcal{C})$ . Let  $n \in \mathbb{N}$ . For any  $i \in \underline{3n}$ , let  $\omega(i)$  be a homogeneous propagating form of  $(C_2(R(\mathcal{C})), \tau)$ . Set*

$$Z_n(\mathcal{C}, L, (\omega(i))) = \sum_{\Gamma \in \mathcal{D}_n^e(\mathcal{L})} \zeta_\Gamma I(\mathcal{C}, L, \Gamma, (\omega(i))_{i \in \underline{3n}}) [\Gamma] \in \mathcal{A}_n(\mathcal{L}),$$

where  $\zeta_\Gamma = \frac{(3n-|E(\Gamma)|)!}{(3n)!2^{|E(\Gamma)|}}$ , with Definition 7.6 for  $\mathcal{D}_n^e(\mathcal{L})$ . Then  $Z_n(\mathcal{C}, L, (\omega(i)))$  is independent of the chosen  $\omega(i)$ . It depends only on  $(\mathcal{C}, L(\mathcal{L}) \cap \mathcal{C})$  up to diffeomorphisms that fix  $\partial\mathcal{C}$  (and  $L(\mathcal{L}) \cap \partial\mathcal{C}$ ), pointwise, on  $p_1(\tau)$ , and on the  $I_\theta(K_j, \tau)$  for the components  $K_j$ ,  $j \in \underline{k}$ , of  $L$ . We denote it by  $Z_n(\mathcal{C}, L, \tau)$ . Set

$$Z(\mathcal{C}, L, \tau) = (Z_n(\mathcal{C}, L, \tau))_{n \in \mathbb{N}} \in \mathcal{A}(\mathcal{L}),$$

and recall the anomalies  $\alpha \in \check{A}(S^1; \mathbb{R})$  and  $\beta \in \mathcal{A}(\emptyset; \mathbb{R})$  from Sections 10.3 and 10.2. Then the expression

$$\exp\left(-\frac{1}{4}p_1(\tau)\beta\right) \prod_{j=1}^k \left(\exp(-I_\theta(K_j, \tau)\alpha)\#_j\right) Z(\mathcal{C}, L, \tau)$$

depends only on the boundary-preserving diffeomorphism class of  $(\mathcal{C}, L)$ .<sup>2</sup> We denote it by  $\mathcal{Z}(\mathcal{C}, L)$ .

## 12.2 Definition of $\mathcal{Z}^f$ for framed tangles

Recall Definition 5.35 of knot parallels.

**Definition 12.8.** A parallel  $K_\parallel$  of an embedding of a long component  $K: \mathbb{R} \hookrightarrow \check{R}(\mathcal{C})$  parametrized so that  $K(\mathbb{R}) \cap \mathcal{C} = K([0, 1])$  is the image of an embedding  $K_\parallel: \mathbb{R} \hookrightarrow \check{R}(\mathcal{C})$  such that there exists an embedding

$$k: [-1, 1] \times \mathbb{R} \rightarrow R(\mathcal{C}) \setminus (L(\mathcal{L}) \setminus K(\mathbb{R}))$$

such that  $K = k|_{\{0\} \times \mathbb{R}}$ ,  $K_\parallel = k|_{\{1\} \times \mathbb{R}}$  and  $k(u, t) = K(t) + u\varepsilon(t)(1, 0, 0)$  for any  $(u, t) \in [-1, 1] \times (\mathbb{R} \setminus ]0, 1[)$ , for a small smooth function  $\varepsilon: \mathbb{R} \setminus ]0, 1[ \rightarrow \mathbb{R} \setminus \{0\}$  such that  $\varepsilon(0)\varepsilon(1)$  is positive for components going from bottom to top or from top to bottom, and negative for components going from bottom to bottom or from top to top. (We push in one of the two horizontal real directions in a way consistent with the orientation.) Parallels are considered up to isotopies that stay within these parallels and up to the exchange of  $k|_{\{1\} \times \mathbb{R}}$  and  $k|_{\{-1\} \times \mathbb{R}}$ . A component  $K$  of an LTR is *framed* if  $K$  is equipped with such a class of parallels, called a *framing* of  $K$ . An LTR is *framed* if all its components are. The *self-linking number* of a circle component  $K$  in a framed LTR is the linking number  $lk(K, K_\parallel)$  of  $K$  and its parallel  $K_\parallel$ .

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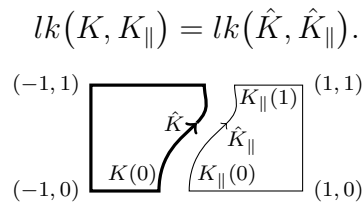
<sup>2</sup>Again, the subscript  $j$  of  $\#_j$  indicates that  $\exp(-I_\theta(K_j, \tau)\alpha)$  is inserted on the component of  $K_j$  of the domain  $\mathcal{L}$  of  $L$ .

For a long component  $K$  equipped with a parallel, we define its *self-linking number*  $lk(K, K_{\parallel})$  in Definitions 12.9 and 12.10 below, and in Definition 13.4, which covers the remaining cases (when  $K$  goes from bottom to bottom or from top to top and when  $K(1) - K(0)$  is not in the direction of the real line).

**Definition 12.9.** When  $K$  goes from bottom to top, and when  $\varepsilon(0)$  is positive, let  $[K_{\parallel}(1), (1, 1)]$  be the straight segment from  $K_{\parallel}(1)$  to  $(1, 1) \in D_1 \times \{1\}$ . Similarly define  $[K(1), (-1, 1)]$ ,  $[(1, 0), K_{\parallel}(0)]$  and  $[(-1, 0), K(0)]$ , and note that they are pairwise disjoint. Define the topological circle embeddings

$$\begin{aligned} \hat{K} &= K([0, 1]) \cup [K(1), (-1, 1)] \cup (-\{-1\} \times [0, 1]) \cup [(-1, 0), K(0)] \\ \hat{K}_{\parallel} &= K_{\parallel}([0, 1]) \cup [K_{\parallel}(1), (1, 1)] \cup (-\{1\} \times [0, 1]) \cup [(1, 0), K_{\parallel}(0)], \end{aligned}$$

as in the figure below,<sup>3</sup> and set



When  $\varepsilon(0)$  is negative, the above definition determines  $lk(K_{\parallel}, K)$ . Then set

$$lk(K, K_{\parallel}) = lk(K_{\parallel}, K).$$

So  $lk(K, K_{\parallel})$  is defined when  $K$  goes from bottom to top. When  $K$  goes from top to bottom,  $(-K)$  goes from bottom to top, and  $-K_{\parallel}$  is a parallel of  $(-K)$ . Then set  $lk(K, K_{\parallel}) = lk(-K, -K_{\parallel})$ .

**Definition 12.10.** When  $K$  goes from bottom to bottom or from top to top, and when  $K(1) - K(0)$  is equal to  $(v, 0, 0)$  for  $v \in \mathbb{R} \setminus \{0\}$ , we define the self-linking number  $lk(K, K_{\parallel})$  as follows. Let us first assume that  $v > 0$  and that  $\varepsilon(0) > 0$ , as in Figure 12.4. In this case, define topological circle embeddings  $\hat{K}_{\parallel} = K_{\parallel}([0, 1]) \cup [K_{\parallel}(1), K_{\parallel}(0)]$ , where  $[K_{\parallel}(1), K_{\parallel}(0)]$  is the straight segment from  $K_{\parallel}(1)$  to  $K_{\parallel}(0)$  in  $D_1 \times \{0\}$  or in  $D_1 \times \{1\}$ , and  $\hat{K} = K([0, 1]) \cup \gamma([0, 1])$  for an arbitrary path  $\gamma$  from  $\gamma(0) = K(1)$  to  $\gamma(1) = K(0)$  such that  $\gamma(]0, 1[) \subset \hat{R}(\mathcal{C}) \setminus \mathcal{C}$  as in Figure 12.4, and set

$$lk(K, K_{\parallel}) = lk(\hat{K}, \hat{K}_{\parallel}).$$

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<sup>3</sup>As we often abusively do, we identify embeddings with their images.

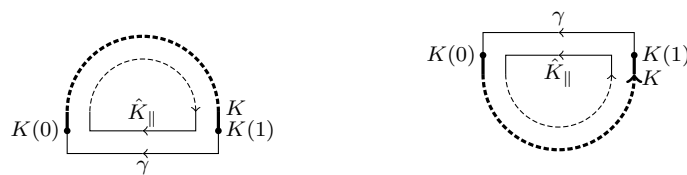


Figure 12.4: Pictures of  $\gamma$  and  $\hat{K}_{\parallel}$

In the other cases, in which  $K$  goes from bottom to bottom or from top to top, and  $K(1) - K(0)$  is equal to  $(v, 0, 0)$  for  $v \in \mathbb{R} \setminus \{0\}$ ,  $lk(K, K_{\parallel})$  is defined so that we have again  $lk(K, K_{\parallel}) = lk(K_{\parallel}, K) = lk(-K, -K_{\parallel})$ .

**Example 12.11.** Let  $K$  be a framed component of an LTR in  $\mathbb{R}^3$ , with a regular projection on  $(\mathbb{R} \subset \mathbb{C}) \times \mathbb{R}$ , and such that  $K(1) - K(0)$  is equal to  $(v, 0, 0)$  for  $v \in \mathbb{R} \setminus \{0\}$  if  $K$  goes from bottom to bottom or from top to top. The *self-linking number* of  $K$  is its *writhe*, which is the *algebraic number* of its crossings, i.e., the number of positive crossings minus the number of negative crossings in its above regular projection. (Check it as an exercise.)

**Definition 12.12.** When a long tangle representative  $L = (K_j)_{j \in \underline{k}}$  is framed by some  $L_{\parallel} = (K_{j\parallel})_{j \in \underline{k}}$ , set

$$\mathcal{Z}^f(\mathcal{C}, (L, L_{\parallel})) = \prod_{j=1}^k \left( \exp(lk(K_j, K_{j\parallel})\alpha) \#_j \right) \mathcal{Z}(\mathcal{C}, L)$$

with the notation of Theorem 12.7.

We will give general relations between  $I_{\theta}$  and self-linking numbers in Section 16.3. See Proposition 16.13.

A *tangle representative* is a pair  $(\mathcal{C}, L(\mathcal{L}) \cap \mathcal{C})$  for a rational homology cylinder  $\mathcal{C}$  and a long tangle representative  $L: \mathcal{L} \hookrightarrow \check{R}(\mathcal{C})$  as in Definition 12.1. Tangle representatives and LTR are in natural one-to-one correspondence, and we also view  $\mathcal{Z}$  as a function of tangle representatives.

A tangle representative  $(\mathcal{C}_1, L_1)$  is *right-composable* by a tangle representative  $(\mathcal{C}_2, L_2)$  when the top configuration of  $(\mathcal{C}_1, L_1)$  coincides with the bottom configuration of  $(\mathcal{C}_2, L_2)$ . In this case, the *product*

$$(\mathcal{C}_1 \mathcal{C}_2, L_1 L_2) = (\mathcal{C}_1, L_1)(\mathcal{C}_2, L_2)$$

is obtained by stacking  $(\mathcal{C}_2, L_2)$  above  $(\mathcal{C}_1, L_1)$ , after affine reparametrizations of  $D_1 \times [0, 1]$ , which becomes  $D_1 \times [0, 1/2]$  for  $(\mathcal{C}_1, L_1)$  and  $D_1 \times [1/2, 1]$  for  $(\mathcal{C}_2, L_2)$ .

The product of two framed tangle representatives is naturally framed. We will prove the following functoriality theorem for  $\mathcal{Z}^f$  in Section 17.2.

**Theorem 12.13.**  $\mathcal{Z}^f$  is functorial: For two framed tangle representatives  $L_1 = (\mathcal{C}_1, L_1)$  and  $L_2 = (\mathcal{C}_2, L_2)$  such that the top configuration of  $L_1$  coincides with the bottom configuration of  $L_2$ , we have

$$\mathcal{Z}^f(L_1 L_2) = \mathcal{Z}^f \left( \begin{array}{|c|} \hline L_2 \\ \hline L_1 \\ \hline \end{array} \right) = \frac{\mathcal{Z}^f(L_2)}{\mathcal{Z}^f(L_1)} = \mathcal{Z}^f(L_1) \mathcal{Z}^f(L_2).$$

The product  $\mathcal{Z}^f(L_1) \mathcal{Z}^f(L_2)$  is the natural product of Section 6.4. When applied to the case in which  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are empty, the above theorem implies that the invariant  $\mathcal{Z}$  of  $\mathbb{Q}$ -spheres is multiplicative under connected sum.

**Remark 12.14.** The multiplicativity of Theorem 12.13 does not hold for the unframed version  $\mathcal{Z}$  of  $\mathcal{Z}^f$ . Indeed, as an unframed tangle, the vertical product



equals  $\cap$ . However, the reader can prove

$$\mathcal{Z} \left( \begin{array}{|c|} \hline \text{loop} \\ \hline \cap \\ \hline \end{array} \right) \mathcal{Z}(\cap) \neq \mathcal{Z}(\cap)$$

as an exercise, using Lemma 12.3 and the behavior of  $\mathcal{Z}^f$  under component orientation reversals described in Theorem 13.12. (Here, the image of any involved boundary planar two-point configuration is  $\{-1/2, 1/2\} \subset \mathbb{C}$ .)

Lemma 13.7 will imply that  $\mathcal{Z}$  is invariant under any isotopy of a tangle representative in a rational homology cylinder during which the bottom and top configurations are constant up to translation and dilation.

**Definition 12.15.** In this book, a *tangle* is an equivalence class of tangle representatives under the equivalence relation that identifies two representatives if and only if they can be obtained from one another by a diffeomorphism  $h$  from the pair  $(\mathcal{C}, L)$  to another such  $(\mathcal{C}, L')$ ,

- which fixes a neighborhood of  $\partial(D_1 \times [0, 1])$  setwise,
- which fixes a neighborhood of  $(\partial D_1) \times [0, 1]$  pointwise,
- such that  $h(c^-(B^-) \times \{0\}) \subset D_1 \times \{0\}$  coincides with  $c^-(B^-) \times \{0\}$  up to translation and dilation (i.e., as a planar configuration of  $\check{\mathcal{S}}_{c^-(B^-)}(\mathbb{C} \times \{0\})$ ), and the classes of  $h(c^+(B^+) \times \{1\}) \subset D_1 \times \{1\}$  and  $c^+(B^+) \times \{1\}$  in  $\check{\mathcal{S}}_{c^+(B^+)}(\mathbb{C} \times \{1\})$  coincide, too,

and which is isotopic to the identity map through such diffeomorphisms.

### 12.3 Defining $\mathcal{Z}^f$ for combinatorial $q$ -tangles

The tangles of Definition 12.15 can be framed by parallels of their components as before to become *framed tangles*. These framed tangles are framed cobordisms (up to isotopy) in  $\mathbb{Q}$ -cylinders from an injective configuration  $c^-(B^-) \in \check{\mathcal{S}}_{c^-(B^-)}(\mathbb{C})$  of points in  $\mathbb{C}$ , up to dilation and translation, to another planar configuration  $c^+(B^+) \in \check{\mathcal{S}}_{c^+(B^+)}(\mathbb{C})$ . Then  $\mathcal{Z}^f$  is an invariant of these framed tangles. In Section 16.5, we extend  $\mathcal{Z}^f$  to framed cobordisms between limit configurations in the compactifications  $\mathcal{S}_{c(B)}(\mathbb{C})$  of the spaces  $\check{\mathcal{S}}_{c(B)}(\mathbb{C})$  introduced in Subsection 8.1.2 and studied in Section 8.3. The category of these limit cobordisms called *q-tangles* is equipped with interesting cabling operations described in Section 13.1. As stated in Theorem 13.12,  $\mathcal{Z}^f$  behaves nicely under these cabling operations. In this section, we define  $\mathcal{Z}^f$  for particular  $q$ -tangles, called *combinatorial q-tangles*, defined below, and we give two examples of cabling properties for these combinatorial  $q$ -tangles, to motivate and introduce our more general presentation of  $q$ -tangles in Section 13.1. Dror Bar-Natan, Thang Lê, and Jun Murakami used these combinatorial  $q$ -tangles to define a functorial extension of the Kontsevich integral of framed oriented links in [BN97] and [LM96].

**Definition 12.16.** A *combinatorial q-tangle* is a triple  $(L, w^-, w^+)$ , where

- $L$  is a framed tangle representative whose bottom and top configurations are on the real line, up to isotopies of  $\mathcal{C}$  that globally preserve the intersection of the bottom disk  $D_1 \times \{0\}$  with  $\mathbb{R} \times \{0\}$  and the intersection of the top disk  $D_1 \times \{1\}$  with  $\mathbb{R} \times \{1\}$ ,
- $w^-$  and  $w^+$  are nonassociative words in the letter  $\bullet$  (as in Example 8.3) such that the letters of  $w^-$  (resp.  $w^+$ ) are in canonical one-to-one correspondence with the elements of the domain  $B^-$  (resp.  $B^+$ ) of the bottom (resp. top) configuration of  $L$ .

The nonassociative words  $w^-$  and  $w^+$  are respectively called the *bottom and top configurations* of the combinatorial  $q$ -tangle  $(L, w^-, w^+)$ . The combinatorial  $q$ -tangle  $(L, w^-, w^+)$  is also called the combinatorial  $q$ -tangle  $L$  from  $w^-$  to  $w^+$ .

**Examples 12.17.** Combinatorial  $q$ -tangles in the standard cylinder are unambiguously represented by one of their regular projections to  $(\mathbb{R} \subset \mathbb{C}) \times [0, 1]$  such that the parallels of their components are parallel in the figures, together with their bottom and top configurations. Examples of these combinatorial  $q$ -tangles include

$$\begin{matrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{matrix}, \begin{matrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{matrix}, \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix}, \text{ and } \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix}.$$



Recall from Example 8.3 that the involved nonassociative words are corners of  $\mathcal{S}_{<,k}(\mathbb{R}) \subset \mathcal{S}_k(\mathbb{C})$ . A combinatorial  $q$ -tangle  $(L, w^-, w^+)$  from a bottom word  $w^-$  to a top word  $w^+$  is thought of as the limit, when  $t$  tends to 0, of framed tangles  $L(t)$  whose bottom and top configurations are the configurations  $w^-(t)$  and  $w^+(t)$  defined in Example 8.3, in the isotopy class of  $L$  with respect to the isotopies of Definition 12.16.

In Theorem 13.8 and Remark 13.11, following Sylvain Poirier [Poi00], we prove that  $\lim_{t \rightarrow 0} \mathcal{Z}^f(L(t))$  exists and that the formula

$$\mathcal{Z}^f(L) = \lim_{t \rightarrow 0} \mathcal{Z}^f(L(t))$$

defines an isotopy invariant  $\mathcal{Z}^f$  of these (framed) combinatorial  $q$ -tangles. The isotopy invariant  $\mathcal{Z}^f$  will thus behave naturally with respect to the deletion of components. If  $L'$  is a subtangle of a tangle  $L$  with domain  $\mathcal{L}'$ , then  $\mathcal{Z}^f(L')$  will be obtained from  $\mathcal{Z}^f(L)$  by forgetting  $\mathcal{L} \setminus \mathcal{L}'$  and all the diagrams with univalent vertices on  $\mathcal{L} \setminus \mathcal{L}'$ .

As a first example, let us compute  $\mathcal{Z}^f$  for the combinatorial  $q$ -tangle  $\uparrow\uparrow$ .

**Lemma 12.18.** *We have*

$$\mathcal{Z}^f(\uparrow\uparrow) = 1 = [\uparrow\uparrow].$$

PROOF: By definition, the left-hand side is the limit, when  $t$  tends to 0, of the evaluation of  $Z$  of the LTR whose image is  $\{0, t\} \times \mathbb{R}$ . There is an action of  $\mathbb{R}$  by vertical translation on the involved configuration spaces. The integrated forms factor through the quotients by this action of  $\mathbb{R}$  of the configuration spaces, whose dimensions are smaller (by one) than the degrees of the integrated forms. So the integrals vanish for all nonempty diagrams.  $\square$

As a second example, we compute  $\mathcal{Z}_{\leq 1}^f$ , which is the truncation of  $\mathcal{Z}^f$  in degrees lower than 2, for the combinatorial  $q$ -tangle  $\uparrow\downarrow$ .

**Lemma 12.19.** *We have*

$$\mathcal{Z}_{\leq 1}^f \left( \uparrow\downarrow \right) = \left[ \begin{array}{c} \uparrow \quad \uparrow \\ \vdots \quad \vdots \\ \vdots \quad \vdots \end{array} \right] + \left[ \begin{array}{c} \uparrow \quad \uparrow \\ \bullet \quad \bullet \\ \vdots \quad \vdots \end{array} \right].$$

PROOF: Since  $\mathcal{Z}$  is invariant under the isotopies that preserve the bottom and the top configurations, the contributions of the nonempty diagrams whose univalent vertices are on one strand of the tangle vanish. So we are

left with the contribution of the numbered graphs with one vertex on each strand, treated in Lemma 12.3.  $\square$

The obtained invariant  $\mathcal{Z}^f$  is still multiplicative under vertical composition as in Theorem 12.13, and we can now define other interesting operations.

For two combinatorial  $q$ -tangles  $L_1 = (\mathcal{C}_1, L_1)$  from  $w_1^-$  to  $w_1^+$ , and  $L_2 = (\mathcal{C}_2, L_2)$  from  $w_2^-$  to  $w_2^+$ , define the product  $L_1 \otimes L_2$ , from the bottom configuration  $w_1^- w_2^-$  to the top configuration  $w_1^+ w_2^+$ , by shrinking  $\mathcal{C}_1$  and  $\mathcal{C}_2$  to make them respectively replace the products by  $[0, 1]$  of the horizontal disks with radius  $\frac{1}{4}$  and respective centers  $-\frac{1}{2}$  and  $\frac{1}{2}$ .

**Examples 12.20.** We have

$$\begin{array}{c} (\bullet) \\ \diagup \diagdown \\ (\bullet) \end{array} \otimes \begin{array}{c} (\bullet) \\ \diagup \diagdown \\ (\bullet) \end{array} = \begin{array}{c} (\bullet) \\ \diagup \diagdown \\ (\bullet) \end{array} \begin{array}{c} (\bullet) \\ \diagup \diagdown \\ (\bullet) \end{array}, \text{ and } \begin{array}{c} (\bullet) \\ \diagup \diagdown \\ (\bullet) \end{array} \otimes \begin{array}{c} (\bullet) \\ \diagup \diagdown \\ (\bullet) \end{array} = \begin{array}{c} (\bullet) \\ \diagup \diagdown \\ (\bullet) \end{array} \begin{array}{c} (\bullet) \\ \diagup \diagdown \\ (\bullet) \end{array}.$$

We can easily deduce the following theorem from the cabling property and the functoriality property of Theorem 13.12.

**Theorem 12.21.**  $\mathcal{Z}^f$  is monoidal: For two combinatorial  $q$ -tangles  $L_1$  and  $L_2$ , we have

$$\mathcal{Z}^f(L_1 \otimes L_2) = \mathcal{Z}^f\left(\begin{array}{|c|c|} \hline L_1 & L_2 \\ \hline \end{array}\right) = \begin{array}{|c|c|} \hline \mathcal{Z}^f(L_1) & \mathcal{Z}^f(L_2) \\ \hline \end{array} = \mathcal{Z}^f(L_1) \otimes \mathcal{Z}^f(L_2),$$

where  $\mathcal{Z}^f(L_1) \otimes \mathcal{Z}^f(L_2)$  denotes the image of  $\mathcal{Z}^f(L_1) \otimes \mathcal{Z}^f(L_2)$  under the natural product from  $\mathcal{A}(\mathcal{L}_1) \otimes_{\mathbb{R}} \mathcal{A}(\mathcal{L}_2)$  to  $\mathcal{A}(\mathcal{L}_1 \sqcup \mathcal{L}_2)$  induced by the disjoint union of diagrams.

We can also *double* a component  $K$  according to its parallelization in a combinatorial  $q$ -tangle  $L$ . This operation replaces a component by two parallel components, and if this component has boundary points, it replaces the corresponding letters in the nonassociative words with  $(\bullet)$ . The obtained combinatorial  $q$ -tangle is denoted by  $L(2 \times K)$ . For example, we have

$$\begin{array}{c} (\bullet) \\ \diagup \diagdown \\ (\bullet) \end{array} (2 \times \nearrow) = \begin{array}{c} (\bullet) \\ \diagup \diagdown \\ (\bullet) \end{array}$$

The following duplication property for  $\mathcal{Z}^f$  is a part of Theorem 13.12, which is proved in Section 17.4.

**Theorem 12.22.** Let  $K$  be a component of a combinatorial  $q$ -tangle  $L$ .<sup>4</sup> Recall Notation 6.31. Then we have

$$\mathcal{Z}^f(L(2 \times K)) = \pi(2 \times K)^* \mathcal{Z}^f(L).$$

These results are some particular cases of the properties of  $\mathcal{Z}^f$ , which are listed in Theorem 13.12 and proved in Chapter 17.

<sup>4</sup>The theorem holds for components going from bottom to bottom or from top to top. It would not hold in this generality if  $\mathcal{Z}^f$  were replaced by certain functorial extensions of the Kontsevich integral.

## 12.4 Good monoidal functors for combinatorial $q$ -tangles

Recall Notation 6.19. In this section, all combinatorial  $q$ -tangles are combinatorial  $q$ -tangles in  $\mathbb{R}^3$  or in the standard cylinder, and coefficients for spaces of Jacobi diagrams are in  $\mathbb{K} = \mathbb{C}$ . In [Poi00], Sylvain Poirier extended the natural projection  $\check{\mathcal{Z}}^f(\mathbb{R}^3, \cdot)$  of  $\mathcal{Z}^f(\mathbb{R}^3, \cdot)$  in  $\check{\mathcal{A}}(\cdot; \mathbb{R}) \subset \check{\mathcal{A}}(\cdot; \mathbb{C})$ , from framed links of  $\mathbb{R}^3$  to these combinatorial  $q$ -tangles, in an elegant way, and he proved that his extension  $Z^l$  is a *good monoidal functor* with respect to the definition below. Good monoidal functors on the category of combinatorial  $q$ -tangles (in  $\mathbb{R}^3$ ) are characterized in [Les02]. This section reviews these results of [Poi00] and [Les02]. The quoted results of Sylvain Poirier will be reproved (with much more details) and generalized in this book. In contrast, the proofs of the results of [Les02] will not be reproduced in this book since they do not involve analysis on configuration spaces.

**Definition 12.23.** A *good monoidal functor* from the category of combinatorial  $q$ -tangles (in  $\mathbb{R}^3$ ) to the category of spaces of Jacobi diagrams is a map  $Y$  sending a combinatorial  $q$ -tangle  $L$  with domain  $\mathcal{L}$  to an element of the space  $\check{\mathcal{A}}(\mathcal{L}; \mathbb{C})$  of Notation 6.19 with the following properties.

- For any combinatorial  $q$ -tangle  $L$ , the degree zero part  $Y_0(L)$  of  $Y(L)$  is 1, which is the class of the empty diagram.
- $Y$  is *functorial*: For two combinatorial  $q$ -tangles  $L_1$  and  $L_2$  such that the top configuration  $\text{top}(L_1)$  of  $L_1$  coincides with the bottom configuration  $\text{bot}(L_2)$  of  $L_2$ , we have

$$Y(L_1 L_2) = Y \left( \begin{array}{|c|} \hline L_2 \\ \hline L_1 \\ \hline \end{array} \right) = \begin{array}{|c|} \hline Y(L_2) \\ \hline Y(L_1) \\ \hline \end{array} = Y(L_1) Y(L_2).$$

- $Y$  is *monoidal*: For two combinatorial  $q$ -tangles  $L_1$  and  $L_2$ , we have

$$Y(L_1 \otimes L_2) = Y \left( \begin{array}{|c|c|} \hline L_1 & L_2 \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline Y(L_1) & Y(L_2) \\ \hline \end{array} = Y(L_1) \otimes Y(L_2).$$

- $Y$  is *compatible with the deletion* of a component: If  $L'$  is a subtangle of  $L$  with domain  $\mathcal{L}'$ , then  $Y(L')$  is obtained from  $Y(L)$  by forgetting  $\mathcal{L} \setminus \mathcal{L}'$  and all the diagrams with univalent vertices on  $\mathcal{L} \setminus \mathcal{L}'$ .

- $Y$  is compatible with the duplication of a regular component, which is a component that can be represented without horizontal tangent vectors: For such a component  $K$  of a combinatorial  $q$ -tangle  $L$ , we have

$$Y(L(2 \times K)) = \pi(2 \times K)^*Y(L)$$

with respect to Notation 6.31.

- $Y$  is invariant under the 180-degree rotation around a vertical axis through the real line.
- Let  $s_{\frac{1}{2}}$  be the reflection with respect to the horizontal plane at height  $\frac{1}{2}$ . Let  $\sigma_{\frac{1}{2}}$  be the linear endomorphism of the topological vector space  $\check{\mathcal{A}}(S^1)$  such that  $\sigma_{\frac{1}{2}}([\Gamma]) = (-1)^d[\Gamma]$ , for any element  $[\Gamma]$  of  $\check{\mathcal{A}}_d(S^1)$ . For any framed knot  $K = K(S^1)$ ,  $Y(K)$  belongs to  $\check{\mathcal{A}}(S^1; \mathbb{R})$  and we have

$$Y \circ s_{\frac{1}{2}}(K) = \sigma_{\frac{1}{2}} \circ Y(K).$$

- $Y$  behaves as in Proposition 10.23 with respect to reversals of component orientations. It can be defined as an invariant of unoriented tangles valued in a space of diagrams whose support is the unoriented domain of the tangle, as in Definitions 6.13 and 6.16.
- The degree one part  $a_1^Y$  of the element  $a^Y \in \check{\mathcal{A}}([0, 1])$  such that  $a_0^Y = 0$  and

$$Y \left( \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right) = \exp(a^Y)Y(\cap)$$

is

$$a_1^Y = \frac{1}{2} \left[ \hat{\mathcal{C}} \right].$$

As shown in [Les02, Proposition 4.2, 10], a good monoidal functor  $Y$  is determined by its values

$$Y \left( \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \right), Y \left( \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \right), \text{ and } Y \left( \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \right).$$

Moreover,  $Y$  can be computed combinatorially from these three ingredients. See also [LM96, BN97]. The value  $Y \left( \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \right)$  is called an *associator*.

In [LM96], Thang Lê and Jun Murakami constructed the first example of such a good monoidal functor from the Kontsevich integral of links in  $\mathbb{R}^3$  described in [CD01].<sup>5</sup> See also [Les99]. We denote the Lê–Murakami functor

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<sup>5</sup>They also explain [LM96, Section 5] how to construct such a functor combinatorially, from an element of  $\mathcal{A}(\| \|)$  satisfying some equations [LM96, Section 3, A1–A4], which allow it to be the associator of a good monoidal functor. See also [BN97].

by  $Z^K$  and call it the *Kontsevich integral* of combinatorial  $q$ -tangles. This Kontsevich integral furthermore satisfies

$$Z^K \left( \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right) = \exp \left( \begin{array}{c} \hat{\curvearrowright} \\ \hat{\curvearrowleft} \end{array} \right) = \begin{array}{c} \hat{\curvearrowright} \\ \hat{\curvearrowleft} \end{array} + \left[ \begin{array}{c} \hat{\curvearrowright} \\ \hat{\curvearrowleft} \end{array} \right] + \frac{1}{2} \left[ \begin{array}{c} \hat{\curvearrowright} \\ \hat{\curvearrowleft} \end{array} \right] + \frac{1}{6} \left[ \begin{array}{c} \hat{\curvearrowright} \\ \hat{\curvearrowleft} \end{array} \right] + \dots$$

This easily implies that the element  $a^{Z^K}$  of the above definition is

$$a^{Z^K} = \frac{1}{2} \left[ \begin{array}{c} \hat{\curvearrowright} \\ \hat{\curvearrowleft} \end{array} \right].$$

So it vanishes in all degrees greater than one.

In [Poi00], Sylvain Poirier extended  $\check{Z}^f(\mathbb{R}^3, \cdot)$  from framed links of  $\mathbb{R}^3$  to combinatorial  $q$ -tangles of  $\mathbb{R}^3$  and he proved that his extension  $Z^l$  satisfies the above properties with

$$a^{Z^l} = \alpha,$$

where  $\alpha$  is the anomaly of Section 10.3.

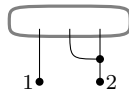
**Remark 12.24.** The published version [Poi02] of [Poi00] does not contain the cited important results of [Poi00], which will be generalized and proved with much more details in the present book.

**Definition 12.25.** Say that an element  $\gamma = (\gamma_n)_{n \in \mathbb{N}}$  in  $\check{\mathcal{A}}([0, 1])$  is a *two-leg element* if, for any  $n \in \mathbb{N}$ ,  $\gamma_n$  is a combination of diagrams with two univalent vertices.

Forgetting  $[0, 1]$  from such a two-leg element gives rise to a unique series  $\gamma^s$  of diagrams with two distinguished univalent vertices  $v_1$  and  $v_2$ , such that  $\gamma^s$  is symmetric with respect to the exchange of  $v_1$  and  $v_2$ , according to the following lemma due to Pierre Vogel. See [Vog11, Corollary 4.2].

**Lemma 12.26** (Vogel). *Two-leg Jacobi diagrams are symmetric with respect to the exchange of their two legs in a diagram space quotiented by the AS and Jacobi relations.*

PROOF: Since a chord is obviously symmetric, we can restrict ourselves to a two-leg diagram with at least one trivalent vertex, and whose two univalent vertices are respectively numbered by 1 and 2. We draw it as



where we do not represent the trivalent part inside the disk bounded by the thick gray topological circle. Applying Lemma 6.23, when the annulus



**Note 12.30.** Theorem 12.28 would not hold if framed links were replaced by arbitrary tangles. For pure combinatorial  $q$ -tangles from a nonassociative word to itself, whose components go from top to bottom or from bottom to top and connect a letter of the bottom word to the corresponding letter of the top word, it is true, up to conjugation by a *twisting function* of nonassociative words, introduced by Thang Lê and Jun Murakami in [LM96, Section 7]. A generalization of Theorem 12.28 to all combinatorial  $q$ -tangles whose components go from top to bottom or from bottom to top is given in [Les02, Theorem 8.5]. It involves the Lê–Murakami twisting functions.

# Chapter 13

## More on the functor $\mathcal{Z}^f$

In this chapter, we state our general Theorem 13.12, which ensures that  $\mathcal{Z}^f$  is a functor behaving naturally with respect to various structures of the category of  $q$ -tangles, such as cabling or duplication. We first describe the category of  $q$ -tangles in Section 13.1 before stating Theorem 13.12 in Section 13.3.

We describe the main steps of the generalization of the construction of  $\mathcal{Z}^f$  to  $q$ -tangles in Section 13.2. We will give the details of these steps in Chapters 14, 15, and 16. We will finish the proof of Theorem 13.12 in Chapter 17.

### 13.1 Tangles and $q$ -tangles

Recall that a tangle representative is a pair  $(\mathcal{C}, L(\mathcal{L}) \cap \mathcal{C})$  for a rational homology cylinder  $\mathcal{C}$  and a long tangle representative  $L: \mathcal{L} \hookrightarrow \check{R}(\mathcal{C})$  as in Definition 12.1.

**Definition 13.1.** In this book, a *braid representative* is a tangle representative  $T(\tilde{\gamma})$  of the standard cylinder  $D_1 \times [0, 1]$  whose components called *strands* may be expressed as

$$\left\{ (\tilde{\gamma}_b(t), t) : t \in [0, 1] \right\},$$

for an element  $b$  of a finite set  $B$ , which labels the strands. In the above expression  $\tilde{\gamma}_b: [0, 1] \rightarrow \mathring{D}_1$  is a path, and, for any  $t \in [0, 1]$  and any pair  $(b, b')$  of distinct elements of  $B$ , we have  $\tilde{\gamma}_b(t) \neq \tilde{\gamma}_{b'}(t)$  as in Figure 13.1. Such a braid representative can be viewed naturally as a path  $\tilde{\gamma}: [0, 1] \rightarrow \check{C}_B[\mathring{D}_1]$ , where  $\check{C}_B[\mathring{D}_1]$  is the space of injections of  $B$  into  $\mathring{D}_1$ , defined in the beginning of Section 8.6.



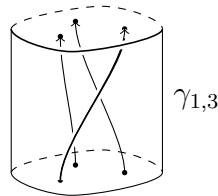


Figure 13.1: A braid representative with three strands

In this book, a *braid* (resp. a *q-braid*) is a homotopy class of paths  $\gamma: [0, 1] \rightarrow \check{\mathcal{S}}_B(\mathbb{C})$  (resp. of paths  $\gamma: [0, 1] \rightarrow \mathcal{S}_B(\mathbb{C})$ ) for some finite set  $B$ , where  $\mathcal{S}_B(\mathbb{C})$  is the compactification of  $\check{\mathcal{S}}_B(\mathbb{C})$  described in Theorem 8.11. A braid  $\gamma$  induces the tangle  $T(\gamma)$ , which is also called a braid, as above. The path  $\bar{\gamma}$  is the path such that  $\bar{\gamma}(t) = \gamma(1 - t)$ . A braid is naturally framed by the parallels obtained by pushing it in the direction of the real line of  $\mathbb{C}$ .

Tangles, as in Definition 12.15, can be multiplied if they have representatives that can be, i.e., if the top configuration of the first tangle agrees with the bottom configuration of the second one, up to dilation and translation. The product is associative. Framed tangles multiply vertically to give rise to framed tangles.

A *q-tangle* is a framed tangle whose bottom and top configurations are allowed to be limit configurations in some  $\mathcal{S}_{B^-}(\mathbb{C})$  and in some  $\mathcal{S}_{B^+}(\mathbb{C})$ . More precisely, a *q-tangle* is represented by a product

$$T(\gamma^-)(\mathcal{C}, L)T(\gamma^+),$$

as in Figure 13.2, where  $\gamma^-$  and  $\gamma^+$  are *q*-braids,  $(\mathcal{C}, L)$  is a framed tangle whose bottom configuration is  $\gamma^-(1)$  and whose top configuration is  $\gamma^+(0)$ , and the strands of  $T(\gamma^-)$  and  $T(\gamma^+)$  get their orientations from the orientation of  $L$ . For consistency, we allow braids with 0 or 1 strand, and we agree that  $\mathcal{S}_\emptyset(\mathbb{C})$  and  $\mathcal{S}_{\{b\}}(\mathbb{C})$  each have one element, which is the unique configuration of one point in  $\mathbb{C}$  up to translation in the latter case. Note that the restriction of a *q-tangle* to one of its components is a framed tangle since configurations of at most two points are always injective. The components of a *q-tangle* representative are framed since braids are.

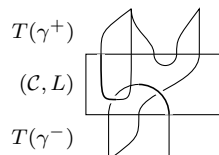


Figure 13.2: A *q-tangle* representative

Now,  $q$ -tangles are classes of these representatives under the equivalence relation that identifies  $T(\gamma^-)(\mathcal{C}, L)T(\gamma^+)$  with  $T(\gamma'^-)(\mathcal{C}', L')T(\gamma'^+)$  if and only if  $\gamma^-(0) = \gamma'^-(0)$ ,  $\gamma^+(1) = \gamma'^+(1)$ , and the framed tangles  $(\mathcal{C}, L)$  and  $T(\alpha)(\mathcal{C}', L')T(\beta)$  represent the same framed tangle for any braids  $\alpha$  and  $\beta$  such that

- the composition  $T(\alpha)(\mathcal{C}', L')T(\beta)$  is well-defined,
- the path  $\alpha$  of  $\check{\mathcal{S}}_{B^-}(\mathbb{C})$  is homotopic to  $\overline{\gamma'^-}\gamma'^-$  in  $\mathcal{S}_{B^-}(\mathbb{C})$ , and
- the path  $\beta$  of  $\check{\mathcal{S}}_{B^+}(\mathbb{C})$  is homotopic to  $\gamma'^+\overline{\gamma'^+}$  in  $\mathcal{S}_{B^+}(\mathbb{C})$  (by homotopies that fix the boundary points).

The *domain* of a  $q$ -tangle (represented by)  $T(\gamma^-)(\mathcal{C}, L)T(\gamma^+)$  is (identified with) the domain  $\mathcal{L}$  of  $L: \mathcal{L} \hookrightarrow \mathcal{C}$ , its *bottom configuration* is  $\gamma^-(0)$ , and its *top configuration* is  $\gamma^+(1)$ .

**Examples 13.2.** A *combinatorial configuration* is a nonassociative word in the letter  $\bullet$ . The combinatorial  $q$ -tangles  $(L, w^-, w^+)$  of Definition 12.16 are examples of  $q$ -tangles. They are the  $q$ -tangles of the standard cylinder whose bottom and top configurations are combinatorial. Recall the notation from Example 8.3. A combinatorial  $q$ -tangle  $(L, w^-, w^+)$  may be represented as  $T(\gamma^-)(\mathcal{C}, L(1/4))T(\gamma^+)$  for a  $q$ -braid  $\gamma^-$  from  $\gamma^-(0) = w^-$  to  $\gamma^-(1) = w^-(1/4)$  such that  $\gamma^-([0, 1])$  is  $w^-([0, 1/4])$ , a  $q$ -braid  $\gamma^+$  from  $\gamma^+(0) = w^+(1/4)$  to  $\gamma^+(1) = w^+$  such that the sets  $\gamma^+([0, 1])$  and  $w^+([0, 1/4])$  are equal, and a representative  $L(1/4)$  of  $L$  from  $w^-(1/4)$  to  $w^+(1/4)$  in the isotopy class of  $L$  of Definition 12.16.

We may omit the external pair of parentheses in a combinatorial configuration since it is always present. We may similarly omit the only possible two-point combinatorial configuration  $(\bullet\bullet)$  from the notation in combinatorial  $q$ -tangles. So we may represent the examples of Example 12.17 by

$$\bowtie, \begin{array}{c} \bullet(\bullet\bullet) \\ \uparrow \\ \bullet\bullet \end{array}, \cup, \text{ and } \cap.$$

When the boundary of a  $q$ -tangle is empty, the  $q$ -tangle is a framed link in  $\check{R}(\mathcal{C})$ . Conversely, any asymptotically standard  $\mathbb{Q}$ -homology  $\mathbb{R}^3$  equipped with a framed link may be obtained in this way, up to diffeomorphism.

In addition to the (vertical) product, which extends to  $q$ -tangles naturally,  $q$ -tangles support a *cabling* operation of a component  $K$  of a  $q$ -tangle  $T_m$  (the main tangle) by a  $q$ -tangle  $T_i$  (the inserted tangle), which produces a  $q$ -tangle  $T_m(T_i/K)$ . This operation roughly consists in replacing the strand  $K$  in  $T_m$  by a tangle  $T_i$  with respect to the framing of  $K$ , as in Figure 13.3.

For example, if  $T_i$  is the trivial braid  $|_1|_2$  with two strands and if  $T_m$  is a combinatorial  $q$ -tangle, then  $T_m(T_i/K)$  is the tangle  $T_m(2 \times K)$  described before Theorem 12.22. As another example, the product  $L_1 \otimes L_2$  described before Example 12.20 can be written as  $(L_1 \otimes |_2)(L_2/|_2)$ , where  $L_1 \otimes |_2 = (|_1|_2)(L_1/|_1)$ .

Let us describe the cabling operation in general in (lengthy) detail.<sup>1</sup>

**Notation 13.3.** A *semi-pure  $q$ -tangle* is a  $q$ -tangle with identical bottom and top configuration (up to dilation and translation). A *pure  $q$ -tangle* is a semi-pure  $q$ -tangle whose interval components connect a bottom configuration point to the top configuration's corresponding point. The cabling operation that produces  $T_m(T_i/K)$  is defined for any pair  $(T_m, T_i)$  of  $q$ -tangles equipped with an interval component  $K$  of  $T_m$  going from bottom to top. It is also defined for any pair  $(T_m, T_i)$  of  $q$ -tangles equipped with a framed circle component  $K$  of  $T_m$  provided that  $T_i$  is semi-pure.

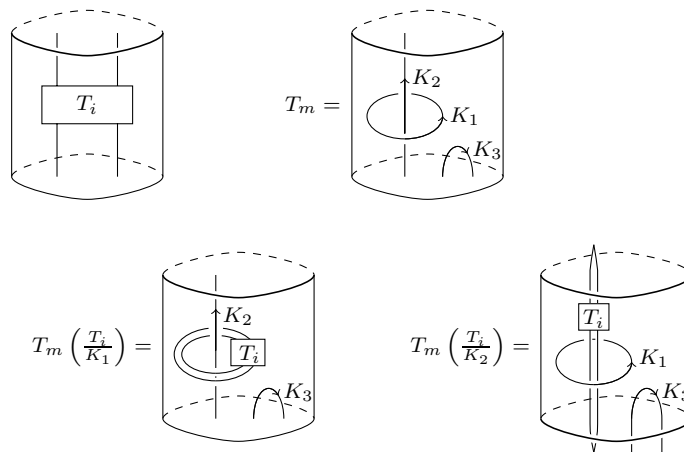


Figure 13.3: Examples of cablings

We begin with the details when  $K$  is a circle, because they are lighter in this case. When  $K$  is a closed component and  $T_i$  is a framed tangle  $(\mathcal{C}_i, L_i)$  with identical injective bottom and top configuration, pick a tubular neighborhood  $D^2 \times K$  of  $K$ , trivialized with respect to the parallelization of  $K$ , that does not meet the other components of  $T_m$ . Write this neighborhood as  $D^2 \times [0, 1]/(0 \sim 1)$ , and replace it by  $(\mathcal{C}_i, L_i)$  using the identification of  $N(\partial(D_1 \times [0, 1]))$  with a neighborhood of  $\partial(D_1 \times [0, 1])$  in order to obtain  $T_m(T_i/K)$ . Note that when  $\gamma$  is a braid such that  $\gamma(1)$  is the bottom

<sup>1</sup>Notation 13.3 ends 4 lines before Definition 13.4.

configuration of  $T_i$ , we have

$$T_m \left( \frac{T_i}{K} \right) = T_m \left( \frac{T(\gamma)T_iT(\bar{\gamma})}{K} \right).$$

Any semi-pure  $q$ -tangle  $T_q$  can be written as  $T(\gamma)(\mathcal{C}_i, L_i)T(\bar{\gamma})$ , for some  $q$ -braid  $\gamma$  and some framed tangle  $(\mathcal{C}_i, L_i)$ . For such a tangle, set  $T_m(T_q/K) = T_m((\mathcal{C}_i, L_i)/K)$ . It is easy to check that this definition is consistent.

Let us now define *cabling* or *duplication* for configurations. Let  $B$  and  $B_i$  be nonempty finite sets, let  $b \in B$ , and let  $B(B_i/b) = (B \setminus \{b\}) \cup B_i$ . Let  $c_m$  be an element of  $\mathcal{S}_B(\mathbb{C})$ , and let  $c_i$  be an element of  $\mathcal{S}_{B_i}(\mathbb{C})$ . The configuration  $c_m(c_i/b)$  of  $\mathcal{S}_{B(B_i/b)}(\mathbb{C})$  is the configuration obtained by letting  $c_i$  replace  $b$ . Up to translation, there is only one configuration of a set of one element. So  $c_m(c_i/b)$  is this unique configuration if  $|B(B_i/b)| = 1$ . If  $|B_i| = 1$ , then  $c_m(c_i/b) = c_m$ . If  $|B| = 1$ , then  $c_m(c_i/b) = c_i$  (with natural identifications). Assume that  $|B_i| \geq 2$  and  $|B| \geq 2$ . When  $c_m$  and  $c_i$  are both combinatorial configurations, it makes natural sense to let  $c_i$  replace  $b$  to produce  $c_m(c_i/b)$ . In general, recall Definition 8.24 and define the  $\Delta$ -parenthesization  $\tau(c_m)(\tau(c_i)/b)$  of  $B(B_i/b)$  from the respective  $\Delta$ -parenthesizations  $\tau(c_m)$  and  $\tau(c_i)$  of  $c_m$  and  $c_i$  by the following one-to-one correspondence

$$\begin{aligned} \phi: \quad \tau(c_m) \sqcup \tau(c_i) &\rightarrow \tau(c_m)(\tau(c_i)/b) \\ A &\mapsto \begin{cases} A & \text{if } A \in \tau(c_i) \text{ or } (A \in \tau(c_m) \text{ and } b \notin A) \\ A(B_i/b) & \text{if } A \in \tau(c_m) \text{ and } b \in A. \end{cases} \end{aligned}$$

With the notation of Theorem 8.28, the configuration  $c_m(c_i/b)$  is the configuration of  $\mathcal{S}_{B(B_i/b), \tau(c_m)(\tau(c_i)/b)}(\mathbb{C})$  that restricts to  $B_i$  as  $c_i$ , and to  $B(\{b'\}/b)$  as  $c_m$  for any element  $b'$  of  $B_i$ .

Assume that  $K$  is an interval component going from bottom to top of a  $q$ -tangle

$$T_m = T(\gamma_m^-)(\mathcal{C}_m, L_m)T(\gamma_m^+),$$

and define  $T_m(T_i/K)$  for a  $q$ -tangle  $T_i = T(\gamma_i^-)(\mathcal{C}_i, L_i)T(\gamma_i^+)$ . Let  $B_i^-$ , (resp.  $B_i^+$ ,  $B^-, B^+$ ) denote the set of strand indices of  $\gamma_i^-$  (resp.  $\gamma_i^+$ ,  $\gamma_m^-, \gamma_m^+$ ). Let  $b_K^-$  (resp.  $b_K^+$ ) denote the strand index of  $K$  in  $B^-$  (resp.  $B^+$ ). Let  $c_m^-$  (resp.  $c_i^-$ ) denote the bottom configuration of  $T_m$  (resp.  $T_i$ ). Let  $c_m^+$  (resp.  $c_i^+$ ) denote the top configuration of  $T_m$  (resp.  $T_i$ ). Assume that  $\gamma_m^-([0, 1]) \subset \check{\mathcal{S}}_{B^-}(\mathbb{C})$ ,  $\gamma_i^-([0, 1]) \subset \check{\mathcal{S}}_{B_i^-}(\mathbb{C})$ ,  $\gamma_m^+([0, 1]) \subset \check{\mathcal{S}}_{B^+}(\mathbb{C})$ , and  $\gamma_i^+([0, 1]) \subset \check{\mathcal{S}}_{B_i^+}(\mathbb{C})$ . Let  $I_K$  denote the intersection of  $K$  with  $\mathcal{C}_m$ . Identify  $I_K$  with  $[0, 1]$ . Let  $D^{(i)}$  be a copy of the disk  $D^2$ . Let  $D^{(i)} \times [0, 1]$  be a tubular neighborhood of  $I_K$  in  $\mathcal{C}_m$  that does not meet the other components of  $L_m$  and that meets  $\partial\mathcal{C}_m$  along  $D^{(i)} \times \partial[0, 1]$  inside  $D_1 \times \partial[0, 1]$ , such that  $(\{\pm 1\} \times [0, 1]) \subset (\partial D^{(i)} \times [0, 1])$

is the given parallel of  $I_K$ . Replace  $D^{(i)} \times [0, 1]$  by  $(\mathcal{C}_i, L_i)$  in order to get a tangle  $(\mathcal{C}_m, L_m) ((\mathcal{C}_i, L_i)/I_K)$ .

Let  $\gamma_m^-(\gamma_i^-/K)$  be the path composition  $\gamma_m^-(c_i^-/K) (\gamma_m^-(1)(\gamma_i^-/K))$  of the paths  $\gamma_m^-(c_i^-/K)$  and  $\gamma_m^-(1)(\gamma_i^-/K)$  in  $\mathcal{S}_{B^-(B_i^-/b_K^-)}(\mathbb{C})$ , where  $\gamma_m^-(c_i^-/K)(t) = \gamma_m^-(t)(c_i^-/b_K^-)$  for any  $t \in [0, 1]$ , and the restriction to  $]0, 1]$  of  $\gamma_m^-(1)(\gamma_i^-/K)$  is represented by a map from  $]0, 1]$  to  $\check{C}_{B^-(B_i^-/b_K^-)}[\mathring{D}_1]$ ,

- which maps 1 to the bottom configuration of  $(\mathcal{C}_m, L_m) ((\mathcal{C}_i, L_i)/I_K)$ ,
- whose restriction to  $B^- \setminus \{b_K^-\}$  is constant,<sup>2</sup> and,
- whose restriction to  $B_i^-$  is a lift of  $\gamma_i^-|_{]0,1]}$  in  $\check{C}_{B_i^-}[p_{\mathbb{C}}(D^{(i)} \times \{0\})]$ , such that  $\gamma_m^-(1)(\gamma_i^-/K)$  is composable by  $\gamma_m^-(c_i^-/K)$  on its left.

The tangle  $T_m(T_i/K)$  is defined to be

$$T(\gamma_m^-(\gamma_i^-/K)) \left( (\mathcal{C}_m, L_m) ((\mathcal{C}_i, L_i)/I_K) \right) T(\gamma_m^+(\gamma_i^+/K)),$$

where the definition of  $T(\gamma_m^+(\gamma_i^+/K))$ , similar to that of  $T(\gamma_m^-(\gamma_i^-/K))$ , follows (and can be skipped...).

The path  $\gamma_m^+(\gamma_i^+/K)$  is the path composition  $(\gamma_m^+(0)(\gamma_i^+/K)) \gamma_m^+(c_i^+/K)$  in  $\mathcal{S}_{B^+(B_i^+/b_K^+)}(\mathbb{C})$ , where  $\gamma_m^+(c_i^+/K)(t) = \gamma_m^+(t)(c_i^+/b_K^+)$  for any  $t \in [0, 1]$ , and the restriction to  $[0, 1[$  of  $\gamma_m^+(0)(\gamma_i^+/K)$  is represented by a map from  $[0, 1[$  to  $\check{C}_{B^+(B_i^+/b_K^+)}[\mathring{D}_1]$ , mapping 0 to the top configuration of  $(\mathcal{C}_m, L_m) ((\mathcal{C}_i, L_i)/I_K)$ , whose restriction to  $B^+ \setminus \{b_K^+\}$  is constant, and whose restriction to  $B_i^+$  is a lift of  $\gamma_i^+|_{[0,1[}$  in  $\check{C}_{B_i^+}[p_{\mathbb{C}}(D^{(i)} \times \{1\})]$ , such that  $\gamma_m^+(0)(\gamma_i^+/K)$  is composable by  $\gamma_m^+(c_i^+/K)$  on its right.

A particular case of cablings is the case in which the inserted  $q$ -tangle  $T_i$  is just the  $q$ -tangle  $y \times [0, 1]$  associated to the constant path of  $\mathcal{S}_{B_i}(\mathbb{C})$  that maps  $[0, 1]$  to a configuration  $y$  of  $\mathcal{S}_{B_i}(\mathbb{C})$ . (Formally, this  $q$ -tangle is represented by  $T(\gamma) (\gamma(1) \times [0, 1]) T(\bar{\gamma})$  for some path  $\gamma$  of  $\mathcal{S}_{B_i}(\mathbb{C})$  such that  $\gamma(0) = y$  and  $\gamma(1) \in \check{\mathcal{S}}_{B_i}(\mathbb{C})$ .) Set

$$T_m(y \times K) = T_m((y \times [0, 1])/K).$$

If  $K$  is a closed component, then  $T_m(y \times K)$  depends only on  $|B_i|$ . It is denoted by  $T_m(|B_i| \times K)$ . If  $K$  is an interval component and  $y$  is the unique configuration of  $\mathcal{S}_{\{1,2\}}(\mathbb{R})$ , then  $T_m(y \times K)$  is again denoted by  $T_m(2 \times K)$ . These special cablings introduced before Theorem 12.22 are called *duplications* or *doublings*. Our functor  $\mathcal{Z}^f$  will behave well under all cablings.

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<sup>2</sup>It is thus located in  $\check{C}_{B^- \setminus \{b_K^-\}}[\mathring{D}_1 \setminus p_{\mathbb{C}}(D^{(i)} \times \{0\})]$ .

We end this section by completing Definition 12.10 of the self-linking number for a framed  $q$ -tangle component, which goes from bottom to bottom or from top to top.

**Definition 13.4.** The *self-linking number*  $lk(K, K_{\parallel})$  of a framed component  $K$  of a  $q$ -tangle going from bottom to bottom (resp. from top to top) is defined as follows. The self-linking number depends only on the component. So there is no loss of generality in representing  $K$  by a tangle representative with injective bottom (resp. top) configuration whose ends are at a distance bigger than  $2\varepsilon$  for a small positive  $\varepsilon$ . Also assume that  $K_{\parallel}(0) = K(0) + (\varepsilon, 0, 0)$  and  $K_{\parallel}(1) = K(1) - (\varepsilon, 0, 0)$ . (There is no loss of generality in this assumption either. It suffices to choose a parallel that satisfies this assumption to define the self-linking number. Recall Definition 12.8.) Let  $\hat{K} = K([0, 1]) \cup \gamma([0, 1])$  for an arbitrary path  $\gamma$  from  $\gamma(0) = K(1)$  to  $\gamma(1) = K(0)$  such that  $\gamma(]0, 1[) \subset \hat{R}(\mathcal{C}) \setminus \mathcal{C}$ .

Let  $[K(1), K(0)]$  denote the straight segment from  $K(1)$  to  $K(0)$  in  $D_1$ . Let  $\alpha_1: [0, 1] \rightarrow D_1$  be an arc from  $K_{\parallel}(1)$  to a point  $a_1$  inside  $[K(1), K(0)]$  such that  $\alpha_1(t) = K(1) - \varepsilon \exp(2i\pi\theta_1 t)$  for some real number  $\theta_1$ . Similarly, let  $\alpha_0: [0, 1] \rightarrow D_1$  be an arc from  $K_{\parallel}(0)$  to a point  $a_0$  inside  $[K(1), K(0)]$  such that  $\alpha_0(t) = K(0) + \varepsilon \exp(2i\pi\theta_0 t)$  for some  $\theta_0 \in \mathbb{R}$ .

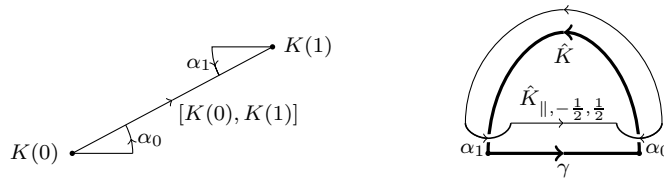


Figure 13.4: A general picture of  $\alpha_0$  and  $\alpha_1$ , and a picture of  $\hat{K}_{\parallel}$  when  $\theta_0 = -\frac{1}{2}$ ,  $\theta_1 = \frac{1}{2}$ , and  $K$  goes from bottom to bottom

If  $K$  goes from bottom to bottom, define

$$\hat{K}_{\parallel, \theta_0, \theta_1} = K_{\parallel}([0, 1]) \cup ((\alpha_1 \cup [a_1, a_0] \cup \bar{\alpha}_0) \times \{0\}),$$

and set

$$lk(K, K_{\parallel}) = lk(\hat{K}, \hat{K}_{\parallel, \theta_0, \theta_1}) + \theta_1 + \theta_0.$$

If  $K$  goes from top to top, define

$$\hat{K}_{\parallel, \theta_0, \theta_1} = K_{\parallel}([0, 1]) \cup ((\alpha_1 \cup [a_1, a_0] \cup \bar{\alpha}_0) \times \{1\})$$

and set

$$lk(K, K_{\parallel}) = lk(\hat{K}, \hat{K}_{\parallel, \theta_0, \theta_1}) - (\theta_1 + \theta_0).$$

Note that these definitions do not depend on the chosen  $\theta_1$  and  $\theta_0$ , which are well-determined modulo  $\mathbb{Z}$ . The angles  $2\pi\theta_1$  and  $2\pi\theta_0$  are both congruent to the angle from the oriented real line to  $\overrightarrow{K(0)K(1)}$  modulo  $2\pi$ . So  $(lk(K, K_{\parallel}) - 2\theta_1) \in \mathbb{Q}$  when  $K$  goes from bottom to bottom, and  $(lk(K, K_{\parallel}) + 2\theta_1) \in \mathbb{Q}$  when  $K$  goes from top to top.

When  $\overrightarrow{K(0)K(1)}$  directs and orients the real line, we can choose  $\theta_0 = \theta_1 = 0$ , and this definition coincides with Definition 12.10. When  $\overrightarrow{K(1)K(0)}$  directs and orients the real line, we can choose  $\theta_1 = \frac{1}{2} = -\theta_0$  so that  $\hat{K}_{\parallel}$  is simply as in Figure 13.4. So the present definition is again consistent with Definition 12.10.

**Lemma 13.5.** *The self-linking number does not depend on the orientations of the components.*

PROOF: This is easy to see for closed components, and this is part of the definition for components going from bottom to top or from top to bottom. When  $K$  goes from top to top, let  $K'$  stand for  $(-K)$ , and let  $K'_{\parallel}$  be the parallel obtained from  $K_{\parallel}$  by a rotation of angle  $\pi$  around  $K$  (and by reversing the orientation). Choose the corresponding angles  $\theta'_0$  and  $\theta'_1$  to be  $\theta'_0 = \theta_1 - \frac{1}{2}$  and  $\theta'_1 = \theta_0 + \frac{1}{2}$ . So  $\hat{K}'_{\parallel, \theta'_0, \theta'_1}$  is isotopic to  $(-\hat{K}_{\parallel, \theta_0, \theta_1})$  in the complement of  $\hat{K}$ . See Figure 13.5. □

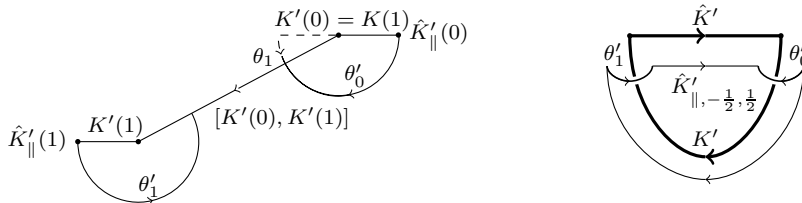


Figure 13.5: A general picture of  $\theta'_0$  and  $\theta'_1$ , and a picture of  $\hat{K}'_{\parallel}$  when  $\theta'_0 = -\frac{1}{2}$ ,  $\theta'_1 = \frac{1}{2}$ , and  $K$  goes from top to top

Proposition 16.13 will show that the real-valued self-linking numbers  $lk(K, K_{\parallel})$  coincide with  $I_{\theta}(K, \tau)$  for the interval components  $K$  of the straight tangles of Definition 16.11. Its proof relies only on the beginning of Section 16.3.

### 13.2 Definition of $\mathcal{Z}^f$ for all $q$ -tangles

Recall Definition 12.12 of the extension of  $\mathcal{Z}^f$  for  $q$ -tangles whose bottom and top configurations are injective. In this section, we extend the definition of  $\mathcal{Z}^f$  to all  $q$ -tangles.

In Chapter 15, we will prove the following particular case of the functoriality property stated in Theorem 12.13. It is a direct corollary of Proposition 15.19.

**Proposition 13.6.** *Let  $(C_1, L_1)$  and  $(C_2, L_2)$  be two framed tangle representatives such that the bottom of  $L_2$  coincides with the top of  $L_1$ . If one of them is a braid, then we have*

$$\mathcal{Z}^f(C_1 C_2, (L_1 L_2)_{\parallel}) = \mathcal{Z}^f(C_1, L_1, L_{1\parallel}) \mathcal{Z}^f(C_2, L_2, L_{2\parallel}).$$

The following lemma allows us to consider the tangles' bottom and top configurations up to translation and dilation. It will also be a direct corollary of Proposition 15.19.

**Lemma 13.7.** *Let  $\gamma: [0, 1] \rightarrow \check{C}_B[\mathring{D}_1]$ . Let  $p_{CS} \circ \gamma: [0, 1] \rightarrow \check{S}_B(\mathbb{C})$  be the composition of  $\gamma$  by the natural projection  $p_{CS}: \check{C}_B[\mathring{D}_1] \rightarrow \check{S}_B(\mathbb{C})$  (which mods out by translations and dilations). Then  $\mathcal{Z}(\gamma)$  and  $\mathcal{Z}^f(\gamma)$  depend only on  $p_{CS} \circ \gamma$ .*

Under the assumptions of the lemma, we set  $\mathcal{Z}(p_{CS} \circ \gamma) = \mathcal{Z}(\gamma)$  and  $\mathcal{Z}^f(p_{CS} \circ \gamma) = \mathcal{Z}^f(\gamma)$ . Recall that  $\mathcal{Z}$  and  $\mathcal{Z}^f$  coincide for braids. We extend the definition of  $\mathcal{Z}$  to piecewise smooth paths of  $\check{S}_B(\mathbb{C})$  so that  $\mathcal{Z}$  is multiplicative with respect to path composition of smooth paths. The following theorem is essentially due to Sylvain Poirier [Poi00]. It allows us to extend  $\mathcal{Z}^f$  to  $q$ -braids, and to  $q$ -tangles in Definition 13.10.

**Theorem 13.8.** *Let  $p_{CS} \circ \gamma: [0, 1] \rightarrow \mathcal{S}_B(\mathbb{C})$  be a path whose restriction to  $]0, 1[$  is the projection of some  $\gamma: ]0, 1[ \rightarrow \check{C}_B[\mathring{D}_1]$  that can be described by a collection of piecewise polynomial continuous maps  $(\gamma_b: [0, 1] \rightarrow \mathring{D}_1)_{b \in B}$ .<sup>3</sup> Then  $\lim_{\varepsilon \rightarrow 0} \mathcal{Z}(p_{CS} \circ \gamma|_{] \varepsilon, 1 - \varepsilon [})$  makes sense, and it depends only on the homotopy class of  $p_{CS} \circ \gamma$  relatively to its boundary. It is denoted by  $\mathcal{Z}(p_{CS} \circ \gamma)$  or  $\mathcal{Z}^f(p_{CS} \circ \gamma)$ .*

Theorem 13.8 and Theorem 16.35, which generalizes its homotopy invariance part, will be proved in Section 16.5.

**Proposition 13.9.** *Any  $q$ -braid  $\gamma: [0, 1] \rightarrow \mathcal{S}_B(\mathbb{C})$  is homotopic relatively to its boundary to a  $q$ -braid  $p_{CS} \circ \tilde{\gamma}$  as in the statement of Theorem 13.8. Setting  $\mathcal{Z}(\gamma) = \mathcal{Z}(p_{CS} \circ \tilde{\gamma})$  consistently extends the definition of  $\mathcal{Z}$  to all  $q$ -braids. Furthermore,  $\mathcal{Z}$  is multiplicative with respect to the  $q$ -braid composition: For two composable paths  $\gamma_1$  and  $\gamma_2$  of  $\mathcal{S}_B(\mathbb{C})$ , we have*

$$\mathcal{Z}(\gamma_1 \gamma_2) = \mathcal{Z}(\gamma_1) \mathcal{Z}(\gamma_2).$$

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<sup>3</sup>Every  $\gamma_b$  is polynomial over a finite number of intervals covering  $[0, 1]$ .



PROOF: Let us first exhibit a  $q$ -braid  $p_{CS} \circ \tilde{\gamma}$  homotopic to a given  $q$ -braid  $\gamma: [0, 1] \rightarrow \mathcal{S}_B(\mathbb{C})$ , with the desired properties. Define a path  $\tilde{\gamma}_1: [0, 1/3] \rightarrow C_B[\mathring{D}_1]$ , such that  $p_{CS} \circ \tilde{\gamma}_1(0) = \gamma(0)$ ,  $\tilde{\gamma}_1([0, 1/3]) \subset \check{C}_B[\mathring{D}_1]$ , and  $\tilde{\gamma}_1$  is a path obtained by replacing all the parameters  $\mu_A$  in the charts of Lemma 8.27 by  $\varepsilon t$  for  $t \in [0, 1/3]$  for some small  $\varepsilon > 0$ . So  $\tilde{\gamma}_1$  is described by a collection of polynomial maps  $(\tilde{\gamma}_{1,b}: [0, 1/3] \rightarrow \mathring{D}_1)_{b \in B}$ . Similarly define a polynomial path  $\tilde{\gamma}_3: [2/3, 1] \rightarrow C_B[\mathring{D}_1]$  such that  $p_{CS} \circ \tilde{\gamma}_3(1) = \gamma(1)$  and  $\tilde{\gamma}_3([2/3, 1]) \subset \check{C}_B[\mathring{D}_1]$ . Define a path  $\tilde{\gamma}'_2: [1/3, 2/3] \rightarrow C_B[\mathring{D}_1]$ , such that  $\tilde{\gamma}'_2(1/3) = \tilde{\gamma}_1(1/3)$ ,  $\tilde{\gamma}'_2(2/3) = \tilde{\gamma}_3(2/3)$ , and  $p_{CS} \circ \tilde{\gamma}'_2$  is a path composition  $(p_{CS} \circ \tilde{\gamma}_1)\gamma(p_{CS} \circ \tilde{\gamma}_3)$ . This path  $\tilde{\gamma}'_2$  of  $C_B[\mathring{D}_1]$  is homotopic to a path of the interior  $\check{C}_B[\mathring{D}_1]$  of the manifold  $C_B[D_1]$  with ridges, and it is homotopic to a polynomial path  $\tilde{\gamma}_2$  in  $\check{C}_B[\mathring{D}_1]$ . Now, the path composition  $\tilde{\gamma} = \tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3$  satisfies the hypotheses of Theorem 13.8, and  $p_{CS} \circ \tilde{\gamma}$  is homotopic to  $\gamma$ , relatively to its boundary. Furthermore, for any other path  $\tilde{\gamma}'$  satisfying these properties,  $p_{CS} \circ \tilde{\gamma}$  and  $p_{CS} \circ \tilde{\gamma}'$  are homotopic relatively to the boundary. So the definition of  $\mathcal{Z}(\gamma)$  is consistent.

Let us prove the multiplicativity. Pick a piecewise polynomial path  $\tilde{\gamma}_1: [0, 1] \rightarrow C_B[\mathring{D}_1]$ , such that  $\gamma_1$  and  $p_{CS} \circ \tilde{\gamma}_1$  are homotopic relatively to the boundary, and  $\tilde{\gamma}_1([0, 1]) \subset \check{C}_B[\mathring{D}_1]$ . Next, pick a piecewise polynomial path  $\tilde{\gamma}_2: [0, 1] \rightarrow C_B[\mathring{D}_1]$ , such that,  $\tilde{\gamma}_2(t) = \tilde{\gamma}_1(1 - t)$  for any  $t \in [0, 1/2]$ ,  $\gamma_2$  and  $p_{CS} \circ \tilde{\gamma}_2$  are homotopic relatively to the boundary, and  $\tilde{\gamma}_2([0, 1]) \subset \check{C}_B[\mathring{D}_1]$ . Thus we have

$$\mathcal{Z}(\gamma_1\gamma_2) = \lim_{\varepsilon \rightarrow 0} \mathcal{Z}\left(p_{CS} \circ (\tilde{\gamma}_1|_{[\varepsilon, 1/2]}\tilde{\gamma}_2|_{[1/2, 1-\varepsilon]})\right).$$

We also have

$$\begin{aligned} \mathcal{Z}(\gamma_1)\mathcal{Z}(\gamma_2) &= \lim_{\varepsilon \rightarrow 0} \mathcal{Z}(p_{CS} \circ \tilde{\gamma}_1|_{[\varepsilon, 1-\varepsilon]})\mathcal{Z}(p_{CS} \circ \tilde{\gamma}_2|_{[\varepsilon, 1-\varepsilon]}) \\ &= \lim_{\varepsilon \rightarrow 0} \mathcal{Z}(p_{CS} \circ \tilde{\gamma}_1|_{[\varepsilon, 1-\varepsilon]}p_{CS} \circ \tilde{\gamma}_2|_{[\varepsilon, 1-\varepsilon]}) \\ &= \lim_{\varepsilon \rightarrow 0} \mathcal{Z}(p_{CS} \circ (\tilde{\gamma}_1|_{[\varepsilon, 1/2]}\tilde{\gamma}_2|_{[1/2, 1-\varepsilon]})), \end{aligned}$$

where the second equality comes from Proposition 13.6. □

Theorem 12.7, Proposition 13.6, Theorem 13.8, and Proposition 13.9 allow us to extend  $\mathcal{Z}^f$  unambiguously to the  $q$ -tangles of Definition 13.1 as follows.

**Definition 13.10.** Let  $\gamma^-$  and  $\gamma^+$  be  $q$ -braids. Let  $(\mathcal{C}, L)$  be a framed tangle whose bottom and top configurations are  $\gamma^-(1)$  and  $\gamma^+(0)$ , respectively. Orient the strands of  $T(\gamma^-)$  and  $T(\gamma^+)$  so that their orientations are consistent with the orientation of  $L$ . Define  $\mathcal{Z}^f(T(\gamma^-)(\mathcal{C}, L)T(\gamma^+))$  to be

$$\mathcal{Z}^f(T(\gamma^-)(\mathcal{C}, L)T(\gamma^+)) = \mathcal{Z}(\gamma^-)\mathcal{Z}^f(\mathcal{C}, L)\mathcal{Z}(\gamma^+),$$

with respect to Definition 12.12 of  $\mathcal{Z}^f(\mathcal{C}, L)$ .<sup>4</sup>

**Remark 13.11.** Let  $(\mathcal{C}, L_q)$  be a  $q$ -tangle from a bottom limit configuration  $c^- \in \mathcal{S}_{B^-}(\mathbb{C})$  to a top configuration  $c^+ \in \mathcal{S}_{B^+}(\mathbb{C})$ . These limit configurations are initial points  $\gamma^\pm(0) = c^\pm$  of polynomial paths  $\gamma^\pm$  of  $\mathcal{S}_{B^\pm}(\mathbb{C})$ , such that  $\gamma^\pm([0, 1]) \subset \check{\mathcal{S}}_{B^\pm}(\mathbb{C})$  (as in the proof of Proposition 13.9). This allows us to regard  $c^\pm$  as a limit

$$c^\pm = \lim_{t \rightarrow 0} \gamma^\pm(t)$$

of injective configurations and view  $L_q$  as a limit of framed tangles between injective configurations. We can indeed view the  $q$ -tangle  $L_q$  as the limit of

$$L_{q,\varepsilon} = T(\overline{\gamma^-|_{[0,\varepsilon]}})L_qT(\gamma^+|_{[0,\varepsilon]})$$

when  $\varepsilon$  tends to 0. Then  $\mathcal{Z}^f(\mathcal{C}, L_q)$  can be defined alternatively to be

$$\mathcal{Z}^f(\mathcal{C}, L_q) = \lim_{\varepsilon \rightarrow 0} \mathcal{Z}^f(L_{q,\varepsilon}).$$

Indeed, the above consistent definition implies

$$\mathcal{Z}^f(\mathcal{C}, L_q) = \mathcal{Z}(\gamma^-|_{[0,\varepsilon]})\mathcal{Z}^f(L_{q,\varepsilon})\mathcal{Z}(\overline{\gamma^+|_{[0,\varepsilon]}}),$$

while Theorem 13.8 and Proposition 13.9 imply

$$\lim_{\varepsilon \rightarrow 0} \mathcal{Z}(\gamma^-|_{[0,\varepsilon]}) = \mathbf{1} \text{ and } \lim_{\varepsilon \rightarrow 0} \mathcal{Z}(\overline{\gamma^+|_{[0,\varepsilon]}}) = \mathbf{1}.$$

We will construct variants of  $\mathcal{Z}^f$  in the spirit of Theorem 7.40 in Section 16.2. These variants will allow us to prove Theorem 13.12 in Chapter 17.

Proving Theorem 13.12 will require lengthy studies of compactifications of configuration spaces, which are not manifolds with boundaries. These studies will help to get all the nice and natural properties of  $\mathcal{Z}^f$ , stated in Theorem 13.12. Robin Koytcheff, Brian Munson, and Ismar Volić proposed another approach to obtain invariants of tangles and avoid our complicated configuration spaces in [KMV13].

In the following three chapters, we will show the construction of  $\mathcal{Z}^f$  for  $q$ -tangles in detail, following the outline of this section.

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<sup>4</sup>Alternatively, we can forget the orientations of  $L$ ,  $\gamma^-$ , and  $\gamma^+$  since  $\mathcal{Z}^f$  depends on the component orientations as in Proposition 10.23 for  $L$ ,  $\gamma^-$ , and  $\gamma^+$ .

### 13.3 Properties of the functor $\mathcal{Z}^f$

In this section, we state the main natural properties of the functor  $\mathcal{Z}^f = (\mathcal{Z}_n^f)_{n \in \mathbb{N}}$ , whose construction is outlined in the previous section.

**Theorem 13.12.** *The invariant  $\mathcal{Z}^f$  of  $q$ -tangles satisfies the following properties.*

- $\mathcal{Z}^f$  coincides with the invariant  $\mathcal{Z}^f$  of Definition 7.41 for framed links in  $\mathbb{Q}$ -spheres.
- $\mathcal{Z}^f$  coincides with the Poirier functor  $Z^l$  of [Poi00] for combinatorial  $q$ -tangles in  $\mathbb{R}^3$ .
- Naturality: If  $L$  is a  $q$ -tangle with domain  $\mathcal{L}$ , then  $\mathcal{Z}^f(L) = (\mathcal{Z}_k^f(L))_{k \in \mathbb{N}}$  is valued in  $\mathcal{A}(\mathcal{L}) = \mathcal{A}(\mathcal{L}; \mathbb{R})$ , and  $\mathcal{Z}_0^f(L)$  is the class of the empty diagram. If  $L'$  is a subtangle of  $L$  with domain  $\mathcal{L}'$ , then  $\mathcal{Z}^f(L')$  is obtained from  $\mathcal{Z}^f(L)$  by mapping all the diagrams with univalent vertices on  $\mathcal{L} \setminus \mathcal{L}'$  to zero and by forgetting  $\mathcal{L} \setminus \mathcal{L}'$ .
- Dependence on component orientations: If  $\mathcal{L}$  and the components of  $L$  are unoriented, then  $\mathcal{Z}^f(L)$  is valued in the space  $\mathcal{A}(\mathcal{L})$  of Definition 6.16, as in Proposition 10.23. Otherwise, component orientation reversals affect  $\mathcal{Z}^f$  as in Proposition 10.23.
- Framing dependence: For a  $q$ -tangle  $L = (\mathcal{C}, \sqcup_{j=1}^k K_j, \sqcup_{j=1}^k K_{j\parallel})$ ,

$$\prod_{j=1}^k \left( \exp(-lk(K_j, K_{j\parallel})\alpha) \#_j \right) \mathcal{Z}^f(L)$$

is independent of the framing of  $L$ . It is denoted by  $\mathcal{Z}(\mathcal{C}, \sqcup_{j=1}^k K_j)$ .<sup>5</sup>

- Functoriality: For two  $q$ -tangles  $L_1$  and  $L_2$  such that the bottom configuration of  $L_2$  coincides with the top configuration of  $L_1$ , we have

$$\mathcal{Z}^f(L_1 L_2) = \mathcal{Z}^f(L_1) \mathcal{Z}^f(L_2),$$

with products obtained by stacking above in natural ways on both sides.

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<sup>5</sup>In particular, when a kink  $\curvearrowright$  is added to the  $j$ th component of a combinatorial  $q$ -tangle  $L_c$ ,  $\mathcal{Z}^f(L_c)$  is changed to

$$\mathcal{Z}^f(\curvearrowright \#_j L_c) = \exp(\alpha) \#_j \mathcal{Z}^f(L_c).$$

We similarly have  $\mathcal{Z}^f(\curvearrowleft \#_j L_c) = \exp(-\alpha) \#_j \mathcal{Z}^f(L_c)$ .

- First duplication property: *Let  $K$  be a component of a  $q$ -tangle  $L$ , then we have*

$$\mathcal{Z}^f(L(2 \times K)) = \pi(2 \times K)^* \mathcal{Z}^f(L)$$

*with respect to Notation 6.31.*

- Second duplication property: *Let  $B$  be a finite set, let  $y$  be an element of  $\mathcal{S}_B(\mathbb{C})$ . Let  $K$  be a component going from bottom to top in a  $q$ -tangle  $L$ . Recall  $L(y \times K) = L((y \times [0, 1]) / K)$  from Notation 13.3. Then we have*

$$\mathcal{Z}^f(L(y \times K)) = \pi(B \times K)^* \mathcal{Z}^f(L).$$

- Dependence on ambient orientation: *Let  $s_{\frac{1}{2}}$  be the orthogonal reflection with respect to the horizontal plane at height  $\frac{1}{2}$ . Extend  $s_{\frac{1}{2}}$  from  $\partial\mathcal{C}$  to an orientation-reversing diffeomorphism  $s_{\frac{1}{2}}$  of  $\mathcal{C}$ . Define the parallels  $s_{\frac{1}{2}}(K)_{\parallel}$  of interval components  $s_{\frac{1}{2}}(K)$  of  $s_{\frac{1}{2}}(L)$  so that they satisfy  $lk_{s_{\frac{1}{2}}(\mathcal{C})}(s_{\frac{1}{2}}(K), s_{\frac{1}{2}}(K)_{\parallel}) = -lk_{\mathcal{C}}(K, K_{\parallel})$ . Then we have*

$$\mathcal{Z}_n^f(s_{\frac{1}{2}}(\mathcal{C}), s_{\frac{1}{2}} \circ L) = (-1)^n \mathcal{Z}_n^f(\mathcal{C}, L)$$

*for all  $n \in \mathbb{N}$ .*

- Symmetry: *Let  $\rho$  be a rotation of  $\mathbb{R}^3$  that preserves the standard homology cylinder  $D_1 \times [0, 1]$  (setwise). Let  $L$  be a  $q$ -tangle of a rational homology cylinder  $\mathcal{C}$ . Extend  $\rho$  from  $\partial\mathcal{C}$  to an orientation-preserving diffeomorphism  $\rho$  of  $\mathcal{C}$ . If the angle of  $\rho$  is different from 0 and  $\pi$ , assume that the interval components of  $L$  go from bottom to top or from top to bottom. Then we have*

$$\mathcal{Z}^f(\rho(\mathcal{C}), \rho \circ L) = \mathcal{Z}^f(\mathcal{C}, L),$$

*where the parallels of interval components  $\rho(K)$  of  $\rho(L)$  are defined so that they satisfy  $lk_{\rho(\mathcal{C})}(\rho(K), \rho(K)_{\parallel}) = lk_{\mathcal{C}}(K, K_{\parallel})$ .*<sup>6</sup>

- Cabling property: *Let  $B$  be a finite set with cardinality greater than 1. Let  $y \in \mathcal{S}_B(\mathbb{C})$ , let  $y \times [0, 1]$  denote the corresponding  $q$ -braid, and let  $K$  be a strand of  $y \times [0, 1]$ . Let  $L$  be a  $q$ -tangle with domain  $\mathcal{L}$ . Then  $\mathcal{Z}^f((y \times [0, 1]) (L/K))$  is obtained from  $\mathcal{Z}^f(L)$  by the natural injection from  $\mathcal{A}(\mathcal{L})$  to  $\mathcal{A}(\sqcup_{b \in B} \mathbb{R}^{\{b\}} (\frac{\mathcal{L}}{K}))$ .*

---

<sup>6</sup>This statement applies to a diffeomorphism  $\rho$  of  $\mathcal{C}$  restricting to  $\partial\mathcal{C}$  as the identity map, in particular.

- Full twist in degree one: *The expansion  $\mathcal{Z}_{\leq 1}^f$  up to degree 1 of  $\mathcal{Z}^f$  satisfies*

$$\mathcal{Z}_{\leq 1}^f \left( \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right) = 1 + \left[ \begin{array}{c} \vdots \quad \vdots \\ \circ \quad \bullet \\ \vdots \quad \vdots \end{array} \right],$$

where the endpoints of the tangle are assumed to lie on  $\mathbb{R} \times \{0, 1\}$ .

- Group-like behavior: *For any  $q$ -tangle  $L$  and any integer  $n$ , we have*

$$\Delta_n (\mathcal{Z}_n^f(L)) = \sum_{i=0}^n \mathcal{Z}_i^f(L) \otimes \mathcal{Z}_{n-i}^f(L)$$

with respect to the coproduct maps  $\Delta_n$  of Section 6.5.

The definition of  $\mathcal{Z}^f$  in Section 13.2 obviously extends the definition of  $\mathcal{Z}^f$  for tangles with empty boundary. Note that the naturality property in Theorem 13.12 easily follows from the definition (which will be justified later). The behavior of  $Z(\mathcal{C}, L, \tau)$  with respect to the coproduct can be observed from the definition, as in the proof of Proposition 10.3. According to Lemmas 6.37 and 6.38, Definition 10.5, and Proposition 10.13, the correction factors are group-like. So the compatibility between the various products and the coproduct ensures that  $\mathcal{Z}^f$  behaves as stated in Theorem 13.12 with respect to the coproduct, for framed tangles between injective configurations. Then Remark 13.11 ensures that this also holds for general  $q$ -tangles.

**Corollary 13.13.** *If  $L$  has at most one component, let  $p^c$  be the projection given by Corollary 6.40 from  $\mathcal{A}(\mathcal{L})$  to the space  $\mathcal{A}^c(\mathcal{L})$  of its primitive elements. Set*

$$\mathfrak{z}^f(\mathcal{C}, L) = p^c (\mathcal{Z}^f(\mathcal{C}, L)).$$

Then we have

$$\mathcal{Z}^f(\mathcal{C}, L) = \exp (\mathfrak{z}^f(\mathcal{C}, L)).$$

PROOF: In these cases, Lemma 6.37 guarantees that  $\mathcal{A}(\mathcal{L})$  is a graded bialgebra, and Theorem 13.12 implies that  $\mathcal{Z}^f(\mathcal{C}, L)$  is group-like. Conclude with Theorem 6.41. □

The proof of Theorem 13.12 will be finished at the end of Section 17.4. The multiplicativity of  $\mathcal{Z}$  under connected sum of Theorem 10.26 is a direct corollary of the functoriality of  $\mathcal{Z}^f$  in the above statement. The functoriality also implies that  $\mathcal{Z}$  and  $\mathcal{Z}^f$  map tangles consisting of vertical segments in the standard cylinder to 1. Consider such a trivial braid consisting of the two vertical segments  $\{-\frac{1}{2}\} \times [0, 1]$  and  $\{\frac{1}{2}\} \times [0, 1]$ . Cable  $\{-\frac{1}{2}\} \times [0, 1]$  by a  $q$ -tangle  $(\mathcal{C}_1, L_1)$ , and cable  $\{\frac{1}{2}\} \times [0, 1]$  by a  $q$ -tangle  $(\mathcal{C}_2, L_2)$ . Call the

resulting  $q$ -tangle  $(\mathcal{C}_1 \otimes \mathcal{C}_2, L_1 \otimes L_2)$ . Formally, this tangle may be expressed as  $\left( \left( \left\{ -\frac{1}{2}, \frac{1}{2} \right\} \times [0, 1] \right) \left( \frac{(\mathcal{C}_1, L_1)}{\left\{ -\frac{1}{2} \right\} \times [0, 1]} \right) \right) \left( \frac{(\mathcal{C}_2, L_2)}{\left\{ \frac{1}{2} \right\} \times [0, 1]} \right)$ .

**Corollary 13.14.** *The functor  $\mathcal{Z}^f$  satisfies the following monoidality property with respect to the above structure:*

$$\mathcal{Z}^f(\mathcal{C}_1 \otimes \mathcal{C}_2, L_1 \otimes L_2) = \mathcal{Z}^f(\mathcal{C}_1, L_1) \otimes \mathcal{Z}^f(\mathcal{C}_2, L_2),$$

where the product  $\otimes$  of the right-hand side is simply induced by the disjoint union of diagrams.

PROOF: This is a consequence of the cabling property and the functoriality in Theorem 13.12.  $\square$

More generally, Theorem 13.12 implies that the Poirier functor  $\mathcal{Z}^l$  is a good monoidal functor. The multiplicativity of  $\mathcal{Z}$  under connected sum of Theorem 10.26 is also a consequence of Corollary 13.14.

**Remark 13.15.** The first duplication property may be iterated. Note that  $\pi(r \times K)^*$  is nothing but the composition of  $(r - 1) \pi(2 \times K)^*$ . Also note that iterating duplications  $\overset{(\bullet\bullet)}{\cup}$  for configurations produces combinatorial configurations as in Example 13.2, i.e., elements in the 0-dimensional strata of some  $\mathcal{S}_B(\mathbb{R})$  discussed in Example 8.3. For example, we have

$$\cup(2 \times \cup) = \overset{(\bullet\bullet)}{\underset{1}{\cup}} \overset{(\bullet\bullet)}{\underset{2}{\cup}} \quad \text{and} \quad \overset{(\bullet\bullet)}{\underset{1}{\cup}} \overset{(\bullet\bullet)}{\underset{2}{\cup}} \left( 2 \times \underset{1}{\cup} \right) = \overset{((\bullet\bullet))}{\underset{1}{\cup}} \overset{((\bullet\bullet))}{\underset{2}{\cup}}.$$

So, iterating twice the first duplication property implies

$$\mathcal{Z}^f \left( \overset{((\bullet\bullet))}{\underset{1}{\cup}} \overset{((\bullet\bullet))}{\underset{2}{\cup}} \right) = \pi(3 \times \cup)^* \mathcal{Z}^f(\cup).$$

**Remark 13.16.** The second duplication property, together with the behavior of  $\mathcal{Z}^f$  under component orientation reversal, yields a similar duplication property for a component  $K$  going from top to bottom. The behavior of  $\mathcal{Z}^f$  under orientation change, the functoriality, and the duplication properties allow us to generalize the cabling property to cablings of components  $K$  going from bottom to top or from top to bottom in arbitrary  $q$ -tangles, by arbitrary  $q$ -tangles. We may similarly generalize the cabling property to components  $K$  going from top to top or from bottom to bottom, cabled by  $q$ -tangles in a rational homology cylinder whose bottom or top configurations are combinatorial configurations (as in Example 8.3). In both cases, we can perform the insertion of the nontrivial part  $T_i$  near an end of  $K$  so that the result is a vertical composition of a tangle obtained by cabling a strand in a trivial vertical braid with  $T_i$  and a possibly iterated duplication of the (main) tangle  $T_m$  in the (main) rational homology cylinder  $\mathcal{C}_m$ , as in Figure 13.6.

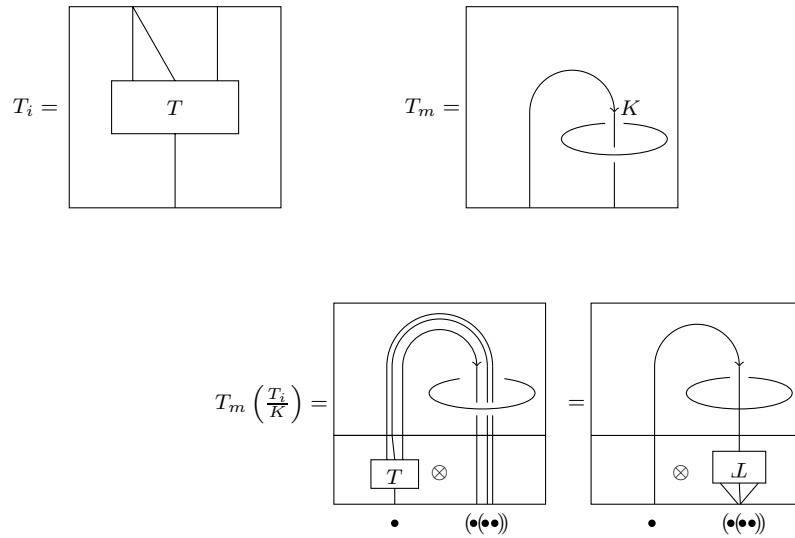


Figure 13.6: Cabling a component going from bottom to bottom in two different ways

**Remark 13.17.** The behavior of  $\mathcal{Z}^f$  when a component  $K$  of a link  $L: \mathcal{L} \rightarrow \check{R}(\mathcal{C}_m)$  is cabled by a semi-pure  $q$ -tangle  $(\mathcal{C}, T_i)$  can be described as follows.

- cut the domain of  $K$  to replace it by a copy of  $\mathbb{R}$  using Proposition 6.27,
- duplicate the corresponding strand and  $\mathcal{Z}^f(R(\mathcal{C}_m), L)$ , accordingly, as in the duplication property above,
- multiply the obtained element by  $\mathcal{Z}^f(\mathcal{C}, T_i)$  i.e., concatenate the diagrams, naturally,
- finally, close the domain of  $L(T/K)$ .

This follows easily from Theorem 13.12, by viewing  $(R, L)$  as a vertical composition of two tangles, where the bottom one is just a cup  $\cup$ . This cup is a trivial strand going from top to top in a standard cylinder. It is a part of  $K$ . We illustrate the process in Figure 13.7. In order to apply the iterated first duplication property, we change  $T_i$  to a conjugate of  $T_i^c$  whose bottom (or top) configuration is combinatorial. The result follows with  $\mathcal{Z}^f(\mathcal{C}, T_i^c)$  instead of  $\mathcal{Z}^f(\mathcal{C}, T_i)$ . Lemma 6.33 guarantees that changing  $\mathcal{Z}^f(\mathcal{C}, T_i^c)$  to  $\mathcal{Z}^f(\mathcal{C}, T_i)$  in the above recipe does not change the result. (This is consistent with the fact that the tangles  $L(T_i^c/K)$  and  $L(T_i/K)$  coincide.)

The first cutting step is not legitimate if  $L$  has interval components. (Recall Exercise 6.28). So the above recipe does not generalize to this case.<sup>7</sup>

<sup>7</sup>For a circle component  $K$  in a general  $q$ -tangle  $T_m$ , and a semi-pure  $q$ -tangle  $T_i$  as

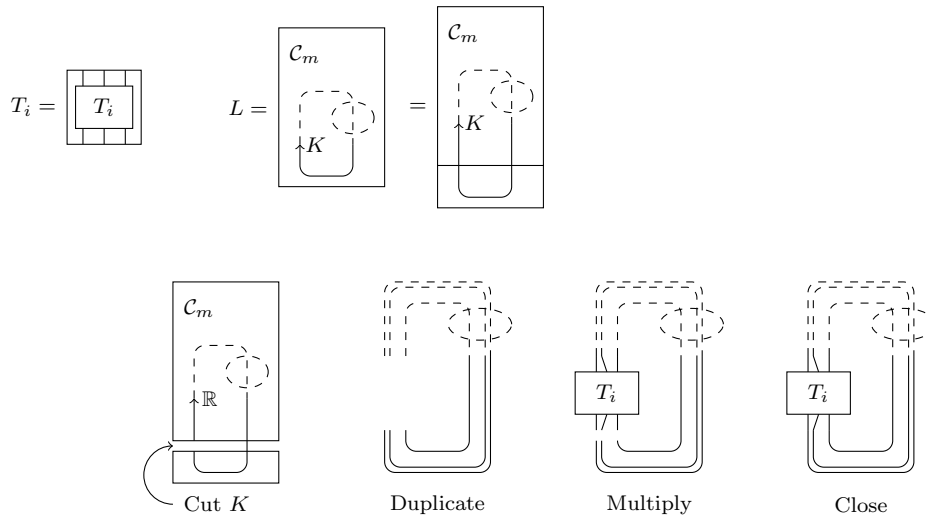


Figure 13.7: Cabling a link component with a semi-pure tangle, step by step

For a  $q$ -braid  $\gamma: [0, 1] \rightarrow \mathcal{S}_B(\mathbb{C})$ ,  $\mathcal{Z}^f(\gamma) = \mathcal{Z}(\gamma)$  stands for

$$\mathcal{Z}^f(T(\gamma)) = \mathcal{Z}^f(D_1 \times [0, 1], T(\gamma)).$$

The following proposition, which leads to interesting cablings, is a corollary of Theorem 13.12.

**Proposition 13.18.** *Let  $q$  be a positive integer. Let  $\gamma_{1,q}$  be a braid represented by the map*

$$\begin{aligned} [0, 1] \times \underline{q} &\rightarrow \mathbb{C} \\ (t, k) &\mapsto \frac{1}{2} \exp\left(\frac{2i\pi(k+\chi(t))}{q}\right) \end{aligned}$$

for a surjective map  $\chi: [0, 1] \rightarrow [0, 1]$  with nonnegative derivative, which is constant in neighborhoods of 0 and 1, as in Figure 13.1 of  $\gamma_{1,3}$ . Recall Notation 6.31 and the anomaly  $\alpha$  of Proposition 10.13. Then we have

$$\mathcal{Z}^f(\gamma_{1,q}) = \exp\left(\pi\left(q \times \hat{\imath}\right)^* \left(\frac{1}{q}\alpha\right)\right) \left(\exp\left(-\frac{1}{q}\alpha\right) \otimes \cdots \otimes \exp\left(-\frac{1}{q}\alpha\right)\right).$$

above,  $T_m(T_i/K)$  is well-defined and it coincides with  $T_m(T_i^c/K)$  for any conjugate  $T_i^c$  of  $T_i$ . We can still write  $T_m$  as a product  $(c_m^- \times [0, 1] \otimes \cup) T'_m$ , where  $\cup$  is a part of  $K$ , pick a conjugate  $T_i^c$  of  $T_i$  whose bottom (or top) configuration is combinatorial, and compute  $\mathcal{Z}^f(T_m(T_i^c/K))$  from  $\mathcal{Z}^f(T'_m)$  and  $\mathcal{Z}^f(T_i^c)$ . Unfortunately,  $\mathcal{Z}^f(T'_m)$  is not determined by  $\mathcal{Z}^f(T_m)$  anymore.



PROOF: Let  $\tilde{\gamma}_{1,1}$  be the trivial one-strand braid  $K$  in the standard cylinder equipped with its parallel  $K_{\parallel}$  such that  $lk(K, K_{\parallel}) = 1$ . The framing dependence property in Theorem 13.12 implies  $\mathcal{Z}^f(\tilde{\gamma}_{1,1}) = \exp(\alpha)$ . Let  $\tilde{\gamma}_{q,q}$  be the  $q$ -tangle obtained by cabling  $\tilde{\gamma}_{1,1}$  as in the second duplication property, by replacing the one-point configuration by the planar configuration of  $\mathbb{C}$  consisting of the  $q$  points  $\frac{1}{2} \exp\left(\frac{2i\pi k}{q}\right)$ , for  $k \in \underline{q}$ . This duplication operation equips each strand  $K_k$  of  $\tilde{\gamma}_{q,q}$  with a parallel  $K_{k\parallel,1}$  such that  $lk(K_k, K_{k\parallel,1}) = 1$ . The  $q$ -tangle  $\tilde{\gamma}_{q,q}$  coincides with  $\gamma_{1,q}^q$  except for the framing since the standard framing of  $\gamma_{1,q}^q$  equips  $K_k$  with a parallel  $K_{k\parallel}$  such that  $lk(K_k, K_{k\parallel}) = 0$ . According to the second duplication property, we have

$$\mathcal{Z}^f(\tilde{\gamma}_{q,q}) = \exp\left(\pi\left(q \times \hat{\cdot}\right)^*(\alpha)\right),$$

whereas the framing dependence property implies

$$\mathcal{Z}^f(\gamma_{1,q}^q) = \mathcal{Z}^f(\tilde{\gamma}_{q,q})(\exp(-\alpha) \otimes \cdots \otimes \exp(-\alpha)).$$

By the invariance of  $\mathcal{Z}^f$  under rotation,  $\mathcal{Z}^f(\gamma_{1,q})$  is invariant under cyclic permutation of the strands. So the functoriality implies  $\mathcal{Z}^f(\gamma_{1,q}^q) = \mathcal{Z}^f(\gamma_{1,q})^q$ . The result follows by unicity of a  $q$ th root of  $\mathcal{Z}^f(\gamma_{1,q}^q)$  with 1 as degree 0 part.  $\square$

Dror Bar-Natan, Thang Lê, and Dylan Thurston computed the Kontsevich integral of the trivial knot  $O$  in [BNLT03]. Thus, Corollary 12.29 allows one to express  $\mathcal{Z}^f$  for the unknot and for the torus knots as a function of the anomaly  $\alpha$ . Note that the symmetry properties imply that  $\mathcal{Z}^f(\cap)$  vanishes in odd degree and that we have  $\mathcal{Z}^f(\cap) = \mathcal{Z}^f(\cup) = \sqrt{\mathcal{Z}^f(O)}$ , where we implicitly use the natural isomorphism of Proposition 6.22.

**Lemma 13.19.** *We have*

$$\mathcal{Z}^f\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right) = \exp\left(\Psi(2\alpha)\left(\begin{array}{c} \hat{\cdot} \\ \hat{\cdot} \\ \vdots \\ \vdots \end{array}\right)\right).$$

PROOF: This lemma can be deduced from Corollary 12.29. Below, as an exercise, we alternatively deduce it from Proposition 13.18 and Theorem 13.12, assuming that  $\alpha$  is a two-leg element of  $\check{\mathcal{A}}(\hat{\cdot})$ , but without assuming Corollary 12.29. Since  $\alpha$  is a two-leg element, we picture it as

$$\alpha = \begin{array}{c} \hat{\cdot} \\ \boxed{\alpha} \\ \hat{\cdot} \end{array}.$$

So, using Lemma 12.26, we get

$$\pi\left(2 \times \hat{\cdot}\right)^*(\alpha) = 2 \begin{array}{c} \hat{\cdot} \\ \boxed{\alpha} \\ \hat{\cdot} \end{array} + \begin{array}{c} \hat{\cdot} \\ \boxed{\alpha} \\ \hat{\cdot} \\ \vdots \\ \vdots \end{array} + \begin{array}{c} \hat{\cdot} \\ \boxed{\alpha} \\ \hat{\cdot} \\ \vdots \\ \vdots \end{array}.$$

Since  $\square_{\hat{\alpha}}$  can be slid along its interval, we obtain

$$\exp\left(\pi(2 \times \hat{\alpha})^*(\alpha)\right) = \exp\left(\Psi(2\alpha)\left(\begin{array}{c} \hat{\alpha} \quad \hat{\alpha} \\ \circ \quad \bullet \\ \vdots \quad \vdots \end{array}\right)\right) (\exp(\alpha) \otimes \exp(\alpha)).$$

□



# Chapter 14

## Invariance of $\mathcal{Z}^f$ for long tangles

In this chapter, we study appropriate compactifications of the configuration space  $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$  associated to a long tangle representative  $L: \mathcal{L} \hookrightarrow \check{R}(\mathcal{C})$  and to a Jacobi diagram  $\Gamma$  with support  $\mathcal{L}$ . These compactifications  $C_L = C(R(\mathcal{C}), L; \Gamma)$  and  $C_L^f = C^f(R(\mathcal{C}), L; \Gamma)$  are introduced in Definition 14.22. They allow us to prove Theorem 12.2, which ensures that the integrals involved in the extension of  $\mathcal{Z}^f$  to long tangles converge, in Section 14.2. They also allow us to prove Theorem 12.7, which ensures the topological invariance of this extension of  $\mathcal{Z}^f$ , in Section 14.3. Our compactifications are locally diffeomorphic to products of smooth manifolds by singular subspaces of  $\mathbb{R}^n$  associated to trees. We study these singular subspaces in Section 14.1 and show how Stokes' theorem applies to them in this preliminary section, which is independent of the rest of the book. We describe the local structure of our compactifications and their codimension-one faces in Theorem 14.23.

### 14.1 Singular models associated to trees

**Definitions 14.1.** In this book, an *oriented tree* is a tree  $\mathcal{T}$  as in Figure 14.1 whose edges are oriented so that  $\mathcal{T}$  satisfies the following properties.

- There is exactly one vertex without outgoing edges. This vertex  $T(\mathcal{T})$  is called the *top* of  $\mathcal{T}$ . It is also simply denoted by  $T$  when  $\mathcal{T}$  is fixed.
- The edges of  $\mathcal{T}$  are oriented towards  $T(\mathcal{T})$ . In other words, for any vertex  $V$  of  $\mathcal{T}$  different from  $T(\mathcal{T})$ , the orientation of the edges in the injective path  $[V, T(\mathcal{T})]$  from  $V$  to  $T(\mathcal{T})$  is induced by the orientation of  $[V, T(\mathcal{T})]$ .

Let  $\mathcal{T}$  be such an oriented tree. A univalent vertex of  $\mathcal{T}$  with one outgoing edge is called a *leaf* of  $\mathcal{T}$ . The set of leaves of  $\mathcal{T}$  is denoted by  $L(\mathcal{T})$ . A *node* of  $\mathcal{T}$  is a vertex with at least two ingoing edges. A *branch* of  $\mathcal{T}$  is an oriented injective path of oriented edges going from a leaf  $\ell$  to a node  $N$  or to the top  $T$ . Such a branch is denoted by  $[\ell, N]$  or by  $[\ell, T]$ . It is viewed as the subset of the set  $E(\mathcal{T})$  of edges of  $\mathcal{T}$  between  $\ell$  and  $N$ , or between  $\ell$  and  $T$ . We denote the edge adjacent to a leaf  $\ell$  by  $\mathbf{e}(\ell)$ . For any two vertices  $N_1, N_2$  on the same branch  $[\ell, T]$ , such that  $N_2$  is closer to  $T$  than  $N_1$ ,  $[N_1, N_2]$  is the set of edges between  $N_1$  and  $N_2$ . These edges may contain  $N_1$  or  $N_2$  as an endpoint.  $]N_1, N_2]$  (resp.  $[N_1, N_2[$ ) denotes the set of edges of  $[N_1, N_2]$  that do not contain  $N_1$  (resp.  $N_2$ ) as an endpoint. For example, the set  $[\ell, N] \setminus \{\mathbf{e}(\ell)\}$  is denoted by  $] \ell, N]$ . Similarly, for any two edges  $e_1, e_2$  on the same branch  $[\ell, T]$ , such that  $e_1$  is closer to  $\ell$  than  $T$ ,  $[e_1, e_2]$  is the set of edges between  $e_1$  and  $e_2$ , including  $e_1$  and  $e_2$ , and  $]e_1, e_2]$  (resp.  $[e_1, e_2[$ ) denotes the set  $[e_1, e_2] \setminus \{e_1\}$  (resp.  $[e_1, e_2] \setminus \{e_2\}$ ). We also mix edges and vertices in this notation. For example,  $]e_1, T]$  is the set of edges between  $e_1$  and  $T$  different from  $e_1$  that may contain  $T$ . The edges are ordered naturally on such an interval of edges. The first is the first encountered when following the orientation of the interval induced by the orientation of the tree.

For two leaves  $\ell_1$  and  $\ell_2$  of  $\mathcal{T}$ ,  $N(\ell_1, \ell_2)$  denotes the node of  $\mathcal{T}$  such that  $[\ell_1, T] \cap [\ell_2, T] = [N(\ell_1, \ell_2), T]$ . For a subset  $\mathcal{E}$  of  $E(\mathcal{T})$ ,  $L(\mathcal{E}) = L(\mathcal{E}, \mathcal{T})$  denotes the set of leaves  $\ell$  of  $\mathcal{T}$  such that  $[\ell, T]$  contains at least one edge of  $\mathcal{E}$ . For a leaf  $\ell$  in  $L(\mathcal{E})$ , the closest edge to  $\ell$  in  $[\ell, T] \cap \mathcal{E}$  is denoted by  $\mathbf{e}(\mathcal{E}, \ell)$ .

**Examples 14.2.** In Figure 14.1, the set of leaves is  $L(\mathcal{T}) = \{\ell_0, \ell_1, \dots, \ell_5\}$ . For  $\mathcal{E} = \{e_6, e_{13}\}$ , we have  $L(\mathcal{E}) = L(\mathcal{T})$ ,  $\mathbf{e}(\mathcal{E}, \ell_0) = \mathbf{e}(\mathcal{E}, \ell_3) = \mathbf{e}(\mathcal{E}, \ell_4) = e_6$ , and  $\mathbf{e}(\mathcal{E}, \ell_5) = e_{13}$ .

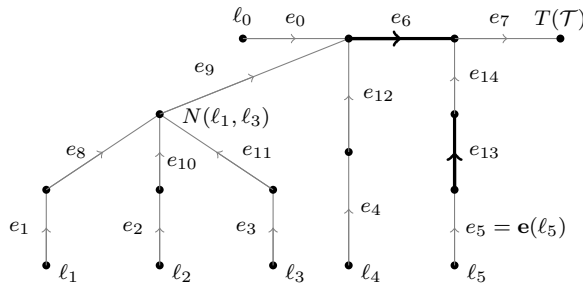


Figure 14.1: A tree  $\mathcal{T}$  with a bold codimension-one system of edges

This section is devoted to the study of the following space  $X(\mathcal{T})$  associated to an oriented tree  $\mathcal{T}$ .

**Definition 14.3.** For  $((u_e)_{e \in E(\mathcal{T})}) \in [0, \infty^{E(\mathcal{T})}$  and for a branch  $[\ell, N]$  of  $\mathcal{T}$ , define

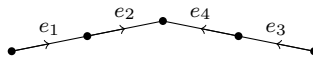
$$U([\ell, N]) = \prod_{e \in [\ell, N]} u_e.$$

Define  $X(\mathcal{T})$  to be the set of the elements  $((u_e)_{e \in E(\mathcal{T})})$  of  $[0, \infty^{E(\mathcal{T})}$  such that the equality

$$*(\ell_1, \ell_2) : U([\ell_1, N(\ell_1, \ell_2)]) = U([\ell_2, N(\ell_1, \ell_2)])$$

holds for any two leaves  $\ell_1$  and  $\ell_2$  of  $\mathcal{T}$ . Set  $\mathring{X}(\mathcal{T}) = X(\mathcal{T}) \cap ]0, \infty^{E(\mathcal{T})}$ .

**Example 14.4.** The space  $X(\mathcal{T})$  associated to the tree



is  $\{(u_1, u_2, u_3, u_4) \in [0, \infty^4 : u_1 u_2 = u_3 u_4\}$ . At its point  $(0, 0, 0, 0)$ , the four half-lines  $\mathbb{R}^+(1, 0, 0, 0)$ ,  $\mathbb{R}^+(0, 1, 0, 0)$ ,  $\mathbb{R}^+(0, 0, 1, 0)$ , and  $\mathbb{R}^+(0, 0, 0, 1)$  embed in  $X(\mathcal{T})$ . Their four independent unit tangent vectors at  $(0, 0, 0, 0)$  generate  $\mathbb{R}^4$ , but the complement of  $X(\mathcal{T})$  is dense in  $[0, \infty^4$ . So  $X(\mathcal{T})$  is not a submanifold with ridges of  $[0, \infty^4$ .

**Remarks 14.5.** Let  $[\ell_1, N]$  and  $[\ell_2, N]$  be two branches of  $\mathcal{T}$  ending at the same vertex  $N$ . If  $((u_e)_{e \in E(\mathcal{T})}) \in X(\mathcal{T})$ , then we have  $U([\ell_1, N]) = U([\ell_2, N])$ .

Let  $\ell_0$  be a leaf of  $\mathcal{T}$ . Then  $\mathring{X}(\mathcal{T})$  is the set of the elements  $((u_e)_{e \in E(\mathcal{T})})$  of  $]0, \infty^{E(\mathcal{T})}$  such that  $U([\ell, T]) = U([\ell_0, T])$  for any leaf  $\ell$  of  $L(\mathcal{T}) \setminus \{\ell_0\}$ . Indeed, the equations  $*(\ell_1, \ell_2)$  of Definition 14.3 are equivalent to  $U([\ell_1, T]) = U([\ell_2, T])$  when no variable is zero.

**Definition 14.6.** A *reducing system of edges* in an oriented tree  $\mathcal{T}$  is a set  $\mathcal{E}_r$  of edges such that  $L(\mathcal{E}_r) = L(\mathcal{T}) \setminus \{\ell_0\}$  for some leaf  $\ell_0$  and  $\mathbf{e}(\mathcal{E}_r, \cdot)$  is a bijection from  $L(\mathcal{E}_r)$  to  $\mathcal{E}_r$ . For such a reducing system  $\mathcal{E}_r$ , the leaf  $\ell_0$  of  $L(\mathcal{T}) \setminus L(\mathcal{E}_r)$  is denoted by  $\ell_0(\mathcal{E}_r)$ . A *maximal free system of edges* of  $\mathcal{T}$  is the complement  $E(\mathcal{T}) \setminus \mathcal{E}_r$  of a reducing system  $\mathcal{E}_r$  of edges of  $\mathcal{T}$  in  $E(\mathcal{T})$ .

**Examples 14.7.** For example, for every leaf  $\ell_0$  of  $\mathcal{T}$ , the set of edges adjacent to the leaves of  $L(\mathcal{T}) \setminus \{\ell_0\}$  is a reducing system of edges. In Figure 14.1,  $\{e_1, e_{10}, e_9, e_{12}, e_{13}\}$  is also a reducing system of edges.

**Lemma 14.8.** *Let  $\mathcal{E}_r$  be a reducing system of edges of  $\mathcal{T}$ . Let  $e_1$  be an edge of  $\mathcal{E}_r$ . Let  $\ell = \ell(\mathcal{E}_r, e_1)$  be the leaf of  $\mathcal{T}$  such that  $e_1 = \mathbf{e}(\mathcal{E}_r, \ell)$ . There exists a unique leaf  $\ell' = \ell'(\mathcal{E}_r, e_1)$  of  $\mathcal{T}$  such that  $e_1 \in [\ell, N(\ell, \ell')]$  and  $e_1$  is the only element of  $\mathcal{E}_r$  in  $[\ell, N(\ell, \ell')] \cup [\ell', N(\ell, \ell')]$ . Then the equation  $\ast(\ell, \ell')$  may be written as*

$$u_{e_1} = \frac{U([\ell', N(\ell, \ell')])}{\prod_{e \in [\ell, N(\ell, \ell')] \setminus \{e_1\}} u_e} = u_{e_1} (E(\mathcal{T}) \setminus \mathcal{E}_r; (u_e)_{e \in E(\mathcal{T}) \setminus \mathcal{E}_r})$$

when  $x = ((u_e)_{e \in E(\mathcal{T})}) \in \mathring{X}(\mathcal{T})$ . So it determines  $u_{e_1}$  in terms of the variables associated to  $E(\mathcal{T}) \setminus \mathcal{E}_r$  in  $\mathring{X}(\mathcal{T})$ . Moreover, we have

$$\frac{du_{e_1}}{u_{e_1}} = \sum_{e \in [\ell', N(\ell, \ell')]} \frac{du_e}{u_e} - \sum_{e \in [\ell, N(\ell, \ell')] \setminus \{e_1\}} \frac{du_e}{u_e}$$

in  $\mathring{X}(\mathcal{T})$ .

PROOF: If there is an edge of  $\mathcal{E}_r$  after  $e_1$  on  $[\ell, T]$ , let  $e_2 = \mathbf{e}(\mathcal{E}_r, \ell')$  be the second edge of  $\mathcal{E}_r$  on  $[\ell, T]$ . This defines  $\ell'$ . Otherwise, set  $\ell' = \ell_0(\mathcal{E}_r)$ . See Figure 14.2. This proves the existence of  $\ell'$  such that  $e_1$  is the only element of  $\mathcal{E}_r$  in  $[\ell, N(\ell, \ell')] \cup [\ell', N(\ell, \ell')]$ . For such an  $\ell' \neq \ell_0(\mathcal{E}_r)$ ,  $\mathbf{e}(\mathcal{E}_r, \ell')$  must be the second edge of  $\mathcal{E}_r$  on  $[\ell, T]$ . So the condition of the statement determines  $\ell'$ .  $\square$

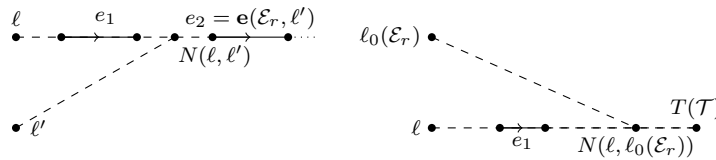


Figure 14.2: The two cases in Lemma 14.8

**Lemma 14.9.** *The set  $\mathring{X}(\mathcal{T})$  is a smooth manifold of dimension*

$$d(\mathcal{T}) = |E(\mathcal{T})| - |L(\mathcal{T})| + 1.$$

For every maximal free system  $\mathcal{E}_b$  of edges of  $\mathcal{T}$ ,  $\mathring{X}(\mathcal{T})$  is freely parametrized by the (variables of) the edges of  $\mathcal{E}_b$ . For a subset  $\mathcal{E}$  of cardinality  $d(\mathcal{T})$ , the form  $\wedge_{e \in \mathcal{E}} du_e$  is a nonvanishing volume form on  $\mathring{X}(\mathcal{T})$  if and only if  $\mathcal{E}$  is a maximal free system of edges of  $\mathcal{T}$ .

PROOF: Let  $\mathcal{E}_b$  be a maximal free system of edges of  $\mathcal{T}$ , and let  $\mathcal{E}_r$  be its complement. An edge  $e$  of  $\mathcal{E}_r$  may be expressed as  $\mathbf{e}(\mathcal{E}_r, \ell(\mathcal{E}_r, e))$  for a unique  $\ell(\mathcal{E}_r, e)$  of  $L(\mathcal{E}_r)$ , and any associated variable  $u_e$  can be expressed in terms of the variables associated to  $\mathcal{E}_b$  as in Lemma 14.8. Furthermore, any  $((u_e)_{e \in E(\mathcal{T})}) \in ]0, \infty[^{E(\mathcal{T})}$  such that

$$U([\ell(\mathcal{E}_r, e), T]) = U([\ell_0(\mathcal{E}_r), T]),$$

for any  $e$  in  $\mathcal{E}_r$ , is in  $\mathring{X}(\mathcal{T})$ . The equation defining  $u_{e_1}$  in Lemma 14.8, for  $e_1 \in \mathcal{E}_r$ , implies  $\ast(\ell(\mathcal{E}_r, e_1), \ell')$ , where  $\ell' = \ell_0(\mathcal{E}_r)$  if  $e_1$  is the only edge of  $\mathcal{E}_r$  in  $[\ell(\mathcal{E}_r, e_1), T]$ . Otherwise, set  $\ell_{e_1} = \ell(\mathcal{E}_r, e_1)$  and let  $e_1, e_2, \dots, e_k$  denote the edges of  $\mathcal{E}_r$  on  $[\ell_{e_1}, T]$ , where  $e_i = \mathbf{e}(\mathcal{E}_r, \ell_{e_i})$  denotes the  $i^{\text{th}}$  edge that we meet from  $\ell_{e_1}$  to  $T$ . Applying Lemma 14.8 to  $e_i$  defines  $u_{e_i}$  by an equation, which is equivalent to  $\ast(\ell_{e_i}, \ell_{e_{i+1}})$  if  $i < k$ , and to  $\ast(\ell_{e_k}, \ell_0(\mathcal{E}_r))$  if  $i = k$ . These equations together imply  $\ast(\ell_{e_1}, \ell_0(\mathcal{E}_r))$  for any  $\ell_{e_1}$  of  $L(\mathcal{T})$ .

Therefore, any  $x = ((u_e)_{e \in E(\mathcal{T})}) \in ]0, \infty[^{E(\mathcal{T})}$  satisfying the equations of Lemma 14.8 defining the  $u_e$  for  $e \in \mathcal{E}_r$  is in  $\mathring{X}(\mathcal{T})$ . Thus the space  $\mathring{X}(\mathcal{T})$  is freely parametrized by the (variables of) the edges of  $\mathcal{E}_b$ . So  $\mathring{X}(\mathcal{T})$  is a smooth submanifold of  $\mathbb{R}^{E(\mathcal{T})}$  of dimension

$$d(\mathcal{T}) = |E(\mathcal{T})| - |L(\mathcal{T})| + 1.$$

Let  $\mathcal{E}$  be a subset of cardinality  $d(\mathcal{T})$ , such that the form  $\wedge_{e \in \mathcal{E}} du_e$  is a nonvanishing volume form on  $\mathring{X}(\mathcal{T})$ . Set  $\mathcal{E}^c = E(\mathcal{T}) \setminus \mathcal{E}$ . Let us show that the map  $\mathbf{e}(\mathcal{E}^c, \cdot): L(\mathcal{E}^c) \rightarrow \mathcal{E}^c$  is injective. If  $\mathbf{e}(\mathcal{E}^c, \ell_1) = \mathbf{e}(\mathcal{E}^c, \ell_2)$ , for two leaves  $\ell_1$  and  $\ell_2$ , then  $[\ell_1, N(\ell_1, \ell_2)] \cup [\ell_2, N(\ell_1, \ell_2)] \subset \mathcal{E}$  and the relation  $U([\ell_1, N(\ell_1, \ell_2)]) = U([\ell_2, N(\ell_1, \ell_2)])$  gives rise to a nontrivial relation between the forms  $du_e$  for  $e \in [\ell_1, N(\ell_1, \ell_2)] \cup [\ell_2, N(\ell_1, \ell_2)]$ .

If  $L \setminus L(\mathcal{E}^c)$  has two distinct elements  $\ell_1$  and  $\ell_2$ , then we similarly have a nontrivial linear relation between coordinate forms  $du_e$  associated to edges of  $[\ell_1, N(\ell_1, \ell_2)] \cup [\ell_2, N(\ell_1, \ell_2)]$ . Therefore, the cardinality of  $L(\mathcal{E}^c)$  is at least  $|L(\mathcal{T})| - 1$ , which is the cardinality of  $\mathcal{E}^c$ . So  $\mathbf{e}(\mathcal{E}^c, \cdot)$  is a bijection.  $\square$

**Definition 14.10.** A *codimension-one system of edges* in an oriented tree  $\mathcal{T}$  is a set  $\mathcal{E}$  of edges such that there is exactly one edge of  $\mathcal{E}$  in any path from a leaf to the top of  $\mathcal{T}$ .

**Examples 14.11.** An example of a codimension-one system of edges of  $\mathcal{T}$  is the set of edges that start at leaves. In Figure 14.1,  $\mathcal{E} = \{e_6, e_{13}\}$ ,  $\{e_7\}$ , and  $\{e_0, e_9, e_{12}, e_5\}$  are other codimension-one systems of edges of  $\mathcal{T}$ .

**Lemma 14.12.** For any codimension-one system  $\mathcal{E}_1$  of  $X(\mathcal{T})$  and any edge  $e_0 \in \mathcal{E}_1$ ,  $\mathcal{E}_1 \setminus \{e_0\}$  can be completed to a reducing system that does not contain  $e_0$ .



PROOF: For each element  $e$  of  $\mathcal{E}_1$ , choose a leaf  $\ell_e$  such that  $\mathbf{e}(\mathcal{E}_1, \ell_e) = e$ . Let  $L_1 = \{\ell_e, e \in \mathcal{E}_1\}$  be the set of these leaves, and set  $L_2 = L(\mathcal{T}) \setminus L_1$ . Let  $\mathcal{E}_2 = \mathbf{e}(L_2)$  be the set of edges adjacent to the leaves of  $L_2$ . Then  $\mathcal{E}_1 \cup \mathcal{E}_2 \setminus \{e_0\}$  is a reducing system of  $\mathcal{T}$ .  $\square$

**Lemma 14.13.** *The closure of  $\mathring{X}(\mathcal{T})$  in  $[0, \infty[^{E(\mathcal{T})}$  is  $X(\mathcal{T})$ .*

Let  $x = ((u_e)_{e \in E(\mathcal{T})})$  be an element of  $X(\mathcal{T}) \setminus \mathring{X}(\mathcal{T})$ . Let  $\mathcal{E}(x)$  denote the set of edges of  $\mathcal{T}$  such that  $u_e = 0$ . Then the image  $\mathcal{E}_1(x)$  of  $(\mathbf{e}(\mathcal{E}(x), \cdot): L(\mathcal{E}(x)) \rightarrow \mathcal{E}(x))$  is a codimension-one system of edges of  $\mathcal{T}$ .

Let  $X_{\mathcal{E}(x)}(\mathcal{T})$  be the set of elements  $x'$  of  $X(\mathcal{T})$  such that  $\mathcal{E}(x') = \mathcal{E}(x)$ . Then  $X_{\mathcal{E}(x)}(\mathcal{T})$  is a smooth manifold of dimension  $d(\mathcal{T}) - 1 - (|\mathcal{E}(x)| - |\mathcal{E}_1(x)|)$ .

PROOF: Let  $x = ((u_e)_{e \in E(\mathcal{T})}) \in X(\mathcal{T}) \setminus \mathring{X}(\mathcal{T})$ . Let us first prove that the image  $\mathcal{E}_1(x)$  of  $\mathbf{e}(\mathcal{E}(x), \cdot): L(\mathcal{E}(x)) \rightarrow \mathcal{E}(x)$  is a codimension-one system of edges of  $\mathcal{T}$ . Since  $\mathcal{E}(x) \neq \emptyset$ , there is a leaf  $\ell$  of  $\mathcal{T}$  such that  $U([\ell, T]) = 0$ . This implies  $U([\ell, T]) = 0$  for all leaves. Thus,  $L(\mathcal{E}(x)) = L(\mathcal{T})$ . Furthermore no branch  $[\ell, T]$  can contain more than one edge of  $\mathcal{E}_1(x)$ . Otherwise, the first two edges of  $\mathcal{E}_1(x)$  on such a branch  $[\ell_1, T]$  would be  $\mathbf{e}(\mathcal{E}(x), \ell_1)$  and  $\mathbf{e}(\mathcal{E}(x), \ell_2)$ , respectively, and  $\mathbf{e}(\mathcal{E}(x), \ell_1)$  would be the only edge of  $\mathcal{E}(x)$  on  $[\ell_1, N(\ell_1, \ell_2)] \cup [\ell_2, N(\ell_1, \ell_2)]$ . Then we would have  $U([\ell_1, N(\ell_1, \ell_2)]) = 0$  and  $U([\ell_2, N(\ell_1, \ell_2)]) \neq 0$ . So  $*(\ell_1, \ell_2)$  would not be satisfied. So  $\mathcal{E}_1(x)$  is a codimension-one system of edges of  $\mathcal{T}$ .

Now, it suffices to prove the following two assertions.

- $x$  is in the closure of  $\mathring{X}(\mathcal{T})$  in  $[0, \infty[^{E(\mathcal{T})}$ , and
- $X_{\mathcal{E}(x)}(\mathcal{T})$  is a smooth manifold of dimension  $d(\mathcal{T}) - 1 - (|\mathcal{E}(x)| - |\mathcal{E}_1(x)|)$ .

Let us first prove them when  $\mathbf{e}(\mathcal{E}(x), \cdot): L(\mathcal{T}) \rightarrow \mathcal{E}_1(x)$  is injective. Define  $x(t) = ((u_e(t))_{e \in E(\mathcal{T})}) \in \mathring{X}(\mathcal{T})$  from  $x = ((u_e)_{e \in E(\mathcal{T})})$  for  $t \in ]0, \infty[$ , as follows. Pick  $\ell_0 \in L(\mathcal{T})$ , and set  $e_0 = \mathbf{e}(\mathcal{E}(x), \ell_0)$ . Replace all the variables  $u_e$  for  $e \in \mathcal{E}(x) \setminus \mathcal{E}_1(x)$  by  $t$ , replace  $u_{e_0}$  by  $t^k$  for some positive integer  $k$ , and leave the variables associated to the edges of  $E(\mathcal{T}) \setminus \mathcal{E}(x)$  (which are not zero) unchanged. For an edge  $f = \mathbf{e}(\mathcal{E}_1(x), \ell_f)$  of  $\mathcal{E}_1(x) \setminus \{e_0\}$ , set

$$u_f(t) = \frac{U([\ell_0, T])(t)}{\prod_{e \in [\ell_f, T] \setminus \{f\}} u_e(t)}.$$

Recall that  $f$  is the only edge of  $\mathcal{E}_1(x)$  on  $[\ell_f, T]$  and that  $e_0$  is the only edge of  $\mathcal{E}_1(x)$  on  $[\ell_0, T]$ . Then  $u_f(t)$  is equal to  $\alpha t^{k+r(f)}$  for some  $\alpha > 0$  and some  $r(f) \in \mathbb{Z}$ . Choose  $k$  so that  $k + r(f) \geq 1$  for any  $f \in \mathcal{E}_1(x)$ . Then  $x(t)$  tends to  $x$  when  $t$  tends to zero. Furthermore, since all  $u_e(t)$  are nonzero, the

defining equations for the  $u_f(t)$  are equivalent to the equations  $U([\ell, T])(t) = U([\ell_0, T])(t)$ . They are satisfied for any  $\ell \in L(\mathcal{T})$ , so  $x(t) \in \overset{\circ}{X}(\mathcal{T})$ . This proves that  $x$  is in the closure of  $\overset{\circ}{X}(\mathcal{T})$ .

The defining equations of  $X(\mathcal{T})$  are satisfied as soon as the  $u_e$ , for  $e \in \mathcal{E}_1(x)$ , vanish (still under the assumption that  $\mathbf{e}(\mathcal{E}(x), \cdot): L(\mathcal{E}(x)) \rightarrow \mathcal{E}(x)$  is injective). Therefore,  $X_{\mathcal{E}(x)}(\mathcal{T})$  is a manifold freely parametrized by the variables corresponding to the edges of  $E(\mathcal{T}) \setminus \mathcal{E}(x)$ . Its dimension is  $|E(\mathcal{T})| - |\mathcal{E}(x)| - |L(\mathcal{T})| + |\mathcal{E}_1(x)| = d(\mathcal{T}) - 1 - (|\mathcal{E}(x)| - |\mathcal{E}_1(x)|)$ . The two assertions are proved when

$$\mathbf{e}(\mathcal{E}(x), \cdot): L(\mathcal{E}(x)) \rightarrow \mathcal{E}(x)$$

is injective.

In general, choose a leaf  $\ell_e$  such that  $\mathbf{e}(\mathcal{E}(x), \ell_e) = e$  for each element  $e$  of  $\mathcal{E}_1(x)$ . Let  $L_1 = \{\ell_e, e \in \mathcal{E}_1(x)\}$  be the set of these leaves. Let  $\mathcal{T}_1$  be the subtree of  $\mathcal{T}$  such that  $E(\mathcal{T}_1) = \cup_{e \in \mathcal{E}_1(x)} [\ell_e, T]$  (so  $L(\mathcal{T}_1) = L_1$ ). Let  $x_1$  be the natural projection of  $x$  in  $]0, \infty[^{E(\mathcal{T}_1)}$ . Note  $\mathcal{E}(x) \subseteq E(\mathcal{T}_1)$  and  $\mathcal{E}(x) = \mathcal{E}(x_1)$ . Also note that the restriction of  $\mathbf{e}(\mathcal{E}(x), \cdot)$  to  $L(\mathcal{T}_1)$  is the map  $\mathbf{e}_{\mathcal{T}_1}(\mathcal{E}(x_1), \cdot)$  associated to  $\mathcal{T}_1$ , which is injective.

In particular, the first part of the proof expresses  $x_1$  as a limit at 0 of some continuous function  $x_1(\cdot): ]0, \infty[ \rightarrow \overset{\circ}{X}(\mathcal{T}_1)$ , such that  $u_e(t)$  is constant for any edge  $e$  of  $\mathcal{E}(\mathcal{T}_1) \setminus \mathcal{E}(x)$ . Define  $x(\cdot): ]0, \infty[ \rightarrow ]0, \infty[^{E(\mathcal{T})}$  so that the variables  $u_e(t)$  for  $e \notin \mathcal{E}(x)$  are constant (and different from zero) and the variables  $u_e(t)$  for  $e \in \mathcal{E}(\mathcal{T}_1)$  are the same for  $x(t)$  and  $x_1(t)$ . Let  $L_2 = L(\mathcal{T}) \setminus L_1$ . For  $\ell_2 \in L_2$ , we have  $\mathbf{e}(\mathcal{E}(x), \ell_2) = \mathbf{e}(\mathcal{E}(x), \ell_1(\ell_2))$  for a unique  $\ell_1(\ell_2)$  of  $L_1$ . The equation  $*(\ell_2, \ell_1(\ell_2))$  between nonvanishing constant products holds for  $x(t)$  for any  $t \in ]0, \infty[$ , and it implies that  $U([\ell_2, T])(t) = U([\ell_1(\ell_2), T])(t) = U([\ell_0, T])(t)$ . So  $x(\cdot)$  is valued in  $\overset{\circ}{X}(\mathcal{T})$ . Its limit at 0 is  $x$ . So  $x$  is in the closure of  $\overset{\circ}{X}(\mathcal{T})$ .

Let  $\mathcal{E}_2 = \mathbf{e}(L_2)$  be the set of edges adjacent to the leaves of  $L_2$ . Set  $\mathcal{E}_3 = E(\mathcal{T}) \setminus (E(\mathcal{T}_1) \cup \mathcal{E}_2)$ . Then any element  $x'$  of  $X_{\mathcal{E}(x)}(\mathcal{T})$  is determined by its projection  $x'_1 \in X_{\mathcal{E}(x)}(\mathcal{T}_1)$  and by the free nonzero variables associated to the edges of  $\mathcal{E}_3$ . More precisely, for an edge  $e = \mathbf{e}(\ell_2 \in L_2)$ , the equation  $*(\ell_2, \ell_1(\ell_2))$  between nonvanishing products determines  $u'_e$  as a function of  $x'_1$  and the free nonzero variables associated to the edges of  $\mathcal{E}_3$ . For elements  $((u_e)_{e \in E(\mathcal{T})}) \in \{0\}^{\mathcal{E}(x)} \times ]0, \infty[^{E(\mathcal{T}) \setminus \mathcal{E}(x)}$ , if the equations  $*(\ell_2, \ell_1(\ell_2))$  are satisfied for all  $\ell_2 \in L_2$ , then all the equations  $*(\ell_2, \ell')$  for  $\ell_2 \in L_2$  and  $\ell' \in L(\mathcal{T})$  are satisfied, as we prove below. Let  $\ell_2 \in L_2$ , and let  $N(\ell_1(\ell_2))$  denote the closest node to  $\mathbf{e}(\mathcal{E}(x), \ell_2) = \mathbf{e}(\mathcal{E}(x), \ell_1(\ell_2))$  in  $[\ell_1(\ell_2), \mathbf{e}(\mathcal{E}(x), \ell_2)]$ . See Figure 14.3. Then  $*(\ell_2, \ell_1(\ell_2))$  is equivalent to  $U([\ell_2, N(\ell_1(\ell_2))]) = U([\ell_1(\ell_2), N(\ell_1(\ell_2))])$ . In particular, if  $\mathbf{e}(\mathcal{E}(x), \ell_2) =$

$\mathbf{e}(\mathcal{E}(x), \ell')$  for  $\ell' \in L(\mathcal{T})$  as in Figure 14.3, then  $*(\ell', \ell_1(\ell_2))$  is equivalent to  $U([\ell', N(\ell_1(\ell_2))]) = U([\ell_1(\ell_2), N(\ell_1(\ell_2))])$ . So  $*(\ell_2, \ell_1(\ell_2))$  and  $*(\ell', \ell_1(\ell_2))$  imply  $*(\ell_2, \ell')$ . If  $\mathbf{e}(\mathcal{E}(x), \ell_2) \neq \mathbf{e}(\mathcal{E}(x), \ell')$ , then  $\mathbf{e}(\mathcal{E}(x), \ell_2) \in [\ell_2, N(\ell_2, \ell')]$  and  $\mathbf{e}(\mathcal{E}(x), \ell') \in [\ell', N(\ell_2, \ell')]$ . So  $*(\ell_2, \ell')$  is equivalent to  $0 = 0$  and is satisfied. Thus  $X_{\mathcal{E}(x)}(\mathcal{T})$  is a smooth manifold whose dimension is  $|\mathcal{E}_3| + |E(\mathcal{T}_1)| - |\mathcal{E}(x)| - |L(\mathcal{T}_1)| + |\mathcal{E}_1(x)| = |E(\mathcal{T})| - |L_2| - |\mathcal{E}(x)| - |L(\mathcal{T}_1)| + |\mathcal{E}_1(x)|$ .  $\square$

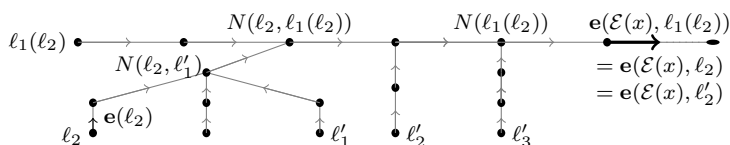


Figure 14.3: Example of leaves  $\ell'_i$  such that  $\mathbf{e}(\mathcal{E}(x), \ell'_i) = \mathbf{e}(\mathcal{E}(x), \ell_2)$  for the proof of Lemma 14.13

**Lemma 14.14.** *The codimension-one faces of  $X(\mathcal{T})$  are in one-to-one correspondence with the codimension-one systems of edges of  $\mathcal{T}$ . In a neighborhood of such an open face,  $X(\mathcal{T})$  has the structure of a smooth manifold with boundary.*

PROOF: According to Lemma 14.13 above, if  $X_{\mathcal{E}}(\mathcal{T})$  is a nonempty manifold of dimension  $d(\mathcal{T}) - 1$ , then  $\mathcal{E}$  is a codimension-one system. Let  $\mathcal{E}_c$  be a codimension-one system of edges of  $\mathcal{T}$ . For any edge of  $e_0 \in \mathcal{E}_c$ ,  $\mathcal{E}_c \setminus \{e_0\}$  can be completed to a reducing system  $\mathcal{E}_r$  that does not contain  $e_0$  as in Lemma 14.12. In particular,  $\overset{\circ}{X}(\mathcal{T})$  is freely parametrized by the variables associated to the edges of  $E(\mathcal{T}) \setminus \mathcal{E}_r$ . When all these variables are nonzero except maybe  $(u_{e_0} \in [0, \varepsilon])$ , we get a local parametrization by  $[0, \varepsilon[ \times ]0, \varepsilon[^{E(\mathcal{T}) \setminus (\mathcal{E}_r \cup \{e_0\})}$  near the locus  $u_{e_0} = 0$ . If  $e \in \mathcal{E}_c \setminus \{e_0\}$  and  $e = \mathbf{e}(\mathcal{E}_r, \ell_e)$ , then the variable  $u_e$  may be expressed as

$$u_e = \frac{\prod_{f \in [\ell_0(\mathcal{E}_r), N(\ell_0(\mathcal{E}_r), \ell_e)] \setminus \{e_0\}} u_f}{\prod_{f \in [\ell_e, N(\ell_0(\mathcal{E}_r), \ell_e)] \setminus \{e\}} u_f} u_{e_0}$$

with respect to this local parametrization.  $\square$

We finish this section by proving a version of Stokes' theorem for spaces modelled by products of  $X(\mathcal{T})$  by a manifold. Let us first introduce its statement given in Theorem 14.16.

**Lemma 14.15.** *Let  $\varepsilon \in ]0, \infty[$ . Let  $X^\varepsilon(\mathcal{T}) = X(\mathcal{T}) \cap [0, \varepsilon]^{E(\mathcal{T})}$ . Assume that  $X(\mathcal{T})$  is oriented. Let  $n \in \mathbb{N}$ . For any smooth form  $\Omega$  on  $[0, \varepsilon]^{E(\mathcal{T})} \times [0, 1]^n$  of degree  $(d(\mathcal{T}) + n)$ , the integral  $\int_{X^\varepsilon(\mathcal{T}) \times [0, 1]^n} \Omega$  of  $\Omega$  along the interior of  $X^\varepsilon(\mathcal{T}) \times [0, 1]^n$  is absolutely convergent.*

For an ordered subset  $\mathcal{E}$  of  $E(\mathcal{T})$ , we set  $\Omega_{\mathcal{E}} = \wedge_{e \in \mathcal{E}} du_e$ , where the factors are ordered with respect to the order of  $\mathcal{E}$ .

**PROOF OF LEMMA 14.15:** Any smooth form  $\Omega$  on  $[0, \varepsilon]^{E(\mathcal{T})} \times [0, 1]^n$  of degree  $(d(\mathcal{T}) + n)$  is a sum of forms  $g_{\mathcal{E}} \Omega_{\mathcal{E}} \wedge (\wedge_{i=1}^n dx_i)$ , for ordered subsets  $\mathcal{E}$  of  $E(\mathcal{T})$  of cardinality  $d(\mathcal{T})$  and for smooth maps  $g_{\mathcal{E}}: [0, \varepsilon]^{E(\mathcal{T})} \times [0, 1]^n \rightarrow \mathbb{R}$ , up to forms that vanish identically on the interior of  $X^\varepsilon(\mathcal{T}) \times [0, 1]^n$ . These forms are bounded on  $[0, \varepsilon]^{E(\mathcal{T})} \times [0, 1]^n$ . They are zero on the interior of  $X^\varepsilon(\mathcal{T}) \times [0, 1]^n$  unless  $\mathcal{E}$  is a maximal free system, according to Lemma 14.9. When  $\mathcal{E}$  is a maximal free system, Lemma 14.9 ensures that  $\overset{\circ}{X}(\mathcal{T})$  is freely parametrized by the variables associated to the edges of  $\mathcal{E}$ .  $\square$

We restrict to the compact subspace  $X^\varepsilon(\mathcal{T})$  of  $X(\mathcal{T})$ . The subspace  $X^\varepsilon(\mathcal{T})$  is the closure of the open manifold  $\overset{\circ}{X}(\mathcal{T}) \cap ]0, \varepsilon]^{E(\mathcal{T})}$ . For a maximal free system  $\mathcal{E}_b$  of edges,  $\overset{\circ}{X}(\mathcal{T})$  is freely parametrized by the variables  $u_f$  for  $f \in \mathcal{E}_b$  as in Lemma 14.9. For  $e_1 \in E(\mathcal{T}) \setminus \mathcal{E}_b$ , the variable  $u_{e_1}$  is the function  $u_{e_1}(\mathcal{E}_b; V \in ]0, \infty]^{E_b})$  of Lemma 14.8. This function is a smooth quotient of monomials in the  $u_f$  for  $f \in \mathcal{E}_b$ . Then  $\overset{\circ}{X}(\mathcal{T}) \cap ]0, \varepsilon]^{E(\mathcal{T})}$  is parametrized by the set of elements  $V \in ]0, \varepsilon]^{E_b}$  such that  $u_e(\mathcal{E}_b; U) \leq \varepsilon$  for any  $e \in E(\mathcal{T}) \setminus \mathcal{E}_b$ . It is a subspace of  $]0, \varepsilon]^{E_b}$  whose boundary can be stratified so that the open strata of the boundary are locally defined by an equation  $u_e(\mathcal{E}_b; V \in ]0, \infty]^{E_b}) = \varepsilon$ . Let  $\partial_\varepsilon X^\varepsilon(\mathcal{T})$  be the union of the corresponding open codimension-one faces of  $X^\varepsilon(\mathcal{T})$ . Then the union  $\check{\partial} X^\varepsilon(\mathcal{T})$  of the oriented open codimension-one faces of  $X^\varepsilon(\mathcal{T})$  is

$$\check{\partial} X^\varepsilon(\mathcal{T}) = \partial_o X^\varepsilon(\mathcal{T}) \cup \partial_\varepsilon X^\varepsilon(\mathcal{T}),$$

where  $\partial_o X^\varepsilon(\mathcal{T})$  is the union of the intersection of the codimension-one faces of  $X(\mathcal{T})$  of Lemma 14.14 with  $X^\varepsilon(\mathcal{T})$ .

**Theorem 14.16.** *Let  $\varepsilon \in ]0, \infty[$ . Recall  $X^\varepsilon(\mathcal{T}) = X(\mathcal{T}) \cap [0, \varepsilon]^{E(\mathcal{T})}$ . Assume that  $X(\mathcal{T})$  is oriented. Let  $n \in \mathbb{N}$ . Let  $\omega$  be a smooth form on  $[0, \varepsilon]^{E(\mathcal{T})} \times [0, 1]^n$  of degree  $(d(\mathcal{T}) - 1 + n)$ . With the notation above, consider the dense part*

$$\check{\partial} \left( X^\varepsilon(\mathcal{T}) \times [0, 1]^n \right) = (\check{\partial} X^\varepsilon(\mathcal{T})) \times [0, 1]^n \cup \left( (-1)^{d(\mathcal{T})} \overset{\circ}{X}^\varepsilon(\mathcal{T}) \times \partial([0, 1]^n) \right)$$

of  $\partial(X^\varepsilon(\mathcal{T}) \times [0, 1]^n)$ .

Then the integral  $\int_{\partial(X^\varepsilon(\mathcal{T}) \times [0,1]^n)} \omega$  of  $\omega$  along  $\partial(X^\varepsilon(\mathcal{T}) \times [0,1]^n)$  is absolutely convergent. Furthermore, Stokes' theorem applies to this setting. So we have

$$\int_{\partial(X^\varepsilon(\mathcal{T}) \times [0,1]^n)} \omega = \int_{X^\varepsilon(\mathcal{T}) \times [0,1]^n} d\omega.$$

Let us prepare the proof of Theorem 14.16 with a few lemmas.

**Lemma 14.17.** *Let  $\mathcal{E}_1$  be an ordered subset of  $E(\mathcal{T})$  of cardinality  $d(\mathcal{T})$ . Let  $g$  be a smooth function on  $[0, \varepsilon]^{E(\mathcal{T})} \times [0, 1]^n$ . Then Theorem 14.16 holds for  $\omega = g\Omega_{\mathcal{E}_1} \wedge (\wedge_{i=2}^n dx_i)$ .*

PROOF: We have

$$\begin{aligned} \int_{X^\varepsilon(\mathcal{T}) \times [0,1]^n} d\omega &= (-1)^{d(\mathcal{T})} \int_{\dot{X}^\varepsilon(\mathcal{T}) \times [0,1]^n} \frac{\partial g}{\partial x_1} \Omega_{\mathcal{E}_1} \wedge (\wedge_{i=1}^n dx_i) \\ &= (-1)^{d(\mathcal{T})} \int_{\dot{X}^\varepsilon(\mathcal{T}) \times \partial[0,1] \times [0,1]^{n-1}} \omega \\ &= \int_{\partial(X^\varepsilon(\mathcal{T}) \times [0,1]^n)} \omega \end{aligned}$$

since  $\dot{X}^\varepsilon(\mathcal{T}) \times \partial[0,1] \times [0,1]^{n-1}$  is the only part of  $\partial(X^\varepsilon(\mathcal{T}) \times [0,1]^n)$  where the integral of  $\omega$  does not vanish.  $\square$

So it suffices to prove Theorem 14.16 for forms  $\omega = g\Omega_{\mathcal{E}_2} \wedge (\wedge_{i=1}^n dx_i)$  for ordered subsets  $\mathcal{E}_2$  of  $E(\mathcal{T})$  of cardinality  $d(\mathcal{T}) - 1$  such that  $\Omega_{\mathcal{E}_2}$  is not zero on  $\dot{X}^\varepsilon(\mathcal{T})$ . (If  $\Omega_{\mathcal{E}_2}$  is zero on  $\dot{X}^\varepsilon(\mathcal{T})$ , then it is also zero on  $X^\varepsilon(\mathcal{T})$ , and both sides of the equality to be proved are zero.)

**Lemma 14.18.** *Let  $\mathcal{E}_2$  be a subspace of  $E(\mathcal{T})$  of cardinality  $d(\mathcal{T}) - 1$  such that  $\Omega_{\mathcal{E}_2}$  is not zero on  $\dot{X}^\varepsilon(\mathcal{T})$ . Let  $\mathcal{E}_2^c = E(\mathcal{T}) \setminus \mathcal{E}_2$ . Then either  $L(\mathcal{E}_2^c) = L(\mathcal{T})$  or  $L(\mathcal{E}_2^c) = L(\mathcal{T}) \setminus \{\ell_0\}$  for a unique  $\ell_0 \in L(\mathcal{T})$ .*

*If  $L(\mathcal{E}_2^c) = L(\mathcal{T})$ , then  $\mathbf{e}(\mathcal{E}_2^c, \cdot)$  is a bijection from  $L(\mathcal{E}_2^c)$  to  $\mathcal{E}_2^c$ .*

*If  $L(\mathcal{E}_2^c) = L(\mathcal{T}) \setminus \{\ell_0\}$ , then  $\mathbf{e}(\mathcal{E}_2^c, \cdot)$  is a bijection from  $L(\mathcal{E}_2^c)$  to  $\mathcal{E}_2^c \setminus \{f\}$  for a unique element  $f$  of  $\mathcal{E}_2^c$ .*

PROOF: The set  $L(\mathcal{T}) \setminus L(\mathcal{E}_2^c)$  cannot contain two distinct leaves  $\ell_3$  and  $\ell_4$ , because Equation  $*(\ell_3, \ell_4)$  would imply that  $\Omega_{\mathcal{E}_2} = 0$ . Similarly,  $\mathbf{e}(\mathcal{E}_2^c, \cdot)$  cannot map two distinct leaves  $\ell_3$  and  $\ell_4$  of  $L(\mathcal{E}_2^c)$  to the same element. So  $\mathbf{e}(\mathcal{E}_2^c, \cdot)$  is injective. Use that the cardinality of  $\mathcal{E}_2^c$  is  $|L(\mathcal{T})|$  to conclude.  $\square$

For a subset  $\mathcal{E}'$  of  $\mathcal{E}$ , let  $p_{\mathcal{E}'}: X^\varepsilon(\mathcal{T}) \rightarrow [0, \varepsilon]^{\mathcal{E}'}$  denote the composition of the inclusion  $X^\varepsilon(\mathcal{T}) \hookrightarrow [0, \varepsilon]^{E(\mathcal{T})}$  with the natural projection.

**Lemma 14.19.** *Recall the assumptions of Lemma 14.18. Let  $\mathcal{E}_3$  be the set of edges  $f$  of  $\mathcal{T}$  such that  $\mathcal{E}_2 \cup \{f\}$  is a maximal free system of edges of  $\mathcal{T}$ . For*

an edge  $f$  of  $\mathcal{E}_3$  and an element  $(u_f, U) \in ]0, \varepsilon]^{f \cup \mathcal{E}_2}$ , let  $x_f(u_f, U)$  denote the element of  $\mathring{X}(\mathcal{T})$  such that  $p_{\{f\} \cup \mathcal{E}_2}(x_f(u_f, U)) = (u_f, U)$ .

Then there exist  $f_2 \in \mathcal{E}_3$  and piecewise smooth functions  $a$  and  $b$  from  $]0, \varepsilon]^{\mathcal{E}_2}$  to  $[0, \varepsilon]$ , such that  $X^\varepsilon(\mathcal{T}) \cap p_{\mathcal{E}_2}^{-1}(]0, \varepsilon]^{\mathcal{E}_2})$  is the set

$$\left\{ x_{f_2}(u_{f_2}, U) : U \in ]0, \varepsilon]^{\mathcal{E}_2} \cap p_{\mathcal{E}_2}(X^\varepsilon(\mathcal{T})), u_{f_2} \in [a(U), b(U)] \right\}.$$

Furthermore, the space  $]0, \varepsilon]^{\mathcal{E}_2} \cap p_{\mathcal{E}_2}(X^\varepsilon(\mathcal{T}))$  is a subspace of  $]0, \varepsilon]^{\mathcal{E}_2}$  determined by the condition  $b(U) - a(U) \geq 0$  and conditions  $u_e(U) \leq \varepsilon$  for  $e \notin \mathcal{E}_3$ , for functions  $u_e$  of  $U$  which are quotients of monomials in the variables  $u_g$  for  $g \in \mathcal{E}_2$ . Moreover, the boundary of  $X^\varepsilon(\mathcal{T})$  consists of

- $(d(\mathcal{T}) - 1)$ -dimensional strata where  $u_z$  vanishes for some  $z \in \mathcal{E}_2$ ,
- (negligible) strata of dimension less than  $(d(\mathcal{T}) - 1)$ , and
- $(d(\mathcal{T}) - 1)$ -dimensional strata of

$$\partial_b(X^\varepsilon(\mathcal{T})) = \left\{ x_{f_2}(b(U), U) : U \in ]0, \varepsilon]^{\mathcal{E}_2} \cap p_{\mathcal{E}_2}(X^\varepsilon(\mathcal{T})) \right\},$$

and

$$-\partial_a(X^\varepsilon(\mathcal{T})) = -\left\{ x_{f_2}(a(U), U) : U \in ]0, \varepsilon]^{\mathcal{E}_2} \cap p_{\mathcal{E}_2}(X^\varepsilon(\mathcal{T})) \right\},$$

which behave as standard codimension-one faces of  $X^\varepsilon(\mathcal{T})$ ,<sup>1</sup>

with respect to some natural stratification.

**PROOF OF THEOREM 14.16 ASSUMING LEMMA 14.19:** Consider a form  $\omega = g\Omega_{\mathcal{E}_2} \wedge (\wedge_{i=1}^n dx_i)$ , for an ordered subset  $\mathcal{E}_2$  of  $E(\mathcal{T})$  of cardinality  $d(\mathcal{T}) - 1$  and a smooth function  $g: [0, \varepsilon]^{E(\mathcal{T})} \times [0, 1]^n \rightarrow \mathbb{R}$ . Assume that the  $du_z$ , for  $z \in \mathcal{E}_2$ , are linearly independent in the  $C^\infty(\mathring{X}^\varepsilon(\mathcal{T}); \mathbb{R})$ -module  $\Omega^1(\mathring{X}^\varepsilon(\mathcal{T}))$ . The integral of

$$d\omega = \sum_{h \in E(\mathcal{T})} \frac{\partial g}{\partial u_h} du_h \wedge \Omega_{\mathcal{E}_2} \wedge (\wedge_{i=1}^n dx_i)$$

is absolutely convergent over the interior of  $X^\varepsilon \times [0, 1]^n$ .

With the notation of Lemma 14.19, we have

$$\int_{\partial(X^\varepsilon(\mathcal{T}) \times [0, 1]^n)} \omega = \int_{\partial_b(X^\varepsilon(\mathcal{T}) \times [0, 1]^n)} \omega - \int_{\partial_a(X^\varepsilon(\mathcal{T}) \times [0, 1]^n)} \omega$$

---

<sup>1</sup>The signs before  $\partial_b$  and  $\partial_a$  correspond to the orientations provided that  $du_{f_2} \wedge \Omega_{\mathcal{E}_2}$  orients  $X^\varepsilon(\mathcal{T})$ , which we assume without loss.

since  $\omega$  vanishes along the  $(d(\mathcal{T}) - 1)$ -dimensional strata where  $u_z$  vanishes for some  $z \in \mathcal{E}_2$ . On the other hand, we have

$$\int_{X^\varepsilon(\mathcal{T}) \times [0,1]^n} d\omega = \int_{[0,1]^n} \left( \int_{\dot{X}^\varepsilon(\mathcal{T})} \frac{\partial g_X}{\partial u_{f_2}} du_{f_2} \wedge \Omega_{\mathcal{E}_2} \right) \wedge_{i=1}^n dx_i,$$

where  $g_X$  is the restriction of  $g$  to  $X^\varepsilon(\mathcal{T}) \times [0,1]^n$ , where  $g_X$  and the  $u_h$  are functions of  $u_{f_2}$  and of the  $u_z$ , for  $z \in \mathcal{E}_2$ . For a fixed implicit  $(x_1, \dots, x_n)$ , we compute

$$\begin{aligned} \int_{\dot{X}^\varepsilon(\mathcal{T})} \frac{\partial g_X}{\partial u_{f_2}} du_{f_2} \wedge \Omega_{\mathcal{E}_2} &= \int_{U \in ]0,\varepsilon] \mathcal{E}_2 \cap p_{\mathcal{E}_2}(X^\varepsilon(\mathcal{T}))} \left( \int_{u_{f_2} \in [a(U), b(U)]} \frac{\partial g_X}{\partial u_{f_2}} du_{f_2} \right) \Omega_{\mathcal{E}_2} \\ &= \int_{U \in ]0,\varepsilon] \mathcal{E}_2 \cap p_{\mathcal{E}_2}(X^\varepsilon(\mathcal{T}))} (g_X(b(U)) - g_X(a(U))) \Omega_{\mathcal{E}_2}. \end{aligned}$$

□

Lemma 14.18 reduces the proof of Lemma 14.19 to the proofs of the following two lemmas.

**Lemma 14.20.** *Lemma 14.19 holds when  $L(\mathcal{E}_2^c) = L(\mathcal{T}) \setminus \{\ell_0\}$ .*

**Lemma 14.21.** *Lemma 14.19 holds when  $L(\mathcal{E}_2^c) = L(\mathcal{T})$ .*

**PROOF OF LEMMA 14.20:** Recall that  $L(\mathcal{E}_2^c) = L(\mathcal{T}) \setminus \{\ell_0\}$  and  $\mathbf{e}(\mathcal{E}_2^c, \cdot)$  is a bijection from  $L(\mathcal{E}_2^c)$  to  $\mathcal{E}_2^c \setminus \{f\}$  for a unique element  $f$  of  $\mathcal{E}_2^c$  from Lemma 14.18. There is a leaf  $\ell_1$  such that  $f \in [\ell_1, T]$ . Let  $f_2$  be the last edge (the closest to  $f$ ) of  $\mathcal{E}_2^c$  in  $[\ell_1, f[$ . Then we have  $f_2 = \mathbf{e}(\mathcal{E}_2^c, \ell_2)$  for some leaf  $\ell_2$  of  $L(\mathcal{E}_2^c)$ . If there is an edge of  $\mathcal{E}_2^c$  in  $]f, T]$ , define the leaf  $\ell_3$  such that  $\mathbf{e}(\mathcal{E}_2^c, \ell_3)$  is the first edge of  $\mathcal{E}_2^c$  in  $]f, T]$  as in Figure 14.4. Otherwise, set  $\ell_3 = \ell_0$ . Note that  $f$  and  $f_2$  are in  $\mathcal{E}_3$ .

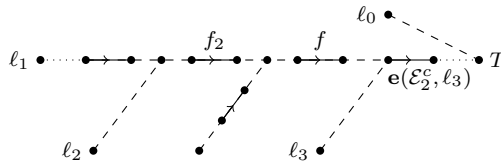


Figure 14.4: The edge  $f$ , when  $L(\mathcal{E}_2^c) \neq L(\mathcal{T})$ , in the proof of Theorem 14.16

The product  $u_{f_2} u_f$  is given by the expression

$$u_{f_2} u_f = \frac{U([\ell_3, N(\ell_2, \ell_3)])}{\prod_{z \in [\ell_2, N(\ell_2, \ell_3)] \setminus \{f, f_2\}} u_z} = \eta_2(U),$$

in  $U \in ]0, \varepsilon]^{E_2} \cap p_{E_2}(X^\varepsilon(\mathcal{T}))$ . In particular, we have  $u_f = \eta_2(U)/u_{f_2}$ , and the conditions  $u_f \leq \varepsilon$  and  $u_{f_2} \leq \varepsilon$  imply

$$\frac{\eta_2(U)}{\varepsilon} \leq u_{f_2} \leq \varepsilon.$$

Let  $e$  be an edge of  $\mathcal{E}_2^c$  different from  $f$ . Then  $e$  may be expressed as  $e = \mathbf{e}(\mathcal{E}_2^c \setminus \{f\}, \ell_e)$  for a unique  $\ell_e = \ell(\mathcal{E}_2^c \setminus \{f\}, e)$ . As in Lemma 14.8, there is a leaf  $\ell' = \ell'(\mathcal{E}_2^c \setminus \{f\}, e)$  of  $\mathcal{T}$  such that  $e$  is the only element of  $\mathcal{E}_2^c \setminus \{f\}$  in  $[\ell_e, N(\ell_e, \ell')] \cup [\ell', N(\ell_e, \ell')]$ . In particular, if  $e \neq f_2$ , then  $f_2$  does not belong to  $[\ell_e, N(\ell_e, \ell')] \cup [\ell', N(\ell_e, \ell')]$ . Note that  $f$  cannot be on  $[\ell', N(\ell_e, \ell')]$ . If  $f \notin [\ell_e, N(\ell_e, \ell_2)]$ , then  $u_e$  is a function of  $U \in ]0, \varepsilon]^{E_2}$ ,  $e \notin \mathcal{E}_3$ , and  $u_e$  is different from zero. If  $f \in [\ell_e, N(\ell_e, \ell_2)]$ , then  $N(\ell_e, \ell_2)$  is on  $]f_2, f[$  (equivalently,

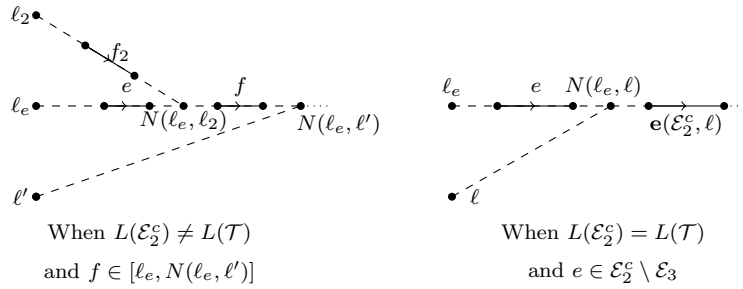


Figure 14.5: More figures for the proof of Theorem 14.16

$f_2 \in [\ell_2, N(\ell_e, \ell_2)]$  and  $f \notin [\ell_2, N(\ell_e, \ell_2)]$ ), as in the left part of Figure 14.5,  $e \in \mathcal{E}_3$ ,  $f_2$  is the unique element of  $\mathcal{E}_2^c$  in  $[\ell_2, N(\ell_e, \ell_2)]$ , and we have

$$u_e = \frac{U([\ell_2, N(\ell_e, \ell_2)])}{\prod_{z \in [\ell_e, N(\ell_e, \ell_2)] \setminus \{e\}} u_z}.$$

So, when  $U \in ]0, \varepsilon]^{E_2} \cap p_{E_2}(X^\varepsilon(\mathcal{T}))$  is fixed,  $u_e$  is a linear function  $u_e = \eta_{e,f_2}(U)u_{f_2}$  of  $u_{f_2}$ . In particular, we have  $u_{f_2} \leq \varepsilon/\eta_{e,f_2}(U)$ . Set  $a(U) = \eta_2(U)/\varepsilon$  and  $b(U) = \min_{e \in \mathcal{E}_3 \setminus \{f\}} (\varepsilon/\eta_{e,f_2}(U))$ , where  $\eta_{f_2,f_2} = 1$ . Then  $U \in ]0, \varepsilon]^{E_2}$  is in  $p_{E_2}(X^\varepsilon(\mathcal{T}))$  if and only if  $a(U) \leq b(U)$  and  $u_e(U) \leq \varepsilon$  for any  $e \in \mathcal{E}_2^c \setminus \mathcal{E}_3$ . Furthermore,  $X^\varepsilon(\mathcal{T}) \cap p_{E_2}^{-1}(]0, \varepsilon]^{E_2})$  is the set

$$\left\{ x_{f_2}(u_{f_2}, U) : U \in ]0, \varepsilon]^{E_2} \cap p_{E_2}(X^\varepsilon(\mathcal{T})), u_{f_2} \in [a(U), b(U)] \right\}.$$

The boundary part  $\partial_a(X^\varepsilon(\mathcal{T}))$ , corresponds to  $u_f = \varepsilon$ . The boundary part  $\partial_b(X^\varepsilon(\mathcal{T}))$ , along which  $u_e = \varepsilon$  for some  $e \in \mathcal{E}_3$ , lies in the intersection of



the nonsingular  $\mathring{X}$  with loci  $(u_e = \varepsilon)$  for some  $e \in \mathcal{E}_3$ . The locus  $(u_e = \varepsilon)$  is transverse to  $\mathring{X}$ , for any  $e \in \mathcal{E}_3$  since  $\mathcal{E}_2 \cup \{e\}$  is a maximal free system for such an  $e$ . So  $\mathring{X} \cap (u_e = \varepsilon)$  is a manifold of dimension  $(d(\mathcal{T}) - 1)$  for all  $e \in \mathcal{E}_3$ . Let  $e$  and  $e'$  be two distinct edges of  $\mathcal{E}_3$ . Then  $u_e = u_{e'} = \varepsilon$  implies a nontrivial equation among variables  $u_h$  associated to edges  $h$  of  $\mathcal{E}_2$  unless  $e$  and  $e'$  are both adjacent to leaves and meet at a node, which implies that  $u_e = u_{e'}$  on  $\mathring{X}$ . We may assume that this configuration never occurs, without loss of generality, because the space  $X(\mathcal{T})$  is canonically diffeomorphic to the space  $X(\mathcal{T}')$  obtained from  $\mathcal{T}$  by erasing  $e'$ . Except for this special configuration, the loci  $\mathring{X} \cap (u_e = u_{e'} = \varepsilon)$  are of dimension less than  $(d(\mathcal{T}) - 1)$ , and we have a stratification of  $\partial_a(X^\varepsilon(\mathcal{T}))$  and  $\partial_b(X^\varepsilon(\mathcal{T}))$ , where the  $(d(\mathcal{T}) - 1)$ -dimensional strata are the loci of  $\partial_a(X^\varepsilon(\mathcal{T}))$  and  $\partial_b(X^\varepsilon(\mathcal{T}))$  where  $u_e = \varepsilon$  for (exactly) one  $e$  of  $\mathcal{E}_3$ . These strata are smooth open  $(d(\mathcal{T}) - 1)$ -manifolds.  $\square$

**PROOF OF LEMMA 14.21:** Let us now assume that  $L(\mathcal{E}_2^c) = L(\mathcal{T})$ , and recall that  $\mathbf{e}(\mathcal{E}_2^c, \cdot)$  is a bijection from  $L(\mathcal{T})$  to  $\mathcal{E}_2^c$  in this case. The elements of  $\mathcal{E}_3$  are the edges  $e$  of  $\mathcal{E}_2^c$  such that there is no edge of  $\mathcal{E}_2$  on  $]e, T]$ . In particular,  $\mathcal{E}_3$  is a codimension-one system of edges of  $\mathcal{T}$ .

Let  $e = \mathbf{e}(\mathcal{E}_2^c, \ell_e)$  be an element of  $\mathcal{E}_2^c \setminus \mathcal{E}_3$ , let  $\ell$  be a leaf of  $\mathcal{T}$  such that the first edge of  $\mathcal{E}_2^c$  on  $]e, T]$  is  $\mathbf{e}(\mathcal{E}_2^c, \ell)$ . Then  $e$  is the only element of  $\mathcal{E}_2^c$  in  $[\ell_e, N(\ell, \ell_e)] \cup [\ell, N(\ell, \ell_e)]$ , as in Figure 14.5. So  $u_e$  depends only on the fixed variables of  $\mathcal{E}_2$ , and it is not zero.

If  $f \in \mathcal{E}_3$ , let  $\ell(f)$  denote the unique leaf such that  $L(\mathcal{E}_2^c \setminus \{f\}) = L(\mathcal{T}) \setminus \{\ell(f)\}$ . The variable  $u_f$  is a linear function  $u_f = \eta_f(U)u_{f_2}$  of  $u_{f_2}$  for one (arbitrary)  $f_2 \in \mathcal{E}_3$ ,

$$\eta_f(U) = \frac{\prod_{z \in [\ell(f_2), T] \setminus \{f_2\}} u_z}{\prod_{z \in [\ell(f), T] \setminus \{f\}} u_z}.$$

Here,  $a(U) = 0$  and  $b(U) = \min_{f \in \mathcal{E}_3} (\varepsilon / \eta_f(U))$ . So  $\partial_a(X^\varepsilon(\mathcal{T}))$  is the codimension-one face associated to  $\mathcal{E}_3$ , as in Lemma 14.14, while  $\partial_b(X^\varepsilon(\mathcal{T}))$  may be stratified as in the previous proof.  $\square$

The proof of Theorem 14.16 is finished. So we have the announced Stokes formula for all forms as in its statement.

## 14.2 Configuration spaces of graphs on long tangles

An  $\infty$ -component of an LTR  $L$  is a connected component of the intersection of the image of  $L$  with  $R(\mathcal{C}) \setminus \mathcal{C}$ . A univalent vertex  $v$  of a Jacobi diagram on the domain  $\mathcal{L}$  of  $L$  approaching  $\infty$  moves on such an  $\infty$ -component. Let  $y_{\pm}(v)$  denote the orthogonal projection on  $\mathbb{C}$  of the corresponding  $\infty$ -component of  $v$ . Then this  $\infty$ -component may be written as  $\{y_{\pm}(v)\} \times ]1, \infty[$  or  $\{y_{\pm}(v)\} \times ]-\infty, 0[$ . The projection  $y_{\pm}(v)$  depends on the considered  $\infty$ -component of  $v$ , in general. When such a component is fixed, we simply denote the projection by  $y(v)$ , and we speak of the  $\infty$ -component of a univalent vertex mapped to  $\infty$ .

**Definition 14.22.** Define the two-point compactification of  $\mathbb{R} = ]-\infty, +\infty[$  to be  $[-\infty, +\infty]$ . For a Jacobi diagram  $\Gamma$  on the domain  $\mathcal{L}$  of  $L$ , let  $U_{ttbb}(\Gamma)$  be the set of vertices  $v$  of  $U(\Gamma)$  such that the component  $L(\mathcal{L}(v))$  of  $v$  goes from top to top or from bottom to bottom. Let  $\overline{\mathcal{L}(v)}$  denote the two-point compactification of the component  $\mathcal{L}(v)$  of  $\mathcal{L}$ .

The open manifold  $\check{C}(R(\mathcal{C}), L; \Gamma)$  embeds naturally in the product

$$C_{V(\Gamma)}(R(\mathcal{C})) \times \prod_{v \in U_{ttbb}(\Gamma)} \overline{\mathcal{L}(v)}.$$

Let  $C_L^f = C^f(R(\mathcal{C}), L; \Gamma)$  be the closure of  $\check{C}(R(\mathcal{C}), L; \Gamma)$  in this product. This closure maps naturally onto the closure  $C_L = C(R(\mathcal{C}), L; \Gamma)$  of  $\check{C}(R(\mathcal{C}), L; \Gamma)$  in  $C_{V(\Gamma)}(R(\mathcal{C}))$ . An element of  $C^f(R(\mathcal{C}), L; \Gamma)$  is a configuration  $c_{V(\Gamma)}$  of  $C(R(\mathcal{C}), L; \Gamma)$  equipped with the additional data of an  $\infty$ -component for each vertex of  $U_{ttbb}(\Gamma) \cap p_b(c_{V(\Gamma)})^{-1}(\infty)$ .<sup>2</sup>

In this section, we prove the following theorem.

**Theorem 14.23.** *Let  $L: \mathcal{L} \hookrightarrow \check{R}(\mathcal{C})$  be a long tangle representative, and let  $\Gamma$  be a numbered degree  $n$  Jacobi diagram with support  $\mathcal{L}$  without looped edges. For any  $c_{V(\Gamma)}^0$  in  $C^f(R(\mathcal{C}), L; \Gamma)$ , there exist*

- a manifold  $W$  with boundary and ridges,
- a small  $\varepsilon > 0$ ,

---

<sup>2</sup>This piece of data is automatically determined by  $c_{V(\Gamma)}$  in most cases. However, it may happen that it is not. For example, when  $\Gamma$  has a unique univalent vertex on a strand going from top to top, and when this vertex is mapped to  $\infty$ , the strand of the vertex may not be determined by the configuration in  $C_{V(\Gamma)}(R(\mathcal{C}))$ .

- an oriented tree  $\mathcal{T}^0 = \mathcal{T}(c_{V(\Gamma)}^0)$  (described in Notation 14.43), with its associated singular space  $X(\mathcal{T}^0)$  defined in Definition 14.3, and
- a smooth map

$$\varphi: [0, \varepsilon]^{E(\mathcal{T}^0)} \times W \rightarrow C_{V(\Gamma)}(R(\mathcal{C})),$$

whose restriction to  $(X(\mathcal{T}^0) \cap [0, \varepsilon]^{E(\mathcal{T}^0)}) \times W$  is injective and induces a map

$$\varphi^f: (X(\mathcal{T}^0) \cap [0, \varepsilon]^{E(\mathcal{T}^0)}) \times W \rightarrow C^f(R(\mathcal{C}), L; \Gamma),$$

such that  $\text{Im}(\varphi^f)$  is an open neighborhood of  $c_{V(\Gamma)}^0$  in  $C^f(R(\mathcal{C}), L; \Gamma)$  where the  $\infty$ -component for each univalent vertex of the  $p_b(c_{V(\Gamma)}^0)^{-1}(\infty)$  is the same as its  $\infty$ -component for  $c_{V(\Gamma)}^0$ .

Let  $\vec{N}$  denote the upward unit vertical vector. The codimension-one open faces of  $C^f(R(\mathcal{C}), L; \Gamma)$  are

- the faces corresponding to the collapse of a subgraph at one point in  $\check{R}(\mathcal{C})$  as before,
- the faces corresponding to a set of vertices mapped to  $\infty$ , for which the configuration up to dilation at  $\infty$  is injective and does not map a point to 0, as before,
- additional faces called *T-faces* (for which  $C(R(\mathcal{C}), L; \Gamma)$  is not transverse to the ridges of  $C_{V(\Gamma)}(R(\mathcal{C}))$ ), where
  - a set of vertices  $B \sqcup \sqcup_{j \in I} B_j$  is mapped to  $\infty$ , for a nonempty set  $I$ ,
  - the corresponding configuration up to dilation from  $B \sqcup \sqcup_{j \in I} B_j$  to  $T_\infty R(\mathcal{C})$  maps each  $B_j$  to a nonzero point of the vertical line, and it injects  $B$  outside zero and the images of the  $B_j$ , which are distinct,
  - each subset  $B_j$  contains univalent vertices of at least 2 distinct  $\infty$ -components,
  - for each  $B_j$ , the infinitesimal configuration of  $B_j$  is an injective configuration of a Jacobi diagram on the lines that extend the half-lines above (resp. below)  $\mathcal{C}$ , if  $B_j$  is mapped to  $\lambda \vec{N}$  for some  $\lambda > 0$  (resp. for some  $\lambda < 0$ ), up to global vertical translation. (No inversion is involved here.)

Together with Lemma 14.15 and Theorem 14.16, Theorem 14.23 implies the following lemma.

**Lemma 14.24.** *Theorem 12.2 is true.*

Recall  $C_L = C(R(\mathcal{C}), L; \Gamma)$  and  $C_L^f = C^f(R(\mathcal{C}), L; \Gamma)$ . Let  $\eta$  be a form of degree  $(\dim(C_L) - 1)$  of  $C_{V(\Gamma)}(R(\mathcal{C}))$ . Then  $\int_{C_L} d\eta$  is the sum  $\sum_F \int_F \eta$ , which runs over the codimension-one faces  $F$  of  $C_L^f$ , oriented as such, and listed in Theorem 14.23.

□

**Example 14.25.** Let  $K: ]0, 1[ \hookrightarrow \check{R}(\mathcal{C})$  be a (long) component of  $L$ . Assume that  $K: ]0, 1[ \hookrightarrow \check{R}(\mathcal{C})$  goes from top to top. Let  $d_1 = -\{z_1\} \times [1, \infty]$  and  $d_2 = \{z_2\} \times [1, \infty]$  denote the vertical half-lines of  $K$  above  $\mathcal{C}$ , where  $d_1$  is before  $d_2$ . Let  $G = \{(h, k) \in ]0, 1] \times \mathbb{R} : k + \frac{1}{h} \geq 1\}$ . Define the diffeomorphism

$$g: \begin{array}{ll} G & \rightarrow d_1 \times d_2 \\ (h, k) & \mapsto \left( (z_1, \frac{1}{h}), (z_2, k + \frac{1}{h}) \right). \end{array}$$

This diffeomorphism  $g$  extends as a continuous map

$$g: G \cup (\{0\} \times \mathbb{R}) \rightarrow C(R(\mathcal{C}), L; \hat{\mathcal{C}}^K).$$

It maps  $(0, k)$  to the limit  $g(0, k)$  at 0 in  $C_2(R(\mathcal{C}))$  of the  $g(]0, \varepsilon] \times k)$ . The image of  $g(0, k)$  under the canonical map from  $C_2(R(\mathcal{C}))$  to  $R(\mathcal{C})^2$  is  $(\infty, \infty)$ . The configuration in  $T_\infty R(\mathcal{C})$  up to dilation corresponding to  $g(0, k)$  maps the two points to the same upward vertical vector. We have  $p_\tau(g(0, k)) = \frac{(z_2 - z_1, k)}{\|(z_2 - z_1, k)\|}$ . The image of  $(-g(\{0\} \times \mathbb{R}))$  under  $p_\tau$  is the open half-circle from  $\vec{N}$  to  $-\vec{N}$  through the direction of  $(z_2 - z_1)$ . This codimension-one face of  $C^f(R(\mathcal{C}), L; \hat{\mathcal{C}}^K)$  is an example of a  $T$ -face, for which  $B \sqcup \sqcup_{j \in I} B_j = B_1$  is the pair of vertices of the graph. Since this codimension-one face sits in a codimension-two face of  $C_2(R(\mathcal{C}))$ ,  $C(R(\mathcal{C}), L; \hat{\mathcal{C}}^K)$  is not transverse to the ridges of  $C_2(R(\mathcal{C}))$ .

Recall that the elements of  $C_L^f = C^f(R(\mathcal{C}), L; \Gamma)$  are elements of the closure  $C_L$  of  $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$  in  $C_{V(\Gamma)}(R(\mathcal{C}))$  equipped with the additional data of the  $\infty$ -components of the univalent vertices sent to  $\infty$ .

First note that the configuration space  $C_L$  intersects  $p_b^{-1}(\check{R}(\mathcal{C})^{V(\Gamma)})$  as a smooth submanifold as in the case of links. The only difference with the case of links occurs when some univalent vertices approach  $\infty$ . Our configuration space is a local product of the space of the restrictions of the configurations

to the points near  $\infty$  and the space of the restrictions of the configurations to the other points, which is a smooth manifold with boundary whose structure has been studied in detail in Chapter 8.

Recall the orientation-reversing embedding  $\phi_\infty$

$$\begin{aligned} \phi_\infty: \mathbb{R}^3 &\longrightarrow S^3 \\ \mu(x \in S^2) &\mapsto \begin{cases} \infty & \text{if } \mu = 0 \\ \frac{1}{\mu}x & \text{otherwise.} \end{cases} \end{aligned}$$

According to Corollary 8.37, with the notation of Chapter 8, and especially those of Section 8.7, an element  $c_{V(\Gamma)}$  of  $C_{V(\Gamma)}(R(\mathcal{C}))$  consists of

- a subset  $V = p_b(c_{V(\Gamma)})^{-1}(\infty)$  of  $V(\Gamma)$ ,
- an element  $c_{V(\Gamma)\setminus V} = c_{V(\Gamma)}|_{V(\Gamma)\setminus V}$  of  $C_{V(\Gamma)\setminus V}[\check{R}(\mathcal{C})]$ ,
- an element  $c = c_{V(\Gamma)}|_V$  of  $C_V(R(\mathcal{C})) \cap p_b^{-1}(\infty^V)$ ,
- a  $\Delta$ -parenthesization  $\mathcal{P}^+ = \mathcal{P}^+(c)$  of  $V^+ = V \sqcup \{v_\infty\}$ ,
- for each  $A \in \mathcal{P}^+$ , an injective configuration

$$T_0\phi_{\infty*} \circ w_A \in \check{S}_{K(A)}(T_\infty R(\mathcal{C})),$$

up to dilation and translation (see Proposition 8.36).

Proposition 8.33 describes the restriction maps, naturally. As reminded above, the configuration space  $C_{V(\Gamma)}(R(\mathcal{C}))$  has a natural stratification induced by  $V = p_b(c)^{-1}(\infty)$ , the parenthesization associated to  $c_{V(\Gamma)\setminus V}$  (as before Proposition 8.34) and the above  $\Delta$ -parenthesization  $\mathcal{P}^+$  of  $V^+$ . Each stratum has a well-defined dimension.

Below, we refine this partition induced on  $C^f(R(\mathcal{C}), L; \Gamma)$  by the stratification of  $C_{V(\Gamma)}(R(\mathcal{C}))$ .

**Notation 14.26.** As in Proposition 8.36, we define the totally ordered subset  $\mathcal{P}_s^+$  to be the set of elements of  $\mathcal{P}^+$  containing  $v_\infty$  and  $\mathcal{P}_s = \{A \setminus \{v_\infty\} : A \in \mathcal{P}_s^+\}$ . We have

$$\mathcal{P}_s = \{V = V(1), V(2), \dots, V(\sigma)\}$$

with  $V(i+1) \subset V(i)$ . We let  $K_d^s(V(i))$  denote the set of kids of  $V(i)^+$  that do not contain  $v_\infty$ . We impose  $w_{V(i)^+}(v_\infty) = 0$  and define  $f_i$  to be the restriction of  $w_{V(i)^+}$  to  $V(i)$ . So  $f_i$  is an injective map from  $K_d^s(V(i))$  to  $\mathbb{R}^3 \setminus \{0\}$  up to dilation. It belongs to  $\check{\mathcal{S}}(\mathbb{R}^3, K_d^s(V(i)))$ .

The sets of  $\mathcal{P}^+$  that contain a univalent vertex are called *univalent*. As in Proposition 8.36, we set  $\mathcal{P}_d = \mathcal{P}^+ \setminus \mathcal{P}_s^+$ . A *possibly separating set* associated to the above parenthesization  $\mathcal{P}^+$  and to the data of the  $\infty$ -components of the elements of  $V$  is a set  $A \in \mathcal{P}_d$  such that

- $A$  has at least two univalent vertices on different  $\infty$ -components and
- each kid of  $A$  has all its univalent vertices on the same  $\infty$ -component.

Let  $\mathcal{P}_X$  denote the set of possibly separating sets associated to the above parenthesization  $\mathcal{P}^+$ . A set  $A$  of  $\mathcal{P}_X$  is *separating* (with respect to  $c_{V(\Gamma)}$ ) if it has at least two univalent kids  $A_1$  and  $A_2$  such that  $w_A(A_1) - w_A(A_2)$  is not vertical. The set of separating sets of  $c_{V(\Gamma)}$  is denoted by  $\mathcal{P}_x$ .

Recall that  $\vec{N}$  denotes the upward unit vertical vector. Let  $p_{\mathbb{C}}: (\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}) \rightarrow \mathbb{C}$  denote the orthogonal projection onto the horizontal plane  $\mathbb{C}$ , and let  $p_{\mathbb{R}}: \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{R}$  denote the orthogonal projection onto the vertical line  $\mathbb{R}$ .

We are going to prove the following two propositions.

**Proposition 14.27.** *The space  $C_L^f = C^f(R(\mathcal{C}), L; \Gamma)$  of Definition 14.22 is the space of configurations  $c_{V(\Gamma)}$  of  $C_{V(\Gamma)}(R(\mathcal{C}))$  as above, equipped with  $\infty$ -components for the univalent vertices of  $V = p_b(c_{V(\Gamma)})^{-1}(\infty)$ , such that the following conditions are satisfied.*

1. *If the configuration  $c_{V(\Gamma)}|_{U(\Gamma) \setminus (V \cap U(\Gamma))}$  is injective, then it factors through the restriction to  $U(\Gamma) \setminus (V \cap U(\Gamma))$  of a representative of  $i_\Gamma$  that maps the univalent vertices of  $V \cap U(\Gamma)$  to their  $\infty$ -components further than the elements of  $U(\Gamma) \setminus (V \cap U(\Gamma))$ . If the configuration  $c_{V(\Gamma)}|_{U(\Gamma) \setminus (V \cap U(\Gamma))}$  is not injective, then it factors through a limit of such restrictions. In any case, the possible infinitesimal configurations of vertices of  $U(\Gamma) \setminus (V \cap U(\Gamma))$  are locally ordered on the tangent space to their component, as in the case of links (see Sections 8.4 and 8.8).*
2. *The  $f_i$  map the elements of  $K_d^s(V(i))$  that contain a univalent vertex on an  $\infty$ -component  $y \times ]1, \infty[$  (resp.  $y \times ]-\infty, 0[$ ) to the half-line  $\mathbb{R}^+ \vec{N}$  (resp.  $\mathbb{R}^+(-\vec{N})$ ).*
3. *If  $v_1 \in A_1$  and  $v_2 \in A_2$  are two univalent vertices of distinct kids  $A_1$  and  $A_2$  of an element  $A \in \mathcal{P}^+$ , fix a normalization of  $w_A$ , and let  $\vec{y} = y(v_2) - y(v_1)$ .*
  - *If  $\vec{y} = 0$  (that is if  $v_1$  and  $v_2$  are on the same  $\infty$ -component), and if  $v_1$  is closer to  $\infty$  than  $v_2$ , then  $w_A(A_2) - w_A(A_1)$  is a nonzero vertical vector, which may be expressed as  $\alpha \vec{N}$ , where  $\alpha$  is positive when the  $\infty$ -component is above  $\mathcal{C}$ , and  $\alpha$  is negative when the  $\infty$ -component is under  $\mathcal{C}$ .*
  - *If  $A \notin \mathcal{P}_X$ , then  $w_A(A_2) - w_A(A_1)$  is also a nonzero vertical vector.*

- If  $A \in \mathcal{P}_X$  and if  $\vec{y} \neq 0$ , then  $w_A(A_2) - w_A(A_1)$  may be expressed as  $(\alpha\vec{N} + \beta\vec{y})$  for some nonzero pair  $(\alpha, \beta)$  of  $\mathbb{R} \times \mathbb{R}^+$ . Furthermore, if  $v_3 \in A_3$  is a univalent vertex of another kid  $A_3$  of  $A$ , then there exists  $\alpha_3 \in \mathbb{R}$  such that  $w_A(A_3) - w_A(A_1)$  is equal to  $(\alpha_3\vec{N} + \beta(y(v_3) - y(v_1)))$ .

**Proposition 14.28.** *The space  $C_L^f$  of Proposition 14.27 is partitioned by the data for a configuration  $c_{V(\Gamma)}$  of*

- the set  $V = p_b(c_{V(\Gamma)})^{-1}(\infty)$ ,
- the parenthesization  $\mathcal{P}(c_{V(\Gamma)}|_{V(\Gamma)\setminus V})$  of  $V(\Gamma)\setminus V$  associated to  $c_{V(\Gamma)}|_{V(\Gamma)\setminus V}$  (as before Proposition 8.34),

and, if  $V \neq \emptyset$ ,<sup>3</sup>

- the  $\Delta$ -parenthesization  $\mathcal{P}^+ = \mathcal{P}^+(c_{V(\Gamma)}|_V)$  of  $V^+ = V \sqcup \{v_\infty\}$
- the data of the  $\infty$ -components of the univalent vertices that are mapped to  $\infty$ , and
- the set  $\mathcal{P}_x$  of separating sets of  $\mathcal{P}^+$ .

The part associated to the above data is a smooth submanifold of  $C_{V(\Gamma)}(R(\mathcal{C}))$  of dimension

$$|U(\Gamma)| + 3|T(\Gamma)| - |\mathcal{P}(c_{V(\Gamma)}|_{V(\Gamma)\setminus V})| - |\mathcal{P}^+| + |\mathcal{P}_x|.$$

This partition is a stratification of  $C_L^f$ .

**Remark 14.29.** Proposition 14.28 implies that the only codimension-one new parts –which come necessarily from strata for which  $V \neq \emptyset$ – come from the parts such that  $\mathcal{P}^+ = \{V^+\} \sqcup \mathcal{P}_x$  and  $\mathcal{P}_d = \mathcal{P}_x$ . They are the  $T$ -faces of Theorem 14.23, for which  $\mathcal{P}_s = \{V\} = \{B \sqcup \sqcup_{j \in I} B_j\}$  and  $\mathcal{P}_d = \mathcal{P}_X = \mathcal{P}_x = \{B_j : j \in I\}$ .

The rest of this section is devoted to the proofs of Theorem 14.23 and the above two propositions.

Let  $c_{V(\Gamma)}^0$  be a configuration of  $C_L^f$  in a stratum as in the statement of Proposition 14.28. Let  $c^0 = c_{V(\Gamma)}^0|_V$  denote the restriction of  $c_{V(\Gamma)}^0$  to  $V(= V(c^0))$ .

It is easy to see that  $c_{V(\Gamma)}^0$  has a neighborhood  $N_\Gamma(c_{V(\Gamma)}^0)$  in  $C_L^f$  consisting of configurations mapping

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<sup>3</sup>When  $V = \emptyset$ , the structure of  $C_L^f$  near  $c_{V(\Gamma)}$  is already known from Chapter 8.

- $V$  to a fixed open neighborhood  $N_\infty$  of  $\infty$  in  $R(\mathcal{C})$ ,
- the univalent vertices of  $V$  to their  $\infty$ -component with respect to  $c^0$ ,
- $V(\Gamma) \setminus V$  to an open subspace  $U$  of  $R(\mathcal{C}) \setminus N_\infty$ .

Let  $C_V^f(N_\infty, L, c^0)$  denote the space of restrictions to  $V$  of configurations of  $C(R(\mathcal{C}), L; \Gamma)$  mapping univalent vertices of  $V$  to  $N_\infty$  and to their  $\infty$ -components (determined by  $c^0$ ), and vertices of  $V(\Gamma) \setminus V$  to  $U$ .

The configuration  $c_{V(\Gamma)}^0$  has a neighborhood  $N_\Gamma(c_{V(\Gamma)}^0)$  in  $C_L^f$  that is a product of  $C_V^f(N_\infty, L, c^0)$  by a smooth submanifold  $N_2$  of  $C_{V(\Gamma) \setminus V}[U]$ . In this product decomposition, the  $N_2$ -part contains the restriction of the configurations  $c_{V(\Gamma)}$  to  $V(\Gamma) \setminus V$ . This space has been studied before (see Proposition 8.6, Section 8.6, and Theorem 8.28), and the manifold  $W$  of the statement of Theorem 14.23 will be a product  $W_V \times N_2$ . This allows us to forget about the  $N_2$ -part. We focus only on  $c^0$  and on a neighborhood  $N(c^0)$  of  $c^0$  in  $C_V(R(\mathcal{C}))$ .

The configuration  $c^0$  is described by

- a  $\Delta$ -parenthesization  $\mathcal{P}^+ = \mathcal{P}^+(c^0) = \mathcal{P}_s^+ \sqcup \mathcal{P}_d$  of  $V^+$ , where  $\mathcal{P}_s = \{V = V(1), V(2), \dots, V(\sigma)\}$  and  $V(i+1) \subset V(i)$ ,
- for any element  $A$  of  $\mathcal{P}^+$ , an element  $w_A^0: K(A) \rightarrow \mathbb{R}^3$  of the manifold  $\tilde{W}_A$  consisting of the injective maps  $w_A: K(A) \rightarrow \mathbb{R}^3$  up to dilation and translation.

**Notation 14.30.** Let  $i \in \bar{\sigma}$ . Recall  $K_d^s(V(i)) = K(V(i)^+) \cap \mathcal{P}_d$ . Set  $K^s(V(\sigma)) = K_d^s(V(\sigma))$ , and  $K^s(V(i)) = K_d^s(V(i)) \sqcup \{V(i+1)\}$  when  $i \neq \sigma$ . We normalize  $w_{V(i)^+}^0$  so that  $w_{V(i)^+}^0(v_\infty) = 0$ , and the restriction  $f_i^0$  of  $w_{V(i)^+}^0$  to  $V(i)$  is in the manifold consisting of the maps  $f_i: K^s(V(i)) \rightarrow \mathbb{R}^3$  such that

- $f_i(V(i+1)) = 0$ , if  $i \neq \sigma$ ,
- $\sum_{A \in K_d^s(V(i))} \|f_i(A)\|^2 = 1$ ,
- $\|f_i(A)\| > \eta$  for any  $i$  and for any  $A \in K_d^s(V(i))$ ,
- $\|f_i(A_2) - f_i(A_1)\| > \eta$  for any two distinct elements  $A_1$  and  $A_2$  of  $K^s(V(i))$ ,

for some real number  $\eta > 0$ .<sup>4</sup> We fix an open neighborhood  $W_i^s$  of  $f_i^0$  in the above manifold.

<sup>4</sup>Recall Notation 14.26. The map  $f_i$  represents an injective configuration  $T_0\phi_\infty \circ f_i$  up to dilation of  $\tilde{\mathcal{S}}(T_\infty R(\mathcal{C}), K_d^s(V(i)))$ .



**Notation 14.31.** We choose univalent basepoints for univalent sets of  $\mathcal{P}_d$ . As usual, our basepoints also satisfy the conditions that for two elements  $A$  and  $B$  of  $\mathcal{P}_d$ , such that  $B \subset A$ , if  $b(A) \in B$ , then  $b(B) = b(A)$ .

For  $A \in \mathcal{P}_d$ , we normalize the configurations of  $\tilde{W}_A$  as follows in a neighborhood  $\tilde{N}_A$  of a given  $w_A^0 \in \tilde{W}_A$ . Choose a kid  $k_n(A)$  such that  $|p_{\mathbb{R}}(w_A^0(k_n(A))) - p_{\mathbb{R}}(w_A^0(b(A)))|$  or  $|p_{\mathbb{C}}(w_A^0(k_n(A))) - p_{\mathbb{C}}(w_A^0(b(A)))|$  is maximal in the set

$$\left\{ \left| p_{\mathbb{R}}(w_A^0(k)) - p_{\mathbb{R}}(w_A^0(b(A))) \right|, \left| p_{\mathbb{C}}(w_A^0(k)) - p_{\mathbb{C}}(w_A^0(b(A))) \right| : k \in K(A) \right\},$$

and call it the *normalizing kid* of  $A$ . If

$$\left| p_{\mathbb{R}}(w_A^0(k_n(A))) - p_{\mathbb{R}}(w_A^0(b(A))) \right| \geq \left| p_{\mathbb{C}}(w_A^0(k_n(A))) - p_{\mathbb{C}}(w_A^0(b(A))) \right|,$$

then we say that  $k_n(A)$  is *vertically normalizing* or *v-normalizing* and normalize the configurations  $w_A$  in a neighborhood of  $w_A^0$  by imposing

$$w_A(b(A)) = 0 \text{ and } |p_{\mathbb{R}}(w_A(k_n(A)))| = 1.$$

Otherwise, we say that  $k_n(A)$  is *horizontally normalizing* or *h-normalizing*, and normalize the configurations  $w_A$  in a neighborhood of  $w_A^0$  by imposing  $w_A(b(A)) = 0$  and  $|p_{\mathbb{C}}(w_A(k_n(A)))| = 1$ . (These normalizations are compatible with the smooth structure of  $C_{V(\Gamma)}(R(\mathcal{C}))$ .)

In our neighborhood  $\tilde{N}_A$ , we also impose that  $\|w_A(k) - w_A^0(k)\| < \varepsilon$ , for a small  $\varepsilon \in ]0, 1[$ . So  $\tilde{N}_A$  is diffeomorphic to the product  $W_A$  of

- the product, over the nonnormalizing kids  $k$  of  $A$  that do not contain  $b(A)$ , of the open balls  $\mathring{B}(w_A^0(k), \varepsilon)$  of radius  $\varepsilon$  centered at  $w_A^0(k)$ , by
- the set of  $w_A(k_n(A))$  in  $\mathring{B}(w_A^0(k_n(A)), \varepsilon)$  such that  $|p_{\mathbb{R}}(w_A(k_n(A)))| = 1$  (resp. such that  $|p_{\mathbb{C}}(w_A(k_n(A)))| = 1$ ) if  $k_n(A)$  is v-normalizing (resp. if  $k_n(A)$  is h-normalizing).

We reduce the  $\eta$  of Notation 14.30 if necessary, and we choose  $\varepsilon$  so that  $\|w_A(B_2) - w_A(B_1)\| > \eta$  for any two distinct kids  $B_1$  and  $B_2$  of  $A$  in our normalized neighborhood  $W_A$ . Note that  $\|w_A(k)\| < 3$  for any  $k \in K(A)$  in this neighborhood.

All the considered maps  $f_i = w_{V(i)+|V(i)}$ ,  $w_A$  are also considered as maps from  $V^+$  to  $\mathbb{R}^3$ , which are constant on the elements of  $K^s(V(i))$  and  $K(A)$ , respectively, and which respectively map  $V^+ \setminus V(i)$  and  $V^+ \setminus A$  to 0.

We use a chart  $\psi$  of  $C_V(R(\mathcal{C}))$  of a neighborhood  $N(c^0)$  of  $c^0$  in  $C_V(R(\mathcal{C}))$  mapping

$$((\mu_A)_{A \in \mathcal{P}^+}, (w_A)_{A \in \mathcal{P}^+}) \in [0, \varepsilon]^{P^+} \times \prod_{i \in \underline{\sigma}} W_i^s \times \prod_{A \in \mathcal{P}_d} W_A$$

to a configuration  $c = \psi((\mu_A)_{A \in \mathcal{P}^+}, (w_A)_{A \in \mathcal{P}^+}) \in C_V(R(\mathcal{C}))$ , such that, when the  $\mu_A$  do not vanish,  $c$  is the injective configuration

$$c = \phi_\infty \circ \left( \sum_{A \in \mathcal{P}^+} \left( \prod_{D \in \mathcal{P}^+ : A \subseteq D} \mu_D \right) w_A \right).$$

With this chart induced by Theorems 8.28, 8.32, and 8.35, we have

$$c^0 = \psi((\mu_A^0 = 0), (w_A^0)).$$

**Notation 14.32.** For  $k \in \underline{\sigma}$ , set  $u_k = \mu_{V(k)^+}$ , and set  $U_k = \prod_{i=1}^k u_i$ . For  $A \in \mathcal{P}_d$ , let  $k(A)$  be the maximal integer among the integers  $k$  such that  $A \subseteq V(k)$ . So  $\psi$  maps

$$((u_i)_{i \in \underline{\sigma}}, (\mu_A)_{A \in \mathcal{P}_d}, (f_i)_{i \in \underline{\sigma}}, (w_A)_{A \in \mathcal{P}_d}) \in [0, \varepsilon]^\sigma \times [0, \varepsilon]^{P_d} \times \prod_{i \in \underline{\sigma}} W_i^s \times \prod_{A \in \mathcal{P}_d} W_A$$

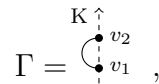
to a configuration  $c = \psi((u_i), (\mu_A), (f_i), (w_A)) \in C_V(R(\mathcal{C}))$ , such that, when the  $u_i$  and the  $\mu_A$  do not vanish,  $c$  is the injective configuration

$$c = \phi_\infty \circ \left( \sum_{V(k) \in \mathcal{P}_s} U_k \left( f_k + \sum_{C \in \mathcal{P}_d : k(C)=k} \left( \prod_{D \in \mathcal{P}_d : C \subseteq D} \mu_D \right) w_C \right) \right).$$

We also write

$$c^0 = \psi((u_i^0 = 0), (\mu_A^0 = 0), (f_i^0), (w_A^0)).$$

**Example 14.33.** In the special case of Example 14.25, with the graph



consider configurations mapping  $v_1$  to  $-\{z_1\} \times [1, \infty]$  and  $v_2$  to  $\{z_2\} \times [1, \infty]$ . When  $V = \{v_1, v_2\}$  and  $\mathcal{P}^+ = \{V^+, V\}$ , we have  $\mathcal{P}_s = \mathcal{P}_d = \{V\}$ . Set  $f = f_1$ ,  $u = u_1$ ,  $\mu = \mu_V$ ,  $b(V) = v_1$ ,  $w = w_V$ . We have  $w(v_1) = 0$ ,  $\|f(v_1)\| = 1$ ,  $f^0(v_1) = \vec{N}$ ,  $c = \phi_\infty \circ (u(f + \mu w))$ ,  $c(v_1) = \phi_\infty(uf(v_1)) = \frac{1}{u}f(v_1)$ , and  $c(v_2) = \phi_\infty(u(f(v_1) + \mu w(v_2)))$ .

Back to the general proof of Theorem 14.23 and Propositions 14.27 and 14.28, we will often reduce  $\varepsilon$  and reduce the spaces  $W_k^s$  and  $W_A$  to smaller manifolds, which are neighborhoods of  $f_k^0$  and  $w_A^0$  in the initial manifolds  $W_k^s$  and  $W_A$ . In particular, we assume that the image  $N(c^0)$  of  $\psi$  is in  $C_V(N_\infty)$ , and we set

$$N_L^f(c^0) = N(c^0) \cap C_V^f(N_\infty, L, c^0).$$

The intersection of  $N_L^f(c^0)$  with  $\check{C}_V(R(\mathcal{C}))$  is determined by the conditions that univalent vertices belong to their  $\infty$ -components and that their order on the  $\infty$ -components is prescribed by the isotopy class of injections from  $U(\Gamma)$  to  $\mathcal{L}$ . These conditions are closed. So they still hold in  $N_L^f(c^0)$ .

In particular, the basepoints  $b(A)$  of the univalent elements  $A$  of  $K_d^s(V(k))$  must go to their  $\infty$ -components. We call this condition the *first condition*. We examine what this “first condition” imposes on the  $f_k$  and prove the following two lemmas.

**Lemma 14.34.** *For any  $k \in \underline{\sigma}$ , for any univalent element  $A$  of  $K_d^s(V(k))$ , we have  $f_k^0(A) = \pm \|f_k^0(A)\| \vec{N}$ , where  $\|f_k^0(A)\| \geq \eta$ , and where the  $\pm$  sign is  $+$  if  $A$  has a univalent vertex above  $\mathcal{C}$  and  $-$  otherwise.*

PROOF: For an element  $A$  of  $K_d^s(V(k))$  and a configuration

$$c = \psi((u_i), (\mu_A), (f_i), (w_A)) \in C_V(R(\mathcal{C}))$$

such that  $U_k = \prod_{i=1}^k u_i \neq 0$ , we have

$$c(b(A)) = \frac{1}{U_k} \frac{f_k(A)}{\|f_k(A)\|^2}.$$

So the condition  $p_{\mathcal{C}}(c(b(A))) = y(b(A))$  is equivalent to the closed condition

$$p_{\mathcal{C}}(f_k(A)) = U_k \|f_k(A)\|^2 y(b(A)), \tag{14.1}$$

where

$$\|f_k(A)\|^2 = \|p_{\mathcal{C}}(f_k(A))\|^2 + \|p_{\mathbb{R}}(f_k(A))\|^2 \leq 1,$$

with our normalization of Notation 14.30.

So Equation 14.1 implies  $\|p_{\mathcal{C}}(f_k(A))\| = O(U_k)$  (meaning that there exists  $C \in \mathbb{R}^{*+}$  such that  $\|p_{\mathcal{C}}(f_k(A))\| \leq CU_k$ ). In particular, since  $U_k = 0$  for  $c^0$ , we have  $p_{\mathcal{C}}(f_k^0(A)) = 0$  and  $f_k^0(A) = \pm \|f_k^0(A)\| \vec{N}$ , where  $\|f_k^0(A)\| \geq \eta$ . Lemma 14.34 follows easily since the sign of  $p_{\mathbb{R}}(f_k(A))$  is constant on  $N_L^f(c^0)$ .  $\square$

In particular, if  $V(\sigma) \in \mathcal{P}_d$ , then  $f_\sigma^0(V(\sigma))$  equals  $\vec{N}$  if  $V(\sigma)$  has a univalent vertex above  $\mathcal{C}$ , and  $f_\sigma^0(V(\sigma))$  equals  $(-\vec{N})$  if  $V(\sigma)$  has a univalent vertex under  $\mathcal{C}$ .

**Lemma 14.35.** *Let  $v_1$  and  $v_2$  be two vertices on some  $\infty$ -component of  $L$ . Assume that  $v_1$  is closer to  $\infty$  than  $v_2$  and that  $v_1 \in A_1$  and  $v_2 \in A_2$ , for two different kids  $A_1$  and  $A_2$  of  $V(k)$ , where  $k \in \underline{\sigma}$ . Then we have*

$$\|f_k^0(A_1)\| < \|f_k^0(A_2)\|. \tag{14.2}$$

**PROOF:** The configuration  $c^0$  is a limit at 0 of a family  $c(t)$  indexed by  $t \in ]0, \varepsilon[$  of injective configurations for which  $\|c(t)(v_1)\| > \|c(t)(v_2)\|$ , with

$$c(t) = c = \psi((u_i), (\mu_A), (f_i), (w_A)).$$

Therefore, we have  $\|f_k^0(A_1)\| \leq \|f_k^0(A_2)\|$ . Since  $f_k^0(A_1) \neq f_k^0(A_2)$ , the result follows.  $\square$

In particular, we have  $|p_{\mathbb{R}}(f_k^0(A_1))| \leq |p_{\mathbb{R}}(f_k^0(A_2))| - \eta$ . We possibly reduce  $W_k^s$  by imposing  $|p_{\mathbb{R}}(f_k(A)) - p_{\mathbb{R}}(f_k^0(A))| < \varepsilon$  for some positive  $\varepsilon$  such that  $\varepsilon < \frac{\eta}{2}$ . This condition ensures that the univalent vertices  $b(A)$ , for the elements  $A$  of  $K_d^s(V(k))$ , are well-ordered on any  $\infty$ -component.

**Lemma 14.36.** *Let  $k \in \underline{\sigma}$  be such that  $V(k) \notin \mathcal{P}_d$ . Recall  $U_k = \prod_{i=1}^k u_i$ . Let  $A_{k,1}, A_{k,2}, \dots, A_{k,\ell(k)}$  denote the univalent elements of  $K_d^s(V(k))$ , where  $\ell(k) \in \mathbb{N}$ . Let  $K_{td}^s(V(k))$  be the set of nonunivalent elements of  $K_d^s(V(k))$ . For  $D \in K_{td}^s(V(k))$ , let  $\mathring{B}_D = \mathring{B}(f_k^0(D), \varepsilon)$  be the open ball of center  $f_k^0(D)$  and of radius  $\varepsilon$  in  $\mathbb{R}^3$ .*

- If  $\ell(k) \geq 1$ , set

$$W_k^L = \prod_{i=1}^{\ell(k)-1} \left[ p_{\mathbb{R}}(f_k^0(A_{k,i})) - \varepsilon, p_{\mathbb{R}}(f_k^0(A_{k,i})) + \varepsilon \right] \times \prod_{D \in K_{td}^s(V(k))} \mathring{B}_D.$$

Up to reducing  $\varepsilon$ , there is a smooth injective map

$$\phi_k : [0, \varepsilon^k[ \times W_k^L \rightarrow W_k^s$$

such that, up to reducing  $N_L^f(c^0)$  (to a smaller open neighborhood of  $c^0$  in  $C_V^f(N_\infty, L, c^0)$ ), all elements

$$c = \psi((u_i)_{i \in \underline{\sigma}}, (\mu_A)_{A \in \mathcal{P}_d}, (f_i)_{i \in \underline{\sigma}}, (w_A)_{A \in \mathcal{P}_d})$$

of  $N_L^f(c^0)$  satisfy the condition

$$f_k = \phi_k \left( \prod_{i=1}^k u_i, \left( p_{\mathbb{R}}(f_k(A_{k,i})) \right)_{i \in \underline{\ell(k)-1}}, \left( f_k(D) \right)_{D \in K_{td}^s(V(k))} \right).$$

This condition is equivalent to

$$p_{\mathbb{C}}\left(c(b(A_{k,i}))\right) = y(b(A_{k,i}))$$

for any  $i \in \underline{\ell(k)}$  when  $c \in \check{C}_V(R(\mathcal{C}))$ , and it implies Equation 14.1 for any  $c$  in  $N(c^0)$ .

- If  $\ell(k) = 0$ , set  $W_k^L = W_k^s$ .

If  $V(\sigma) \in \mathcal{P}_d$  and if  $V(\sigma)$  is not univalent, also set  $W_\sigma^L = W_\sigma^s$ .

If  $V(\sigma) \in \mathcal{P}_d$  and if  $V(\sigma)$  is univalent, then all elements  $c$  of  $N_L^f(c^0)$  satisfy the condition

$$p_{\mathbb{C}}\left(f_\sigma(V(\sigma))\right) = U_\sigma y\left(b(V(\sigma))\right).$$

This condition is equivalent to  $p_{\mathbb{C}}(c(b(V(\sigma)))) = y(b(V(\sigma)))$  when  $c \in \check{C}_V(R(\mathcal{C}))$ . These elements  $c$  also satisfy

$$p_{\mathbb{R}}\left(f_\sigma(V(\sigma))\right) = \sqrt{1 - U_\sigma^2 |y(b(V(\sigma)))|^2} p_{\mathbb{R}}\left(f_\sigma^0(V(\sigma))\right).$$

In this case, set  $W_\sigma^L = \{*_\sigma\}$ .

Let  $N_1$  denote the subspace of  $N(c^0)$ , where the first condition (stated before Lemma 14.34) is satisfied. Then  $N_1$  is a smooth manifold parametrized by

$$[0, \varepsilon]^\sigma \times [0, \varepsilon]^{P_d} \times \prod_{k \in \underline{\sigma}} W_k^L \times \prod_{A \in \mathcal{P}_d} W_A.$$

PROOF: The proof of Lemma 14.34 shows that for  $i \in \underline{\ell(k) - 1}$ ,  $p_{\mathbb{C}}(f_k(A_{k,i}))$  is an implicit function of  $U_k = \prod_{i=1}^k u_i$  and  $p_{\mathbb{R}}(f_k(A_{k,i}))$ , which is close to  $\pm \|f_k^0(A_{k,i})\| \neq 0$  on  $N_1$  and  $N_L^f(c^0)$ . This implicit function is determined by Equation 14.1.

Then the condition that  $\sum_{A \in K_d^s(V(k))} \|f_k(A)\|^2 = 1$  in  $W_k^s$  determines  $\|f_k(A_{k,\ell(k)})\| \neq 0$  as a function of  $U_k$ ,  $(p_{\mathbb{R}}(f_k(A_{k,i})))_{i \in \underline{\ell(k)-1}}$  and of the  $f_k(D)$  for  $D \in K_{id}^s(V(k))$ . Now, Equation 14.1 determines  $p_{\mathbb{C}}(\overline{f_k(A_{k,\ell(k)})})$ , which in turn determines  $p_{\mathbb{R}}(f_k(A_{k,\ell(k)}))$ . This is how the map  $\phi_k$  of the statement is constructed. It is easy to check that  $\phi_k$  has the desired properties and that  $N_1$  is parametrized naturally, as announced, using the maps  $\phi_k$ .  $\square$

In Example 14.33, we have  $p_{\mathbb{C}}(f(v_1)) = uy(v_1)$  and

$$p_{\mathbb{R}}(f(v_1)) = \sqrt{1 - u^2 \|y(v_1)\|^2}.$$

So  $f(v_1)$  is just a smooth function of the small parameter  $u$ .

We now restrict to the submanifold  $N_1$  of  $N(c^0)$  of Lemma 14.36 and take care of the univalent basepoints of the kids of elements of  $\mathcal{P}_d$  in the following lemmas.

**Lemma 14.37.** *For  $X \in S^2$ , let  $s(X)$  denote the orthogonal reflection of  $\mathbb{R}^3$  with respect to the plane orthogonal to  $X$ . Let  $A$  be an element of  $\mathcal{P}_d$ . Recall that  $k(A)$  is the maximal integer  $k$  such that  $A \subseteq V(k)$ . The restriction of  $c^0$  to  $A$  maps  $A$  to*

$$X_A^0 = \frac{f_{k(A)}^0(A)}{\|f_{k(A)}^0(A)\|} \in \partial\mathcal{B}(R(\mathcal{C}), \infty).$$

*It is represented by  $s(X_A^0) \circ w_A^0$  up to translation and dilation, as a configuration of the ambient  $\mathbb{R}^3$  outside  $\mathcal{C}$ . If  $p$  and  $q$  are univalent vertices in two different kids of  $A$ , if they belong to an  $\infty$ -component  $K^+$ , and if  $p$  is closer to  $\infty$  than  $q$ , then there exists  $\alpha^0 \in \mathbb{R}$  such that  $|\alpha^0| > \eta$  and  $w_A^0(q) - w_A^0(p) = \alpha^0 \vec{N}$ , where  $\alpha^0 > 0$  if  $K^+$  is above  $\mathcal{C}$ , and  $\alpha^0 < 0$  otherwise.*

We introduce some notation before the proof.

**Notation 14.38.** For an element  $A$  of  $\mathcal{P}_d$  such that  $k(A) = k$  and a configuration  $c = \psi((u_i), (\mu_A), (f_i), (w_B)) \in N(c^0)$ , set

$$M_A = \prod_{D \in \mathcal{P}_d: A \subseteq D} \mu_D,$$

and, for any element  $q$  of  $A$ , define

$$\tilde{f}_k(q) = f_k(A) + \sum_{C \in \mathcal{P}_d: q \in C} M_C w_C(q),$$

so that we have  $c(q) = \frac{1}{U_k} \frac{\tilde{f}_k(q)}{\|\tilde{f}_k(q)\|^2}$  when  $U_k \neq 0$ . Both  $M_A$  and  $\tilde{f}_k$  depend on the configuration  $c$ .

**PROOF OF LEMMA 14.37:** The configuration  $c_{V(\Gamma)}^0$  is a limit at 0 of a family  $c(t)$  indexed by  $t \in ]0, \varepsilon[$  of injective configurations of  $N_L^f(c^0)$ . We have

$$c = c(t) = \psi((u_i), (\mu_B), (f_i), (w_B)),$$

where the  $u_i$  and the  $\mu_B$  are positive. Set  $k = k(A)$ . Let  $p = b(A)$  be the basepoint of  $A$ , and let  $q$  be the basepoint of a kid of  $A$  that does not contain  $p$ . Since  $\tilde{f}_k(q) - \tilde{f}_k(p) = M_A w_A(q)$ , we have

$$\|\tilde{f}_k(q)\|^2 = \|\tilde{f}_k(p)\|^2 + 2M_A \langle w_A(q), \tilde{f}_k(p) \rangle + M_A^2 \|w_A(q)\|^2.$$

We get

$$\begin{aligned} c(q) - c(p) &= \frac{\|\tilde{f}_k(p)\|^2 \tilde{f}_k(q) - \|\tilde{f}_k(q)\|^2 \tilde{f}_k(p)}{U_k \|\tilde{f}_k(q)\|^2 \|\tilde{f}_k(p)\|^2} \\ &= \frac{M_A \|\tilde{f}_k(p)\|^2 w_A(q)}{U_k \|\tilde{f}_k(q)\|^2 \|\tilde{f}_k(p)\|^2} - \frac{2M_A \langle w_A(q), \tilde{f}_k(p) \rangle + M_A^2 \|w_A(q)\|^2}{U_k \|\tilde{f}_k(q)\|^2 \|\tilde{f}_k(p)\|^2} \tilde{f}_k(p). \end{aligned}$$

When the  $\mu_B$  tend to 0 and when  $\tilde{f}_k$  tends to  $f_k^0$ ,  $\tilde{f}_k(p)$  and  $\tilde{f}_k(q)$  tend to  $f_k^0(A)$ . Thus,

$$\frac{U_k \|\tilde{f}_k(q)\|^2}{M_A} (c(q) - c(p))$$

tends to

$$w_A(q) - 2\langle w_A(q), X_A^0 \rangle X_A^0 = s(X_A^0)(w_A(q)).$$

So  $w_A^0$  is the limit of the  $s(X_A^0) \circ c|_A$  up to dilation and translation.

If  $A$  contains a univalent vertex of an  $\infty$ -component above  $\mathcal{C}$ , then  $X_A^0$  equals  $\vec{N}$  according to Lemma 14.34. In this case, if  $a$  and  $q$  are univalent vertices in two different kids of  $A$ , if they belong to an  $\infty$ -component  $K^+$ , and if  $a$  is closer to  $\infty$  than  $q$ , then we have  $c(q) - c(a) = -\alpha(t)\vec{N}$  for some positive  $\alpha(t)$  for any  $t > 0$ . So  $(w_A^0(q) - w_A^0(a))$ , which is defined up to dilation, is a positive multiple of  $\vec{N}$ .  $\square$

**Notation 14.39.** Let  $\mathcal{P}_{\vec{X}} (= \mathcal{P}_{\vec{X}}(c^0))$  denote the set of elements of  $\mathcal{P}_d (= \mathcal{P}_d(c^0))$  that contain or are equal to an element of  $\mathcal{P}_X (= \mathcal{P}_X(c^0))$ .

**Lemma 14.40.** Let  $A \in \mathcal{P}_d$  be such that  $k(A) = k$ . Let  $c$  be as in Notation 14.38. For any univalent kid  $B$  of  $A$  such that  $y(b(B)) - y(b(A)) = 0$ , if  $c \in N_L^f(c^0)$ , we have

$$p_{\mathcal{C}}(w_A(B)) = U_k \left( 2\langle \tilde{f}_k(b(A)), w_A(B) \rangle + M_A \|w_A(B)\|^2 \right) y(b(B)). \quad (14.3)$$

Furthermore, as soon as  $p_{\mathcal{C}}(c(b(A))) = y(b(A))$  and  $c \in \check{C}_V(R(\mathcal{C}))$ , Equation 14.3 implies  $p_{\mathcal{C}}(c(b(B))) = y(b(B))$  for such a  $B$ .

If  $c \in N_L^f(c^0)$ , then there exists  $(\lambda_A = \lambda_A(c))_{A \in \mathcal{P}_{\vec{X}}} \in (\mathbb{R}^+)^{\mathcal{P}_{\vec{X}}}$  such that the following properties are satisfied.

- For any univalent kid  $B$  of  $A \in \mathcal{P}_{\vec{X}}$ , we have

$$\begin{aligned} p_{\mathcal{C}}(w_A(B)) &= \lambda_A \|\tilde{f}_{k(A)}(b(A))\|^2 \left( y(b(B)) - y(b(A)) \right) \\ &\quad + U_{k(A)} \left( 2\langle \tilde{f}_{k(A)}(b(A)), w_A(B) \rangle + M_A \|w_A(B)\|^2 \right) y(b(B)). \end{aligned} \quad (14.4)$$

- For  $M \in \mathcal{P}_X$ , set  $\tilde{\lambda}_M = \lambda_M \mu_M$ . For any  $A \in \mathcal{P}_{\overline{X}} \setminus \mathcal{P}_X$ , there exists  $M \in \mathcal{P}_X$  such that  $M \subset A$  and we have

$$\lambda_A = \tilde{\lambda}_M \prod_{D \in \mathcal{P}_d: M \subset D \subset A} \mu_D.$$

- The map  $\lambda_A$  is continuous on  $N_\Gamma(c^0)$ .
- We have  $\lambda_A M_A = U_{k(A)}$ .
- For any three elements  $A, B$ , and  $D$  in  $\mathcal{P}_d$  such that  $A$  and  $B$  are in  $\mathcal{P}_{\overline{X}}$  and  $A \cup B \subseteq D$ , we have

$$\lambda_A \prod_{C \in \mathcal{P}_d: A \subseteq C \subset D} \mu_C = \lambda_B \prod_{C \in \mathcal{P}_d: B \subseteq C \subset D} \mu_C.$$

- For a univalent kid  $B$  of  $A \in \mathcal{P}_{\overline{X}}$ , as soon as  $p_{\mathbb{C}}(c(b(A))) = y(b(A))$  holds and  $c \in \check{C}_V(R(\mathcal{C}))$ , Equation 14.4 implies  $p_{\mathbb{C}}(c(b(B))) = y(b(B))$ .

PROOF: Let  $A \in \mathcal{P}_d$  be such that  $k(A) = k$ . Consider a univalent kid  $B$  of  $A$  and assume  $p_{\mathbb{C}}(c(b(A))) = y(b(A))$ . This equality is equivalent to

$$p_{\mathbb{C}}\left(\tilde{f}_k(b(A))\right) = U_k \|\tilde{f}_k(b(A))\|^2 y(b(A)),$$

with  $\tilde{f}_k(b(A)) = f_k(A) + \sum_{C \in \mathcal{P}_d: A \subset C} M_C w_C(A)$  and  $\tilde{f}_k(b(B)) = \tilde{f}_k(b(A)) + M_A w_A(B)$ . So the condition  $p_{\mathbb{C}}(c(b(B))) = y(b(B))$  may be written as

$$p_{\mathbb{C}}\left(\tilde{f}_k(b(A)) + M_A w_A(B)\right) = U_k \|\tilde{f}_k(b(B))\|^2 y(b(B)),$$

which is equivalent to

$$\begin{aligned} M_A p_{\mathbb{C}}(w_A(B)) &= U_k \|\tilde{f}_k(b(A))\|^2 \left(y(b(B)) - y(b(A))\right) \\ &\quad + U_k \left(\|\tilde{f}_k(b(B))\|^2 - \|\tilde{f}_k(b(A))\|^2\right) y(b(B)), \end{aligned} \tag{14.5}$$

with

$$\|\tilde{f}_k(b(B))\|^2 - \|\tilde{f}_k(b(A))\|^2 = M_A \left(2\langle \tilde{f}_k(b(A)), w_A(B) \rangle + M_A \|w_A(B)\|^2\right).$$

When  $y(b(B)) - y(b(A)) = 0$  and  $M_A \neq 0$ , Equation 14.5 simplifies to Equation 14.3, which also holds in the closure  $C_L$ . In particular, we have  $p_{\mathbb{C}}(w_A^0(b(B))) = 0$ . So we also get  $|p_{\mathbb{R}}(w_A^0(b(B)))| \geq \eta$ .



When  $M_A \neq 0$ , Equation 14.5 is equivalent to

$$\begin{aligned} p_{\mathbb{C}}(w_A(B)) - U_k \left( 2 \langle \tilde{f}_k(b(A)), w_A(B) \rangle + M_A \|w_A(B)\|^2 \right) y(b(B)) \\ = \frac{U_k}{M_A} \|\tilde{f}_k(b(A))\|^2 \left( y(b(B)) - y(b(A)) \right). \end{aligned} \quad (14.6)$$

It tells that the left-hand side is colinear to  $\|\tilde{f}_k(b(A))\|^2 (y(b(B)) - y(b(A)))$ , and that the scalar product of these two vectors is nonnegative. This remains true in the closure  $C_L$ . When  $y(b(B)) - y(b(A)) \neq 0$ , this uniquely defines  $\lambda_A = \lambda_A(B, c)$  such that Equation 14.4 holds for  $B$ . Furthermore,  $\lambda_A(B, c)$  is continuous on  $C_L$ , we have

$$\lim_{U_k \rightarrow 0} \lambda_A(B, c) = \frac{\|p_{\mathbb{C}}(w_A(B))\|}{\|\tilde{f}_k(b(A))\|^2 \|y(b(B)) - y(b(A))\|},$$

and  $\lambda_A(B, c) = U_k/M_A$  when  $M_A \neq 0$ . In particular, if  $B'$  is another univalent kid of  $A$  such that  $y(b(B')) - y(b(A)) \neq 0$  and if  $M_A \neq 0$ , then  $\lambda_A(B, c) = \lambda_A(B', c)$ . This remains true in the closure  $C_L$  when  $M_A = 0$ . This allows us to define the continuous  $\lambda_A = \lambda_A(B, c)$  for the subset  $\mathcal{P}'_{\overline{X}}$  of  $\mathcal{P}_{\overline{X}}$  made of the sets  $A$  that have kids with basepoints on different components. These  $\lambda_A$  satisfy  $\lambda_A M_A = U_{k(A)}$  when the parameters  $\mu_D$  do not vanish. This also remains true in the closure  $C_L$  when  $M_A = 0$ . Observe  $\mathcal{P}_X \subseteq \mathcal{P}'_{\overline{X}}$ . For  $A \in \mathcal{P}_{\overline{X}} \setminus \mathcal{P}'_{\overline{X}}$ , there exists  $M \in \mathcal{P}_X$  such that  $M \subset A$ . Define

$$\lambda(A, M) = \tilde{\lambda}_M \prod_{D \in \mathcal{P}_d : M \subset D \subset A} \mu_D.$$

So we have  $\lambda(A, M)M_A = U_{k(A)}$  and  $\lambda(A, M) = \lambda(A, M')$  for any subset  $M'$  of  $A$  in  $\mathcal{P}_X$ , when  $M_A \neq 0$ . So  $\lambda(A, M) = \lambda(A, M')$  on  $C_L$ , and we can set  $\lambda_A = \lambda(A, M)$ . The other properties of the parameters  $\lambda_A$  are obvious when the parameters  $\mu_D$  do not vanish. So they hold in  $C_L$ , and Lemma 14.40 is proved. (Note that the set  $\mathcal{P}_X$  of possibly separating sets is the subset of  $\mathcal{P}_{\overline{X}}$  consisting of its minimal sets with respect to the inclusion.)  $\square$

**Lemma 14.41.** *The configuration*

$$c^0 = \psi \left( (0)_{k \in \underline{\sigma}}, (0)_{A \in \mathcal{P}_d}, (f_k^0)_{k \in \underline{\sigma}}, (w_A^0)_{A \in \mathcal{P}_d} \right)$$

of  $C_V^f(N_\infty, L, c^0)$  is such that

- we have  $p_{\mathbb{C}} \circ w_A^0(B) = 0$  for any univalent kid  $B$  of  $A$  if  $A \in \mathcal{P}_d \setminus \mathcal{P}_X$ ,

- for any  $A \in \mathcal{P}_X$ , there exists  $\lambda_A^0 \geq 0$  such that

$$p_{\mathbb{C}} \circ w_A^0(B) = \lambda_A^0 \|f_{k(A)}^0(A)\|^2 \left( y(b(B)) - y(b(A)) \right)$$

for any univalent kid  $B$  of  $A$ .

PROOF: Lemma 14.40 implies that  $\lambda_A^0 = 0$  when  $A \in \mathcal{P}_{\bar{X}} \setminus \mathcal{P}_X$ . □

**Example 14.42.** Let us go back to Example 14.33, where  $\|y(v_2) - y(v_1)\|$  is nonzero. When  $A = V$  and  $B = \{v_2\}$ , we have  $\tilde{f}(v_2) = f(v_1) + \mu w(v_2)$ . When  $c$  is injective, Equation 14.5 is equivalent to

$$p_{\mathbb{C}}(w(v_2)) = \lambda \|\tilde{f}(v_1)\|^2 (y(v_2) - y(v_1)) + u \left( 2 \langle \tilde{f}(v_1), w(v_2) \rangle + \mu \|w(v_2)\|^2 \right) y(v_2),$$

with  $\lambda = \lambda_V = \frac{u}{\mu}$ . We get  $p_{\mathbb{C}}(w^0(v_2)) = \lambda^0 (y(v_2) - y(v_1))$ .

Let us now define the oriented tree  $\mathcal{T}^0$  (as in Definition 14.1) of the statement of Theorem 14.23.

**Notation 14.43.** The set  $E(\mathcal{T}^0)$  of edges of  $\mathcal{T}^0$  is in one-to-one correspondence with  $\{u_i\}_{i \in \underline{\sigma}} \cup \{\mu_A\}_{A \in \mathcal{P}_{\bar{X}} \setminus \mathcal{P}_x} \cup \{\lambda_A\}_{A \in \mathcal{P}_{\bar{X}} \setminus \mathcal{P}_x} \cup \{\tilde{\lambda}_A = \lambda_A \mu_A\}_{A \in \mathcal{P}_x}$ , and its edges are labeled by these variables. So  $E(\mathcal{T}^0)$  is in one-to-one correspondence with the disjoint union of  $\underline{\sigma}$ ,  $\mathcal{P}_x$ , and two disjoint copies of  $\mathcal{P}_{\bar{X}} \setminus \mathcal{P}_x$ . (Recall that  $\mathcal{P}_{\bar{X}}$  is the set of elements of  $\mathcal{P}_d$  that contain or are equal to an element of  $\mathcal{P}_X$ .)

The set of vertices of  $\mathcal{T}(\mathcal{P})$  is in one-to-one correspondence with the disjoint union

$$\mathcal{P}_{\bar{X}} \sqcup \mathcal{P}_s^+ \sqcup \{r_1\} \sqcup \{r_A : A \in \mathcal{P}_{\bar{X}} \setminus \mathcal{P}_x\}.$$

Its elements label the vertices.

The edge labeled by  $u_i$  ends at the vertex labeled by  $V(i)^+$ . It starts at the vertex labeled by  $V(i-1)^+$  when  $i > 1$ , and at the univalent vertex labeled by  $r_1$  when  $i = 1$ .

For  $A \in \mathcal{P}_{\bar{X}} \setminus \mathcal{P}_x$ , the edge labeled by  $\lambda_A$  starts from the univalent vertex labeled by  $r_A$ , and it goes to the vertex labeled by  $A$ .

For  $A \in \mathcal{P}_{\bar{X}} \setminus \mathcal{P}_x$  (resp. for  $A \in \mathcal{P}_x$ ), the edge labeled by  $\mu_A$  (resp. by  $\tilde{\lambda}_A$ ) starts at the vertex  $A$ , it ends at the mother  $m(A)$  of  $A$ . See Figure 14.6 for an example of a tree  $\mathcal{T}^0$ .

According to Lemma 14.40, the variables of

$$\{u_i\}_{i \in \underline{\sigma}} \cup \{\mu_A\}_{A \in \mathcal{P}_{\bar{X}} \setminus \mathcal{P}_x} \cup \{\lambda_A\}_{A \in \mathcal{P}_{\bar{X}} \setminus \mathcal{P}_x} \cup \{\tilde{\lambda}_A = \lambda_A \mu_A\}_{A \in \mathcal{P}_x}$$

satisfy the equations that define  $X(\mathcal{T}^0)$ .

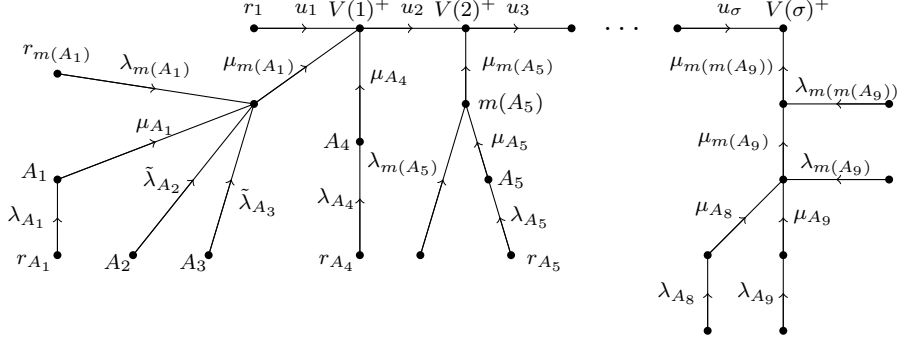


Figure 14.6: A tree  $\mathcal{T}^0 = \mathcal{T}(c^0)$

**Lemma 14.44.** *The dimension of  $X(\mathcal{T}^0)$  is  $d(\mathcal{T}^0) = |\mathcal{P}_s| + |\mathcal{P}_{\overline{X}}| - |\mathcal{P}_x|$ .*

PROOF: Recall  $d(\mathcal{T}^0) = |E(\mathcal{T}^0)| - |L(\mathcal{T}^0)| + 1$  from Lemma 14.9. The set of leaves different from  $r_1$  is in one-to-one correspondence with  $\mathcal{P}_{\overline{X}}$ .  $\square$

**Lemma 14.45.** *Let  $c^0 = \psi((0), (0), (f_i^0), (w_B^0))$  be a configuration of  $C_V(R(\mathcal{C}))$  that satisfies the equations of Lemmas 14.36 and 14.40. Then  $c^0$  belongs to  $C_V^f(N_\infty, L, c^0)$ .*

Let  $A$  be a univalent element of  $\mathcal{P}_d$ . Let  $K_u(A)$  denote the set of univalent kids of  $A$  that do not contain  $b(A)$ , and let  $K_t(A)$  denote the set of nonunivalent kids of  $A$  that do not contain  $b(A)$ . Recall Notation 14.31. For any element  $D$  of  $K_t(A)$ , let  $B'_D$  denote the open ball  $\mathring{B}(w_A^0(D), \varepsilon)$  of radius  $\varepsilon$  with center  $w_A^0(D)$  in  $\mathbb{R}^3$  if  $D$  is not normalizing, and let  $B'_D$  denote the set of elements  $w_A(D)$  of  $\mathring{B}(w_A^0(D), \varepsilon)$  such that  $|p_{\mathbb{R}}(w_A(D))| = 1$  (resp. such that  $|p_{\mathbb{C}}(w_A(D))| = 1$ ) if  $D$  is  $v$ -normalizing (resp. if  $D$  is  $h$ -normalizing). For any element  $D$  of  $K_u(A)$ , let  $J'_D$  denote the interval  $]p_{\mathbb{R}}(w_A^0(D)) - \varepsilon, p_{\mathbb{R}}(w_A^0(D)) + \varepsilon[$  if  $D$  is not  $v$ -normalizing, and set  $J'_D = \{p_{\mathbb{R}}(w_A^0(D))\}$  if  $D$  is  $v$ -normalizing. Set

$$W_A^L = \prod_{D \in K_u(A)} J'_D \times \prod_{D \in K_t(A)} B'_D.$$

When  $A$  is a nonunivalent element of  $\mathcal{P}_d$ , set  $W_A^L = W_A$ . Recall that  $W_k^L$  has been introduced in Lemma 14.36.

Assume that  $2\varepsilon < \lambda_A^0$  for all  $A \in \mathcal{P}_x$ . Let  $\mathcal{P}_{x, hn}$  be the set of elements  $A$

of  $\mathcal{P}_x$  such that  $k_n(A)$  is a univalent horizontally normalizing kid. Set

$$W = [0, \varepsilon]^{[\mathcal{P}_d \setminus \mathcal{P}_{\overline{x}}]} \times \left( \prod_{k \in \underline{\sigma}} W_k^L \right) \times \left( \prod_{A \in \mathcal{P}_d} W_A^L \right) \times \prod_{A \in \mathcal{P}_x \setminus \mathcal{P}_{x, hn}} ] \lambda_A^0 - \varepsilon, \lambda_A^0 + \varepsilon[.$$

Recall Notation 14.43. There exists a smooth map from  $[0, \varepsilon]^{E(\mathcal{T}^0)} \times W$  to  $C_V(R(\mathcal{C}))$  which restricts to  $([0, \varepsilon]^{E(\mathcal{T}^0)} \cap X(\mathcal{T}^0)) \times W$  as a continuous injective map  $\varphi$ , whose image is an open neighborhood  $N_L^f(c^0)$  of  $c^0$  in  $C_V^f(N_\infty, L, c^0)$ .

PROOF: Here and in Notation 14.43, the parameters  $u_i$  and  $\mu_B$  of  $E(\mathcal{T}^0)$  are the initial parameters of  $c$  in the chart  $\psi$  of  $C_V(R(\mathcal{C}))$ . The parameters  $\lambda_A$  are defined in Lemma 14.40. Lemma 14.40 implies that the parameters  $\lambda_A$  satisfy the equations of  $X(\mathcal{T}^0)$  for configurations in  $C_V^f(N_\infty, L, c^0)$ . The factor of  $[0, \varepsilon]^{[\mathcal{P}_d \setminus \mathcal{P}_{\overline{x}}]}$  of  $W$  contains the parameters  $\mu_B$  for  $B \in \mathcal{P}_d \setminus \mathcal{P}_{\overline{x}}$ .

Lemma 14.36 shows how to express the parameter  $f_k$  of  $\psi^{-1}(c)$ , for configurations  $c$  in  $C_V^f(N_\infty, L, c^0)$ , as a smooth function of  $W \times [0, \varepsilon]^{[\mathcal{P}_d]}$ , where the factor  $[0, \varepsilon]^{[\mathcal{P}_d]}$  contains the  $u_i$ .

We now construct the  $w_A$  as smooth maps from  $[0, \varepsilon]^{E(\mathcal{T}^0)} \times W$  to  $\check{\mathcal{S}}_{K(A)}(\mathbb{R}^3)$ , in order to finish constructing a smooth map

$$\varphi: [0, \varepsilon]^{E(\mathcal{T}^0)} \times W \rightarrow C_{V(\Gamma)}(R(\mathcal{C})).$$

The coordinates of the nonunivalent kids of  $A$  and the vertical coordinates of the univalent kids of  $A$  are part of  $W$ , and we do not change them. We only need to determine the horizontal coordinate  $p_C(w_A(B))$  for any univalent kid  $B$  of  $A$  as a smooth map.

Note that  $\lambda_A$  is a free parameter in  $[0, \varepsilon]^{E(\mathcal{T}^0)} \times W$  when  $A \in \mathcal{P}_{\overline{x}} \setminus \mathcal{P}_{x, hn}$ .

Let  $A$  be a univalent element of  $\mathcal{P}_d$ . Set  $k = k(A)$ . By induction, assume that  $w_C$  and  $\mu_C$  have already been constructed as smooth functions of  $[0, \varepsilon]^{E(\mathcal{T}^0)} \times W$  for any  $C \in \mathcal{P}_d$  that contains  $A$ , so that  $\tilde{f}_k(b(A))$  is a smooth function of  $[0, \varepsilon]^{E(\mathcal{T}^0)} \times W$  defined by the expression from Notation 14.38:

$$\tilde{f}_k(b(A)) = f_k(A) + \sum_{C \in \mathcal{P}_d: A \subset C} M_C w_C(A). \tag{14.7}$$

If  $A \notin \mathcal{P}_{\overline{x}}$ , then Equation 14.3 determines  $p_C(w_A(B))$  as a smooth implicit function of  $(p_{\mathbb{R}}(w_A(B)))$ , which is determined by  $W$ ,  $\tilde{f}_k(b(A))$ , the  $u_i$ , and the  $\mu_C$  for the  $C \in \mathcal{P}_d$  such that  $A \subseteq C$ . If  $A \in \mathcal{P}_{\overline{x}}$ , then Equation 14.4 determines  $p_C(w_A(B))$  as a smooth implicit function of the same parameters

$(W, \tilde{f}_k(b(A)),$  the  $u_i,$  and the  $\mu_C$  for the  $C \in \mathcal{P}_d$  such that  $A \subseteq C),$  and  $\lambda_A.$  The parameter  $\mu_A$  is among the parameters unless  $A \in \mathcal{P}_x.$  If  $A \in \mathcal{P}_x \setminus \mathcal{P}_{x,hn},$  then  $\mu_A$  is determined as  $\lambda_A/\lambda_A,$  where  $\lambda_A$  is a parameter of  $W.$  So  $\mu_A$  is this smooth function of our parameters. If  $A \in \mathcal{P}_{x,hn},$  then  $p_C(w_C(k_n(A)))$  is still a smooth implicit function of  $W, \tilde{f}_k(b(A)),$  the  $u_i,$  the  $\mu_C$  for the  $C \in \mathcal{P}_d$  such that  $A \subseteq C,$  and  $\lambda_A,$  and the normalizing condition  $|p_C(w_C(k_n(A)))| = 1$  determines  $\lambda_A$  as a smooth implicit function of the given parameters.

We have constructed a smooth function  $\varphi: [0, \varepsilon]^{E(\mathcal{T}^0)} \times W \rightarrow C_{V(\Gamma)}(R(\mathcal{C})).$  Set

$$\mathring{W} = ]0, \varepsilon[^{\mathcal{P}_d \setminus \mathcal{P}_x} \times \times \left( \prod_{k \in \underline{\sigma}} W_k^L \right) \times \left( \prod_{A \in \mathcal{P}_d} W_A^L \right) \times \prod_{A \in \mathcal{P}_x \setminus \mathcal{P}_{x,hn}} ]\lambda_A^0 - \varepsilon, \lambda_A^0 + \varepsilon[.$$

Lemma 14.40 implies that  $\varphi$  maps  $(]0, \varepsilon[^{E(\mathcal{T}^0)} \cap X(\mathcal{T}^0)) \times \mathring{W}$  to  $C_V^f(N_\infty, L, c^0)$  (up to reducing  $\varepsilon$ ). Since the closure of  $\mathring{X}(\mathcal{T}^0)$  in  $[0, \infty[^{E(\mathcal{T}^0)}$  is  $X(\mathcal{T}^0),$  according to Lemma 14.13, we have

$$\varphi\left(\left([0, \varepsilon]^{E(\mathcal{T}^0)} \cap X(\mathcal{T}^0)\right) \times W\right) \subseteq C_V^f(N_\infty, L, c^0),$$

too. In particular, any  $c^0$  that satisfies the equations of Lemmas 14.36 and 14.40 is in  $C_V^f(N_\infty, L, c^0).$

Lemma 14.40 also implies  $N_L^f(c^0) \subseteq \varphi\left(\left([0, \varepsilon]^{E(\mathcal{T}^0)} \cap X(\mathcal{T}^0)\right) \times W\right)$  for an open neighborhood  $N_L^f(c^0)$  of  $c^0$  in  $C_V^f(N_\infty, L, c^0).$  The injectivity of the restriction of  $\varphi$  to  $\left(\left([0, \varepsilon]^{E(\mathcal{T}^0)} \cap X(\mathcal{T}^0)\right) \times W\right)$  comes from the fact that Equation 14.4 determine the parameters  $\lambda_A$  for  $A \in \mathcal{P}_x$  and the equations of Lemma 14.40 determine the others.<sup>5</sup>  $\square$

**PROOF OF PROPOSITION 14.27:** Lemmas 14.34, 14.35, 14.37, and 14.41 show that a configuration  $c_{V(\Gamma)}^0$  of  $C_L$  must satisfy the conditions of the statement of Proposition 14.27. According to Lemma 14.45, if a configuration  $c_{V(\Gamma)}^0$  satisfies the conditions of the statement of Proposition 14.27, its restriction  $c^0$  to  $V = p_b(c_{V(\Gamma)}^0)^{-1}(\infty)$  is in  $C_V^f(N_\infty, L, c^0),$  so  $c_{V(\Gamma)}^0$  is in  $C_L.$   $\square$

**PROOF OF PROPOSITION 14.28:** The strata of Proposition 14.28 are smooth submanifolds of  $C_{V(\Gamma)}(R(\mathcal{C}))$  since they correspond to the locus where

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<sup>5</sup>We can remove these other parameters  $\lambda_C,$  for  $C \in \mathcal{P}_x \setminus \mathcal{P}_x,$  from our parametrization for the statement. However, it was more convenient to keep them for the proof since the definition of the  $\lambda_A,$  for  $A \in \mathcal{P}_{x,hn},$  involves the  $\lambda_C$  for the  $C$  that contain  $A.$  These  $\lambda_C$  are functions of the  $\lambda_A$  for which  $A \in \mathcal{P}_x.$

all the variables associated to  $\mathcal{T}^0$  in Notation 14.43 are zero. (The parameters  $\lambda_A$  are either zero or not zero on the whole stratum, in the charts of Lemma 14.45.) Let us assume  $V = V(\Gamma)$ . Then the codimension of the stratum of  $c^0$  is  $d(\mathcal{T}^0) + |(\mathcal{P}_d \setminus \mathcal{P}_{\bar{X}})|$ , which is  $|\mathcal{P}_s| + |\mathcal{P}_d| - |\mathcal{P}_x|$ , according to Lemma 14.44.  $\square$

**Lemma 14.46.** *The codimension-one faces of  $C^f(R(\mathcal{C}), L; \Gamma)$  are those listed in Theorem 14.23. In a neighborhood of these faces,  $C^f(R(\mathcal{C}), L; \Gamma)$  has the structure of a smooth manifold with boundary.*

Let  $s = s(\vec{N})$  be the orthogonal reflection of  $\mathbb{R}^3$  with respect to the horizontal plane. A configuration

$$c_{V(\Gamma)}^0|_V = c^0 = \left( T_0\phi_\infty \circ f_1^0, (T_0\phi_{\infty*} \circ w_A^0)_{A \in \mathcal{P}_X}, (\lambda_A^0)_{A \in \mathcal{P}_X} \right)$$

of a  $T$ -face is the limit at  $t = 0$  of a family of injective configurations  $c(t)_{t \in ]0, \varepsilon[}$  on the vertical parts of the tangle, far above or far below, such that  $c(t)|_A = s \circ w_A^0$  up to dilation and translation for any  $A \in \mathcal{P}_X (= \mathcal{P}_d = \mathcal{P}_x)$ . In particular, for an edge  $e = (v_1, v_2)$  whose vertices are in  $A$ , we have

$$p_\tau \circ p_e(c^0) = \frac{s \circ w_A^0(v_2) - s \circ w_A^0(v_1)}{\|s \circ w_A^0(v_2) - s \circ w_A^0(v_1)\|}.$$

For an edge  $e = (v_1, v_2)$  whose vertices are in different kids of  $V$ , we have

$$\begin{aligned} p_\tau \circ p_e(c^0) &= \frac{\phi_\infty \circ f_1^0(v_2) - \phi_\infty \circ f_1^0(v_1)}{\|\phi_\infty \circ f_1^0(v_2) - \phi_\infty \circ f_1^0(v_1)\|} \\ &= \frac{\|f_1^0(v_1)\|^2 f_1^0(v_2) - \|f_1^0(v_2)\|^2 f_1^0(v_1)}{\| \|f_1^0(v_1)\|^2 f_1^0(v_2) - \|f_1^0(v_2)\|^2 f_1^0(v_1) \|}. \end{aligned}$$

For an edge  $e = (v_1, v_2)$  such that  $v_1 \in V$  and  $v_2 \notin V$ , we have

$$p_\tau \circ p_e(c^0) = -\frac{f_1^0(v_1)}{\|f_1^0(v_1)\|}.$$

For an edge  $e = (v_1, v_2)$  such that  $v_2 \in V$  and  $v_1 \notin V$ , we have  $p_\tau \circ p_e(c^0) = \frac{f_1^0(v_2)}{\|f_1^0(v_2)\|}$ .

PROOF: Let  $\partial_\infty(C^f(R(\mathcal{C}), L; \Gamma))$  be the subspace of  $C^f(R(\mathcal{C}), L; \Gamma)$  consisting of the configurations as above that map at least one univalent vertex to  $\infty$ . Outside this subspace, the spaces  $C^f(R(\mathcal{C}), L; \Gamma)$  and  $C(R(\mathcal{C}), L; \Gamma)$  are the same, and  $C^f(R(\mathcal{C}), L; \Gamma)$  has the structure of a smooth manifold with

ridges. Recall from Remark 14.29 that the only codimension-one new parts are the  $T$ -faces of Theorem 14.23, where  $\mathcal{P}_s = \{B \sqcup \sqcup_{j \in I} B_j\}$  and  $\mathcal{P}_d = \mathcal{P}_X = \mathcal{P}_x = \{B_j : j \in I\}$ . Lemmas 14.45 and 14.14 imply that these strata arise as codimension-one faces of  $C^f(R(\mathcal{C}), L; \Gamma)$ , along which  $C^f(R(\mathcal{C}), L; \Gamma)$  is a smooth manifold with boundary.

Lemma 14.34 implies  $f_1^0(b) = \pm \|f_1^0(b)\| \vec{N}$  for any univalent vertex  $b$  of  $B \sqcup \sqcup_{j \in I} B_j$ . Lemma 14.37 implies that the restriction of  $c^0$  to any  $A$  of  $\mathcal{P}_X = \{B_j : j \in I\}$  is represented by  $s \circ w_A^0$ , up to translation and dilation, as a configuration of  $\mathbb{R}^3$ . Furthermore, Lemma 14.41 implies

$$p_{\mathcal{C}}(w_A^0(b)) = \lambda_A^0 \|f_1^0(b(A))\|^2 (y(b) - y(b(A)))$$

for any  $A \in \mathcal{P}_X$  and for any  $b \in A$ . Therefore, the configuration  $c^0$  is the limit of the following family  $c(t)$  of configurations, indexed by  $t \in ]0, \varepsilon[$ , where we have

$$c(t)(b) = \frac{1}{t} \frac{f_1^0(b)}{\|f_1^0(b)\|^2} \text{ for any trivalent vertex of } B,$$

$$c(t)(b) = (y(b), 0) + \frac{1}{t} \frac{f_1^0(b)}{\|f_1^0(b)\|^2} \text{ for any univalent vertex of } B, \text{ and}$$

$$c(t)(b) = (y(b(A)), 0) + \frac{1}{t} \frac{f_1^0(b(A))}{\|f_1^0(b(A))\|^2} + \frac{s \circ w_A^0(b)}{\lambda_A^0 \|f_1^0(b(A))\|^2} \text{ for any vertex } b \text{ of an element } A \text{ of } \mathcal{P}_X = \{B_j : j \in I\}.$$

So  $c^0$  is the limit at  $t = 0$  of the family of injective configurations  $c(t)_{t \in ]0, \varepsilon[}$ , and  $c(t)|_A = s \circ w_A^0$  up to dilation and translation, for any  $A \in \mathcal{P}_X$ .  $\square$

Theorem 14.23 is now proved.  $\square$

### 14.3 Variations of integrals on configuration spaces of long tangles

In this section, we prove Theorem 12.7.

**Lemma 14.47.** *For any two propagating forms  $\omega$  and  $\omega'$  of  $C_2(R)$  (as in Definition 3.11) that coincide on  $\partial C_2(R)$ , there exists a one-form  $\eta$  of  $C_2(R)$  that vanishes on  $\partial C_2(R)$  such that  $\omega' = \omega + d\eta$ . In particular, for any two homogeneous propagating forms  $\omega$  and  $\omega'$  of  $C_2(R)$  as in Definition 3.13 that coincide on  $UB_R$ , there exists a one-form  $\eta$  of  $C_2(R)$  that vanishes on  $\partial C_2(R)$  such that  $\omega' = \omega + d\eta$ .*

PROOF: Exercise. See the proof of Lemma 3.17.  $\square$

**Lemma 14.48.** *The element  $Z_n(\mathcal{C}, L, (\omega(i)))$  of  $\mathcal{A}_n(\mathcal{L})$  is independent of the chosen homogeneous propagating forms  $\omega(i)$  of  $(C_2(R(\mathcal{C})), \tau)$ , under the assumptions of Theorem 12.7.*

*More generally, if the  $\omega(i)$  are only assumed to be homogeneous propagating forms of  $C_2(R(\mathcal{C}))$ , then  $Z_n(\mathcal{C}, L, (\omega(i)))$  depends only on  $(\mathcal{C}, L \cap \mathcal{C}, \tau)$  and on the restrictions of the  $\omega(i)$  to  $UC$ .*

PROOF: By Lemma 14.47, it suffices to prove that  $Z$  does not vary when  $\omega(i)$  is changed to  $\omega(i) + d\eta$  for a one-form  $\eta$  on  $C_2(R(\mathcal{C}))$  that vanishes on  $\partial C_2(R(\mathcal{C}))$ . Let  $\Omega_\Gamma = \bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$  and let  $\tilde{\Omega}_\Gamma$  be obtained from  $\Omega_\Gamma$  by replacing  $\omega(i)$  by  $\eta$ . The variations of the integrals  $\int_{(C(R(\mathcal{C}), L; \Gamma), o(\Gamma))} \Omega_\Gamma$  are computed with Stokes' theorem, as the sum over the codimension-one faces  $F$  of  $C(R(\mathcal{C}), L; \Gamma)$  of  $\int_F \tilde{\Omega}_\Gamma$ , as allowed by Lemma 14.24.

These faces are the faces listed in Theorem 14.23. The arguments of Lemmas 9.9, 9.10, 9.11, 9.13, 9.14 allow us to get rid of all the faces, except the faces in which some vertices are at  $\infty$ , and the faces  $F(\check{\Gamma}_A, L, \Gamma)$  in which  $\check{\Gamma}_A$  is a connected diagram on  $\mathbb{R}$  (these faces are components of  $F(A, L, \Gamma)$  as in the proof of Lemma 10.20) and  $\Gamma$  is a diagram that contains  $\check{\Gamma}_A$  as a subdiagram on a component  $\mathcal{L}_j$  of  $L$ . The contribution of the faces  $F(\check{\Gamma}_A, L, \Gamma)$  is zero when  $i \notin j_E(E(\check{\Gamma}_A))$  for dimension reasons. It is zero when  $i \in j_E(E(\check{\Gamma}_A))$  because  $\eta$  vanishes on  $\partial C_2(R(\mathcal{C}))$ . So we are left with the faces for which some vertices are at  $\infty$ . Let  $F$  be such a face. Let  $V$  be the set of vertices mapped to  $\infty$  in  $F$ , let  $E_\infty$  be the set of edges between elements of  $V$ , and let  $E_m$  denote the set of edges with one end in  $V$ . When  $i \in j_E(E_\infty \cup E_m)$ , the contribution vanishes because  $\eta$  vanishes on  $\partial C_2(R(\mathcal{C}))$ .

Assume  $i \notin j_E(E_\infty \cup E_m)$ . The face  $F$  is diffeomorphic to a product by

$$\check{C}_{V(\Gamma) \setminus V}(\check{R}(\mathcal{C}), L; \Gamma),$$

whose dimension is

$$3|T(\Gamma) \cap (V(\Gamma) \setminus V)| + |U(\Gamma) \cap (V(\Gamma) \setminus V)|,$$

of a space  $C_V$  of dimension  $3|T(\Gamma) \cap V| + |U(\Gamma) \cap V| - 1$ , along which

$$\bigwedge_{e \in E_\infty \cup E_m} p_e^*(\omega(j_E(e)))$$

has to be integrated. The degree  $2|E_\infty \cup E_m|$  of this form is bigger than the dimension of  $C_V$  as a count of half-edges shows. So the faces for which some vertices are at  $\infty$  (including the  $T$ -faces) do not contribute either.  $\square$



**Proposition 14.49.** *Let  $L: \mathcal{C} \hookrightarrow \check{R}(\mathcal{C})$  denote a long tangle representative in a rational homology cylinder. Let  $\tau$  denote a parallelization of  $\mathcal{C}$  as in Definition 12.4. Let  $\omega_0$  and  $\omega_1$  be two homogeneous propagating forms of  $C_2(R(\mathcal{C}))$  (as in Definitions 3.11 and 3.13). Let  $\tilde{\omega}$  be a closed 2-form on  $[0, 1] \times \partial C_2(R(\mathcal{C}))$  whose restriction  $\tilde{\omega}(t)$  to  $\{t\} \times (\partial C_2(R(\mathcal{C})) \setminus UB_{R(\mathcal{C})})$  is  $p_\tau^*(\omega_{S^2})$  for any  $t \in [0, 1]$ , and such that the restriction of  $\omega_i$  to  $\partial C_2(R(\mathcal{C}))$  is  $\tilde{\omega}(i)$  for  $i \in \{0, 1\}$ . For any component  $K_j$  of  $L = \sqcup_{j=1}^k K_j$ , define  $I_j = \sum_{\Gamma_B \in \mathcal{D}^c(\mathbb{R})} \zeta_{\Gamma_B} I(\Gamma_B, K_j, \tilde{\omega})$ , where*

$$I(\Gamma_B, K_j, \tilde{\omega}) = \int_{u \in [0, 1]} \int_{w \in K_j \cap UB_{R(\mathcal{C})}} \int_{\check{S}(T_w \check{R}(\mathcal{C}), \vec{t}_w; \Gamma_B)} \bigwedge_{e \in E(\Gamma_B)} p_e^*(\tilde{\omega}(u)) [\Gamma_B]$$

and  $\vec{t}_w$  denotes the unit tangent vector to  $K_j$  at  $w$ . (The notation  $\check{S}(T_w \check{R}(\mathcal{C}), \vec{t}_w; \Gamma_B)$  is introduced before Lemma 8.16, and  $\mathcal{D}^c(\mathbb{R})$  is introduced at the beginning of Section 10.5.) Define

$$z(\tilde{\omega}) = \sum_{n \in \mathbb{N}} z_n([0, 1] \times UB_{R(\mathcal{C})}; \tilde{\omega})$$

as in Corollary 9.4. Then we have

$$Z(\mathcal{C}, L, \omega_1) = \left( \prod_{j=1}^k \exp(I_j) \#_j \right) Z(\mathcal{C}, L, \omega_0) \exp(z(\tilde{\omega})).$$

PROOF: According to Proposition 10.17, this statement holds when  $L$  is a link, and when  $L$  is the empty link in particular. Using Notation 7.16, it suffices to prove that

$$\check{Z}(\mathcal{C}, L, \omega_1) = \left( \prod_{j=1}^k \exp(I_j) \#_j \right) \check{Z}(\mathcal{C}, L, \omega_0)$$

since  $Z(\mathcal{C}, L, \omega_1) = Z(\check{R}(\mathcal{C}), \emptyset, \omega_1) \check{Z}(\mathcal{C}, L, \omega_1)$  as in Lemma 7.27. As in the proof of Lemma 14.48 above, the only faces contributing to the variation of  $\check{Z}(\mathcal{C}, L, \omega_t)$  are the faces  $F(\check{\Gamma}_A, L, \Gamma)$  for which  $\check{\Gamma}_A$  is a connected diagram on  $\mathbb{R}$  and  $\Gamma$  is a diagram that contains  $\check{\Gamma}_A$  as a subdiagram on a component  $\mathcal{L}_j$  of  $L$ . Their contribution yields the result as in the proof of Lemma 10.18.  $\square$

**Lemma 14.50.** *Recall Definition 12.6 of  $I_\theta(K, \tau)$  for a long component  $K: \mathbb{R} \hookrightarrow \check{R}(\mathcal{C})$  of a tangle in a parallelized  $\mathbb{Q}$ -cylinder  $(\mathcal{C}, \tau)$ . Let  $\mathcal{C}$  be a*

$\mathbb{Q}$ -cylinder. Let  $(\tau_t)_{t \in [0,1]}$  be a smooth homotopy of parallelizations of  $\mathcal{C}$ . For any component  $K$  of a tangle in  $\mathcal{C}$ , we have

$$I_\theta(K, \tau_u) - I_\theta(K, \tau_0) = 2 \int_{\cup_{t \in [0,u]} p_{\tau_t}(U+K)} \omega_{S^2}.$$

PROOF: When  $K$  is closed,  $I_\theta$  is defined in Lemma 7.15, and the lemma follows from Proposition 10.13 and Lemma 10.21. Lemma 9.1 implies the existence of a closed 2-form  $\omega$  on  $[0, 1] \times C_2(R(\mathcal{C}))$  that restricts to  $\{t\} \times C_2(R(\mathcal{C}))$  as a homogeneous propagating form of  $(C_2(R(\mathcal{C})), \tau_t)$  for all  $t \in [0, 1]$ . The integral of this form on  $\partial([0, u] \times C(R(\mathcal{C}), K; \hat{\mathfrak{K}}^K))$  is zero, and it is half the difference between the two sides of the equality to be proved when  $K$  is a long component.  $\square$

PROOF OF THEOREM 12.7: Let  $\omega$  be a homogeneous propagating form of  $(C_2(R(\mathcal{C})), \tau)$ . Let us study the variation of  $\check{Z}(\mathcal{C}, L, \tau) = (\check{Z}_n(\mathcal{C}, L, \omega))_{n \in \mathbb{N}}$  when  $\tau$  varies inside its homotopy class.

Let  $(\tau(t))_{t \in [0,1]}$  be a smooth homotopy of parallelizations of  $\mathcal{C}$ . Set  $\check{Z}(t) = \check{Z}(\mathcal{C}, L, \tau(t))$ . Thanks to Proposition 14.49, we have

$$\frac{\partial}{\partial t} \check{Z}(t) = \sum_{j=1}^k \left( \frac{\partial}{\partial t} \left( 2 \int_{\cup_{u \in [0,t]} p_{\tau(u)}(U+K_j)} \omega \right) \alpha \#_j \right) \check{Z}(t)$$

as in Lemma 10.21. Lemma 14.50 implies

$$I_\theta(K_j, \tau(t)) - I_\theta(K_j, \tau(0)) = 2 \int_{\cup_{u \in [0,t]} p_{\tau(u)}(U+K_j)} \omega$$

for any  $j$ . As in Corollary 10.22, conclude that

$$\prod_{j=1}^k \left( \exp \left( -I_\theta(K_j, \tau(t)) \alpha \right) \#_j \right) \check{Z}(t)$$

is constant, and note  $\check{Z}_1(\mathcal{C}, K_j, \tau(t)) = \frac{1}{2} I_\theta(K_j, \tau(t)) [\hat{\mathfrak{K}}]$  for an interval component  $K_j$ .

Proposition 14.49 and Lemma 9.1 imply that changing the trivialization  $\tau$  in a ball  $B_\tau$  that does not meet the tangle does not change  $\check{Z}$  (where the form  $\omega^\partial$  of Lemma 9.1 is easily assumed to pull back through the projection of  $[0, 1] \times (\partial C_2(R(\mathcal{C})) \setminus U(B_\tau))$  onto  $\partial C_2(R(\mathcal{C})) \setminus U(B_\tau)$  on  $[0, 1] \times (\partial C_2(R(\mathcal{C})) \setminus U(B_\tau))$ ). Then the proof of Theorem 12.7 can be concluded like the proof of Theorem 7.20 at the end of Section 10.5, with the following additional argument for the strands going from bottom to bottom or from top to top. In

the proof of Theorem 7.20, we assumed that  $p_\tau(U^+K_j) = v$  for some  $v \in S^2$ , and that  $g$  maps  $K_j$  to rotations with axis  $v$  for any  $j \in \underline{k}$ , in order to ensure that  ${}^\tau\psi(g^{-1})$  induces a diffeomorphism of  $\cup_{w \in K_j} \check{\mathcal{S}}(T_w\check{R}, \vec{t}_w; \check{\Gamma}_B)$ . Without loss of generality, we instead assume  $v = \vec{N}$  and  $p_\tau(U^+K_j) = \pm\vec{N}$  for all components  $K_j$  of  $L$ , except possibly in a neighborhood of the boundary of  $\mathcal{C}$ , which is mapped to 1 by  $g$  (so that  $p_\tau(U^+K_j)$  can move from  $\pm\vec{N}$  to  $\mp\vec{N}$  in this neighborhood).  $\square$

# Chapter 15

## The invariant $Z$ as a holonomy for braids

In this chapter, we interpret the extension of  $Z^f$  to long tangles of the previous chapter as a holonomy for long braids, and we study it as such.

Recall the compactification  $\mathcal{S}_V(T)$  of the space  $\check{\mathcal{S}}_V(T)$  of injective maps from a finite set  $V$  to a vector space  $T$  up to translation and dilation, from Theorem 8.11. Let  $B$  be a finite set of cardinality at least 2. Let  $\Gamma$  be a Jacobi diagram on a disjoint union of lines  $\mathbb{R}_b$  indexed by elements  $b$  of  $B$ . Let  $p_B: U(\Gamma) \rightarrow B$  be the natural map induced by  $i_\Gamma$ . We assume that  $p_B$  is onto. Let  $U_b = U_b(\Gamma) = p_B^{-1}(b)$  be the set of univalent vertices of  $\Gamma$  sent to  $\mathbb{R}_b$  by  $i_\Gamma$ . Let  $\check{\mathcal{V}}(\Gamma) \subset \check{\mathcal{S}}_{V(\Gamma)}(\mathbb{R}^3)$  be the quotient by the translations and the dilations of the space of injective maps  $c$  from  $V(\Gamma)$  to  $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$  that map  $U_b(\Gamma)$  to a vertical line  $y(c, b) \times \mathbb{R}$  for each  $b \in B$ , with respect to the order induced by  $i_\Gamma$ , so that the planar configuration  $y(c, \cdot): B \rightarrow \mathbb{C}$  is injective. Let  $\mathcal{V}(\Gamma)$  denote the closure of the image of  $\check{\mathcal{V}}(\Gamma)$  in  $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3) \times \mathcal{S}_B(\mathbb{C})$  under the map  $(c \mapsto (c, y(c, \cdot)))$ .

### 15.1 On the structure of $\mathcal{V}(\Gamma)$

In this section, we investigate the structure of  $\mathcal{V}(\Gamma)$ , as we did in Section 14.2 for  $C(R(\mathcal{C}), L; \Gamma)$ .

**Lemma 15.1.** *An element  $(c, y)$  of  $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3) \times \mathcal{S}_B(\mathbb{C})$  is in  $\mathcal{V}(\Gamma)$  if and only if*

- *for any  $b \in B$  and for any  $(v_1, v_2) \in U_b^2$ , the restriction of  $c$  to  $\{v_1, v_2\}$  is vertical, and its direction is that prescribed by  $i_\Gamma$ ,*

- for any pair  $(b_1, b_2)$  of distinct elements of  $B$  and for any  $(v_1, v_2)$  in  $U_{b_1} \times U_{b_2}$ , there exists  $\beta \in \mathbb{R}^+$  such that the restriction  $c|_{\{v_1, v_2\}}$  of  $c$  to  $\{v_1, v_2\}$  satisfies

$$p_{\mathbb{C}}(c|_{\{v_1, v_2\}}(v_2) - c|_{\{v_1, v_2\}}(v_1)) = \beta(y|_{\{b_1, b_2\}}(b_2) - y|_{\{b_1, b_2\}}(b_1))$$

(where  $\beta$  depends on chosen representatives of  $c$  and  $y$ ), and

- for any triple  $(b_1, b_2, b_3)$  of distinct elements of  $B$  and for any  $(v_1, v_2, v_3) \in U_{b_1} \times U_{b_2} \times U_{b_3}$ , there exists  $\beta \in \mathbb{R}^+$  such that

$$\begin{aligned} & \left( p_{\mathbb{C}}(c|_{\{v_1, v_2, v_3\}}(v_2) - c|_{\{v_1, v_2, v_3\}}(v_1)), p_{\mathbb{C}}(c|_{\{v_1, v_2, v_3\}}(v_3) - c|_{\{v_1, v_2, v_3\}}(v_1)) \right) \\ & = \beta \left( y|_{\{b_1, b_2, b_3\}}(b_2) - y|_{\{b_1, b_2, b_3\}}(b_1), y|_{\{b_1, b_2, b_3\}}(b_3) - y|_{\{b_1, b_2, b_3\}}(b_1) \right) \end{aligned}$$

in  $\mathbb{C}^2$ .

It is easy to see that an element of  $\mathcal{V}(\Gamma)$  must satisfy these conditions since they are closed and satisfied on  $\check{\mathcal{V}}(\Gamma)$ . We will prove the converse after Lemma 15.6.

Let  $(c^0, y^0)$  satisfy the conditions of Lemma 15.1. We are going to study how a neighborhood  $N(c^0, y^0)$  of  $(c^0, y^0)$  in  $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3) \times \mathcal{S}_B(\mathbb{C})$  intersects  $\check{\mathcal{V}}(\Gamma)$ .

Let us introduce notation in the following long notation paragraph, which ends just before Lemma 15.3.

**Notation 15.2.** As in Theorem 8.28, the configuration  $c^0$  in  $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$  is described by a  $\Delta$ -parenthesization  $\mathcal{P}$  of  $V = V(\Gamma)$  (as in Definition 8.24) and

$$(c_Z^0 \in \check{\mathcal{S}}_{K(Z)}(\mathbb{R}^3))_{Z \in \mathcal{P}}.$$

The configurations  $c$  in a neighborhood of  $c^0$  in  $\mathcal{S}_V(\mathbb{R}^3)$  may be expressed as

$$\begin{aligned} c((\mu_Z), (c_Z)) &= \sum_{Z \in \mathcal{P}} \left( \prod_{Y \in \mathcal{P}: Z \subseteq Y \subset V} \mu_Y \right) c_Z \\ &= c_V + \sum_{Z \in D(V)} \mu_Z (c_Z + \sum_{Y \in D(Z)} \mu_Y (c_Y + \dots)) \end{aligned}$$

for  $((\mu_Z)_{Z \in \mathcal{P} \setminus \{V\}}, (c_Z)_{Z \in \mathcal{P}}) \in [0, \varepsilon]^{P \setminus \{V\}} \times \prod_{Z \in \mathcal{P}} W_Z$ , as in Lemma 8.27.

Similarly, the configuration  $y^0$  in  $\mathcal{S}_B(\mathbb{C})$  may be written as

$$(y_D^0 \in \check{\mathcal{S}}_{K(D)}(\mathbb{C}))_{D \in \mathcal{P}_B}$$

for a  $\Delta$ -parenthesization  $\mathcal{P}_B$  of  $B$ , and the configurations  $y$  in a neighborhood of  $y^0$  in  $\mathcal{S}_B(\mathbb{C})$  may be expressed as

$$\begin{aligned} y((u_D), (y_D)) &= \sum_{D \in \mathcal{P}_B} \left( \prod_{E \in \mathcal{P}_B: D \subseteq E \subset B} u_E \right) y_D \\ &= y_B + \sum_{D \in D(B)} u_D (y_D + \sum_{E \in D(D)} u_E (y_E + \dots)) \end{aligned}$$

for  $((u_D)_{D \in \mathcal{P}_B \setminus \{B\}}, (y_D)_{D \in \mathcal{P}_B}) \in [0, \varepsilon]^{|\mathcal{P}_B \setminus \{B\}|} \times \prod_{D \in \mathcal{P}_B} N_D$ .

We normalize the  $y_D$  by choosing basepoints  $b(D)$  for the  $D \in \mathcal{P}_B$  and imposing  $y_D(b(D)) = 0$  and  $\sum_{E \in K(D)} \|y_D(E)\|^2 = 1$ .

For a set  $Y$  of  $\mathcal{P}$ , we choose a basepoint  $b(Y)$ , which is univalent if there is a univalent vertex in  $Y$ . We also choose a kid  $k_n(Y)$  such that  $|p_{\mathbb{R}}(c_Y^0(k_n(Y))) - p_{\mathbb{R}}(c_Y^0(b(Y)))|$  or  $|p_{\mathbb{C}}(c_Y^0(k_n(Y))) - p_{\mathbb{C}}(c_Y^0(b(Y)))|$  is maximal in the set

$$\left\{ \left| p_{\mathbb{R}}(c_Y^0(k)) - p_{\mathbb{R}}(c_Y^0(b(Y))) \right|, \left| p_{\mathbb{C}}(c_Y^0(k)) - p_{\mathbb{C}}(c_Y^0(b(Y))) \right| : k \in K(Y) \right\}$$

and we call it the *normalizing kid* of  $Y$ . If

$$\left| p_{\mathbb{R}}(c_Y^0(k_n(Y))) - p_{\mathbb{R}}(c_Y^0(b(Y))) \right| \geq \left| p_{\mathbb{C}}(c_Y^0(k_n(Y))) - p_{\mathbb{C}}(c_Y^0(b(Y))) \right|,$$

then we say that  $k_n(Y)$  is *vertically normalizing* or *v-normalizing*, and we normalize the configurations  $c_Y$  in a neighborhood of  $c_Y^0$  by imposing  $c_Y(b(Y)) = 0$  and  $|p_{\mathbb{R}}(c_Y(k_n(Y)))| = 1$ . Otherwise, we say that  $k_n(Y)$  is *horizontally normalizing* or *h-normalizing*, and we first normalize the configurations  $c_Y$  in a neighborhood of  $c_Y^0$  by imposing

$$c_Y(b(Y)) = 0 \text{ and } \left| p_{\mathbb{C}}(c_Y(k_n(Y))) \right| = 1.$$

(These normalizations are compatible with the smooth structure of  $\mathcal{S}_V(\mathbb{R}^3)$ . In the case of a horizontally normalizing kid, they will be changed in [Notation 15.4](#).)

In our neighborhood, we also impose  $\|c_Y(k) - c_Y^0(k)\| < \varepsilon$  for any kid  $k$  of  $Y$ , for a small  $\varepsilon \in ]0, 1[$ . So the manifold  $W_Y$  is diffeomorphic to the product of the product, over the nonnormalizing kids  $k$  of  $Y$  that do not contain  $b(Y)$ , of balls  $\mathring{B}(c_Y^0(k), \varepsilon)$ , by the set of elements  $c_Y(k_n(Y))$  of  $\mathring{B}(c_Y^0(k_n(Y)), \varepsilon)$ , such that  $|p_{\mathbb{R}}(c_Y(k_n(Y)))| = 1$  (resp. such that  $|p_{\mathbb{C}}(c_Y(k_n(Y)))| = 1$ ) if  $k$  is v-normalizing (resp. if  $k$  is h-normalizing). Note that  $|p_{\mathbb{C}}(c_Y(k))| < 2$  for any  $k \in K(Y)$ .

For a set  $Y$  of  $\mathcal{P}$ , let  $B(Y) \subseteq B$  be the set of (labels of) the components of its univalent vertices, and let  $\hat{B}(Y)$  denote the smallest element of  $\mathcal{P}_B$  such that  $B(Y) \subseteq \hat{B}(Y)$  if  $|B(Y)| \geq 2$ . If  $|B(Y)| = 1$ , set  $\hat{B}(Y) = B(Y)$ . If  $B(Y) \neq \emptyset$ , the set  $Y$  is called *univalent*.

For  $D \in \mathcal{P}_B$ , define the set  $\mathcal{P}_{X,D}$  of elements  $Y$  of  $\mathcal{P}$  such that  $\hat{B}(Y) = D$  and  $\hat{B}(Z) \neq D$  for every daughter  $Z$  of  $Y$ . Note that any element  $Y$  of  $\mathcal{P}_{X,D}$  has at least two kids  $Y_a$  and  $Y_b$  such that  $B(Y_a) \neq D$  and  $B(Y_b) \neq D$ . Set

$$\mathcal{P}'_X = \cup_{D \in \mathcal{P}_B} \mathcal{P}_{X,D}.$$

Let  $\widehat{\mathcal{P}}'_X$  be the subset of  $\mathcal{P}$  consisting of the univalent sets  $Z$  of  $\mathcal{P}$  such that  $|B(Z)| \geq 2$ .

**Lemma 15.3.** *Let  $(c^0, y^0)$  be a configuration parametrized as above, which satisfies the conditions of Lemma 15.1.*

*For any  $Y \in \widehat{\mathcal{P}}'_X$ , there exists a unique  $\lambda^0(Y) \in \mathbb{R}^+$  such that for any two univalent kids  $Y_1$  and  $Y_2$  of  $Y$  such that  $B(Y_1) \subset D_1$  and  $B(Y_2) \subset D_2$ , where  $D_1$  and  $D_2$  are kids of  $\hat{B}(Y)$ , we have*

$$p_{\mathbb{C}}(c_Y^0(Y_2) - c_Y^0(Y_1)) = \lambda^0(Y)(y_{\hat{B}(Y)}^0(D_2) - y_{\hat{B}(Y)}^0(D_1)).^1$$

*For any  $Y \in \widehat{\mathcal{P}}'_X \setminus \mathcal{P}'_X$ , we have  $\lambda^0(Y) = 0$ . Conversely, if an element  $(c^0, y^0)$  satisfies the above properties and the first condition of Lemma 15.1, then it satisfies the conditions of Lemma 15.1.*

PROOF: Let  $Y \in \mathcal{P}$ . Assume  $|B(Y)| \geq 2$ . The first condition of Lemma 15.1 ensures that when  $v$  and  $v'$  belong to  $Y \cap U_b$ , we have

$$p_{\mathbb{C}} \circ c_Y^0(v) = p_{\mathbb{C}} \circ c_Y^0(v') = (p_{\mathbb{C}} \circ c_Y^0)(b \in B).$$

So, we view  $p_{\mathbb{C}} \circ c_Y^0$  as a map from  $B(Y)$  to  $\mathbb{C}$ .

Assume  $Y \in \widehat{\mathcal{P}}'_X$ . Then  $y_{\hat{B}(Y)}^0|_{B(Y)}$  is not constant. So, the second and third conditions ensure that there exists a unique  $\lambda^0(Y) \in \mathbb{R}^+$  such that  $p_{\mathbb{C}} \circ c_Y^0$  coincides with  $\lambda^0(Y)y_{\hat{B}(Y)}^0|_{B(Y)}$  up to translation. Since  $p_{\mathbb{C}} \circ c_Y^0$  is constant on  $B(Z)$  for any kid  $Z$  of  $Y$ , if there exists such a kid  $Z$  such that  $\hat{B}(Z) = \hat{B}(Y)$ , then  $\lambda^0(Y)$  must be equal to zero.

The last assertion is an easy exercise. □

In order to finish the proof of Lemma 15.1, we are going to prove that the configurations that satisfy the conditions of its statement are in  $\mathcal{V}(\Gamma)$ . We take a closer look at the structure of  $\mathcal{V}(\Gamma)$ .

**Notation 15.4.** For a univalent  $Y \in \mathcal{P}$ , define  $d(Y) \in B$  such that  $b(Y) \in U_{d(Y)}$ . Also assume that  $p_B(b(V))$  is the basepoint  $b_0$  of  $B$ . As always, our basepoints satisfy that if  $Z \subset Y$  and if  $b(Y) \in Z$ , then  $b(Z) = b(Y)$ . For  $D \in \mathcal{P}_B$ , set  $U_D = \prod_{E \in \mathcal{P}_B \setminus \{B\} : D \subseteq E} u_E$ . For  $d \in D$ , set

$$\tilde{y}_D(d) = y_D(d) + \sum_{E \in \mathcal{P}_B : d \in E \subset D} \left( \prod_{F \in \mathcal{P}_B : E \subseteq F \subset D} u_F \right) y_E(d).$$

For  $Y \in \mathcal{P}_X$ , set  $M_Y = \prod_{Z \in \mathcal{P} \setminus \{V\} : Y \subseteq Z} \mu_Z$ .

---

<sup>1</sup> $\lambda^0(Y)$  depends on the fixed normalizations of  $c_Y^0$  and  $y_{\hat{B}(Y)}^0$ .

Let  $\mathcal{P}'_{X,hn}$  be the set of elements  $Z$  of  $\widehat{\mathcal{P}}'_X$  such that  $k_n(Z)$  is a univalent horizontally normalizing kid. For an element  $Z$  of  $\mathcal{P}'_{X,hn}$ , we have  $\lambda^0(Z) \neq 0$ , which implies  $\mathcal{P}'_{X,hn} \subseteq \mathcal{P}'_X$ . We change the normalizations of the  $c_Z$  for the elements  $Z$  of  $\mathcal{P}'_{X,hn}$  in our neighborhood  $N(c^0, y^0)$  of  $(c^0, y^0)$  in  $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3) \times \mathcal{S}_B(\mathbb{C})$  (which we reduce if needed) in  $\mathcal{S}_B(\mathbb{C})$  so that  $c_Z(b(Z)) = 0$  and

$$p_{\mathbb{C}}\left(c_Z(k_n(Z))\right) = \lambda^0(Z) \left( \tilde{y}_{\hat{B}(Z)}\left(d(k_n(Z))\right) - \tilde{y}_{\hat{B}(Z)}\left(d(Z)\right) \right). \quad (15.1)$$

With this normalization, Lemma 15.3 is still valid, and we have  $\lambda(Z) = \lambda^0(Z)$  in our neighborhood.

**Lemma 15.5.** *With the above normalizations and notation, there exist continuous maps  $\lambda$  and  $\lambda(Y)$ , for  $Y \in \mathcal{P}'_X$ , from  $N(c^0, y^0) \cap \mathcal{V}(\Gamma)$  to  $\mathbb{R}^+$  satisfying the following properties. For any  $Y \in \mathcal{P}'_X$ , we have  $\lambda(Y)(c^0) = \lambda^0(Y)$  with the notation of Lemma 15.3 and, for any configuration  $(c, y)$  of  $N(c^0, y^0) \cap \mathcal{V}(\Gamma)$ ,*

- we have  $p_{\mathbb{C}} \circ c(v) = \lambda y(d)$  for any  $d \in B$  and for any  $v \in U_d$ ,
- for  $D \in \mathcal{P}_B$  and  $Y \in \mathcal{P}_{X,D}$ ,
  - if  $Y_a$  and  $Y_b$  are two univalent kids of  $Y$ , then we have

$$p_{\mathbb{C}}(c_Y(Y_b) - c_Y(Y_a)) = \lambda(Y) \left( \tilde{y}_D(d(Y_b)) - \tilde{y}_D(d(Y_a)) \right),$$

– and the equality

$$*(Y) : \lambda(Y)M_Y = \lambda U_{\hat{B}(Y)}$$

holds.

When  $V \in \mathcal{P}'_X$ ,  $*(V)$  is equivalent to  $\lambda(V) = \lambda$ . When  $V \notin \mathcal{P}'_X$ , we also let  $\lambda(V)$  and  $\lambda$  both denote  $\lambda$ , depending on the context.

PROOF: In  $N(c^0, y^0) \cap \check{\mathcal{V}}(\Gamma)$ , where  $\prod_{D \in \mathcal{P}_B \setminus \{B\}} u_D \times \prod_{W \in \mathcal{P} \setminus \{V\}} \mu_W \neq 0$ , there exists some  $\lambda > 0$  such that  $p_{\mathbb{C}} \circ c(v) = \lambda y(b)$  for any  $b \in U_b$  and for any  $v \in U_b$ . This map  $\lambda$ , starting from  $N(c^0, y^0) \cap \check{\mathcal{V}}(\Gamma)$ , can be extended continuously on  $N(c^0, y^0) \cap \mathcal{V}(\Gamma)$ , as follows. Let  $b \in B$  be such that  $b$  and  $b_0$  belong to different kids of  $B$ , and let  $v \in U_b$ . Then we have  $p_{\mathbb{C}}(c(v)) = \lambda y(b)$  on  $N(c^0, y^0) \cap \check{\mathcal{V}}(\Gamma)$ , and  $y(b)$  does not vanish on  $N(c^0, y^0)$ . The closed condition that  $p_{\mathbb{C}}(c(v))$  and  $y(b)$  are colinear and that their scalar product is nonnegative is satisfied on  $N(c^0, y^0) \cap \mathcal{V}(\Gamma)$ . It allows us to define  $\lambda(v)$  such that  $p_{\mathbb{C}}(c(v)) = \lambda(v)y(b)$  on  $N(c^0, y^0) \cap \mathcal{V}(\Gamma)$  and  $\lambda(v)$  is continuous. Now, since  $\lambda(v) = \lambda$  is independent of  $v$  as above on  $N(c^0, y^0) \cap \check{\mathcal{V}}(\Gamma)$ , it is



also on  $N(c^0, y^0) \cap \mathcal{V}(\Gamma)$ . Set  $\lambda = \lambda(v)$ . Then we have  $p_{\mathbb{C}} \circ c(v) = \lambda y(b)$  for any  $b \in U_b$  and for any  $v \in U_b$  on  $N(c^0, y^0) \cap \check{\mathcal{V}}(\Gamma)$ . This is also true on  $N(c^0, y^0) \cap \mathcal{V}(\Gamma)$ .

Let  $Y \in \mathcal{P}$ . Let  $Y_a$  and  $Y_b$  be two univalent kids of  $Y$ . If  $c \in N(c^0, y^0) \cap \check{\mathcal{V}}(\Gamma)$ , then we have

$$M_Y \left( p_{\mathbb{C}}(c_Y(Y_b) - c_Y(Y_a)) \right) = \lambda U_{\hat{B}(Y)} \left( \tilde{y}_{\hat{B}(Y)}(d(Y_b)) - \tilde{y}_{\hat{B}(Y)}(d(Y_a)) \right).$$

In particular,  $p_{\mathbb{C}}(c_Y(Y_b) - c_Y(Y_a))$  and  $(\tilde{y}_{\hat{B}(Y)}(d(Y_b)) - \tilde{y}_{\hat{B}(Y)}(d(Y_a)))$  are colinear, and their scalar product is nonnegative on  $N(c^0, y^0) \cap \check{\mathcal{V}}(\Gamma)$ . As above, as soon as there exist two kids  $Y_a$  and  $Y_b$  as above, such that  $d(Y_a)$  and  $d(Y_b)$  are in two distinct kids of  $\hat{B}(Y)$ , we can define the continuous function  $\lambda(Y)$  such that

$$p_{\mathbb{C}}(c_Y(Y_b) - c_Y(Y_a)) = \lambda(Y) \left( \tilde{y}_{\hat{B}(Y)}(d(Y_b)) - \tilde{y}_{\hat{B}(Y)}(d(Y_a)) \right)$$

for any two univalent kids  $Y_a$  and  $Y_b$  of  $Y$ , and we have  $\lambda(Y)M_Y = \lambda U_{\hat{B}(Y)}$ . □

Let  $\mathcal{P}'_B$  be the set of elements  $D$  of  $\mathcal{P}_B$  such that  $\mathcal{P}_{X,D} \neq \emptyset$ . Let  $\widehat{\mathcal{P}}'_B$  be the set of elements  $D$  of  $\mathcal{P}_B$  that contain, or are equal to, an element of  $\mathcal{P}'_B$ .

For any collections  $(B_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ ,  $(B_i^+)_{i \in \mathbb{Z}/n\mathbb{Z}}$ ,  $(B'_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$  of (possibly equal) sets of  $\widehat{\mathcal{P}}'_B$  such that

$$(B_{i-1}^+ \cup B_i) \subseteq B'_i,$$

for any collections  $(Y_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ ,  $(Y_i^+)_{i \in \mathbb{Z}/n\mathbb{Z}}$ ,  $(Y'_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$  such that

$$Y_i \in \mathcal{P}_{X, B_i}, Y_i^+ \in \mathcal{P}_{X, B_i^+}, Y'_i \in \widehat{\mathcal{P}}'_X, \text{ and } (Y_i^+ \cup Y_i) \subseteq Y'_i,$$

we have

$$\prod_{i=1}^n \left( \lambda(Y_i) \frac{M_{Y_i}}{M_{Y'_i}} \right) \prod_{i=1}^n \frac{U_{B_i^+}}{U_{B'_{i+1}}} = \prod_{i=1}^n \frac{U_{B_i}}{U_{B'_i}} \prod_{i=1}^n \left( \lambda(Y_i^+) \frac{M_{Y_i^+}}{M_{Y'_i}} \right)$$

for configurations of  $N(c^0, y^0) \cap \check{\mathcal{V}}(\Gamma)$ . This equation is equivalent to the following equation, which also holds in  $N(c^0, y^0) \cap \mathcal{V}(\Gamma)$ :

$$\begin{aligned} & \prod_{i=1}^n \left( \lambda(Y_i) \prod_{Z \in \mathcal{P}: Y_i \subseteq Z \subset Y'_i} \mu_Z \right) \prod_{i=1}^n \left( \prod_{E \in \mathcal{P}_B: B_i^+ \subseteq E \subset B'_{i+1}} u_E \right) \\ &= \prod_{i=1}^n \left( \lambda(Y_i^+) \prod_{Z \in \mathcal{P}: Y_i^+ \subseteq Z \subset Y'_i} \mu_Z \right) \prod_{i=1}^n \left( \prod_{E \in \mathcal{P}_B: B_i \subseteq E \subset B'_i} u_E \right). \end{aligned}$$

For  $Y \in \mathcal{P}$ , let  $K_u(Y)$  denote the set of univalent kids of  $Y$  that do not contain  $b(Y)$ , and let  $K_t(Y)$  denote the set of nonunivalent kids of  $Y$  that do not contain  $b(Y)$ .

We are going to prove the following lemma.

**Lemma 15.6.** *Let  $(c^0, y^0)$  satisfy the conditions of Lemma 15.1 (and hence of Lemma 15.3). There is a neighborhood of  $(c^0, y^0)$  in  $\mathcal{V}(\Gamma)$  parametrized by the following variables:*

1.  $((u_D)_{D \in \mathcal{P}_B \setminus \{B\}}, (y_D)_{D \in \mathcal{P}_B}) \in [0, \varepsilon]^{|\mathcal{P}_B \setminus \{B\}|} \times \prod_{D \in \mathcal{P}_B} N_D$ ,
2.  $(\mu_Z)_{Z \in \mathcal{P} \setminus \{V\}} \in [0, \varepsilon]^{|\mathcal{P} \setminus \{V\}|}$ ,
3. the  $c_Z$  for nonunivalent sets  $Z$  of  $\mathcal{P}$ ,
4. for univalent sets  $Z$  of  $\mathcal{P}$ , the  $c_Z(Y)$  for the kids  $Y \in K_t(Z)$  of  $Z$ , and the  $p_{\mathbb{R}} \circ c_Z(Y)$  for the kids  $Y \in K_u(Z)$  of  $Z$ ,
5. the parameter  $\lambda$ , and
6. for every element  $Y$  of  $\mathcal{P}'_X$ , the parameter  $\lambda(Y)$  defined in Lemma 15.5.

These variables satisfy the following constraints and determine the  $p_{\mathbb{C}} \circ c_Z(Y)$  for the univalent kids  $Y \in K_u(Z)$  of a set  $Z$  of  $\mathcal{P}$  as follows.

1. For a vertically normalizing kid  $Y$  of an element  $Z$  of  $\mathcal{P}$ , we have  $p_{\mathbb{R}} \circ c_Z(Y) = p_{\mathbb{R}} \circ c_Z^0(Y) = \pm 1$ .
2. For a horizontally normalizing nonunivalent kid  $Y$  of an element  $Z$  of  $\mathcal{P}$ , we have  $|p_{\mathbb{C}} \circ c_Z(Y)| = 1$ .
3. For a horizontally normalizing univalent kid  $Y$  of an element  $Z$  of  $\mathcal{P}$ , we have

$$p_{\mathbb{C}}(c_Z(Y)) = \lambda^0(Z) \left( \tilde{y}_{\hat{B}(Z)}(d(Y)) - \tilde{y}_{\hat{B}(Z)}(d(Z)) \right).$$

4. For any element  $Y$  of  $\mathcal{P}_{X,D}$ , we have

$$*(Y) : M_Y \lambda(Y) = \lambda U_D$$

where  $M_Y = \prod_{Z \in \mathcal{P} \setminus \{V\} : Y \subseteq Z} \mu_Z$  and  $U_D = \prod_{E \in \mathcal{P}_B \setminus \{B\} : D \subseteq E} u_E$ .

5. For any collections  $(B_i)_{i \in \mathbb{Z}/n\mathbb{Z}}, (B_i^+)_{i \in \mathbb{Z}/n\mathbb{Z}}, (B'_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$  of (possibly equal) sets of  $\widehat{\mathcal{P}}'_B$  such that

$$(B_{i-1}^+ \cup B_i) \subseteq B'_i,$$

for any collections  $(Y_i)_{i \in \mathbb{Z}/n\mathbb{Z}}, (Y_i^+)_{i \in \mathbb{Z}/n\mathbb{Z}}, (Y'_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$  such that

$$Y_i \in \mathcal{P}_{X, B_i}, Y_i^+ \in \mathcal{P}_{X, B_i^+}, Y'_i \in \widehat{\mathcal{P}}'_X, \text{ and } (Y_i^+ \cup Y_i) \subseteq Y'_i,$$

we have

$$\begin{aligned} & \prod_{i=1}^n \left( \lambda(Y_i) \prod_{Z \in \mathcal{P}: Y_i \subseteq Z \subseteq Y'_i} \mu_Z \right) \prod_{i=1}^n \left( \prod_{E \in \mathcal{P}_B: B_i^+ \subseteq E \subseteq B'_{i+1}} u_E \right) \\ &= \prod_{i=1}^n \left( \lambda(Y_i^+) \prod_{Z \in \mathcal{P}: Y_i^+ \subseteq Z \subseteq Y'_i} \mu_Z \right) \prod_{i=1}^n \left( \prod_{E \in \mathcal{P}_B: B_i \subseteq E \subseteq B'_i} u_E \right). \end{aligned} \tag{15.2}$$

6. For any univalent kid  $Z_b$  of an element  $Z$  of  $\mathcal{P}$  such that  $d(Z_b) = d(Z)$ , we have

$$p_{\mathbb{C}} \circ c_Z(Z_b) = p_{\mathbb{C}} \circ c_Z(b(Z)) = 0.$$

7. For any univalent element  $Z$  of  $\mathcal{P}$  such that  $|B(Z)| \geq 2$ , for any maximal element  $Y$  of  $\mathcal{P}'_X$  such that  $Y \subseteq Z$  and  $\hat{B}(Z) = \hat{B}(Y)$ ,<sup>2</sup> for any univalent kid  $Z_b$  of  $Z$ ,  $p_{\mathbb{C}} \circ c_Z(Z_b)$  is equal to

$$p_{\mathbb{C}} \circ c_Z(Z_b) = \frac{M_Y}{M_Z} \lambda(Y) \left( \check{y}_{\hat{B}(Z)}(d(Z_b)) - \check{y}_{\hat{B}(Z)}(d(Z)) \right). \tag{15.3}$$

The analysis performed before Lemma 15.6 ensures that all the elements in our neighborhood  $N(c^0, y^0)$  of  $(c^0, y^0)$  in  $\mathcal{S}_V(\mathbb{R}^3) \times \mathcal{S}_B(\mathbb{C})$  that are in  $\mathcal{V}(\Gamma)$  are described by the parameters described in the statement and that they satisfy all the equations of the statement. (See the proof of Lemma 15.5 for the last one.) In order to prove Lemma 15.6, we are going to prove that, conversely, the elements described in its statement are in  $\mathcal{V}(\Gamma)$ . Note that the elements described in this statement such that neither  $\lambda$ , nor the  $\mu_Z$  nor the  $u_D$  vanish, correspond to elements of  $\check{\mathcal{S}}_V(\mathbb{R}^3) \times \check{\mathcal{S}}_B(\mathbb{C})$  and are in  $\check{\mathcal{V}}(\Gamma)$ .

We define a tree  $\mathcal{T}(\mathcal{P}, \mathcal{P}_B)$  with oriented edges from the tree whose vertices are the elements of  $\widehat{\mathcal{P}}'_X$  and whose edges start at an element  $Z$  of  $\widehat{\mathcal{P}}'_X \setminus \{V\}$ , end at its mother  $m(Z)$ , and are labeled by  $\mu_Z$ ,

---

<sup>2</sup>Note that the minimal univalent elements  $Z$  of  $\mathcal{P}$  such that  $|B(Z)| \geq 2$  are in  $\mathcal{P}'_X$ .

- by gluing an edge, which arrives at the vertex  $Y$ , starts at a vertex labeled by  $O_Y$ , and which is labeled by  $\lambda(Y)$ , at the vertex  $Y$ , for any  $Y$  of  $\mathcal{P}'_X$ ,
- by gluing an edge, which arrives at the vertex  $V$ , starts at a vertex labeled by  $B$ , and is labeled by  $\lambda$ , at the vertex  $V$ ,
- by gluing to the root  $B$  of this edge the subtree of  $\mathcal{P}_B$  whose vertices are the elements of  $\widehat{\mathcal{P}}'_B$  and whose edges are labeled by the  $u_D$  for  $D \in \widehat{\mathcal{P}}'_B \setminus \{B\}$ , where the edge labeled by  $u_D$  starts at  $D$  and ends at its mother  $m(D)$ .

An example is drawn in Figure 15.1.

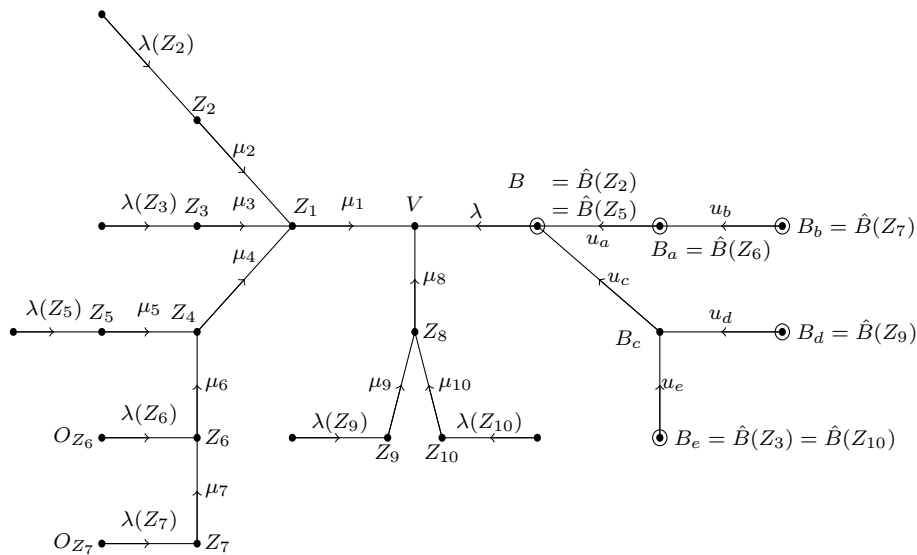


Figure 15.1: The tree  $\mathcal{T}(\mathcal{P}, \mathcal{P}_B)$  associated to a configuration  $c$

**PROOF OF LEMMA 15.1:** We prove that our configuration  $(c^0, y^0)$  of Lemma 15.3 and 15.6 is in  $\mathcal{V}(\Gamma)$  by exhibiting a family  $(c(t), y(t))$  of configurations of  $\check{\mathcal{V}}(\Gamma)$ , for  $t \in ]0, \varepsilon[$  tending to 0, which tends to  $(c^0, y^0)$ . Our family will be described by nonvanishing  $u_D(t)$ ,  $\mu_Z(t)$  tending to 0, nonvanishing  $\lambda(Y)(t)$  tending to our given  $\lambda^0(Y)$ , and nonvanishing  $\lambda(t)$  tending to our given  $\lambda^0$ , such that the equations

$$*(Y)(t): M_Y(t)\lambda(Y)(t) = \lambda(t)U_D(t)$$

are satisfied for any element  $Y$  of  $\mathcal{P}_{X,D}$ . The parameters  $(y_D)_{D \in \mathcal{P}_B}$ ,  $c_Z(Y)$  for nonunivalent kids  $Y$  of elements  $Z$  of  $\mathcal{P}$ ,  $p_{\mathbb{R}}(c_Z(Y))$  for univalent kids  $Y$  of elements  $Z$  of  $\mathcal{P}$ , of the statement of Lemma 15.6 will be defined to be the same as those of  $(c^0, y^0)$ .

Set  $\mu_Y(t) = t$  for any  $Y \in \mathcal{P} \setminus (\mathcal{P}'_X \cup \{V\})$ , and  $u_D(t) = t$  for any  $D \in \mathcal{P}_B \setminus (\mathcal{P}'_B \cup \{B\})$ .

Denote  $\lambda(t)$  by  $u_B(t)$  to make notation homogeneous. See Figure 15.1. Recall that we have  $\lambda(V) = \lambda$  when  $V \in \mathcal{P}_{X,B}$ .

For  $D \in \mathcal{P}'_B$ , we are going to define some integer  $g(D) \geq 1$  and set  $u_D(t) = t^{g(D)}$  when  $D \neq B$ . We are going to set  $u_B(t) = \lambda(t) = t^{g(B)}$  if  $\lambda^0 = 0$ , and  $u_B(t) = \lambda(t)$ , where  $\lambda(t) = \lambda^0$  if  $\lambda^0 \neq 0$ .

Let  $Y \in \mathcal{P}'_X$ . If  $Y$  is maximal in  $\mathcal{P}'_X$ , set  $\lambda'(Y) = 1$ . If  $Y$  is not maximal in  $\mathcal{P}'_X$ , let  $s(Y)$  be the minimal element of  $\mathcal{P}'_X$  that contains  $Y$  strictly. If  $\lambda^0(s(Y)) = 0$ , set  $\lambda'(Y)(t) = 1$ . If  $\lambda^0(s(Y)) \neq 0$ , set  $\lambda'(Y)(t) = \lambda(s(Y))(t)$ . For  $Y \in \mathcal{P}'_X$ , we are going to define some integer  $g(Y) \geq 1$  and set

- $\lambda(Y)(t) = t$  and  $\mu_Y(t) = t^{g(Y)}\lambda'(Y)$  if  $\lambda^0(Y) = 0$ ,
- $\lambda(Y)(t) = \lambda^0(Y)$  and  $\mu_Y(t) = \frac{t^{g(Y)+1}}{\lambda^0(Y)}\lambda'(Y)(t)$  if  $\lambda^0(Y) \neq 0$ .

(If  $Y = V$ , just forget about  $\mu_V$ , which is useless.)

Order the elements of  $\mathcal{P}'_B$ , by calling them  $D_1, D_2, \dots, D_{|\mathcal{P}'_B|}$  so that  $D_1$  is maximal in  $\mathcal{P}'_B$  (with respect to the inclusion) and  $D_{i+1}$  is maximal in  $\mathcal{P}'_B \setminus \{D_1, D_2, \dots, D_i\}$ . Note that  $B \in \mathcal{P}'_B$ , so  $D_1 = B$ .

Recall  $\lambda(t) = t^{g(B)}$  if  $\lambda^0 = 0$  and  $\lambda(t) \neq 0$  if  $\lambda^0 \neq 0$ . Set  $g(D) = 1$  for any  $D \in \widehat{\mathcal{P}'_B} \setminus \mathcal{P}'_B$ .

We are going to define the integers  $g(D_i)$  and the integers  $g(Y)$  for every  $Y \in \mathcal{P}_{X,D_i}$ , for  $i \in \underline{|\mathcal{P}'_B|}$ , inductively, so that the following assertion  $(*(i))$  holds for every  $i$ .

$(*(i))$ : for every  $Y \in \mathcal{P}_{X,D_i}$ , we have  $M_Y(t)\lambda(Y)(t) = \lambda(t)U_{D_i}(t) = \lambda(t)t^{f(i)}$  for all  $t \in ]0, \varepsilon[$ , where  $f(i) = \sum_{D \in \widehat{\mathcal{P}'_B}: D_i \subseteq D \subset B} g(D)$ .

Recall  $D_1 = B$  and note  $f(1) = 0$ . If  $V \in \mathcal{P}_{X,B}$ , set  $g(D_1) = 1$ . So  $(*(1))$  is satisfied. (If  $\lambda^0 \neq 0$ , then  $g(D_1)$  is not used, and  $(*(1))$  is satisfied.) If  $V \notin \mathcal{P}_{X,B}$ , then  $\lambda^0 = 0$ , according to Lemma 15.3. Define  $g(D_1)$  to be the maximum over the elements  $Y$  of  $\mathcal{P}_{X,B}$  of the number of elements of  $\mathcal{P}$  that contain  $Y$  strictly, plus one. Then define the  $g(Y)$  for the elements  $Y$  of  $\mathcal{P}_{X,B}$  so that  $(*(1))$  holds.

Let  $k \in \underline{|\mathcal{P}'_B|}$ .

Assume that we have defined  $g(D_i)$  and the  $g(Y)$  for all  $Y \in \mathcal{P}_{X,D_i}$ , for all  $i < k$ , so that the assertions  $(*(i))$  hold for any  $i < k$ . Let us define the  $g(Y)$  for all  $Y \in \mathcal{P}_{X,D_k}$  and  $g(D_k)$  so that  $(*(k))$  holds. For any

$Y \in \mathcal{P}_{X, D_k}$ ,  $M_Y(t)\lambda(Y)(t)$  is equal to  $\lambda(t)t^{g(Y)+h(Y)}$  for some integer  $h(Y)$ , and  $\lambda(t)U_{D_k}(t)$  is equal to  $\lambda(t)t^{g(D_k)+h(D_k)}$  for some integer  $h(D_k)$ . Let  $H(k)$  be the maximum over  $h(D_k)$  and all the integers  $h(Y)$  for  $Y \in \mathcal{P}_{X, D_k}$ . Set  $g(D_k) = H(k) + 1 - h(D_k)$  and  $g(Y) = H(k) + 1 - h(Y)$ , so that  $(*(k))$  holds with  $f(k) = H(k) + 1$ .

This process is illustrated in Figure 15.2. Its result does not depend on the arbitrary order that respects our condition on the  $D_i$ .

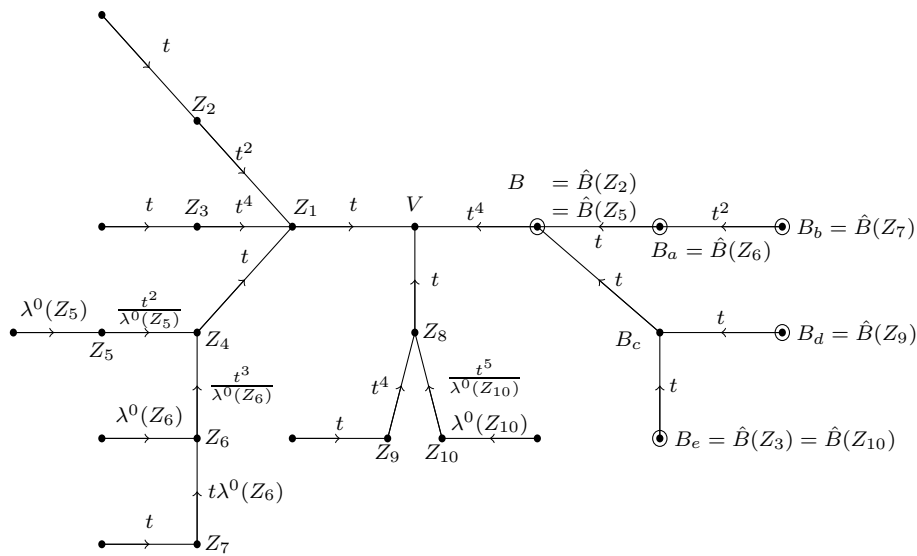


Figure 15.2: Interior configurations tending to our limit configuration  $c^0$ , whose associated tree  $\mathcal{T}(\mathcal{P}, \mathcal{P}_B)$  is as in Figure 15.1, where  $\lambda^0(Z_2) = \lambda^0(Z_3) = \lambda^0(Z_7) = \lambda^0(Z_9) = 0$  and the other  $\lambda^0(Z_i)$  are not zero.

□

Let us now begin the longer proof of Lemma 15.6, which ends after Sublemma 15.11. The elements of  $\mathcal{V}(\Gamma)$  obviously satisfy the equations of the statement of Lemma 15.6. Let us conversely prove that an element  $c$  parametrized as in this statement belongs to  $\mathcal{V}(\Gamma)$  by exhibiting a sequence of configurations  $c(t)$  of  $\check{\mathcal{V}}(\Gamma)$  that approaches it. Again, we index our family of configurations  $c(t)$  tending to the given one  $c$  by a variable  $t$  tending to 0.

For  $Y \in \mathcal{P} \setminus (\widehat{\mathcal{P}}'_X \cup \{V\})$ , set  $\mu_Y = \mu_Y(c)$  and

$$\mu_Y(t) = \begin{cases} t & \text{if } \mu_Y = 0 \\ \mu_Y & \text{if } \mu_Y \neq 0. \end{cases}$$

Similarly, for  $D \in \mathcal{P}_B \setminus (\widehat{\mathcal{P}}'_B \cup \{B\})$ , set  $u_D = u_D(c)$  and

$$u_D(t) = \begin{cases} t & \text{if } u_D = 0 \\ u_D & \text{if } u_D \neq 0. \end{cases}$$

We will again focus on the parameters  $\mu_Z$  for  $Z \in \widehat{\mathcal{P}}'_X \setminus \{V\}$ ,  $u_D$  for  $D \in \widehat{\mathcal{P}}'_B \setminus \{B\}$ ,  $\lambda(Y)$  for  $Y \in \mathcal{P}'_X$ , and  $\lambda$ . All of these parameters correspond to edges of our tree  $\mathcal{T}(\mathcal{P}, \mathcal{P}_B)$ . We denote the parameter corresponding to an edge  $e$  by  $\lambda(e)$  and the corresponding parameter for the family by  $\lambda(e)(t)$ . The other parameters of Lemma 15.6 are fixed as in the proof of Lemma 15.1.

When  $\lambda(e) \neq 0$ , set  $\lambda(e)(t) = \lambda(e)$ .

Let  $\mathcal{P}'_{X,c}$  be the set of elements  $Y$  of  $\mathcal{P}'_X$  such that  $\lambda(Y)M_Y = 0$ . If  $Y \in \mathcal{P}'_X \setminus \mathcal{P}'_{X,c}$ , then  $(\lambda(Y)M_Y = \lambda U_{\hat{B}(Y)})$  is not zero. Let  $\mathcal{P}'_{B,c}$  be the set of elements  $D$  of  $\mathcal{P}'_B$  such that  $\lambda U_D = 0$ .

The main equations

$$*(Y)(t) : \lambda(Y)(t)M_Y(t) = \lambda(t)U_{\hat{B}(Y)}(t)$$

associated to elements of  $\mathcal{P}'_X \setminus \mathcal{P}'_{X,c}$  are obviously satisfied.

For  $Y \in \mathcal{P}'_X$ , recall that  $O_Y$  denotes the initial vertex of the edge of  $\lambda(Y)$  in the tree  $\mathcal{T}(\mathcal{P}, \mathcal{P}_B)$ . For  $Y \in \mathcal{P}'_{X,c}$ , define  $e(Y)$  to be the closest edge to  $O_Y$  such that  $\lambda(e(Y)) = 0$  between  $O_Y$  and the vertex  $V$  in  $\mathcal{T}(\mathcal{P}, \mathcal{P}_B)$ . For  $D \in \mathcal{P}'_{B,c}$ , define  $e(D)$  to be the closest edge to  $D$  such that  $\lambda(e(D)) = 0$  between  $D$  and the vertex  $V$ .

For edges  $e$  of  $\mathcal{T}(\mathcal{P}, \mathcal{P}_B)$  such that  $\lambda(e) = 0$  that are not in  $e(\mathcal{P}'_{X,c} \cup \mathcal{P}'_{B,c})$ , set  $\lambda(e)(t) = t$ . For edges  $e \in e(\mathcal{P}'_{X,c} \cup \mathcal{P}'_{B,c})$ , set  $\lambda(e)(t) = r(e)t^{k(e)}$ , where  $(r(e), k(e)) \in \mathbb{R}^+ \times \mathbb{N}$ ,  $r(e) \neq 0$ , and  $k(e) \neq 0$ . We are going to show how to define pairs  $(r(e), k(e))$ , so that the equations  $*(Y)(t)$  are satisfied, for all  $t$  and for all  $Y$  such that  $Y \in \mathcal{P}'_X$ . Since they imply the equations 15.2, because all the coefficients  $\lambda(e)(t)$  are different from zero, this will conclude the proof.

If  $\mathcal{P}'_{B,c} = \emptyset$ , then no parameter  $\lambda(e)$  vanishes. So there is nothing to prove.

For a set  $I$  of edges of  $\mathcal{T}(\mathcal{P}, \mathcal{P}_B)$ ,  $\lambda(I)$  denotes the product over the edges  $e$  of  $I$  of the  $\lambda(e)$ , while  $\lambda_t(I)$  denotes the product over the edges  $e$  of  $I$  of the  $\lambda(e)(t)$ . With the notation of Definition 14.1, the main equations  $*(Y)(t)$  may be written as

$$*(Y)(t) : \lambda_t([O_Y, V]) = \lambda_t([\hat{B}(Y), V]).$$

We also have the obvious sublemma.

**Sublemma 15.7.** *Condition 15.2 of Lemma 15.6 may be rewritten as follows. For any collections  $(Y_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ ,  $(Y_i^+)_{i \in \mathbb{Z}/n\mathbb{Z}}$ ,  $(Y'_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$  such that  $Y_i \in \mathcal{P}'_X$ ,  $Y_i^+ \in \mathcal{P}'_X$ ,  $Y'_i \in \widehat{\mathcal{P}}'_X$ , and  $(Y_i^+ \cup Y_i) \subseteq Y'_i$ , and for any collection  $(B'_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$  of sets of  $\widehat{\mathcal{P}}'_B$  such that*

$$(\hat{B}(Y_{i-1}^+) \cup \hat{B}(Y_i)) \subseteq B'_i,$$

*the products of the numbers over the arrows in one direction equals the product of the numbers over the arrows in the opposite direction in the cycle of Figure 15.3.*

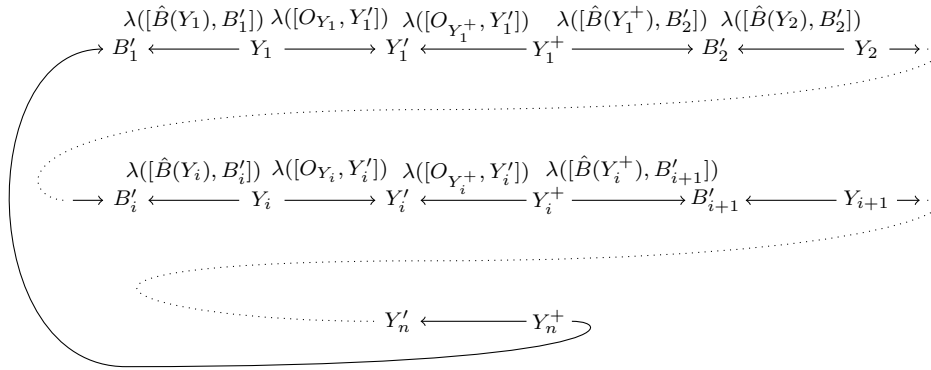


Figure 15.3: The cycle of Condition 15.2 of Lemma 15.6.

We can redraw a typical part of the cycle of Figure 15.3 as in Figure 15.4 to emphasize the different natures of its segments and show how the sets at their ends determine the coefficients over the arrows.

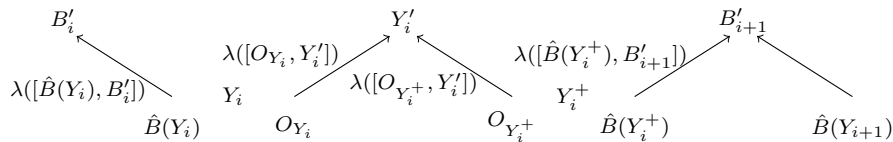


Figure 15.4: Another representation of a part of the cycle of Figure 15.3.

For  $g \in e(\mathcal{P}'_{B,c})$  and for  $f \in e(\mathcal{P}'_{X,c})$ , let  $\mathcal{P}(f, g)$  be the set of elements  $Y \in \mathcal{P}'_X$  such that  $e(Y) = f$  and  $e(\hat{B}(Y)) = g$ . Note the following sublemma.



**Sublemma 15.8.** *Recall the notation of Definition 14.1 for sets of edges of a tree. Let  $g \in e(\mathcal{P}'_{B,c})$ , let  $f \in e(\mathcal{P}'_{X,c})$ . Assume that there exists an element  $Y \in \mathcal{P}(f, g)$ . Set*

$$c(f, g) = \frac{\lambda([\hat{B}(Y), g])}{\lambda([O_Y, f])}.$$

*Then  $c(f, g)$  is a positive coefficient independent of the chosen element  $Y$  of  $\mathcal{P}(f, g)$ . The equation*

$$*_t(f, g) : \lambda_t([f, V]) = c(f, g)\lambda_t([g, V])$$

*must be satisfied for our sequence of configurations  $c(t)$  to be in  $\check{V}(\Gamma)$ . Furthermore, if Equation  $*_t(f, g)$  is satisfied for all  $t$ , then for any  $Y^+ \in \mathcal{P}(f, g)$ , Equation  $*(Y^+)(t)$  is satisfied for all  $t$ .*

PROOF: With the given coefficient  $c(f, g)$ , Equation  $*_t(f, g)$  is equivalent to Equation  $*(Y)(t)$ . For any other element  $Y^+$  of  $\mathcal{P}(f, g)$ , according to Condition 15.2 of Lemma 15.6—where  $n = 1$ ,  $Y'_1$  is the origin of  $f$ , and  $B'_1$  is the origin of  $g$ , we have

$$\lambda([O_Y, f])\lambda([\hat{B}(Y^+), g]) = \lambda([O_{Y^+}, f])\lambda([\hat{B}(Y), g]).$$

□

Define the equivalence relation  $\sim$  on  $e(\mathcal{P}'_{X,c} \cup \mathcal{P}'_{B,c})$  to be the relation generated by the equivalences: Whenever  $g \in e(\mathcal{P}'_{B,c})$  and  $f \in e(\mathcal{P}'_{X,c})$ , if  $\mathcal{P}(f, g) \neq \emptyset$ , then we have  $g \sim f$ . When  $Y \in \mathcal{P}(f, g)$ , such a generating elementary equivalence will also be denoted by  $g \sim_Y f$ , and its inverse will be denoted by  $f \sim_Y g$ . In this case, set  $c(g, f) = c(f, g)^{-1}$ .

**Sublemma 15.9.** *For any cycle*

$$g = g_1 \sim_{Y_1} f_1 \sim_{Y_1^+} g_2 \sim_{Y_2} f_2 \sim_{Y_2^+} g_3 \cdots f_n \sim_{Y_n^+} g_{n+1} = g$$

*of elementary equivalences for sequences  $(g_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$  of edges of  $e(\mathcal{P}'_{B,c})$  and  $(f_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$  of edges of  $e(\mathcal{P}'_{X,c})$ , we have*

$$\prod_{i=1}^n c(g_i, f_i)c(f_i, g_{i+1}) = 1.$$

PROOF: Apply Condition 15.2 of Lemma 15.6, as stated in Sublemma 15.7, to the above sequences  $(Y_i)$  and  $(Y_i^+)$ , where  $Y'_i$  is the origin  $o(f_i)$  of  $f_i$  and  $B'_i$  is the origin  $o(g_i)$  of  $g_i$ . Note that Sublemma 15.8 implies that, in each triangle of Figure 15.5, the product of the coefficients over the edges of the

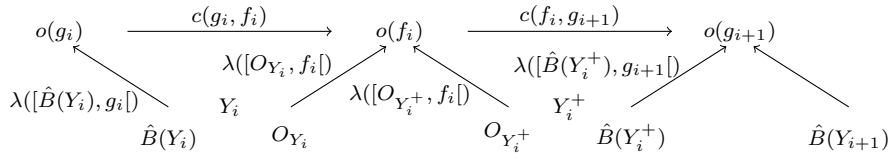


Figure 15.5: Definition of  $c(g_i, f_i)$  and  $c(f_i, g_{i+1})$ .

boundary of the triangle that are oriented as part of that boundary is equal to the product of the coefficients over the edges with the opposite orientation.  $\square$

The following sublemma is an easy corollary of the previous one.

**Sublemma 15.10.** *Let  $e$  and  $e'$  be two elements of  $e(\mathcal{P}'_{X,c} \cup \mathcal{P}'_{B,c})$  such that  $e \sim e'$ . There exists a sequence  $(e_i)_{i \in \underline{m}}$  of edges of  $e(\mathcal{P}'_{X,c} \cup \mathcal{P}'_{B,c})$  such that  $e_1 = e$ ,  $e_m = e'$ , and  $e_i \sim_{Z_i} e_{i+1}$  for any  $i \in \underline{m-1}$ . For such a sequence, set*

$$c(e, e') = \prod_{i=1}^{m-1} c(e_i, e_{i+1}).$$

*Then  $c(e, e')$  is a positive coefficient independent of the chosen sequences as above.*

*The equation*

$$*_t(e, e') : \lambda_t([e, V]) = c(e, e') \lambda_t([e', V])$$

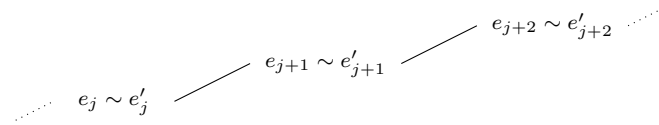
*must be satisfied for our sequence of configurations  $c(t)$  to be in  $\check{V}(\Gamma)$ . Furthermore, for any three elements  $e, e_0$ , and  $e'$  of  $e(\mathcal{P}'_{B,c})$  in the same equivalence class under  $\sim$ , the equations  $*_t(e, e')$  and  $*_t(e', e)$  are equivalent, and the equations  $*_t(e, e_0)$  and  $*_t(e', e_0)$  imply  $*_t(e, e')$ .*

$\square$

The following sublemma will allow us to define a partial order on the set of equivalence classes under  $\sim$ .

**Sublemma 15.11.** *Let  $k \in \mathbb{N}$ . It is not possible to find edges  $e_1, \dots, e_k, e'_1, \dots, e'_k$  of  $e(\mathcal{P}'_{X,c} \cup \mathcal{P}'_{B,c})$  such that  $e_j \sim e'_j$  and  $e_{j+1} \in ]e'_j, V]$  for all  $j \in \mathbb{Z}/k\mathbb{Z}$ .*

**PROOF:** Assume that there exist edges  $e_1, \dots, e_k, e'_1, \dots, e'_k$  of  $e(\mathcal{P}'_{X,c} \cup \mathcal{P}'_{B,c})$  such that  $e_j \sim e'_j$  and  $e_{j+1} \in ]e'_j, V]$  for all  $j \in \mathbb{Z}/k\mathbb{Z}$ . Let us picture this hypothesis as follows.



Let us now construct a cycle as in Figure 15.3 under this hypothesis. An elementary equivalence  $g \sim_Y f$  where  $Y \in \mathcal{P}(f, g)$  is represented by a path  $E(g, f)$

$$o(g) \xleftarrow{\lambda([\hat{B}(Y), g])} Y \xrightarrow{\lambda([O_Y, f])} o(f)$$

where  $o(g)$  is the set associated to the origin of  $g$  and  $o(f)$  is associated to the origin of  $f$ . (It is equal to  $O_Y$  if  $\lambda(Y) = 0$ , and it is the set associated to the initial point of  $f$  otherwise.) The inverse equivalence  $f \sim_Y g$  is similarly represented by the following path  $E(f, g)$

$$o(f) \xleftarrow{\lambda([O_Y, f])} Y \xrightarrow{\lambda([\hat{B}(Y), g])} o(g)$$

Again, the coefficients over the arrows are determined by the labels of the ends. In these cases, they do not vanish, and we picture the paths  $E(g, f)$  and  $E(f, g)$  as  $o(g) \leftarrow Y \hookrightarrow o(f)$  and  $o(f) \leftarrow Y \hookrightarrow o(g)$ , respectively. Note that according to our assumptions, no  $e_j$  can start at some  $O_Y$ .

If  $e_j \neq e'_j$  for all  $j \in \mathbb{Z}/k\mathbb{Z}$ , then our cycle of arrows as above is obtained by assembling the following paths of arrows  $A(e_j, e_{j+1})$  from  $o(e_j)$  to  $o(e_{j+1})$ . The path  $A(e_j, e_{j+1})$  is obtained from a sequence  $E(e_j, e'_j)$  of paths  $E(e, e')$  of arrows associated to elementary equivalences by replacing the last arrow  $X_j \hookrightarrow o(e'_j)$ , which ends at  $o(e'_j)$ , by an arrow  $X_j \rightarrow o(e_{j+1})$  ending at  $o(e_{j+1})$ . The coefficient of this arrow is obtained from the coefficient of  $X_j \hookrightarrow o(e'_j)$  by multiplication by  $\lambda([e'_j, e_{j+1}])$ , which vanishes in our case because it has a factor  $\lambda(e'_j)$ . According to the recalled criterion, the product of the coefficients over the arrows in the direction of our cycle must be equal to the product of the coefficients over the arrows in the opposite direction. The first product is zero because of its factors  $\lambda(e'_j)$ . The second one is nonzero because it only contains nonzero factors associated to equivalences. Therefore, the lemma is proved when  $e_j \neq e'_j$  for any  $j \in \mathbb{Z}/k\mathbb{Z}$ . This case is ruled out.

We cannot have  $e_j = e'_j$  for all  $j \in \mathbb{Z}/k\mathbb{Z}$ .

Up to permuting our indices cyclically, it suffices to rule out the case in which  $k \geq 2$  and there exists  $r \geq 1$  such that  $e_r \neq e'_r$ ,  $e_j = e'_j$  for all  $j$  such that  $r + 1 \leq j \leq k$ , and  $e_1 \neq e'_1$ . In this case, we define a path  $A(e_r, e_1)$  by replacing the last arrow  $X_r \hookrightarrow o(e'_r)$  in  $E(e_r, e'_r)$  with  $X_r \rightarrow o(e_1)$ , and

multiplying the corresponding coefficient by  $\lambda([e'_r, e_1]) = \prod_{j=r}^k \lambda([e'_j, e_{j+1}])$ . Similarly define paths  $A(e_s, e_t)$ , for all pair  $(s, t)$  of integers such that  $s < t < k$ ,  $e_s \neq e'_s$ ,  $e_t \neq e'_t$ , and  $e_j = e'_j$  for all  $j$  such that  $s + 1 \leq j \leq t - 1$ . Define the cycle yielding the contradiction by composing these paths, which include our former paths  $A(e_j, e_{j+1})$  for which  $e_j \neq e'_j$  and  $e_{j+1} \neq e'_{j+1}$ .  $\square$

END OF THE PROOF OF LEMMA 15.6: Sublemma 15.11 allows us to define the partial order  $\succeq$  on the set  $E(\sim)$  of equivalence classes of the relation  $\sim$  on  $e(\mathcal{P}'_{X,c} \cup \mathcal{P}'_{B,c})$  such that two equivalence classes  $\bar{e}$  and  $\bar{f}$  of  $E(\sim)$  satisfy  $\bar{f} \succeq \bar{e}$  if and only if there exist a positive integer  $k \geq 2$  and two sequences  $(e_i)_{i \in \underline{k} \setminus \{1\}}$  and  $(e'_i)_{i \in \underline{k-1}}$  such that  $e'_1 \in \bar{e}$ ,  $e_k \in \bar{f}$ ,  $e_j \sim e'_j$  for all  $j \in \underline{k-1} \setminus \{1\}$ , and  $e_{j+1} \in [e'_j, V]$  (i.e.,  $e_{j+1}$  is between the initial point of  $e'_j$  and  $V$ ) for all  $j \in \underline{k-1}$ . For example, if  $e_2 \in [e'_1, V]$ , then  $\bar{e}_2 \succeq \bar{e}'_1$ .

Fix an arbitrary total order on  $E(\sim)$  compatible with the above partial order by writing

$$E(\sim) = \{\bar{g}_i : i \in \underline{m}\}$$

so that for any  $(i, j) \in \underline{m}^2$  such that  $\bar{g}_j \succeq \bar{g}_i$ , we have  $j \leq i$ .

We will pick one representative  $g_i \in e(\mathcal{P}'_{B,c})$  in each equivalence class  $\bar{g}_i$  of  $E(\sim)$  and define  $\lambda(g_i)(t) = r(g_i)t^{k(g_i)}$ , for  $i = 1, \dots, m$ , inductively.

For  $e \in \bar{g}_i$ , the equation

$$*_t(e, g_i) : \lambda_t([e, V]) = c(e, g_i)\lambda_t([g_i, V]),$$

which must be satisfied, determines  $\lambda(e)(t) = r(e)t^{k(e)}$  with  $(r(e), k(e)) \in \mathbb{R}^{+*} \times (\mathbb{N} \setminus \{0\})$  as a function of  $\lambda(g_i)(t)$  and of the  $\lambda(e')(t) = r(e')t^{k(e')}$  for edges  $e'$  that belong to  $\cup_{j=1}^{i-1} \bar{g}_j$ . (Recall that we fixed the coefficients of the other edges.) More precisely,  $k(e) - k(g_i)$  is a degree one polynomial in the variables  $k(e')$  for edges  $e'$  that belong to  $\cup_{j=1}^{i-1} \bar{g}_j$ , which are already defined by induction. In particular, if we choose  $k(g_i)$  to be a sufficiently large integer and fix  $r(g_i) = 1$ , then the  $\lambda(e)(t)$  are uniquely determined so that the equations  $*_t(e, g_i)$  are satisfied with  $k(e) > 0$  for any  $e \in \bar{g}_i$ .

Thus, once the induction is achieved, according to Sublemmas 15.8 and 15.10, Equation  $*(Y)(t)$  is satisfied for all  $t$  for any  $Y \in \mathcal{P}$ . This shows that the elements described in the statement of Lemma 15.6 are in  $\mathcal{V}(\Gamma)$  and finishes the proof of Lemma 15.6.  $\square$

Let  $\mathcal{P}_u$  denote the set of univalent elements of  $\mathcal{P}$ , which are the elements of  $\mathcal{P}$  that contain at least one univalent vertex.

For a configuration  $y \in \mathcal{S}_B(\mathbb{C})$  and a Jacobi diagram  $\Gamma$  on a disjoint union of lines  $\mathbb{R}_b$  indexed by elements  $b$  of  $B$ , let  $\mathcal{V}(y, \Gamma)$  denote the preimage of  $y$  under

$$p_{\mathcal{S}_B} : \mathcal{V}(\Gamma) \rightarrow \mathcal{S}_B(\mathbb{C}).$$

**Lemma 15.12.** *With respect to the notation introduced before Lemma 15.3 and in Lemma 15.3, let  $\mathcal{P}'_x$  be the subset of  $\mathcal{P}'_X$  consisting of its elements  $Y$  such that  $\lambda^0(Y) \neq 0$ . The dimension of the stratum of  $c^0$  in the fiber  $\mathcal{V}(y^0, \Gamma)$  over the configuration  $y^0 \in \mathcal{S}_B(\mathbb{C})$  is  $|U(\Gamma)| + 3|T(\Gamma)| - |\mathcal{P}| + |\mathcal{P}'_x| - 1$ .*

PROOF: For any  $Y \in \mathcal{P} \setminus \mathcal{P}_u$ , the configuration  $c_Y$  is defined up to global translation and dilation. (With our normalizations, the quotient by translations is replaced by the fact that we send basepoints to zero, and our horizontal or vertical normalization condition replaces the quotient by dilation.) For any  $Y \in \mathcal{P}_u \setminus \mathcal{P}'_x$ , the configuration  $c_Y$ , whose restriction to the set of univalent vertices is vertical, is defined up to global vertical translation and dilation. For any  $Y \in \mathcal{P}'_x$ , we still have these two codimension-one normalization conditions on the configuration  $c_Y$  when  $\lambda(Y)$  is fixed. But varying the parameter  $\lambda(Y)$  adds one to the dimension.

Say that a subset of  $V(\Gamma)$  is *trivalent* if it contains no univalent vertex. Write the set  $K(A)$  of kids of an element  $A$  of  $\mathcal{P}$  as the union of its set  $K_u^0(A)$  of univalent kids and its set  $K_t^0(A)$  of trivalent kids (including the kid of the basepoint).

$$K(A) = K_u^0(A) \sqcup K_t^0(A).$$

Then the dimension of the involved space of maps up to dilation and (possibly vertical) translation from  $K(A)$  to  $\mathbb{R}^3$  is

- $3|K_t^0(A)| - 4$  if  $A \in \mathcal{P} \setminus \mathcal{P}_u$ ,
- $|K_u^0(A)| + 3|K_t^0(A)| - 2$  if  $A \in \mathcal{P}_u \setminus \mathcal{P}'_x$ ,
- $|K_u^0(A)| + 3|K_t^0(A)| - 1$  if  $A \in \mathcal{P}'_x$ .

Let  $\mathcal{P}_{ext}$  denote the union of  $\mathcal{P}$  with the set of singletons of elements of  $V$ . So  $\mathcal{P}_{ext}$  contains all the kids of elements of  $\mathcal{P}$ . The only element of  $\mathcal{P}_{ext}$  that is not a kid is  $V$ . Let  $\mathcal{P}_{ext,u}$  denote the set of univalent sets of  $\mathcal{P}_{ext}$ , and let  $\mathcal{P}_{ext,t}$  denote the set of trivalent sets of  $\mathcal{P}_{ext}$ . The sum over  $A \in \mathcal{P}$  of the above dimensions is equal to

$$3|\mathcal{P}_{ext,t}| + |\mathcal{P}_{ext,u}| - 1 - 3|\mathcal{P} \setminus \mathcal{P}_u| - |\mathcal{P}_u| - |\mathcal{P} \setminus \mathcal{P}'_x|,$$

where  $|\mathcal{P}_{ext,t}| = |T(\Gamma)| + |\mathcal{P} \setminus \mathcal{P}_u|$ , and  $|\mathcal{P}_{ext,u}| = |U(\Gamma)| + |\mathcal{P}_u|$ . □

Lemma 15.6 simplifies when  $y^0 \in \check{\mathcal{S}}_B(\mathbb{C})$ . It allows us to describe the structure of  $\mathcal{V}(\Gamma)$  over  $\check{\mathcal{S}}_B(\mathbb{C})$  in the following lemma.

**Lemma 15.13.** *Let  $y^0 \in \check{\mathcal{S}}_B(\mathbb{C})$ . For any  $(c^0, y^0) \in \mathcal{V}(\Gamma)$ , there exist*

- *a manifold  $W$  with boundary and ridges,*

- an open neighborhood  $N(y^0)$  of  $y^0$  in  $\check{\mathcal{S}}_B(\mathbb{C})$ ,
- a small  $\varepsilon > 0$ ,
- an oriented tree  $\mathcal{T}_V(c^0)$ , and
- a smooth map  $\varphi_1: [0, \varepsilon]^{E(\mathcal{T}_V(c^0))} \times W \times N(y^0) \rightarrow \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$ ,

such that the product of  $\varphi_1$  by the natural projection  $p_{N(y^0)}: [0, \varepsilon]^{E(\mathcal{T}_V(c^0))} \times W \times N(y^0) \rightarrow N(y^0)$  has the following property. The map

$$\varphi_1 \times p_{N(y^0)}: [0, \varepsilon]^{E(\mathcal{T}_V(c^0))} \times W \times N(y^0) \rightarrow \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3) \times N(y^0)$$

restricts to  $(X(\mathcal{T}_V(c^0)) \cap [0, \varepsilon]^{E(\mathcal{T}_V(c^0))}) \times W \times N(y^0)$  as a bijection onto an open neighborhood of  $(c^0, y^0)$  in  $\mathcal{V}(\Gamma)$ , where  $X(\mathcal{T}_V(c^0))$  was introduced in Definition 14.3.

PROOF: According to Lemma 15.1,  $(c^0, y^0)$  has a neighborhood parametrized as in Lemma 15.6. In such a parametrization, no  $u_D$  is involved since  $y^0 = y_V^0 \in \check{\mathcal{S}}_B(\mathbb{C})$ . Also note  $\mathcal{P}'_X = \mathcal{P}_{X,B}$ . In particular, the elements of  $\mathcal{P}'_X$  are pairwise disjoint sets. Define the tree  $\mathcal{T}_V(c^0)$  associated to  $(c^0, y^0)$  to be the tree obtained from  $\mathcal{T}(\mathcal{P}, \mathcal{P}_B)$  by replacing the pair of edges respectively labeled by  $\lambda(Y)$  and  $\mu_Y$  by a single edge labeled by  $\tilde{\lambda}(Y) = \lambda(Y)\mu_Y$  for each  $Y \in \mathcal{P}'_x$ . (In this tree, the right  $B$ -part in Figure 15.1 is reduced to one edge labeled by  $\lambda$ , and the edges labeled by  $\tilde{\lambda}(Y)$  for  $Y \in \mathcal{P}'_x$  start at univalent vertices.) The lemma is an easy consequence of Lemma 15.6.  $\square$

**Lemma 15.14.** *Let  $y \in \check{\mathcal{S}}_B(\mathbb{C})$ . A codimension-one open face of  $\mathcal{V}(y, \Gamma)$  is a stratum  $(\mathcal{P}, \mathcal{P}'_x)$  as in Lemma 15.12 such that*

- either we have  $V(\Gamma) \in \mathcal{P}'_x$ ,  $\lambda \neq 0$ ,  $\mathcal{P}'_x = \{V(\Gamma)\}$ , and  $\mathcal{P} = \{V(\Gamma), C\}$  for some  $C \subset V(\Gamma)$ ,
- or we have  $V(\Gamma) \notin \mathcal{P}'_x$ ,  $\lambda = 0$ , and  $\mathcal{P} = \{V(\Gamma)\} \cup \mathcal{P}'_x$  (where  $\mathcal{P}'_x$  can be empty).

PROOF: According to Lemma 15.12, the faces of the  $\partial\mathcal{V}(y, \Gamma)$  with maximal dimension are such that  $|\mathcal{P}| = |\mathcal{P}'_x| + 1$ , where  $V(\Gamma) \in \mathcal{P}$ .  $\square$

Lemmas 14.14 and 15.13 guarantee that  $\mathcal{V}(y, \Gamma)$  behaves as a codimension-one face of a manifold with boundary along a face as in Lemma 15.14.

## 15.2 A one-form on $\check{\mathcal{S}}_B(\mathbb{C})$

For a finite set  $B$  of cardinality at least 2, a configuration  $y \in \mathcal{S}_B(\mathbb{C})$ , and a Jacobi diagram  $\Gamma$  on a disjoint union of lines  $\mathbb{R}_b$  indexed by elements  $b$  of  $B$  such that  $p_B: U(\Gamma) \rightarrow B$  is onto, recall that  $\mathcal{V}(y, \Gamma)$  is the preimage of  $y$  under  $p_{\mathcal{S}_B}: \mathcal{V}(\Gamma) \rightarrow \mathcal{S}_B(\mathbb{C})$ . When  $y \in \check{\mathcal{S}}_B(\mathbb{C})$ , let  $\check{\mathcal{V}}(y, \Gamma)$  denote the quotient by vertical translations of  $\check{C}(\mathbb{R}^3, y^0 \times \mathbb{R}; \Gamma)$  for a representative  $y^0 \in \check{C}_B[D_1]$  of  $y$ . Note that  $\check{\mathcal{V}}(y, \Gamma)$  is an open  $T$ -face of  $C(R(\mathcal{C}), L; \Gamma)$  for any tangle  $L$  whose top configuration is  $y$ , as in Theorem 14.23, where the set  $B$  of Theorem 14.23 is empty,  $I = \{j\}$ , and  $\mathcal{P}_s = \mathcal{P}_d = \mathcal{P}_x = \{V(\Gamma)\}$  with Notation 14.26. Assume that  $\Gamma$  is equipped with a vertex-orientation  $o(\Gamma)$  as in Definition 6.13 and with an edge-orientation  $o_E(\Gamma)$  of  $H(\Gamma)$  as before Lemma 7.1. The space  $\check{\mathcal{V}}(y, \Gamma)$  is a smooth manifold of dimension  $|U(\Gamma)| + 3|T(\Gamma)| - 1$ . It is oriented as the part of the boundary of  $C(R(\mathcal{C}), L; \Gamma)$  that is the  $T$ -face in which all univalent vertices of  $\Gamma$  tend to  $\infty$  above  $\mathcal{C}$ , for a tangle  $L$  whose top configuration is  $y$ . (Here,  $\Gamma$  is also viewed as a diagram on the domain  $\mathcal{L}$  of  $L$  by representing the original  $[i_\Gamma]$  by a map  $i_\Gamma: U(\Gamma) \hookrightarrow (\sqcup_{b \in B} [1, \infty[_b \subset \mathcal{L})$ .) Note that  $\check{\mathcal{V}}(y, \Gamma)$  is therefore oriented as the part of the boundary of  $(-C(R(\mathcal{C}), L; \Gamma))$  that is (minus) the  $T$ -face in which all univalent vertices of  $\Gamma$  tend to  $\infty$  below  $\mathcal{C}$ , for a tangle  $L$  whose bottom configuration is  $y$ . The orientation of  $\check{\mathcal{V}}(y, \Gamma)$  depends on  $o(\Gamma)$  and on  $o_E(\Gamma)$ , but it does not depend on the global orientations of the lines  $\mathbb{R}_b$ , which are only locally oriented by  $o(\Gamma)$  near the images of the univalent vertices of  $\Gamma$  as in Definition 6.13.<sup>3</sup>

Below, we define a one-form  $\eta_\Gamma = \eta_{\Gamma, o(\Gamma)}$  on  $\check{\mathcal{S}}_B(\mathbb{C})$  to be the integral of  $\bigwedge_{e \in E(\Gamma)} p_{e, S^2}^*(\omega_{S^2})$  along the interiors  $\check{\mathcal{V}}(y, \Gamma)$  of the compact fibers  $\mathcal{V}(y, \Gamma)$ . We agree that the integral along the fiber of  $dx \wedge \omega$  for a volume form  $\omega$  of the fiber is  $(\int_{\text{fiber}} \omega) dx$ .

**Proposition 15.15.** *The integral of  $\bigwedge_{e \in E(\Gamma)} p_{e, S^2}^*(\omega_{S^2})$  along the interior  $\check{\mathcal{V}}(y, \Gamma)$  of the fiber  $\mathcal{V}(y, \Gamma)$  is absolutely convergent. It defines a smooth one-form  $\eta_\Gamma$  on  $\check{\mathcal{S}}_B(\mathbb{C})$ .<sup>4</sup> The definition of  $\eta_\Gamma$  extends naturally to diagrams  $\Gamma$  on  $\sqcup_{b \in B} \mathbb{R}_b$  such that  $p_B: U(\Gamma) \rightarrow B$  is not onto.<sup>5</sup>*

Let  $\gamma: [0, 1] \rightarrow \check{\mathcal{S}}_B(\mathbb{C})$  be a smooth map. Orient  $p_{\mathcal{S}_B}^{-1}(\gamma([0, 1]))$  as the local

<sup>3</sup>The reader who prefers working with oriented strands can assume that the lines  $\mathbb{R}_b$  are oriented from bottom to top and consider braids  $L$  instead of tangles  $L$ , above, for the moment. However, since we will later need to allow various orientations for our strands  $\mathbb{R}_b$ , it is better to work with unoriented strands as much as possible.

<sup>4</sup>Again,  $\eta_\Gamma$  depends on the arbitrary vertex-orientation  $o(\Gamma)$  of  $\Gamma$ , but the product  $\eta_\Gamma[\Gamma]$  is independent of  $o(\Gamma)$ .

<sup>5</sup>In general,  $\eta_\Gamma$  pulls back through  $\check{\mathcal{S}}_{p_B(U(\Gamma))}(\mathbb{C})$ . So  $\eta_\Gamma = 0$  if  $|p_B(U(\Gamma))| < 2$ .

product  $[0, 1] \times \text{fiber}$ .<sup>6</sup> Then the integral

$$\int_{[0,1]} \gamma^*(\eta_\Gamma) = \int_{p_{S^B}^{-1}(\gamma([0,1]))} \bigwedge_{e \in E(\Gamma)} p_{e,S^2}^*(\omega_{S^2})$$

is absolutely convergent. The map  $(t \mapsto \int_{[0,t]} \gamma^*(\eta_\Gamma))$  is differentiable, and we have

$$\frac{\partial}{\partial t} \left( \int_{[0,t]} \gamma^*(\eta_\Gamma) \right) (u) = \eta_\Gamma \left( \gamma(u), \frac{\partial}{\partial t} \gamma_u \right).$$

PROOF: In the proof, we assume that  $p_B$  is onto, without loss of generality. Lemma 15.13 implies that the integral  $\int_{p_{S^B}^{-1}(\gamma([0,1]))} \bigwedge_{e \in E(\Gamma)} p_{e,S^2}^*(\omega_{S^2})$  is absolutely convergent. See the proof of Theorem 14.16. Let us prove that the integral of  $\bigwedge_{e \in E(\Gamma)} p_{e,S^2}^*(\omega_{S^2})$  along the interior  $\check{\mathcal{V}}(y, \Gamma)$  of the fiber  $\mathcal{V}(y, \Gamma)$  is absolutely convergent and that it defines a smooth one-form  $\eta_\Gamma$  on  $\check{\mathcal{S}}_B(\mathbb{C})$ .

Let  $y^0 \in \check{\mathcal{S}}_B(\mathbb{C})$ , let  $N(y^0)$  be small neighborhood of  $y^0$  in  $\check{\mathcal{S}}_B(\mathbb{C})$ , and let

$$(\zeta_r : N(y^0) \rightarrow \mathbb{R})_{r \in \underline{2|B|-3}}$$

be a system of coordinates on  $N(y^0)$ . These coordinates give rise to associated smooth one-forms  $d\zeta_r = dp_{N(y^0)} \circ \zeta_r$  on  $\check{\mathcal{V}}(N(y^0), \Gamma) = \check{\mathcal{V}}(\Gamma) \cap p_{S^B}^{-1}(N(y^0))$ . For a local system  $(f_1, \dots, f_k)$  of coordinates of the interior of a fiber  $\check{\mathcal{V}}(y, \Gamma)$ , and a local product structure with the base, we also have associated forms  $df_i$ , which depend on the product structure. (Changing this product structure adds some combination of the  $d\zeta_r$  and  $df_j$  to  $df_i$ ). We also have an associated volume form of the fiber  $\omega_F = df_1 \wedge \dots \wedge df_k$ , which depends on the product structure too. A  $(k + 1)$ -form  $\Omega$  on  $\check{\mathcal{V}}(N(y^0), \Gamma)$  may be expressed as  $\Omega = \sum_{r=1}^{2|B|-3} d\zeta_r \wedge (g_r \omega_F) + \sum_{i=1}^k \omega_{i,v}$ , where  $\omega_{i,v}$  vanishes at the tangent vector  $\xi_i$  to a curve of the fiber whose coordinates  $f_j$  for  $j \in \underline{k} \setminus \{i\}$  are constant ( $\omega_{i,v}$  is expressed as a wedge product of coordinates forms that does not involve  $df_i$ , this decomposition is not canonical). In order to check the convergence of the integral of the pull-back  $\Omega$  of the form  $\bigwedge_{e \in E(\Gamma)} p_{e,S^2}^*(\omega_{S^2})$  on  $\check{\mathcal{V}}(N(y^0), \Gamma)$  along the fiber, it suffices to cover the fiber by finitely many neighborhoods as above, to express  $\Omega$  as above with respect to the corresponding charts, and to check that the  $g_r \omega_F$  (which are well-defined, up to forms whose integrals along the fiber vanish) and their derivatives with respect to the coordinates  $\zeta_j$  are bounded in each of these neighborhoods. Lemma 15.13 implies that  $\Omega$  is the restriction to

$$\left( X(\mathcal{T}_\mathcal{V}(c^0)) \cap [0, \varepsilon^{E(\mathcal{T}_\mathcal{V}(c^0))}] \right) \times W \times N(y^0)$$

---

<sup>6</sup>Note that it amounts to say that the  $[0, 1]$  factor replaces the upward vertical translation parameter of the quotient  $\check{\mathcal{V}}(y, \Gamma)$ , as far as orientations are concerned.



of a smooth form on  $[0, \varepsilon[^{E(\mathcal{T}_V(c^0))} \times W \times N(y^0)$ , locally. The same holds for the  $g_r \omega_F$  and all their iterated partial derivatives with respect to the  $\zeta_j$ . These forms are bounded in a neighborhood of an arbitrary element  $(c^0, y^0)$  of  $\mathcal{V}(y^0, \Gamma)$  as in Lemma 15.13. Since the fiber  $\mathcal{V}(y^0, \Gamma)$  is compact, the integral of  $g_r \omega_F$  along the fiber is absolutely convergent, for any  $r \in \overline{2|B| - 3}$ . So are the integrals of its iterated partial derivatives with respect to the  $\zeta_j$  on  $\check{\mathcal{V}}(N(y^0), \Gamma)$  for a small neighborhood  $N(y^0)$  of  $y^0$  in  $\check{\mathcal{S}}_B(\mathbb{C})$ . This proves that  $\eta_\Gamma$  is a well-defined smooth one-form on  $\check{\mathcal{S}}_B(\mathbb{C})$ .  $\square$

**Definition 15.16.** For  $k \in \mathbb{N} \setminus \{0\}$ , set

$$\eta_{k,B} = \sum_{\Gamma \in \mathcal{D}_k^e(\sqcup_{b \in B} \mathbb{R}_b)} \frac{(3k - |E(\Gamma)|)!}{(3k)! 2^{|E(\Gamma)|}} \eta_\Gamma \quad [\Gamma \in \Omega^1(\mathcal{S}_B(\mathbb{C}); \mathcal{A}_k(\sqcup_{b \in B} \mathbb{R}_b))],$$

where  $\eta_\Gamma = 0$  if  $|p_B(U(\Gamma))| < 2$  or if  $\Gamma$  is not connected, and

$$\eta_B = \sum_{k \in \mathbb{N} \setminus \{0\}} \eta_{k,B}.$$

The form  $\eta_B$  is a one-form on  $\check{\mathcal{S}}_B(\mathbb{C})$  with coefficients in the space  $\mathcal{A}(\sqcup_{b \in B} \mathbb{R}_b)$  of Jacobi diagrams on  $\sqcup_{b \in B} \mathbb{R}_b$ , which is treated as an unoriented manifold as in Definition 6.16. The form  $\eta_B$  will be regarded as a *connection*. Let  $p_{CS}: \check{C}_B[D_1] \rightarrow \mathcal{S}_B(\mathbb{C})$  denote the natural projection. For a path  $\gamma: [a, b] \rightarrow \check{C}_B[D_1]$ , define the *holonomy*  $\text{hol}_\gamma(\eta_B)$  of  $\eta_B$  along  $\gamma$  to be

$$\text{hol}_\gamma(\eta_B) = \sum_{r=0}^\infty \int_{(t_1, \dots, t_r) \in [a, b]^r : t_1 \leq t_2 \leq \dots \leq t_r} \bigwedge_{i=1}^r (p_{CS} \circ \gamma \circ p_i)^*(\eta_B),$$

where  $p_i(t_1, \dots, t_r) = t_i$ , the wedge product of forms is performed as usual, and the diagrams are multiplied from bottom to top (from left to right) with respect to their order of appearance.

The degree 0 part of  $\text{hol}_\gamma(\eta_B)$  is the unit  $[\emptyset]$  of  $\mathcal{A}_k(\sqcup_{b \in B} \mathbb{R}_b)$ , and we have

$$\text{hol}_\gamma(\eta_B) = [\emptyset] + \sum_{r=1}^\infty \int_{(t_1, \dots, t_r) \in [a, b]^r : t_1 \leq t_2 \leq \dots \leq t_r} \bigwedge_{i=1}^r (p_{CS} \circ \gamma \circ p_i)^*(\eta_B).$$

This holonomy is valued in a space of diagrams on an unoriented domain as in Definition 6.16, Proposition 10.23, and Remark 10.24. It satisfies the following properties.

- For an orientation-preserving diffeomorphism  $\psi: [c, d] \rightarrow [a, b]$ , we have

$$\text{hol}_{\gamma \circ \psi}(\eta_B) = \text{hol}_\gamma(\eta_B).$$

- When  $\gamma_1\gamma_2$  is the path composition of  $\gamma_1$  and  $\gamma_2$ , we have

$$\text{hol}_{\gamma_1\gamma_2}(\eta_B) = \text{hol}_{\gamma_1}(\eta_B) \text{hol}_{\gamma_2}(\eta_B).$$

- We have  $\frac{\partial}{\partial t} \text{hol}_{\gamma|_{[a,t]}}(\eta_B) = \text{hol}_{\gamma|_{[a,t]}}(\eta_B)\eta_B(\gamma'(t))$ .
- We have  $\frac{\partial}{\partial t} \text{hol}_{\gamma|_{[t,b]}}(\eta_B) = -\eta_B(\gamma'(t)) \text{hol}_{\gamma|_{[t,b]}}(\eta_B)$ .

The following lemma expresses the variation of the invariant  $\mathcal{Z}$  of long tangles under isotopies that do not fix the bottom and top configurations. It uses the above holonomy and the anomaly  $\alpha$  of Section 10.3.

**Lemma 15.17.** *Let  $(h_t)_{t \in [0,1]}$  be an isotopy of  $\check{R}(\mathcal{C})$  such that  $h_t$  is the identity map on  $(\mathbb{C} \setminus D_1) \times \mathbb{R}$  for any  $t$ ,  $h_t$  may be expressed as  $h_t^- \times \mathbf{1}_{]-\infty, 0]}$  on  $\mathbb{C} \times ]-\infty, 0]$  for a planar isotopy  $(h_t^-)_{t \in [0,1]}$ , and  $h_t$  may be expressed as  $h_t^+ \times \mathbf{1}_{[1, +\infty[}$  on  $\mathbb{C} \times [1, +\infty[$  for a planar isotopy  $(h_t^+)_{t \in [0,1]}$ . Assume that  $h_0 = \mathbf{1}$  and note that  $h_t$  preserves  $\mathcal{C}$  setwise. Let  $L$  be a long tangle representative of  $\check{R}(\mathcal{C})$  whose bottom (resp. top) configuration is represented by a map  $y^-: B^- \rightarrow D_1$  (resp.  $y^+: B^+ \rightarrow D_1$ ). Let  $J_{bb}$  denote the set of components of  $L$  going from bottom to bottom, and let  $J_{tt}$  denote the set of components of  $L$  going from top to top. Set  $\varepsilon(K_j) = -$  for  $K_j \in J_{bb}$ , and  $\varepsilon(K_j) = +$  for  $K_j \in J_{tt}$ . For a component  $K_j$  of  $J_{bb} \cup J_{tt}$ , let  $\theta_j: [0, 1] \rightarrow \mathbb{R}$  be a path such that the difference  $(h_t^{\varepsilon(K_j)}(y^{\varepsilon(K_j)}(K_j(1))) - h_t^{\varepsilon(K_j)}(y^{\varepsilon(K_j)}(K_j(0))))$  is a positive multiple of the complex direction  $\exp(i2\pi\theta_j(t))$ . With the notation of Theorem 12.7, set  $\mathcal{Z}(t) = \mathcal{Z}(\mathcal{C}, h_t(L))$ . Then we have*

$$\begin{aligned} & \left( \prod_{K_j \in J_{bb} \cup J_{tt}} \left( \exp(-2\varepsilon(K_j)(\theta_j(t) - \theta_j(0))\alpha) \#_j \right) \right) \mathcal{Z}(t) \\ & = \text{hol}_{h^-|_{[t,0]} \circ y^-}(\eta_{B^-}) \mathcal{Z}(0) \text{hol}_{h^+|_{[0,t]} \circ y^+}(\eta_{B^+}). \end{aligned}$$

PROOF: Let  $\tau$  be a parallelization of  $\mathcal{C}$ . Set  $L = (K_j)_{j \in \mathbf{k}}$ , and recall

$$\mathcal{Z}(t) = \exp\left(-\frac{1}{4}p_1(\tau)\beta\right) \left( \prod_{j=1}^{\mathbf{k}} \left( \exp(-I_\theta(K_j(t), \tau)\alpha) \#_j \right) \right) Z(\mathcal{C}, h_t(L), \tau).$$

The algebraic boundary of the chain  $\cup_{t \in [t_0, t_1]} C(R(\mathcal{C}), h_t(L); \Gamma)$  is

$$C(R(\mathcal{C}), h_{t_1}(L); \Gamma) - C(R(\mathcal{C}), h_{t_0}(L); \Gamma) - \sum (\cup_{t \in [t_0, t_1]} F_t),$$

where the sum runs over the codimension-one faces  $F_t$  of  $C(R(\mathcal{C}), h_t(L); \Gamma)$ .<sup>7</sup> Faces cancel as in Section 14.3 except for the anomaly  $\alpha$  faces and the faces for which some vertices are at  $\infty$ .

The variations due to the anomaly  $\alpha$  faces contribute to  $\frac{\partial}{\partial t} \tilde{Z}(\mathcal{C}, h_t(L), \tau)$  as

$$\left( \sum_{j=1}^k \frac{\partial}{\partial t} \left( 2 \int_{[0,t] \times U^{+K_j}} p_\tau^*(\omega_{S^2}) \right) \alpha \#_j \right) \tilde{Z}(\check{R}, h_t(L), \tau)$$

as in Lemma 10.21.

When the bottom and top configurations are not fixed and when  $K_j \in J_{bb} \cup J_{tt}$ , we have

$$I_\theta(K_j(u), \tau) - I_\theta(K_j(0), \tau) = 2 \int_{\cup_{t \in [0,u]} p_\tau(U^{+K_j(t)} \cup S(K_j(t)))} \omega_{S^2},$$

as in the proof of Lemma 14.50, where  $S(K_j(t))$  denotes the half-circle from  $\varepsilon(K_j)\vec{N}$  to  $-\varepsilon(K_j)\vec{N}$  through  $\exp(2i\pi\theta_j(t))$  as in Lemma 12.5. So we get

$$I_\theta(K_j(t), \tau) - I_\theta(K_j(0), \tau) = 2 \int_{\cup_{u \in [0,t]} p_\tau(U^{+K_j(u)})} \omega_{S^2} - 2\varepsilon(K_j)(\theta_j(t) - \theta_j(0)).$$

Thus

$$\tilde{Z}(t) = \left( \prod_{K_j \in J_{bb} \cup J_{tt}} \left( \exp(-2\varepsilon(K_j)(\theta_j(t) - \theta_j(0))\alpha) \#_j \right) \right) \mathcal{Z}(t)$$

gets no variation from the anomaly  $\alpha$  faces as in Corollary 10.22.

Let  $F_t$  be a codimension-one face of  $C(R(\mathcal{C}), h_t(L); \Gamma)$  for which a subset  $V$  of  $V(\Gamma)$  is mapped to  $\infty$ . Such a face is either a  $T$ -face as in Theorem 14.23 and Lemma 14.46, or a face  $F_\infty(V, L, \Gamma)$  as around Notation 8.18. Set  $F = \cup_{t \in [t_0, t_1]} F_t$ .

In both cases, an element  $c^0$  of the face involves an injective configuration  $T_0\phi_\infty \circ f_1^0$  from the kids of  $V$  to  $(T_\infty R(\mathcal{C}) \setminus 0)$  up to dilation. Let  $e = (v_1, v_2)$  be an edge. If the vertices  $v_1$  and  $v_2$  are in different kids of  $V$ , then we have

$$\begin{aligned} p_\tau \circ p_e(c^0) &= \frac{\phi_\infty \circ f_1^0(v_2) - \phi_\infty \circ f_1^0(v_1)}{\|\phi_\infty \circ f_1^0(v_2) - \phi_\infty \circ f_1^0(v_1)\|} \\ &= \frac{\|f_1^0(v_1)\|^2 f_1^0(v_2) - \|f_1^0(v_2)\|^2 f_1^0(v_1)}{\| \|f_1^0(v_1)\|^2 f_1^0(v_2) - \|f_1^0(v_2)\|^2 f_1^0(v_1) \|}. \end{aligned}$$

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<sup>7</sup>This chain is locally modelled by open subsets of  $C(R(\mathcal{C}), h_t(L); \Gamma) \times ]t - \varepsilon, t + \varepsilon[$  unless the isotopies  $h_t^\pm$  are degenerate. See Theorem 14.23 and Lemma 14.14. Stokes' theorem applies thanks to Theorem 14.16.

When  $v_1 \in V$  and  $v_2 \notin V$ , then we have

$$p_\tau \circ p_e(c^0) = -\frac{f_1^0(v_1)}{\|f_1^0(v_1)\|}.$$

If  $v_2 \in V$  and  $v_1 \notin V$ , we similarly have

$$p_\tau \circ p_e(c^0) = \frac{f_1^0(v_2)}{\|f_1^0(v_2)\|}.$$

Let  $E_\infty$  be the set of edges between elements of the set  $V$  of vertices mapped to  $\infty$  in  $F$ , and let  $E_m$  denote the set of edges with one end in  $V$ . The face  $F_t$  is diffeomorphic to a product by  $\check{C}_{V(\Gamma)\setminus V}(\check{R}(\mathcal{C}), h_t(L); \Gamma)$ , whose dimension is

$$3|T(\Gamma) \cap (V(\Gamma) \setminus V)| + |U(\Gamma) \cap (V(\Gamma) \setminus V)|,$$

of a space  $C_{V,t}$  of dimension

$$3|T(\Gamma) \cap V| + |U(\Gamma) \cap V| - 1 = 2|E_\infty| + |E_m| - 1,$$

and  $\bigwedge_{e \in E_\infty \cup E_m} p_e^*(\omega(j_E(e)))$  has to be integrated along  $\cup_{t \in [0,1]} C_{V,t}$ , according to the above expression of  $p_\tau \circ p_e$  for edges of  $E_\infty \cup E_m$ . The degree of this form is  $2|E_\infty \cup E_m|$ . So the face  $F$  cannot contribute unless  $E_m = \emptyset$ .

Now the expression  $p_\tau \circ p_e$  for edges of  $E_\infty$  makes also clear that, if  $f_1^0$  is changed to  $\phi_\infty \circ T \circ \phi_\infty \circ f_1^0$ , for a vertical translation  $T$  such that 0 is not in the image of  $\phi_\infty \circ T \circ \phi_\infty \circ f_1^0$ , then the image under  $\prod_{e \in E_\infty} p_e$  is unchanged. So we have a one-parameter group acting on our face  $F$  such that  $\prod_{e \in E_\infty} p_e$  factors through this action. Unless  $V^+$  has only one kid, this action is not trivial, and the quotient of the face by this action is of dimension strictly less than the face dimension.

Therefore, for the contributing faces, we have  $E_m = \emptyset$  and  $V^+$  has only one kid. Thus, according to Lemma 14.46, we are left with the  $T$ -faces of Theorem 14.23 (for which  $B = \emptyset$  and  $I$  has one element) for which  $\mathcal{P}_x = \{V\}$ . These faces yield the derivative  $\frac{\partial}{\partial t} \tilde{\mathcal{Z}} = d\tilde{\mathcal{Z}} \left( \frac{\partial}{\partial t} \right)$  with

$$d\tilde{\mathcal{Z}} = -((t \mapsto h_t^- \circ y^-)^*(\eta_{B^-}))\tilde{\mathcal{Z}} + \tilde{\mathcal{Z}}((t \mapsto h_t^+ \circ y^+)^*(\eta_{B^+})).$$

This proves the equality

$$\tilde{\mathcal{Z}}(t) = \text{hol}_{h^-|_{[t,0]} \circ y^-}(\eta_{B^-})\tilde{\mathcal{Z}}(0) \text{hol}_{h^+|_{[0,t]} \circ y^+}(\eta_{B^+})$$

and leads to the formula for  $\mathcal{Z}$ . □

**Corollary 15.18.** *Under the hypotheses of Lemma 15.17, let  $L_{\parallel}$  be a parallel of  $L$ . Let  $h_t(L)_{\parallel} = (h_t(K_j)_{\parallel})_j$  be the parallel of  $h_t(L)$  such that*

$$lk(h_t(K_j), h_t(K_j)_{\parallel}) - lk(K_j, K_{j\parallel}) = -\varepsilon(K_j)2(\theta_j(t) - \theta_j(0))$$

for any component  $K_j$  of  $J_{bb} \cup J_{tt}$ , and  $lk(h_t(K), h_t(K)_{\parallel}) = lk(K, K_{\parallel})$  for any other component  $K$  of  $L$ . Use Definition 12.12 to set

$$\mathcal{Z}^f(t) = \mathcal{Z}^f(\mathcal{C}, h_t(L), h_t(L)_{\parallel}).$$

Then we have

$$\mathcal{Z}^f(t) = \text{hol}_{h^-|_{[t,0]} \circ y^-}(\eta_{B^-})\mathcal{Z}^f(0) \text{hol}_{h^+|_{[0,t]} \circ y^+}(\eta_{B^+}).^8$$

PROOF: See Definition 13.4. □

A connection is *flat* if its holonomy along a null-homotopic loop is trivial.

**Proposition 15.19.** *The connection  $\eta_B$  is flat on  $\check{\mathcal{S}}_B(\mathbb{C})$ . When  $\gamma: [0, 1] \rightarrow \check{C}_B[D_1]$  is smooth with vanishing derivatives at 0 and 1, the image  $T(\gamma)$  of the graph  $\{(\gamma(t), t)\}$  of  $\gamma$  in  $D_1 \times [0, 1]$  is a tangle in  $D_1 \times [0, 1]$ , and we have*

$$\mathcal{Z}^f(T(\gamma)) = \mathcal{Z}(T(\gamma)) = \text{hol}_{p_{CS} \circ \gamma}(\eta_B),$$

where  $p_{CS} \circ \gamma$  is the composition of  $\gamma$  by the natural projection  $p_{CS}: \check{C}_B[D_1] \rightarrow \mathcal{S}_B(\mathbb{C})$ . For two framed tangles  $(\mathcal{C}_1, L_1)$  and  $(\mathcal{C}_2, L_2)$  such that the bottom of  $L_2$  coincides with the top of  $L_1$ , if one of them is a braid  $T(\gamma)$  as above, we have

$$\mathcal{Z}^f(\mathcal{C}_1\mathcal{C}_2, L_1L_2) = \mathcal{Z}^f(\mathcal{C}_1, L_1)\mathcal{Z}^f(\mathcal{C}_2, L_2),$$

with products obtained by stacking above in natural ways on both sides, reading from left to right.

PROOF: Applying Lemma 15.17 when  $L$  is a trivial braid,  $h_t^- = h_0^-$  is constant, and  $h_t^+ \circ y^+ = \gamma(t)$  shows  $\mathcal{Z}(T(\gamma)) = \text{hol}_{p_{CS} \circ \gamma}(\eta_B)$ . Then the isotopy invariance of  $\mathcal{Z}$  shows that  $\eta_B$  is flat on  $\check{\mathcal{S}}_B(\mathbb{C})$ . Applying Corollary 15.18 when  $h_t^- = h_0^-$  is constant and  $\gamma(t) = h_t^+ \circ y^+$  proves

$$\mathcal{Z}^f(\mathcal{C}, LT(\gamma)) = \mathcal{Z}^f(\mathcal{C}, L)\mathcal{Z}^f(T(\gamma)).$$

So we have  $\mathcal{Z}^f(T(\gamma_1)T(\gamma_2)) = \mathcal{Z}^f(T(\gamma_1))\mathcal{Z}^f(T(\gamma_2))$  for braids.

Applying Corollary 15.18 when  $h_t^+ = h_0^+$  is constant and  $\gamma(t) = h_{1-t}^- \circ y^-$  proves  $\mathcal{Z}^f(\mathcal{C}_1\mathcal{C}_2, L_1L_2) = \mathcal{Z}^f(\mathcal{C}_1, L_1)\mathcal{Z}^f(\mathcal{C}_2, L_2)$  when  $(\mathcal{C}_1, L_1)$  is a braid and  $(\mathcal{C}_2, L_2)$  is a framed tangle, too. □

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<sup>8</sup>Again, the holonomies are considered as valued in spaces of diagrams on unoriented domains, where the vertex-orientation of Jacobi diagrams includes local orientations of strands, which can be made consistent with a global orientation induced by  $L$ , as in Definition 6.16.

**Note 15.20.** Recall the space  $\check{C}_B[\mathbb{C}]$  of injective planar configurations  $y: B \hookrightarrow \mathbb{C}$ . Let  $\gamma: [0, 1] \rightarrow \check{C}_B[\mathbb{C}]$  be a braid. Proposition 15.19 expresses  $\mathcal{Z}(\gamma) = \mathcal{Z}(T(\gamma))$  as the holonomy of the flat connection  $\eta_B$  with coefficients in  $\mathcal{A}(\sqcup_{b \in B} \mathbb{R}_b)$  along  $p_{CS} \circ \gamma$ . Kontsevich defined his integral  $Z^K$  as the holonomy of another flat connection  $\eta_{KZ}$  along  $\gamma$  for such a braid [BN95a, Section 4]. The involved *Knizhnik–Zamolodchikov connection*  $\eta_{KZ}$  is defined on the space  $\check{C}_B[\mathbb{C}]$ . The coefficients of  $\eta_{KZ}$  belong to the tensor product of the same diagram space  $\mathcal{A}(\sqcup_{b \in B} \mathbb{R}_b)$  by  $\mathbb{C}$ .

Assume  $B = \{b_1, b_2\}$ . Let  $\gamma^c: ]0, 1[ \rightarrow \check{C}_B[\mathbb{C}]$  map  $t$  to  $\gamma^c(t)$  with  $\gamma^c(t)(b_1) = 0$  and  $\gamma^c(t)(b_2) = t$ . For  $\varepsilon \in ]0, 1[$ , set  $\gamma_\varepsilon^c = \gamma^c|_{] \varepsilon, 1[}$ . Then the holonomy of  $\eta_B$  along  $\gamma_\varepsilon^c$  is trivial. However, the holonomy of  $\eta_{KZ}$  along  $\gamma_\varepsilon^c$  is not. It does not even converge when  $\varepsilon$  tends to zero. In particular, the connection  $\eta_{KZ}$  does not factor through  $p_{CS}$ .

Nevertheless, the holonomy of  $\eta_{KZ}$  can be regularized. It lead Thang Lê and Jun Murakami to a definition of the Kontsevich integral for combinatorial  $q$ -braids  $\gamma$  in [LM96]. This definition is related to the limit  $\mathcal{Z}^f(T(\gamma))$  of the holonomy of  $\eta_B$  as in Note 12.30.

In the above proposition, we proved that  $\eta_B$  is flat by proving that its holonomy is  $\mathbf{1} = [\emptyset]$  on null-homotopic loops. The flatness of a differentiable connection  $\eta$  is often established by proving that its *curvature*  $(d\eta + \eta \wedge \eta)$  vanishes, instead. The following lemma shows how the curvature vanishing implies the homotopy invariance of the holonomy.

**Lemma 15.21.** *Set  $\Delta^{(r)} = \{(t_1, \dots, t_r) \in [0, 1]^r : t_1 \leq t_2 \leq \dots \leq t_r\}$ . Let  $\eta$  be a one-form on  $\check{S}_B(\mathbb{C})$  with coefficients in the space  $\mathcal{A}(\sqcup_{b \in B} \mathbb{R}_b)$ . Let  $\gamma: [0, 1] \times [0, 1] \rightarrow \check{S}_B(\mathbb{C})$  be a homotopy mapping  $([0, 1] \times \{0, 1\}) \cup \{0\} \times [0, 1]$  to a point. Set  $\gamma_u(t) = \gamma(u, t)$ . Then the holonomy of  $\eta$  along  $\gamma_1$  is*

$$\mathbf{1} + \sum_{r=1}^{\infty} \int_{[0,1] \times \Delta^{(r)}} \left( \sum_{j=1}^r (-1)^{j-1} \bigwedge_{i=1}^r (\gamma \circ p_i)^*(\eta) \left( \frac{(\gamma \circ p_j)^*(d\eta + \eta \wedge \eta)}{(\gamma \circ p_j)^*(\eta)} \right) \right),$$

where the fraction means that the denominator is replaced by the numerator in the preceding expression.

In particular, if  $d\eta + \eta \wedge \eta$  vanishes, then the holonomy of  $\eta$  is trivial along any null-homotopic loop.

**PROOF:** Stokes’ theorem allows us to compute

$$\text{hol}_{\gamma_1}(\eta) = \mathbf{1} + \sum_{r=1}^{\infty} \int_{\Delta^{(r)}} \bigwedge_{i=1}^r (\gamma_1 \circ p_i)^*(\eta)$$

by integrating  $d \bigwedge_{i=1}^r (\gamma_u \circ p_i)^*(\eta)$  over  $[0, 1] \times \Delta^{(r)}$  as follows. Set  $F_0(\Delta^{(r)}) = \{(0, t_2, \dots, t_r) \in \Delta^{(r)}\}$ ,  $F_r(\Delta^{(r)}) = \{(t_1, t_2, \dots, t_{r-1}, 1) \in \Delta^{(r)}\}$ , and

$$F_j(\Delta^{(r)}) = \{(t_1, \dots, t_r) \in \Delta^{(r)} : t_j = t_{j+1}\}.$$

for  $j \in \underline{r-1}$ . Then we have

$$\partial \Delta^{(r)} = \sum_{j=0}^r (-1)^{j+1} F_j(\Delta^{(r)}),$$

and

$$\partial ([0, 1] \times \Delta^{(r)}) = \left( (\partial [0, 1]) \times \Delta^{(r)} \right) \cup \left( [0, 1] \times \partial(-\Delta^{(r)}) \right).$$

So we get

$$\begin{aligned} & \int_{\Delta^{(r)}} \bigwedge_{i=1}^r (\gamma_1 \circ p_i)^*(\eta) - \int_{\Delta^{(r)}} \bigwedge_{i=1}^r (\gamma_0 \circ p_i)^*(\eta) \\ & + \sum_{j=0}^r (-1)^j \int_{[0,1] \times F_j(\Delta^{(r)})} \bigwedge_{i=1}^r (\gamma \circ p_i)^*(\eta) = \int_{[0,1] \times \Delta^{(r)}} d \left( \bigwedge_{i=1}^r (\gamma \circ p_i)^*(\eta) \right), \end{aligned}$$

where the faces  $F_0$  and  $F_r$  do not contribute since  $\gamma$  maps  $([0, 1] \times \{0, 1\})$  to a point, and  $\int_{\Delta^{(r)}} \bigwedge_{i=1}^r (\gamma_0 \circ p_i)^*(\eta)$  similarly vanishes. We obtain

$$\begin{aligned} & \int_{\Delta^{(r)}} \bigwedge_{i=1}^r (\gamma_1 \circ p_i)^*(\eta) \\ & = \sum_{j=1}^{r-1} (-1)^{j-1} \int_{[0,1] \times \Delta^{(r-1)}} \bigwedge_{i=1}^{r-1} (\gamma \circ p_i)^*(\eta) \left( \frac{(\gamma \circ p_j)^*(\eta \wedge \eta)}{(\gamma \circ p_j)^*(\eta)} \right) \\ & \quad + \sum_{j=1}^r (-1)^{j-1} \int_{[0,1] \times \Delta^{(r)}} \left( \bigwedge_{i=1}^r (\gamma \circ p_i)^*(\eta) \right) \left( \frac{(\gamma \circ p_j)^*(d\eta)}{(\gamma \circ p_j)^*(\eta)} \right), \end{aligned}$$

which yields the result.  $\square$

**Corollary 15.22.** *We have  $d\eta_B + \eta_B \wedge \eta_B = 0$ .*

PROOF: Let us prove that the degree  $k$  part  $(d\eta_B + \eta_B \wedge \eta_B)_k$  of  $(d\eta_B + \eta_B \wedge \eta_B)$  vanishes for any  $k \in \mathbb{N}$ , by induction on the degree  $k$ . This is obviously true for  $k = 0$ . Let us assume that  $(d\eta_B + \eta_B \wedge \eta_B)_i$  vanishes for  $i < k$ . For any disk  $D = \gamma([0, 1] \times \Delta^{(1)})$  as in Lemma 15.21, the degree

$k$  part of the holonomy of  $\eta_B$  along  $\partial D$ , which vanishes, is the integral of  $(d\eta_B + \eta_B \wedge \eta_B)_k$  along  $D$  according to Lemma 15.21. Therefore, the integral of  $(d\eta_B + \eta_B \wedge \eta_B)_k$  vanishes along any disk  $D$ , and  $(d\eta_B + \eta_B \wedge \eta_B)_k$  is zero.  $\square$

Below, we compute  $d\eta_B$  and sketch an alternative proof for the equality of Corollary 15.22.

**Lemma 15.23.** *Along the open codimension-one faces of  $\mathcal{V}(y, \Gamma)$ , the integral of  $(-\bigwedge_{e \in E(\Gamma)} p_{e, S^2}^*(\omega_{S^2}))$  is absolutely convergent. This allows us to define the smooth two-form  $(y \mapsto d\eta_\Gamma(y))$  on  $\check{\mathcal{S}}_B(\mathbb{C})$  so that  $d\eta_\Gamma(y)$  is the sum of these integrals along the open codimension-one faces of  $\mathcal{V}(y, \Gamma)$ .*

PROOF: The integral of  $\bigwedge_{e \in E(\Gamma)} p_{e, S^2}^*(\omega_{S^2})$  along these codimension-one faces of  $\mathcal{V}(y, \Gamma)$  is absolutely convergent and it defines a smooth two-form on  $\check{\mathcal{S}}_B(\mathbb{C})$  as in the proof of Proposition 15.15.

To see that this two-form is  $d\eta_\Gamma$ , use a chart  $\psi: \mathbb{R}^s \rightarrow N(y)$  of a neighborhood of  $y$  in  $\check{\mathcal{S}}_B(\mathbb{C})$ , with  $s = 2|B| - 3$ . Let  $\zeta_i$  denote the composition  $p_i \circ \psi^{-1}$ . So we have  $\eta_\Gamma = \sum_{i=1}^s \eta_i d\zeta_i$  and

$$d\eta_\Gamma = \sum_{(i,j) \in \underline{s}^2 : i < j} \left( \frac{\partial}{\partial \zeta_i} \eta_j - \frac{\partial}{\partial \zeta_j} \eta_i \right) d\zeta_i \wedge d\zeta_j,$$

where

$$\frac{\partial}{\partial \zeta_2} \eta_1 = \lim_{\substack{t_2 \rightarrow 0 \\ t_2 \in ]0, \infty[}} \frac{1}{t_2} \left( \lim_{\substack{t_1 \rightarrow 0 \\ t_1 \in ]0, \infty[}} \frac{1}{t_1} \int_{\psi([0, t_1] \times \{(t_2, 0, \dots, 0)\}) - \psi([0, t_1] \times \{(0, 0, \dots, 0)\})} \eta_\Gamma \right).$$

We thus have

$$\frac{\partial}{\partial \zeta_1} \eta_2 - \frac{\partial}{\partial \zeta_2} \eta_1 = \lim_{\substack{(t_1, t_2) \rightarrow 0 \\ (t_1, t_2) \in ]0, \infty[^2}} \frac{1}{t_1 t_2} \int_{\partial N(t_1, t_2)} \eta_\Gamma,$$

where  $N(t_1, t_2) = \psi([0, t_1] \times [0, t_2] \times \{0\})$ . Furthermore, we have

$$\int_{\partial N(t_1, t_2)} \eta_\Gamma = \int_{\partial p_{S^B}^{-1}(N(t_1, t_2)) \setminus (\cup_{y \in N(t_1, t_2)} \partial \mathcal{V}(y, \Gamma))} \bigwedge_{e \in E(\Gamma)} p_{e, S^2}^*(\omega_{S^2}).$$

Lemma 15.13 and Theorem 14.16 imply that Stokes' theorem applies to the closed form  $\bigwedge_{e \in E(\Gamma)} p_{e, S^2}^*(\omega_{S^2})$ . So we have  $\int_{\partial p_{S^B}^{-1}(N(t_1, t_2))} \bigwedge_{e \in E(\Gamma)} p_{e, S^2}^*(\omega_{S^2}) = 0$  and

$$\int_{N(t_1, t_2)} d\eta_\Gamma = - \int_{y \in N(t_1, t_2)} \int_{\partial \mathcal{V}(y, \Gamma)} \bigwedge_{e \in E(\Gamma)} p_{e, S^2}^*(\omega_{S^2}).$$

$\square$



**Remark 15.24.** The flatness condition  $d\eta_B + \eta_B \wedge \eta_B = 0$  can be proved from the definition of  $d\eta_B$ , which can be extracted from Definition 15.16 and Lemma 15.23, as an exercise along the following lines.

Let us use Lemma 15.23 to compute  $d\eta_B$ . Let  $y \in \check{\mathcal{S}}_B(\mathbb{C})$ . According to Lemma 15.14, for a connected Jacobi diagram  $\Gamma \in \mathcal{D}_k^e(\sqcup_{b \in B} \mathbb{R}_b)$ , the faces of the  $\partial\mathcal{V}(y, \Gamma)$  with maximal dimension are of two types. Either  $V(\Gamma) \notin \mathcal{P}'_x$ , or  $V(\Gamma) \in \mathcal{P}'_x$  and  $\mathcal{P} = \{V(\Gamma), C\}$ .

If  $V(\Gamma) \notin \mathcal{P}'_x$ , then we have  $\mathcal{P} = \{V(\Gamma)\} \sqcup \mathcal{P}'_x$ , and the configuration of the kids of  $V(\Gamma)$  maps the univalent kids of  $V(\Gamma)$  to the vertical line through the origin. It is defined up to vertical translation and dilation. Since  $\Gamma$  is connected, if a univalent daughter  $Y$  of  $V(\Gamma)$  contains a trivalent vertex, then it contains a trivalent vertex that is bivalent, univalent, or 0-valent in  $\Gamma_Y$ . This type of face does not contribute to  $d\eta_B$  (as in Sections 9.2 and 9.3). (Note that the univalent daughters of  $V(\Gamma)$  are in  $\mathcal{P}'_x$ . So they must have vertices on at least two strands, and  $\Gamma_Y$  cannot be an edge between a univalent vertex and a trivalent one.) For the faces in which all the daughters  $Y$  of  $V(\Gamma)$  contain only univalent vertices, the integrated form is determined by the configuration space of the kids of  $V(\Gamma)$  up to dilation and (conjugates of) vertical translations since  $\Gamma_Y$  does not contain chords. The dimension of this space is smaller than  $3|T(\Gamma)| + |U(\Gamma)| - 2$  if  $\mathcal{P}'_x$  is not empty. If  $\mathcal{P}'_x$  is empty, then the face is independent of the planar configuration  $y$ . So it does not contribute to  $d\eta_B$  either.

Thus, the only faces of the  $\partial\mathcal{V}(y, \Gamma)$  contributing to  $d\eta_B$  are such that  $\mathcal{P} = \{V(\Gamma), C\}$  and the univalent vertices of  $C$  are on one strand. Most of these faces cancel as an analysis similar to that performed in Section 9.3 shows. The only contributing faces are the STU-faces that involve a collapse of an edge containing one univalent vertex such that the other two diagrams involved in the corresponding STU relation are not connected. Then these other two diagrams are disjoint unions of two components  $\Gamma_1$  and  $\Gamma_2$  on  $\sqcup_{b \in B} \mathbb{R}_b$ . (If the involved diagrams were connected, then the corresponding faces would cancel by STU.) Consider a configuration  $c_1$  of  $\Gamma_1$  on  $\sqcup_{b \in B} y(b) \times \mathbb{R}_b$  and a configuration  $c_2$  of  $\Gamma_2$  on  $\sqcup_{b \in B} y(b) \times \mathbb{R}_b$  for a planar configuration  $y$ . View  $\Gamma_1$  far below  $\Gamma_2$ , and slide it vertically until it is far above. During this sliding, there will be heights at which one univalent vertex of  $\Gamma_1$  coincides with one univalent vertex of  $\Gamma_2$ . (For a generic pair  $(c_1, c_2)$ , there are no heights at which more than one univalent vertex of  $\Gamma_1$  coincides with one univalent vertex of  $\Gamma_2$ .) Each collision corresponds to a configuration contributing to an incomplete STU relation in  $d\eta_B$ . Furthermore the sum of the corresponding graph classes is  $([\Gamma_1][\Gamma_2] - [\Gamma_2][\Gamma_1])$ . This roughly shows how  $d\eta_B = -\eta_B \wedge \eta_B$ .

# Chapter 16

## Discretizable variants of $\mathcal{Z}^f$ and extensions to $q$ -tangles

We introduce and study variants of  $\mathcal{Z}^f$  involving nonhomogeneous propagating forms in Sections 16.1, 16.2, and 16.3. These variants allow discrete computations from algebraic intersections as in Chapter 11. We will use them in the proofs of important properties of  $\mathcal{Z}^f$  in Chapter 17, where we will finish the proof of Theorem 13.12. Section 16.5 is devoted to the extension of  $\mathcal{Z}^f$  and its variants to  $q$ -tangles. This extension relies on the theory of semi-algebraic sets. We review known facts about semi-algebraic structures and extract useful lemmas for our purposes in Section 16.4.

Throughout this chapter,  $N$  denotes some (large) fixed integer  $N$ ,  $N \geq 2$ , and, for  $i \in \underline{3N}$ ,  $\tilde{\omega}(i, S^2) = (\tilde{\omega}(i, t, S^2))_{t \in [0,1]}$  is a closed 2-form on  $[0, 1] \times S^2$  such that  $\tilde{\omega}(i, 0, S^2)$  is a volume-one form of  $S^2$ . As in Definition 7.6, for a finite set  $A$ , an  $A$ -numbered Jacobi diagram is a Jacobi diagram  $\Gamma$  with oriented edges equipped with an injection  $j_E: E(\Gamma) \hookrightarrow A$ , which numbers the edges. Let  $\mathcal{D}_{n,A}^e(\mathcal{L})$  denote the set of  $A$ -numbered degree  $n$  Jacobi diagrams with support  $\mathcal{L}$  without looped edges.

### 16.1 Discretizable holonomies

Let  $B$  be a finite set. Let  $\Gamma$  be a  $\underline{3N}$ -numbered Jacobi diagram on  $\sqcup_{b \in B} \mathbb{R}_b$ . For an edge  $e$  of  $\Gamma$  from a vertex  $v(e, 1)$  to a vertex  $v(e, 2)$ , let

$$p_{e,S^2}: [0, 1] \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3) \rightarrow [0, 1] \times S^2$$

send  $(t, c)$  to  $(t, p_{S^2}((c(v(e, 1)), c(v(e, 2))))$ ). The maps  $p_{e,S^2}$  provide the form

$$\bigwedge_{e \in E(\Gamma)} p_{e,S^2}^* \left( \tilde{\omega}(j_E(e), S^2) \right)$$

over  $[0, 1] \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$ . This form pulls back to provide smooth forms on the smooth strata of  $[0, 1] \times \mathcal{V}(\Gamma)$ . Define the smooth form  $\eta_\Gamma = \eta_\Gamma((\tilde{\omega}(i, S^2))_{i \in \underline{3N}})$  on  $[0, 1] \times \check{\mathcal{S}}_B(\mathbb{C})$  so that  $\eta_\Gamma(t, y)$  is the integral of  $\bigwedge_{e \in E(\Gamma)} P_{e, S^2}^*(\tilde{\omega}(j_E(e), S^2))$  along the interiors  $\{t\} \times \check{\mathcal{V}}(y, \Gamma)$  of the fibers  $\{t\} \times \mathcal{V}(y, \Gamma)$  of  $[0, 1] \times \mathcal{V}(\Gamma)$ , as in Proposition 15.15.

**Definitions 16.1.** When  $A$  is a subset of  $\underline{3N}$  with cardinality  $3n$ , with  $n > 0$ , set

$$\eta_{B,A} = \sum_{\Gamma \in \mathcal{D}_{n,A}^e(\sqcup_{b \in B} \mathbb{R}_b)} \zeta_\Gamma \eta_\Gamma [\Gamma] \in \Omega^1\left([0, 1] \times \check{\mathcal{S}}_B(\mathbb{C}); \mathcal{A}_n(\sqcup_{b \in B} \mathbb{R}_b)\right),$$

where  $\zeta_\Gamma = \frac{(|A| - |E(\Gamma)|)!}{|A|! 2^{|E(\Gamma)|}}$ . The form

$$\eta_{B,A} = \eta_{B,A}\left((\tilde{\omega}(i, S^2))_{i \in A}\right)$$

pulls back to a one-form on  $[0, 1] \times \check{C}_B[D_1]$  still denoted by  $\eta_{B,A}$  with coefficients in  $\mathcal{A}_n(\sqcup_{b \in B} \mathbb{R}_b)$ , which is again viewed as a space of diagrams on an un-oriented domain as in Definition 6.16, Proposition 10.23, and Remark 10.24. Set  $\eta_{B,\emptyset} = 0$ .

Let  $\eta_{B,\underline{3N}}$  denote  $\eta_{B,\underline{3N}}$ . If  $\tilde{\omega}(i, 0, S^2) = \omega_{S^2}$ , then the restriction of  $\eta_{B,\underline{3N}}$  to  $\{0\} \times \check{\mathcal{S}}_B(\mathbb{C})$  is the form  $\eta_{N,B}$  of Definition 15.16.

For an integer  $r$ , and for a set  $A$  of cardinality  $3n$ , let  $P_r(A)$  denote the set of  $r$ -tuples  $(A_1, A_2, \dots, A_r)$ , where  $A_i$  is a subset of  $A$  with a cardinality multiple of 3, the  $A_i$  are pairwise disjoint, and their union is  $A$ .

Let  $n$  be a positive integer. Let  $A$  be a subset of cardinality  $3n$  of  $\underline{3N}$ . For a path  $\gamma: [0, 1] \rightarrow [0, 1] \times \check{\mathcal{S}}_B(\mathbb{C})$ , define the  $A$ -holonomy  $\text{hol}_\gamma(\eta_{B,A})$  of  $\eta_{B,A}$  along  $\gamma$  to be

$$\begin{aligned} & \widetilde{\text{hol}}_\gamma(\eta_{B,A}) \\ &= \sum_{r=0}^{\infty} \sum_{(A_1, \dots, A_r) \in P_r(A)} \frac{\prod_{i=1}^r |A_i|!}{|A|!} \int_{(t_1, \dots, t_r) \in [0, 1]^r : t_1 \leq t_2 \leq \dots \leq t_r} \bigwedge_{i=1}^r (\gamma \circ p_i)^*(\eta_{B,A_i}), \end{aligned}$$

with  $p_i(t_1, \dots, t_r) = t_i$  and  $\widetilde{\text{hol}}_\gamma(\eta_{B,\emptyset}) = [\emptyset]$ . Then we have

$$\widetilde{\text{hol}}_{\gamma_1 \gamma_2}(\eta_{B,A}) = \sum_{(A_1, A_2) \in P_2(A)} \frac{|A_1|! |A_2|!}{|A|!} \widetilde{\text{hol}}_{\gamma_1}(\eta_{B,A_1}) \widetilde{\text{hol}}_{\gamma_2}(\eta_{B,A_2}),$$

$$\frac{\partial}{\partial t} \widetilde{\text{hol}}_{\gamma|_{[0,t]}}(\eta_{B,A}) = \sum_{(A_1, A_2) \in P_2(A)} \frac{|A_1|! |A_2|!}{|A|!} \widetilde{\text{hol}}_{\gamma|_{[0,t]}}(\eta_{B,A_1}) \eta_{B,A_2}(\gamma(t), \gamma'(t)),$$

and

$$\frac{\partial}{\partial t} \widetilde{\text{hol}}_{\gamma|_{[t,1]}}(\eta_{B,A}) = - \sum_{(A_1,A_2) \in P_2(A)} \frac{|A_1|!|A_2|!}{|A|!} \eta_{B,A_1}(\gamma(t), \gamma'(t)) \widetilde{\text{hol}}_{\gamma|_{[t,1]}}(\eta_{B,A_2}).$$

Note the following lemma about the behavior of the coefficients.

**Lemma 16.2.** *Let  $\Gamma = \Gamma_1 \sqcup \Gamma_2$  be a Jacobi diagram equipped with an injection*

$$j_E: E(\Gamma) \hookrightarrow A,$$

where  $|A| = 3 \deg(\Gamma)$ . We have

$$\zeta_{\Gamma_1 \cup \Gamma_2} = \sum_{\substack{(A_1,A_2) \in P_2(A), \\ j_E(E(\Gamma_1)) \subseteq A_1, j_E(E(\Gamma_2)) \subseteq A_2, \\ |A_1|=3 \deg(\Gamma_1), |A_2|=3 \deg(\Gamma_2)}} \frac{|A_1|!|A_2|!}{|A|!} \zeta_{\Gamma_1} \zeta_{\Gamma_2}$$

PROOF: We have

$$\zeta_{\Gamma_1 \cup \Gamma_2} = \frac{(|A| - |E(\Gamma_1)| - |E(\Gamma_2)|)!}{|A|! 2^{|E(\Gamma_1 \cup \Gamma_2)|}},$$

and the number of pairs  $(A_1, A_2)$  in the sum is

$$\frac{(|A| - |E(\Gamma_1)| - |E(\Gamma_2)|)!}{(|A_1| - |E(\Gamma_1)|)! (|A_2| - |E(\Gamma_2)|)!}.$$

□

**Remark 16.3.** We fixed the cardinality of the sets  $A_i$  of edge indices to be  $f_0(n_i) = 3n_i$  for degree  $n_i$  graphs. We could have replaced  $f_0$  by any  $f: \mathbb{N} \rightarrow \mathbb{N}$  such that  $f(n) \geq 3n - 2$  (so that  $f(n) \geq |j_E(E(\Gamma))|$  for any degree  $n$  Jacobi diagram with at least two univalent vertices) and such that  $f(n) + f(m) = f(n + m)$ , and we would have obtained the same equalities as above. However, we had to fix such a map  $f$  to get these equalities.

If  $\tilde{\omega}(i, 0, S^2) = \omega_{S^2}$ , then for a subset  $A$  of  $\underline{3N}$  with cardinality  $3n$  and for a path  $\gamma: [0, 1] \rightarrow \{0\} \times \check{\mathcal{S}}_B(\mathbb{C})$ ,  $\widetilde{\text{hol}}_{\gamma}(\eta_{B,A})$  is the degree  $n$  part of  $\text{hol}_{\gamma}(\eta_B)$ .

**Notation 16.4.** For a finite set  $A$ , let  $\mathcal{D}_{k,A}^c(\mathbb{R})$  denote the set of connected  $A$ -numbered degree  $k$  Jacobi diagrams with support  $\mathbb{R}$  without looped edges.

For a vertex-oriented graph  $\check{\Gamma} \in \mathcal{D}_{k,A}^c(\mathbb{R})$ , define the two-form  $\omega(\check{\Gamma})$  on  $[0, 1] \times S^2$  to be

$$\omega(\check{\Gamma})(t, v) = \int_{\{t\} \times Q(v; \check{\Gamma})} \bigwedge_{e \in E(\check{\Gamma})} p_{e, S^2}^*(\tilde{\omega}(j_E(e), S^2)),$$

where  $p_{e, S^2}$  again abusively denotes  $\mathbf{1}_{[0,1]} \times p_{e, S^2}$  and  $Q(v; \check{\Gamma})$  is defined in Section 10.3. For a subset  $A$  of  $\underline{3N}$  of cardinality  $3k$ , define the two-form  $\omega(A)$  on  $[0, 1] \times S^2$  with coefficients in  $\check{\mathcal{A}}_k(\mathbb{R})$  to be

$$\omega(A) = \omega\left(A, (\tilde{\omega}(i, S^2))_{i \in \underline{3N}}\right) = \sum_{\check{\Gamma} \in \mathcal{D}_{k,A}^c(\mathbb{R})} \zeta_{\check{\Gamma}} \omega(\check{\Gamma}) [\check{\Gamma}].$$

Here, we view  $\check{\mathcal{A}}(\mathbb{R})$  as a space of Jacobi diagrams on the oriented  $\mathbb{R}$ . So the sets  $U(\Gamma)$  of univalent vertices in involved Jacobi diagrams  $\Gamma$  are ordered by  $i_\Gamma$ , and the univalent vertices are oriented by the orientation of  $\mathbb{R}$ . Let  $s_*: \check{\mathcal{A}}(\mathbb{R}) \rightarrow \check{\mathcal{A}}(\mathbb{R})$  be the map that sends (the class of) a Jacobi diagram  $\Gamma$  on  $\mathbb{R}$  to (the class of) the Jacobi diagram  $s(\Gamma)$  obtained from  $\Gamma$  by multiplying it by  $(-1)^{|U(\Gamma)|}$  and by reversing the order of  $U(\Gamma)$ .<sup>1</sup>

**Lemma 16.5.** *Let  $\iota: [0, 1] \times S^2 \rightarrow [0, 1] \times S^2$  map  $(t, v)$  to  $(t, \iota_{S^2}(v))$ , where  $\iota_{S^2}$  is the antipodal map of  $S^2$ . With the above notation, the form*

$$\omega(A) = \omega\left(A, (\tilde{\omega}(i, S^2))_{i \in \underline{3N}}\right)$$

*is a closed form of  $[0, 1] \times S^2$  with coefficients in  $\check{\mathcal{A}}_k(\mathbb{R})$ , and we have  $\iota^*(\omega(A)) = -s_*(\omega(A))$  and  $\iota^*(\omega(A)) = (-1)^k(\omega(A))$ .*

**PROOF:** In order to prove that  $\omega(A)$  is closed, it suffices to prove that its integral vanishes on the boundary of any 3-ball  $B$  of  $[0, 1] \times S^2$ . View

$$Q_k(t, v) = \sum_{\check{\Gamma} \in \mathcal{D}_{k,A}^c(\mathbb{R})} \zeta_{\check{\Gamma}} [\check{\Gamma}] \left( \{t\} \times Q(v; \check{\Gamma}) \times (S^2)^{A \setminus j_E(E(\check{\Gamma}))} \right)$$

as a  $(6k - 2)$ -chain with coefficients in  $\check{\mathcal{A}}_k(\mathbb{R})$ . So the integral  $\int_{\partial B} \omega(A)$  is the integral of the pull-back of the closed form  $\bigwedge_{a \in A} \tilde{\omega}(a, S^2)$  under a natural map  $P$  from  $\cup_{(t,v) \in \partial B} Q_k(t, v)$  to  $[0, 1] \times (S^2)^A$ .

The analysis of the boundary of  $Q(v; \check{\Gamma})$  in the proof of Proposition 10.13 shows that the codimension-one faces of the boundary of  $P(Q_k(t, v))$  can be glued. So the boundary of  $P(Q_k(t, v))$  vanishes algebraically.

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<sup>1</sup>To my knowledge, the map  $s_*$  might be the identity map. If it is, real-valued Vassiliev invariants cannot distinguish an oriented knot from that obtained by reversing its orientation, as Greg Kuperberg proved in [Kup96]. In other words, there would be no odd Vassiliev invariants. Recall the first two questions in Section 1.6.

Therefore, since the cycle  $\cup_{(t,v) \in \partial B} P(Q_k(t, v))$  bounds  $\cup_{(t,v) \in B} \check{P}(Q_k(t, v))$  in  $[0, 1] \times (S^2)^A$ , the integral  $\int_{\partial B} \omega(A)$  vanishes, and  $\omega(A)$  is closed.

For  $\check{\Gamma} \in \mathcal{D}_{k,A}^c(\mathbb{R})$ , recall that the class of  $s(\check{\Gamma})$  is obtained from the class of  $\check{\Gamma}$  by multiplying it by  $(-1)^{|U(\check{\Gamma})|}$  and by reversing the order of  $U(\check{\Gamma})$ . A configuration of  $Q(v; \check{\Gamma})$  is naturally a configuration of  $Q(-v; s(\check{\Gamma}))$ . Below, we consider  $\check{\Gamma}$  and  $s(\check{\Gamma})$  as diagrams on unoriented domains whose univalent vertices are equipped with matching local orientations of the domain at univalent vertices. The natural identification from  $Q(v; \check{\Gamma})$  to  $Q(-v; s(\check{\Gamma}))$  reverses the orientation since the local orientations at univalent vertices coincide, the quotients by dilation coincide, but the translations act in opposite directions. Therefore, for any 2-chain  $\Delta$  of  $[0, 1] \times S^2$ , we have  $\int_{\Delta} \omega(\check{\Gamma}) = - \int_{\iota(\Delta)} \omega(s(\check{\Gamma}))$ , where  $\iota(\Delta)$  is equipped with the orientation of  $\Delta$ . We get

$$\begin{aligned} \sum_{\check{\Gamma} \in \mathcal{D}_{k,A}^c(\mathbb{R})} \zeta_{\check{\Gamma}} \int_{\Delta} \omega(\check{\Gamma}) [\check{\Gamma}] &= - \sum_{\check{\Gamma} \in \mathcal{D}_{k,A}^c(\mathbb{R})} \zeta_{s(\check{\Gamma})} \int_{\iota(\Delta)} \omega(s(\check{\Gamma})) [s(\check{\Gamma})] \\ &= - \sum_{\check{\Gamma} \in \mathcal{D}_{k,A}^c(\mathbb{R})} \zeta_{\check{\Gamma}} \int_{\iota(\Delta)} \omega(\check{\Gamma}) [s(\check{\Gamma})] \end{aligned}$$

and  $\omega(A) = -\iota^*(s_*(\omega(A)))$ .

In order to prove  $\iota^*(\omega(A)) = (-1)^k(\omega(A))$ , use the notation and the arguments of the proof that  $\alpha_{2n} = 0$  in the proof of Proposition 10.13 to prove that for any 2-chain  $\Delta$  of  $[0, 1] \times S^2$  and any  $\check{\Gamma} \in \mathcal{D}_{k,A}^c(\mathbb{R})$ , we have

$$\int_{\Delta} \omega(\check{\Gamma}) [\check{\Gamma}] = (-1)^k \int_{\iota(\Delta)} \omega(\check{\Gamma}^{eo}) [\check{\Gamma}^{eo}].$$

□

**Definition 16.6.** For any oriented tangle component  $K$ , let  $U^+K$  denote the fiber space over  $K$  consisting of the tangent vectors to the knot  $K$  of  $\check{R}(\mathcal{C})$  that orient  $K$  up to dilation, as in Section 7.3. When the ambient manifold is equipped with a parallelization  $\tau$ , define the one-form  $\eta(A, p_{\tau}(U^+K)) = \eta(A, (\tilde{\omega}(i, S^2))_{i \in \underline{3N}}, p_{\tau}(U^+K))$  on  $[0, 1]$  valued in  $\check{\mathcal{A}}(\mathbb{R})$  to be

$$\eta(A, p_{\tau}(U^+K))(t) = \eta(A, p_{\tau}(U^+K), \tau)(t) = \int_{\{t\} \times p_{\tau}(U^+K)} \omega\left(A, (\tilde{\omega}(i, S^2))_{i \in \underline{3N}}\right).$$

More precisely, we have

$$\eta(A, p_{\tau}(U^+K))\left(u, \frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t} \left( \int_{[0,t] \times p_{\tau}(U^+K)} \omega(A) \right) (u)$$

as in Proposition 15.15. Define the  $A$ -holonomy of  $\eta(\cdot, p_\tau(U^+K))$  along  $[a, b] \subseteq [0, 1]$  to be

$$\widetilde{\text{hol}}_{[a,b]} \left( \eta(A, p_\tau(U^+K)) \right) = \sum_{r=0}^{\infty} \sum_{(A_1, \dots, A_r) \in P_\tau(A)} \frac{\prod_{i=1}^r |A_i|!}{|A|!} \int_{(t_1, \dots, t_r) \in [a,b]^r : t_1 \leq t_2 \leq \dots \leq t_r} \bigwedge_{i=1}^r p_i^* \left( \eta(A_i, p_\tau(U^+K)) \right)$$

with  $p_i(t_1, \dots, t_r) = t_i$  and  $\widetilde{\text{hol}}_{[a,b]}(\eta(\emptyset, p_\tau(U^+K))) = \mathbf{1} = [\emptyset]$ .

**Lemma 16.7.** *For any oriented tangle component  $K$ , set  $U^-K = U^+(-K)$ . We have*

$$\begin{aligned} \widetilde{\text{hol}}_{[a,b]} \left( \eta(A, -p_\tau(U^-K)) \right) &= -\eta(A, p_\tau(U^-K)) \\ &= s_* \left( \widetilde{\text{hol}}_{[a,b]} \left( \eta(A, p_\tau(U^+K)) \right) \right). \end{aligned}$$

PROOF: Lemma 16.5 implies  $\eta(A, -p_\tau(U^-K)) = s_* \left( \eta(A, p_\tau(U^+K)) \right)$ .  $\square$

Also note the following lemma for later use.

**Lemma 16.8.** *Let  $K(\mathcal{L})$  be a connected oriented one-dimensional submanifold of a 3-manifold  $R(\mathcal{C})$  equipped with a parallelization  $\tau$ . Assume that  $K = K(\mathcal{L})$  is a finite union  $K = \cup_{j=1}^k K_j$  of intervals  $K_j$  with disjoint interiors. Let  $A$  be a subset of  $\underline{3N}$  of cardinality multiple of 3. Then we have  $\eta(A, p_\tau(U^+K)) = \sum_{j=1}^k \eta(A, p_\tau(U^+K_j))$  and*

$$\begin{aligned} \widetilde{\text{hol}}_{[0,1]} \left( \eta(A, p_\tau(U^+K)) \right) &= \sum_{(B_1, \dots, B_k) \in P_k(A)} \frac{\prod_{j=1}^k |B_j|!}{|A|!} \prod_{j=1}^k \widetilde{\text{hol}}_{[0,1]} \left( \eta(B_j, p_\tau(U^+K_j)) \right) \end{aligned}$$

in  $\check{\mathcal{A}}(\mathcal{L})$ .

PROOF: It suffices to prove the lemma when  $k$  equals 2. Recall

$$\widetilde{\text{hol}}_{[0,1]} \left( \sum_{j=1}^2 \eta(\cdot, p_\tau(U^+K_j)) \right) = \sum_{r=0}^{\infty} \sum_{(A_1, \dots, A_r) \in P_\tau(A)} \frac{\prod_{i=1}^r |A_i|!}{|A|!} I(A_1, \dots, A_r),$$

with

$$\begin{aligned} &I(A_1, \dots, A_r) \\ &= \int_{(t_1, \dots, t_r) \in [0,1]^r : t_1 \leq t_2 \leq \dots \leq t_r} \bigwedge_{i=1}^r p_i^* \left( \eta(A_i, p_\tau(U^+K_1)) + \eta(A_i, p_\tau(U^+K_2)) \right). \end{aligned}$$

Write

$$\begin{aligned} \bigwedge_{i=1}^r p_i^* \left( \eta(A_i, p_\tau(U^+ K_1)) + \eta(A_i, p_\tau(U^+ K_2)) \right) \\ = \sum_{f: \{1, \dots, r\} \rightarrow \{1, 2\}} \bigwedge_{i=1}^r p_i^* \left( \eta(A_i, p_\tau(U^+ K_{f(i)})) \right). \end{aligned}$$

For  $f$  as above, also write

$$\begin{aligned} I_j((A_i)_{i \in f^{-1}(j)}) \\ = \int_{(t_i)_{i \in f^{-1}(j)} \in [0, 1]^{f^{-1}(j)} : t_i \leq t_s \text{ when } i \leq s} \bigwedge_{i \in f^{-1}(j)} p_i^* \left( \eta(A_i, p_\tau(U^+ K_j)) \right). \end{aligned}$$

Decompose the terms  $I_1((A_i)_{i \in f^{-1}(1)}) I_2((A_i)_{i \in f^{-1}(2)})$  of the right-hand side of the equality to be proved with respect to the relative orders of the involved  $t_i$  in  $I_1$  and  $I_2$ .

The contribution to  $\widetilde{\text{hol}}_{[0, 1]}(\sum_{j=1}^2 \eta(\cdot, p_\tau(U^+ K_j)))$  of the terms such that

- $\{A_i : i \in \underline{r}\}$  is fixed as an unordered set,
- $f$  is fixed as a map from this unordered set to  $\{1, 2\}$ , and
- the partial order induced by the numbering in  $\underline{r}$  is fixed over the sets  $\{A_i : i \in f^{-1}(j)\}$  for  $j = 1, 2$ ,

is also a sum over the possible total orders on the  $\{A_i : i \in \underline{r}\}$  that induce the given orders on the two subsets.

We can easily identify the involved coefficients of the above similar contributions to prove the result when  $s$  equals 2. □

## 16.2 Variants of $\mathcal{Z}^f$ for tangles

We now present alternative definitions of  $\mathcal{Z}^f$  involving nonhomogeneous propagating forms associated to volume forms  $\tilde{\omega}(i, 1, S^2) = \omega_{S^2} + d\eta(i, S^2)$ , where  $\omega_{S^2}$  is the homogeneous volume-one form of  $S^2$ .

Let  $\mathcal{C}$  be a rational homology cylinder equipped with a parallelization  $\tau$ . Let  $L: \mathcal{L} \hookrightarrow R(\mathcal{C})$  denote a long tangle representative. Recall that we fixed some (large) integer  $N$ ,  $N \geq 2$ , and that  $(\tilde{\omega}(i, S^2) = (\tilde{\omega}(i, t, S^2))_{t \in [0, 1]})_{i \in \underline{3N}}$  is



a family of a closed 2-forms on  $[0, 1] \times S^2$  such that  $\tilde{\omega}(i, 0, S^2)$  is a volume-one form of  $S^2$ . For  $i \in \underline{3N}$ , let  $\tilde{\omega}(i) = (\tilde{\omega}(i, t))_{t \in [0,1]}$  be a closed 2-form on  $[0, 1] \times C_2(R(\mathcal{C}))$  such that  $\tilde{\omega}(i) = (\mathbf{1}_{[0,1]} \times p_\tau)^*(\tilde{\omega}(i, S^2))$  on  $[0, 1] \times \partial C_2(R(\mathcal{C}))$ .

For a diagram  $\Gamma \in \mathcal{D}_{k, \underline{3N}}^e(\mathcal{L})$ , define

$$I(\mathcal{C}, L, \Gamma, o(\Gamma), (\tilde{\omega}(i, t))_{i \in \underline{3N}}) = \int_{(\check{C}(\check{R}(\mathcal{C}), L; \Gamma), o(\Gamma))} \bigwedge_{e \in E(\Gamma)} p_e^*(\tilde{\omega}(j_E(e), t)),$$

which converges, according to Theorem 12.2.

**Theorem 16.9.** *Let  $L: \mathcal{L} \hookrightarrow R(\mathcal{C})$  denote the long tangle associated to a tangle in a rational homology cylinder equipped with a parallelization  $\tau$ . Let  $\{K_j\}_{j \in I}$  be the set of components of  $L$ . Assume that the bottom and top configurations of  $L$  are respectively represented by maps  $y^-: B^- \rightarrow D_1$  and  $y^+: B^+ \rightarrow D_1$ .*

*Let  $N \in \mathbb{N}$ . For  $i \in \underline{3N}$ , let  $\tilde{\omega}(i, S^2) = (\tilde{\omega}(i, t, S^2))_{t \in [0,1]}$  be a closed 2-form on  $[0, 1] \times S^2$  such that  $\tilde{\omega}(i, 0, S^2)$  is a volume-one form of  $S^2$ ,<sup>2</sup> and let  $\tilde{\omega}(i) = (\tilde{\omega}(i, t))_{t \in [0,1]}$  be a closed 2-form on  $[0, 1] \times C_2(R(\mathcal{C}))$  such that  $\tilde{\omega}(i) = (\mathbf{1}_{[0,1]} \times p_\tau)^*(\tilde{\omega}(i, S^2))$  on  $[0, 1] \times \partial C_2(R(\mathcal{C}))$ . For a subset  $A$  of  $\underline{3N}$  with cardinality  $3k$ , set*

$$Z(\mathcal{C}, L, \tau, A, (\tilde{\omega}(i, t))_{i \in A}) = \sum_{\Gamma \in \mathcal{D}_{k, A}^e(\mathcal{L})} \zeta_\Gamma I(\mathcal{C}, L, \Gamma, (\tilde{\omega}(i, t))_{i \in A}) [\Gamma] \in \mathcal{A}_k(\mathcal{L})$$

and

$$Z(\mathcal{C}, L, \tau, A)(t) = Z(\mathcal{C}, L, \tau, A, (\tilde{\omega}(i, t))_{i \in A}).$$

Then we have

$$Z(\mathcal{C}, L, \tau, A)(t) = \sum_{\mathfrak{N}=(A_1, A_2, A_3, (A_{K_j})_{j \in I}) \in P_{3+|I|}(A)} \zeta_d(\mathfrak{N}) Z(\mathfrak{N}, t),$$

with

$$\zeta_d(\mathfrak{N}) = \frac{|A_1|! |A_2|! |A_3|! \prod_{j \in I} |A_{K_j}|!}{|A|!}$$

and

$$Z(\mathfrak{N}, t) = \left( \prod_{j \in I} \widetilde{\text{hol}}_{[0,t]}(\eta(A_{K_j}, p_\tau(U^+ K_j))) \#_j \right) \widetilde{\text{hol}}_{[t,0] \times y^-}(\eta_{B^-, A_1}) Z(\mathcal{C}, L, \tau, A_2)(0) \widetilde{\text{hol}}_{[0,t] \times y^+}(\eta_{B^+, A_3}).$$

---

<sup>2</sup>In this chapter, we apply the theorem only when  $\tilde{\omega}(i, 0, S^2)$  is the standard homogeneous volume-one form  $\omega_{S^2}$  on  $S^2$ . However, we use this general statement in the next chapter.

The terms of this formula belong to spaces of diagrams on unoriented one-manifolds as in Definition 6.16, except for the term  $\text{hol}_{[0,t]}(\eta(A_{K_j}, p_\tau(U^+ K_j)))$  and its action  $\#_j$  for which we first pick an orientation of the  $K_j$ , which we may forget afterwards.<sup>3</sup> The formula implies that  $Z(\mathcal{C}, L, \tau, A)(t)$  depends only on  $(\tilde{\omega}(i, t, S^2))_{i \in A}$  for any  $t$ . (It also depends on  $\tau$  and on the specific embedding  $L$ .) It will be denoted by  $Z(\mathcal{C}, L, \tau, A, (\tilde{\omega}(i, t, S^2))_{i \in A})$ .

PROOF: Compute  $\frac{\partial}{\partial t} Z(A, t) = dZ(A, \cdot) \left( \frac{\partial}{\partial t} \right)$  as in Lemma 10.20 with the help of Proposition 9.2, using the same analysis of faces as in the proof of Lemma 15.17, to find

$$\begin{aligned} dZ(A, \cdot) &= \sum_{(A_1, A_2) \in P_2(A)} \frac{|A_1|!|A_2|!}{|A|!} \sum_{j \in I} (\eta(A_1, p_\tau(U^+ K_j)) \#_j) Z(A_2, t) \\ &\quad + \sum_{(A_1, A_2) \in P_2(A)} \frac{|A_1|!|A_2|!}{|A|!} Z(A_1, t) (\{t\} \times y^+)^*(\eta_{B^+, A_2}) \\ &\quad - \sum_{(A_1, A_2) \in P_2(A)} \frac{|A_1|!|A_2|!}{|A|!} (\{t\} \times y^-)^*(\eta_{B^-, A_1}) Z(A_2, t), \end{aligned}$$

with  $\eta(\emptyset, p_\tau(U^+ K_j)) = 0$  and  $\eta_{B^+, \emptyset} = 0$ . (Recall Lemma 16.2 for the behavior of the coefficients.) This shows that both sides of the equality to be proved vary in the same way when  $t$  varies. Since they take the same value at  $t = 0$ , the formula is proved. Apply the formula when  $\tilde{\omega}(i, 0, S^2)$  is the standard form  $\omega_{S^2}$ , and use Lemma 9.1 together with the isotopy invariance of  $Z$  of Theorem 12.7 to see that  $Z(\mathcal{C}, L, \tau, A)(t)$  depends only on the  $\tilde{\omega}(i, S^2)$  for  $i \in A$ . So it depends only on the  $\tilde{\omega}(i, t, S^2)$ .  $\square$

Let us introduce some notation in order to rephrase Theorem 16.9. View  $Z(\mathcal{C}, L, \tau, \cdot)(t)$  as a map from the set  $\mathcal{P}_{(3)}(\underline{3N})$  of subsets of  $\underline{3N}$  with cardinality multiple of 3 to  $\mathcal{A}_{\leq N}(\mathcal{L}) = \bigoplus_{k=0}^N \mathcal{A}_k(\mathcal{L})$ , which maps  $\emptyset$  to the class of the empty diagram. Similarly, consider  $\text{hol}_{[0,t]}(\eta(\cdot, p_\tau(U^+ K_j)))$ ,  $\text{hol}_{[t,0] \times y^-}(\eta_{B^-, \cdot})$  and  $\text{hol}_{[0,t] \times y^+}(\eta_{B^+, \cdot})$  as maps from  $\mathcal{P}_{(3)}(\underline{3N})$  to spaces of diagrams, which map the empty set to the class of the empty diagram. The values of these maps can be multiplied as in the statement of the theorem using the structures of the space of diagrams.

**Definition 16.10.** For such maps  $z_1$  and  $z_2$  from  $\mathcal{P}_{(3)}(\underline{3N})$  to spaces of diagrams, define their product  $(z_1 z_2)_\sqcup$  to be the map with domain  $\mathcal{P}_{(3)}(\underline{3N})$  such that

$$(z_1 z_2)_\sqcup(A) = \sum_{(A_1, A_2) \in P_2(A)} \frac{|A_1|!|A_2|!}{|A|!} z_1(A_1) z_2(A_2)$$

whenever the products  $z_1(A_1) z_2(A_2)$  make sense.

---

<sup>3</sup>Both sides are independent of the domain orientations in the sense of the last sentence of Proposition 10.23 and Remark 10.24, thanks to Lemma 16.7.

This product is associative, and  $(z_1 z_2 z_3)_\sqcup$  denotes

$$((z_1 z_2)_\sqcup z_3)_\sqcup = (z_1 (z_2 z_3)_\sqcup)_\sqcup.$$

The maps that send all nonempty elements of  $P_{(3)}(\underline{3N})$  to 0 and the empty element to the class of the empty diagram are neutral for this product. They are denoted by  $\mathbf{1}$ . With the above definition, we may write the equality of Theorem 16.9 as

$$Z(\cdot)(t) = \left( \left( \prod_{j \in I} \widetilde{\text{hol}}_{[0,t]}(\eta(\cdot, U_j^+)) \#_j \right) \widetilde{\text{hol}}_{[t,0] \times y^-}(\eta_{B^-, \cdot}) Z(\cdot)(0) \widetilde{\text{hol}}_{[0,t] \times y^+}(\eta_{B^+, \cdot}) \right)_\sqcup$$

with  $Z(\cdot)(t) = Z(\mathcal{C}, L, \tau, \cdot)(t)$  and  $U_j^+ = p_\tau(U^+ K_j)$ .

### 16.3 Straight tangles

**Definition 16.11.** Recall that  $\vec{N}$  denotes the vertical unit vector. A tangle  $L: \mathcal{L} \hookrightarrow \mathcal{C}$  is *straight* with respect to  $\tau$  if

- $p_\tau(U^+ K) \subset \{-\vec{N}, \vec{N}\}$  for closed components  $K$  and components  $K$  going from bottom to top or from top to bottom,
- for any interval component  $\mathcal{L}_j$ , the map  $p_\tau$  sends the unit tangent vectors to  $L(\mathcal{L}_j)$  to the vertical half great circle  $S_{WE}$  from  $-\vec{N}$  to  $\vec{N}$  that contains the west-east direction (to the right), or to the vertical half great circle  $S_{EW}$  from  $-\vec{N}$  to  $\vec{N}$  that contains the east-west direction (to the left). See Figure 16.1.

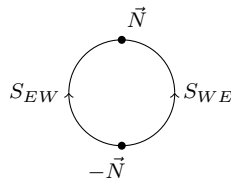


Figure 16.1: The half-circles  $S_{WE}$  and  $S_{EW}$ .

Orient  $S_{WE}$  and  $S_{EW}$  from  $-\vec{N}$  to  $\vec{N}$ . Straight tangles with respect to  $\tau$  get the following framing induced by  $\tau$ . For any  $k \in K$ ,  $p_\tau(U_k^+ K)$  is an

element  $v_k$  of the vertical circle  $S_{WE} \cup (-S_{EW})$ , which intersects the horizontal plane  $\mathbb{C}$  in the real line. Let  $\rho_{i,\pi/2}(v_k)$  be the image of  $v_k$  under the rotation of angle  $\pi/2$  about the axis  $i$  ( $i \in \mathbb{C}$  points toward the sheet). Then  $K_{\parallel}$  is the parallel of  $K$  obtained by pushing  $K$  in the direction of the section  $(k \mapsto \tau(\rho_{i,\pi/2}(v_k)))$  of the unit normal bundle of  $K$ . (This is consistent with the conventions of Definition 12.8.)

**Remark 16.12.** A boundaryless straight tangle is a straight link in the sense of Definition 7.37. However, the converse is not true. The present definition is more restrictive.

The following proposition generalizes Lemma 7.35 to interval components.

**Proposition 16.13.** *Let  $K$  be a component of a straight  $q$ -tangle in a parallelized homology cylinder  $(\mathcal{C}, \tau)$ . Then we have*

$$I_{\theta}(K, \tau) = I_{\theta}(-K, \tau) = lk(K, K_{\parallel})$$

with the notation of Definitions 12.9 and 13.4 for  $lk(K, K_{\parallel})$ , and of Definition 12.6 and Lemma 7.15 for  $I_{\theta}$ .

In order to prove Proposition 16.13, we describe some propagating forms, which will also be useful in the next chapter.

**Notation 16.14.** For an interval  $I$  of  $\mathbb{R}$  that contains  $[0, 1]$  and for a real number  $x \in [1, +\infty[$ ,  $R_{x,I}(\mathcal{C})$  denotes the part that replaces  $D_x \times I$  in  $R(\mathcal{C})$ , and  $R_{x,I}^c(\mathcal{C}) = S^3 \setminus (\overset{\circ}{D}_x \times \overset{\circ}{I})$  denotes the closure of its complement.

Let  $\chi_{\mathcal{C}}$  be a smooth map from  $\check{R}(\mathcal{C})$  to  $[0, 1]$  that sends  $\mathcal{C} = R_{1,[0,1]}(\mathcal{C})$  to 1 and  $R_{2,[-1,2]}^c(\mathcal{C})$  to 0. Define

$$\begin{aligned} \pi_{\mathcal{C}}: \quad \check{R}(\mathcal{C}) &\rightarrow \mathbb{R}^3 \\ x &\mapsto (1 - \chi_{\mathcal{C}}(x))x, \end{aligned}$$

with  $0x = 0$ . The map

$$\begin{aligned} p: \quad \left( \check{R}(\mathcal{C}) \right)^2 \setminus \left( \check{R}_{2,[-1,2]}(\mathcal{C})^2 \cup \text{diag} \left( \check{R}_{2,[-1,2]}^c(\mathcal{C})^2 \right) \right) &\rightarrow S^2 \\ (x, y) &\mapsto \frac{\pi_{\mathcal{C}}(y) - \pi_{\mathcal{C}}(x)}{\|\pi_{\mathcal{C}}(y) - \pi_{\mathcal{C}}(x)\|} \end{aligned}$$

extends to

$$D(p) = C_2(R(\mathcal{C})) \setminus \overset{\circ}{C}_2(R_{2,[-1,2]}(\mathcal{C})).$$

When a parallelization  $\tau$  of  $\mathcal{C}$  is given, the corresponding extension of  $p$  to

$$D(p_{\tau}) = D(p) \cup UR_{2,[-1,2]}(\mathcal{C})$$

is denoted by  $p_{\tau}$ .

**Lemma 16.15.** *For any volume-one 2-form  $\omega(S^2)$  on  $S^2$  and any parallelization  $\tau$  of  $\mathcal{C}$  (as in Definition 12.4), there exists a propagating form  $\omega$  of  $(C_2(\check{R}(\mathcal{C})), \tau)$  that restricts to  $D(p_\tau)$  as  $p_\tau^*(\omega(S^2))$ . For any  $X \in S^2$ , there exists a propagating chain  $F$  of  $(C_2(\check{R}(\mathcal{C})), \tau)$  that restricts to  $D(p_\tau)$  as  $p_\tau^{-1}(X)$ .*

PROOF: We again need an extension to the interior of  $C_2(R_{2,[-1,2]}(\mathcal{C}))$  of a closed 2-form defined on the boundary. Since this space is a 6-manifold with ridges with the same homology as  $S^2$ , the form extends as a closed form.  $\square$

PROOF OF PROPOSITION 16.13: Lemma 7.35 leaves us with the case of interval components  $K$ . In this case, we have  $I_\theta(K, \tau) = 2I(\hat{\mathcal{K}}^K, \omega_{S^2})$  and  $I_\theta(-K, \tau) = I_\theta(K, \tau)$ .

First assume that  $K$  goes from bottom to top. In this case, Lemma 12.5 guarantees that  $I(\hat{\mathcal{K}}^K, \omega(S^2))$  is independent of the chosen volume-one form  $\omega(S^2)$  of  $S^2$ . Choose a volume-one form  $\omega_{0,0}$  of  $S^2$   $\varepsilon$ -dual to  $p_\tau^{-1}(i)$  for the complex horizontal direction  $i$  and for a small positive number  $\varepsilon$ . (Recall Definition 11.6.) Set  $\omega_0(S^2) = \frac{1}{2}(\omega_{0,0} - \iota^*(\omega_{0,0}))$ , and choose a propagating form  $\omega_0$  that restricts to  $D(p_\tau)$  as  $p_\tau^*(\omega_0(S^2))$  as in Lemma 16.15. Assume that  $\varepsilon$  is sufficiently small so that  $p_\tau$  maps  $(K \times K_\parallel) \cap D(p_\tau)$  outside the support of  $\omega_0(S^2)$ . Define knots  $C$  and  $C_\parallel$  such that

- the knots  $C$  and  $C_\parallel$  respectively coincide with  $K$  and  $K_\parallel$  on  $R_{9,[-9,9]}(\mathcal{C})$ ,
- $(C, C_\parallel)$  is isotopic to the pair  $(\hat{K}, \hat{K}_\parallel)$  of Definition 12.9 of  $lk(K, K_\parallel)$  so that we have  $lk(K, K_\parallel) = lk(C, C_\parallel)$ , and
- $p_\tau$  maps  $(C \times C_\parallel) \cap D(p_\tau)$  outside the support of  $\omega_0(S^2)$  so that we have  $\int_{C \times C_\parallel} \omega_0 = \int_{K \times K_\parallel} \omega_0$ .

Our choice of  $\omega_0$  also allows us to let  $K_\parallel$  approach and replace  $K$  without changing the rational integral  $\int_{K \times K_\parallel} \omega_0$ . So we have  $\int_{K \times K_\parallel} \omega_0 = \int_{K \times K \setminus \Delta} \omega_0$ . Then we get

$$\begin{aligned} lk(K, K_\parallel) &= \int_{C \times C_\parallel} \omega_0 = \int_{K \times K_\parallel} \omega_0 = \int_{K \times K \setminus \Delta} \omega_0 \\ &= I(\hat{\mathcal{K}}^K, \omega_0(S^2)) + I(\hat{\mathcal{K}}^K, \omega_0(S^2)) \\ &= I(\hat{\mathcal{K}}^K, \omega_0(S^2)) + I(\hat{\mathcal{K}}^K, -\iota^*(\omega_0(S^2))) = 2I(\hat{\mathcal{K}}^K, \omega_0(S^2)). \end{aligned}$$

This proves Proposition 16.13 when  $K$  goes from bottom to top. Since we have  $I_\theta(-K, \tau) = I_\theta(K, \tau)$ , Definition 12.9 allows us to deduce it when  $K$  goes from top to bottom.

Let us now assume that  $K$  goes from bottom to bottom (resp. from top to top). Lemma 13.5 reduces the proof to the case  $p_\tau(U^+K) \subseteq S_{WE}$ .

Let  $S(K)$  be the half-circle from  $-\vec{N}$  to  $\vec{N}$  (resp. from  $\vec{N}$  to  $-\vec{N}$ ) through the direction of  $[K(0), K(1)]$ . Lemma 12.5 implies that  $I(\hat{\xi}^K, \omega(S^2))$  depends only on the integrals of the restriction of  $\omega(S^2)$  to the components of  $S^2 \setminus (p_\tau(U^+K) \cup S(K))$ , with the notation of Lemma 12.5. In particular, when  $S(K) = \pm S_{WE}$  (i.e., when the segment  $[K(0), K(1)]$  is directed and oriented as the real line, as in Figure 12.4), it does not depend on  $\omega(S^2)$ , and we conclude as above.

Otherwise,  $S^2 \setminus (p_\tau(U^+K) \cup S(K))$  has two connected components  $A_1$  and  $A_2$  such that  $\partial \overline{A_1} = p_\tau(U^+K) \cup S(K) = -\partial \overline{A_2}$ . For  $j \in \underline{2}$ , let  $I_j = I(\hat{\xi}^K, \omega_j(S^2))$ , where  $\omega_j(S^2)$  is a volume-one form supported on  $A_j$ . Then  $I_2$  is a rational number since it is the intersection of a propagating chain with boundary  $p_\tau^{-1}(X)$  for  $X \in A_2$  with  $C(R(\mathcal{C}, K; \hat{\xi}^K))$  in  $C_2(R(\mathcal{C}))$ .<sup>4</sup> According to Lemma 12.5, for any volume-one form  $\omega(S^2)$  of  $S^2$ , we have

$$\begin{aligned} I(\hat{\xi}^K, \omega(S^2)) &= I_1 + \int_{A_1} (\omega(S^2) - \omega_1(S^2)) = I_1 + \int_{A_1} \omega(S^2) - 1 \\ &= I_2 + \int_{A_1} (\omega(S^2) - \omega_2(S^2)) = I_2 + \int_{A_1} \omega(S^2). \end{aligned}$$

The rational numbers  $I_1$  and  $I_2$  are not changed when  $(K, \tau)$  moves continuously so that the angle from the real positive half-line to  $\overrightarrow{K(0)K(1)}$  varies in  $]0, 2\pi[$  and the trivialization varies accordingly so that  $K$  remains straight. Therefore,  $I(\hat{\xi}^K, \omega_{S^2})$  varies like the area of  $A_1$ , which we compute now. Assume that the direction of  $[K(0), K(1)]$  coincides with the direction of  $\exp(2i\pi\theta)$  for  $\theta \in ]0, 1[$ .

If  $K$  goes from bottom to bottom, then  $A_1$  is the set of vectors  $v$  of  $S^2$  such that  $p_{\mathcal{C}}(v) = \lambda \exp(2i\pi\alpha)$  for some  $(\lambda, \alpha) \in ]0, +\infty[ \times ]0, \theta[$ , and the area of  $A_1$  is  $\theta$ . So we have  $I_\theta(K, \tau) = I_1 + I_2 + 2\theta - 1$ .

If  $K$  goes from top to top, then  $A_1$  is the set of vectors  $v$  of  $S^2$  such that  $p_{\mathcal{C}}(v) = \lambda \exp(2i\pi\alpha)$  for some  $(\lambda, \alpha) \in ]0, +\infty[ \times ]\theta, 1[$ , and the area of  $A_1$  is  $1 - \theta$ . So we have  $I_\theta(K, \tau) = I_1 + I_2 + 1 - 2\theta$ .

Let us treat the case  $\theta = \frac{1}{2}$  when  $K$  goes from bottom to bottom. With the notation of Definition 13.4, set  $\theta_1 = -\theta_0 = \frac{1}{2}$ . So  $\widehat{K}_{\parallel, \theta_0, \theta_1}$  looks as in Figure 13.4, and it is isotopic to the parallel  $\widehat{(-K)}_{\parallel}$ , which looks as in Figure 12.4. In this case, the interior of  $A_1$  contains the direction  $i$ . We can assume that  $\omega_1(S^2) = \omega_{0,0}$  and compute  $lk(K, K_{\parallel}) = lk(C, C_{\parallel})$  for a pair  $(C, C_{\parallel})$  of parallel closed curves that coincides with  $(K, \widehat{(-K)}_{\parallel})$  in a big neighborhood of  $\mathcal{C}$  and that lies in a vertical plane orthogonal to  $i$  outside that big neighborhood. We get

$$\begin{aligned} lk(K, K_{\parallel}) &= lk(C, C_{\parallel}) = \int_{K \times K \setminus \Delta} \omega_{0,0} = I(\hat{\xi}^K, \omega_{0,0}) + I(\hat{\xi}^K, \omega_{0,0}) \\ &= I(\hat{\xi}^K, \omega_{0,0}) + I(\hat{\xi}^K, -\iota^*(\omega_{0,0})) = I_1 + I_2 = I_\theta(K, \tau). \end{aligned}$$

---

<sup>4</sup>When  $\mathcal{C} = D_1 \times [0, 1]$  and  $\tau = \tau_s$ , the map  $p_\tau$  extends to  $C_2(R(\mathcal{C}))$  and  $I_2$  is the integral local degree of this extended  $p_\tau$  at a point of  $A_2$ .

So Proposition 16.13 holds in the case  $\theta = \frac{1}{2}$  when  $K$  goes from bottom to bottom. It similarly holds in the case  $\theta = \frac{1}{2}$  when  $K$  goes from top to top.

We now deduce the case in which  $\theta \in ]0, 1[$  from the case  $\theta = \frac{1}{2}$ . Recall that the rational numbers  $I_1$  and  $I_2$  are unchanged under isotopies that make  $\theta$  vary continuously in  $]0, 1[$ . Define  $\theta_1 = \theta$  and  $\theta_0 = \theta - 1$ , so that the arcs  $\alpha_0$  and  $\alpha_1$  of Definition 13.4 vary continuously as  $\theta$  varies from 0 to 1. Then the isotopy class of the pair  $(\widehat{K(\theta)}, \widehat{K(\theta)}_{\parallel, \theta_0, \theta_1})$  is unchanged. So when  $K$  goes from bottom to bottom, we have  $lk(K(\theta), K(\theta)_{\parallel}) = lk(K(\frac{1}{2}), K(\frac{1}{2})_{\parallel}) + 2\theta - 1$  and  $I_{\theta}(K(\theta), \tau) = I_{\theta}(K(\frac{1}{2}), \tau) + 2\theta - 1$ . When  $K$  goes from top to top,  $lk(K(\theta), K(\theta)_{\parallel}) - I_{\theta}(K, \tau)$  is similarly fixed when  $\theta$  varies. So Proposition 16.13 holds in any case.  $\square$

View the anomaly  $\beta$  of Section 10.2 as the map from  $P_{(3)}(\underline{3N})$  to  $\check{\mathcal{A}}(\mathbb{R})$  that sends any subset of  $\underline{3N}$  with cardinality  $3k$  to  $\beta_k$ .

With the notation of Definition 16.10, we get the following corollary of Theorem 16.9, Theorem 12.7, and Proposition 16.13.

**Theorem 16.16.** *Let  $L$  be a straight tangle with respect to a parallelization  $\tau$ . Let  $J_{bb} = J_{bb}(L)$  (resp.  $J_{tt} = J_{tt}(L)$ ) denote the set of components of  $L$  going from bottom to bottom (resp. from top to top). For  $K \in J_{bb} \cup J_{tt}$ , the orientation of  $K$  induced by  $\tau$  is the orientation of  $K$  such that  $p_{\tau}(U^+K) \subseteq S_{WE}$ . Under the assumptions of Theorem 16.9,*

$$\left( Z\left(\mathcal{C}, L, \tau, \cdot, (\tilde{\omega}(i, 1))_{i \in \underline{3N}}\right) \exp\left(-\frac{1}{4}p_1(\tau)\beta(\cdot)\right) \right)_{\sqcup}$$

*depends only on the  $\tilde{\omega}(i, 1, S^2)$ , on the boundary-fixing diffeomorphism class of  $(\mathcal{C}, L)$ , on the orientations of the components of  $J_{bb} \cup J_{tt}$  induced by  $\tau$ , and on the parallel  $L_{\parallel}$  of  $L$  induced by  $\tau$ .*

*It is denoted by  $\mathcal{Z}^f(\mathcal{C}, L, L_{\parallel}, \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}})$ . Set*

$$\mathcal{Z}_{\leq N}^f(\mathcal{C}, L, L_{\parallel}, \cdot) = \mathcal{Z}^f(\mathcal{C}, L, L_{\parallel}, \cdot, (\omega_{S^2})_{i \in \underline{3N}}).$$

*Then  $\mathcal{Z}_{\leq N}^f(\mathcal{C}, L, L_{\parallel}, \cdot)$  maps any subset of  $\underline{3N}$  of cardinality  $3k$  to the degree  $k$  part  $\mathcal{Z}_k^f(\mathcal{C}, L, L_{\parallel})$  of the invariant  $\mathcal{Z}^f(\mathcal{C}, L, L_{\parallel})$  of Definition 12.12. We will drop the subscript “ $\leq N$ ” from  $\mathcal{Z}_{\leq N}^f(\mathcal{C}, L, L_{\parallel}, \cdot)$ .*

**PROOF:** First note that  $\mathcal{Z}^f(\mathcal{C}, L, L_{\parallel}, \cdot, (\omega_{S^2})_{i \in \underline{3N}})$  maps any subset of  $\underline{3N}$  of cardinality  $3k$  to  $\mathcal{Z}_k^f(\mathcal{C}, L, L_{\parallel})$  as stated.

Let  $K$  be a component of  $L$ . If  $K$  is a knot, or a component going from bottom to top or from top to bottom, then the form  $\eta(A_K, p_{\tau}(U^+K))$  of Definition 16.6 vanishes. Since the components of  $J_{bb} \cup J_{tt}$  are equipped with the

orientation induced by  $\tau$ ,  $\eta(A_K, p_\tau(U^+K))$  is the same for all components  $K$  of  $J_{tt}$ , and it is independent of  $L$ . It is denoted by  $\eta(A_K, S_{WE})$ . Similarly, we have  $\eta(A_K, p_\tau(U^+K)) = -\eta(A_K, S_{WE})$  for all components  $K$  of  $J_{bb}$ . Therefore, the factor  $\left(\prod_{j \in I} \widetilde{\text{hol}}_{[0,1]}(\eta(A_{K_j}, p_\tau(U^+K_j))) \#_j\right)$  in Theorem 16.9 is equal to

$$\left(\prod_{K_j \in J_{bb}} \widetilde{\text{hol}}_{[0,1]}(-\eta(A_{K_j}, S_{WE})) \#_j\right) \left(\prod_{K_j \in J_{tt}} \widetilde{\text{hol}}_{[0,1]}(\eta(A_{K_j}, S_{WE})) \#_j\right),$$

and  $\mathcal{Z}^f(\mathcal{C}, L, L_\parallel, \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}})$  is determined by  $\mathcal{Z}^f(\mathcal{C}, L, L_\parallel)$  and by the holonomies, which depend only on the  $\tilde{\omega}(i, \cdot, S^2)$ .  $\square$

**Definition 16.17.** With the notation of Theorem 16.16 and under its assumptions, Theorems 16.9 and 12.7 together with Proposition 16.13 imply that

$$\left(\prod_{j=1}^k \left(\exp(-lk(K_j, K_{j\parallel})\alpha) \#_j\right) \mathcal{Z}^f(\mathcal{C}, L, L_\parallel, \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}})\right)_{\sqcup}$$

is independent of the framing of  $L$  with  $L = (K_j)_{j \in \underline{k}}$  and  $L_\parallel = (K_{j\parallel})_{j \in \underline{k}}$ . It is denoted by  $\mathcal{Z}(\mathcal{C}, L, \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}})$ . It a priori depends on the orientations of the components of  $J_{bb} \cup J_{tt}$ .

The data of an orientation for the components of  $J_{bb} \cup J_{tt}$  is called a  $J_{bb,tt}$ -orientation, and  $L$  is said to be  $J_{bb,tt}$ -oriented when it is equipped with such an orientation.

All the involved products are as in Definition 16.10, and  $lk(K_j, K_{j\parallel})\alpha$  is considered as a function of subsets of  $\underline{3N}$  with cardinality multiple of 3, which depends only on the degree. View the invariant of Theorem 12.7 as such a function  $\mathcal{Z}(\mathcal{C}, L)(\cdot)$ . Then, according to Theorem 16.9—applied when  $\tilde{\omega}(i, 0, S^2) = \omega_{S^2}$ , we get

$$\begin{aligned} &\mathcal{Z}\left(\mathcal{C}, L, \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}}\right) = \\ &\left(\left(\prod_{K_j \in J_{bb}} \widetilde{\text{hol}}_{[0,1]}(-\eta(\cdot, S_{WE})) \#_j\right) \left(\prod_{K_j \in J_{tt}} \widetilde{\text{hol}}_{[0,1]}(\eta(\cdot, S_{WE})) \#_j\right) \right)_{\sqcup} \\ &\quad \widetilde{\text{hol}}_{[1,0] \times y^-}(\eta_{B^-}, \cdot) \mathcal{Z}(\mathcal{C}, L)(\cdot) \widetilde{\text{hol}}_{[0,1] \times y^+}(\eta_{B^+}, \cdot) \end{aligned} \tag{16.1}$$

This allows us to extend the definition of  $\mathcal{Z}^f(\cdot, \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}})$  for  $J_{bb,tt}$ -oriented framed tangles that are not represented by straight tangles so that



the equality

$$\begin{aligned} & \mathcal{Z}^f(\mathcal{C}, L, L_{\parallel}, \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}}) \\ &= \left( \prod_{j=1}^k \left( \exp(\text{lk}(K_j, K_{j\parallel})\alpha) \#_j \right) \mathcal{Z}(\mathcal{C}, L, \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}}) \right)_{\sqcup} \end{aligned}$$

holds for all these  $J_{bb,tt}$ -oriented framed tangles, and we have

$$\begin{aligned} & \mathcal{Z}^f(\mathcal{C}, L, L_{\parallel}, \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}}) \\ &= \left( \left( \prod_{K_j \in J_{bb}} \widetilde{\text{hol}}_{[0,1]}(-\eta(\cdot, S_{WE})) \#_j \right) \left( \prod_{K_j \in J_{tt}} \widetilde{\text{hol}}_{[0,1]}(\eta(\cdot, S_{WE})) \#_j \right) \right. \\ & \quad \left. \widetilde{\text{hol}}_{[1,0] \times y^-}(\eta_{B^- \cdot}) \mathcal{Z}^f(\mathcal{C}, L, L_{\parallel})(\cdot) \widetilde{\text{hol}}_{[0,1] \times y^+}(\eta_{B^+ \cdot}) \right)_{\sqcup}. \end{aligned} \tag{16.2}$$

In order to compute  $\mathcal{Z}^f(\mathcal{C}, L, L_{\parallel}, \cdot, \cdot)$  from the discretizable definition of  $Z(\mathcal{C}, L, \tau, \cdot, \cdot)$  in Theorem 16.9, we first represent  $L$  as a straight tangle with another induced parallel  $L' = (K'_j)_{j \in \underline{k}}$  (but with the same  $J_{bb,tt}$ -orientation induced by the parallelization), and we correct by setting

$$\begin{aligned} & \mathcal{Z}^f(\mathcal{C}, L, L_{\parallel}, \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}}) \\ &= \left( \prod_{j=1}^k \left( \exp(\text{lk}(K_j, K_{j\parallel} - K'_j)\alpha) \#_j \right) \mathcal{Z}^f(\mathcal{C}, L, L', \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}}) \right)_{\sqcup}. \end{aligned}$$

**Remark 16.18.** Definition 16.17 is not canonical because of the arbitrary choice of  $S_{WE}$ . The defined invariant may not have the usual natural dependence on the component orientations (as in Proposition 10.23). Indeed, Definition 16.17 involves the  $J_{bb,tt}$ -orientation. So it is not symmetric under the reversal of a component orientation. See Remark 16.48 for further explanations.

**Lemma 16.19.** *Let  $(\omega(i))_{i \in \underline{3N}}$  denote a fixed family of propagating forms of  $(C_2(R(\mathcal{C})), \tau)$ . These propagators may be expressed as  $\tilde{\omega}(i, 1)$  for forms  $\tilde{\omega}(i)$  as in Theorem 16.9 (thanks to Lemma 9.1). Set*

$$Z(\mathcal{C}, L, \tau, A) = Z(\mathcal{C}, L, \tau, A)(1)$$

with the notation of Theorem 16.9. Let  $h_t$  be an isotopy of  $\check{R}(\mathcal{C})$  that is the identity on  $(\mathbb{C} \setminus D_1) \times \mathbb{R}$  for any  $t$  and that restricts to  $\mathbb{C} \times ]-\infty, 0]$  and  $\mathbb{C} \times [1, +\infty[$  as isotopies  $h_t^- \times \mathbf{1}_{]-\infty, 0]}$  and  $h_t^+ \times \mathbf{1}_{[1, +\infty[}$ , for planar isotopies  $h_t^-$

and  $h_t^+$ . Assume  $h_0 = \mathbf{1}$ . Let  $L$  be a long tangle of  $\check{R}(\mathcal{C})$  whose bottom (resp. top) configuration is represented by a map  $y^-: B^- \rightarrow D_1$  (resp.  $y^+: B^+ \rightarrow D_1$ ). Let  $(\tau_t)_{t \in [0,1]}$  be a smooth homotopy of parallelizations of  $\mathcal{C}$  such that  $p_{\tau_t}|_{U^+(L)}$  is constant with respect to  $t$ . With the notation of Definition 16.10, for  $A \in P_{(3)}(\underline{3N})$ , set  $Z(t, A) = Z(\mathcal{C}, h_t(L), \tau_t, A)$ . Then we have

$$Z(t, \cdot) = \left( \widetilde{\text{hol}}_{h_{[t,0]}^- \circ y^-}(\eta_{B^-, \cdot}) Z(0, \cdot) \widetilde{\text{hol}}_{h_{[0,t]}^+ \circ y^+}(\eta_{B^+, \cdot}) \right)_{\square}.$$

PROOF: The proof is similar to that of Lemma 15.17. □

Lemma 16.19 implies that  $\left( \widetilde{\text{hol}}_{h_{[t,0]}^- \circ y^-}(\eta_{B^-, \cdot}) \widetilde{\text{hol}}_{h_{[0,t]}^- \circ y^-}(\eta_{B^-, \cdot}) \right)_{\square}$  is neutral for the product of Definition 16.10.

The following proposition can be proved as Proposition 15.19.

**Proposition 16.20.** *With the notation and assumptions of Theorem 16.9, when*

$$\gamma: [0, 1] \rightarrow \check{C}_B[D_1]$$

is smooth with vanishing derivatives at 0 and 1, deform the standard parallelization of  $\mathbb{R}^3$  to a homotopic parallelization  $\tau$  such that  $T(\gamma)$  is straight with respect to  $\tau$  at any time of the homotopy. For any subset  $A$  of  $\underline{3N}$  with cardinality  $3k$ , we have

$$Z\left(\mathcal{C}_0 = D_1 \times [0, 1], T(\gamma), \tau, A, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}}\right) = \widetilde{\text{hol}}_{\{1\} \times p_{CS} \circ \gamma}(\eta_{B,A}),$$

where  $p_{CS}$  is the natural projection  $\check{C}_B[D_1] \rightarrow \mathcal{S}_B(\mathbb{C})$ . Thus we also have

$$\begin{aligned} Z\left(\gamma, A, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}}\right) &= Z^f\left(\mathcal{C}_0, T(\gamma), T(\gamma)_{\parallel}, A, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}}\right) \\ &= \widetilde{\text{hol}}_{\{1\} \times p_{CS} \circ \gamma}(\eta_{B,A}). \end{aligned}$$

Proposition 15.19 could be fully generalized to this setting, too, but we will prove more general functoriality properties in Section 17.2.

**Lemma 16.21.** *Let  $u \in [0, 1]$ . Let  $\gamma: [0, 1] \rightarrow \{u\} \times \check{\mathcal{S}}_B(\mathbb{C})$  be a smooth path. We have the following properties.*

- $\widetilde{\text{hol}}_{\gamma}(\eta_{B,A})$  depends only on  $\gamma(0)$ ,  $\gamma(1)$ , the  $\tilde{\omega}(i, S^2)$  for  $i \in A$ , and the homotopy class of  $\gamma$  relatively to  $\partial\gamma$  in  $\{u\} \times \check{\mathcal{S}}_B(\mathbb{C})$ .
- If  $(\bar{\gamma}: t \mapsto \gamma(1 - t))$  denotes the inverse of  $\gamma$  with respect to the path composition, then  $\left( \widetilde{\text{hol}}_{\gamma}(\eta_{B, \cdot}) \widetilde{\text{hol}}_{\bar{\gamma}}(\eta_{B, \cdot}) \right)_{\square}$  is neutral with respect to the product of Definition 16.10.

- Let  $w$ ,  $t$ , and  $\varepsilon$  be three elements of  $[0, 1[$ , such that  $[w, w + \varepsilon]$  and  $[t, t + \varepsilon]$  are subsets of  $[0, 1]$ . Let  $\ell$  be the boundary of the square  $[w, w + \varepsilon] \times p_{CS} \circ \gamma([t, t + \varepsilon])$  of  $[0, 1] \times \tilde{\mathcal{S}}_B(\mathbb{C})$ , then  $\widetilde{\text{hol}}_\ell(\eta_{B,\cdot})$  is trivial.

PROOF: The first assertion is a direct consequence of Proposition 16.20. It implies that for any  $u \in [0, 1]$ , for any homotopically trivial loop  $\ell$  of  $\{u\} \times \tilde{\mathcal{S}}_B(\mathbb{C})$ ,  $\widetilde{\text{hol}}_\ell(\eta_{B,\cdot})$  is neutral with respect to the product of Definition 16.10. Since  $\gamma\bar{\gamma}$  is such a loop, it implies the second assertion. The third assertion is a direct consequence of Lemma 16.19 and Proposition 16.20.  $\square$

## 16.4 Semi-algebraic structures on some configuration spaces

We would like to extend the definitions of our connections  $\eta$  of Sections 15.2 and 16.1 on  $\mathcal{S}_B(\mathbb{C})$ , in order to extend the definition of  $\mathcal{Z}^f$  to  $q$ -tangles. Unfortunately, I do not know whether the connections  $\eta$  extend as differentiable forms on  $\mathcal{S}_B(\mathbb{C})$ . However, we will be able to extend the definitions of their holonomies and prove that these holonomies along paths make sense (as  $\sqrt{t^{-1}}$  may be integrated on  $[0, 1]$  though  $\sqrt{t^{-1}}$  is not defined at 0). In order to do that, we will need to prove that integrals over singular spaces converge absolutely. Our proofs rely on the theory of semi-algebraic sets. We review the results of this theory that we will use below. Our primary reference is [BCR98, Section 1.4 and Chapter 2] by Jacek Bochnak, Michel Coste, and Marie-Françoise Roy.

**Definition 16.22.** [BCR98, Definition 2.1.4] A *semi-algebraic subset* of  $\mathbb{R}^n$  is a subset of the form

$$\bigcup_{i=1}^s \left( \bigcap_{j=1}^{r_i} \{x \in \mathbb{R}^n : f_{i,j}(x) < 0\} \cap \bigcap_{j=r_i+1}^{s_i} \{x \in \mathbb{R}^n : f_{i,j}(x) = 0\} \right)$$

for an integer  $s$ ,  $2s$  integers  $r_1, \dots, r_s, s_1, \dots, s_s$ , such that  $s_i \geq r_i$  for any  $i \in \underline{s}$ , and  $\sum_{i=1}^s s_i$  real polynomials  $f_{i,j}$  in the natural coordinates of  $x$ . A *semi-algebraic set* is a semi-algebraic subset of  $\mathbb{R}^n$  for some  $n \in \mathbb{N}$ .

The set of semi-algebraic subsets of  $\mathbb{R}^n$  is obviously stable under finite union, finite intersection, and taking complements. The set of semi-algebraic sets is stable under finite products.

**Theorem 16.23.** *Semi-algebraic sets also satisfy the following deeper properties, proved in [BCR98].*

[BCR98, Theorem 2.2.1] Let  $S$  be a semi-algebraic subset of  $\mathbb{R}^{n+1}$ . Let  $\Pi: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$  be the projection onto the space of the first  $n$  coordinates. Then  $\Pi(S)$  is a semi-algebraic subset of  $\mathbb{R}^n$ .

[BCR98, Proposition 2.2.2] The closure and the interior of a semi-algebraic set are semi-algebraic sets.

□

**Definition 16.24.** [BCR98, Definition 2.2.5] A map from a semi-algebraic subset of  $\mathbb{R}^n$  to a semi-algebraic subset of  $\mathbb{R}^m$  is *semi-algebraic* if its graph is semi-algebraic in  $\mathbb{R}^{n+m}$ .

The following proposition [BCR98, Proposition 2.2.7] can be deduced from Theorem 16.23 above as an exercise.

**Proposition 16.25.** Let  $f$  be a semi-algebraic map from a semi-algebraic set  $A$  to a semi-algebraic subset of  $\mathbb{R}^n$ . For any semi-algebraic subset  $S$  of  $A$ ,  $f(S)$  is semi-algebraic. For any semi-algebraic subset  $S$  of  $\mathbb{R}^n$ ,  $f^{-1}(S)$  is semi-algebraic. The composition of two composable semi-algebraic maps is semi-algebraic.

□

As an example, which will be useful very soon, we prove the following proposition.

**Proposition 16.26.** Let  $V$  denote a finite set of cardinality at least 2. Let  $T$  be a vector space of dimension  $\delta$ . The manifold  $\mathcal{S}_V(T)$  of Theorem 8.11 has a canonical structure of a semi-algebraic set. For any subset  $A$  of  $V$  with  $|A| \geq 2$ , the restriction map  $\mathcal{S}_V(T) \rightarrow \mathcal{S}_A(T)$  is semi-algebraic with respect to the canonical structures.

PROOF: The charts of Lemma 8.9 provide canonical semi-algebraic structures on  $\overline{\mathcal{S}}_V(T)$  and  $\check{\mathcal{S}}_V(T)$ . The restriction maps from  $\check{\mathcal{S}}_V(T)$  to  $\check{\mathcal{S}}_A(T)$  are semi-algebraic with respect to these structures. The description of  $\mathcal{S}_V(T)$  as the closure of the image of  $\check{\mathcal{S}}_V(T)$  in  $\prod_{A \in \mathcal{P}_{\geq 2}} \overline{\mathcal{S}}_A(T)$  of Lemma 8.39 makes clear that  $\mathcal{S}_A(T)$  has a natural semi-algebraic structure, thanks to Theorem 16.23.

□

We also have the following easy lemma.

**Lemma 16.27.** The space  $\check{\mathcal{V}}(\Gamma)$  of Chapter 15 and its compactification  $\mathcal{V}(\Gamma)$  carry natural structures of semi-algebraic sets. The projection  $p_{\mathcal{S}_B(\mathbb{C})}$  from  $\mathcal{V}(\Gamma)$  to  $\mathcal{S}_B(\mathbb{C})$  and its projections to the  $\mathcal{S}_e(\mathbb{R}^3)$  for ordered pairs  $e$  of  $V(\Gamma)$  are semi-algebraic maps with respect to these structures. For any configuration  $y \in \mathcal{S}_B(\mathbb{C})$ , the spaces  $\check{\mathcal{V}}(y, \Gamma)$  and  $\mathcal{V}(y, \Gamma)$  are semi-algebraic.

□

**Lemma 16.28.** *Let  $f: ]a, b[^d \rightarrow \mathbb{R}$  be a  $C^1$  semi-algebraic map. Then its partial derivatives  $\frac{\partial f}{\partial x_i}$  are semi-algebraic functions.*

PROOF: Let  $(e_1, \dots, e_d)$  be a basis of  $\mathbb{R}^d = \{\sum_{i=1}^d x_i e_i\}$ . As in [BCR98, Proposition 2.9.1], note that the set

$$\left\{ \left( t, x, f(x), (f(x + te_i) - f(x))/t \right) : t \in ]0, 1], x \in ]a, b[^d, x + te_i \in ]a, b[^d \right\}$$

is semi-algebraic. So are its closure, the locus ( $t = 0$ ) of this closure, and its projection to the graph of the partial derivative of  $f$  with respect to  $x_i$ . □

**Lemma 16.29.** *Let  $f$  be a semi-algebraic smooth map from an open hypercube  $]0, 1[^d$  to  $\mathbb{R}^n$ . Then the critical set of  $f$ , which is the subset of  $]0, 1[^d$  for which  $f$  is not a submersion, is semi-algebraic.*

PROOF: According to Lemma 16.28, the partial derivatives  $\frac{\partial p_j \circ f}{\partial x_i}$  with respect to the factors of  $\mathbb{R}^d$  of the  $p_j \circ f$  for the projections  $p_j$  on the factors of  $\mathbb{R}^n$  are semi-algebraic. It is easy to see that the product and the sum of two real-valued semi-algebraic maps are semi-algebraic. Being in the critical set may be written as “For any subset  $I$  of  $\underline{d}$  of cardinality  $n$ , the determinant  $\det \left[ \frac{\partial p_j \circ f}{\partial x_i}(x) \right]_{i \in I, j \in \underline{n}}$  is equal to zero.” □

An essential property of semi-algebraic sets, which we are going to use, is the following decomposition theorem [BCR98, Proposition 2.9.10].

**Theorem 16.30.** *Let  $S$  be a semi-algebraic subset of  $\mathbb{R}^n$ . Then, as a set,  $S$  is the disjoint union of finitely many smooth semi-algebraic submanifolds, each semi-algebraically diffeomorphic to an open hypercube  $]0, 1[^d$ .*

The *dimension* of a semi-algebraic set is the maximal dimension of a hypercube in a decomposition as above. It is proved in [BCR98, Section 2.8] that it does not depend on the decomposition. According to [BCR98, Theorem 2.8.8], the dimension of the image of a semi-algebraic set of dimension  $d$  under a semi-algebraic map is smaller than or equal to  $d$ . According to [BCR98, Proposition 2.8.13], if  $A$  is a semi-algebraic set of dimension  $\dim(A)$ , then we have  $\dim(\overline{A} \setminus A) < \dim(A)$ .

The following lemma is a corollary of Theorem 16.30.

**Lemma 16.31.** *Let  $f$  be a continuous semi-algebraic map from a compact semi-algebraic set  $A$  of dimension  $d$  to a semi-algebraic smooth manifold  $B$*

with boundary. Let  $\omega$  be a smooth differential form of degree  $d$  on  $B$ . Assume that the restriction of  $f$  to each piece of a decomposition as in Theorem 16.30 is smooth. Then the integrals  $\int_{\Delta} f^*(\omega)$  of  $f^*(\omega)$  over the open pieces  $\Delta$  of dimension  $d$  of such a decomposition converge absolutely, and  $\int_A f^*(\omega)$  is well defined to be the sum of these  $\int_{\Delta} f^*(\omega)$ .

PROOF: It suffices to prove the lemma when  $\omega$  is supported on a subset  $[-1, 1]^n$  of an open subset of  $B$  semi-algebraically diffeomorphic to  $B_{k,n} = ]-2, 1]^k \times ]-2, 2]^{n-k}$ . Indeed, using a partition of unity allows us to write  $\omega$  as a finite sum of such forms around the compact  $f(A)$ . This allows us to reduce the proof to the case  $B = B_{k,n}$ . Now a degree  $d$  differential form on  $B_{k,n}$  is a sum over the parts  $J$  of cardinality  $d$  of  $\underline{n}$  of pull-backs of degree  $d$  forms on  $B_J = B_{J,k,n} = ]-2, 1]^{k \cap J} \times ]-2, 2]^{(\underline{n} \setminus k) \cap J}$  multiplied by smooth functions on  $B_{k,n}$ , which are bounded on their compact supports. This allows us to reduce the proof to the case in which  $\omega$  is such a pull-back of a form  $\omega_J$  on  $B_J$ , under the projection  $p_J: B_{k,n} \rightarrow B_J$ , multiplied by a bounded function  $g_J$  on  $B_{k,n}$ .

Decompose  $A$  as in Theorem 16.30. It suffices to prove that the integral of  $\omega$  over each hypercube  $H$  of dimension  $d$  converges absolutely. Let  $f_J$  denote  $p_J \circ f$ . Consider the closure  $\overline{H} \subset A$  of the hypercube  $H$  in  $A$ , set  $\partial H = \overline{H} \setminus H$ . Then  $\partial H$  and its image  $f_J(\partial H)$  in  $B_J$  are algebraic subsets of  $B_J$  of dimension less than  $d$ , according to the dimension properties recalled before the lemma. Therefore, the form  $f_J^*(\omega_J)$  vanishes on the dimension  $d$  pieces of the intersection of  $H$  with the semi-algebraic compact set  $f_J^{-1}(f_J(\partial H))$ . Let  $\Sigma(f_J)$  be the set of critical points of  $f_J|_H$ . According to Lemma 16.29,  $\Sigma(f_J)$  is semi-algebraic. According to the Morse–Sard theorem 1.4,  $f_J(\Sigma(f_J))$ , which is semi-algebraic, is of zero measure. Therefore, its dimension is less than  $d$ . Now,  $B_J \setminus (f_J(\Sigma(f_J)) \cup \partial H) \cup \partial B_J$  is an open semi-algebraic subset of  $B_J$ , which therefore has a finite number of connected components according to Theorem 16.30. On each of these connected components, the local degree of  $f_J$  is finite because  $\overline{H}$  is compact and the points in the preimage of a regular value are isolated. Our assumptions make this local degree locally constant. Indeed, for a point  $y$  in such a component, there exists a small  $d$ -dimensional disk  $D(y)$  around  $y$  whose preimage contains disk neighborhoods of the points of the preimage, each mapped diffeomorphically to  $D(y)$ . The image of  $\overline{H}$  minus these open disks is a compact that does not meet  $y$ . Therefore, there is a smaller disk around  $y$  that is not met by this compact.

Then  $\int_{\overline{H}} f^*(\omega)$  is the integral of  $\omega_J$  weighted by this bounded local degree and by a multiplication by  $g_J \circ f$ . So it is absolutely convergent.  $\square$

Recall that an *open simplex* in  $\mathbb{R}^n$  is a subset of the form  $v_1 \dots v_k =$

$\{\sum_{i=1}^k t_i v_i : t_i \in ]0, 1], \sum_{i=1}^k t_i = 1\}$ , where  $v_1, \dots, v_k$  are affinely independent points in  $\mathbb{R}^n$ . The faces of  $v_1 \dots v_k$  are the simplices  $v_{i_1} \dots v_{i_j}$  for subsets  $\{i_1, \dots, i_j\} \subset \underline{k}$ . A *locally finite simplicial complex* in  $\mathbb{R}^n$  is a locally finite collection  $K$  of disjoint open simplices such that each face of a simplex of  $K$  belongs to  $K$ . For such a complex  $K$ ,  $|K|$  denotes the union of the simplices of  $K$ .

The following Lojasiewicz triangulation theorem [Loj64, Theorem 1, p. 463, §3] ensures that a compact semi-algebraic set may be viewed as a topological chain (as in Subsection 2.1.5).

**Theorem 16.32.** *For any locally finite collection  $\{B_i\}$  of semi-algebraic subsets of  $\mathbb{R}^n$ , there exist a locally finite simplicial complex  $K$  of  $\mathbb{R}^n$  such that  $|K| = \mathbb{R}^n$  and a homeomorphism  $\tau$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  such that*

- *for any open simplex  $\sigma$  of  $K$ ,  $\tau(\sigma)$  is an analytic submanifold of  $\mathbb{R}^n$ , and  $\tau|_\sigma$  is an analytic isomorphism from  $\sigma$  to  $\tau(\sigma)$ ,*
- *for any open simplex  $\sigma$  of  $K$  and any  $B_i$  of the collection  $\{B_i\}$ , we have  $\tau(\sigma) \subset B_i$  or  $\tau(\sigma) \subset \mathbb{R}^n \setminus B_i$ .*

## 16.5 Extending $\mathcal{Z}^f$ to $q$ -tangles

In Chapter 15, the behavior of  $\mathcal{Z}$  on braids, which are paths in  $\check{\mathcal{S}}_B(\mathbb{C})$  for some finite set  $B$  was discussed. Recall that  $\mathcal{Z}$  and  $\mathcal{Z}^f$  coincide for braids. In this section, we extend  $\mathcal{Z}^f$  and its variants of Section 16.2 to paths of  $\mathcal{S}_B(\mathbb{C})$ , where  $B$  is a finite set. This is already mostly done in [Poi00], where the main ideas come from. However, our presentation is different, and it provides additional statements and explanations.

Our extension to paths of  $\mathcal{S}_B(\mathbb{C})$  will allow us to define the extension of  $\mathcal{Z}^f$  to  $q$ -tangles in rational homology cylinders so that Proposition 15.19 is still valid in the setting of  $q$ -tangles.

Recall the semi-algebraic subsets  $\mathcal{V}(\Gamma)$  and  $\mathcal{V}(y, \Gamma)$  of  $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3) \times \mathcal{S}_B(\mathbb{C})$ , introduced in Chapter 15, for a  $3N$ -numbered Jacobi diagram  $\Gamma$  on  $\sqcup_{b \in B} \mathbb{R}_b$ . Both  $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$  and  $\mathcal{S}_B(\mathbb{C})$  are stratified by  $\Delta$ -parenthesizations according to Theorem 8.28. Let  $\mathcal{P}_B$  be a  $\Delta$ -parenthesization of  $B$ . Let  $\mathcal{P}$  be a  $\Delta$ -parenthesization of  $V(\Gamma)$ . Set

$$\mathcal{V}_{\mathcal{P}_B, \mathcal{P}}(\Gamma) = \mathcal{V}(\Gamma) \cap (\mathcal{S}_{V(\Gamma), \mathcal{P}}(\mathbb{R}^3) \times \mathcal{S}_{B, \mathcal{P}_B}(\mathbb{C})).$$

An element of  $\mathcal{S}_{V(\Gamma), \mathcal{P}}(\mathbb{R}^3)$  is denoted by  $(c_Y)_{Y \in \mathcal{P}}$ , where  $c_Y \in \check{\mathcal{S}}_{K(Y)}(\mathbb{R}^3)$ . An element of  $\mathcal{S}_{B, \mathcal{P}_B}(\mathbb{C})$  is denoted by  $(y_D)_{D \in \mathcal{P}_B}$ , where  $y_D \in \check{\mathcal{S}}_{K(D)}(\mathbb{C})$ . Fix  $\Gamma$  and  $\mathcal{P}_B$ . Recall the natural map  $p_B: U(\Gamma) \rightarrow B$  induced by  $i_\Gamma$ . Let

$c = ((c_Y)_{Y \in \mathcal{P}}, (y_D)_{D \in \mathcal{P}_B}) \in \mathcal{V}_{\mathcal{P}_B, \mathcal{P}}(\Gamma)$ , and let  $Y$  be in the set  $\widehat{\mathcal{P}}'_X$  introduced in Notation 15.2. Then we have

$$p_{\mathbb{C}} \circ c_Y|_{U(\Gamma) \cap Y} = \lambda(Y) \left( y_{\hat{B}(Y)} \circ p_B - y_{\hat{B}(Y)} \circ p_B(b(Y)) \right)$$

for some  $\lambda(Y) \geq 0$ , with the normalizations of Notation 15.2. Recall the set  $\mathcal{P}'_x(c)$  of separating sets  $Y$  for  $c$  from Lemma 15.12. Such a separating set is an element  $Y$  of  $\widehat{\mathcal{P}}'_X$  with  $\lambda(Y) \neq 0$ . For a subset  $\mathcal{P}'_x$  of  $\widehat{\mathcal{P}}'_X$ , set  $\mathcal{V}_{\mathcal{P}_B, \mathcal{P}, \mathcal{P}'_x}(\Gamma) = \{c \in \mathcal{V}_{\mathcal{P}_B, \mathcal{P}}(\Gamma) : \mathcal{P}'_x(c) = \mathcal{P}'_x\}$ . We use the data  $(\mathcal{P}_B, \mathcal{P}, \mathcal{P}'_x)$  to stratify  $\mathcal{V}(\Gamma)$  (or  $[0, 1] \times \mathcal{V}(\Gamma)$ , whose strata will be the products by  $[0, 1]$  of the strata of  $\mathcal{V}(\Gamma)$  by definition). Recall that for any  $D \in \mathcal{P}_B$ , the elements  $Y$  of  $\mathcal{P}'_x$  with  $\hat{B}(Y) = D$  are minimal with respect to the inclusion among the elements of  $\mathcal{P}$  with  $\hat{B}(Y) = D$ .

For  $y \in \mathcal{S}_{B, \mathcal{P}_B}(\mathbb{C})$ , for a  $\Delta$ -parenthesization  $\mathcal{P}$  of  $V(\Gamma)$ , and for a subset  $\mathcal{P}'_x$  of  $\mathcal{P}$ , set

$$\mathcal{V}(y, \Gamma, \mathcal{P}, \mathcal{P}'_x) = \mathcal{V}(y, \Gamma) \cap \mathcal{V}_{\mathcal{P}_B, \mathcal{P}, \mathcal{P}'_x}(\Gamma).$$

Recall from Lemma 15.12 that when  $\mathcal{V}(y, \Gamma, \mathcal{P}, \mathcal{P}'_x)$  is not empty, its dimension is  $|U(\Gamma)| + 3|T(\Gamma)| - 1 - |\mathcal{P} \setminus \mathcal{P}'_x|$ . In particular, the dimension of  $\mathcal{V}(y, \Gamma)$  is at most  $2|E(\Gamma)| - 1$ .

Fix the family  $(\tilde{\omega}(i, S^2) = (\tilde{\omega}(i, t, S^2))_{t \in [0, 1]})_{i \in \underline{3N}}$  of closed 2-forms on  $[0, 1] \times S^2$  once for all in this section, and assume that  $\tilde{\omega}(i, 0, S^2) = \omega_{S^2}$  for all  $i$ . For an edge  $e$  of  $\Gamma$ , recall the map

$$p_{e, S^2}: [0, 1] \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3) \rightarrow [0, 1] \times S^2,$$

sending  $(t, c \in \check{\mathcal{S}}_{V(\Gamma)}(\mathbb{R}^3))$  to  $(t, p_{S^2}((c(v(e, 1)), c(v(e, 2))))$ ). Also recall the  $(2|E(\Gamma)|)$ -form

$$\Omega_{\Gamma} = \bigwedge_{e \in E(\Gamma)} p_{e, S^2}^* \left( \tilde{\omega}(j_E(e), S^2) \right)$$

over  $[0, 1] \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$ . This form pulls back to provide smooth forms on the smooth strata of  $[0, 1] \times \mathcal{V}(\Gamma)$ .

Let  $A$  denote a subset of  $\underline{3N}$  with cardinality  $3n$ . An ordered  $r$ -component  $A$ -numbered Jacobi diagram  $\Gamma^{(r)}$  on  $\sqcup_{b \in B} \mathbb{R}_b$  is a degree  $n$   $A$ -numbered Jacobi diagram  $\Gamma^{(r)}$  on  $\sqcup_{b \in B} \mathbb{R}_b$  that has  $r$  connected components  $\Gamma_1, \dots, \Gamma_r$  and such that  $i_{\Gamma}$  is represented by an injection of  $V(\Gamma)$  that maps all univalent vertices of  $\Gamma_i$  before (or below) the univalent vertices of  $\Gamma_{i+1}$  for any  $i \in \underline{r-1}$ . The data of such an ordered  $r$ -component  $A$ -numbered Jacobi diagram  $\Gamma^{(r)}$  is equivalent to the data of an  $r$ -tuple  $(\Gamma_1, \dots, \Gamma_r)$  of  $A$ -numbered connected Jacobi diagrams with pairwise disjoint  $j_E(E(\Gamma_i))$  such that the sum of the



degrees of the  $\Gamma_i$  is  $n$ . Let  $\mathcal{D}_{n,A}^{e,r}(\sqcup_{b \in B} \mathbb{R}_b)$  denote the set of these ordered  $r$ -component  $A$ -numbered Jacobi diagrams  $\Gamma^{(r)}$  on  $\sqcup_{b \in B} \mathbb{R}_b$ .<sup>5</sup>

Such a diagram provides the  $(2|E(\Gamma^{(r)})|)$ -form

$$\Omega_{\Gamma^{(r)}} = \bigwedge_{i=1}^r P_i^*(\Omega_{\Gamma_i})$$

on (the smooth strata of)  $\prod_{i=1}^r ([0, 1] \times \mathcal{V}(\Gamma_i))$ , where

$$P_i: \prod_{i=1}^r ([0, 1] \times \mathcal{V}(\Gamma_i)) \rightarrow [0, 1] \times \mathcal{V}(\Gamma_i)$$

is the projection onto the  $i$ th factor. This form is also the pullback of a smooth form on  $[0, 1]^r \times (S^2)^{E(\Gamma^{(r)})}$ , by a semi-algebraic map. Recall  $\Delta^{(r)} = \{(t_1, \dots, t_r) \in [0, 1]^r : 0 \leq t_1 \leq t_2 \leq \dots \leq t_r\}$ . A semi-algebraic path  $\gamma: [0, 1] \rightarrow [0, 1] \times \mathcal{S}_B(\mathbb{C})$  induces the semi-algebraic map

$$\begin{aligned} \gamma^{(r)}: \Delta^{(r)} &\rightarrow ([0, 1] \times \mathcal{S}_B(\mathbb{C}))^r \\ (t_1, \dots, t_r) &\mapsto (\gamma(t_1), \dots, \gamma(t_r)). \end{aligned}$$

Consider the product  $P_{\Gamma^{(r)}}: \prod_{i=1}^r ([0, 1] \times \mathcal{V}(\Gamma_i)) \rightarrow ([0, 1] \times \mathcal{S}_B(\mathbb{C}))^r$  of natural projections. Assume that  $\gamma$  is injective. Set  $C(\Gamma^{(r)}, \gamma) = P_{\Gamma^{(r)}}^{-1}(\gamma^{(r)}(\Delta^{(r)}))$ . Then  $C(\Gamma^{(r)}, \gamma)$  is a semi-algebraic subset of  $\prod_{i=1}^r ([0, 1] \times \mathcal{V}(\Gamma_i))$  of dimension at most  $2|E(\Gamma^{(r)})|$  whose  $2|E(\Gamma^{(r)})|$ -dimensional strata are oriented canonically, as soon as the Jacobi diagrams  $\Gamma_i$  are: Fix an arbitrary vertex-orientation for the  $\Gamma_i$ . The set  $C(\Gamma^{(r)}, \gamma)$  is locally oriented as the product of the  $C(\Gamma_i^{(1)}, \gamma)$  for  $i \in \underline{r}$ . The parameter  $t_i$  replaces the translation parameter in  $\mathcal{V}(\gamma(t_i), \Gamma_i)$ .

Define the  $A$ -holonomy  $\widetilde{\text{hol}}(\eta_{B,A})$  along injective semi-algebraic paths  $\gamma$  of  $[0, 1] \times \mathcal{S}_B(\mathbb{C})$ , with respect to our family  $(\widetilde{\omega}(i, S^2))_{i \in \underline{3N}}$ , to be

$$\widetilde{\text{hol}}_\gamma(\eta_{B,A}) = [\emptyset] + \sum_{r=1}^{\infty} \sum_{\Gamma^{(r)} \in \mathcal{D}_{n,A}^{e,r}(\sqcup_{b \in B} \mathbb{R}_b)} \zeta_{\Gamma^{(r)}} \int_{C(\Gamma^{(r)}, \gamma)} \Omega_{\Gamma^{(r)}}[\Gamma^{(r)}],$$

with

$$\zeta_{\Gamma^{(r)}} = \frac{(|A| - |E(\Gamma^{(r)})|)!}{|A|! 2^{|E(\Gamma^{(r)})|}}.$$

---

<sup>5</sup>The notation  $\mathcal{D}_{n,A}^{e,r}$  has a redundancy since the cardinality of  $A$  is  $3n$ . We keep the redundancy for consistency here because we will use other spaces of numbered Jacobi diagrams, where the degree is not determined by the cardinality of the set of indices, in Chapter 17.

(Again, we fix an arbitrary vertex-orientation for the components  $\Gamma_i$  of each  $\Gamma^{(r)}$ , and  $\widetilde{\text{hol}}_\gamma(\eta_{B,A})$  is independent of our choices.)

The involved integrals make sense as soon as  $\gamma$  is semi-algebraic, thanks to Lemma 16.31, which also justifies the following lemma.

**Lemma 16.33.** *For any injective semi-algebraic path  $\gamma$  of  $[0, 1] \times \mathcal{S}_B(\mathbb{C})$ ,*

$$\lim_{\varepsilon \rightarrow 0} \widetilde{\text{hol}}_{\gamma|_{[\varepsilon, 1-\varepsilon]}}(\eta_{B,A})$$

*makes sense, and it is equal to  $\widetilde{\text{hol}}_\gamma(\eta_{B,A})$ .*

□

Together with the identification  $\mathcal{Z}(T(\gamma)) = \text{hol}_{p_{CS} \circ \gamma}(\eta_B)$  for braids provided by Proposition 15.19, Lemma 16.33 implies the convergence part of Theorem 13.8. The above convergent integrals extend Definition 16.1 of  $\widetilde{\text{hol}}_\gamma(\eta_{B,A})$  for injective semi-algebraic paths in  $[0, 1] \times \check{\mathcal{S}}_B(\mathbb{C})$ . Note the following easy lemma.

**Lemma 16.34.** *The  $A$ -holonomy  $\widetilde{\text{hol}}_\gamma(\eta_{B,\cdot})$ , which is valued in  $\mathcal{A}_n(\sqcup_{b \in B} \mathbb{R}_b)$ , extends naturally to noninjective semi-algebraic paths. This holonomy is multiplicative under path composition with respect to the product of Definition 16.10.*

□

Recall that  $\tilde{\omega}(i, 0, S^2)$  is the standard homogeneous volume-one form on  $S^2$ . When  $\gamma$  is valued in  $\{0\} \times \mathcal{S}_B(\mathbb{C})$ , there is no need to number the diagram edges since  $\tilde{\omega}(i, 0, S^2) = \omega_{S^2}$  for all  $i$ , and we simply have

$$\widetilde{\text{hol}}_\gamma(\eta_B) = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\Gamma^{(r)} \in \mathcal{D}_{k, \underline{3k}}^{\varepsilon, r}(\sqcup_{b \in B} \mathbb{R}_b)} \zeta_{\Gamma^{(r)}} \int_{C(\Gamma^{(r)}, \gamma)} \Omega_{\Gamma^{(r)}}[\Gamma^{(r)}] \in \mathcal{A}(\sqcup_{b \in B} \mathbb{R}_b).$$

In this case,  $\widetilde{\text{hol}}_\gamma(\eta_B)$  is nothing but the Poirier functor  $Z^l$  of [Poi00, Section 1.4] applied to  $\gamma$ . The projection in  $\mathcal{A}_n(\sqcup_{b \in B} \mathbb{R}_b)$  of  $\widetilde{\text{hol}}_\gamma(\eta_B)$ , which coincides with the holonomy  $\text{hol}_\gamma(\eta_B)$  defined in Section 15.2 when  $\gamma$  is valued in  $\{0\} \times \check{\mathcal{S}}_B(\mathbb{C})$ , coincides with  $\widetilde{\text{hol}}_\gamma(\eta_{B,A})$ , when  $|A| = 3n$ , in this case of homogeneous forms.

Recall that  $\mathcal{S}_B(\mathbb{C})$  is a smooth manifold with ridges, which can also be equipped with a semi-algebraic structure for which the local charts provided in Theorem 8.28 are semi-algebraic maps. In such a trivialized open simply connected subspace, any two points can be connected by a semi-algebraic

path. In particular, any two points  $a$  and  $b$  of  $\mathcal{S}_B(\mathbb{C})$  are connected by a semi-algebraic path  $\gamma: [0, 1] \rightarrow \mathcal{S}_B(\mathbb{C})$  such that  $\gamma(0) = a$  and  $\gamma(1) = b$ . Furthermore, any path from  $a$  to  $b$  of  $\mathcal{S}_B(\mathbb{C})$  can be  $C^0$ -approximated by a homotopic semi-algebraic path. So, any homotopy class of paths from  $a$  to  $b$  has a semi-algebraic representative.

Now Theorem 13.8 is a direct corollary of the following one, which will be proved after Lemma 16.41. This theorem is a mild generalization of [Poi00, Proposition 9.2], thanks to Lemma 16.33.

**Theorem 16.35.** *Let  $\gamma: [0, 1] \rightarrow [0, 1] \times \mathcal{S}_B(\mathbb{C})$  be a semi-algebraic path. Then  $\widetilde{\text{hol}}_\gamma(\eta_{B,A})$  depends only on  $\gamma(0)$ ,  $\gamma(1)$ , the  $\tilde{\omega}(i, S^2)$  for  $i \in A$ , and the homotopy class of  $\gamma$  relatively to  $\partial\gamma$ .*

According to Lemmas 16.21 and 16.34, Theorem 16.35 holds for smooth paths  $\gamma$  of  $\{u\} \times \check{\mathcal{S}}_B(\mathbb{C})$  and their piecewise smooth compositions. We now prove the following other particular case of Theorem 16.35.

**Lemma 16.36.** *Let  $\gamma: [0, 1]^2 \rightarrow [0, 1] \times \mathcal{S}_B(\mathbb{C})$  be a semi-algebraic homotopy such that*

- $\gamma$  is injective on  $[0, 1] \times ]0, 1[$ ,
- $\gamma_u(t) = \gamma(u, t)$  belongs to  $[0, 1] \times \check{\mathcal{S}}_B(\mathbb{C})$  for any  $(u, t) \in ]0, 1[ \times ]0, 1[$ ,
- we have  $\gamma_u(0) = \gamma_0(0)$  and  $\gamma_u(1) = \gamma_0(1)$  for all  $u \in [0, 1]$ ,
- $\gamma_0(t)$  is in a fixed stratum of  $[0, 1] \times \mathcal{S}_B(\mathbb{C})$  for  $t \in ]0, 1[$ , where a stratum of  $[0, 1] \times \mathcal{S}_B(\mathbb{C})$  is the product by  $[0, 1]$  of a stratum of  $\mathcal{S}_B(\mathbb{C})$  associated to a  $\Delta$ -parenthesization.

Then we have

$$\widetilde{\text{hol}}_{\gamma_0}(\eta_{B,A}) = \widetilde{\text{hol}}_{\gamma_1}(\eta_{B,A}).$$

Lemma 16.36 is the direct consequence of Lemmas 16.39 and 16.40 below. The proof of Lemma 16.39 uses Lemma 16.38 and the following sublemma.

**Sublemma 16.37.** *Under the assumptions of Lemma 16.36, let  $\Gamma^{(r)}$  be an element of  $\mathcal{D}_{n,A}^{e,r}(\sqcup_{b \in B} \mathbb{R}_b)$ , recall  $C(\Gamma^{(r)}, \gamma_u) = P_{\Gamma^{(r)}}^{-1}(\gamma_u^{(r)}(\Delta^{(r)}))$ , and let*

$$C(\Gamma^{(r)}, (\gamma_u)) = \cup_{u \in [0,1]} C(\Gamma^{(r)}, \gamma_u)$$

be the associated semi-algebraic set of dimension  $2|E(\Gamma^{(r)})| + 1$ . Then the codimension-one boundary of  $C(\Gamma^{(r)}, (\gamma_u))$  is

$$C(\Gamma^{(r)}, \gamma_1) - C(\Gamma^{(r)}, \gamma_0) - \cup_{u \in [0,1]} \partial C(\Gamma^{(r)}, \gamma_u),$$

with

$$\begin{aligned} \partial C(\Gamma^{(r)}, \gamma_u) &= \partial_C C(\Gamma^{(r)}, \gamma_u) + \partial_\Delta C(\Gamma^{(r)}, \gamma_u), \\ \partial_C C(\Gamma^{(r)}, \gamma_u) &= \pm \cup_{(t_1, \dots, t_r) \in \Delta^{(r)}} \partial P_{\Gamma^{(r)}}^{-1} \left( \gamma_u^{(r)}((t_1, \dots, t_r)) \right), \end{aligned}$$

and

$$\partial_\Delta C(\Gamma^{(r)}, \gamma_u) = \pm P_{\Gamma^{(r)}}^{-1} \left( \gamma_u^{(r)}(\partial \Delta^{(r)}) \right).$$

PROOF: When the image of  $(\gamma_u)$  is in  $[0, 1] \times \check{\mathcal{S}}_B(\mathbb{C})$ , it follows from Lemmas 15.13 and 14.14. Let us prove that it is still true for our homotopies  $(\gamma_u)$ . The part  $C(\Gamma^{(r)}, \gamma_1)$  is in the boundary as before. We can ignore the contributions of the extremities of  $\gamma_1$  since they belong to parts of dimension at most  $2|E(\Gamma^{(r)})| - 1$ , thanks to Lemma 15.12. For the part coming from  $\cup_u \partial C(\Gamma^{(r)}, \gamma_u)$  in the  $2|E(\Gamma)|$ -dimensional boundary, we may restrict to  $u \in ]0, 1[$  for dimension reasons, which we do. So this part is in the boundary as before, too

The part over  $\gamma_0$  of the codimension-one boundary of  $C(\Gamma^{(r)}, (\gamma_u))$  is included in  $C(\Gamma^{(r)}, \gamma_0)$ . Let us prove that the corresponding algebraic boundary is indeed  $-C(\Gamma^{(r)}, \gamma_0)$  when  $\gamma_0(]0, 1[)$  is in some stratum of  $[0, 1] \times \partial \mathcal{S}_B(\mathbb{C})$ , associated to a parenthesization  $\mathcal{P}_B$  of  $B$ .

Let  $t_i \in ]0, 1[$ . In a neighborhood  $[0, \eta[ \times N(t_i)$  of  $(0, t_i)$  in  $[0, 1]^2$ ,  $\gamma_u(t) = \gamma(u, t)$  is expressed as

$$\gamma_u(t) = \left( (y_D(u, t))_{D \in \mathcal{P}_B}, (u_D(u, t))_{D \in \mathcal{P}_B \setminus \{B\}} \right),$$

where we have  $u_D(0, t) = 0$  for all  $t \in N(t_i)$ .

Let  $c^0$  be a point of the  $(2|E(\Gamma_i)| - 1)$ -dimensional open part of  $\mathcal{V}(\gamma_0(t_i), \Gamma_i)$  with  $\gamma'_0(t_i) \neq 0$ . According to Lemma 15.12,  $\mathcal{P}(c^0)$  equals  $\mathcal{P}'_x(c^0)$ . We express a neighborhood of  $c^0$  in  $\cup_{(u,t) \in [0, \eta[ \times N(t_i)} \mathcal{V}(\gamma_u(t), \Gamma_i)$ , as a product by  $[0, \eta[ \times N(t_i)$ , as follows. We use the above coordinates  $y_D(u, t)$ ,  $u_D(u, t)$  of the base, and the parameters  $c_Z$ ,  $p_{\mathbb{R}} \circ c_Z(Y)$ ,  $\lambda(Y)$  listed in the fourth, fifth and sixth sets of variables of Lemma 15.6. We have  $\lambda(V(\Gamma))(c^0) = \lambda(c^0)$  and  $\lambda(c^0) \neq 0$  since we have  $\mathcal{P} = \mathcal{P}'_x$ . Let  $Y$  be an element of  $\mathcal{P} \setminus \{V(\Gamma)\}$ . We similarly have  $\lambda(Y)(c^0) \neq 0$ . Let  $Y^+$  be the smallest set of  $\mathcal{P}$  that strictly contains  $Y$ . Set  $Y' = Y^+$ . Set  $B_1 = \hat{B}(Y)$  and  $B'_1 = B_1^+ = \hat{B}(Y^+)$ . Equation 15.2 in Lemma 15.6 applied to these sets when  $n = 1$  may be written as  $\lambda(Y)\mu_Y = \lambda(Y^+)u_{\hat{B}(Y)}$ . It implies that  $\mu_Y = (\lambda(Y^+)/\lambda(Y))u_{\hat{B}(Y)}$  is determined by the listed parameters, and that all  $\mu_Y$  are zero over  $\gamma_0(N(t_i))$  in our neighborhood of  $c^0$ . So we have  $\mathcal{P} = \mathcal{P}'_x = \mathcal{P}(c^0)$  over  $\gamma_0(N(t_i))$  in our neighborhood of  $c^0$ . In particular, if  $Y \subsetneq Y'$ , then  $\hat{B}(Y) \subsetneq \hat{B}(Y')$ , and all equations 15.2 are satisfied, there. Over  $\gamma(]0, \eta[ \times N(t_i))$ , no  $u_D$  vanishes so that no  $\mu_Y$  vanishes either in our neighborhood of  $c^0$ , and the

equations 15.2 are implied by the equations  $*(Y)$ , which are implied by the equations  $\mu_Y = (\lambda(Y^+)/\lambda(Y))u_{\hat{B}(Y)}$  and  $\lambda(V(\Gamma)) = \lambda$ . So we get a neighborhood of  $c^0$  in  $\cup_{(u,t) \in [0,\eta] \times N(t_i)} \mathcal{V}(\gamma_u(t), \Gamma_i)$  parametrized by  $(u, t)$  and by the parameters  $c_Z, p_{\mathbb{R}} \circ c_Z(Y), \lambda(Y)$  in the fourth, fifth and sixth lists of variables of Lemma 15.6. For any horizontally normalizing kid  $Y$  of  $\mathcal{P}$ , remove  $\lambda(Y) = \lambda^0(Y)$  from the parameters. Use the first and second constraints of Lemma 15.6 to remove other superfluous parameters  $p_{\mathbb{R}} \circ c_Z(Y), |p_{\mathbb{C}} \circ c_Z(Y)|$  and get a free system of parameters. Thus, we obtain an open  $(2|E(\Gamma_i)|)$ -dimensional neighborhood  $\mathcal{O}$  of  $c^0$  in  $\cup_{t \in N(t_i)} \mathcal{V}(\gamma_0(t), \Gamma_i)$  and a local open embedding of the product of  $[0, \eta] \times \mathcal{O}$  into  $\cup_{(u,t) \in [0,\eta] \times N(t_i)} \mathcal{V}(\gamma_u(t), \Gamma_i)$ . So  $(-C(\Gamma^{(r)}, \gamma_0))$  is the algebraic boundary of  $C(\Gamma^{(r)}, (\gamma_u))$  over  $\gamma_0$ .  $\square$

**Lemma 16.38.** *For a Jacobi diagram in  $\mathcal{D}_{n,A}^e(\sqcup_{b \in B} \mathbb{R}_b)$ , for an element  $(t, y)$  of  $[0, 1] \times \check{\mathcal{S}}_B(\mathbb{C})$ ,  $d\eta_{\Gamma}(t, y)$  is the integral of  $(-\bigwedge_{e \in E(\Gamma)} p_{e,S^2}^*(\tilde{\omega}(j_E(e), S^2)))$  along the interiors of the codimension-one faces of  $\{t\} \times \mathcal{V}(y, \Gamma)$ .*

PROOF: See the proof of Lemma 15.23.  $\square$

Recall from the beginning of Section 15.2 that the fiber  $\mathcal{V}(y, \Gamma)$  is oriented so that the orientation of  $\mathcal{V}(y, \Gamma)$  preceded by the upward translation parameter—which replaces the parametrization of the paths along which we integrate—matches the usual orientation of our configuration spaces, induced as in Lemma 7.1.

For a subset  $A_i$  of  $A$ , we have

$$d\eta_{B,A_i} = \sum_{\Gamma_i \in \mathcal{D}_{n_i,A_i}^e(\sqcup_{b \in B} \mathbb{R}_b)} \zeta_{\Gamma_i} d\eta_{\Gamma_i} [\Gamma_i],$$

and we set

$$(\eta \wedge \eta)_{B,A_i} = \sum_{(\Gamma, \Gamma') \in \mathcal{D}_{n_i,A_i}^{e,2}(\sqcup_{b \in B} \mathbb{R}_b)} \zeta_{\Gamma \sqcup \Gamma'} \eta_{\Gamma} \wedge \eta_{\Gamma'} [\Gamma] [\Gamma'].$$

**Lemma 16.39.** *Let  $(\gamma_u(t))_{u \in [0,1]}$  be a semi-algebraic homotopy satisfying the assumptions of Lemma 16.36 or a smooth homotopy valued in  $[0, 1] \times \check{\mathcal{S}}_B(\mathbb{C})$ . Assume that  $\gamma_u(0)$  and  $\gamma_u(1)$  do not depend on  $u$ .*

*Then we have*

$$\begin{aligned} & \widetilde{\text{hol}}_{\gamma_1}(\eta_{B,A}) - \widetilde{\text{hol}}_{\gamma_0}(\eta_{B,A}) \\ &= \sum_{r=0}^{\infty} \sum_{(A_1, \dots, A_r) \in P_r(A)} \frac{\prod_{i=1}^r |A_i|!}{|A|!} \int_{[0,1] \times \Delta^{(r)}} \sum_{i=1}^r \delta(i, A_1, \dots, A_r), \end{aligned}$$

with

$$\delta(i, A_1, \dots, A_r) = (-1)^{i-1} \bigwedge_{j=1}^r (\gamma \circ p_j)^*(\eta_{B,A_j}) \left( \frac{(\gamma \circ p_i)^*(d\eta_{B,A_i} + (\eta \wedge \eta)_{B,A_i})}{(\gamma \circ p_i)^*(\eta_{B,A_i})} \right),$$

where the fraction means that  $(\gamma \circ p_i)^*(\eta_{B,A_i})$  is replaced by  $(\gamma \circ p_i)^*(d\eta_{B,A_i} + (\eta \wedge \eta)_{B,A_i})$ .

PROOF: Set  $\partial_C C(\Gamma^{(r)}, (\gamma_u)) = -\cup_{u \in ]0,1[} \partial_C C(\Gamma^{(r)}, \gamma_u)$  and

$$\partial_\Delta C(\Gamma^{(r)}, (\gamma_u)) = -\cup_{u \in ]0,1[} \partial_\Delta C(\Gamma^{(r)}, \gamma_u).$$

Since

$$\partial C(\Gamma^{(r)}, (\gamma_u)) = C(\Gamma^{(r)}, \gamma_1) - C(\Gamma^{(r)}, \gamma_0) + \partial_C C(\Gamma^{(r)}, (\gamma_u)) + \partial_\Delta C(\Gamma^{(r)}, (\gamma_u))$$

is a null-homologous cycle, we get

$$\widetilde{\text{hol}}_{\gamma_1}(\eta_{B,A}) - \widetilde{\text{hol}}_{\gamma_0}(\eta_{B,A}) = \widetilde{\delta}_{(\gamma_u)}(d\eta_{B,A}) + \widetilde{\delta}_{(\gamma_u)}((\eta \wedge \eta)_{B,A}),$$

with

$$\widetilde{\delta}_{(\gamma_u)}(d\eta_{B,A}) \stackrel{\text{def}}{=} - \sum_{r=0}^{\infty} \sum_{\Gamma^{(r)} \in \mathcal{D}_{n,A}^{e,r}(\sqcup_{b \in B} \mathbb{R}_b)} \zeta_{\Gamma^{(r)}} \int_{\partial_C C(\Gamma^{(r)}, (\gamma_u))} \Omega_{\Gamma^{(r)}} [\Gamma^{(r)}]$$

and

$$\widetilde{\delta}_{(\gamma_u)}((\eta \wedge \eta)_{B,A}) \stackrel{\text{def}}{=} - \sum_{r=0}^{\infty} \sum_{\Gamma^{(r)} \in \mathcal{D}_{n,A}^{e,r}(\sqcup_{b \in B} \mathbb{R}_b)} \zeta_{\Gamma^{(r)}} \int_{\partial_\Delta C(\Gamma^{(r)}, (\gamma_u))} \Omega_{\Gamma^{(r)}} [\Gamma^{(r)}].$$

Let us study these terms. Since

$$P_{\Gamma^{(r)}}^{-1}(\gamma_u^{(r)}((t_1, \dots, t_r))) = \pm \prod_{i=1}^r \left( \left\{ p_{[0,1]}(\gamma_u(t_i)) \right\} \times \mathcal{V}(p_{\mathfrak{S}_B(\mathbb{C})}(\gamma_u(t_i)), \Gamma_i) \right)$$

is diffeomorphic to  $\pm \prod_{i=1}^r \mathcal{V}(\gamma_u(t_i), \Gamma_i)$  (forgetting the natural  $p_{\mathfrak{S}_B(\mathbb{C})}$ ), we have

$$\partial P_{\Gamma^{(r)}}^{-1}(\gamma_u^{(r)}((t_1, \dots, t_r))) = \pm \sum_{i=1}^r \partial \mathcal{V}(\gamma_u(t_i), \Gamma_i) \times \prod_{j \in \underline{n} \setminus \{i\}} \mathcal{V}(\gamma_u(t_j), \Gamma_j).$$

According to Lemma 16.38, we have

$$\widetilde{\delta}_{(\gamma_u)}(d\eta_{B,A}) = \sum_{r=0}^{\infty} \sum_{(A_1, \dots, A_r) \in P_r(A)} \frac{\prod_{i=1}^r |A_i|!}{|A|!} \int_{[0,1] \times \Delta^{(r)}} \sum_{i=1}^r \alpha(i, A_1, \dots, A_r),$$

with

$$\alpha(i, A_1, \dots, A_r) = (-1)^{i-1} \bigwedge_{j=1}^r (\gamma \circ p_j)^*(\eta_{B,A_j}) \left( \frac{(\gamma \circ p_i)^*(d\eta_{B,A_i})}{(\gamma \circ p_i)^*(\eta_{B,A_i})} \right)$$

and  $d\eta_{B,\emptyset} = 0$ .

Let us give a few details about the involved signs. Recall that the space  $C(\Gamma^{(r)}, \gamma_u)$  is oriented locally as the product of the  $C(\Gamma_i^{(1)}, \gamma_u)$  for  $i \in \underline{r}$ , which are oriented so that the parameter  $t$  replaces the translation parameter in  $\mathcal{V}(\gamma_u(t), \Gamma_i)$ . Thus, the boundary  $\partial_C$ , along which we integrate, is locally diffeomorphic to

$$\left( - \cup_{u \in [0,1]} \left( \prod_{j=1}^{i-1} C(\Gamma_j^{(1)}, \gamma_u) \right) \times \partial_C C(\Gamma_i^{(1)}, \gamma_u) \times \left( \prod_{j=i+1}^r C(\Gamma_j^{(1)}, \gamma_u) \right) \right),$$

where the dimension of  $C(\Gamma_j^{(1)}, \gamma_u)$  is even. When rewriting such an integral as an integral over  $[0, 1] \times \Delta^r$  of the two-form  $(\gamma \circ p_i)^*(d\eta_{B,A_i})$  and one-forms  $(\gamma \circ p_j)^*(\eta_{B,A_j})$ , one must take into account the fact that the two-form will be integrated along the product by  $[0, 1]$  of the interval parametrized by  $t_i$ . This gives rise to the factor  $(-1)^{i-1}$ .

Recall  $\partial\Delta^{(r)} = \sum_{i=0}^r (-1)^{i+1} F_i(\Delta^{(r)})$ , with  $F_0(\Delta^{(r)}) = \{(0, t_2, \dots, t_r) \in \Delta^r\}$ ,  $F_r(\Delta^{(r)}) = \{(t_1, t_2, \dots, t_{r-1}, 1) \in \Delta^{(r)}\}$ , and

$$F_i(\Delta^{(r)}) = \{(t_1, \dots, t_i, t_i, t_{i+1}, \dots, t_{r-1}) \in \Delta^{(r)}\}$$

for  $i \in \underline{r-1}$ . Observe that the faces  $F_0$  and  $F_r$  do not contribute to  $\tilde{\delta}_{(\gamma_u)}((\eta \wedge \eta)_{B,A})$ . Indeed for  $F_0$ , the directions of the edges of  $\Gamma_1$  are in the image of  $\mathcal{V}(\gamma_0(0), \Gamma_1)$ , which is  $(2|E(\Gamma_1)| - 1)$ -dimensional. The contribution of the faces  $F_i$  yields

$$\begin{aligned} & \tilde{\delta}_{(\gamma_u)}((\eta \wedge \eta)_{B,A}) \\ &= \sum_{r=1}^{\infty} \sum_{(A_1, \dots, A_{r-1}) \in P_{r-1}(A)} \frac{\prod_{i=1}^{r-1} |A_i|!}{|A|!} \int_{[0,1] \times \Delta^{r-1}} \sum_{i=1}^{r-1} \tilde{\delta}_d(i, A_1, \dots, A_{r-1}), \end{aligned}$$

with

$$\tilde{\delta}_d(i, A_1, \dots, A_{r-1}) = (-1)^{i-1} \bigwedge_{j=1}^{r-1} (\gamma \circ p_j)^*(\eta_{B,A_j}) \left( \frac{(\gamma \circ p_i)^*((\eta \wedge \eta)_{B,A_i})}{(\gamma \circ p_i)^*(\eta_{B,A_i})} \right).$$

□

Note that  $(\eta \wedge \eta)_{B,A_i} = \sum_{(\Gamma, \Gamma') \in \mathcal{D}_{n_i, A_i}^{e,2}(\sqcup_{b \in B} \mathbb{R}_b)} \zeta_{\Gamma \sqcup \Gamma'} \eta_{\Gamma} \wedge \eta_{\Gamma'} [\Gamma] [\Gamma']$  is valued in the space of primitive elements of  $\mathcal{A}_{n_i}(\sqcup_{b \in B} \mathbb{R}_b)$ . (Use  $\eta_{\Gamma} \wedge \eta_{\Gamma'} [\Gamma] [\Gamma'] + \eta_{\Gamma'} \wedge \eta_{\Gamma} [\Gamma] [\Gamma'] = \eta_{\Gamma} \wedge \eta_{\Gamma'} ([\Gamma] [\Gamma'] - [\Gamma'] [\Gamma])$ .)

**Lemma 16.40.** *The form  $d\eta_{B,A} + (\eta \wedge \eta)_{B,A}$  on  $[0, 1] \times \check{\mathcal{S}}_B(\mathbb{C})$  vanishes identically for any subset  $A$  of  $\underline{3N}$  with cardinality  $3n$ .*

PROOF: We proceed by induction on  $n$  as in the proof of Corollary 15.22, using Lemma 16.39 and Lemma 16.21, which guarantees that  $\text{hol}(\eta_{B,A})$  vanishes along homotopically trivial loops in  $\{u\} \times \check{\mathcal{S}}_B(\mathbb{C})$  and along boundaries of squares  $[w, w + \varepsilon] \times p_{CS} \circ \gamma([t, t + \varepsilon])$ . (This flatness condition can also be proved directly as an exercise, as in Remark 15.24.)  $\square$

Lemma 16.36 is proved.  $\square$

We now generalize Lemma 16.36 as follows.

**Lemma 16.41.** *Let  $\gamma$  and  $\delta$  be two semi-algebraic paths of  $[0, 1] \times \mathcal{S}_B(\mathbb{C})$  homotopic relatively to  $\{0, 1\}$ . Assume  $\gamma(]0, 1[) \subset \check{\mathcal{S}}_B(\mathbb{C})$  and  $\delta(]0, 1[) \subset \check{\mathcal{S}}_B(\mathbb{C})$ . Then we have*

$$\widetilde{\text{hol}}(\eta_{B,A})(\gamma) = \widetilde{\text{hol}}(\eta_{B,A})(\delta).$$

PROOF: According to Lemma 16.21, it suffices to take care of homotopies near the endpoints. Thanks to Lemma 16.34, it suffices to prove that there exist  $t, t' > 0$ , and a path  $\varepsilon$  from  $\gamma(t)$  to  $\delta(t')$  in  $[0, 1] \times \check{\mathcal{S}}_B(\mathbb{C})$  such that  $\gamma|_{[0,t]} \varepsilon \delta|_{[0,t']}$  is null-homotopic and the holonomy along  $\gamma|_{[0,t]} \varepsilon \delta|_{[0,t']}$  is one. When  $t$  and  $t'$  are small enough, the images of  $\gamma|_{[0,t]}$  and  $\delta|_{[0,t']}$  lie in a subset equipped with a local semi-algebraic chart as in Theorem 8.28, from which it is easy to construct semi-algebraic interpolations in products of sphere pieces and intervals. Furthermore, there is no loss of generality in assuming that  $\gamma|_{[0,t]} \varepsilon$  and  $\delta|_{[0,t']}$  meet only at  $\gamma(0)$  and  $\delta(t')$ , and that straight interpolation provides a boundary-fixing semi-algebraic homotopy from  $\gamma|_{[0,t]} \varepsilon$  to  $\delta|_{[0,t']}$ , which satisfies the injectivity hypotheses of Lemma 16.36. (Otherwise, we could use an intermediate  $\gamma'|_{[0,t]}$ .) Thus, Lemma 16.36 allows us to prove  $\widetilde{\text{hol}}_{\gamma|_{[0,t]} \varepsilon}(\eta_{B,A}) = \widetilde{\text{hol}}_{\delta|_{[0,t]'}}(\eta_{B,A})$ .  $\square$

PROOF OF THEOREM 16.35: Lemma 16.41 allows us to define a map  $\tilde{h}_A$  induced by  $\widetilde{\text{hol}}(\eta_{B,A})$  from homotopy classes of paths with fixed boundaries of  $[0, 1] \times \mathcal{S}_B(\mathbb{C})$  to  $\mathcal{A}_{|A|/3}(\sqcup_{b \in B} \mathbb{R}_b)$  as follows. For a path  $\gamma: [0, 1] \rightarrow \mathcal{S}_B(\mathbb{C})$ , set  $\tilde{h}_A(\gamma) = \widetilde{\text{hol}}(\eta_{B,A})(\delta)$  for any semi-algebraic path  $\delta$  of  $[0, 1] \times \mathcal{S}_B(\mathbb{C})$  that is homotopic to  $\gamma$  relatively to  $\{0, 1\}$  and such that  $\delta(]0, 1[) \subset \check{\mathcal{S}}_B(\mathbb{C})$ .

Now, it suffices to prove that  $\text{hol}_\gamma(\eta_{B,A})$  coincides with  $\tilde{h}_A(\gamma)$  for any semi-algebraic path  $\gamma: [0, 1] \rightarrow [0, 1] \times \mathcal{S}_B(\mathbb{C})$ . Recall that a stratum of  $[0, 1] \times \mathcal{S}_B(\mathbb{C})$  is the product by  $[0, 1]$  of a stratum of  $\mathcal{S}_B(\mathbb{C})$  associated to a  $\Delta$ -parenthesization. The preimage under  $\gamma$  of such a stratum of  $[0, 1] \times \mathcal{S}_B(\mathbb{C})$  is semi-algebraic. So  $\gamma$  is a finite composition of paths whose interiors lie in a



fixed stratum of  $\mathcal{S}_B(\mathbb{C})$  (according to the Lojasiewicz theorem 16.32). Thus, thanks to Lemma 16.34, it suffices to prove that  $\widetilde{\text{hol}}_\gamma(\eta_{B,A})$  coincides with  $\tilde{h}_A(\gamma)$  for any injective semi-algebraic path  $\gamma$  whose interior lies in a fixed stratum of  $\mathcal{S}_B(\mathbb{C})$  and in a subset equipped with a local semi-algebraic chart as in Theorem 8.28. Such a path can be deformed by sending the vanishing coordinates of  $\gamma(t)$  in the  $[0, \varepsilon[$  factors in such a chart to  $\varepsilon(\frac{1}{2} - |\frac{1}{2} - t|)u$  for  $u \in [0, 1]$  giving rise to a semi-algebraic homotopy  $(\gamma_u(t))_{u \in [0, 1]}$  such that  $\tilde{h}_A(\gamma) = \widetilde{\text{hol}}_{\gamma_1}(\eta_{B,A})$ . So Lemma 16.36 implies  $\widetilde{\text{hol}}_{\gamma_0}(\eta_{B,A}) = \widetilde{\text{hol}}_{\gamma_1}(\eta_{B,A})$ .  $\square$

Theorem 16.35 allows us to set the following definition.

**Definition 16.42.** For any continuous path  $\gamma: [0, 1] \rightarrow [0, 1] \times \mathcal{S}_B(\mathbb{C})$ , define  $\widetilde{\text{hol}}(\eta_{B,A})(\gamma)$  to be

$$\widetilde{\text{hol}}(\eta_{B,A})(\gamma) = \widetilde{\text{hol}}(\eta_{B,A})(\delta)$$

for any semi-algebraic path  $\delta$  of  $[0, 1] \times \mathcal{S}_B(\mathbb{C})$  homotopic to  $\gamma$  relatively to  $\{0, 1\}$ .

We can now generalize [Poi00, Proposition 1.18] for braids.

**Proposition 16.43.** *Let  $B$  and  $C$  be two finite sets. Let  $b_0 \in B$ .*

*Let  $\gamma_B$  be a path of  $[0, 1] \times \mathcal{S}_B(\mathbb{C})$  and let  $\gamma_C$  be a path of  $[0, 1] \times \mathcal{S}_C(\mathbb{C})$ . Let  $\gamma_B(\gamma_C/b_0) = \gamma_B(\gamma_C/K_{b_0})$  be the  $q$ -braid obtained by cabling the strand  $K_{b_0}$  of  $b_0$  in  $T(\gamma_B)$  by  $T(\gamma_C)$ , as in Notation 13.3. Then we have*

$$\widetilde{\text{hol}}_{\gamma_B(\gamma_C/b_0)}(\eta_{B(\frac{C}{b_0}),.}) = \left( \pi(C \times b_0)^* (\widetilde{\text{hol}}_{\gamma_B}(\eta_{B,.})) \widetilde{\text{hol}}_{\gamma_C}(\eta_{C,.}) \right)_{\square},$$

where  $\pi(C \times b_0)^*$  denotes the duplication of the strand  $\mathbb{R}_{b_0}$  for diagrams as in Notation 6.31 and we use the product of Definition 16.10.

**PROOF:** Thanks to Definition 16.42, we assume that  $\gamma_C$  and  $\gamma_B$  are semi-algebraic, without loss of generality. Since a semi-algebraic path is a path composition of finitely many semi-algebraic paths whose interiors lie in a fixed stratum of  $[0, 1] \times \mathcal{S}(\mathbb{C})$  (according to the Lojasiewicz theorem 16.32), the functoriality of Lemma 16.34 allows us to assume furthermore that the image of the interior of  $\gamma_C$  lies in a fixed stratum of  $[0, 1] \times \mathcal{S}_C(\mathbb{C})$  and that the image of the interior of  $\gamma_B$  lies in a fixed stratum of  $[0, 1] \times \mathcal{S}_B(\mathbb{C})$ , for the proof. (Recall the commutation lemma 6.33.)

Let  $B(C/b_0)$  be the set obtained from  $B$  by replacing  $b_0$  by  $C$ . We refer to Lemma 15.12 for the description of the stratification of  $\mathcal{V}(\Gamma)$  and the dimensions of the fibers for connected diagrams on  $B(C/b_0) \times \mathbb{R}$ . When computing the “holonomy” of  $\eta_{B(C/b_0),.}$ , we integrate over products of one-parameter families  $\mathcal{V}(y, \Gamma)$  with  $y \in \gamma_B(\gamma_C/b_0) ([t_i - \varepsilon, t_i + \varepsilon])$ , locally. We

may restrict to the strata of  $\mathcal{V}(y, \Gamma)$  of dimension  $(2|E(\Gamma)| - 1)$ , which are described in Lemma 15.12.

Consider a connected diagram  $\Gamma$  on  $B(C/b_0) \times \mathbb{R}$  together with a  $\Delta$ -parenthesization  $\mathcal{P}$  of its vertices corresponding to such a stratum of configurations. Since  $\mathcal{P}$  is equal to  $\mathcal{P}'_x$ , all elements of  $\mathcal{P}$  are univalent. Assume  $\mathcal{P} \neq \{V(\Gamma)\}$ . Let  $\Gamma_{\mathcal{P}}$  be obtained from  $\Gamma$  by identifying all the vertices in a daughter  $A$  of  $V(\Gamma)$  to a single vertex  $v_A$  and by erasing the edges between two elements in  $A$ , for each  $A$  of  $D(V(\Gamma))$ . Then  $\Gamma_{\mathcal{P}}$  is connected and its vertices  $v_A$  move along vertical lines. Let  $U(\Gamma_{\mathcal{P}})$  and  $T(\Gamma_{\mathcal{P}})$  respectively denote the set of univalent vertices of  $\Gamma_{\mathcal{P}}$  distinct from the  $v_A$  and the set of trivalent vertices of  $\Gamma_{\mathcal{P}}$  distinct from the  $v_A$ . The dimension of the one-parameter family of configurations of the vertices of  $\Gamma_{\mathcal{P}}$  up to vertical translation is  $|D(V(\Gamma))| + |U(\Gamma_{\mathcal{P}})| + 3|T(\Gamma_{\mathcal{P}})|$ . The form  $\bigwedge_{e \in E(\Gamma_{\mathcal{P}})} p_{e, S^2}^*(\tilde{\omega}(j_E(e), S^2))$  factors through this one-parameter family. Thus, a count of half-edges shows that the stratum cannot contribute unless the vertices  $v_A$  are univalent in  $\Gamma_{\mathcal{P}}$ .

Now assume that all the vertices  $v_A$  are univalent in  $\Gamma_{\mathcal{P}}$ . Let  $e_A$  be the edge of  $\Gamma_{\mathcal{P}}$  with maximal label such that  $e_A$  is adjacent to a vertex  $v_A$  for some  $A \in D(V(\Gamma))$ . The subgraph  $\Gamma_A$  of  $\Gamma$  consisting of the vertices of  $A$  and the edges of  $\Gamma$  between two such vertices is connected. It has one bivalent vertex  $b$  in  $\Gamma_A$  (which is the end of  $e_A$  in  $\Gamma \cap A$ ). Its configurations are considered up to vertical translations. Their contribution is opposite to that of the configurations of the graph  $\Gamma'_A$  obtained from  $\Gamma_A$  by exchanging the labels and possibly the orientations of the two edges of  $\Gamma_A$  that contain  $b$ , as in Lemma 9.11.

Thus, the  $\Delta$ -parenthesization of  $\Gamma$  is  $\{V(\Gamma)\}$  in the strata that may contribute. If  $p_{B(C/b_0)}(U(\Gamma)) \subset C$ , then  $\Gamma$  is a diagram on  $C \times \mathbb{R}$ , which contributes as in  $\text{hol}_{\gamma_C}(\eta_C, \cdot)$ . Otherwise, the projection to the horizontal plane of the vertices of  $U(\Gamma) \cap p_{B(C/b_0)}^{-1}(C)$  is reduced to a point. So all diagrams obtained from these diagrams  $\Gamma$  by changing the map from  $U(\Gamma) \cap p_{B(C/b_0)}^{-1}(C)$  to  $C$  arbitrarily contribute together to  $\pi(C \times b_0)^*(\widetilde{\text{hol}}_{\gamma_B}(\eta_B, \cdot))$  as desired, locally. We get the proposition since the two kinds of diagrams commute thanks to Lemma 6.33. □

**Definition 16.44.** Recall Proposition 16.20 and Definition 16.42. For a  $q$ -braid (representative)  $\gamma: [0, 1] \rightarrow \mathcal{S}_B(\mathbb{C})$ , set

$$\mathcal{Z}^f(\gamma, \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}}) = \widetilde{\text{hol}}_{\{1\} \times \gamma}(\eta_B, \cdot).$$

For a  $q$ -tangle

$$T = T(\gamma^-)(\mathcal{C}, L, L_{\parallel})T(\gamma^+)$$

such that  $\gamma^-$  and  $\gamma^+$  are  $q$ -braids, and  $(\mathcal{C}, L)$  is a  $J_{bb,tt}$ -oriented framed tangle whose bottom configuration is  $\gamma^-(1)$  and whose top configuration is  $\gamma^+(0)$ , as in Definition 13.1, for  $N \in \mathbb{N}$ , and for a family  $(\omega(i, S^2))_{i \in \underline{3N}}$  of volume-one forms of  $S^2$ , set

$$\begin{aligned} & \mathcal{Z}^f\left(T, \cdot, (\omega(i, S^2))_{i \in \underline{3N}}\right) \\ &= \left( \mathcal{Z}^f\left(\gamma^-, \cdot, (\omega(i, S^2))\right) \mathcal{Z}^f\left(\mathcal{C}, L, L_{\parallel}, \cdot, (\omega(i, S^2))\right) \mathcal{Z}^f\left(\gamma^+, \cdot, (\omega(i, S^2))\right) \right)_{\square} \end{aligned}$$

with the notation of Definition 16.10 and Definition 16.17.

Lemma 16.34 and the isotopy invariance of Theorems 16.16 and 16.35 ensure that the definition above is consistent.

Theorem 16.9 allows us to express the variation of

$$\mathcal{Z}^f\left(T, \cdot, (\omega(i, S^2))_{i \in \underline{3N}}\right)$$

when  $(\omega(i, S^2))_{i \in \underline{3N}}$  varies for framed straight tangles with injective top and bottom configurations.<sup>6</sup> As a corollary of Theorem 16.35, this expression generalizes to  $q$ -tangles. We get the following theorem.

**Theorem 16.45.** *Let  $N \in \mathbb{N}$ . For  $i \in \underline{3N}$ , let  $\tilde{\omega}(i, S^2) = (\tilde{\omega}(i, t, S^2))_{t \in [0,1]}$  be a closed 2-form on  $[0, 1] \times S^2$  with  $\int_{\{0\} \times S^2} \tilde{\omega}(i, 0, S^2) = 1$ . Let  $T$  denote a  $J_{bb,tt}$ -oriented  $q$ -tangle. Assume that the bottom and top configurations of  $T$  are elements  $y^-$  of  $\mathcal{S}_{B^-}(\mathbb{C})$  and  $y^+$  of  $\mathcal{S}_{B^+}(\mathbb{C})$ , respectively. Then we have*

$$\begin{aligned} & \mathcal{Z}^f\left(T, \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}}\right) \\ &= \left( \left( \prod_{K_j \in J_{bb}} \widetilde{\text{hol}}_{[0,1]}(-\eta(\cdot, S_{WE})) \#_j \right) \left( \prod_{K_j \in J_{tt}} \widetilde{\text{hol}}_{[0,1]}(\eta(\cdot, S_{WE})) \#_j \right) \right. \\ & \quad \left. \widetilde{\text{hol}}_{[1,0] \times y^-}(\eta_{B^-, \cdot}) \mathcal{Z}^f\left(T, \cdot, (\tilde{\omega}(i, 0, S^2))\right) \widetilde{\text{hol}}_{[0,1] \times y^+}(\eta_{B^+, \cdot}) \right)_{\square}. \end{aligned}$$

□

Recall that the forms  $\eta$  and their holonomies introduced in Definitions 16.1 and 16.6 depend on the forms  $\tilde{\omega}(i, S^2)$ . In the end of this chapter, we will apply Theorem 16.45, only when  $\tilde{\omega}(i, 0, S^2)$  is the standard form  $\omega_{S^2}$ . So  $\mathcal{Z}^f(T, \cdot, (\tilde{\omega}(i, 0, S^2)))$  is simply  $\mathcal{Z}^f(T, \cdot)$ .

The following lemma is also easy to prove.

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<sup>6</sup>See also the proof of Theorem 16.16.

**Lemma 16.46.** *The behavior of  $\mathcal{Z}^f$  under reversing the orientation of a closed component is the same as that described in Proposition 10.23. Its behavior under reversing the orientation of component going from bottom to top or from top to bottom in a  $q$ -tangle  $T$  is similar: Let  $K$  be an oriented component going from bottom to top or from top to bottom in a  $q$ -tangle  $T$ . For a Jacobi diagram  $\Gamma$  on the domain  $\mathcal{L}$  of  $T$ , let  $U_K(\Gamma)$  denote the set of univalent vertices of  $\Gamma$  mapped to the domain  $\mathbb{R}_K$  of  $K$ . This set is ordered by the orientation of  $\mathbb{R}_K$ . When the orientation of  $K$  is changed,  $\mathcal{Z}^f(T)$  is modified by reversing the orientation of  $\mathbb{R}_K$  (that is reversing the order of  $U_K(\Gamma)$ ) in classes  $[\Gamma]$  of diagrams  $\Gamma$  on  $\mathcal{L}$ , and multiplying these classes by  $(-1)^{|U_K(\Gamma)|}$  in  $\mathcal{A}(\mathcal{L})$ , simultaneously.*

□

In other words, we can forget the orientation of closed components and components going from bottom to top or from top to bottom, and regard  $\mathcal{Z}^f(L)$  as valued in spaces of diagrams where the domains of these components are not oriented, as in Definitions 6.13 and 6.16. However, we a priori need a  $J_{bb,tt}$ -orientation as Lemma 16.47 below indicates.

**Lemma 16.47.** *Let  $A$  be a set of cardinality  $3k$ . Recall Definition 16.6. The form  $\eta(A, S_{WE} \cup (-S_{EW}))$  is zero when  $k$  is odd. If  $k$  is even, then we have*

$$\eta(A, S_{WE} \cup (-S_{EW})) = -s_* \left( \eta(A, S_{WE} \cup (-S_{EW})) \right).^7$$

*If  $\eta(A, S_{WE} \cup (-S_{EW}))$  is zero, then  $\mathcal{Z}^f(\cdot, \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \mathbb{3}\mathbb{N}})$  is independent of the  $J_{bb,tt}$ -orientation of the tangles (as in Proposition 10.23).*

PROOF: The circle  $S_{WE} \cup (-S_{EW})$  is the great circle  $\partial D$  of  $S^2$  that is the boundary of the hemisphere  $D$  of  $S^2$  centered at  $(-i)$ . By Definition 16.6, we have  $\eta(A, \partial D)(t) = \int_{\{t\} \times \partial D} \omega(A)$  with

$$\int_{[0,t] \times \partial D} \omega(A) = \int_{(\partial[0,t]) \times D} \omega(A) = \int_{\{t\} \times D} \omega(A) - \int_{\{0\} \times D} \omega(A)$$

and

$$\int_{\{t\} \times S^2} \omega(A) = \int_{\{t\} \times D} \left( \omega(A) - \iota^*(\omega(A)) \right) = (1 + (-1)^{k+1}) \int_{\{t\} \times D} \omega(A)$$

according to Lemma 16.5. In particular, when  $k$  is odd, the integral

$$\int_{\{t\} \times D} \omega(A) = \frac{1}{2} \int_{\{t\} \times S^2} \omega(A)$$

---

<sup>7</sup>In particular,  $\eta(A, S_{WE} \cup (-S_{EW}))$  is also zero when  $k$  is even, if  $s_*$  is the identity map. But this is unknown to me.

is independent of  $t$  and  $\eta(A, \partial D)$  is zero. When  $k$  is even, Lemma 16.5 implies  $\omega(A) = -s_*(\omega(A))$ .

Changing the orientation of a component  $K$  of  $J_{bb}$  amounts to construct the invariant by imposing the condition  $p_\tau(U^+K) \subseteq S_{EW}$  rather than  $p_\tau(U^+K) \subseteq S_{WE}$ . So this replaces the factor  $\widetilde{\text{hol}}_{[0,1]}(-\eta(\cdot, S_{WE}))$  associated to  $K$  with  $\widetilde{\text{hol}}_{[0,1]}(-\eta(\cdot, S_{EW}))$  in the formula of Theorem 16.45. See Theorem 16.16 and Definition 16.17. This amounts to multiply by  $\widetilde{\text{hol}}_{[0,1]}(\eta(\cdot, S_{WE} \cup -S_{EW}))$  on the component of  $K$ .  $\square$

**Remarks 16.48.** Similarly, if we had imposed that  $p_\tau$  maps the unit tangent vectors to components of  $J_{bb} \cup J_{tt}$  to the vertical half great circle  $S(\theta)$  from  $-\vec{N}$  to  $\vec{N}$  that contains the complex direction  $\exp(2i\pi\theta)$ , for  $\theta \in ]0, 1[$ , in our definition of straight tangles in Section 16.3, then  $S(\theta)$  would replace  $S_{WE}$  in the formula of Theorem 16.45, and  $\mathcal{Z}^f(T, \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}})$  would have been multiplied by

$$\left( \prod_{K_j \in J_{bb}} \widetilde{\text{hol}}_{[0,1]} \eta(\cdot, S_{WE} \cup (-S(\theta))) \#_j \right) \left( \prod_{K_j \in J_{tt}} \widetilde{\text{hol}}_{[0,1]} \eta(\cdot, S(\theta) \cup (-S_{WE})) \#_j \right).$$

In particular, when  $T$  is a tangle with only one component going from bottom to bottom,  $\mathcal{Z}^f(T, \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}})$  would have been multiplied by  $\widetilde{\text{hol}}_{[0,1]} \eta(\cdot, S_{WE} \cup (-S(\theta)))$ .

With the notation of Definition 16.6 and Lemma 16.5,

$$\begin{aligned} \widetilde{\text{hol}}_{[0,1]} \eta(\underline{3}, S_{WE} \cup (-S(\theta))) &= \int_{[0,1] \times (S_{WE} \cup (-S(\theta)))} \omega(\underline{3}) \\ &= \frac{1}{6} \sum_{i=1}^3 \int_{[0,1] \times (S_{WE} \cup (-S(\theta)))} (\tilde{\omega}(i, S^2) - \iota^* \tilde{\omega}(i, S^2)) [\hat{\zeta}] \end{aligned}$$

would have been added to  $\mathcal{Z}^f(T, \underline{3}, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3}})$ . For a chain  $D$  of  $S^2$  bounded by  $(S_{WE} \cup (-S(\theta)))$ , we have

$$\int_{[0,1] \times (S_{WE} \cup (-S(\theta)))} \tilde{\omega}(i, S^2) = \int_{(\partial[0,1]) \times D} \tilde{\omega}(i, S^2).$$

In particular, if  $\theta$  is in  $]1/2, 1[$ , then we can choose  $D$  and  $\tilde{\omega}(i, 1, S^2)$  so that  $(\tilde{\omega}(i, 1, S^2) - \iota^* \tilde{\omega}(i, 1, S^2))$  is supported outside  $D \cup \iota_{S^2}(D)$  for any  $i$ . In this

case, we get

$$\widetilde{\text{hol}}_{[0,1]}\eta\left(\underline{\mathfrak{z}}, S_{WE} \cup (-S(\theta))\right) = - \int_{\{0\} \times D} \omega_{S^2} = \theta - 1.$$

Thus, as claimed in Remark 16.18, Definition 16.17 is not canonical.

The above calculation does not rule out the alternative choice of the vertical half great circle  $S(\frac{1}{2}) = S_{EW}$  for our definition. This choice would multiply  $\mathcal{Z}^f(T, \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}})$  by  $\widetilde{\text{hol}}_{[0,1]}(\eta(\cdot, S_{WE} \cup (-S_{EW})))$ , which is zero in degree 1, according to Lemma 16.47.

It might be tempting to modify the definition of  $\mathcal{Z}^f(T, \cdot, (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3N}})$ , by multiplying it by

$$\left( \prod_{K_j \in J_{bb}} \widetilde{\text{hol}}_{[0,1]}(\eta(\cdot, S_{WE})) \#_j \right) \left( \prod_{K_j \in J_{tt}} \widetilde{\text{hol}}_{[0,1]}(-\eta(\cdot, S_{WE})) \#_j \right).$$

Unfortunately,  $\widetilde{\text{hol}}_{[0,1]}(\eta(\cdot, S_{WE}))$  depends on the closed 2-forms  $\tilde{\omega}(i, S^2)$  of  $[0, 1] \times S^2$ , and not only on the  $\tilde{\omega}(i, 1, S^2)$ . Indeed, assume that all  $\tilde{\omega}(i, S^2)$  coincide with each other and change all of them by adding  $d\eta_S$ , for a one-form  $\eta_S$  of  $[0, 1] \times S^2$  supported on the product of  $[1/4, 3/4]$  by a small neighborhood of  $\vec{N}$ . Then the variation of the degree one part of  $2\widetilde{\text{hol}}_{[0,1]}(\eta(\cdot, S_{WE}))$  maps  $\underline{\mathfrak{z}}$  to  $\int_{[0,1] \times S_{WE}} (d\eta_S - \iota^*(d\eta_S))$ , with

$$\int_{[0,1] \times S_{WE}} d\eta_S = \int_{\partial([0,1] \times S_{WE})} \eta_S = - \int_{[0,1] \times \{\vec{N}\}} \eta_S$$

and

$$\begin{aligned} \int_{[0,1] \times S_{WE}} -\iota^*(d\eta_S) &= - \int_{[0,1] \times \iota_{S^2}(S_{WE})} d\eta_S = \int_{\partial([0,1] \times \iota_{S^2}(S_{WE}))} \eta_S \\ &= - \int_{[0,1] \times \{\vec{N}\}} \eta_S. \end{aligned}$$

(In Theorem 16.9, the factors  $\widetilde{\text{hol}}_{[t,0] \times y^-}(\eta_{B^-, \cdot})$  and  $\widetilde{\text{hol}}_{[0,t] \times y^+}(\eta_{B^+, \cdot})$  also depend on  $\tilde{\omega}(i, S^2)$ . However, both types of dependences cancel each other.)



# Chapter 17

## Justifying the properties of $\mathcal{Z}^f$

Recall Definition 13.10 of the invariant  $\mathcal{Z}^f$ . So far we have succeeded in constructing this invariant  $\mathcal{Z}^f$  of  $q$ -tangles, invariant under boundary-fixing diffeomorphisms, generalizing both the invariant  $\mathcal{Z}^f$  for framed links in  $\mathbb{Q}$ -spheres and the Poirier functor  $Z^l$  for  $q$ -tangles in  $\mathbb{R}^3$ . The framing dependence of Theorem 13.12 comes from Definition 12.12.

The behavior of  $\mathcal{Z}$  and  $\mathcal{Z}^f$  under orientation changes of the components described in the statement of Theorem 13.12 can be justified as in the case of links in rational homology spheres treated in Section 10.6.

**Lemma 17.1.** *The invariant  $\mathcal{Z}^f$  behaves as prescribed by Theorem 13.12 under the diffeomorphisms  $s_{\frac{1}{2}}$  and  $\rho$ .*

PROOF: Let  $L$  be a tangle representative as in Theorem 12.7. We proceed as in the proof of Proposition 10.1, except that we need to take care of the facts that, for  $\psi = s_{\frac{1}{2}}$  or  $\psi = \rho$ ,  $\psi_*(\tau)$  is not asymptotically standard, and that  $s_{\frac{1}{2}}$  reverses the orientation. Therefore, we use  $\tau' = \psi_*(\tau) \circ (\mathbf{1}_{\tilde{R}(\mathcal{C})} \times \psi_{\mathbb{R}^3}^{-1}) = T\psi \circ \tau \circ (\psi^{-1} \times \psi_{\mathbb{R}^3}^{-1})$ , as an asymptotically standard parallelization of  $R = R(\mathcal{C})$ . So we have  $(\rho_*^{-1})^* p_\tau^*(\omega_{S^2}) = p_{\tau'}^*(\omega_{S^2})$  and  $(\sigma_{\frac{1}{2}*}^{-1})^* p_\tau^*(\omega_{S^2}) = -p_{\tau'}^*(\omega_{S^2})$ . If  $\omega$  is a homogeneous propagating form of  $(C_2(R), \tau)$ , then  $(\rho_*^{-1})^*(\omega)$  is a homogeneous propagating form of  $(C_2(\rho(R)), \tau')$  and  $(-\sigma_{\frac{1}{2}*}^{-1})^*(\omega)$  is a homogeneous propagating form of  $(C_2(s_{\frac{1}{2}}(R)), \tau')$ . Let us now focus on the case  $\psi = s_{\frac{1}{2}}$ , since the case  $\psi = \rho$  is similar, but simpler. For any Jacobi diagram  $\Gamma$  on the domain of  $L$ , equipped with an implicit orientation  $o(\Gamma)$ , compute  $I = I(s_{\frac{1}{2}}(R), s_{\frac{1}{2}}(L), \Gamma, (-\sigma_{\frac{1}{2}*}^{-1})^*(\omega))$ . We have

$$\begin{aligned} I &= \int_{\tilde{C}(s_{\frac{1}{2}}(R), s_{\frac{1}{2}}(L); \Gamma)} \bigwedge_{e \in E(\Gamma)} p_e^* \left( -(\sigma_{\frac{1}{2}*}^{-1})^*(\omega) \right) \\ &= (-1)^{|E(\Gamma)|} \int_{\tilde{C}(s_{\frac{1}{2}}(R), s_{\frac{1}{2}}(L); \Gamma)} (\sigma_{\frac{1}{2}*}^{-1})^* \left( \bigwedge_{e \in E(\Gamma)} p_e^*(\omega) \right) \\ &= (-1)^{|E(\Gamma)| + |T(\Gamma)|} I(R, L, \Gamma, \omega). \end{aligned}$$



Thus, we get  $Z_n(s_{\frac{1}{2}}(\mathcal{C}), s_{\frac{1}{2}}(L), \tau') = (-1)^n Z_n(\mathcal{C}, L, \tau)$  for all  $n \in \mathbb{N}$ . In particular, Corollary 4.8 and Propositions 5.15 and 7.17 imply  $p_1(\tau') = -p_1(\tau)$  and  $I_\theta(K_j, \tau') = -I_\theta(K_j, \tau)$  for any component  $K_j$  of  $L$ . So we have  $Z_n(s_{\frac{1}{2}}(\mathcal{C}), s_{\frac{1}{2}}(L)) = (-1)^n Z_n(\mathcal{C}, L)$  for all  $n \in \mathbb{N}$  since the anomalies  $\alpha$  and  $\beta$  vanish in even degrees, thanks to Propositions 10.7 and 10.13. Now, recall Proposition 16.13, and note that if a component  $K$  is straight with respect to  $\tau$ , then  $s_{\frac{1}{2}}(K)$  is straight with respect to  $\tau'$ . In particular, the condition

$$lk_{s_{\frac{1}{2}}(\mathcal{C})} \left( s_{\frac{1}{2}}(K), s_{\frac{1}{2}}(K)_{\parallel} \right) = -lk_{\mathcal{C}}(K, K_{\parallel})$$

is realizable and natural, and we get the desired equality for a framed tangle from an injective bottom configuration to an injective top configuration. Thanks to Remark 13.11, it is still true for a  $q$ -tangle.  $\square$

We are thus left with the proofs of the functoriality, the duplication properties, and the cabling property to finish the proof of Theorem 13.12. These proofs will occupy four sections of this chapter, which will end with a section describing other properties of  $\mathcal{Z}^f$ . The corresponding properties of variants of  $\mathcal{Z}$  and  $\mathcal{Z}^f$  involving nonhomogeneous propagating forms will be treated simultaneously since they are often easier to prove, and since we are going to use them to prove some of the results for homogeneous propagating forms.

## 17.1 Transversality and rationality

In this section, we generalize the rationality results of Chapter 11 to the tangle case. The generalization will be useful in the proofs of the properties later.

Let  $S_H^2$  denote the subset of  $S^2$  consisting of the vectors whose vertical coordinate is in  $] -\frac{1}{2}, \frac{1}{2}[$ .

**Proposition 17.2.** *Let  $(\mathcal{C}, \tau)$  be a parallelized rational homology cylinder. Let  $L: \mathcal{L} \hookrightarrow \mathcal{C}$  be a long tangle of  $\check{R}(\mathcal{C})$ . Let  $N$  be an integer greater than 1. Then there exist  $(X_1, X_2, \dots, X_{3N}) \in (S_H^2)^{3N}$ ,  $M \in [1, +\infty[$ , and propagating chains  $P(i)$  of  $(C_2(\check{R}(\mathcal{C})), \tau)$  for  $i \in \underline{3N}$  such that*

- $P(i)$  intersects the domain  $D(p_\tau)$  of Notation 16.14 as  $p_\tau^{-1}(X_i)$ ,
- the  $P(i) \cap C_2(R_{M,[-M,M]}(\mathcal{C}))$  are in general  $3N$ -position with respect to  $L$ , with the natural generalization of the notion of Definition 11.3 (where  $R_{M,[-M,M]}(\mathcal{C})$  replaces  $R$ , with Notation 16.14),

- the intersections

$$I_S\left(\Gamma, (P(i))_{i \in \underline{3N}}\right) = \bigcap_{e \in E(\Gamma)} p_e^{-1}\left(P(j_E(e))\right)$$

in  $C(R(\mathcal{C}), L; \Gamma)$  are transverse and located in  $C(R_{M, [-M, M]}(\mathcal{C}), L; \Gamma)$  for any  $\Gamma \in \mathcal{D}_{\underline{3N}}^e(\mathcal{L}) = \cup_{k \in \mathbb{N}} \mathcal{D}_{k, \underline{3N}}^e(\mathcal{L}) = \cup_{k=0}^{3N} \mathcal{D}_{k, \underline{3N}}^e(\mathcal{L})$ ,

- for any  $\alpha > 0$ , there exists  $\beta > 0$  such that

$$\bigcap_{e \in E(\Gamma)} p_e^{-1}\left(N_\beta\left(P(j_E(e))\right)\right) \subset N_\alpha\left(I_S\left(\Gamma, (P(i))_{i \in \underline{3N}}\right)\right)$$

for any  $\Gamma \in \mathcal{D}_{\underline{3N}}^e(\mathcal{L})$ , with the notation of Definition 11.6,

- there exists an open ball  $B_X$  around  $(X_1, X_2, \dots, X_{3N})$  in  $(S^2)^{\underline{3N}}$  such that  $B_X \subset (S_H^2)^{\underline{3N}}$  and for any  $(Y_1, Y_2, \dots, Y_{3N}) \in B_X$ , there exist propagating chains  $P(i)(Y_i)$  of  $(C_2(\check{R}(\mathcal{C})), \tau)$  satisfying all the above conditions with respect to  $Y_i$  with the same  $M$ .

The set of  $(X_1, X_2, \dots, X_{3N}) \in (S_H^2)^{\underline{3N}}$  such that there exist

$$M = M(X_1, X_2, \dots, X_{3N}) \in [1, +\infty[$$

and propagating chains  $P(i)$  of  $(C_2(\check{R}(\mathcal{C})), \tau)$  satisfying the above conditions is dense in  $(S_H^2)^{\underline{3N}}$ .

In order to prove the proposition, we begin by producing some

$$(W_1, W_2, \dots, W_{3N}) \in (S_H^2)^{\underline{3N}}$$

(in a given neighborhood of some  $(W_1^0, W_2^0, \dots, W_{3N}^0)$  in  $(S_H^2)^{\underline{3N}}$ ) and

$$M(W_1, W_2, \dots, W_{3N}) \in [1, +\infty[.$$

For a one-manifold  $\mathcal{L}$  and a finite set  $A$ , the set of connected  $A$ -numbered degree  $k$  Jacobi diagrams with support  $\mathcal{L}$  without looped edges is denoted by  $\mathcal{D}_{k, A}^c(\mathcal{L})$ .

**Lemma 17.3.** *Let  $N$  be an integer greater than 1. Let  $B$  be a finite set. Let  $y: B \hookrightarrow D_1$  be a planar configuration. For a  $\underline{3N}$ -numbered Jacobi diagram  $\Gamma$  on  $\sqcup_{b \in B} \mathbb{R}_b$ , define the semi-algebraic map*

$$g(\Gamma): C(S^3, y(B) \times \mathbb{R}; \Gamma) \times (S^2)^{\underline{3N} \setminus j_E(E(\Gamma))} \rightarrow (S^2)^{\underline{3N}}$$

to be the product  $(\prod_{e \in E(\Gamma)} p_e, S^2) \times \mathbf{1}((S^2)^{3N \setminus j_E(E(\Gamma))})$ .

The subset  $\mathcal{O}(N, y)$  of  $(S_H^2)^{3N}$  of points that are in the complement of the images of the maps  $g(\Gamma)$  for all  $\Gamma \in \cup_{k=1}^{3N} \mathcal{D}_{k, 3N}^c(\sqcup_{b \in B} \mathbb{R}_b)$  is dense and open.<sup>1</sup>

PROOF: It suffices to prove that the complement of the image of the map  $g(\Gamma)$  is open and dense for any of the finitely many graphs  $\Gamma \in \cup_{k=1}^{3N} \mathcal{D}_{k, 3N}^c(\mathbb{R})$ . The dimension of  $C(S^3, y(B) \times \mathbb{R}; \Gamma)$  is the same as the dimension of  $(S^2)^{j_E(E(\Gamma))}$ . The quotient of  $\check{C}(S^3, y(B) \times \mathbb{R}; \Gamma)$  by global vertical translations is a semi-algebraic set with dimension one less. Thus, the image of  $g(\Gamma)$  is a compact semi-algebraic subset of  $(S_H^2)^{3N}$  of codimension at least one. Its complement is thus an open dense semi-algebraic subset of  $(S_H^2)^{3N}$ .  $\square$

Note the following easy lemma.

**Lemma 17.4.** *Let  $a$  and  $h$  denote two vectors of  $\mathbb{R}^n$  such that  $a$  and  $a + h$  are different from 0. Then we have*

$$\left\| \frac{1}{\|a+h\|}(a+h) - \frac{1}{\|a\|}a \right\| \leq 2 \frac{\|h\|}{\|a\|}.$$

PROOF: The left-hand side can be written as

$$\left\| \frac{1}{\|a\|}h + \left( \frac{1}{\|a+h\|} - \frac{1}{\|a\|} \right) (a+h) \right\|.$$

So it is less than or equal to

$$\left( \frac{\|h\|}{\|a\|} + \frac{\| \|a\| - \|a+h\| \|}{\|a\|} \right).$$

$\square$

**Lemma 17.5.** *Under the assumptions of Lemma 17.3, assume that the image of the configuration  $y: B \hookrightarrow D_1$  contains 0. Equip  $(S_H^2)^{3N}$  with the distance coming from the Euclidean norm of  $(\mathbb{R}^3)^{3N}$ . Let  $\varepsilon \in ]0, \frac{1}{20^{12N}}[$  be such that the ball  $B(W_1, W_2, \dots, W_{3N})$  centered at  $(W_1, W_2, \dots, W_{3N})$  of radius  $24N\varepsilon^{\frac{1}{12N}}$  of  $(S_H^2)^{3N}$  sits in the subset  $\mathcal{O}(N, y)$  of Lemma 17.3.*

Let  $L: \mathcal{L} \hookrightarrow \mathcal{C}$  be a long tangle of  $\check{R}(\mathcal{C})$  whose bottom and top planar configurations are subconfigurations (i.e., restrictions) of  $y$ .

Let  $D(W_i, \varepsilon)$  be the disk of radius  $\varepsilon$  centered at  $W_i$  in  $S^2$ . For  $i \in \underline{3N}$  and  $Y_i \in D(W_i, \varepsilon)$ , let  $P(Y_i)$  be a propagating chain of  $(C_2(R(\mathcal{C})), \tau)$  that

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<sup>1</sup>Though we are going to study  $\mathcal{Z}^f$  up to degree  $N$ , higher degree diagrams will occur in our proofs. See the proof of Lemma 17.5.

coincides with  $p_\tau^{-1}(Y_i)$  on the domain  $D(p_\tau)$  of Notation 16.14. Then for any  $\Gamma \in \cup_{k=1}^{3N} \mathcal{D}_{k,3N}^c(\mathcal{L})$ , we have

$$\bigcap_{e \in E(\Gamma)} p_e^{-1} \left( \cup_{Y_{j_E(e)} \in B(W_{j_E(e)}, \varepsilon)} P(Y_{j_E(e)}) \right) \subset C \left( R_{\frac{1}{\sqrt{\varepsilon}}, [-\frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}}]}(\mathcal{C}), L; \Gamma \right).$$

For  $i \in \underline{3N}$ , let  $\omega_{i,S^2}$  be a two-form of  $S^2$  supported in  $D(W_i, \varepsilon)$ . Then for any family  $(\omega_i)_{i \in \underline{3N}}$  of closed propagating forms  $\omega_i$  of  $(C_2(R(\mathcal{C})), \tau)$  restricting to  $D(p_\tau)$  as  $p_\tau^*(\omega_{i,S^2})$ , the support of  $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega_{j_E(e)})$  is included in  $C(R_{\frac{1}{\sqrt{\varepsilon}}, [-\frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}}]}(\mathcal{C}), L; \Gamma)$ .

PROOF: Fix a connected  $\underline{3N}$ -numbered Jacobi diagram  $\Gamma$  on the domain  $\mathcal{L}$  of  $L$ . Note that  $\Gamma$  has at most  $6N$  vertices. Define a sequence  $\alpha_1, \dots, \alpha_{6N}$  by

$$\alpha_k = \varepsilon^{\frac{1-k}{12N}}.$$

Since  $\varepsilon$  is smaller than  $\frac{1}{20^{12N}}$ , we have  $\varepsilon^{-\frac{1}{12N}} > 20$  and  $\alpha_2 > 20$ .

Define an open covering of  $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$  associated to colorings of the vertices of  $\Gamma$  by colors blue and  $k$ , for  $k \in \underline{6N}$ , such that

- vertices of color 1 go to (or belong to)  $\mathring{R}_{3,[-2,3]}(\mathcal{C})$ ,
- blue vertices and vertices of color  $k \geq 2$  do not go to  $R_{2,[-1,2]}(\mathcal{C})$ , with Notation 16.14,
- any vertex of color 2 is connected to a vertex of color 1 by an edge of  $\Gamma$ ,
- any vertex of color 2 is at a distance smaller than  $5\alpha_2$  from  $(0, 0) \in \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$  (with respect to the Euclidean norm of  $\mathbb{R}^3$ ),
- for  $k$  such that  $2 \leq k \leq 6N - 1$ , any vertex of color  $(k + 1)$  is connected to a vertex of color  $k$  by an edge of length smaller than  $5\alpha_{k+1}$ ,<sup>2</sup>
- when there is an edge of  $\Gamma$  between a blue vertex and a vertex colored by 1, the distance between the blue vertex and  $(0, 0)$  is greater than  $3\alpha_2$  (with respect to the Euclidean distance of  $\mathbb{R}^3$ ),
- when there is an edge of  $\Gamma$  between a blue vertex and a vertex colored by  $k$  for  $2 \leq k \leq 6N - 1$ , the distance between the two vertices is greater than  $3\alpha_{k+1}$ . (Since  $\Gamma$  has at most  $6N$  vertices, if there is a blue vertex, then no vertex can be colored by  $6N$ .)

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<sup>2</sup>This edge length makes sense since vertices of color  $k \geq 2$  belong to  $R_{2,[-1,2]}^c(\mathcal{C}) \subset \mathbb{R}^3$ .

The subset  $U(\mathbf{c})$  of  $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$ , consisting of the configurations that satisfy the above conditions with respect to a coloring  $\mathbf{c}$  of the vertices is open. Let us prove that  $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$  is covered by these sets. For a configuration  $c$ , color by 1 its vertices that are in  $\check{R}_{3,[-2,3]}(\mathcal{C})$ . Then color by 2 all the still uncolored vertices  $v$  connected to a vertex of color 1 (by an edge of  $\Gamma$ ) such that  $d(v, (0, 0)) < 5\alpha_2$ . Continue by coloring all the possible uncolored vertices connected to a vertex of color 2 by an edge of length smaller than  $5\alpha_3$  by 3, and so on, in order to end up with a coloring, which obviously satisfies the above conditions, by coloring the uncolored vertices blue.

Note that the distance between a vertex colored by  $k \geq 2$  and the point  $(0, 0) \in \mathbb{R}^3$  is less than

$$\left( 5 \sum_{i=2}^k \alpha_i = 5 \frac{\alpha_{k+1} - \alpha_2}{\varepsilon^{-\frac{1}{12N}} - 1} \right) < 5 \frac{\alpha_{k+1}}{\frac{5}{6} \varepsilon^{-\frac{1}{12N}}} \leq 6\alpha_k \leq 6\alpha_{6N},$$

where since  $\varepsilon^{-\frac{1}{12N}}$  is greater than 20, we have

$$\left( 6\alpha_{6N} = 6(\varepsilon^{-\frac{1}{12N}})^{6N-1} \right) \leq \left( (\varepsilon^{-\frac{1}{12N}})^{6N} = \frac{1}{\sqrt{\varepsilon}} \right).$$

Thus, the vertices colored by some  $k$  are in  $R_{\frac{1}{\sqrt{\varepsilon}}, [-\frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}}]}(\mathcal{C})$ .

Let us prove that an open set  $U(\mathbf{c})$  associated to a coloring  $\mathbf{c}$  for which the color blue appears cannot intersect  $\bigcap_{e \in E(\Gamma)} p_e^{-1}(\cup_{Y_{j_E(e)} \in B(W_{j_E(e), \varepsilon})} P(Y_{j_E(e)}))$ . Fix such a coloring  $\mathbf{c}$ . Remove from  $\Gamma$  all the edges that do not contain a blue vertex (without removing their ends). Let  $\Gamma_b$  be a connected component with at least one blue vertex of the obtained graph. It has blue vertices, which are trivalent or univalent (in  $\Gamma$  and  $\Gamma_b$ ). The blue univalent vertices go to  $y \times \mathbb{R}$ . Color its other vertices colored by some  $k$  red. Red vertices may have 1, 2, or 3 adjacent edges in  $\Gamma_b$ . Let  $\Gamma'_b$  be the uni-trivalent graph obtained by blowing up  $\Gamma_b$  at its red vertices by replacing such a vertex by a red univalent vertex for each adjacent edge. Color the edges between blue vertices blue, and the edges between a blue vertex and a red one purple. To a configuration of  $U(\mathbf{c})$  in  $\bigcap_{e \in E(\Gamma)} p_e^{-1}(\cup_{Y_{j_E(e)} \in B(W_{j_E(e), \varepsilon})} P(Y_{j_E(e)}))$  associate the configuration of  $\Gamma'_b$  obtained by sending all the red vertices to  $\mathbf{o} = (0, 0)$ , leaving the positions of the blue vertices unchanged. Thus,

- the direction of a blue edge numbered by  $i$  is at a distance less than  $\varepsilon$  from  $W_i$ ,
- the direction of a purple edge numbered by  $p$  is at a distance less than  $(\varepsilon + 4\varepsilon^{\frac{1}{12N}})$  from  $W_p$ .

Let us justify the second assertion. Let  $\mathbf{b}$  denote the blue vertex of the purple edge numbered by  $p$ , and let  $\mathbf{r}$  be its red vertex in the configured graph  $\Gamma$ . Assume that  $\mathbf{r}$  is colored by  $k$  with  $k \geq 2$ . Then we have

$$d(\mathbf{b}, \mathbf{r}) > 3\alpha_{k+1}, (\mathbf{o}, \mathbf{b}) \in D(p_\tau), (\mathbf{r}, \mathbf{b}) \in D(p_\tau), \text{ and } d(\mathbf{o}, \mathbf{r}) < 6\alpha_k.$$

Since the configuration consisting of  $\mathbf{b}$  and  $\mathbf{r}$  in  $\Gamma$  is in some  $P(Y_p)$ , we have

$$\left| \pm \frac{1}{\|\vec{\mathbf{r}\mathbf{b}}\|} \vec{\mathbf{r}\mathbf{b}} - W_p \right| \leq \varepsilon.$$

Apply Lemma 17.4 with  $a = \vec{\mathbf{r}\mathbf{b}}$  and  $h = \vec{\mathbf{o}\mathbf{r}}$  to obtain

$$\left\| \frac{1}{\|\vec{\mathbf{o}\mathbf{b}}\|} \vec{\mathbf{o}\mathbf{b}} - \frac{1}{\|\vec{\mathbf{r}\mathbf{b}}\|} \vec{\mathbf{r}\mathbf{b}} \right\| \leq \frac{2\|\vec{\mathbf{o}\mathbf{r}}\|}{\|\vec{\mathbf{r}\mathbf{b}}\|} < \frac{4\alpha_k}{\alpha_{k+1}} \leq 4\varepsilon^{\frac{1}{12N}}.$$

This proves the second assertion when  $\mathbf{r}$  is colored by  $k \geq 2$ .

Assume that  $\mathbf{r}$  is colored by 1. Then we have  $d(\mathbf{b}, \mathbf{o}) > 3\alpha_2$ . When the configuration consisting of  $\mathbf{b}$  and  $\mathbf{r}$  in  $\Gamma$  is in some  $P(Y_p)$ , there exists an  $\mathbf{s}$  in  $D_3 \times [-2, 3]$  such that the direction of  $\pm \vec{\mathbf{s}\mathbf{b}}$  is at a distance less than  $\varepsilon$  from  $W_p$ . Here, we have  $d(\mathbf{o}, \mathbf{s}) < 5\alpha_1$ . So the direction of  $\pm \vec{\mathbf{o}\mathbf{b}}$  is still at a distance less than  $(\varepsilon + 4\varepsilon^{\frac{1}{12N}})$  from  $W_p$ . Indeed Lemma 17.4 applied with  $a = \vec{\mathbf{o}\mathbf{b}}$  and  $h = \vec{\mathbf{s}\mathbf{o}}$  yields

$$\left\| \frac{1}{\|\vec{\mathbf{o}\mathbf{b}}\|} \vec{\mathbf{o}\mathbf{b}} - \frac{1}{\|\vec{\mathbf{s}\mathbf{b}}\|} \vec{\mathbf{s}\mathbf{b}} \right\| \leq \frac{2\|\vec{\mathbf{o}\mathbf{s}}\|}{\|\vec{\mathbf{o}\mathbf{b}}\|} \leq \frac{10\alpha_1}{3\alpha_2} < 4\varepsilon^{\frac{1}{12N}}.$$

Therefore, the directions of the edges numbered by  $i$  of the configured graph  $\Gamma'_b$  are at a distance less than  $(\varepsilon + 4\varepsilon^{\frac{1}{12N}} \leq 8\varepsilon^{\frac{1}{12N}})$  from the  $W_i$ . But the directions of these edges cannot be in the image of  $\prod_{e \in E(\Gamma'_b)} p_{e, S^2}$  according to our conditions. Indeed, together with  $(W_i)_{i \in 3N \setminus j_E(E(\Gamma'_b))}$ , they form a  $3N$ -tuple that is at a distance less than  $3N \times (8\varepsilon^{\frac{1}{12N}})$  from  $(W_i)_{i \in 3N}$ .

Thus  $\bigcap_{e \in E(\Gamma)} p_e^{-1}(\cup_{Y_{j_E(e)} \in B(W_{j_E(e)}, \varepsilon)} P(Y_{j_E(e)}))$  does not intersect the open subsets of the coverings that use the blue color.

It is now easy to conclude. □

**PROOF OF PROPOSITION 17.2:** Fix an  $\varepsilon$  as in Lemma 17.5. For a diagram  $\Gamma$  of  $\mathcal{D}_{3N}^\varepsilon(\mathcal{L})$  and a subset  $E$  of  $E(\Gamma)$ , the map

$$q(\Gamma, E) = \prod_{e \in E} p_\tau \circ p_e : C\left(R_{\frac{1}{\varepsilon}, [-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]}(\mathcal{C}), L; \Gamma\right) \cap \bigcap_{e \in E} p_e^{-1}(D(p_\tau)) \rightarrow (S^2)^E$$

has an open dense set of regular values. The product of this set by  $(S^2)^{\underline{3N} \setminus E}$  is also open and dense. So is the intersection  $\mathcal{I}_q$  over all such pairs  $(\Gamma, E)$ . Thus, there exist  $(X_1, X_2, \dots, X_{3N})$  in this intersection and  $\alpha \in ]0, \varepsilon]$  such that  $\prod_{i=1}^{3N} B(X_i, \alpha) \subset \mathcal{I}_q \cap \prod_{i=1}^{3N} B(W_i, \varepsilon)$  for the  $B(W_i, \varepsilon)$  of Lemma 17.5. For  $(Y_1, Y_2, \dots, Y_{3N}) \in \prod_{i=1}^{3N} B(X_i, \alpha)$ , let  $P(Y_i)$  be a propagator of  $(C_2(\check{R}(\mathcal{C})), \tau)$  restricting to  $D(p_\tau)$  as  $p_\tau^{-1}(Y_i)$ , for each  $i \in \underline{3N}$ , as in Lemma 16.15. Then the  $P(Y_i)$  can be put in general  $3N$  position as in Section 11.3, by changing them only on  $\check{C}_2(R_{2,[-1,2]}(\mathcal{C}))$  since they satisfy the general position conditions on the boundaries. Thus, Proposition 17.2 holds with  $M = \frac{1}{\sqrt{\varepsilon}}$ . For a given  $\alpha > 0$ , the existence of a  $\beta > 0$  such that

$$\bigcap_{e \in E(\Gamma)} p_e^{-1} \left( N_\beta \left( P(j_E(e)) \right) \right) \subset N_\alpha \left( I_S \left( \Gamma, (P(i))_{i \in \underline{3N}} \right) \right)$$

for any  $\Gamma \in \mathcal{D}_{\underline{3N}}^\varepsilon(\mathcal{L})$  can be proved as in the end of the proof of Lemma 11.13.  $\square$

**Corollary 17.6.** *For forms  $\omega_i$   $\beta$ -dual (as in Definition 11.6) to the  $P(i)$  of Proposition 17.2, for any subset  $A$  of  $\underline{3N}$  with cardinality  $3k$ ,*

$$Z(\mathcal{C}, L, \tau, A, (\omega_i)_{i \in \underline{3N}}),$$

*which is defined in Theorem 16.9, is rational.*

PROOF: As in Lemma 11.7, the involved configuration space integrals can be computed as algebraic intersections of rational preimages of the  $P(i)$ .  $\square$

## 17.2 Functoriality

In this chapter, we prove the functoriality of  $\mathcal{Z}^f$ , which implies the multiplicativity of  $\mathcal{Z}$  under connected sum. A reader only interested in the latter proof can read the proof by replacing the set  $B$  of strands by  $\{0\}$  and by viewing  $\check{R}(\mathcal{C}_j)$  as an asymptotically standard  $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ , identified with  $\mathbb{C} \times \mathbb{R}$  outside  $D_1 \times [0, 1]$ , where  $D_1$  is the unit disk of  $\mathbb{C}$ , for  $j \in \underline{2}$ .

**Proposition 17.7.** *Let  $N \in \mathbb{N}$ . Let  $(\mathcal{C}_1, L_1, \tau_1)$  and  $(\mathcal{C}_2, L_2, \tau_2)$  be two composable tangles. There exist volume-one forms  $\omega(i, S^2)$  of  $S^2$  for  $i \in \underline{3N}$  such that*

$$Z(\mathcal{C}_1 \mathcal{C}_2, L_1 L_2, \tau_1 \tau_2, A, (\omega(i, S^2))_{i \in A}) = \sum_{(A_1, A_2) \in P_2(A)} \frac{|A_1|! |A_2|!}{|A|!} Z(\mathcal{C}_1, L_1, \tau_1, A_1, (\omega(i, S^2))) Z(\mathcal{C}_2, L_2, \tau_2, A_2, (\omega(i, S^2)))$$

for any subset  $A$  of  $\underline{3N}$  with cardinality  $3n$ , with the notation of Theorem 16.9.

PROOF: Let  $y: B \hookrightarrow D_1$  be a planar configuration whose image contains 0 and the images of the bottom and top configurations of  $L_1$  and  $L_2$ . Let  $B(X_1, X_2, \dots, X_{3N})$  be a ball centered at  $(X_1, X_2, \dots, X_{3N})$  of radius  $24N\varepsilon^{\frac{1}{12N}}$  of  $(S_H^2)^{3N}$  sitting in the subset  $\mathcal{O}(N, y)$  of Lemma 17.3, with  $\varepsilon \in ]0, \frac{1}{20^{12N}}[$  as in Lemma 17.5. For  $i \in \underline{3N}$ , let  $\omega(i, S^2)$  be a volume-one form on  $S^2$  supported on a disk  $D(X_i, \varepsilon)$  of  $S_H^2$  of radius  $\varepsilon$  centered at  $X_i$ . Define an extension  $\omega_1(i)$  of  $p_{\tau_1}^*(\omega(i, S^2))$  to  $C_2(R(\mathcal{C}_1))$  as in Lemma 16.15. Let  $m_\varepsilon$  denote the multiplication by  $\varepsilon$  in  $\mathbb{R}^3$ . Let  $m_{1,\varepsilon}$  be a diffeomorphism from  $\check{R}(\mathcal{C}_1)$  to a manifold denoted by  $\check{R}(\varepsilon\mathcal{C}_1)$  such that  $m_{1,\varepsilon}$  coincides with  $m_\varepsilon$  on  $\mathbb{R}^3 \setminus \text{Int}(D_1) \times ]0, 1[$ . Call  $(\varepsilon\mathcal{C}_1, \varepsilon L_1)$  the intersection of  $m_{1,\varepsilon}((\mathcal{C}_1, L_1))$  with the part that replaces  $D_1 \times [0, 1]$  in  $\check{R}(\varepsilon\mathcal{C}_1)$ . Note that  $\check{R}(\varepsilon\mathcal{C}_1)$  is standard outside  $D_\varepsilon \times [0, \varepsilon]$ . Use forms  $(m_{1,\varepsilon}^{-1})^*(\omega_1(i))$  for  $(\varepsilon\mathcal{C}_1, \varepsilon L_1)$ . So we have

$$Z\left(\varepsilon\mathcal{C}_1, \varepsilon L_1, m_{1,\varepsilon*}(\tau_1), A_1, (\omega(i, S^2))\right) = Z\left(\mathcal{C}_1, L_1, \tau_1, A_1, (\omega(i, S^2))\right)$$

for any subset  $A_1$  of  $\underline{3N}$  with cardinality multiple of 3, with  $m_{1,\varepsilon*}(\tau_1) = Tm_{1,\varepsilon} \circ \tau_1 \circ (m_{1,\varepsilon}^{-1} \times m_\varepsilon^{-1})$ . Define  $\omega_2(i)$  on  $C_2(\check{R}(\mathcal{C}_2))$ , so that  $\omega_2(i)$  extends  $p_{\tau_2}^*(\omega(i, S^2))$ , as in Lemma 16.15. Let  $T_{2,\varepsilon}$  be the vertical translation of  $\mathbb{R}^3$  by  $(0, 0, 1 - \varepsilon)$ . Let  $m_{2,\varepsilon}$  be a diffeomorphism from  $\check{R}(\mathcal{C}_2)$  to a manifold denoted by  $\check{R}(\varepsilon\mathcal{C}_2)$  such that  $m_{2,\varepsilon}$  coincides with the composition  $T_{2,\varepsilon} \circ m_\varepsilon$  on  $\mathbb{R}^3 \setminus \text{Int}(D_1) \times ]0, 1[$ . Call  $(\varepsilon\mathcal{C}_2, \varepsilon L_2)$  the intersection of  $m_{2,\varepsilon}((\mathcal{C}_2, L_2))$  with the part that replaces  $D_1 \times [0, 1]$  in  $\check{R}(\varepsilon\mathcal{C}_2)$ . Note that  $\check{R}(\varepsilon\mathcal{C}_2)$  is standard outside  $D_\varepsilon \times [1 - \varepsilon, 1]$ . Use forms  $(m_{2,\varepsilon}^{-1})^*(\omega_2(i))$  for  $(\varepsilon\mathcal{C}_2, \varepsilon L_2)$ . So we have

$$Z\left(\varepsilon\mathcal{C}_2, \varepsilon L_2, m_{2,\varepsilon*}(\tau_2), A_2, (\omega(i, S^2))\right) = Z\left(\mathcal{C}_2, L_2, \tau_2, A_2, (\omega(i, S^2))\right)$$

for any subset  $A_2$  of  $\underline{3N}$  with cardinality multiple of 3, where  $m_{2,\varepsilon*}(\tau_2) = Tm_{2,\varepsilon} \circ \tau_2 \circ (m_{2,\varepsilon}^{-1} \times m_\varepsilon^{-1})$ .

Now, let  $(\varepsilon\mathcal{C}_1\mathcal{C}_2, \varepsilon L_1 L_2)$  be obtained from  $(\varepsilon\mathcal{C}_1, \varepsilon L_1)$  by inserting

$$(\varepsilon\mathcal{C}_2, \varepsilon L_2) \cap R_{\varepsilon, [1-\varepsilon, 1]}(\varepsilon\mathcal{C}_2)$$

instead of  $D_\varepsilon \times [1 - \varepsilon, 1]$ . Here and below, we use a natural extension of the  $R_{\dots}$  notation introduced in 16.14. Define the propagator  $\omega(i)$  of  $(R(\varepsilon\mathcal{C}_1\mathcal{C}_2), \tau_1\tau_2)$ ,

- to coincide with  $(m_{1,\varepsilon}^{-1})^*(\omega_1(i))$  on  $C_2(R(\varepsilon\mathcal{C}_1\mathcal{C}_2) \setminus R_{2\varepsilon, [1-2\varepsilon, 1+\varepsilon]}(\varepsilon\mathcal{C}_1\mathcal{C}_2))$ ,
- to coincide with  $(m_{2,\varepsilon}^{-1})^*(\omega_2(i))$  on  $C_2(R(\varepsilon\mathcal{C}_1\mathcal{C}_2) \setminus R_{2\varepsilon, [-\varepsilon, 2\varepsilon]}(\varepsilon\mathcal{C}_1\mathcal{C}_2))$ , and



- to be zero on  $R_{.1,[-.1,.1]}(\varepsilon\mathcal{C}_1\mathcal{C}_2) \times R_{.1,[.9,1.1]}(\varepsilon\mathcal{C}_1\mathcal{C}_2)$  and

$$R_{.1,[.9,1.1]}(\varepsilon\mathcal{C}_1\mathcal{C}_2) \times R_{.1,[-.1,.1]}(\varepsilon\mathcal{C}_1\mathcal{C}_2).$$

This definition is consistent because the form  $\omega(i, S^2)$  is supported in  $S_H^2$ . Then compute  $Z(\mathcal{C}_1\mathcal{C}_2, L_1L_2, \tau_1\tau_2, A, (\omega(i, S^2))_{i \in A})$  as

$$Z\left(\varepsilon\mathcal{C}_1\mathcal{C}_2, \varepsilon L_1L_2, \tau_1\tau_2, A, (\omega(i, S^2))_{i \in A}\right)$$

with these propagators  $\omega(i)$ . We are going to prove the following lemma under the above hypotheses.

**Lemma 17.8.** *For any  $\underline{3N}$ -numbered Jacobi diagram  $\Gamma_1$  on the domain of  $L_1$  of degree at most  $N$ , the form on  $C(R(\varepsilon\mathcal{C}_1), \varepsilon L_1; \Gamma_1)$*

$$\bigwedge_{e \in E(\Gamma_1)} (p_e m_{1,\varepsilon}^{-1})^* \left( \omega_1(j_E(e)) \right)$$

is supported on  $C_{V(\Gamma_1)}(R_{.1,[-.1,.1]}(\varepsilon\mathcal{C}_1)) \cap C(R(\varepsilon\mathcal{C}_1), \varepsilon L_1; \Gamma_1)$ .

For any  $\underline{3N}$ -numbered Jacobi diagram  $\Gamma_2$  on the domain of  $L_2$  of degree at most  $N$ , the form  $\bigwedge_{e \in E(\Gamma_2)} (p_e m_{2,\varepsilon}^{-1})^* (\omega_2(j_E(e)))$  on  $C(R(\varepsilon\mathcal{C}_2), \varepsilon L_2; \Gamma_2)$  is supported on  $C_{V(\Gamma_2)}(R_{.1,[.9,1.1]}(\varepsilon\mathcal{C}_2)) \cap C(R(\varepsilon\mathcal{C}_2), \varepsilon L_2; \Gamma_2)$ .

For any  $\underline{3N}$ -numbered Jacobi diagram  $\Gamma$  on the domain of  $L_1L_2$  of degree at most  $N$ , the form  $\bigwedge_{e \in E(\Gamma)} p_e^* (\omega(j_E(e)))$  on  $C(R(\varepsilon\mathcal{C}_1\mathcal{C}_2), \varepsilon L_1L_2; \Gamma)$  is supported on

$$\cup_{\{V_1, V_2\} \in P_2(\Gamma)} C_{V_1}(R_{.1,[-.1,.1]}(\varepsilon\mathcal{C}_1\mathcal{C}_2)) \times C_{V_2}(R_{.1,[.9,1.1]}(\varepsilon\mathcal{C}_1\mathcal{C}_2)),$$

where  $P_2(\Gamma)$  denotes the set of partitions  $\{V_1, V_2\}$  of  $V(\Gamma)$  into two disjoint subsets  $V_1$  and  $V_2$  such that no edge of  $\Gamma$  has one vertex in  $V_1$  and the other in  $V_2$ .

Assuming Lemma 17.8, we can conclude the proof of Proposition 17.7 as follows. Lemma 17.8 implies that both sides of the equality to be proved are sums over pairs  $(\Gamma_1, \Gamma_2)$  of  $A$ -numbered diagrams such that  $\Gamma_1$  is a diagram on the domain of  $L_1$ ,  $\Gamma_2$  is a diagram on the domain of  $L_2$ , and  $j_E(E(\Gamma_1)) \cap j_E(E(\Gamma_2)) = \emptyset$ , of terms

$$I\left(\mathcal{C}_1, L_1, \Gamma_1, (\omega_1(i))_{i \in A}\right) I\left(\mathcal{C}_2, L_2, \Gamma_2, (\omega_2(i))_{i \in A}\right) [\Gamma_1] [\Gamma_2].$$

Using Lemma 16.2 to identify the coefficients finishes the proof of Proposition 17.7 up to the proof of Lemma 17.8, which follows.  $\square$

**PROOF OF LEMMA 17.8:** The first two assertions follow from Lemma 17.5. Let us focus on the third one. Fix a  $\underline{3N}$ -numbered Jacobi diagram  $\Gamma$  of degree at most  $N$  on the domain of  $L_1L_2$ . For  $i \in \underline{6N}$ , set  $\beta_i = \varepsilon\alpha_i$  with the sequence  $\alpha_i = \varepsilon^{\frac{1-i}{12N}}$  of the proof of Lemma 17.5. Define an open covering of  $\check{C}(\check{R}(\varepsilon\mathcal{C}_1\mathcal{C}_2), \varepsilon\mathcal{C}_1\mathcal{C}_2; \Gamma)$  associated to colorings of the vertices by colors blue,  $(1, k)$ , and  $(2, k)$ , with  $k \in \underline{6N}$ , such that

- blue vertices and vertices of color  $(j, k)$  with  $j \in \underline{2}$  and  $k \geq 2$  do not go to  $R_{2\varepsilon, [-\varepsilon, 2\varepsilon]}(\varepsilon\mathcal{C}_1\mathcal{C}_2) \cup R_{2\varepsilon, [1-2\varepsilon, 1+\varepsilon]}(\varepsilon\mathcal{C}_1\mathcal{C}_2)$ ,
- vertices of color  $(1, 1)$  go to  $\mathring{R}_{3\varepsilon, [-2\varepsilon, 3\varepsilon]}(\varepsilon\mathcal{C}_1\mathcal{C}_2)$ ,
- vertices of color  $(2, 1)$  go to  $\mathring{R}_{3\varepsilon, [1-3\varepsilon, 1+2\varepsilon]}(\varepsilon\mathcal{C}_1\mathcal{C}_2)$ ,
- for  $j \in \underline{2}$ , any vertex of color  $(j, 2)$  is connected by an edge of  $\Gamma$  to a vertex of color  $(j, 1)$  and is at a distance smaller than  $5\beta_2$  from  $(0, j-1)$  (with respect to the Euclidean norm of  $\mathbb{R}^3$ ),
- for  $j \in \underline{2}$  and  $k$  such that  $2 \leq k \leq 6N-1$ , any vertex of color  $(j, k+1)$  is connected to a vertex of color  $(j, k)$  by an edge of length smaller than  $5\beta_{k+1}$ ,
- when there is an edge of  $\Gamma$  between a blue vertex and a vertex colored by  $(j, 1)$  for  $j \in \underline{2}$ , the distance between the blue vertex and  $(0, j-1)$  is greater than  $3\beta_2$ , and
- when there is an edge of  $\Gamma$  between a blue vertex and a vertex colored by  $(j, k)$  with  $j \in \underline{2}$  and  $2 \leq k \leq 6N-1$ , the distance between the two vertices is greater than  $3\beta_{k+1}$ .

The subset  $U(\mathbf{c})$  of  $\check{C}(\check{R}(\varepsilon\mathcal{C}_1\mathcal{C}_2), \varepsilon\mathcal{C}_1\mathcal{C}_2; \Gamma)$  consisting of the configurations that satisfy the above conditions with respect to a coloring  $\mathbf{c}$  of the vertices is open, and  $\check{C}(\check{R}(\varepsilon\mathcal{C}_1\mathcal{C}_2), \varepsilon\mathcal{C}_1\mathcal{C}_2; \Gamma)$  is covered by these sets as in the proof of Lemma 17.5. The only additional thing to notice is that a vertex could not be simultaneously colored by  $(1, k)$  and by  $(2, k')$  since a vertex colored by  $(j, k)$  is at a distance less than

$$6\beta_k \leq \sqrt{\varepsilon} \leq \frac{1}{20^6}$$

from  $(0, j-1)$ . In particular, the vertices colored by  $(1, k)$  are in

$$R_{.1, [-.1, .1]}(\varepsilon\mathcal{C}_1\mathcal{C}_2),$$

and the vertices colored by  $(2, k)$  are in  $R_{.1, [.9, 1.1]}(\varepsilon\mathcal{C}_1\mathcal{C}_2)$ .

So the form  $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$  vanishes on open sets corresponding to colorings for which a vertex  $(1, k)$  is connected to a vertex  $(2, k')$  (by some edge of  $\Gamma$ ), according to the conditions before Lemma 17.8.

As in the proof of Lemma 17.5, we prove that our form vanishes on open sets  $U(\mathbf{c})$  associated to colorings for which the color blue appears. Fix such a coloring  $\mathbf{c}$  and remove from  $\Gamma$  all the edges that do not contain a blue vertex. Let  $\Gamma_b$  be a connected component of this graph with at least one blue vertex. It has blue vertices, which are trivalent or univalent in  $\Gamma$  and  $\Gamma_b$ . The blue univalent vertices lie on  $\varepsilon y \times \mathbb{R}$ . The other vertices of  $\Gamma_b$  are either colored by some  $(1, k)$ , in which case we color them yellow, or by some  $(2, k)$ , in which case we color them red. Red and yellow vertices may have 1, 2, or 3 adjacent edges in  $\Gamma_b$ . Let  $\Gamma'_b$  be the uni-trivalent graph obtained by blowing up  $\Gamma_b$  at its yellow and red vertices by replacing such a vertex with a univalent vertex of the same color for each adjacent edge. Color the edges between blue vertices blue, the edges between a blue vertex and a yellow one green, and the edges between a blue vertex and a red one purple. To a configuration of  $U(\mathbf{c})$  in the support of  $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$ , associate the configuration of  $\Gamma'_b$  obtained by sending all the yellow vertices to  $\mathbf{o}$  and all the red ones to  $(0, 1)$ , leaving the positions of the blue vertices unchanged. Thus,

- the direction of a blue edge numbered by  $i$  is in the support of  $\omega(i, S^2)$  at a distance less than  $\varepsilon$  from  $X_i$ ,
- the direction of a green edge numbered by  $g$  is at a distance less than  $(\varepsilon + 4\varepsilon^{\frac{1}{12N}})$  from  $X_g$ , (as in the proof of Lemma 17.5),
- the direction of a purple edge numbered by  $p$  is at a distance less than  $(\varepsilon + 4\varepsilon^{\frac{1}{12N}})$  from  $X_p$ .

However, the directions of the edges of  $\Gamma'_b$  cannot be in the image of  $\prod_{e \in E(\Gamma)} p_e, S^2$  according to our conditions in the beginning of the proof of Proposition 17.7 (the  $\varepsilon$  rescaling of  $y$  does not change the image). Therefore, the support of  $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$  does not intersect the open subsets of the covering that use the blue color. It is now easy to conclude. □

For an integer  $N$ ,  $Z_{\leq N}$  denotes the truncation of  $Z$  valued in  $\mathcal{A}_{\leq N}(\mathcal{L}) = \prod_{j=0}^N \mathcal{A}_j(\mathcal{L})$ .

**Theorem 17.9.** *Let  $N$  be a natural number, and let  $(\omega_{i, S^2})_{i \in 3N}$  be a family of volume-one forms of  $S^2$ . For any two composable tangles  $(\mathcal{C}_1, L_1, \tau_1)$  and*

$(\mathcal{C}_2, L_2, \tau_2)$  in parallelized rational homology cylinders, we have

$$\begin{aligned} Z(\mathcal{C}_1\mathcal{C}_2, L_1L_2, \tau_1\tau_2, \cdot, (\omega_{i,S^2})_{i \in \underline{3N}}) \\ = \left( Z(\mathcal{C}_1, L_1, \tau_1, \cdot, (\omega_{i,S^2})) Z(\mathcal{C}_2, L_2, \tau_2, \cdot, (\omega_{i,S^2})) \right)_{\sqcup} \end{aligned}$$

with the notation of Theorem 16.9 and Definition 16.10.

For any two composable  $J_{bb,tt}$ -oriented  $q$ -tangles  $T_1$  and  $T_2$ , we also have

$$\mathcal{Z}^f(T_1T_2, \cdot, (\omega_{i,S^2})_{i \in \underline{3N}}) = \left( \mathcal{Z}^f(T_1, \cdot, (\omega_{i,S^2})) \mathcal{Z}^f(T_2, \cdot, (\omega_{i,S^2})) \right)_{\sqcup},$$

with the notation of Definition 16.44, and

$$\mathcal{Z}^f(T_1)\mathcal{Z}^f(T_2) = \mathcal{Z}^f(T_1T_2).$$

PROOF: Let us prove the first assertion. Apply Theorem 16.9, with

$$\tilde{\omega}(i, 1, S^2) = \omega_{i,S^2} \quad \text{and} \quad \tilde{\omega}(i, 0, S^2) = \omega(i, S^2)$$

with the form  $\omega(i, S^2)$  of Proposition 17.7. We get

$$\begin{aligned} Z_{<N}(\mathcal{C}_1\mathcal{C}_2, L_1L_2, \tau_1\tau_2, \cdot, (\omega_{i,S^2})) \\ = \left( \left( \prod_{j \in I} \widetilde{\text{hol}}_{[0,1]}(\eta(\cdot, p_\tau(U^+K_j))) \right) \#_j \right. \\ \left. \widetilde{\text{hol}}_{[1,0] \times y_1^-}(\eta_{B^-, \cdot}) Z(\mathcal{C}_1\mathcal{C}_2, L_1L_2, \tau_1\tau_2, \cdot, (\omega(i, S^2))) \widetilde{\text{hol}}_{[0,1] \times y_2^+}(\eta_{B^+, \cdot}) \right)_{\sqcup}, \end{aligned}$$

where  $y_i^-$  (resp.  $y_i^+$ ) represents the bottom (resp. top) configuration of  $L_i$ , with  $y_2^- = y_1^+$  and

$$\begin{aligned} Z(\mathcal{C}_1\mathcal{C}_2, L_1L_2, \tau_1\tau_2, \cdot, (\omega(i, S^2))) \\ = \left( Z(\mathcal{C}_1, L_1, \tau_1, \cdot, (\omega(i, S^2))) Z(\mathcal{C}_2, L_2, \tau_2, \cdot, (\omega(i, S^2))) \right)_{\sqcup}. \end{aligned}$$

A neutral factor  $(\widetilde{\text{hol}}_{[0,1] \times y_1^+}(\eta_{B_1^+, \cdot}) \widetilde{\text{hol}}_{[1,0] \times y_2^-}(\eta_{B_2^-, \cdot}))_{\sqcup}$  can be inserted in the middle. So the first equality of the statement becomes clear, up to the behavior of the factors  $\widetilde{\text{hol}}_{[0,1]}(\eta(\cdot, p_\tau(U^+K_j)))$  of Definition 16.6. For these factors, note that a component  $K$  of  $L_1L_2$  consists of a bunch of components  $K_k$  from  $L_1$  and  $L_2$ , for  $k$  in a finite set  $E$ , and that  $\eta(A, p_\tau(U^+K))$  is the sum of the corresponding  $\eta(A, p_\tau(U^+K_k))$ . Lemma 16.8 ensures that we have

$$\widetilde{\text{hol}}_{[0,1]} \left( \sum_{k \in E} \eta(\cdot, p_\tau(U^+K_k)) \right) = \left( \prod_{k \in E} \widetilde{\text{hol}}_{[0,1]} \eta(\cdot, p_\tau(U^+K_k)) \right)_{\sqcup}$$

in the commutative algebra  $\check{\mathcal{A}}(\mathbb{R})$ . This finishes the proof of the first assertion of Theorem 17.9.

Orient  $L_1$  and  $L_2$  in a compatible way. Recall  $p_1(\tau_1\tau_2) = p_1(\tau_1) + p_1(\tau_2)$ . Also recall the associativity of the product  $(\ )_{\sqcup}$ . According to the first assertion applied to straight tangles with the induced parallelization, if  $L_1$  and  $L_2$  are framed by parallels  $L_{1\parallel}$  and  $L_{2\parallel}$  induced by parallelizations  $\tau_k$  such that  $p_{\tau_k}(U^+L_k) \subset S_{WE}$  for  $k \in \underline{2}$ , then we have

$$\begin{aligned} &\mathcal{Z}^f(\mathcal{C}_1\mathcal{C}_2, L_1L_2, (L_1L_2)_{\parallel}, \cdot, (\omega_{i,S^2})_{i \in \underline{3N}}) \\ &= \left( \mathcal{Z}^f(\mathcal{C}_1, L_1, L_{1\parallel}, \cdot, (\omega_{i,S^2})) \mathcal{Z}^f(\mathcal{C}_2, L_2, L_{2\parallel}, \cdot, (\omega_{i,S^2})) \right)_{\sqcup}. \end{aligned}$$

This generalizes to any pair  $((L_1, L_{1\parallel}), (L_2, L_{2\parallel}))$  of parallelized  $J_{bb,tt}$ -oriented tangles with the invariant  $\mathcal{Z}^f$  of framed tangles of Definition 16.17, as follows. When  $(L_1, L_{1\parallel})$  is not representable as a straight tangle with respect to a parallelization, then  $(L_1, L_{1\parallel+1})$  is, where  $L_{1\parallel+1}$  is the parallel of  $L_1$  such that  $(L_{1\parallel+1} - L_{1\parallel})$  is homologous to a positive meridian of  $L_1$  in a tubular neighborhood of  $L_1$  deprived of  $L_1$ . Thus, the known behavior of  $\mathcal{Z}^f$  under such a framing change yields the second equality of the statement when  $T_1$  and  $T_2$  have injective bottom and top configurations. According to Definitions 16.44 and 16.42, the second equality is also true when  $T_1$  and  $T_2$  are  $q$ -braids, thanks to the multiplicativity of  $\text{hol}(\cdot)$  with respect to the product of Definition 16.10 in Lemma 16.34. Thus, Definition 16.44 of  $\mathcal{Z}^f$  implies the second equality for general  $q$ -tangles. The third equality is a direct consequence of the second one when  $\omega_{i,S^2} = \omega_{S^2}$  for all  $i$ .  $\square$

### 17.3 Insertion of a tangle in a trivial $q$ -braid.

In this section, we prove the following result, which is the cabling property of Theorem 13.12 generalized to all variants of the invariant  $\mathcal{Z}^f$  of Definition 16.44.

**Proposition 17.10.** *Let  $B$  be a finite set with cardinality greater than 1. Let  $y \in \mathcal{S}_B(\mathbb{C})$ , let  $y \times [0, 1]$  denote the corresponding  $q$ -braid, and let  $K$  be a strand of  $y \times [0, 1]$ . Let  $L$  be a  $q$ -tangle with domain  $\mathcal{L}$ . Then*

$$\mathcal{Z}^f((y \times [0, 1]) (L/K))$$

*is obtained from  $\mathcal{Z}^f(L)$  by the natural injection from  $\mathcal{A}(\mathcal{L})$  to  $\mathcal{A}((B \times \mathbb{R})(\mathcal{L}/K))$ .*

Furthermore, if  $L$  is  $J_{bb,tt}$ -oriented, for any  $N \in \mathbb{N}$ , for any subset  $A$  of  $\underline{3N}$  whose cardinality is a multiple of 3, and for any family of volume-one forms  $(\omega_{i,S^2})_{i \in \underline{3N}}$ ,  $\mathcal{Z}^f((y \times [0, 1]) (L/K), A, (\omega_{i,S^2})_{i \in \underline{3N}})$  is obtained from  $\mathcal{Z}^f(L, A, (\omega_{i,S^2})_{i \in \underline{3N}})$  by the natural injection from  $\mathcal{A}(\mathcal{L})$  to  $\mathcal{A}((B \times \mathbb{R})(\mathcal{L}/K))$ .

We first prove the following particular case of Proposition 17.10.

**Lemma 17.11.** *Under the hypotheses of Proposition 17.10, furthermore assume that  $y \in \check{\mathcal{S}}_B(\mathbb{C})$  and  $L$  is a  $J_{bb,tt}$ -oriented framed tangle represented by a tangle embedding*

$$L: \mathcal{L} \hookrightarrow \mathcal{C}$$

with injective bottom and top configurations. Let  $N$  be a positive integer. Then there exists a family  $(\omega(i, S^2))_{i \in \underline{3N}}$  of volume-one forms of  $S^2$  such that

$$\mathcal{Z}^f\left((y \times [0, 1]) (L/K), A, (\omega(i, S^2))_{i \in \underline{3N}}\right)$$

is obtained from  $\mathcal{Z}^f(L, A, (\omega(i, S^2))_{i \in \underline{3N}})$  by the natural injection from  $\mathcal{A}(\mathcal{L})$  to  $\mathcal{A}((B \times \mathbb{R})(\mathcal{L}/K))$  for any subset  $A$  of  $\underline{3N}$  whose cardinality is a multiple of 3.

**PROOF:** Without loss of generality, translate and rescale  $y$  so that  $K = \{0\} \times [0, 1]$ . Let  $\eta \in ]0, 1[$  be the distance between  $K$  and the other strands of  $y \times [0, 1]$ . Because of the known variation of  $\mathcal{Z}^f$  under framing changes, there is no loss of generality in assuming that  $L$  is straight with respect to a parallelization  $\tau$ , which we do. Let  $y_1: B \hookrightarrow D_1$  be a planar configuration whose image contains  $y$  and the images of the bottom and top configurations of  $L$ . Let  $B(X_1, X_2, \dots, X_{3N})$  be a ball centered at  $(X_1, X_2, \dots, X_{3N})$  of radius  $24N\varepsilon^{\frac{1}{12N}}$  of  $(S^2_H)^{3N}$  sitting in the subset  $\mathcal{O}(N, y_1)$  of Lemma 17.3, with  $\varepsilon \in ]0, \frac{1}{20^{12N}}[$  as in Lemma 17.5. For  $i \in \underline{3N}$ , let  $\omega(i, S^2)$  be a volume-one form on  $S^2$  supported on a disk  $D(X_i, \varepsilon)$  of  $S^2_H$ . Define  $\omega_1(i)$  on  $C_2(R(\mathcal{C}))$  so that  $\omega_1(i)$  coincides with  $p_\tau^*(\omega(i, S^2))$  on  $D(p_\tau)$ , as in Lemma 16.15. Perform a global homothety  $m_{1,\eta\varepsilon}$  of  $\check{R}(\mathcal{C})$ , of ratio  $\eta\varepsilon$ , where  $D_1 \times [0, 1]$  is consequently changed to  $D_{\eta\varepsilon} \times [0, \eta\varepsilon]$ . Call  $(\eta\varepsilon\mathcal{C}, \eta\varepsilon L)$  the intersection of the image of the long tangle  $(\mathcal{C}, L)$  by this homothety with the part that replaces  $D_1 \times [0, 1]$ , which is now standard outside  $D_{\eta\varepsilon} \times [0, 1]$ . Use forms  $(m_{1,\eta\varepsilon}^{-1})^*(\omega_1(i))$  for  $(\eta\varepsilon\mathcal{C}, \eta\varepsilon L)$ . So we have

$$\mathcal{Z}^f\left(\eta\varepsilon\mathcal{C}, \eta\varepsilon L, m_{1,\eta\varepsilon}^*(\tau), \cdot, (\omega(i, S^2))\right) = \mathcal{Z}^f\left(\mathcal{C}, L, \tau, \cdot, (\omega(i, S^2))\right).$$

Let  $(y \times [0, 1]) (\eta\varepsilon L/K)$  be the tangle obtained from  $(y \times [0, 1])$  by letting  $R_{\eta\varepsilon, [0,1]}(\eta\varepsilon\mathcal{C})$  replace  $D_{\eta\varepsilon} \times [0, 1]$ . Graphs that do not involve vertices of

$R_{2\eta\varepsilon, [-\eta\varepsilon, 2\eta\varepsilon]}(\eta\varepsilon\mathcal{C})$  cannot contribute to

$$\mathcal{Z}^f\left(\eta\varepsilon\mathcal{C}, (y \times [0, 1]) (\eta\varepsilon L/K), A, (\omega(i, S^2))_{i \in \underline{3N}}\right).$$

As in the proof of Lemma 17.5, the only contributing graphs are located in  $\check{R}_{\eta\sqrt{\varepsilon}, [-\eta\sqrt{\varepsilon}, \eta\sqrt{\varepsilon}]}(\eta\varepsilon\mathcal{C})$ .

We conclude that  $\mathcal{Z}^f(\eta\varepsilon\mathcal{C}, (y \times [0, 1]) (\eta\varepsilon L/K), A, (\omega(i, S^2))_{i \in \underline{3N}})$  is obtained from  $\mathcal{Z}^f(\mathcal{C}, L, A, (\omega(i, S^2))_{i \in \underline{3N}})$  by the natural injection from  $\mathcal{A}(\mathcal{L})$  to  $\mathcal{A}((B \times \mathbb{R})(\mathcal{L}/K))$ . Since this is also true when  $\eta$  is replaced by a smaller  $\eta'$ , this is also true when  $\eta\varepsilon L$  is replaced by a legal composition  $\gamma^-(\eta\varepsilon L)\gamma^+$  for braids  $\gamma^-$  and  $\gamma^+$  with constant projections in  $\mathcal{S}_{B^\pm}(CC)$ , which respectively go from  $\eta'y^-$  to  $\eta y^-$ , and from  $\eta y^+$  to  $\eta'y^+$  (up to adjusting the parallelizations), thanks to the isotopy invariance of  $\mathcal{Z}^f$ . Therefore, this is also true at the limit, when  $\eta'$  tends to zero, thanks to Lemma 16.33. See Definition 16.44.  $\square$

**Corollary 17.12.** *Proposition 17.10 is true when  $y \in \check{\mathcal{S}}_B(\mathbb{C})$ .*

PROOF: Recall Theorem 16.45, which expresses the variation of

$$\mathcal{Z}^f\left((y \times [0, 1])(L/K), A, (\omega_{i, S^2})_{i \in \underline{3N}}\right)$$

when  $(\omega_{i, S^2})_{i \in \underline{3N}}$  varies, for  $q$ -tangles. This variation is given by the insertion of factors on components going from bottom to bottom or from top to top, which are identical in both sides of the implicit equality to be proved, and  $D$ -holonomies for the bottom and top configurations, for  $D \subseteq A$ . The  $D$ -holonomies satisfy the duplication property of Proposition 16.43. The  $D$ -holonomies of the bottom and top configurations of  $L$  contribute in the same way to both sides of the equality. The  $D$ -holonomies of the bottom and top configurations of  $y \times [0, 1]$  are inverse to each other. After the insertion, they are duplicated both at the top and at the bottom on possibly different numbers of strands. Let  $B^+$  (resp.  $B^-$ ) be the set of upper (resp. lower)  $\infty$ -components of  $L$ . Lemma 6.23 ensures that for any diagram  $\Gamma$  on  $\mathcal{L}$  and for any duplication  $\pi(B^+ \times K^+)^*$  (resp.  $\pi(B^- \times K^-)^*$ ) of the upper part  $K^+$  (resp. lower part  $K^-$ ) of the long strand of  $K$  by  $B^+ \times [1, +\infty[$  (resp.  $B^- \times ]-\infty, 0]$ ) of a diagram  $\Gamma'$  on  $B \times \mathbb{R}$ , we have

$$\Gamma\pi(B^+ \times K^+)^*(\Gamma') = \pi(B^- \times K^-)^*(\Gamma')\Gamma$$

in  $\mathcal{A}((B \times \mathbb{R})(\mathcal{L}/K))$ . Thus, the “holonomies”  $\pi(B^- \times K^-)^*\widetilde{\text{hol}}_{[1,0] \times y}(\eta_{B, \cdot})$  and  $\pi(B^+ \times K^+)^*\widetilde{\text{hol}}_{[0,1] \times y}(\eta_{B, \cdot})$  cancel, and Proposition 17.10 is true when

$y \in \check{\mathcal{S}}_B(\mathbb{C})$  as soon as the bottom and top configurations of  $L$  may be represented by injective configurations. When  $L = T(\gamma^-)(\mathcal{C}, L, L_{\parallel})T(\gamma^+)$  is a general  $q$ -tangle and  $\gamma^-$  and  $\gamma^+$  are paths of configurations,  $(y \times [0, 1]) \left(\frac{L}{K}\right)$  is equal to

$$(y \times [0, 1]) \left(\frac{T(\gamma^-)}{K}\right) (y \times [0, 1]) \left(\frac{(\mathcal{C}, L, L_{\parallel})}{K}\right) (y \times [0, 1]) \left(\frac{T(\gamma^+)}{K}\right).$$

So the result follows using the Functoriality theorem 17.9 and the cabling theorem for  $q$ -braids (Proposition 16.43).  $\square$

**PROOF OF PROPOSITION 17.10:** Corollary 17.12 leaves us with the case in which  $y$  is a limit configuration. To treat this case, pick a path  $\gamma: [0, 1] \rightarrow \mathcal{S}_B(\mathbb{C})$  such that  $\gamma(1) = y$  and  $\gamma([0, 1]) \subset \check{\mathcal{S}}_B(\mathbb{C})$ , view  $y \times [0, 1]$  as the path composition  $\bar{\gamma}\gamma(0)\gamma$  where  $\gamma(0)$  is thought of as a constant map. If  $y^-$  and  $y^+$  respectively denote the bottom and top configurations of  $L$  and if  $K^-$  and  $K^+$  denote the strand of  $K$  in  $\bar{\gamma}$  and in  $\gamma$ , respectively, then  $(y \times [0, 1]) \left(\frac{L}{K}\right)$  may be expressed as

$$T(\bar{\gamma}) \left(\frac{y^- \times [0, 1]}{K^-}\right) (\gamma(0) \times [0, 1]) \left(\frac{L}{K}\right) T(\gamma) \left(\frac{y^+ \times [0, 1]}{K^+}\right).$$

Use the functoriality theorem 17.9, Corollary 17.12, the cabling theorem for  $q$ -braids (Proposition 16.43), and the commutation argument in the above proof to conclude.  $\square$

## 17.4 Duplication property

We are about to show how  $\mathcal{Z}^f$  and all its variants behave under a general parallel duplication of a component going from bottom to top in a tangle.

**Proposition 17.13.** *Let  $K$  be a component going from bottom to top or from top to bottom in a  $q$ -tangle  $L$  in a rational homology cylinder  $\mathcal{C}$ . Let  $y$  be an element of  $\mathcal{S}_B(\mathbb{C})$  for a finite set  $B$ . Let  $L(y \times K)$  be the tangle obtained by duplicating  $K$  as in Section 13.1. Then we have*

$$\mathcal{Z}^f(L(y \times K)) = \pi(B \times K)^* \mathcal{Z}^f(L)$$

with a natural extension of Notation 6.31. Furthermore, for any  $N \in \mathbb{N}$ , for any family  $(\omega_{i,S^2})_{i \in \underline{3N}}$  of volume-one forms of  $S^2$ , and for any subset  $A$  of  $\underline{3N}$  whose cardinality is a multiple of 3, we have

$$\mathcal{Z}^f(\mathcal{C}, L(y \times K), A, (\omega_{i,S^2})_{i \in \underline{3N}}) = \pi(B \times K)^* \mathcal{Z}^f(\mathcal{C}, L, A, (\omega_{i,S^2})_{i \in \underline{3N}})$$

for any  $J_{bb,tt}$ -orientation of  $L$ .



In order to prove this proposition, we are going to prove the following lemmas.

**Lemma 17.14.** *Let  $L: \mathcal{L} \hookrightarrow \mathcal{C}$  be a straight tangle in a parallelized rational homology cylinder  $(\mathcal{C}, \tau)$ . Let  $K$  be a component of  $L$  going from bottom to top. Let  $y$  be an element of  $\check{\mathcal{S}}_B(\mathbb{C})$  for a finite set  $B$ . Let  $N \in \mathbb{N}$ . There exists a family of volume-one forms  $(\omega(i, S^2))_{i \in \underline{3N}}$  such that*

$$Z\left(\mathcal{C}, L(y \times K), \tau, A, (\omega(i, S^2))_{i \in \underline{3N}}\right) = \pi(B \times K)^* Z\left(\mathcal{C}, L, \tau, A, (\omega(i, S^2))_{i \in \underline{3N}}\right)$$

for any subset  $A$  of  $\underline{3N}$  whose cardinality is a multiple of 3.

**Lemma 17.15.** *Lemma 17.14 implies Proposition 17.13.*

PROOF: The known behavior of  $Z$  under strand orientation changes for components going from bottom to top of Lemma 16.46 allows us to reduce the proof to the case in which  $K$  goes from bottom to top. Lemma 17.14, Theorem 16.45, and Proposition 16.43 imply that Proposition 17.13 holds when  $L$  is a straight tangle (with injective bottom and top configurations) and when  $y \in \check{\mathcal{S}}_B(\mathbb{C})$ . Then the duplication property for braids of Proposition 16.43 and the functoriality imply that Proposition 17.13 holds if  $y \in \check{\mathcal{S}}_B(\mathbb{C})$  for any  $J_{bb,tt}$ -oriented  $q$ -tangle  $L$  for which  $(K, K_{\parallel})$  can be represented by a straight knot with respect to a parallelization  $\tau$  of  $\mathcal{C}$  and its associated parallel. Therefore, Proposition 17.13 is also true if  $y \in \mathcal{S}_B(\mathbb{C})$  by iterating the duplication process as soon as  $(K, K_{\parallel})$  is representable by a straight knot. In particular, it is true when  $K$  is a strand of a trivial braid whose framing has been changed so that  $lk(K, K_{\parallel}) = 2$ . (Recall Lemma 7.39 and Proposition 16.13.) Thanks to the functoriality of  $\mathcal{Z}^f$ , since an element whose degree 0 part is 1 is determined by its square, Proposition 17.13 is also true when  $K$  is a strand of a trivial braid whose framing has been changed so that  $lk(K, K_{\parallel}) = 1$ . If our general  $(K, K_{\parallel})$  is not representable, then Proposition 17.13 is true when  $L$  is composed by a trivial braid such that the framing of the strand  $I$  that extends  $K$  is changed so that  $lk(I, I_{\parallel}) = -1$ . So it is also true for  $L$ .  $\square$

Let us introduce some notation for the proof of Lemma 17.14. Choose a tubular neighborhood

$$N_{\eta_0}(K) = D_{\eta_0} \times \mathbb{R}_K$$

of  $K = \{0\} \times \mathbb{R}_K$  in  $R(\mathcal{C}) \setminus (L(\mathcal{L}) \setminus K)$  for some  $\eta_0$  such that  $0 < 10\eta_0 < 1$ , where  $D_{\eta_0}$  denotes the disk of radius  $\eta_0$  centered at 0 in  $\mathbb{C}$ . Assume that the trivialization  $\tau$  maps  $(d \in D_{\eta_0}, k \in \mathbb{R}_K, e_1 = \vec{N})$  to an oriented tangent vector to  $d \times K$ , and that  $\tau$  maps  $(d, k, (e_2, e_3))$  to the standard frame  $(1, i)$  of  $D_{\eta_0}(\times k) \subset \mathbb{C}$ . Pick a representative  $y$  of  $y$  in  $\check{C}_B[D_{1/2}]$ . For  $\eta \in ]0, \eta_0]$ ,

let  $L(\eta^2 y \times K)$  denote the tangle obtained from  $L$  by replacing  $\{0\} \times \mathbb{R}_K$  by  $\eta^2 y \times \mathbb{R}_K$  in  $D_{\eta_0} \times \mathbb{R}_K$ .

Let us now reduce the proof of Lemma 17.14 to the proof of the following lemma.

**Lemma 17.16.** *There exist  $\eta_1 \in ]0, \eta_0]$  and volume-one forms  $(\omega(i, S^2))$  of  $S^2$  for  $i \in \underline{3N}$  such that*

$$Z\left(\mathcal{C}, L(\eta^2 y \times K), \tau, A, (\omega(i, S^2))_{i \in \underline{3N}}\right) = \pi(B \times K)^* Z\left(\mathcal{C}, L, \tau, A, (\omega(i, S^2))_{i \in \underline{3N}}\right)$$

for any  $\eta \in ]0, \eta_1]$  and any subset  $A$  of  $\underline{3N}$  whose cardinality is a multiple of three.

**Lemma 17.17.** *Lemma 17.16 implies Lemma 17.14.*

PROOF: Definition 16.44 and Lemma 16.33 allow us to write

$$\begin{aligned} \mathcal{Z}^f\left(\mathcal{C}, L(y \times K), A, (\omega(i, S^2))_{i \in \underline{3N}}\right) &= \lim_{\eta \rightarrow 0} \mathcal{Z}^f\left(\mathcal{C}, L(\eta^2 y \times K), A, (\omega(i, S^2))_{i \in \underline{3N}}\right) \\ &= \pi(B \times K)^* \mathcal{Z}^f\left(\mathcal{C}, L, A, (\omega(i, S^2))_{i \in \underline{3N}}\right). \end{aligned}$$

□

To prove Lemma 17.16, we need some preliminary lemmas, which involve the following new type of Jacobi diagram. A *special Jacobi  $\underline{3N}$ -diagram* on  $B \times \mathbb{R}$  is a connected graph  $\Gamma_s$  without looped edges with univalent vertices, trivalent vertices, and one bivalent vertex, equipped with an injection  $j_E$  from its set  $E(\Gamma_s)$  of edges into  $\underline{3N}$  and with an isotopy class of injections  $j_{\Gamma_s}$  from its set  $U(\Gamma_s)$  of univalent vertices into  $B \times \mathbb{R}$ . The space of these diagrams is denoted by  $\mathcal{D}_{\underline{3N}}^{e, \text{special}}(B \times \mathbb{R})$ . For a special Jacobi  $\underline{3N}$ -diagram  $\Gamma_s$  with univalent vertices on at least two strands, the space of configurations of  $\Gamma_s$  with respect to  $y$  is the space  $\check{\mathcal{V}}(y, \Gamma_s)$  of injections of the set  $V(\Gamma_s)$  of vertices of  $\Gamma_s$  into  $\mathbb{C} \times \mathbb{R}$  whose restriction to  $U(\Gamma_s)$  is the composition of  $y \times 1_{\mathbb{R}}$  with an injection from  $U(\Gamma_s)$  into  $B \times \mathbb{R}$  in the isotopy class  $[j_{\Gamma_s}]$ , up to vertical translation, for our representative  $y \in \check{C}_B[D_{\frac{1}{2}}]$ . This space  $\check{\mathcal{V}}(y, \Gamma_s)$  is similar to former spaces  $\check{\mathcal{V}}(y, \Gamma)$  of Section 15.2 and is compactified as in Chapter 15. See also Lemma 16.27. Its compactification is its closure in  $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$ . If  $\Gamma_s$  has no univalent vertices or univalent vertices on one strand, then configurations are also considered up to dilation, and the compactification is again the closure in  $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$ . The configurations of  $\check{\mathcal{V}}(y, \Gamma_s)$  are normalized so that a vertex of  $\Gamma_s$  is sent to  $D_{1/2} \times \{0\}$ .

**Lemma 17.18.** *Let  $B_\infty$  be a finite set. Let  $y_\infty: B_\infty \hookrightarrow D_1$  be a planar configuration. Let  $N$  be a natural number. The set  $\mathcal{O}_s(N, y_\infty)$  of points  $(X_i)_{i \in \underline{3N}}$  of  $(S_H^2)^{3N}$  that are regular values of*

- *the maps  $g(\Gamma)$  of Lemma 17.3 associated to  $\underline{3N}$ -numbered Jacobi diagrams  $\Gamma$  on  $\sqcup_{b \in B_\infty} \mathbb{R}_b$  and to the configuration  $y_\infty$  and*
- *similar maps  $g(\Gamma_s)$  associated to special  $\underline{3N}$ -numbered Jacobi diagrams  $\Gamma_s$  (as above with one bivalent vertex) on  $\sqcup_{b \in B_\infty} \mathbb{R}_b$  and to the configuration  $y_\infty$*

*is a dense open subset of  $(S_H^2)^{3N}$ .*

PROOF: The arguments of Lemma 17.3 allow us to prove that the images of the above maps  $g(\Gamma)$  and of the maps  $g(\Gamma_s)$ , when the univalent vertices of  $\Gamma_s$  are on one strand or when  $\Gamma_s$  has no univalent vertices, are compact semi-algebraic subsets of  $(S_H^2)^{3N}$  of codimension at least one. The complement of the union of these images is therefore open and dense.

For a special  $\Gamma_s$  with univalent vertices on at least 2 strands, the images under  $g(\Gamma_s)$  of the boundary of the configuration space and of the parts where  $g(\Gamma_s)$  is not a submersion are also compact semi-algebraic subsets of  $(S_H^2)^{3N}$  of codimension at least one. □

**Lemma 17.19.** *Assume that the configuration  $y_\infty$  of Lemma 17.18 contains the configuration  $y: B \hookrightarrow D_1$ , the bottom configuration  $y_-$  of  $L$ , the top configuration  $y_+$  of  $L$ , and 0 as subconfigurations.*

*Let  $(X_i)_{i \in \underline{3N}}$  be the center of a tiny ball of radius  $\varepsilon_0 > 0$  in the set  $\mathcal{O}_s(N, y_\infty)$  defined in Lemma 17.18. There exist  $\varepsilon_1 \in ]0, \frac{\varepsilon_0}{\sqrt{3N}}[$  and  $M_1 \in ]1, +\infty[$  such that for any family  $(\omega(i, S^2))_{i \in \underline{3N}}$  of volume-one forms of  $S^2$  supported inside disks  $D(X_i, \varepsilon)$  of  $S^2$  centered at  $X_i$  of radius  $\varepsilon \in ]0, \varepsilon_1]$ ,*

- *for any Jacobi diagram  $\Gamma \in \mathcal{D}_{\underline{3N}}^e(B_\infty \times \mathbb{R})$  and for any special Jacobi diagram  $\Gamma \in \mathcal{D}_{\underline{3N}}^{e, \text{special}}(B \times \mathbb{R})$  with univalent vertices on at most one strand, the support of  $\bigwedge_{e \in E(\Gamma)} (p_{S^2} \circ p_e)^*(\omega(j_E(e), S^2))$  in  $C(\mathbb{R}^3, y_\infty \times K; \Gamma)$  is empty,*
- *for any special Jacobi diagram  $\Gamma \in \mathcal{D}_{\underline{3N}}^{e, \text{special}}(B \times \mathbb{R})$ ,*
  - *the support of  $\bigwedge_{e \in E(\Gamma)} (p_{S^2} \circ p_e)^*(\omega(j_E(e), S^2))$  in the space  $\check{V}(y, \Gamma)$  introduced before Lemma 17.18 is contained in disjoint open subsets where  $\prod_{e \in E(\Gamma)} (p_{S^2} \circ p_e)$  is a diffeomorphism onto*

$$\prod_{e \in E(\Gamma)} \mathring{D}(X_{j_E(e)}, \varepsilon)$$

and the distance of the images of a vertex under two configurations is at most  $M_1\varepsilon$ ,<sup>3</sup> and

- the images of the vertices under the configurations of the support are contained in  $D_{M_1} \times [-M_1, M_1]$ .

PROOF: The definition of  $\mathcal{O}_s(N, y_\infty)$  in Lemma 17.18 and the hypotheses on  $(X_i)_{i \in \underline{3N}}$  in Lemma 17.20 guarantee that the first assertion is satisfied for all  $\varepsilon_1 \in ]0, \frac{\varepsilon_0}{\sqrt{3N}}[$ . Proceed as in Section 11.4, to reduce the support to disjoint subsets, where  $\prod_{e \in E(\Gamma)}(p_{S^2} \circ p_e)$  is a diffeomorphism onto  $\prod_{e \in E(\Gamma)} \mathring{D}(X_{j_E(e)}, \varepsilon_1)$ . (It is simpler here.) Then there exists an  $M$  such that the distance of the images of a vertex under two configurations is at most  $M\varepsilon$  on the preimage of  $\prod_{e \in E(\Gamma)} \mathring{D}(X_{j_E(e)}, \varepsilon)$  under such a diffeomorphism, for any  $\varepsilon \in ]0, \varepsilon_1]$ .  $\square$

Choose a Riemannian metric on  $\check{R}(\mathcal{C})$ , which coincides with the standard metric of  $\mathbb{R}^3$  outside  $\check{R}_{1, [0, 1]}(\mathcal{C})$ , and assume that this Riemannian metric restricts as the natural product metric on  $N_{\eta_0}(K)$ , locally. (Reduce  $\eta_0$  if necessary.)

Let  $C_{2, \leq 10\eta_0}(N_{\eta_0}(K))$  denote the closure in  $C_2(R(\mathcal{C}))$  of the space of pairs of points  $(x_1, x_2) \in N_{\eta_0}(K)^2$  at a distance less than  $10\eta_0$  from each other. Naturally extend  $p_\tau$  to  $C_{2, \leq 10\eta_0}(N_{\eta_0}(K))$  by viewing  $D_{\eta_0} \times \mathbb{R}_K$  as a subspace of  $\mathbb{R}^3$ , locally, with the usual formula  $p_\tau = \frac{1}{\|x_2 - x_1\|}(x_2 - x_1)$ .

Let  $\Gamma \in \mathcal{D}_{\underline{3N}}^e(\mathcal{L})$ . Set  $U_K(\Gamma) = j_\Gamma^{-1}(\mathbb{R}_K)$ , where  $\mathbb{R}_K$  is viewed as the domain of  $K$ . For  $\eta \in ]0, \eta_0]$ , let  $C(R(\mathcal{C}), L, \eta; \Gamma)$  be the configuration space obtained from  $C(R(\mathcal{C}), L; \Gamma)$  by replacing the condition that the restriction of the configurations to the set  $U_K(\Gamma)$  of univalent vertices on  $\mathbb{R}_K$  factors through  $K \circ j_\Gamma$ , for a  $\Gamma$ -compatible injection  $j_\Gamma$  from  $U(\Gamma)$  to  $\mathcal{L}$ , with the condition that  $U_K(\Gamma)$  is mapped to the interior of  $N_\eta(K)$ . (In other words, the conditions on the restriction of a configuration  $c$  to  $U(\Gamma)$  now only impose that  $c(U_K(\Gamma)) \subset \mathring{N}_\eta(K)$ , and that  $c|_{U(\Gamma) \setminus U_K(\Gamma)}$  may be expressed as  $L \circ j_\Gamma|_{U(\Gamma) \setminus U_K(\Gamma)}$  for some  $\Gamma$ -compatible  $j_\Gamma$ .) There is a natural projection  $p_K$  from this configuration space  $C(R(\mathcal{C}), L, \eta; \Gamma)$  to  $\mathring{D}_\eta^{U_K(\Gamma)}$ , and  $C(R(\mathcal{C}), L; \Gamma)$  is contained in the preimage  $p_K^{-1}((0)^{U_K(\Gamma)})$  in  $C(R(\mathcal{C}), L, \eta; \Gamma)$ . (This preimage also contains  $C(R(\mathcal{C}), L; \tilde{\Gamma})$  for Jacobi diagrams  $\tilde{\Gamma}$  different from  $\Gamma$  because the linear order of the vertices of  $U_K(\Gamma)$  is not induced by  $j_\Gamma$ .)

**Lemma 17.20.** *Let  $(X_i)_{i \in \underline{3N}}$  and  $\varepsilon_1$  be as in Lemma 17.19. There exist propagating chains  $P(i)$  of  $(C_2(R(\mathcal{C})), \tau)$  and  $\varepsilon_1$ -dual propagating forms  $\omega(i)$*

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<sup>3</sup>In  $\check{V}(y, \Gamma)$ , configurations are normalized so that univalent vertices go to  $y(B) \times \mathbb{R}$  for our fixed representative of  $y$  and one vertex goes to  $y(B) \times \{0\}$ . The distance is the Euclidean distance of  $\mathbb{R}^3$ .

of  $(C_2(R(\mathcal{C})), \tau)$ , for  $i \in \underline{3N}$ , and  $\eta_2 \in ]0, \eta_0]$  such that

- we have  $P(i) \cap D(p_\tau) = p_\tau^{-1}(X_i) \cap D(p_\tau)$  for any  $i \in \underline{3N}$ ,
- we have  $P(i) \cap C_{2, \leq 10\eta_2}(N_{\eta_2}(K)) = p_\tau^{-1}(X_i) \cap C_{2, \leq 10\eta_2}(N_{\eta_2}(K))$ , for the above natural extension of  $p_\tau$  on  $C_{2, \leq 10\eta_2}(N_{\eta_2}(K))$ ,
- the  $P(i)$  are in general  $3N$ -position with respect to  $(\check{R}(\mathcal{C}), L, \tau)$  (again as in Definition 11.3), and
- $\omega(i)$  restricts to  $D(p_\tau) \cup C_{2, \leq 10\eta_2}(N_{\eta_2}(K))$  as  $p_\tau^*(\omega(i, S^2))$  for some form  $\omega(i, S^2)$  as in Lemma 17.19.

Let  $\Gamma \in \mathcal{D}_{\underline{3N}}^e(\mathcal{L})$ .

For  $(\omega(i))_{i \in \underline{3N}}$  as above and  $\eta \in ]0, \eta_2]$ , let  $\text{Supp}(\Gamma, \eta; (\omega(i)))$  denote the support of  $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$  in  $C(R(\mathcal{C}), L, \eta; \Gamma)$ .

For  $(P(i))_{i \in \underline{3N}}$  as above, for any configuration  $c$  in the discrete set

$$I_S(L, \Gamma, (P(i))) = C(R(\mathcal{C}), L; \Gamma) \bigcap \bigcap_{e \in E(\Gamma)} p_e^{-1}(P(j_E(e))),$$

and for any edge  $e$  of  $E(\Gamma)$ ,  $p_e(c)$  is in the interior of a 4-cell  $\Delta_e(c)$  of  $P(j_E(e))$ , and  $C_2(R(\mathcal{C}))$  is diffeomorphic to  $D_{\varepsilon_1} \times \Delta_e(c)$  near  $p_e(c)$ , where  $D_{\varepsilon_1}$  is the local fiber of a tubular neighborhood of  $\Delta_e(c)$ . Let  $p_{c,e}$  be the associated projection onto  $D_{\varepsilon_1}$  in a neighborhood of  $c$  in  $C(R(\mathcal{C}), L, \eta; \Gamma)$ . Without loss of generality, assume that  $\omega(j_E(e))$  may be expressed as  $p_{c,e}^*(\omega_\varepsilon(c, e))$  locally, for a 2-form  $\omega_\varepsilon(c, e)$  supported on  $D_\varepsilon$ , such that  $\int_{D_\varepsilon} \omega_\varepsilon(c, e)$  is the rational coefficient of the above 4-cell  $\Delta_e(c)$  in  $P(j_E(e))$ .

With this notation, there exist  $\varepsilon \in ]0, \varepsilon_1]$ , propagating chains  $P(i)$ ,  $\varepsilon$ -dual propagating forms  $\omega(i)$  as above,  $\eta_3 \in ]0, \eta_2]$ , and  $M_2 > 1$  such that for any  $\eta \in ]0, \eta_3]$  and for any Jacobi diagram  $\Gamma \in \mathcal{D}_{\underline{3N}}^e(\mathcal{L})$ ,

- $\text{Supp}(\Gamma, \eta; (\omega(i)))$  is contained in disjoint submanifolds, indexed by the configurations  $c$  of  $I_S(\Gamma, (P(i)))$ , and which contain those, where  $p_K \times (p_E = \prod_{e \in E(\Gamma)} p_{c,e})$  is a diffeomorphism onto  $\mathring{D}_\eta^{U_K(\Gamma)} \times \mathring{D}_\varepsilon^{E(\Gamma)}$ , and where the distance of the images of a vertex under two configurations of  $(p_K \times p_E)^{-1}(\mathring{D}_\eta^{U_K(\Gamma)} \times \{W\})$  is at most  $M_2\eta$  for any  $W \in \mathring{D}_\varepsilon^{E(\Gamma)}$ ,
- the involved configurations map vertices of  $V(\Gamma) \setminus U_K(\Gamma)$  at a distance greater than  $9M_2\eta$  from  $K$ , and
- they map two distinct vertices of  $U_K(\Gamma)$  at a distance greater than  $9M_2\eta$  from each other.

PROOF: The existence of the  $P(i)$  in general  $3N$ -position with prescribed behavior near the boundary can be proved as in Section 11.3. Fix a graph  $\Gamma \in \mathcal{D}_{3N}^e(\mathcal{L})$ . Once the  $P(i)$  are in general  $3N$ -position, for such a given graph  $\Gamma$ ,  $I_S(\Gamma, (P(i)))$  consists of a finite number of isolated intersection points at which trivalent vertices cannot be on  $K$ . Indeed, this would correspond to a degenerate configuration for a graph in which 3 univalent vertices replace the trivalent vertex on  $K$ .

The existence of a family  $(\omega(i))_{i \in 3N}$  of propagating forms of  $R(\mathcal{C})$ ,  $\varepsilon_1$ -dual to  $P(i)$ , which may be expressed as  $p_\tau^*(\omega(i, S^2))$  on  $D(p_\tau) \cup C_{2, \leq 10\eta_2}(N_{\eta_2}(K))$ , with respect to a family  $(\omega(i, S^2))_{i \in 3N}$  of 2-forms of volume 1 supported inside a disk  $D(X_i, \varepsilon)$ , as in Lemma 17.19, can be proved as in Section 11.4.

Let  $c \in I_S(\Gamma, (P(i)))$ . Transversality implies that the restriction to a neighborhood of  $c$  in  $C(R(\mathcal{C}), L; \Gamma)$  of  $p_E$  is a submersion, with the notation of the statement. This implies that the restriction of  $p_K \times p_E$  to a neighborhood of  $c$  in  $C(R(\mathcal{C}), L, \eta; \Gamma)$  is a submersion, too, so that it is a local diffeomorphism in a neighborhood of  $c$  in  $C(R(\mathcal{C}), L, \eta; \Gamma)$ . After reducing  $\eta$  and  $\varepsilon$ , we obtain a neighborhood  $N(c)$  of  $c$  in  $C(R(\mathcal{C}), L, \eta; \Gamma)$  such that  $p_K \times p_E$  is a diffeomorphism from  $N(c)$  onto  $\overset{\circ}{D}_\eta^{U_K(\Gamma)} \times \overset{\circ}{D}_\varepsilon^{E(\Gamma)}$  for all  $c \in I_S(\Gamma, (P(i)))$ . The compact intersection of  $C(R(\mathcal{C}), L; \Gamma)$  with the complement of  $\cup_{c \in I_S(\Gamma, (P(i)))} N(c)$  is mapped outside  $\prod_{e \in E(\Gamma)} P(j_E(e))$  by  $\prod_{e \in E(\Gamma)} p_e$ . Therefore, its image avoids a neighborhood of  $\prod P(j_E(e))$ . Reducing  $\varepsilon$  allows us to assume that it avoids the compact closure of the support of  $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$ . We may now reduce  $\eta$  so that  $C(R(\mathcal{C}), L, \eta; \Gamma)$  does not meet this compact closure outside  $\cup_{c \in I_S(\Gamma, (P(i)))} N(c)$ . This can be achieved simultaneously for all the finitely many considered  $\Gamma$ .

Now, the Jacobians of the corresponding inverse local diffeomorphisms (viewed as maps from  $\overset{\circ}{D}_\eta^{U_K(\Gamma)} \times \prod_{e \in E(\Gamma)} D_\varepsilon$  to  $R(\mathcal{C})^{V(\Gamma)} \setminus \text{diag}$ ) are bounded (after reducing  $\varepsilon$  and  $\eta$  if necessary).

So we get a  $M_2$  for which the distance of the images of a vertex under two configurations of any  $(p_K \times p_E)^{-1}(\overset{\circ}{D}_\eta^{U_K(\Gamma)} \times \{W\})$  in a connected component of  $\text{Supp}(\Gamma, \eta; (\omega(i)))$  is at most  $M_2\eta$ , for all the finitely many considered  $\Gamma$ . It is easy to reduce  $\eta_3$  so that the last two conditions are satisfied with our given  $M_2$ . □

Set  $\mathcal{D}_{\leq N, 3N}^e(\cdot) = \cup_{k=0}^N \mathcal{D}_{k, 3N}^e(\cdot)$ . For  $\Gamma \in \mathcal{D}_{\leq N, 3N}^e(\mathcal{L}(B \times \mathbb{R}_K))$ , let

$$I_S(L(\eta^2 y \times K), \Gamma, (P(i))_{i \in 3N})$$

denote the set of configurations  $c$  of

$$C(R(\mathcal{C}), L(\eta^2 y \times K); \Gamma) \bigcap \bigcap_{e \in E(\Gamma)} p_e^{-1}(P(j_E(e)))$$

with respect to the propagating chains  $P(i)$ .

The following crucial lemma justifies the introduction of special Jacobi diagrams.

**Lemma 17.21.** *Let  $\eta_1$  be the minimum in the set  $\{\eta_3, \frac{1}{(2N-1)8NM_1}, \frac{1}{100NM_2}\}$  of positive numbers introduced in Lemmas 17.19 and 17.20. For any family  $(P(i))_{i \in \underline{3N}}$  of propagating chains as in Lemma 17.20, for any*

$$\Gamma_B \in \mathcal{D}_{\leq N, \underline{3N}}^e(\mathcal{L}(B \times \mathbb{R}_K)),$$

for any configuration  $c_{\eta_1}$  of the set  $I_S(L(\eta_1^2 y \times K), \Gamma_B, (P(i))_{i \in \underline{3N}})$ , there exists a continuous map

$$\begin{aligned} ]0, \eta_1] &\rightarrow C_V(\Gamma_B)(R(\mathcal{C})) \\ \eta &\mapsto c_\eta \end{aligned}$$

such that, for any  $\eta \in ]0, \eta_1]$ ,  $c_\eta$  belongs to  $I_S(L(\eta^2 y \times K), \Gamma_B, (P(i))_{i \in \underline{3N}})$  and the graph  $\Gamma_B$  configured by  $c_\eta$  is the union of

- (small red) special Jacobi diagrams  $\Gamma_s$  on  $B \times \mathbb{R}_K$  of diameter less than  $10\eta$  configured on  $\eta^2 y \times K$  in  $N_\eta(K)$  (with univalent vertices on at least two strands of  $B \times \mathbb{R}_K$ ), and
- a uni-trivalent (blue and purple) graph  $\Gamma$  on the domain  $\mathcal{L}$  of  $L$  configured so that its univalent vertices are
  - either univalent vertices of  $\Gamma_B$  on  $(L \setminus K) \cup (\eta^2 y \times K)$
  - or trivalent vertices of  $\Gamma_B$  attached to a bivalent vertex of a (small red) special graph  $\Gamma_s$ , as in Figure 17.1.

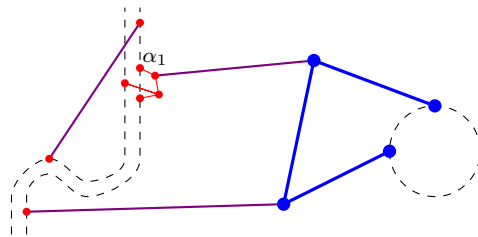


Figure 17.1: A configured diagram  $\Gamma_B$  with its small red vertices, its small thin red edges, its big blue vertices (the rightmost four), and its thick blue edges (the rightmost three)

Furthermore, the configuration  $c_\eta$  arises as a transverse intersection point, and the intersections that involve at least one (red) special Jacobi diagram

cancel algebraically. For a fixed  $\eta$ , the remaining configurations are in natural one-to-one correspondence—independent of  $\eta$ —with triples  $(\Gamma, f, c')$ , where  $\Gamma \in \mathcal{D}_{\leq N, 3N}^e(\mathcal{L})$ ,  $U_K(\Gamma)$  is the set of univalent vertices of  $\Gamma_B$  on  $B \times \mathbb{R}_K$ ,  $f \in B^{U_K(\Gamma)}$ , and  $c' \in I_S(L, \Gamma, (P(i))_{i \in \underline{3N}})$ . The inverse of this natural one-to-one correspondence maps  $(\Gamma, f, c')$  to a pair  $(\Gamma_f, c_\eta)$ , where

- $\Gamma_f$  is a Jacobi diagram on  $\mathcal{L}(B \times \mathbb{R}_K)$  obtained from  $\Gamma$  by changing the (isotopy class of the) injection from  $U_K(\Gamma)$  to  $\mathbb{R}_K$  to the injection from  $U_K(\Gamma)$  to  $B \times \mathbb{R}_K$  that maps a vertex  $u$  to  $f(u) \times \mathbb{R}_K$  so that the order of vertices on each strand of  $B \times \mathbb{R}_K$  is induced by their former order on  $\mathbb{R}_K$ ,
- $c_\eta$  belongs to  $I_S(L(\eta^2 y \times K), \Gamma_f, (P(i))_{i \in \underline{3N}})$ ,
- $d(c_\eta(v), c'(v))$  is smaller than  $\eta$  for any vertex  $v$  of  $\Gamma$ ,

and the sign of the algebraic intersection at  $c_\eta$  is the same as the sign of the algebraic intersection at  $c'$ , with respect to consistent vertex-orientations of  $\Gamma$  and  $\Gamma_f$ .

Furthermore, for any  $\eta \in ]0, \eta_1]$ , the support  $\text{Supp}(L(\eta^2 y \times K), \Gamma_B; (\omega(i)))$  of  $\bigwedge_{e \in E(\Gamma_B)} p_e^*(\omega(j_E(e)))$  in  $C(R(\mathcal{C}), L(\eta^2 y \times K); \Gamma_B)$  consists of disjoint neighborhoods  $N_{L(\eta^2 y \times K)}(c_\eta)$  of configurations  $c_\eta$  as above, where projections  $p_{c_\eta, e}$  as in the statement of Lemma 17.20 make sense, and such that the restriction of  $\prod_{e \in E(\Gamma_B)} p_{c_\eta, e}$  to  $N_{L(\eta^2 y \times K)}(c_\eta)$  is a diffeomorphism onto  $\mathring{D}_\varepsilon^{E(\Gamma_B)}$ .

PROOF: Let  $\Gamma_B \in \mathcal{D}_{\leq N, 3N}^e(\mathcal{L}(B \times \mathbb{R}_K))$ . Instead of starting with a configuration  $c_{\eta_1} \in I_S(L(\eta_1^2 y \times K), \Gamma_B, (P(i))_{i \in \underline{3N}})$ , as in the statement, we consider a configuration  $c$  in  $\text{Supp}(L(\eta_1^2 y \times K), \Gamma_B; (\omega(i)))$ , for some  $\eta_1 \in ]0, \eta_1]$ . View  $\Gamma_B$  as a graph configured by  $c$ . So its vertices become elements of  $R(\mathcal{C})$ . Color the vertices of  $\Gamma_B$  in  $N_{\eta_1^2}(K)$  with (red, 1). Next color by (red, 2) its uncolored vertices at a distance less than  $4\eta_1^2 M_1$  from a vertex colored by (red, 1). For  $k \geq 2$ , inductively color the still uncolored vertices that are at a distance less than  $4\eta_1^2 M_1$  from a vertex  $v$  colored by (red,  $k$ ), by (red,  $k + 1$ ).

Define the map

$$\begin{aligned} r: \check{R}(\mathcal{C}) & \rightarrow [0, \eta_1] \\ (z_D, t) \in D_{\eta_1} \times \mathbb{R}_K & \mapsto |z_D| \\ x \in R(\mathcal{C}) \setminus (D_{\eta_1} \times \mathbb{R}_K) & \mapsto \eta_1. \end{aligned}$$

Note that a vertex  $v$  colored by (red,  $k$ ) with  $k \geq 2$  satisfies  $\eta_1^2 \leq r(v) \leq 4k\eta_1^2 M_1$ , by induction. So we have  $r(v) \leq 8N\eta_1^2 M_1 \leq \eta_1$  for such a vertex.

Color the vertices that are still uncolored after this algorithm blue. Color the edges between two blue vertices blue. Color the edges between a red



vertex and a blue one purple. Also color the edges between red vertices at a distance greater than  $8N\eta_1^2M_1$  purple. Color the remaining edges between two red vertices red.

Remove the red edges between red vertices, and the red vertices that do not belong to a purple edge from  $\Gamma_B$ . Blow up the obtained graph  $\tilde{\Gamma}$  at red vertices that belong to at least two purple edges, so that these red vertices become univalent. A red vertex that belongs to  $r$  purple edges is transformed into  $r$  red vertices during this process. Let  $\Gamma'$  be the obtained configured uni-trivalent graph with blue and purple edges. Its red vertices are in  $N_{\eta_1}(K)$ . Let  $U_K(\Gamma')$  denote the set of red vertices of  $\Gamma'$ . The restriction of  $c$  to  $V(\Gamma')$  is in  $C(R(\mathcal{C}), L, \eta_1; \Gamma') \cap \text{Supp}(\bigwedge_{e \in E(\Gamma')} p_e^*(\omega(j_E(e))))$ . So, according to Lemma 17.20, it is in one of the disjoint submanifolds where  $p_K \times p_E$  restricts as a diffeomorphism onto  $\mathring{D}_{\eta_1}^{U_K(\Gamma')} \times \mathring{D}_\varepsilon^{E(\Gamma')}$ , with  $p_E = \prod_{e \in E(\Gamma')} p_{c,e}$ , and where the distance of the images of a vertex under two configurations of  $(p_K \times p_E)^{-1}(\mathring{D}_{\eta_1}^{U_K(\Gamma')} \times \{W\})$  is at most  $M_2\eta$  for any  $W \in \mathring{D}_\varepsilon^{E(\Gamma')}$ . Set  $W_0 = p_E(c)$ . (For  $c \in I_S(L(\eta_4^2y \times K), \Gamma_B, (P(i)))$ , we have  $W_0 = 0 = (0)_{e \in E(\Gamma')}$ .) Then  $c' = (p_K \times p_E)^{-1}((0)_{U_K(\Gamma')}, W_0)$  is a configuration of a graph  $\Gamma$  on  $\mathcal{L}$ , obtained from  $\Gamma'$  by adding the data of an isotopy of injections of  $U_K(\Gamma')$  into  $\mathbb{R}_K$ , where  $U_K(\Gamma) = U_K(\Gamma')$ . According to Lemma 17.20, no collision of vertices of  $\Gamma$  can occur. So the red vertices of  $\Gamma'$  were univalent in  $\tilde{\Gamma}$  (there was no need to blow them up) and  $\tilde{\Gamma} = \Gamma'$ .

Furthermore,  $c'$  maps two red vertices at a distance at least  $9M_2\eta_1$  from each other. In particular, two red vertices of  $\tilde{\Gamma}$  are at a distance at least  $7M_2\eta_1$  from each other, with respect to  $c$ .

The univalent vertices of  $\tilde{\Gamma}$  are either univalent vertices of  $\Gamma_B$ , sent to  $\eta_4^2y \times K$  or to  $L \setminus K$  by  $c$ , or trivalent vertices of  $\Gamma_B$ , which belong to a bivalent vertex of a red subgraph of  $\Gamma_B$ . Let  $\Gamma_R$  be the subgraph of  $\Gamma_B$  consisting of its red vertices and of its red edges.

Let  $\Gamma_{R,1}$  be a connected component of  $\Gamma_R$  such that  $\Gamma_{R,1}$  is not reduced to a univalent vertex. Since two vertices of  $\Gamma_{R,1}$  are at a distance at most  $(2N - 1)8N\eta_1^2M_1 \leq \eta_1$  from each other, there is at most one red vertex of  $\tilde{\Gamma}$  in  $\Gamma_{R,1}$ , and  $c$  sends  $\Gamma_{R,1}$  to a part of  $N_{\eta_1}(K)$  identified with a part of  $\mathbb{R}^3$  of diameter less than  $10\eta_1$ . So such a  $\Gamma_{R,1}$  configured by  $c$  may be viewed as a graph with straight edges, directed by the  $X_i$  if  $c \in I_S(L(\eta_4^2y \times K), \Gamma_B, (P(i)))$ , and by the  $\tilde{W}_i \in D(X_i, \varepsilon)$  in general. In particular, Lemma 17.19 implies that  $\Gamma_{R,1}$  must be a configured special Jacobi diagram in  $D_{\eta_1^2M_1} \times [x - \eta_1^2M_1, x + \eta_1^2M_1]$ . Its configuration  $\eta_4^2c_{R,1}$  is determined up to translation along  $\mathbb{R}_K$ . The projection of the bivalent vertex  $\alpha_1$  of  $\Gamma_{R,1}$  to  $D_{\eta_1^2M_1}$  is  $p_{\mathbb{C}}(\eta_4^2c_{R,1}(\alpha_1))$ .

Let  $\mathcal{A}$  denote the set of bivalent vertices of  $\Gamma_R$ . Write the corresponding

configured special Jacobi diagrams  $(\Gamma_{R,\alpha}, c_{R,\alpha})_{\alpha \in \mathcal{A}}$ . Let  $U_K(\Gamma_B) = j_{\Gamma_B}^{-1}(B \times \mathbb{R}_K)$ . Note the natural inclusions  $U_K(\Gamma) \subseteq U_K(\Gamma_B) \sqcup \mathcal{A}$  and  $\mathcal{A} \subseteq U_K(\Gamma)$ . Let  $f_{\Gamma_B}: U_K(\Gamma_B) \rightarrow B$  be the map that sends  $u \in j_{\Gamma_B}^{-1}(\{b\} \times \mathbb{R}_K)$  to  $b$ . Let  $f$  denote the restriction of  $f_{\Gamma_B}$  to  $U_K(\Gamma) \setminus \mathcal{A}$ .

So far, our analysis allows us to associate

$$\Phi(\Gamma_B, c) = \left( \Gamma, c', \mathcal{A} \subset U_K(\Gamma), f: U_K(\Gamma) \setminus \mathcal{A} \rightarrow B, (\Gamma_{R,\alpha}, c_{R,\alpha})_{\alpha \in \mathcal{A}} \right)$$

to our configured graph  $(\Gamma_B, c)$  as above, where

- $\Gamma \in \mathcal{D}_{\leq N, \underline{3N}}^e(\mathcal{L})$ ,
- $c' \in \text{Supp}(L, \Gamma; (\omega(i))) = C(R(\mathcal{C}), L; \Gamma) \cap \text{Supp}(\bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e))))$ ,
- $p_E(c') = p_E(c) = W_0$ ,
- $c' \in I_S(L, \Gamma, (P(i)))$  when  $c \in I_S(L(\eta_4^2 y \times K), \Gamma_B, (P(i)))$ ,
- the  $\Gamma_{R,\alpha}$  are special Jacobi diagrams on  $B \times \mathbb{R}_K$  whose disjoint union is numbered in  $\underline{3N} \setminus j_E(E(\Gamma))$ ,
- the  $c_{R,\alpha}$  are configurations of these diagrams with respect to  $y \times \mathbb{R}$  in

$$\bigcap_{e \in E(\Gamma_{R,\alpha})} (p_{S^2} \circ p_e^{-1})(\tilde{W}_{j_E(e)}),$$

(or in  $\bigcap_{e \in E(\Gamma_{R,\alpha})} (p_{S^2} \circ p_e^{-1})(X_{j_E(e)})$  when  $c \in I_S(L(\eta_4^2 y \times K), \Gamma_B, (P(i)))$ ).

Let us now show how to reconstruct  $(\Gamma_B, c)$  from the data

$$(\Gamma, c', \mathcal{A} \subset U_K(\Gamma), f: U_K(\Gamma) \setminus \mathcal{A} \rightarrow B, (\Gamma_{R,\alpha}, c_{R,\alpha})_{\alpha \in \mathcal{A}}).$$

The graph  $\Gamma_B$  is obtained from  $\Gamma$  and the  $\Gamma_{R,\alpha}$  by gluing  $\Gamma$  and the  $\Gamma_{R,\alpha}$  at the vertices  $\alpha$ . The components of its univalent vertices are determined by those of the univalent vertices of  $\Gamma_{R,\alpha}$  and of  $\Gamma$  outside  $\mathcal{A}$ , and by  $f$ . Their order on a strand of  $B \times \mathbb{R}_K$  is the restriction to the vertices of such a strand of the order induced on  $U_K(\Gamma_B)$  by letting the ordered set of the univalent vertices of  $\Gamma_{R,\alpha}$  replace  $\alpha$ , for every  $\alpha \in \mathcal{A}$ , in the ordered set of the univalent vertices of  $\Gamma$ .

Below, we construct a smooth embedding

$$\mathbf{c}: ]0, \eta_1] \times \mathring{D}_\varepsilon^{E(\Gamma_B)} \hookrightarrow C_{V(\Gamma_B)}(R(\mathcal{C}))$$

such that  $c(\eta, W_B)$  belongs to  $C(R(\mathcal{C}), L(\eta^2 y \times K); \Gamma_B)$  for any  $(\eta, W_B) \in ]0, \eta_1] \times \mathring{D}_\varepsilon^{E(\Gamma_B)}$  and our configuration  $c$  is equal to  $\mathbf{c}(\eta_4, W_{B,0})$ .

Lemma 17.19 guarantees that  $p_{E,\alpha} = \prod_{e \in E(\Gamma_{R,\alpha})} (p_{S^2} \circ p_e)$  is a diffeomorphism from a neighborhood of  $c_{R,\alpha}$  in the configuration space of configurations of  $\Gamma_{R,\alpha}$  on  $y \times \mathbb{R}_K$  up to translation along  $\mathbb{R}_K$  onto  $\prod_{e \in E(\Gamma_{R,\alpha})} \mathring{D}(X_{j_E(e)}, \varepsilon)$ , for every  $\alpha \in \mathcal{A}$ . Use an implicit diffeomorphism  $\phi_e: D(X_{j_E(e)}, \varepsilon) \rightarrow D_\varepsilon$  to identify  $D(X_{j_E(e)}, \varepsilon)$  with  $D_\varepsilon$ . Set  $W_{\alpha,0} = (\phi_e \circ p_{S^2} \circ p_e(c_{R,\alpha}))_{e \in E(\Gamma_{R,\alpha})} = p_{E,\alpha}(c_{R,\alpha})$ . Lemma 17.20 provides a neighborhood of  $c'$  in  $C(R(\mathcal{C}), L, \eta_1; \Gamma)$ , where  $p_K \times p_E$  is a diffeomorphism onto  $\mathring{D}_{\eta_1}^{U_K(\Gamma)} \times \mathring{D}_\varepsilon^{E(\Gamma)}$ . Recall  $p_E(c) = W_0$ . These diffeomorphisms assemble to form a diffeomorphism  $\Psi_\eta$  from a neighborhood of  $c(\eta, W)$  in  $C(R(\mathcal{C}), L(\eta^2 y \times K); \Gamma_B)$  to  $\mathring{D}_\varepsilon^{E(\Gamma_B)}$ , whose inverse  $\Psi_\eta^{-1} = \mathbf{c}(\eta, \cdot)$  is described below. Write  $W_B = (W_e)_{e \in E(\Gamma_B)} = ((W_\alpha)_{\alpha \in \mathcal{A}}, W)$ , with  $W_\alpha = (W_e)_{e \in E(\Gamma_{R,\alpha})}$  and  $W = (W_e)_{e \in E(\Gamma)}$ .  $\Psi_\eta^{-1}(W)$  is constructed from representatives  $\eta^2 c_{R,\alpha}(\alpha)$  (where the vertical translation parameter is fixed) of the  $\eta^2 p_{E,\alpha}^{-1}(W_\alpha)$  by assembling them with

$$c_1(\eta, W) = (p_K \times p_E)^{-1} \left( \left( \left( \eta^2 y(f(u)) \right)_{u \in U_K(\Gamma) \setminus \mathcal{A}}, \left( p_{\mathbb{C}}(\eta^2 c_{R,\alpha}(\alpha)) \right)_{\alpha \in \mathcal{A}} \right), W \right)$$

so that the height (projection onto  $\mathbb{R}_K$ )  $p_{\mathbb{R}}(\eta^2 c_{R,\alpha}(\alpha))$  of  $\alpha$  in  $\eta^2 c_{R,\alpha}(\alpha)$  coincides with  $p_{\mathbb{R}}(c_1(\eta, W)(\alpha))$ .

This ensures that  $\mathbf{c}(\eta, W_B) = \Psi_\eta^{-1}(W_B)$  arises as a transverse intersection, for any  $W_B$ . This is true when  $W_B = 0$ , in particular. So it is true for the configuration  $c_{\eta_1}$  of the statement, which may be expressed as  $\mathbf{c}(\eta_1, 0)$ . The family  $c_\eta$  is the continuous family  $\Psi_\eta^{-1}(0)$ , in this case. The sign of the corresponding intersection is the sign of the Jacobian of  $\Psi_\eta$ . Since we started with an arbitrary configuration in  $\text{Supp}(L(\eta_4^2 y \times K), \Gamma_B; (\omega(i)))$  for some  $\eta_4 \in ]0, \eta_1]$ , the above arguments also prove the final assertion of the lemma. Let us finally focus on the claimed algebraic cancellation. From now on, we set  $W = 0$ .

Assume that  $\mathcal{A} \neq \emptyset$ . Every  $\alpha \in \mathcal{A}$  is contained in one edge  $e(\alpha)$  of  $\Gamma$ . Choose  $\alpha_0$  in  $\mathcal{A}$  such that  $j_E(e(\alpha_0))$  is minimal. Let  $s(\Gamma_{R,\alpha_0})$  be obtained from  $\Gamma_{R,\alpha_0}$  by exchanging the labels of the two edges  $e_1$  and  $e_2$  of  $\Gamma_{R,\alpha_0}$  that contain  $\alpha_0$ , and by reversing their orientations if they both come from  $\alpha_0$  or go to  $\alpha_0$  as in Lemma 9.11. Let  $s_{\alpha_0}(\Gamma_B)$  be obtained from  $\Gamma_B$  by performing the same changes. Let  $s(c_{R,\alpha_0})$  be obtained from  $c_{R,\alpha_0}$  by changing the position of  $c_{R,\alpha_0}(\alpha_0)$  by a central symmetry with respect to the middle of the two other ends of  $e_1$  and  $e_2$ . The intersection point associated to the configured graph

$$(s_{\alpha_0}(\Gamma_B), c_2) = \Phi^{-1} \left( \Gamma, c', \mathcal{A}, f, (\Gamma_{R,\alpha}, c_{R,\alpha})_{\alpha \in \mathcal{A} \setminus \{\alpha_0\}}, s(\Gamma_{R,\alpha_0}), s(c_{R,\alpha_0}), 0 \right)$$

and the intersection point associated to  $(\Gamma_B, c)$  cancel algebraically, as in

Lemma 9.11. (Our process defines an involution on the configured graphs  $(\Gamma_B, c)$  with  $\mathcal{A} \neq \emptyset$  such that a configured graph and its image cancel.)

Therefore, the configured graphs  $(\Gamma_B, c)$  that contribute to the intersection are the graphs for which  $\mathcal{A} = \emptyset$ . They are obtained from some  $\Gamma \in \mathcal{D}_{\leq N, 3N}^e(\mathcal{L})$ , some  $c'$ , and some  $f: U_K(\Gamma) \rightarrow B$  as in the statement. □

PROOF OF LEMMA 17.16: Lemma 17.21 implies that

- for any  $\Gamma \in \mathcal{D}_{\leq N, 3N}^e(\mathcal{L})$ ,  $I(\mathcal{C}, L, \Gamma, o(\Gamma), (\omega(i))_{i \in 3N})$  is the algebraic intersection  $I(\mathcal{C}, \overline{L}, \overline{\Gamma}, o(\Gamma), (P(i))_{i \in 3N})$  of the preimages of the propagating chains  $P(i)$  in  $C(R(\mathcal{C}), L; \Gamma)$  with respect to  $\Gamma$ ,
- for any  $\eta \in ]0, \eta_1]$  and for any  $\Gamma_B \in \mathcal{D}_{\leq N, 3N}^e(\mathcal{L}(B \times \mathbb{R}_K))$ , we have

$$I(\mathcal{C}, L(\eta^2 y \times K), \Gamma_B, o(\Gamma_B), (\omega(i))) = I(\mathcal{C}, L(\eta^2 y \times K), \Gamma_B, o(\Gamma_B), (P(i))),$$

and

- for any  $\eta \in ]0, \eta_1]$  and for any subset  $A$  of  $3N$  with cardinality  $3k$ , we have

$$\begin{aligned} & \sum_{\Gamma \in \mathcal{D}_{k, A}^e(\mathcal{L})} \zeta_{\Gamma} I(\mathcal{C}, L, \Gamma, (P(i))_{i \in A}) \pi(B \times K)^*([\Gamma]) \\ &= \sum_{\Gamma_B \in \mathcal{D}_{k, A}^e(\mathcal{L}(B \times \mathbb{R}_K))} \zeta_{\Gamma_B} I(\mathcal{C}, L(\eta^2 y \times K), \Gamma_B, (P(i))_{i \in A}) [\Gamma_B]. \end{aligned}$$

□

Now, both the second duplication property and the first duplication property for components going from bottom to top or from top to bottom of Theorem 13.12 are proved. Below, we prove the first duplication property, more generally, in the doubling case.

**Lemma 17.22.** *Let  $\nu$  be the element of  $\mathcal{A}(\mathbb{R})$  obtained from*

$$\mathcal{Z}^f \left( \begin{array}{c} \begin{array}{ccc} \begin{array}{c} \bullet \\ \diagup \end{array} & & \begin{array}{c} \bullet \\ \diagdown \end{array} \\ \begin{array}{c} \uparrow \\ \bullet \end{array} & \begin{array}{c} \uparrow \\ \bullet \end{array} & \begin{array}{c} \uparrow \\ \bullet \end{array} \\ \begin{array}{c} \bullet \\ \diagdown \end{array} & & \begin{array}{c} \bullet \\ \diagup \end{array} \end{array} \end{array} \right) \in \mathcal{A} \left( \begin{array}{c} \uparrow \\ \bullet \end{array} \uparrow \begin{array}{c} \uparrow \\ \bullet \end{array} \uparrow \begin{array}{c} \uparrow \\ \bullet \end{array} \right)$$

by inserting  $\uparrow \uparrow \uparrow$  in  $\mathbb{R}$  as indicated by the picture  $\dots \xrightarrow{1} \dots \xrightarrow{2} \dots \xrightarrow{3} \dots$ . Let  $\nu^{-\frac{1}{2}}$  be the unique element of  $\mathcal{A}(\mathbb{R})$  whose degree 0 part is 1 such that  $(\nu^{-\frac{1}{2}})^2 \nu$  equals 1. Then we have

$$\mathcal{Z}^f \left( \begin{array}{c} \bullet \\ \diagup \end{array} \uparrow \begin{array}{c} \bullet \\ \diagdown \end{array} \right) = \mathcal{Z}^f \left( \begin{array}{c} \bullet \\ \diagup \end{array} \uparrow \begin{array}{c} \bullet \\ \diagdown \end{array} \right) = \nu^{-\frac{1}{2}}.$$

PROOF: Use Theorem 13.12 except for the first duplication property, which is about to be proved. The symmetry and the isotopy invariance respectively imply

$$\mathcal{Z}^f(\curvearrowright) = \mathcal{Z}^f(\curvearrowleft) \text{ and } \mathcal{Z}^f\left(\begin{array}{c} \uparrow \\ \downarrow \end{array}\right) = \mathcal{Z}^f\left(\begin{array}{c} \downarrow \\ \uparrow \end{array}\right) = 1.$$

The functoriality implies

$$\mathcal{Z}^f\left(\begin{array}{c} \uparrow \\ \downarrow \end{array}\right) = \mathcal{Z}^f\left(\begin{array}{c} \uparrow \\ \downarrow \\ \uparrow \\ \downarrow \end{array}\right) = \mathcal{Z}^f\left(\begin{array}{c} \uparrow \\ \downarrow \end{array}\right) \mathcal{Z}^f\left(\begin{array}{c} \uparrow \\ \downarrow \end{array}\right) \mathcal{Z}^f\left(\begin{array}{c} \uparrow \\ \downarrow \end{array}\right) \mathcal{Z}^f\left(\begin{array}{c} \uparrow \\ \downarrow \end{array}\right).$$

The cabling property implies that  $\mathcal{Z}^f(\begin{array}{c} \uparrow \\ \downarrow \end{array})$  is obtained from  $\mathcal{Z}^f(\curvearrowright)$  by the map induced by the natural injection from  $\curvearrowright$  to  $\begin{array}{c} \uparrow \\ \downarrow \end{array}$ .  $\mathcal{Z}^f(\begin{array}{c} \uparrow \\ \downarrow \end{array})$  is obtained similarly from  $\mathcal{Z}^f(\curvearrowleft)$ . Since the insertions of  $\mathcal{Z}^f(\curvearrowright)$  and  $\mathcal{Z}^f(\curvearrowleft)$  can be performed at arbitrary places according to Proposition 6.22, we have

$$\mathcal{Z}^f(\curvearrowright) \mathcal{Z}^f(\curvearrowleft) \nu = 1$$

in the algebra  $\mathcal{A}(\mathbb{R})$ . In this algebra, an element whose degree 0 part is the class of the empty diagram is determined by its inverse. Its square also determines it.  $\square$

**Lemma 17.23.** *The first duplication property of Theorem 13.12 is true when  $K$  is the unique component of the tangle  $\curvearrowright$  or the unique component of  $\curvearrowleft$ . In other words, we have*

$$\mathcal{Z}^f(2 \times \curvearrowright) = \pi(2 \times \mathbb{R})^* \left( \mathcal{Z}^f(\curvearrowright) \right)$$

and  $\mathcal{Z}^f(2 \times \curvearrowleft) = \pi(2 \times \mathbb{R})^* \left( \mathcal{Z}^f(\curvearrowleft) \right)$ .

PROOF: Again we have  $\mathcal{Z}^f(2 \times \curvearrowright) = \mathcal{Z}^f(2 \times \curvearrowleft)$  by symmetry. Thus,  $\mathcal{Z}^f(2 \times \curvearrowright)$  can be computed from  $\mathcal{Z}^f\left(2 \times \begin{array}{c} \uparrow \\ \downarrow \end{array}\right)$  as  $\mathcal{Z}^f(\curvearrowright)$  is computed from  $\mathcal{Z}^f\left(\begin{array}{c} \uparrow \\ \downarrow \end{array}\right)$  in the proof of Lemma 17.22. Indeed the boxes  $\mathcal{Z}^f(2 \times \curvearrowright)$  and  $\mathcal{Z}^f(2 \times \curvearrowleft)$  can slide across the duplicated strands of

$$\begin{aligned} \pi\left(2 \times \begin{array}{c} \uparrow \\ \downarrow \end{array}\right)^* \left( \mathcal{Z}^f\left(\begin{array}{c} \uparrow \\ \downarrow \end{array}\right) \right) &= \mathcal{Z}^f\left(\begin{array}{c} \uparrow \\ \downarrow \end{array}\right) \left(2 \times \begin{array}{c} \uparrow \\ \downarrow \end{array}\right) \left(2 \times \begin{array}{c} \uparrow \\ \downarrow \end{array}\right) \left(2 \times \begin{array}{c} \uparrow \\ \downarrow \end{array}\right) \\ &= \pi\left(2 \times \begin{array}{c} \uparrow \\ \downarrow \end{array}\right)^* \pi\left(2 \times \begin{array}{c} \uparrow \\ \downarrow \end{array}\right)^* \pi\left(2 \times \begin{array}{c} \uparrow \\ \downarrow \end{array}\right)^* \left( \mathcal{Z}^f\left(\begin{array}{c} \uparrow \\ \downarrow \end{array}\right) \right), \end{aligned}$$

according to Lemma 6.33, so that we get

$$\mathcal{Z}^f(2 \times \textcircled{\curvearrowright})^2 \pi(2 \times \mathbb{R})^*(\nu) = 1$$

in the algebra  $\mathcal{A}(\mathbb{R} \sqcup \mathbb{R})$ . Since  $\pi(2 \times \mathbb{R})^*$  is an algebra morphism from  $\mathcal{A}(\mathbb{R})$  to  $\mathcal{A}(\mathbb{R} \sqcup \mathbb{R})$ , Lemma 17.22 implies

$$\pi(2 \times \mathbb{R})^* \left( \mathcal{Z}^f(\textcircled{\curvearrowright})^2 \right) \pi(2 \times \mathbb{R})^*(\nu) = 1$$

in the algebra  $\mathcal{A}(\mathbb{R} \sqcup \mathbb{R})$ . Since the multiplication by an element whose degree 0 part is 1 is injective, and since an element whose degree 0 part is 1 is determined by its square, we get  $\mathcal{Z}^f(2 \times \textcircled{\curvearrowright}) = \pi(2 \times \mathbb{R})^*(\mathcal{Z}^f(\textcircled{\curvearrowright}))$  as desired.  $\square$

**Proposition 17.24.** *Let  $K$  be a component of a  $q$ -tangle  $L$  in a rational homology cylinder  $\mathcal{C}$ . Let  $L(2 \times K)$  be the tangle obtained by duplicating  $K$  as in Section 13.1. Then we have*

$$\mathcal{Z}^f(L(2 \times K)) = \pi(2 \times K)^* \mathcal{Z}^f(L).$$

PROOF: A tangle  $L$ , with a strand  $K$  going from bottom to bottom, can be written as a composition

$$L_1 L_2 = \boxed{\begin{array}{c} \curvearrowright \\ | \\ L_1 \\ | \\ | \\ | \end{array}}$$

of some tangle  $L_1$ , with a cabling  $L_2$  of a trivial braid by the replacement of a strand by  $\textcircled{\curvearrowright}$ , where  $K$  is the concatenation of one strand of  $L_1$  going from bottom to top,  $\textcircled{\curvearrowright}$ , and another strand of  $L_1$ , which goes from top to bottom. The statement for such a pair follows from Lemma 17.23, Proposition 17.10, and Proposition 17.13, using functoriality. The case in which  $K$  goes from top to top can be treated similarly, by sending  $\textcircled{\curvearrowright}$  below. So the proposition is proved.  $\square$

Theorem 13.12 is now proved.  $\square$

**Remark 17.25.** Theorem 16.45 and Proposition 16.43 do not allow me to prove

$$\mathcal{Z}^f(\mathcal{C}, L(2 \times K), A, (\omega(i, S^2))_{i \in \underline{3N}}) = \pi(2 \times K)^* \mathcal{Z}^f(\mathcal{C}, L, A, (\omega(i, S^2))_{i \in \underline{3N}})$$

for a  $J_{bb,tt}$ -oriented  $q$ -tangle  $L$ , an integer  $N$ , a subset  $A$  of  $\underline{3N}$  whose cardinality is a multiple of 3, and a family of volume-one forms  $(\omega(i, S^2))_{i \in \underline{3N}}$ . Indeed, I can see no reason to believe that  $\pi(2 \times \mathbb{R})^*(\widetilde{\text{hol}}_{[0,t]}(\eta(\cdot, S_{WE})))$  is the product of twice  $\widetilde{\text{hol}}_{[0,t]}(\eta(\cdot, S_{WE}))$  on the two strands of  $2 \times \mathbb{R}$ . Unfortunately, as noticed in Remark 16.48, I do not know how to get rid of our noncanonical normalization of  $\mathcal{Z}^f(\mathcal{C}, L, A, (\omega(i, S^2))_{i \in \underline{3N}})$  and of the corresponding factors  $\widetilde{\text{hol}}_{[0,t]}(\eta(\cdot, S_{WE}))$ , which might not behave well under duplication.

## 17.5 Behavior of $\mathcal{Z}^f$ with respect to the co-product

The behavior of  $\mathcal{Z}^f$  with respect to the coproduct described in Theorem 13.12 is justified after the statement of Theorem 13.12. Proposition 17.27 below shows how this behavior generalizes to the variants of  $\mathcal{Z}^f$ .

Before stating and proving it, let us prove that the compatibility between product and coproduct implies the following preliminary lemma.

**Lemma 17.26.** *Say that a map  $F$  from the set  $\mathcal{P}_{(3)}(\underline{3N})$  of subsets of  $\underline{3N}$  whose cardinalities are multiple of 3 to a space of Jacobi diagrams is group-like if*

$$\Delta(F(B)) = \sum_{(B_1, B_2) \in \mathcal{P}_2(B)} \frac{|B_1|! |B_2|!}{|B|!} F(B_1) \otimes F(B_2)$$

for any element  $B$  of  $\mathcal{P}_{(3)}(\underline{3N})$ .

Let  $F$  and  $G$  be two maps from  $\mathcal{P}_{(3)}(\underline{3N})$  to spaces  $\mathcal{A}_F$  and  $\mathcal{A}_G$  of Jacobi diagrams, such that there is a product from  $\mathcal{A}_F \times \mathcal{A}_G$  to a space of Jacobi diagrams  $\mathcal{A}_{FG}$ .<sup>4</sup> If  $F$  and  $G$  are group-like, then  $(FG)_\sqcup$  is group-like, too.

PROOF: Let  $A \in \mathcal{P}_{(3)}(\underline{3N})$ . We have

$$\begin{aligned} \Delta((FG)_\sqcup(A)) &= \sum_{(B, C) \in \mathcal{P}_2(A)} \frac{|B|! |C|!}{|A|!} \Delta(F(B)) \Delta(G(C)) \\ &= \sum_{(B_1, B_2, C_1, C_2) \in \mathcal{P}_4(A)} \frac{|B_1|! |B_2|! |C_1|! |C_2|!}{|A|!} (F(B_1)G(C_1) \otimes F(B_2)G(C_2)), \end{aligned}$$

with

$$(FG)_\sqcup(A_1) = \sum_{(B_1, C_1) \in \mathcal{P}_2(A_1)} \frac{|B_1|! |C_1|!}{|A_1|!} F(B_1)G(C_1).$$

So we get

$$\Delta((FG)_\sqcup(A)) = \sum_{(A_1, A_2) \in \mathcal{P}_2(A)} \frac{|A_1|! |A_2|!}{|A|!} (FG)_\sqcup(A_1) \otimes (FG)_\sqcup(A_2).$$

□

---

<sup>4</sup>This assumption guarantees that the product  $(FG)_\sqcup$  of Definition 16.10 makes sense.

**Proposition 17.27.** *For any  $q$ -tangle  $L$  in a rational homology cylinder  $\mathcal{C}$ , for any  $N \in \mathbb{N}$ , for any subset  $A$  of cardinality  $3k$  of  $\underline{3N}$ , and for any family  $(\omega(i, S^2))_{i \in \underline{3N}}$  of volume-one forms of  $S^2$ , we have*

$$\Delta_k \left( \mathcal{Z}^f(\mathcal{C}, L, A, (\omega(i, S^2))_{i \in \underline{3N}}) \right) = \sum_{j=0}^k \sum_{\substack{A_1 \subset A, \\ |A_1|=3j, \\ A_2=A \setminus A_1}} \frac{|A_1|! |A_2|!}{|A|!} \mathcal{Z}^f(\mathcal{C}, L, A_1, (\omega(i, S^2))) \otimes \mathcal{Z}^f(\mathcal{C}, L, A_2, (\omega(i, S^2)))$$

with the coproduct maps  $\Delta_n$  defined in Section 6.5.

PROOF: Observe that the statement of Proposition 17.27 is valid for  $q$ -tangles that can be represented as straight tangles with respect to a parallelization  $\tau$  such that  $p_1(\tau) = 0$ , by Theorem 16.16. The coefficients are treated as in Lemma 16.2.

Say that a map  $F$  from  $\mathcal{P}_{(3)}(\underline{3N})$  to a space  $\mathcal{A}_F$  of Jacobi diagrams is *cardinality-determined* if it maps any element  $A$  of  $\mathcal{P}_{(3)}(\underline{3N})$  to a degree  $|A|/3$  element  $F_{|A|/3}$  which depends only on the cardinality of  $A$ . The truncation  $F_{\leq N} = (F_k)_{k \in \mathbb{N}: k \leq N}$  until degree  $N$  of any element  $(F_k)_{k \in \mathbb{N}}$  of  $\mathcal{A}_F$  can be viewed as such a cardinality-determined map. Note that such a truncation of a group-like element is group-like in the sense of Lemma 17.26.

Let  $T$  be a trivial  $q$ -braid (represented by a constant path) except for the framing of one of its strands  $K$ , which is  $lk(K, K_{\parallel}) = 1$  instead of 0. Definition 16.17 implies  $\mathcal{Z}^f(D_1 \times [0, 1], T, \cdot, (\omega(i, S^2))) = \exp(\alpha) \#_K [\emptyset]$ . In particular, this expression does not depend on  $(\omega(i, S^2))_{i \in \underline{3N}}$ , and  $\mathcal{Z}^f(D_1 \times [0, 1], T, \cdot, (\omega(i, S^2)))$  is group-like. Similarly,  $\exp_{\leq N}(-\frac{1}{4}p_1(\tau)\beta(\cdot))_{\sqcup}$  is group-like. Therefore, Lemma 17.26 allows us to conclude the proof of Proposition 17.27 for framed tangles with injective bottom and top configurations. Use Lemma 16.33 to conclude for general  $q$ -tangles.  $\square$

**Proposition 17.28.** *Let  $(\omega(i, S^2))_{i \in \underline{3N}}$  be a family of volume-one forms of  $S^2$ . Let  $\mathcal{C}$  be a rational homology cylinder. Let  $L$  be a  $J_{bb,tt}$ -oriented  $q$ -tangle of  $\mathcal{C}$ . Let  $\check{\mathcal{Z}}^f(\mathcal{C}, L, \cdot, (\omega(i, S^2)))$  denote the projection of  $\mathcal{Z}^f(\mathcal{C}, L, \cdot, (\omega(i, S^2)))$  on  $\check{\mathcal{A}}(\mathcal{C})$ . Then we have*

$$\mathcal{Z}^f(\mathcal{C}, L, \cdot, (\omega(i, S^2))_{i \in \underline{3N}}) = \left( \check{\mathcal{Z}}^f(\mathcal{C}, L, \cdot, (\omega(i, S^2))_{i \in \underline{3N}}) \mathcal{Z}_{\leq N}^f(\mathcal{C}, \emptyset) \right)_{\sqcup}.$$

PROOF:  $\mathcal{Z}^f(\mathcal{C}, \emptyset, \cdot, (\omega(i, S^2))_{i \in \underline{3N}}) = \mathcal{Z}(R(\mathcal{C}))$  is independent of  $(\omega(i, S^2))_{i \in \underline{3N}}$ . See Theorems 7.40 and 16.16.  $\square$



## 17.6 A proof of universality

In this section, we apply the properties of  $\mathcal{Z}^f$  to study its variation under crossing changes and generalize Theorem 6.9.

Define a *singular tangle representative with  $n$  double points* to be an oriented 1-dimensional manifold  $L(\mathcal{L})$  immersed in  $\mathcal{C}$  such that

- the boundary of  $L(\mathcal{L})$  sits in the interior of  $D_1 \times \{0, 1\}$ ,
- $L(\mathcal{L})$  meets a neighborhood  $N(\partial(D_1 \times [0, 1]))$  as vertical segments, and
- the only singular points of the immersion are  $n$  double points  $\bowtie$ , for which the directions of the two meeting branches generate a *tangent plane*.

Define a *singular tangle with  $n$  double points* to be an equivalence class of such representatives under the equivalence relation defined as in the nonsingular case in Definition 12.15, by adding the adjective singular.

Extend the invariant  $\mathcal{Z}$  to unframed singular tangles by the local rule

$$\mathcal{Z}\left(\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array}\right) = \mathcal{Z}\left(\begin{array}{c} \circlearrowleft \\ \circlearrowleft \end{array}\right) - \mathcal{Z}\left(\begin{array}{c} \circlearrowright \\ \circlearrowright \end{array}\right).$$

This local rule relates the invariants  $\mathcal{Z}$  of three singular tangles that coincide outside the represented ball and are as in the pictures in this ball.

Define the *chord diagram*  $\Gamma_C(L)$  associated to an unframed singular tangle  $L$  with  $n$  double points to be the following diagram on the domain  $\mathcal{L}$  of  $L$ . The diagram  $\Gamma_C(L)$  has  $2n$  vertices. Its vertices are univalent and located at the preimages of the double points. It has  $n$  edges, one between each pair of preimages of a double point. These edges are called *chords*. (The chords are attached on the left-hand side of the oriented domain  $\mathcal{L}$ , when orientations of univalent vertices are needed, as in Definition 6.16.)

**Notation 17.29.** Respectively denote the images of  $\mathcal{Z}$  and of  $\check{p} \circ \mathcal{Z} = \check{\mathcal{Z}}$  under the quotient by the  $1T$ -relation by  $\overline{\mathcal{Z}}$  and  $\overline{\check{\mathcal{Z}}}$ . (So  $\overline{\mathcal{Z}}(L)$  and  $\overline{\check{\mathcal{Z}}}(L)$  respectively belong to  $\mathcal{A}(\mathcal{L})/(1T)$  and  $\check{\mathcal{A}}(\mathcal{L})/(1T)$ .)

In this section, we prove the following theorem, which is a generalization of Theorem 6.9 from knots to links and tangles.

**Theorem 17.30.** *Let  $n$  be a natural number. For any unframed singular tangle  $L$  with  $n$  double points in a rational homology cylinder  $\mathcal{C}$ , the expansion  $\overline{\mathcal{Z}}_{\leq n-1}(L)$  up to degree  $n - 1$  of  $\overline{\mathcal{Z}}(L)$  vanishes, and its expansion  $\overline{\mathcal{Z}}_{\leq n}(L)$  up to degree  $n$  is equal to*

$$\overline{\mathcal{Z}}_{\leq n}(L) = [\Gamma_C(L)]$$

*in  $\mathcal{A}(\mathcal{L})/(1T)$ . So we have  $\overline{\check{\mathcal{Z}}}_{\leq n}(L) = [\Gamma_C(L)]$  in  $\check{\mathcal{A}}(\mathcal{L})/(1T)$ .*

Theorem 6.9 and its proof, whose easiest part is presented in Section 6.2, generalize to  $k$ -component oriented links, with numbered components, to produce the following corollary to Theorem 17.30. The isomorphism of the corollary was first shown by Dror Bar-Natan and Maxim Kontsevich [BN95a].

**Corollary 17.31.** *With the notation of Section 6.1, for any  $(k, n) \in \mathbb{N}^2$ ,  $\overline{\mathcal{Z}}_{\leq n}$  induces an isomorphism from  $\frac{\mathcal{F}_n(\mathcal{K}_k; \mathbb{Q})}{\mathcal{F}_{n+1}(\mathcal{K}_k; \mathbb{Q})}$  to  $\check{\mathcal{A}}(\sqcup_{i=1}^k (S^1)_i)/(1T)$ , where  $(S^1)_i$  is the copy of  $S^1$  associated to the  $i^{\text{th}}$  component of a link.*

□

In order to prove Theorem 17.30, we first define framed singular tangles and extend  $\mathcal{Z}^f$  to these tangles. A *parallelization* of a singular tangle is an isotopy class of parallels as in the nonsingular case, with the same restrictions near the boundary, where the parallel of a neighborhood of a double point is on one side of the tangent plane of the double point  $\bowtie$ . Recall that there are two ways of desingularizing  $\bowtie$ , the *positive* one for which  $\bowtie$  is replaced by  $\nearrow$ , and the *negative* one for which  $\bowtie$  is replaced by  $\searrow$ . In particular, every desingularization of such a singular tangle gets a natural parallelization from the parallelization of the singular tangle. Locally, the parallel of each branch is well-defined.

In general, we define the self-linking number of a component of a singular framed tangle as before, where the *components* of a singular framed tangle are in natural one-to-one correspondence with the components of its domain. Let  $p$  be a double point for which both branches belong to a component  $K_j$ . Let  $L(p, +)$  and  $L(p, -)$  respectively denote the positive and negative desingularizations of  $L$  at  $p$ . Then the self-linking numbers of  $K_j$  in  $L$  and in these two desingularizations are related by

$$lk((K_j, K_{j\parallel}) \subset L(p, +)) = lk((K_j, K_{j\parallel}) \subset L) + 1$$

and  $lk((K_j, K_{j\parallel}) \subset L(p, -)) = lk((K_j, K_{j\parallel}) \subset L) - 1$ .

We formally extend  $\mathcal{Z}^f$  to (framed) singular  $q$ -tangles by the formula

$$\mathcal{Z}^f\left(\text{diagram of } \bowtie\right) = \mathcal{Z}^f\left(\text{diagram of } \nearrow\right) - \mathcal{Z}^f\left(\text{diagram of } \searrow\right),$$

where the parallels of the three tangles are supposed to be behind and to match on the boundary of the ball.

As an example, we have

$$\mathcal{Z}^f\left(\text{diagram of } \bowtie\right) = \mathcal{Z}^f\left(\text{diagram of } \nearrow\right) - \mathcal{Z}^f\left(\text{diagram of } \searrow\right) = \mathcal{Z}^f\left(\text{diagram of } \nearrow\right) - \mathcal{Z}^f\left(\text{diagram of } \uparrow\uparrow\right),$$

where the endpoints of the tangles are assumed to lie on the real line. So Lemmas 12.18 and 12.19 yield

$$\mathcal{Z}_{\leq 1}^f \left( \text{diagram of a double point} \right) = \left[ \text{diagram of a double point with arrows} \right].$$

Note that  $\mathcal{Z}^f$  is now a functor on the category of singular  $q$ -tangles, which satisfies the cabling property and the duplication properties of Theorem 13.12 provided that the components involved in a double point are not duplicated.

**Proposition 17.32.** *Let  $n$  be a natural number. For any singular  $q$ -tangle  $L$  with  $n$  double points, we have*

$$\mathcal{Z}_{\leq n}^f (L) = [\Gamma_C (L)].$$

PROOF: In the proof below, we evaluate the lowest degree part that does not vanish in  $\mathcal{Z}^f(L)$  for various singular  $q$ -tangles. Note that this part is unchanged when such a singular  $q$ -tangle is multiplied by a nonsingular  $q$ -tangle (except for the modification of the domains) since the lowest degree part that does not vanish for a nonsingular  $q$ -tangle is the class of the empty diagram. In particular, the lowest degree nonvanishing part is independent of the bottom and top configurations of our  $q$ -tangles, which will not be specified.

Applying the cabling property of  $\mathcal{Z}^f$  to the following cable of a trivial braid with three strands yields

$$\mathcal{Z}_{\leq 1}^f \left( \text{diagram of a cable of a trivial braid} \right) = \text{diagram of a cable of a trivial braid with arrows}.$$

We thus get

$$\mathcal{Z}_{\leq 1}^f \left( \text{diagram of a U-tangle} \right) = \text{diagram of a U-tangle with arrows}.$$

by functoriality as desired for such a  $q$ -tangle.

Starting with a trivial braid and successively cabling some of its strands

by replacing  $n$  of them by  $\text{diagram of a U-tangle}$ , we find

$$\mathcal{Z}_{\leq n}^f \left( \text{diagram of a braid with n U-tangles} \right) = \text{diagram of a braid with n U-tangles with arrows}.$$

(Here, we first apply the cabling property to the first strand of a trivial braid  $\mathbf{1}_{n+k}$  and regard the resulting tangle  $T_1$  as a product  $\mathbf{1}_{n+k-1}T_1$ , where  $\mathbf{1}_{n+k-1}$  is a trivial braid. We next apply the cabling property to the first strand of  $\mathbf{1}_{n+k-1}$ . We keep going to find the obtained formula with the help of the functoriality property.)

Since every singular  $q$ -tangle with  $n$  double points can be written as a product of a  $q$ -tangle as above and a nonsingular  $q$ -tangle by moving the double points below, the proposition follows.  $\square$

We are ready to deduce Theorem 17.30 from Proposition 17.32.

PROOF OF THEOREM 17.30: Assume that the singular  $q$ -tangle  $L$  with  $n$  double points has  $k$  components  $K_i$  for  $i = 1, \dots, k$ .

The case  $n = 0$  is obvious. Assume that  $n = 1$ . Let  $s_i$  denote the self-linking number of  $K_i$  in  $L$ . Let  $L^+$  be the positive desingularization of  $L$ , and let  $L^-$  be the negative desingularization of  $L$ . Let  $s_i^+$  (resp.  $s_i^-$ ) denote the self-linking number of the  $i^{\text{th}}$  component in  $L^+$  (resp. in  $L^-$ ). Recall that  $\overline{\mathcal{Z}}$  is the image of  $\mathcal{Z}$  under the quotient by the  $1T$ -relation. Also recall

$$\mathcal{Z}(L^+) = \prod_{j=1}^k (\exp(-s_j^+ \alpha) \#_j) \mathcal{Z}^f(L^+).$$

If the two strands involved in the double point belong to two distinct components, then we have  $s_j^+ = s_j = s_j^-$  for any  $j \in \underline{k}$  and

$$\mathcal{Z}(\text{crossing}) - \mathcal{Z}(\text{crossing}) = \prod_{j=1}^k (\exp(-s_j \alpha) \#_j) \left( \mathcal{Z}^f(\text{crossing}) - \mathcal{Z}^f(\text{crossing}) \right).$$

So the result follows since the lowest degree part of any  $\exp(-s_j \alpha)$  is the class of the empty diagram.

If the two strands involved in the double point belong to the same component  $K_i$ , then we have  $s_i^+ = s_i^- + 2$  and

$$\begin{aligned} \mathcal{Z}(\text{crossing}) - \mathcal{Z}(\text{crossing}) &= \prod_{j=1}^k (\exp(-s_j^+ \alpha) \#_j) \left( \mathcal{Z}^f(\text{crossing}) - \mathcal{Z}^f(\text{crossing}) \right) \\ &\quad + \left( \prod_{j=1}^k (\exp(-s_j^+ \alpha) \#_j) - \prod_{j=1}^k (\exp(-s_j^- \alpha) \#_j) \right) \mathcal{Z}^f(\text{crossing}), \end{aligned}$$

where the  $\left( \prod_{j=1}^k (\exp(-s_j^+ \alpha) \#_j) - \prod_{j=1}^k (\exp(-s_j^- \alpha) \#_j) \right)$  “factor” begins with its degree one part, which is  $-2\alpha_1 = -[\hat{\zeta}]$ . We get

$$\overline{\mathcal{Z}}_{\leq 1}(\text{crossing}) - \overline{\mathcal{Z}}_{\leq 1}(\text{crossing}) = \left[ \begin{array}{c} \hat{\zeta} \\ \text{crossing} \end{array} \right]$$

as desired. Note that this equality would be wrong if  $\overline{\mathcal{Z}}$  were replaced by  $\mathcal{Z}$ , that is, without modding out by  $1T$ .

Let us now conclude the proof by induction on  $n$ . Assume that the result is known for singular  $q$ -tangles with less than  $n$  double points. Let  $P$  denote the set of double points of  $L$ . For  $i \in \underline{k}$ , let  $P_i$  denote the set of double points for which both branches belong to  $K_i$ . For a subset  $I$  of  $P$ , the  $q$ -tangle obtained from  $L$  by performing negative desingularizations on double points of  $I$  and positive ones on double points of  $P \setminus I$  is denoted by  $L_I$ , and  $s_{j,I}$  denotes the self-linking number of the component  $K_{j,I}$  in  $L_I$ .

Then  $\overline{\mathcal{Z}}(L)$  is equal to

$$\begin{aligned} \sum_{I \subseteq P} (-1)^{|I|} \overline{\mathcal{Z}}(L_I) &= \sum_{I \subseteq P} (-1)^{|I|} \prod_{j=1}^k (\exp(-s_{j,I}\alpha) \#_j) \mathcal{Z}^f(L_I) \text{ mod } 1T \\ &= \sum_{I \subseteq P} (-1)^{|I|} \left( \prod_{j=1}^k (\exp(-s_{j,I}\alpha) \#_j) - \prod_{j=1}^k (\exp(-s_{j,\emptyset}\alpha) \#_j) \right) \mathcal{Z}^f(L_I) \\ &\quad + \prod_{j=1}^k (\exp(-s_{j,\emptyset}\alpha) \#_j) \sum_{I \subseteq P} (-1)^{|I|} \mathcal{Z}^f(L_I) \text{ mod } 1T. \end{aligned}$$

The lowest degree term of the last line is  $[\Gamma_C(L)]$ . So it suffices to prove that the previous line does not contain terms of degree less than  $n + 1 \pmod{1T}$ . This previous line can be rewritten as

$$T_2 = \sum_{I \subseteq P} (-1)^{|I|} \left( 1 - \prod_{j=1}^k (\exp((s_{j,I} - s_{j,\emptyset})\alpha) \#_j) \right) \mathcal{Z}(L_I) \text{ mod } 1T,$$

with  $s_{j,I} - s_{j,\emptyset} = -2|I \cap P_j| = \sum_{p \in I \cap P_j} (-2)$ . So  $T_2$  can be rewritten as

$$T_2 = \sum_{I \subseteq P} (-1)^{|I|} \left( 1 - \prod_{j=1}^k \left( \prod_{p \in I \cap P_j} \exp(-2\alpha_p) \#_j \right) \right) \overline{\mathcal{Z}}(L_I),$$

where  $\alpha_p$  is a copy of  $\alpha$ .

Let  $F$  be the set of maps  $f: \cup_{j=1}^k P_j \rightarrow \mathbb{N}$  such that  $f(\cup_{j=1}^k P_j) \neq \{0\}$ . Let  $F(I)$  be the set of maps  $f$  of  $F$  such that  $f(p) = 0$  for any  $p \notin I$ . Then we have

$$1 - \prod_{j=1}^k \left( \prod_{p \in I \cap P_j} \exp(-2\alpha_p) \#_j \right) = - \sum_{f \in F(I)} D_f$$

with

$$D_f = \prod_{j=1}^k \left( \left( \prod_{p \in P_j} \frac{(-2\alpha)^{f(p)}}{f(p)!} \right) \#_j \right).$$

Now,  $T_2$  can be rewritten as

$$\sum_{f \in F} D_f \sum_{I: I \subseteq P, f \in F(I)} (-1)^{|I|+1} \overline{\mathcal{Z}}(L_I).$$

Set  $N(f) = \{p : p \in \cup_{j=1}^k P_j, f(p) \neq 0\}$ . Let  $L(N(f), -)$  denote the singular  $q$ -tangle with  $(n - |N(f)|)$  double points obtained from  $L$  by desingularizing the double points of  $N(f)$  in a negative way. Since the condition  $f \in F(I)$  is equivalent to the condition  $N(f) \subseteq I$ , we get

$$\sum_{I: I \subseteq P, f \in F(I)} (-1)^{|I|+1} \overline{\mathcal{Z}}(L_I) = (-1)^{|N(f)|+1} \overline{\mathcal{Z}}(L(N(f), -)).$$

Note that  $(-2\alpha)^{f(p)}$  is of degree at least 3 as soon as  $f(p) \neq 0$  (when working modulo  $1\Gamma$  since  $\alpha_1 = \alpha_2 = 0$ ). So the degree of  $D_f$  is at least  $3|N(f)|$ . By induction, the degree of  $\overline{\mathcal{Z}}(L(N(f), -))$  is  $n - |N(f)|$ . Therefore, the parts of  $T_2$  of degree at most  $n$  vanish.  $\square$



**Part IV**  
**Universality**





# Chapter 18

## The main universality statements and their corollaries

### 18.1 Universality with respect to Lagrangian-preserving surgeries

Let us recall some definitions quickly surveyed in the book introduction.

**Definition 18.1.** An *integer (resp. rational) homology handlebody* of genus  $g$  is a compact oriented 3-manifold  $A$  with the same integral (resp. rational) homology as the usual solid handlebody  $H_g$  of Figure 1.1. The *Lagrangian*  $\mathcal{L}_A$  of a compact 3-manifold  $A$  is the kernel of the map induced by the inclusion from  $H_1(\partial A; \mathbb{Q})$  to  $H_1(A; \mathbb{Q})$ .

**Exercise 18.2.** Show that if  $A$  is a rational homology handlebody of genus  $g$ , then  $\partial A$  is a connected genus  $g$  surface. (See Appendix A.1, where some basic properties of homology are recalled.)

In Figure 1.1, the Lagrangian of  $H_g$  is freely generated by the classes of the curves  $a_i$ .

**Definition 18.3.** An *integral (resp. rational) Lagrangian-Preserving (or LP) surgery*  $(A'/A)$  is the replacement of an integer (resp. rational) homology handlebody  $A$ , embedded in the interior of a 3-manifold  $M$ , by another such  $A'$ , whose boundary  $\partial A'$  is identified with  $\partial A$  by an orientation-preserving diffeomorphism sending  $\mathcal{L}_A$  to  $\mathcal{L}_{A'}$ . The manifold  $M(A'/A)$  obtained by such an LP-surgery is

$$M(A'/A) = (M \setminus \text{Int}(A)) \cup_{\partial A} A'.$$

(This only defines the topological structure of  $M(A'/A)$ , but  $M(A'/A)$  is equipped with its unique smooth structure.)

We present an interesting example of an integral Lagrangian-preserving surgery in Subsection 18.3.1. The Matveev Borromean surgery of Section 18.4 is another example of integral LP-surgery.

**Lemma 18.4.** *If  $(A'/A)$  is an integral (resp. rational) LP-surgery, then the homology of  $M(A'/A)$  with  $\mathbb{Z}$ -coefficients (resp. with  $\mathbb{Q}$ -coefficients) is canonically isomorphic to  $H_*(M; \mathbb{Z})$  (resp. to  $H_*(M; \mathbb{Q})$ ). If  $M$  is a  $\mathbb{Q}$ -sphere, if  $(A'/A)$  is a rational LP-surgery, and if  $(J, K)$  is a two-component link of  $M \setminus A$ , then the linking number of  $J$  and  $K$  in  $M$  and the linking number of  $J$  and  $K$  in  $M(A'/A)$  coincide.*

PROOF: Exercise. □

Let  $(A'/A)$  be a rational LP-surgery in a punctured rational homology sphere  $\tilde{R}$ . Let

$$\partial_{MV} : H_2(A \cup_{\partial A} -A'; \mathbb{Q}) \rightarrow \mathcal{L}_A$$

be the morphism that maps the class of a closed surface in the closed 3-manifold  $(A \cup_{\partial A} -A')$  to the boundary of its intersection with  $A$ . The Mayer–Vietoris long exact sequence (see Theorem A.11) shows that the above canonical morphism  $\partial_{MV}$  is an isomorphism. This isomorphism carries the algebraic triple intersection of surfaces to a trilinear antisymmetric form  $\mathcal{I}_{AA'}$  on  $\mathcal{L}_A$ . Explicitly, we have

$$\mathcal{I}_{AA'}(a_i, a_j, a_k) = \langle \partial_{MV}^{-1}(a_i), \partial_{MV}^{-1}(a_j), \partial_{MV}^{-1}(a_k) \rangle_{A \cup -A'}.$$

Let  $(a_1, a_2, \dots, a_g)$  be a basis of  $\mathcal{L}_A$  and let  $z_1, \dots, z_g$  be (curves representing) homology classes of  $\partial A$  such that the system  $(z_1, z_2, \dots, z_g)$  is dual to  $(a_1, a_2, \dots, a_g)$  with respect to  $\langle, \rangle_{\partial A}$ :

$$\langle a_i, z_j \rangle_{\partial A} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Note that  $(z_1, \dots, z_g)$  is a basis of  $H_1(A; \mathbb{Q})$ .

Represent  $\mathcal{I}_{AA'}$  by the following combination  $T(\mathcal{I}_{AA'})$  of tripods whose three univalent vertices form an ordered set

$$T(\mathcal{I}_{AA'}) = \sum_{(i,j,k) \in g^3 : i < j < k} \mathcal{I}_{AA'}(a_i, a_j, a_k) \begin{array}{c} \leftarrow z_k \\ \leftarrow z_j \\ \leftarrow z_i \end{array},$$

where the tripods are considered up to the relations

$$\begin{array}{c} \leftarrow z \\ \leftarrow y \\ \leftarrow x \end{array} = \begin{array}{c} \leftarrow x \\ \leftarrow z \\ \leftarrow y \end{array} = - \begin{array}{c} \leftarrow z \\ \leftarrow x \\ \leftarrow y \end{array} = - \begin{array}{c} \leftarrow z \\ \leftarrow y \\ \leftarrow x \end{array}.$$

**Notation 18.5.** Let  $G$  be a graph with  $2k$  oriented trivalent vertices and with univalent vertices. Assume that the univalent vertices of  $G$  are decorated with disjoint curves of a punctured  $\mathbb{Q}$ -sphere  $\check{R}$ . Let  $P(G)$  be the set of partitions of the set of univalent vertices of  $G$  into disjoint pairs.

For  $p \in P(G)$ , identifying the two vertices of each pair provides a vertex-oriented trivalent Jacobi diagram  $\Gamma_p$ . Let  $\ell(p)$  be the product, over the disjoint pairs of  $p$ , of the linking numbers of the curves corresponding to the two vertices in a pair. We get an element  $[\ell(p)\Gamma_p]$  of  $\mathcal{A}_k(\emptyset)$ . Define

$$\langle\langle G \rangle\rangle = \sum_{p \in P(G)} [\ell(p)\Gamma_p].$$

The contraction  $\langle\langle \cdot \rangle\rangle$  is linearly extended to linear combinations of graphs. The disjoint union of combinations of graphs is bilinear.

The universality theorem with respect to Lagrangian-preserving surgeries is the following one. It was proved in [Les04b] for the invariant  $\mathcal{Z}$  of rational homology spheres. The statement below is more general since it applies to the invariant  $\mathcal{Z}$  of Theorem 12.7, which satisfies the properties stated in Theorem 13.12. Nevertheless, its proof reproduced in this book, is identical to the proof of the preprint [Les04b], except for some editorial improvements. ([Les04b] has never been submitted for publication.)

**Theorem 18.6.** *Let  $L$  be a  $q$ -tangle representative in a rational homology cylinder  $\mathcal{C}$ . Let  $x$  be a positive integer. Let  $\sqcup_{i=1}^x A^{(i)}$  be a disjoint union of rational homology handlebodies embedded in  $\mathcal{C} \setminus L$ . Let  $(A^{(i)'}/A^{(i)})$  be rational LP-surgeries in  $\mathcal{C}$ . For a subset  $I$  of  $\underline{x}$ , let  $\mathcal{C}_I = \mathcal{C}((A^{(i)'}/A^{(i)})_{i \in I})$  be the rational homology cylinder obtained from  $\mathcal{C}$  by performing the LP-surgeries that replace  $A^{(i)}$  by  $A^{(i)'}$  for  $i \in I$ . Set  $X = [\mathcal{C}, L; (A^{(i)'}/A^{(i)})_{i \in \underline{x}}]$  and*

$$\mathcal{Z}_n(X) = \sum_{I \subseteq \underline{x}} (-1)^{x+|I|} \mathcal{Z}_n(\mathcal{C}_I, L).$$

*If  $2n < x$ , then  $\mathcal{Z}_n(X)$  vanishes. If  $2n = x$ , then we have*

$$\mathcal{Z}_n(X) = \left[ \left\langle \left\langle \left\langle \bigsqcup_{i \in \underline{x}} T(\mathcal{I}_{A^{(i)} A^{(i)'}}) \right\rangle \right\rangle \right].$$

Before proving Theorem 18.6, we discuss some of its consequences and variants. Section 18.3 shows that Theorem 18.6 yields a direct proof of a surgery formula for the Theta invariant, as in [Les09, Section 9]. The article [Les09] presents many other surgery formulae implied by Theorem 18.6, which are not reproduced in this book.

Section 18.4 shows how Theorem 18.6 implies that  $\mathcal{Z}$  restricts to a universal finite type invariant of integer homology 3-spheres. In Section 18.5, we review the Moussard classification of finite type invariants of rational homology 3-spheres [Mou12], and we show how  $\mathcal{Z}$  can be augmented to provide a universal finite type invariant of rational homology 3-spheres, too, following an idea of Gwénaél Massuyeau. Section 18.6 shows how Theorem 18.6 also implies that the invariant  $\frac{1}{6}\Theta$  is the Casson–Walker invariant. Assuming this identification, the surgery formula of Section 18.3 is nothing but a consequence of the Casson–Walker surgery formula of [Wal92]. So a reader who does not need examples can skip Section 18.3, at first.

We sketch the proof of Theorem 18.6 in Section 18.7. We complete the details of the proof in the following two chapters.

Theorem 18.6 and the universality theorems 6.9 and 17.30 for knots or tangles are put together in Theorem 18.35, which generalizes all of them.

Section 18.8 shows how the main ingredients of the proof of Theorem 18.6 also lead to Theorem 18.39, which allows us to compute the degree 2 part of  $\check{\mathcal{Z}}$ , for any null-homologous knot, in Theorem 18.43, with the help of the contents of Section 18.3.

Section 18.2 below gives some background about Dehn surgeries. We use this background in Sections 18.3 and 18.4.

## 18.2 On Dehn surgeries

In this section, we define the manifold  $R_{(K;p/q)}$  obtained by  $p/q$ -surgery on a  $\mathbb{Q}$ -sphere  $R$  along a knot  $K$ . We also introduce the lens spaces  $L(p, q)$  and give some examples of surgeries on links used in Section 18.4.

Let  $K$  be a knot in a 3-manifold  $M$ , and let  $N(K)$  be a tubular neighborhood of  $K$ . The *exterior*  $E(K)$  of  $K$  is the closure of the complement of  $N(K)$  in  $M$ . Let  $\mu$  be a nonseparating simple closed curve of the boundary  $\partial N(K)$  of  $N(K)$ . The manifold obtained from  $M$  by *Dehn surgery* on  $K$  with respect to  $\mu$  is the union  $E(K) \cup_{\partial N(K)} T$  of  $E(K)$  and a solid torus  $T$ , where  $E(K)$  and  $T$  are glued along  $\partial N(K)$  by an orientation-reversing homeomorphism from  $\partial T$  to  $\partial N(K)$ , which maps a meridian of  $T$  to  $\mu$ . The result is then smoothed in a standard way. (Since the gluing of  $T$  can be achieved by gluing a meridian disk of  $T$  along  $\mu$ , thickening it, and gluing a 3-dimensional ball to the resulting boundary, this surgery operation is well-defined.<sup>1</sup>)

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<sup>1</sup>The operation of gluing a 3-ball  $B^3$  along  $S^2$  is well-defined, because any homeomorphism  $f$  from  $S^2$  to  $S^2$  extends to  $B^3$  as the homeomorphism that maps  $tx$  to  $tf(x)$  for  $t \in [0, 1]$  and  $x \in S^2$ .

**Example 18.7.** As the reader can check, the manifold obtained by Dehn surgery on the unknot  $U$  of  $S^3$  with respect to its meridian  $m(U)$  is  $S^2 \times S^1$ .

Let  $K$  be a knot in a rational homology sphere. If  $K$  is null-homologous, then  $K$  has a unique parallel  $\ell(K)$  such that  $lk(K, \ell(K)) = 0$ . This parallel is called the *preferred longitude* of  $K$ . (A parallel of  $K$  is also called a *longitude* of  $K$ .) Let  $\mu$  be a simple closed curve in the boundary  $\partial N(K)$  of a tubular neighborhood of  $K$  such that  $\mu$  does not separate  $\partial N(K)$ . The class of the curve  $\mu$  in  $H_1(\partial N(K))$  may be expressed as  $pm(K) + q\ell(K)$ , where  $m(K)$  is the meridian of  $K$ . The coefficient of the Dehn surgery along  $K$  with respect to  $\mu$  is  $p/q$ . We refer to this Dehn surgery as the  $p/q$ -surgery on  $K$ . This coefficient  $p/q$  may be expressed as  $lk(K, \mu)/\langle m(K), \mu \rangle_{\partial N(K)}$ . The  $p/q$ -surgery along a non-necessarily null-homologous knot  $K$  in a rational homology 3-sphere  $R$  is the Dehn surgery with respect to a nonseparating simple closed curve  $\mu$  of  $\partial N(K)$  such that

$$\frac{lk(K, \mu)}{\langle m(K), \mu \rangle_{\partial N(K)}} = \frac{p}{q}.$$

Let  $R_{(K;p/q)}$  denote the result of a  $p/q$ -Dehn surgery on  $R$  along  $K$ . As shown in Example 18.7, we have  $S^3_{(U;0)} = S^2 \times S^1$ .

According to a theorem independently proved by Raymond Lickorish [Lic62] and Andrew Wallace [Wal60] in 1960, every closed oriented 3-manifold can be obtained from  $S^3$  by surgery along a link of  $S^3$  whose components are equipped with integers. (Surgeries are performed simultaneously along all the components of the link.) In [Rou85], Colin Rourke gave a quick and elementary proof of this result.

**Examples 18.8.** As the reader can check, the manifold obtained by Dehn surgery on the trivial link of  $S^3$  with  $g$  components, all equipped with the coefficient 0, is the connected sum of  $g$  copies of  $S^2 \times S^1$ . Furthermore, this connected sum is homeomorphic to the manifold  $H_g \cup_{\mathbf{1}_{\partial H_g}} (-H_g)$ .



Figure 18.1: Borromean link

As a more challenging exercise, the reader can prove the following fact. The manifold obtained by Dehn surgery on the *Borromean link* of  $S^3$ , represented in Figure 18.1, whose components are equipped with the coefficient 0, is diffeomorphic to  $(S^1)^3$ . A hint can be found in [Thu78, Example 13.1.5].

Let  $p$  and  $q$  be two coprime integers,  $p > 0$ . View  $S^3$  as the unit sphere of  $\mathbb{C}^2$ . The *lens space*  $L(p, q)$  is the quotient of  $S^3$  by the action of  $\mathbb{Z}/p\mathbb{Z}$  on  $S^3$ , where the generator  $[1]$  of  $\mathbb{Z}/p\mathbb{Z}$  acts on a unit vector  $(z_1, z_2)$  of  $\mathbb{C}^2$  by mapping it to  $(\exp(2i\pi/p)z_1, \exp(2i\pi q/p)z_2)$ .

Let us study more surgeries along *unknots* or *trivial knots*, which are knots that bound an embedded disk, and prove the following well-known lemma.

**Lemma 18.9.** *Let  $k$  be an integer. Let  $U$  be a trivial knot. Then  $S^3_{(U;1/k)} \cong S^3$ . More generally, for any pair  $(a, b)$  of coprime integers such that  $a > 0$ ,  $S^3_{(U;a/(b+ka))}$  is diffeomorphic to  $S^3_{(U;a/b)}$ , and we have*

$$S^3_{(U;a/b)} = L(a, -b).$$

If  $U$  is a trivial knot of a 3-manifold  $M$ , then  $M_{(U;a/b)}$  is the connected sum  $M \# L(a, -b)$ .

**PROOF:** The exterior  $E$  of the unknot  $U$  in  $S^3$  is a solid torus whose meridian  $m(E)$  is the preferred longitude  $\ell(U)$  of  $U$ . The meridian  $m(U)$  is a longitude  $\ell(E)$  of  $E$ . Performing  $(a/b)$ -surgery along  $U$  on  $S^3$  amounts to gluing a solid torus with meridian

$$\mu = am(U) + b\ell(U) = bm(E) + a\ell(E)$$

to  $E$ , where  $\langle m(E), \mu \rangle_{\partial E} = a$  and  $\langle \mu, \ell(E) \rangle_{\partial E} = b$ . The manifold  $S^3_{(U;a/b)}$  is the union of two solid tori  $E$  and  $T$  glued by a homeomorphism from  $(-\partial T)$  to  $\partial E$  mapping the meridian of  $T$  to a curve  $\mu$  as above.

Let  $k$  be an integer. Then  $(\ell(E) - km(E))$  is another longitude of  $E$ . This shows that for any coprime integers  $a$  and  $b$ , the manifold  $S^3_{(U;a/(b+ka))}$  is diffeomorphic to  $S^3_{(U;a/b)}$ . In particular, we have  $S^3_{(U;1/k)} \cong S^3$ .

For a trivial knot  $U$  in a 3-manifold  $M$ , the manifold  $M_{(U;a/b)}$  is the connected sum of  $S^3_{(U;a/b)}$  and  $M$ . (The connected sum replaces a ball in the interior of the above solid torus  $E$  by the exterior of a ball containing  $U$  in  $M$ .)

Below,  $S^3$  is viewed as the sphere of  $\mathbb{C}^2$  with radius  $\sqrt{2}$ . The action of  $\mathbb{Z}/p\mathbb{Z}$  on  $S^3$  that defines the lens space  $L(p, q)$  preserves the solid torus  $|z_1| \leq |z_2|$  and the solid torus  $|z_1| \geq |z_2|$ . Let  $E$  be the quotient of the second torus. We have

$$E = \frac{\left\{ \left( \exp\left(\frac{2i\pi t}{p}\right) \sqrt{2 - |z_2|^2}, z_2 \right) : t \in [0, 1], z_2 \in \mathbb{C}, |z_2| \leq 1 \right\}}{\left( \sqrt{2 - |z_2|^2}, z_2 \right) \sim \left( \exp\left(\frac{2i\pi}{p}\right) \sqrt{2 - |z_2|^2}, z_2 \exp\left(\frac{2i\pi q}{p}\right) \right)}.$$

The meridian of the solid torus  $E$  is

$$m(E) = \left\{ (1, \exp(2i\pi u)) : u \in [0, 1] \right\}$$

and the possible (homology classes of) longitudes of  $E$  are all the  $\ell(E) + km(E)$  for  $k \in \mathbb{Z}$ , where

$$\ell(E) = \left\{ \left( \exp\left(\frac{2i\pi t}{p}\right), \exp\left(\frac{2i\pi tq}{p}\right) \right) : t \in [0, 1] \right\}.$$

The boundary of  $E$  is oriented as  $(-S^1) \times S^1$ . The quotient of the solid torus  $|z_1| \leq |z_2|$  is also a solid torus whose meridian is

$$\begin{aligned} \mu &= \left\{ (\exp(2i\pi s), 1) : s \in [0, 1] \right\} \\ &= \bigcup_{j=0}^{p-1} \left\{ \left( \exp\left(\frac{2i\pi(t+j)}{p}\right), 1 \right) : t \in [0, 1] \right\} \\ &= \bigcup_{j=0}^{p-1} \left\{ \left( \exp\left(\frac{2i\pi t}{p}\right), \exp\left(-\frac{2i\pi qj}{p}\right) \right) : t \in [0, 1] \right\}. \end{aligned}$$

So we have  $\langle m(E), \mu \rangle_{\partial E} = p$  and  $\langle \mu, \ell(E) \rangle_{\partial E} = -q$ . (There are  $|q|$  pairs  $(t, j)$  with  $t \in [0, 1[$  and  $j \in \{0, 1, \dots, p-1\}$  such that  $(t+j) \in \frac{p}{q}\mathbb{Z}$ .)  $\square$

**Remark 18.10.** A homeomorphism from  $M$  to  $M_{(U; 1/k)}$  can also be directly described as follows. Let  $D$  be a disk bounded by  $U$ , and let  $d$  be a smaller disk inside  $D$ . The disk  $D$  is parametrized by the disk of radius 2 in  $\mathbb{C}$ , and the unit disk parametrizes  $d$ . The exterior  $E$  of  $U$  is homeomorphic to

$$\left( M \setminus (\mathring{D} \times ]0, 2\pi[) \right) \cup_{d \times \{0, 2\pi\}} (d \times [0, 2\pi])$$

by a homeomorphism mapping the meridian of  $U$  to

$$(\{1\} \times [0, 2\pi]) \cup ([1, 2] \times \{2\pi\}) \cup (-\{2\} \times [0, 2\pi]) \cup (-[1, 2] \times \{0\}).$$

See Figure 18.2.

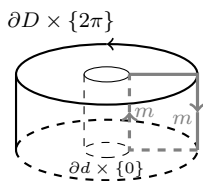


Figure 18.2: The gray image  $m$  of  $m(U)$  in  $D \times [0, 2\pi]$

The homeomorphism of  $E$  restricting to  $(M \setminus (\mathring{D} \times [0, 2\pi]))$  as the identity map and sending  $(z, \theta) \in d \times [0, 2\pi]$  to  $(z \exp(ik\theta), \theta)$  sends the above meridian  $m(U)$  to a curve homologous to  $m(U) + k\ell(U)$ . So this homeomorphism extends to provide a homeomorphism from  $M$  to  $M_{(U; \frac{1}{k})}$ .



As another example, we prove the following standard lemma.

**Lemma 18.11.** *Let  $m$  be a meridian of a knot  $K$  in a 3-manifold  $A$ . Equip  $m$  with its preferred longitude  $\ell(m)$ .<sup>2</sup> Equip  $K$  with a curve  $\mu$  parallel to  $K$ . Then the Dehn surgery on  $((K; \mu), (m; \ell(m)))$  does not change the 3-manifold  $A$ .*

PROOF: Before and after the surgery, the two involved tori can be glued together along an annulus, whose core is a meridian of one of the knots and a longitude of the other, to form a solid torus in which  $\ell(m)$  bounds a disk.  $\square$

We refer the reader to the book [Rol90, Chapter 9, G, H] by Dale Rolfsen for many other examples of surgeries.

### 18.3 Direct proof of a surgery formula for $\Theta$

In this section, we apply Theorem 18.6 to compute  $\Theta(R_{(K;p/q)}) - \Theta(R) + \Theta(L(p, q))$  for any null-homologous knot  $K$  in Proposition 18.12. In order to prove Proposition 18.12, we describe a special LP-surgery introduced in [Les09, Section 9]. We will also use this special LP-surgery to compute the degree 2 part of  $\check{Z}$  for a null-homologous knot in Theorem 18.43.

Let  $K$  be a null-homologous knot  $K$  in a 3-manifold  $M$ . A *Seifert surface* of  $K$  is a compact connected oriented surface  $\Sigma$  embedded in  $M$  such that the boundary  $\partial\Sigma$  of  $\Sigma$  is  $K$ . A *symplectic basis* for the  $H_1$  of such a Seifert surface is a basis  $(x_1, y_1, \dots, x_g, y_g)$ , as in Figure 18.3, where  $\langle x_i, y_i \rangle_\Sigma = 1$ , for  $i \in \underline{g}$ .

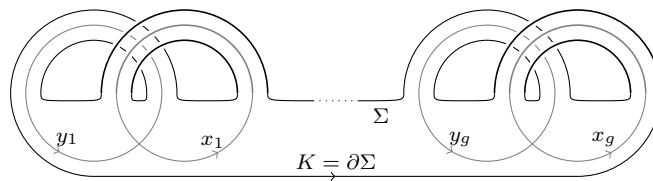


Figure 18.3: Symplectic basis of a Seifert surface

**Proposition 18.12.** *Let  $K$  be a null-homologous knot in a rational homology sphere  $R$ . Let  $\Sigma$  be a Seifert surface of  $K$  in  $R$ , and let  $(x_1, y_1, \dots, x_g, y_g)$  be*

<sup>2</sup>It is equivalent to equip  $m$  with the coefficient 0 since  $m$  lies in a ball.

a symplectic basis of  $\Sigma$ . For a curve  $c$  of  $\Sigma$ , let  $c^+$  denote its push-off in the direction of the positive normal to  $\Sigma$ . Set

$$a_2(\Sigma) = \sum_{(i,j) \in \underline{g}^2} (lk(x_i, x_j^+)lk(y_i, y_j^+) - lk(x_i, y_j^+)lk(y_i, x_j^+)).$$

Then we have

$$\Theta(R_{(K;p/q)}) = \Theta(R) - \Theta(L(p, q)) + 6\frac{q}{p}a_2(\Sigma).$$

We will prove this proposition in Subsection 18.3.2 in these words. Thus, this proposition implies that  $a_2(\Sigma)$  is an invariant of  $K$ . This invariant will be denoted by  $a_2(K)$ . It is equal to  $\frac{1}{2}\Delta_K''(1)$ , where  $\Delta_K$  is the Alexander polynomial of  $K$  normalized so that  $\Delta_K(t) = \Delta_K(t^{-1})$  and  $\Delta_K(1) = 1$ .

**Definition 18.13.** Here is a possible quick definition of the *Alexander polynomial*  $\Delta_K$  of the null-homologous knot  $K$ . Rewrite the symplectic basis  $(x_i, y_i)_{i \in \underline{g}}$  as the basis  $(z_j)_{j \in \underline{2g}}$  such that  $z_{2i-1} = x_i$  and  $z_{2i} = y_i$  for  $i \in \underline{g}$ . Let  $V = [lk(z_j, z_k^+)]_{(j,k) \in \underline{2g}^2}$  denote the associated Seifert matrix, and let  ${}^tV$  denote its transpose, then we have

$$\Delta_K(t) = \det(t^{1/2}V - t^{-1/2}{}^tV).$$

See [Ale28] or [Les96, Chapter 2] for other definitions of the Alexander polynomial, which will be mentioned later but not used in this book anymore.

**Remark 18.14.** Proposition 18.12 is also a consequence of the identification of  $\Theta$  with  $6\lambda_{CW}$  in Theorem 18.31, which is proved independently in Section 18.6, and of the Walker surgery formula proved in [Wal92, Theorem 5.1].

### 18.3.1 A Lagrangian-preserving surgery associated to a Seifert surface

**Definition 18.15.** Let  $c(S^1)$  be a curve embedded in the interior of an oriented surface  $F$ , and let  $c(S^1) \times [-1, 1]$  be a collar neighborhood of  $c(S^1)$  in  $F$ . A *right-handed* (resp. *left-handed*) *Dehn twist* about the curve  $c(S^1)$  is a homeomorphism of  $F$  that coincides with the identity map of  $F$  outside  $c(S^1) \times [-1, 1]$  and that maps

$$(c(z), t) \in c(S^1) \times [-1, 1] \text{ to } \left( c\left( z \exp(if(t)) \right), t \right)$$

for  $f(t) = \pi(t + 1)$  (resp. for  $f(t) = -\pi(t + 1)$ ).

Let  $\Sigma$  be a Seifert surface of a knot  $K$  in a manifold  $M$ . Consider an annular neighborhood  $[-3, 0] \times K$  of  $(\{0\} \times K =) K = \partial\Sigma$  in  $\Sigma$ , a small disk  $D$  inside  $] -2, -1[ \times K$ , and a small open disk  $d$  whose closure is in the interior of  $D$ . Let  $F = \Sigma \setminus d$ . Let  $h_F$  be the composition of the two left-handed Dehn twists on  $F$  about  $c = \partial D$  and  $K_2 = \{-2\} \times K$  with the right-handed one about  $K_1 = \{-1\} \times K$ . See Figure 18.4.

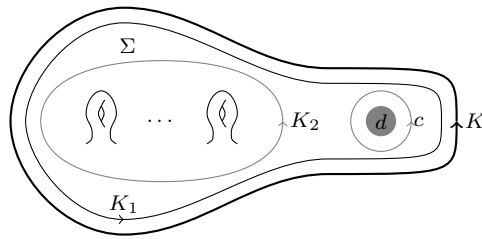


Figure 18.4:  $K, \Sigma, F, c, K_1,$  and  $K_2$

View  $F$  as  $F \times \{0\}$  in the boundary of a handlebody  $A_F = F \times [-1, 0]$  of  $M$ . Extend  $h_F$  to a homeomorphism  $h_A$  of  $\partial A_F$  that is the identity map outside  $F \times \{0\}$ . Let  $A'_F$  be a copy of  $A_F$ . Identify  $\partial A'_F$  with  $\partial A_F$  with

$$h_A: \partial A'_F \rightarrow \partial A_F.$$

Define the *surgery associated to  $\Sigma$*  to be the surgery  $(A'_F/A_F)$  associated to  $(A_F, A'_F; h_A)$ . If  $j$  denotes the embedding from  $\partial A_F$  to  $M$ . This surgery replaces

$$M = A_F \cup_j (M \setminus \mathring{A}_F)$$

by

$$M_F = A'_F \cup_{j \circ h_A} (M \setminus \mathring{A}_F).$$

**Proposition 18.16.** *With the above notation, the surgery  $(A'_F/A_F)$  associated to  $\Sigma$  is a Lagrangian-preserving surgery with the following properties. There is a homeomorphism from  $M_F$  to  $M$ ,*

- which extends the identity map of

$$M \setminus ([-3, 0] \times K \times [-1, 0]),$$

- which transforms a curve passing through  $d \times [-1, 0]$  by a band sum with  $K$ ,

- which transforms a 0-framed meridian  $m$  of  $K$  passing through  $d \times [-1, 0]$ , viewed as a curve of  $M \setminus A_F$  (which may be expressed as  $h_A^{-1}(m)$  in  $A_F$ ), to a 0-framed copy of  $K$  isotopic to the framed curve  $h_A^{-1}(m)$  of Figure 18.5 (with the framing induced by  $\partial A_F$ ).

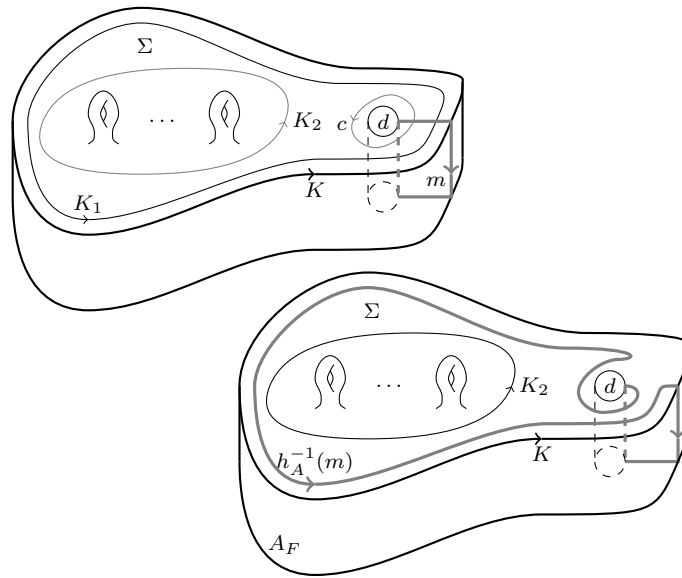


Figure 18.5:  $m$  and  $h_A^{-1}(m)$

PROOF: Observe that  $h_A|_{(F \times \{-1, 0\}) \cup (\partial \Sigma \times [-1, 0])}$  extends to  $\Sigma \times [-1, 0]$  as

$$\begin{aligned}
 h: \quad \Sigma \times [-1, 0] &\rightarrow \Sigma \times [-1, 0] \\
 (\sigma, t) &\mapsto h(\sigma, t) = (h_t(\sigma), t),
 \end{aligned}$$

where  $h_{-1}$  is the identity map of  $\Sigma$ ,  $h_0$  is the extension of  $h_F$  by the identity map on  $d$ ,<sup>3</sup>  $h_t$  coincides with the identity map outside  $[-5/2, -1/2] \times K(S^1)$ , and  $h_t$  is defined as follows on  $[-5/2, -1/2] \times K(S^1)$ .

- When  $t \leq -1/2$ ,  $h_t$  coincides with the identity map  $h_{-1}$  outside the disk  $D$ , whose elements are written as  $D(z \in \mathbb{C})$ , with  $|z| \leq 1$ . The elements of  $d$  are the  $D(z)$  for  $|z| < 1/2$ . On  $D$ ,  $h_t$  describes the isotopy between the identity map and the left-handed Dehn twist about  $\partial D$  located on  $\{D(z) : 1/2 \leq |z| \leq 1\}$ . We have

$$\begin{aligned}
 h_t(z \in D) &= z \exp(i\pi(2t + 2)4(|z| - 1)) && \text{if } |z| \geq 1/2 \\
 h_t(z \in D) &= z \exp(-2i\pi(2t + 2)) && \text{if } |z| \leq 1/2.
 \end{aligned}$$

<sup>3</sup>This extension is isotopic to the identity map of  $\Sigma$ .

- When  $t \geq -1/2$ ,  $h_t$  describes the following isotopy between  $h_{-1/2}$  and the composition  $h_0$  of  $h_{-1/2}$  with the left-handed Dehn twist about  $K_2$  and the right-handed Dehn twist about  $K_1$ , where the first twist is supported on  $[-5/2, -2] \times K(S^1)$  and the second one is supported on  $[-1, -1/2] \times K(S^1)$ :

$$\begin{aligned}
 h_t(u, K(z)) &= \left( u, K \left( z \exp \left( i(2t + 1)(4\pi(u + 5/2)) \right) \right) \right) && \text{if } -5/2 \leq u \leq -2, \\
 h_t(u, K(z)) &= h_{-1/2} \left( u, K \left( z \exp \left( i(2t + 1)(2\pi) \right) \right) \right) && \text{if } -2 \leq u \leq -1, \\
 h_t(u, K(z)) &= \left( u, K \left( z \exp \left( i(2t + 1)(4\pi(-u - 1/2)) \right) \right) \right) && \text{if } -1 \leq u \leq -1/2.
 \end{aligned}$$

Now,  $M_F$  is naturally homeomorphic to

$$\left( A'_F \cup_{h|_{\partial A'_F \setminus (\partial d \times [-1, 0])}} \left( M \setminus \text{Int}(\Sigma \times [-1, 0]) \right) \right) \cup_{\partial(d \times [-1, 0])} (d \times [-1, 0]),$$

and hence to

$$(\Sigma \times [-1, 0]) \cup_{h|_{\partial(\Sigma \times [-1, 0])}} \left( M \setminus \text{Int}(\Sigma \times [-1, 0]) \right),$$

which is mapped homeomorphically to  $M$  by the identity map outside  $\Sigma \times [-1, 0]$ , and by  $h$  on  $\Sigma \times [-1, 0]$ . Therefore, we indeed have a homeomorphism from  $M_F$  to  $M$ . This homeomorphism is the identity map outside  $[-3, 0] \times K \times [-1, 0]$ . It maps  $d \times [-1, 0]$  to a cylinder running along  $K$  after being twisted negatively.

View the meridian  $m$  as a curve of  $M \setminus \mathring{A}_F$  with its framing induced by the boundary of  $A_F$ . Thicken  $m$  as a band  $[0, 1] \times m$  in  $\partial A_F$ . Assume that a part of this band lies in the vertical boundary  $\partial d \times [-1, 0]$  of  $d \times [-1, 0]$  and can be written as a rectangle  $[\theta, \theta'] \times [-1, 0] \subset \partial d \times [-1, 0]$ . Then  $h$  sends  $[\theta, \theta'] \times [-1, -1/2]$  to some other rectangle in  $\partial d \times [-1, -1/2]$ . The image under  $h$  of  $[\theta, \theta'] \times [-1/2, 0]$  together with a small additional piece of the thickened meridian can be isotoped in a tubular neighborhood of  $K$  to a band on  $\partial A_F$ , which is first vertical in  $\partial d \times [-1/2, 0]$ , and which then runs along  $K$ . So  $h$  sends the framed meridian  $m$  to a curve isotopic to  $h_A^{-1}(m)$  in a tubular neighborhood of  $K$  with the framing induced by the boundary of  $A_F$ . See Figure 18.5.

Now,  $H_1(\partial A_F)$  is generated by the generators of  $H_1(\Sigma) \times \{0\}$ , the generators of  $H_1(\Sigma) \times \{-1\}$ , and the homology classes of  $c = \partial D$  and  $m$ . Among

them, the only generator that could be affected by  $h_A$  is the class of  $m$ , which is not. Thus  $h_A$  acts trivially on  $H_1(\partial A_F)$ , and the defined surgery is an LP-surgery.  $\square$

Let  $\Sigma \times [-1, 2]$  be an extension of the previous neighborhood of  $\Sigma$ . Set  $B_F = F \times [1, 2]$ . Define the homeomorphism  $h_B$  of  $\partial B_F$  to be the identity map anywhere except on  $F \times \{1\}$ , where it coincides with the homeomorphism  $h_F$  of  $F$ , with the obvious identification.

Let  $B'_F$  be a copy of  $B_F$ . Identify  $\partial B'_F$  with  $\partial B_F$  with

$$h_B: \partial B'_F \rightarrow \partial B_F.$$

Define the *inverse surgery associated to  $\Sigma$*  to be the surgery associated to  $(B_F, B'_F)$  (or  $(B_F, B'_F; h_B)$ ). We can apply the previous study to this surgery by using the central symmetry of  $[-1, 2]$ .

The following obvious lemma, which we will not prove, justifies the terminology.

**Lemma 18.17.** *With the above notation, performing both surgeries  $(B'_F/B_F)$  and  $(A'_F/A_F)$  affects neither  $M$  nor the curves in the complement of  $F \times [-1, 2]$  (up to isotopy). Performing  $(A'_F/A_F)$  (resp.  $(B'_F/B_F)$ ) changes a 0-framed meridian of  $K$  passing through  $d \times [-1, 2]$  into a 0-framed copy of  $K$  (resp.  $(-K)$ ).*

**Lemma 18.18.** *Let  $(x_i, y_i)_{i=1, \dots, g}$  be a symplectic basis of  $\Sigma$ . Then the tripod combination  $T(\mathcal{I}_{A_F A'_F})$  associated to the surgery  $(A'_F/A_F)$  is*

$$T(\mathcal{I}_{A_F A'_F}) = \sum_{i=1}^g \begin{matrix} x_i \\ \swarrow \\ y_i \\ \searrow \\ c = \partial D \end{matrix}.$$

For a curve  $\gamma$  of  $F$ , let  $\gamma^+$  denote  $\gamma \times \{1\}$ . The tripod combination  $T(\mathcal{I}_{B_F B'_F})$  associated to the surgery  $(B'_F/B_F)$  is

$$T(\mathcal{I}_{B_F B'_F}) = - \sum_{i=1}^g \begin{matrix} x_i^+ \\ \swarrow \\ y_i^+ \\ \searrow \\ c^+ \end{matrix}.$$

PROOF: For a curve  $\gamma$  of  $F$ ,  $\gamma^-$  denotes  $\gamma \times \{-1\}$ . In order to compute the intersection form of  $(A_F \cup -A'_F)$ , use the basis  $(m, (x_i - x_i^-)_{i \in \underline{g}}, (y_i - y_i^-)_{i \in \underline{g}})$  of the Lagrangian of  $A_F$ . The system  $(c, (y_i)_{i \in \underline{g}}, (-x_i)_{i \in \underline{g}})$  is dual to this basis. The only curve of the Lagrangian basis modified by  $h_A$  is  $m$ , and  $h_A^{-1}(m)$

may be expressed as a path composition  $mc^{-1}K_2$ . Let  $D_m$  be a disk of  $A_F$  bounded by  $m$ . The isomorphism  $\partial_{MV}^{-1}$  from  $\mathcal{L}_{A_F}$  to  $H_2(A_F \cup -A'_F)$  satisfies:

$$\begin{aligned} \partial_{MV}^{-1}(x_i - x_i^-) &= S(x_i) = -(x_i \times [-1, 0]) \cup (x_i \times [-1, 0] \subset A'_F), \\ \partial_{MV}^{-1}(y_i - y_i^-) &= S(y_i) = -(y_i \times [-1, 0]) \cup (y_i \times [-1, 0] \subset A'_F), \\ \partial_{MV}^{-1}(m) &= S_A(m) = D_m - \left( (\Sigma \setminus ([-2, 0] \times K)) \cup (D_m \subset A'_F) \right), \end{aligned}$$

where the given expression of  $\partial_{MV}^{-1}(m)$  must be completed in

$$\partial A_F \cap ([-2, 0] \times K \times [-1, 0])$$

so that the boundary of  $\partial_{MV}^{-1}(m)$  actually vanishes, as it does algebraically.

Since  $x_i$  intersects only  $y_i$  among the curves  $x_j$  and  $y_j$  for  $j \in \underline{g}$ , the surface  $S(x_i)$  intersects only  $S(y_i)$  and  $S_A(m)$  in our basis of  $H_2(A_F \cup -A'_F)$ . The algebraic intersection of  $S(x_i)$ ,  $S(y_i)$ , and  $S_A(m)$  is  $-1$ .

For the surgery  $(B'_F/B_F)$ , use the reflection  $T: [-1, 2] \rightarrow [-1, 2]$  such that  $T(x) = 1 - x$ . The induced reflection  $T_F = 1_F \times T$  of  $F \times [-1, 2]$  maps  $A_F$  onto  $-B_F$ . Use the image by  $T_F$  of the above basis of  $\mathcal{L}_{A_F}$  for  $\mathcal{L}_{B_F}$ . The system

$$\left( -c^+, (-y_i)_{i \in \underline{g}}, (x_i^+)_{i \in \underline{g}} \right)$$

is dual to the obtained basis of  $\mathcal{L}_{B_F}$ . (Since  $T_F$  reverses the orientation, the intersection numbers on  $\partial A_F$  are multiplied by  $-1$ .) Use the images under  $T_F$  of the former surfaces. Their triple intersection numbers are the same since their positive normals and the ambient orientation are reversed.  $\square$

### 18.3.2 A direct proof of the Casson surgery formula

In this subsection, we prove Proposition 18.12, assuming Theorem 18.6, which will be proved independently.

Note the following easy, well-known lemma.

**Lemma 18.19.** *The variation of the linking number of two knots  $J$  and  $K$  in a rational homology 3-sphere  $R$  after a  $p/q$ -surgery on a knot  $V$  disjoint from  $J \sqcup K$  in  $R$  is given by the following formula.*

$$lk_{R(V;p/q)}(J, K) = lk_R(J, K) - \frac{q}{p} lk_R(V, J) lk_R(V, K).$$

**PROOF:** The  $p/q$ -surgery on  $V$  is the surgery with respect to a curve  $\mu_V \subset \partial N(V)$ . Set  $q_V = \langle m(V), \mu_V \rangle_{\partial N(V)}$  and  $p_V = lk(V, \mu_V)$ . We have  $\frac{p}{q} = \frac{p_V}{q_V}$ . In  $H_1(R \setminus (V \cup K); \mathbb{Q})$ , we have

$$J = lk_R(J, K)m(K) + lk_R(V, J)m(V) \text{ and } \mu_V = p_V m(V) + q_V lk_R(V, K)m(K).$$

Since  $\mu_V$  vanishes in  $H_1(R_{(V;p/q)} \setminus K; \mathbb{Q})$ , we get

$$J = lk_R(J, K)m(K) - \frac{q}{p}lk_R(V, K)lk_R(V, J)m(K)$$

in  $H_1(R_{(V;p/q)} \setminus K; \mathbb{Q})$ . □

**PROOF OF PROPOSITION 18.12 ASSUMING THEOREM 18.6:** Recall that  $K$  bounds a Seifert surface  $\Sigma$  in a rational homology sphere  $R$ . Let  $\Sigma \times [-1, 2]$  be a collar of  $\Sigma$  in  $R$ , and let  $(A'/A) = (A'_F/A_F)$  and  $(B'/B) = (B'_F/B_F)$  be the LP-surgeries of Subsection 18.3.1. Let  $U$  be a meridian of  $K$  passing through  $d \times [-1, 2]$ , such that performing one of the two surgeries transforms  $U$  into  $\pm K$  and performing both or none of them leaves  $U$  unchanged. We have

$$\begin{aligned} \mathcal{Z}_1\left([R_{(U;p/q)}, \emptyset; A'/A, B'/B\right] &= 2\mathcal{Z}_1(R_{(U;p/q)}) - 2\mathcal{Z}_1(R_{(K;p/q)}) \\ &= \left[\langle\langle T(\mathcal{I}_{AA'}) \sqcup T(\mathcal{I}_{BB'}) \rangle\rangle_{R_{(U;p/q)}}\right]. \end{aligned}$$

According to Lemma 18.18, the tripods associated to the surgery  $(A, A')$  and to the surgery  $(B, B')$ , are

$$\sum_{i=1}^g \left\langle \begin{matrix} x_i \\ y_i \\ c \end{matrix} \right\rangle \quad \text{and} \quad \sum_{j=1}^g \left\langle \begin{matrix} x_j^+ \\ y_j^+ \\ c^+ \end{matrix} \right\rangle,$$

respectively. Among the curves of the tripods in the right-hand side, the only curve linking  $c$  algebraically in  $R_{(U;p/q)_{i \in N}}$  is  $c^+$  with a linking number  $-q/p$ . Therefore, the vertices labeled by  $c$  and  $c^+$  must be paired together with coefficient  $-q/p$ . We get

$$\left\langle \left\langle \left\langle \begin{matrix} x_i & x_j^+ \\ y_i & y_j^+ \\ c & c^+ \end{matrix} \right\rangle \right\rangle \right\rangle = -\frac{q}{p} (lk(x_i, x_j^+)lk(y_i, y_j^+) - lk(x_i, y_j^+)lk(y_i, x_j^+)) [\Theta].$$

So we have

$$\left[\langle\langle T(\mathcal{I}_{AA'}) \sqcup T(\mathcal{I}_{BB'}) \rangle\rangle_{R_{(U;p/q)}}\right] = -\frac{q}{p}a_2(\Sigma) [\Theta]$$

and

$$\mathcal{Z}_1(R_{(K;p/q)}) = \mathcal{Z}_1(R_{(U;p/q)}) + \frac{q}{2p}a_2(\Sigma) [\Theta].$$

Corollary 10.11 implies  $\mathcal{Z}_1(R) = \frac{1}{12}\Theta(R) [\Theta]$ , where  $[\Theta] \neq 0$  in  $\mathcal{A}(\emptyset)$ . Also recall that  $\Theta$  is additive under connected sum according to Corollary 10.27, and that Proposition 5.15 implies  $\Theta(L(p, -q)) = -\Theta(L(p, q))$ . The result follows, thanks to Lemma 18.9. □



### 18.4 Finite type invariants of $\mathbb{Z}$ -spheres

In this section, we state the fundamental theorem of finite type invariants for integer homology 3-spheres due to Thang Lê [Lê97], and we show how we may use Theorem 18.6 in its proof. This shows in what sense Theorem 18.6 implies that  $\mathcal{Z}$  restricts to a *universal* finite type invariant of integer homology 3-spheres. In order to do this, we first follow Mikhail Goussarov [GGP01] and Kazuo Habiro [Hab00] and construct surjective maps from  $\mathcal{A}_n(\emptyset)$  to  $\mathcal{F}_{2n}(\mathcal{M})/\mathcal{F}_{2n+1}(\mathcal{M})$ .

**Mapping  $\mathcal{A}_n(\emptyset)$  to  $\mathcal{F}_{2n}(\mathcal{M})/\mathcal{F}_{2n+1}(\mathcal{M})$ .** Let  $\Gamma$  be a degree  $n$  trivalent Jacobi diagram whose vertices are numbered in  $\underline{2n}$ . Let  $\Sigma(\Gamma)$  be an oriented surface containing  $\Gamma$  in its interior such that  $\Sigma(\Gamma)$  is a regular neighborhood of  $\Gamma$  in  $\Sigma(\Gamma)$ . Equip  $\Gamma$  with its vertex-orientation induced by the orientation of  $\Sigma(\Gamma)$ . Embed  $\Sigma(\Gamma)$  in a ball inside  $\mathbb{R}^3$ . Replace neighborhoods



Thus,  $\Sigma(\Gamma)$  is transformed into a collection of disjoint oriented surfaces

$$\Sigma(Y) = \text{graph with three leaves},$$

one for each trivalent vertex. The graph equipped with its framing induced by  $\Sigma(Y)$  is called a *Y-graph*. Its looped edges are called *leaves*. Thickening the  $\Sigma(Y)$  transforms each of them into a standard genus 3 handlebody  $H$ . The handlebody  $H$  has three handles with meridians  $m_j$  and longitudes  $\ell_j$ , such that  $\langle m_i, \ell_j \rangle_{\partial H} = \delta_{ij}$ . Its longitudes  $\ell_j$  are on  $\Sigma(Y)$  as in Figure 18.6.

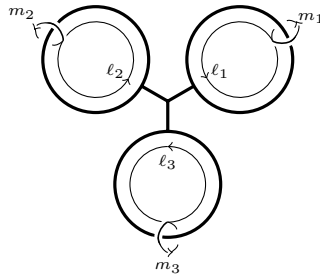


Figure 18.6: Meridians and longitudes of the genus 3 handlebody  $H$

The Matveev *Borromean surgery* on  $H$  is the Dehn surgery on the six-component link  $L_6$  inside  $H$  with respect to the parallels of its components that are parallel in Figure 18.7. Sergei Matveev studied it in [Mat87].

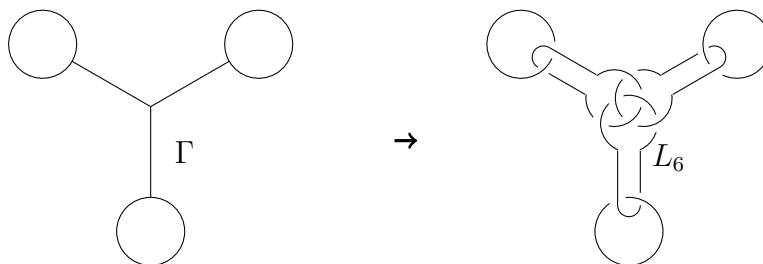


Figure 18.7: Y-graph and associated LP-surgery

**Lemma 18.20.** *The Matveev Borromean surgery changes the handlebody  $H$  to an integer homology handlebody  $H'$  with the same boundary and Lagrangian as  $H$ . The manifold  $H \cup_{\partial} (-H')$  is diffeomorphic to  $(S^1)^3$ , and there exists  $\varepsilon_S = \pm 1$  such that*

$$T(\mathcal{I}_{HH'}) = \varepsilon_S \begin{matrix} \ell_3 \\ \ell_2 \\ \ell_1 \end{matrix}.$$

(The sign  $\varepsilon_S$  is well-determined by the data. We do not need its explicit value.)

**PROOF:** First observe that the Lagrangian of  $H'$  is the same as the Lagrangian of  $H$ . As in Example 18.8,  $H \cup_{\partial} (-H)$  is obtained from  $S^3$  by surgery on three 0-framed meridians of three handles of  $H$ , where  $H$  is embedded in  $S^3$  in a standard way. So  $H' \cup_{\partial} (-H)$  is obtained by surgery on the zero-framed nine-component link obtained from the six-component link  $L_6$  of Figure 18.7 by adding a meridian for each outermost component of  $L_6$ . Lemma 18.11 implies that  $H' \cup_{\partial} (-H)$  is obtained by surgery on the zero-framed Borromean link. Therefore, according to Example 18.8,  $H' \cup_{\partial} (-H)$  is diffeomorphic to  $(S^1)^3$ . So is  $H \cup_{\partial} (-H')$ . Easy homological computations imply that  $H'$  is an integer homology handlebody.  $\square$

Denote the Borromean surgeries associated to the Y-graphs corresponding to the vertices of  $\Gamma$  by  $(A^{(i)'}/A^{(i)})$ . With the notation of Section 6.1, define  $\psi_n(\Gamma)$  to be the class of

$$[S^3; (A^{(i)'}/A^{(i)})_{i \in \underline{2n}}] = \sum_{I \subseteq \underline{2n}} (-1)^{|I|} S^3((A^{(i)'}/A^{(i)})_{i \in I})$$

in  $\mathcal{F}_{2n}(\mathcal{M})/\mathcal{F}_{2n+1}(\mathcal{M})$ . The coefficient field  $\mathbb{K}$  of Section 6.1 for  $\mathcal{F}_{2n}(\mathcal{M})$  is  $\mathbb{R}$ , from now on.<sup>4</sup>

In [GGP01, Theorem 4.13, Section 4], Stavros Garoufalidis, Mikhail Goussarov, and Michael Polyak proved the following theorem.

**Theorem 18.21** (Garoufalidis, Goussarov, Polyak). *Let  $n \in \mathbb{N}$ . For a degree  $n$  trivalent Jacobi diagram  $\Gamma$ , the element  $\psi_n(\Gamma)$  of  $\mathcal{F}_{2n}(\mathcal{M})/\mathcal{F}_{2n+1}(\mathcal{M})$  constructed above depends only on the class of  $\Gamma$  in  $\mathcal{A}_n(\emptyset)$ , and the map*

$$\psi_n: \mathcal{A}_n(\emptyset) \rightarrow \frac{\mathcal{F}_{2n}(\mathcal{M})}{\mathcal{F}_{2n+1}(\mathcal{M})}$$

is surjective. Furthermore, we have  $\frac{\mathcal{F}_{2n+1}(\mathcal{M})}{\mathcal{F}_{2n+2}(\mathcal{M})} = \{0\}$ .

Assuming the above theorem, the following *Lê fundamental theorem on finite type invariants of  $\mathbb{Z}$ -spheres* becomes a corollary of Theorem 18.6.

**Theorem 18.22** (Lê). *There exists a family  $(Y_n: \mathcal{F}_0(\mathcal{M}) \rightarrow \mathcal{A}_n(\emptyset))_{n \in \mathbb{N}}$  of linear maps such that*

- $Y_n(\mathcal{F}_{2n+1}(\mathcal{M})) = 0$ ,
- the restriction  $\bar{Y}_n$  to  $\mathcal{F}_{2n}(\mathcal{M})/\mathcal{F}_{2n+1}(\mathcal{M})$  of the morphism induced by  $Y_n$  on  $\mathcal{F}_0(\mathcal{M})/\mathcal{F}_{2n+1}(\mathcal{M})$  to  $\mathcal{A}_n(\emptyset)$  is a left inverse of  $\psi_n$ .

In particular, for any  $n \in \mathbb{N}$ , we have

$$\frac{\mathcal{F}_{2n}(\mathcal{M})}{\mathcal{F}_{2n+1}(\mathcal{M})} \cong \mathcal{A}_n(\emptyset) \text{ and } \frac{\mathcal{I}_{2n}(\mathcal{M})}{\mathcal{I}_{2n-1}(\mathcal{M})} \cong \mathcal{A}_n^*(\emptyset).$$

An invariant  $Y$  satisfying the properties in the statement of Theorem 18.22 above is called a *universal finite type invariant of  $\mathbb{Z}$ -spheres*. In order to prove Theorem 18.22, Thang Lê proved that the Lê–Murakami–Ohtsuki invariant  $Z^{LMO} = (Z_n^{LMO})_{n \in \mathbb{N}}$  of [LMO98] is a universal finite type invariant of  $\mathbb{Z}$ -spheres in [L97].

As a corollary of Theorem 18.6, we get the following Kuperberg–Thurston theorem [KT99].

**Theorem 18.23** (Kuperberg, Thurston). *The restriction of  $\mathcal{Z}$  to  $\mathbb{Z}$ -spheres is a universal finite type invariant of  $\mathbb{Z}$ -spheres.*

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<sup>4</sup>For the statements involving only invariants  $\mathcal{Z}$  valued in spaces of Jacobi diagrams with rational coefficients (when no interval components are involved), we can fix the coefficient field to be  $\mathbb{Q}$ , provided that we also restrict the coefficient field of our related spaces of Jacobi diagrams to be  $\mathbb{Q}$ .

PROOF: Theorem 18.6 ensures  $\mathcal{Z}_n(\mathcal{F}_{2n+1}(\mathcal{M})) = 0$ . This reduces the proof of Theorem 18.23 to the proof of the following lemma.  $\square$

**Lemma 18.24.** *For any trivalent Jacobi diagram  $\Gamma$ , we have  $\mathcal{Z}_n \circ \psi_n([\Gamma]) = [\Gamma]$ .*

PROOF: Let us show how this lemma follows from Theorem 18.6. Number the vertices of  $\Gamma$  in  $\underline{2n}$ . Call  $(A^{(i)'}/A^{(i)})$  the Borromean surgery associated to the vertex  $i$ . The associated tripod is

$$T(\mathcal{I}_{A^{(i)'}/A^{(i)}}) = \varepsilon_S \begin{array}{c} \ell_3^{(i)} \\ \swarrow \\ \ell_2^{(i)} \\ \searrow \\ \ell_1^{(i)} \end{array}$$

for the fixed  $\varepsilon_S = \pm 1$  of Lemma 18.20. Embed the tripods  $T(\mathcal{I}_{A^{(i)'}/A^{(i)}})$  into the graph  $\Gamma$ , naturally, so that the half-edge of  $\ell_k^{(i)}$  is on the half-edge that gave rise to the leaf of  $\ell_k^{(i)}$  in the  $Y$ -graph associated to  $i$ . In order to contribute to

$$\left\langle \left\langle \bigsqcup_{i \in \underline{2n}} T(\mathcal{I}_{A^{(i)'}/A^{(i)}}) \right\rangle \right\rangle_n,$$

a partition must pair a half-edge associated to a leaf of some  $\ell_k^{(i)}$  with the half-edge of the only leaf that links  $\ell_k^{(i)}$ , which is the other half-edge of the same edge. We get

$$\left[ \left\langle \left\langle \bigsqcup_{i \in \underline{2n}} T(\mathcal{I}_{A^{(i)'}/A^{(i)}}) \right\rangle \right\rangle_n \right] = [\Gamma].$$

$\square$

**Remark 18.25.** In the original work of Thang Lê [Lê97] and in the article [GGP01], the primary filtration used for the space of  $\mathbb{Z}$ -spheres is defined from Borromean surgeries rather than from integral LP-surgeries. In [AL05], Emmanuel Auclair and I proved that the two filtrations coincide. We also proved that a universal finite type invariant of  $\mathbb{Z}$ -spheres automatically satisfies the more general formula of Theorem 18.6 for any  $X = [\tilde{R}; (A^{(i)'}/A^{(i)})_{i \in \underline{x}}]$  such that  $R$  is a  $\mathbb{Z}$ -sphere and the  $(A^{(i)'}/A^{(i)})$  are integral LP-surgeries in  $\tilde{R}$ . In [GGP01], Stavros Garoufalidis, Mikhail Goussarov, and Michael Polyak compare other filtrations of the space of  $\mathbb{Z}$ -spheres, including the original filtration of Tomotada Ohtsuki using surgeries on algebraically split links. This original Ohtsuki filtration defined in his introduction of finite invariants of  $\mathbb{Z}$ -spheres [Oht96] gives rise to the same notion of real-valued finite-type invariants.

## 18.5 Finite type invariants of $\mathbb{Q}$ -spheres

For a  $\mathbb{Q}$ -sphere  $R$ , the cardinality of  $H_1(R; \mathbb{Z})$  is the product over the prime numbers  $p$  of  $p^{\nu_p(R)}$ , where  $\nu_p(R)$  is called the  $p$ -valuation of the order of  $H_1(R; \mathbb{Z})$ . In [Mou12, Proposition 1.9], Delphine Moussard proved that  $\nu_p$  is a degree 1 invariant of  $\mathbb{Q}$ -spheres with respect to  $\mathcal{O}_{\mathbb{Z}}^{\mathbb{Q}}$ , which is defined in Section 6.1. She also proved [Mou12, Corollary 1.10] that the degree 1 invariants of  $\mathbb{Q}$ -spheres with respect to  $\mathcal{O}_{\mathbb{Z}}^{\mathbb{Q}}$  are (possibly infinite) linear combinations of the invariants  $\nu_p$  and of a constant map.

Define an *augmented trivalent Jacobi diagram* to be the disjoint union of a trivalent Jacobi diagram and a finite number of isolated 0-valent vertices equipped with prime numbers. The *degree* of such a diagram is half the number of its vertices. It is a half-integer. For a half-integer  $h$ , let  $\mathcal{A}_h^{\text{aug}}$  denote the quotient of the  $\mathbb{Q}$ -vector space generated by degree  $h$  augmented trivalent Jacobi diagrams, by the Jacobi relation and the antisymmetry relation. The product induced by the disjoint union turns  $\mathcal{A}^{\text{aug}} = \prod_{h \in \frac{1}{2}\mathbb{N}} \mathcal{A}_h^{\text{aug}}$  to a graded algebra. In [Mou12], Delphine Moussard proved that

$$\frac{\mathcal{F}_n(\mathcal{M}_{\mathbb{Q}})}{\mathcal{F}_{n+1}(\mathcal{M}_{\mathbb{Q}})} \cong \mathcal{A}_{n/2}^{\text{aug}}$$

for any integer  $n$ . Her proof used the configuration space integral  $Z_{KKT}$ , described in [KT99] and [Les04a], and the splitting formulae of [Les04b] stated in Theorem 18.6. See [Mou12, Theorem 1.7]. The invariant  $Z_{KKT}$  is the restriction to  $\mathbb{Q}$ -spheres of the invariant  $\mathcal{Z}$  described in this book.

The maps  $\psi_n$  of Section 18.4 can be generalized to canonical maps

$$\psi_h: \mathcal{A}_h^{\text{aug}} \rightarrow \frac{\mathcal{F}_{2h}(\mathcal{M}_{\mathbb{Q}})}{\mathcal{F}_{2h+1}(\mathcal{M}_{\mathbb{Q}})}$$

as follows. For any prime number  $p$ , let  $B_p$  be a rational homology ball such that  $|H_1(B_p; \mathbb{Z})| = p$ . Let  $\Gamma^a$  be the disjoint union of a degree  $k$  trivalent Jacobi diagram  $\Gamma$  and  $r$  isolated 0-valent vertices  $v_j$  equipped with prime numbers  $p_j$  for  $j \in \underline{r}$ . Embed  $\Gamma^a$  in  $\mathbb{R}^3$ . Thicken it, replace  $\Gamma$  by  $2k$  genus 3 handlebodies  $A^{(i)}$  associated to the vertices of  $\Gamma$  as in Section 18.4, and replace each vertex  $v_j$  by a small ball  $B(v_j)$  around it so that the  $B(v_j)$  and the  $A^{(i)}$  form a family of  $2k + r$  disjoint rational homology handlebodies. Define  $\psi_{k+r/2}(\Gamma^a)$  to be the class of

$$\left[ S^3; (A^{(i)'}/A^{(i)})_{i \in \underline{2k}}, (B_{p_j}/B(v_j))_{j \in \underline{r}} \right] \text{ in } \frac{\mathcal{F}_{2k+r}(\mathcal{M}_{\mathbb{Q}})}{\mathcal{F}_{2k+r+1}(\mathcal{M}_{\mathbb{Q}})},$$

where the  $(A^{(i)'}/A^{(i)})$  are the Borromean surgeries associated to the  $Y$ -graphs corresponding to the vertices of  $\Gamma$  as in Section 18.4.

**Lemma 18.26.** *The map  $\psi_h$  is well-defined.*

PROOF: According to [Mou12, Lemma 6.11], if  $B'_{p_j}$  is a  $\mathbb{Q}$ -ball whose  $H_1(\cdot; \mathbb{Z})$  has the same cardinality as  $H_1(B_{p_j}; \mathbb{Z})$ , then

$$(S^3(B'_{p_j}/B^3) - S^3(B_{p_j}/B^3))$$

belongs to  $\mathcal{F}_2(\mathcal{M}_{\mathbb{Q}})$ . This guarantees that  $\psi_{r/2+k}(\Gamma^a)$  does not depend on the chosen balls  $B_{p_j}$ . Thus, Theorem 18.21 implies that  $\psi_h$  is well-defined.  $\square$

This map  $\psi_h$  is canonical. The generators of  $\mathcal{F}_{2h}(\mathcal{M}_{\mathbb{Q}})/\mathcal{F}_{2h+1}(\mathcal{M}_{\mathbb{Q}})$  exhibited in [Mou12, Section 6.2 and Proposition 6.9] are in the image of  $\psi_h$ . So  $\psi_h$  is surjective.

Let  $\mathcal{A}_h^{\text{aug,c}}$  denote the subspace of  $\mathcal{A}_h^{\text{aug}}$  generated by connected degree  $h$  diagrams. So, if  $\mathcal{A}_h^{\text{aug,c}} \neq 0$ , then  $h \in \mathbb{N}$  or  $h = 1/2$ . Set  $\mathcal{A}^{\text{aug,c}} = \prod_{h \in \frac{1}{2}\mathbb{N}} \mathcal{A}_h^{\text{aug,c}}$ . Let  $z^{\text{aug}}$  denote the  $\mathcal{A}^{\text{aug,c}}$ -valued invariant  $z^{\text{aug}}$  of  $\mathbb{Q}$ -spheres such that, for any  $\mathbb{Q}$ -sphere  $R$ , we have

- $z_0^{\text{aug}}(R) = 0$ ,
- $z_{1/2}^{\text{aug}}(R) = \sum_{p \text{ prime}} \nu_p(R) \bullet_p$ , and
- $z_n^{\text{aug}}(R)$  is the natural projection  $z_n^{\text{aug}}(R, \emptyset) = p^c(\mathcal{Z}_n(R, \emptyset))$  of  $\mathcal{Z}_n(R, \emptyset)$  to the subspace  $\mathcal{A}_n^c(\emptyset)$  of  $\mathcal{A}_n(\emptyset)$  generated by connected diagrams. (Recall from Notation 7.16 that the projection  $p^c$  maps disconnected diagrams to 0.)

Define an  $\mathcal{A}^{\text{aug}}$ -valued invariant  $Z^{\text{aug}} = (Z_n^{\text{aug}})_{n \in \frac{1}{2}\mathbb{N}}$  to be  $Z^{\text{aug}} = \exp(z^{\text{aug}})$  for the  $\mathcal{A}^{\text{aug,c}}$ -valued invariant  $z^{\text{aug}}$  (meaning  $Z^{\text{aug}}(R) = \exp(z^{\text{aug}}(R))$  for any  $\mathbb{Q}$ -sphere  $R$ ).

As noticed by Gwénaél Massuyeau, the Moussard fundamental theorem for *finite type invariants of  $\mathbb{Q}$ -spheres* can be stated as follows.

**Theorem 18.27** (Moussard). *The family  $(Z_h^{\text{aug}}: \mathcal{F}_0(\mathcal{M}_{\mathbb{Q}}) \rightarrow \mathcal{A}_h^{\text{aug}})_{h \in \frac{1}{2}\mathbb{N}}$  of linear maps is such that, for any  $h \in \frac{1}{2}\mathbb{N}$ ,*

- we have  $Z_h^{\text{aug}}(\mathcal{F}_{2h+1}(\mathcal{M}_{\mathbb{Q}})) = 0$ , and
- $Z_h^{\text{aug}}$  induces a left inverse to  $\psi_h$  from  $\frac{\mathcal{F}_{2h}(\mathcal{M}_{\mathbb{Q}})}{\mathcal{F}_{2h+1}(\mathcal{M}_{\mathbb{Q}})}$  to  $\mathcal{A}_h^{\text{aug}}$ .

In particular, we have  $\frac{\mathcal{F}_n(\mathcal{M}_{\mathbb{Q}})}{\mathcal{F}_{n+1}(\mathcal{M}_{\mathbb{Q}})} \cong \mathcal{A}_{n/2}^{\text{aug}}$  and  $\frac{\mathcal{I}_n(\mathcal{M}_{\mathbb{Q}})}{\mathcal{I}_{n-1}(\mathcal{M}_{\mathbb{Q}})} \cong (\mathcal{A}_{n/2}^{\text{aug}})^*$  for any  $n \in \mathbb{N}$ .

PROOF: The first assertion follows from Theorem 18.6 and Lemma 6.3. Let  $\Gamma^a$  be the disjoint union of a degree  $k$  trivalent Jacobi diagram  $\Gamma$  and  $r$  isolated 0-valent vertices  $v_j$  equipped with prime numbers  $p_j$  for  $j \in \underline{r}$ . Let  $\psi_\Gamma$  be a representative of  $\psi_{k+r/2}(\Gamma^a)$  in  $\mathcal{F}_{2k+r}(\mathcal{M}_\mathbb{Q})$ . Let us prove

$$Z_{\leq k+r/2}^{\text{aug}}(\psi_\Gamma) = [\Gamma^a]$$

Write  $\psi_\Gamma$  as  $[S^3; (A^{(i)'}/A^{(i)})_{i \in \underline{2k}}, (B_{p_j}/B(v_j))_{j \in \underline{r}}]$ . If  $r = 0$ , then  $\psi_\Gamma$  belongs to  $\mathcal{F}_{2k+r}(\mathcal{M})$  and  $Z^{\text{aug}}(\psi_\Gamma)$  is equal to  $\mathcal{Z}(\psi_\Gamma)$ . So we have  $Z_k^{\text{aug}}(\psi_\Gamma) = [\Gamma]$ , thanks to Lemma 18.24. (All the involved manifolds are  $\mathbb{Z}$ -spheres.) The general case follows by induction. If  $r > 0$ , let  $\Gamma'$  be obtained from  $\Gamma^a$  by forgetting the vertex  $v_r$ . Let  $\psi_{\Gamma'}$  be obtained from  $\psi_\Gamma$  by forgetting the surgery  $(B_{p_r}/B(v_r))$ . This surgery is nothing but a connected sum with  $S_{p_r} = B_{p_r} \cup_{S^2} B^3$ . Since  $Z^{\text{aug}}$  is multiplicative under connected sum according to Theorem 10.26, we have

$$Z^{\text{aug}}(\psi_\Gamma) = Z^{\text{aug}}(\psi_{\Gamma'}) (Z^{\text{aug}}(S_{p_r}) - 1).$$

Then, identifying the nonvanishing terms with minimal degree yields

$$Z_{\leq k+r/2}^{\text{aug}}(\psi_\Gamma) = Z_{k+(r-1)/2}^{\text{aug}}(\psi_{\Gamma'}) [\bullet_{p_r}],$$

which allows us to conclude the proof. Indeed, any  $\lambda \in (\mathcal{F}_n(\mathcal{M}_\mathbb{Q})/\mathcal{F}_{n+1}(\mathcal{M}_\mathbb{Q}))^*$  extends to the linear form  $\lambda \circ \psi_{n/2} \circ Z_{n/2}^{\text{aug}}$  of  $\mathcal{I}_n(\mathcal{M}_\mathbb{Q})$ . So the natural injection

$$\frac{\mathcal{I}_n(\mathcal{M}_\mathbb{Q})}{\mathcal{I}_{n-1}(\mathcal{M}_\mathbb{Q})} \hookrightarrow \left( \frac{\mathcal{F}_n(\mathcal{M}_\mathbb{Q})}{\mathcal{F}_{n+1}(\mathcal{M}_\mathbb{Q})} \right)^*$$

is surjective. □

According to Corollary 10.11, we have

$$\mathcal{Z}_1(R, \emptyset) = \frac{1}{12} \Theta(R) [\Theta].$$

In particular, the invariant  $\Theta$  is of degree at most 2 with respect to  $\mathcal{O}_\mathcal{L}^\mathbb{Q}$  according to Theorem 18.6. Furthermore, Lemma 18.24 implies

$$\Theta(\psi_1(\Theta)) = 12.$$

The following easy corollary of Theorem 18.27 can be proved as Corollary 6.10.

**Corollary 18.28.** *For any real-valued invariant  $\nu$  of  $\mathbb{Q}$ -spheres of degree at most 2 with respect to  $\mathcal{O}_{\mathcal{L}}^{\mathbb{Q}}$ , there exist real numbers  $a_{\theta}$ ,  $a_0$ ,  $a_p$  for any prime number  $p$ , and  $a_{p,q}$  for any pair  $(p, q)$  of prime numbers such that  $p \leq q$ , such that*

$$\nu(R) = a_0 + \sum_{p \text{ prime}} a_p \nu_p(R) + \sum_{p, q \text{ prime}: p \leq q} a_{p,q} \nu_p(R) \nu_q(R) + a_{\theta} \Theta(R)$$

for any  $\mathbb{Q}$ -sphere  $R$ .

Note that the above infinite sums of the statement do not cause problems since they are finite when applied to a  $\mathbb{Q}$ -sphere  $R$ .

According to Proposition 5.15 (or to Theorem 10.29), we have  $\Theta(-R) = -\Theta(R)$  for any  $\mathbb{Q}$ -sphere  $R$ .

**Theorem 18.29.** *Let  $\nu$  be a real-valued invariant of  $\mathbb{Q}$ -spheres such that*

- *the invariant  $\nu$  is of degree at most 2 with respect to  $\mathcal{O}_{\mathcal{L}}^{\mathbb{Q}}$  and*
- *we have  $\nu(-R) = -\nu(R)$  for any  $\mathbb{Q}$ -sphere  $R$ .*

*Then there exists a real number  $a_{\theta}$  such that  $\nu = a_{\theta} \Theta$ .*

PROOF: Apply Corollary 18.28. We have

$$(\nu - a_{\theta} \Theta)(-R) = (\nu - a_{\theta} \Theta)(R) = -(\nu - a_{\theta} \Theta)(R)$$

for any  $\mathbb{Q}$ -sphere  $R$ . We get  $\nu = a_{\theta} \Theta$ . □

**Remark 18.30.** A similar result was proved in [Les04b, Proposition 6.2] without using the Moussard theorem.

## 18.6 Identifying $\Theta$ with the Casson–Walker invariant

In 1984, Andrew Casson introduced an invariant of  $\mathbb{Z}$ -spheres, which counts the conjugacy classes of irreducible  $SU(2)$ -representations of their fundamental groups using Heegaard splittings. See [AM90, GM92, Mar88]. This invariant lifts the Rohlin  $\mu$ -invariant of Definition 5.29 from  $\mathbb{Z}/2\mathbb{Z}$  to  $\mathbb{Z}$ . In 1988, Kevin Walker generalized the Casson invariant to  $\mathbb{Q}$ -spheres in [Wal92]. Here, the Casson–Walker invariant  $\lambda_{CW}$  is normalized as in [AM90, GM92, Mar88] for integer homology 3-spheres, and as  $\frac{1}{2}\lambda_W$  for rational homology 3-spheres,



where  $\lambda_W$  is the Walker normalisation in [Wal92]. [Wal92, Lemma 3.1] implies

$$\lambda_{CW}(-R) = -\lambda_{CW}(R)$$

for any  $\mathbb{Q}$ -sphere  $R$ . [Les98] shows that the Casson–Walker generalization satisfies the same splitting formulae as  $\frac{1}{6}\Theta$ . So  $\lambda_{CW}$  is of degree at most 2 with respect to  $\mathcal{O}_{\mathcal{C}}^{\mathbb{Q}}$ , and we have

$$\lambda_{CW}(\psi_1(\Theta)) = 2.$$

(This is a consequence of [Les98, Theorem 1.3].)

As a direct corollary of Theorem 18.29, we obtain the following theorem first proved by Greg Kuperberg and Dylan Thurston in [KT99] for  $\mathbb{Z}$ -spheres, and generalized to  $\mathbb{Q}$ -spheres in [Les04b, Section 6]. See [Les04b, Theorem 2.6].

**Theorem 18.31.** *We have  $\Theta = 6\lambda_{CW}$ .*

PROOF: Recall that Lemma 18.24 and Corollary 10.11 imply  $\Theta(\psi_1(\Theta)) = 12$ . □

## 18.7 Sketch of the proof of Theorem 18.6

Fix  $\sqcup_{i=1}^x A^{(i)}$  and a representative of  $L$  in  $\mathcal{C}$  as in the statement of Theorem 18.6. We have  $\sqcup_{i=1}^x A^{(i)} \subset \mathcal{C} \setminus L$ .

For  $I \subseteq \underline{x}$ , recall  $\mathcal{C}_I = \mathcal{C} \left( (A^{(i)'}/A^{(i)})_{i \in I} \right)$ . Set  $\check{R}_I = \check{R}(\mathcal{C}_I)$ ,  $R_I = R(\mathcal{C}_I)$ ,  $\check{R} = \check{R}_{\emptyset} = \check{R}(\mathcal{C})$ , and  $R = R_{\emptyset} = R(\mathcal{C})$ .

For any part  $X$  of  $R_I$ ,  $C_2(X)$  denotes the preimage of  $X^2$  under the blowdown map from  $C_2(R_I)$  to  $R_I^2$ .

In order to prove Theorem 18.6, we will compute the  $\mathcal{Z}_n(\mathcal{C}_I, L)$  with antisymmetric homogeneous propagating forms  $\omega_I$  on the  $C_2(R_I)$  such that the  $\omega_I$  coincide with each other as much as possible (with respect to Definition 3.14 of antisymmetric propagating forms). More precisely, for any subsets  $I$  and  $J$  of  $\underline{x}$ , our forms will satisfy

$$\omega_I = \omega_J \text{ on } C_2\left(\left(R \setminus \cup_{i \in I \cup J} \text{Int}(A^{(i)})\right) \cup_{i \in I \cap J} A^{(i)'}\right).$$

When dealing with integral LP-surgeries, such forms will be associated with parallelizations  $\tau_I$  of the  $\mathcal{C}_I$  (as in Definition 12.4), which coincide as much as possible with each other, i.e., such that

$$\tau_I = \tau_J \text{ on } \left(\left(R \setminus \cup_{i \in I \cup J} \text{Int}(A^{(i)})\right) \cup_{i \in I \cap J} A^{(i)'}\right) \times \mathbb{R}^3.$$

Unfortunately, such a consistent choice of parallelizations is not always possible for rational LP-surgeries. See Section 19.1 and Example 19.8. To remedy this problem, we will make the definition of  $\mathcal{Z}$  more flexible by allowing more general propagating forms associated with generalizations of parallelizations, called *pseudo-parallelizations*.

We define these pseudo-parallelizations in Chapter 19, and we show that they satisfy the following properties.

- They generalize parallelizations. They are genuine parallelizations outside a link tubular neighborhood, inside which they can be thought of as an average of genuine parallelizations.
- A parallelization defined near the boundary of a rational homology handlebody always extends to this rational homology handlebody as a pseudo-parallelization. (See Lemma 19.10.)
- When  $\check{R}$  is an asymptotic rational homology  $\mathbb{R}^3$ , a pseudo-parallelization  $\tilde{\tau}$  of  $\check{R}$  induces a homotopy class of special complex trivializations  $\tilde{\tau}_{\mathbb{C}}$  of  $T\check{R} \otimes_{\mathbb{R}} \mathbb{C}$ , which has a Pontrjagin number  $p_1(\tilde{\tau}_{\mathbb{C}})$ . Outside the link tubular neighborhood considered above,  $\tilde{\tau}_{\mathbb{C}}$  is  $\tilde{\tau} \otimes_{\mathbb{R}} 1_{\mathbb{C}}$ . We set  $p_1(\tilde{\tau}) = p_1(\tilde{\tau}_{\mathbb{C}})$ . (See Definitions 19.11 and 19.12.)
- The notion of a *homogeneous propagating form* of  $(C_2(R), \tilde{\tau})$  is presented in Definition 19.15. This definition allows us to extend the definition of  $\mathcal{Z}$  of Theorem 12.7 using pseudo-parallelizations instead of parallelizations as follows. For any long tangle representative

$$L: \mathcal{L} \hookrightarrow \check{R}(\mathcal{C})$$

in a rational homology cylinder equipped with a pseudo-parallelization  $\tilde{\tau}$  restricting to a neighborhood of the image of  $L$  as a genuine parallelization, for any  $n \in \mathbb{N}$ , and for any family  $(\omega(i))_{i \in \underline{3n}}$  of homogeneous propagating forms of  $(C_2(R(\mathcal{C})), \tilde{\tau})$ , the sum

$$Z_n(\mathcal{C}, L, (\omega(i))) = \sum_{\Gamma \in \mathcal{D}_n^e(\mathcal{L})} \zeta_{\Gamma} I(\mathcal{C}, L, \Gamma, (\omega(i))_{i \in \underline{3n}}) [\Gamma] \in \mathcal{A}_n(\mathcal{L})$$

depends only on  $(\mathcal{C}, L, p_1(\tilde{\tau}_{\mathbb{C}}))$  and on the  $I_{\theta}(K_j, \tilde{\tau})$ , which are defined as in Lemma 7.15 and Definition 12.6, for the components  $K_j$ ,  $j \in \underline{k}$ , of  $L$ . It is denoted by  $Z_n(\mathcal{C}, L, \tilde{\tau})$ . Set

$$Z(\mathcal{C}, L, \tilde{\tau}) = (Z_n(\mathcal{C}, L, \tilde{\tau}))_{n \in \mathbb{N}} \in \mathcal{A}(\mathcal{L}).$$

Then Theorem 19.17 ensures that we have

$$\mathcal{Z}(\mathcal{C}, L) = \exp\left(-\frac{1}{4}p_1(\tilde{\tau}_{\mathcal{C}})\beta\right) \prod_{j=1}^k \left(\exp(-I_{\theta}(K_j, \tilde{\tau})\alpha) \#_j\right) Z(\mathcal{C}, L, \tilde{\tau}).$$

In the general case, we begin by choosing pseudo-parallelizations  $\tau_I$  of the  $\mathcal{C}_I$  such that  $\tau_I$  and  $\tau_J$  coincide on

$$\left((R \setminus \cup_{i \in I \cup J} \text{Int}(A^{(i)})) \cup_{i \in I \cap J} A^{(i)'}\right) \times \mathbb{R}^3.$$

A reader only interested in the cases for which pseudo-parallelizations are not necessary, as in the applications of Sections 18.3 and 18.4, can skip Chapter 19, and substitute the word pseudo-parallelization with parallelization in the rest of the proof below and in Chapter 20.

Set  $\tau_{\emptyset} = \tau$ . The following lemma relates the  $p_1(\tau_I)$ .

**Lemma 18.32.** *Set  $p(i) = p_1(\tau_{\{i\}}) - p_1(\tau)$ . For any subset  $I$  of  $\underline{x}$ , we have*

$$p_1(\tau_I) = p_1(\tau) + \sum_{i \in I} p(i).$$

PROOF: Proceed by induction on the cardinality of  $I$ . The lemma is obviously true if  $|I|$  is zero or one. Assume that  $|I| \geq 2$ . Let  $j \in I$ . It suffices to prove  $p_1(\tau_I) - p_1(\tau_{I \setminus \{j\}}) = p_1(\tau_{\{j\}}) - p_1(\tau)$ . This follows by applying twice the second part of Proposition 5.26, where  $M_0 = A^{(j)}$ ,  $M_1 = A^{(j)'}$ , and  $D = \mathcal{C}$  or  $D = \mathcal{C}_{I \setminus \{j\}}$ . The first application identifies  $(p_1(\tau_{\{j\}}) - p_1(\tau))$  with  $p_1(\tau|_{A^{(j)}}, \tau_{\{j\}}|_{A^{(j)'}})$ . The second one yields the conclusion.  $\square$

For any  $i \in \underline{x}$ , fix pairwise disjoint simple closed curves  $(a_j^i)_{j=1, \dots, g_i}$  and pairwise disjoint simple closed curves  $(z_j^i)_{j=1, \dots, g_i}$  on  $\partial A^{(i)}$  such that

$$\mathcal{L}_{A^{(i)}} = \oplus_{j=1}^{g_i} [a_j^i]$$

and

$$\langle a_j^i, z_k^i \rangle_{\partial A^{(i)}} = \delta_{jk} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$$

Let  $[-4, 4] \times (\sqcup_{i \in N} \partial A^{(i)})$  be a tubular neighborhood of  $(\sqcup_{i \in N} \partial A^{(i)})$  in  $\mathcal{C}$ . This neighborhood intersects  $A^{(i)}$  as  $[-4, 0] \times \partial A^{(i)}$ . Let  $[-4, 0] \times \partial A^{(i)}$  be a neighborhood of  $\partial A^{(i)'} = \partial A^{(i)}$  in  $A^{(i)'}$ . The manifold  $\mathcal{C}_{\{i\}} = \mathcal{C}_i$  is obtained from  $\mathcal{C}$  by removing  $(A^{(i)} \setminus (]-4, 0] \times \partial A^{(i)}))$  and by gluing back  $A^{(i)'}$  along  $]-4, 0] \times \partial A^{(i)}$ .

Let  $\eta_{[-1,1]}$  be a one-form with compact support in  $] -1, 1[$  such that

$$\int_{[-1,1]} \eta_{[-1,1]} = 1.$$

Let  $a_j^i \times [-1, 1]$  be a tubular neighborhood of  $a_j^i$  in  $\partial A^{(i)}$ . Let  $\eta(a_j^i)$  be a closed one-form on  $A^{(i)}$  such that the support of  $\eta(a_j^i)$  intersects  $[-4, 0] \times \partial A^{(i)}$  inside  $[-4, 0] \times (a_j^i \times [-1, 1])$ , where  $\eta(a_j^i)$  can be written as

$$\eta(a_j^i) = p_{[-1,1]}^*(\eta_{[-1,1]}),$$

with the projection  $p_{[-1,1]}: [-4, 0] \times (a_j^i \times [-1, 1]) \rightarrow [-1, 1]$  to the  $[-1, 1]$  factor. Let  $\eta(a_j^i)$  also denote a closed one-form on  $A^{(i)'}$  which can be written in the same way on  $[-4, 0] \times \partial A^{(i)}$ . Note that the forms  $\eta(a_j^i)$  on  $A^{(i)}$  and  $A^{(i)'}$  induce a closed one-form on  $(A^{(i)} \cup_{\partial A^{(i)}} -A^{(i)'})$  that restrict to the previous ones on  $A^{(i)}$  and  $(-A^{(i)'})$ . This one-form is also denoted by  $\eta(a_j^i)$ . The form  $\eta(a_j^i)$  on  $(A^{(i)} \cup_{\partial A^{(i)}} -A^{(i)'})$  is Poincaré dual to the homology class  $\partial_{MV}^{-1}(a_j^i)$  in  $(A^{(i)} \cup_{\partial A^{(i)}} -A^{(i)'})$ , with the notation introduced before Theorem 18.6. Define the part  $A_I^{(i)}$  of  $\mathcal{C}_I$  to be  $A_I^{(i)} = A^{(i)}$  if  $i \notin I$  and  $A_I^{(i)} = A^{(i)'}$  if  $i \in I$ .

The following proposition is the key to the proof of Theorem 18.6. Its proof is more complicated than I expected. We give it in Chapter 20.

**Proposition 18.33.** *There exist homogeneous antisymmetric propagating forms  $\omega_I$  of  $(C_2(R_I), \tau_I)$  with the following properties.*

- For any subsets  $I$  and  $J$  of  $\underline{x}$ ,  $\omega_I$  and  $\omega_J$  coincide on

$$C_2\left(\left(R \setminus \cup_{i \in I \cup J} \text{Int}(A^{(i)})\right) \cup_{i \in I \cap J} A^{(i)'}\right),$$

- For any  $(i, k) \in \underline{x}^2$  such that  $i \neq k$ ,  $\omega_I$  can be written as follows on  $A_I^{(i)} \times A_I^{(k)}$ :

$$\omega_I = \sum_{\substack{j=1, \dots, g_i \\ \ell=1, \dots, g_k}} lk(z_j^i, z_\ell^k) p_{A_I^{(i)}}^*(\eta(a_j^i)) \wedge p_{A_I^{(k)}}^*(\eta(a_\ell^k)),$$

where  $p_{A_I^{(i)}}: A_I^{(i)} \times A_I^{(k)} \rightarrow A_I^{(i)}$  and  $p_{A_I^{(k)}}: A_I^{(i)} \times A_I^{(k)} \rightarrow A_I^{(k)}$  again denote the natural projections onto the factor corresponding to the subscript.

Recall Notation 18.5. Let  $\Gamma$  be an oriented Jacobi diagram without univalent vertices. When  $x$  is even and when  $G = \bigsqcup_{i \in \underline{x}} T(\mathcal{I}_{A^{(i)} A^{(i)'}})$ , define

$$\langle\langle G \rangle\rangle_\Gamma = \sum_{p \in P(G): \Gamma_p \text{ isomorphic to } \Gamma} [\ell(p)\Gamma_p],$$

where the sum runs over the  $p$  such that  $\Gamma_p$  is isomorphic to  $\Gamma$  as a nonoriented trivalent graph.

Assuming Proposition 18.33, one can prove the following lemma.

**Lemma 18.34.** *Let  $\omega_I$  be forms as in Proposition 18.33. Let  $\Gamma$  be an oriented Jacobi diagram on  $\mathcal{L}$ . If  $\Gamma$  has less than  $x$  trivalent vertices, then we have*

$$\sum_{I \subseteq \underline{x}} (-1)^{|I|} I(R_I, L, \Gamma, o(\Gamma), (\omega_I)) = 0.$$

If  $\Gamma$  is a trivalent Jacobi diagram with  $x$  vertices, then we have

$$\sum_{I \subseteq \underline{x}} (-1)^{|I|} I(R_I, L, \Gamma, (\omega_I)) [\Gamma] = |\text{Aut}(\Gamma)| \left\langle \left\langle \bigsqcup_{i \in \underline{x}} T(\mathcal{I}_{A^{(i)} A^{(i)'}}) \right\rangle \right\rangle_{\Gamma}.$$

PROOF: Let  $n \in \mathbb{N}$ . Let  $\Gamma$  be an oriented degree  $n$  Jacobi diagram on  $\mathcal{L}$ . Let us compute

$$\Delta = \sum_{I \subseteq \underline{x}} (-1)^{|I|} I(R_I, L, \Gamma, o(\Gamma), (\omega_I)).$$

Number the vertices of  $\Gamma$  in  $\underline{2n}$  arbitrarily. So the open configuration space  $\check{C}(\check{R}_I, L; \Gamma)$  becomes a submanifold of  $\check{R}_I^{2n}$ . The order of the vertices orders the oriented local factors (some of which are tangle components) of  $\check{C}(\check{R}_I, L; \Gamma)$ . Thus, it orients  $\check{C}(\check{R}_I, L; \Gamma)$ . Orient the edges of  $\Gamma$  so that the edge-orientation of  $H(\Gamma)$  and the vertex-orientation of  $\Gamma$  induce the above orientation of  $\check{C}(\check{R}_I, L; \Gamma)$  as in Lemma 7.1.

Let  $i \in \underline{x}$ . The forms  $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega_I)$  over

$$\check{C}(\check{R}_I, L; \Gamma) \cap (\check{R}_I \setminus A_I^{(i)})^{2n}$$

are identical for  $I = K$  and  $I = K \cup \{i\}$  for any  $K \subseteq \underline{x} \setminus \{i\}$ . Since their integrals enter the sum  $\Delta$  with opposite signs, they cancel each other. This argument allows us to get rid of the contributions of the integrals over the  $\check{C}(\check{R}_I, L; \Gamma) \cap (\check{R}_I \setminus A_I^{(1)})^{2n}$  for any  $I \subseteq \underline{x}$ . The contributions over the

$$\left( \check{C}(\check{R}_I, L; \Gamma) \setminus \left( \check{C}(\check{R}_I, L; \Gamma) \cap (\check{R}_I \setminus A_I^{(1)})^{2n} \right) \cap (\check{R}_I \setminus A_I^{(2)})^{2n} \right)$$

cancel in the same way. Iterating, we get rid of the contributions of the integrals over the  $\check{C}(\check{R}_I, L; \Gamma) \cap (\check{R}_I \setminus A_I^{(i)})^{2n}$  for any  $i \in \underline{x}$ , and for any  $I \subseteq \underline{x}$ . Thus, we are left with the contributions of the integrals over the subsets  $Q_I$  of

$$\check{C}(\check{R}_I, L; \Gamma) \subset \check{R}_I^{2n}$$

with the following property:

For any  $i \in \underline{x}$ , any element of  $Q_I$  is sent to  $A_I^{(i)}$  under at least one of the  $(2n)$  projections onto  $\check{R}_I$ . These subsets  $Q_I$  are empty if  $2n < x$ . Thus, the lemma is proved when  $\Gamma$  has less than  $x$  trivalent vertices.

Assume  $\Gamma$  is trivalent and  $x = 2n$ . Then  $Q_I$  is equal to

$$\cup_{\sigma \in \mathfrak{S}_{2n}} \prod_{i=1}^{2n} A_I^{(\sigma(i))},$$

where  $\mathfrak{S}_{2n}$  is the set of permutations of  $\underline{2n}$ . We get

$$\Delta = \sum_{\sigma \in \mathfrak{S}_{2n}} \Delta_\sigma$$

with

$$\Delta_\sigma = \sum_{I \subseteq \underline{2n}} (-1)^{|I|} \int_{\prod_{i=1}^{2n} A_I^{(\sigma(i))}} \bigwedge_{e \in E(\Gamma)} p_e^*(\omega_I).$$

Let us compute  $\Delta_\sigma$ . For any  $i \in \underline{x}$ ,  $p_i: \check{C}(\check{R}_I, L; \Gamma) \rightarrow \check{R}_I$  denotes the projection onto the  $i$ th factor. When  $e$  is an oriented edge from the vertex  $x(e) \in V(\Gamma)$  to  $y(e) \in V(\Gamma)$ , we have

$$p_e^*(\omega_I) \Big|_{\prod_{i=1}^{2n} A_I^{(\sigma(i))}} = \sum_{\substack{j=1, \dots, g_\sigma(x(e)) \\ \ell=1, \dots, g_\sigma(y(e))}} lk\left(z_j^{\sigma(x(e))}, z_\ell^{\sigma(y(e))}\right) p_{x(e)}^*\left(\eta\left(a_j^{\sigma(x(e))}\right)\right) \wedge p_{y(e)}^*\left(\eta\left(a_\ell^{\sigma(y(e))}\right)\right),$$

where the vertices are regarded as elements of  $\underline{2n}$  via the numbering. Recall the sets  $E(\Gamma)$  and  $H(\Gamma)$  of edges and half-edges of  $\Gamma$ . For a half-edge  $c$ , let  $v(c)$  denote the label of the vertex contained in  $c$ .

Let  $F_\sigma$  denote the set of maps  $f$  from  $H(\Gamma)$  to  $\mathbb{N}$  such that  $f(c) \in \{1, 2, \dots, g_\sigma(v(c))\}$  for any  $c \in H(\Gamma)$ . For such a map  $f$ ,  $f(x(e))$  (resp.  $f(y(e))$ ) denotes the value of  $f$  at the half-edge of  $e$  that contains  $x(e)$  (resp.  $y(e)$ ). We have

$$\Delta_\sigma = \sum_{f \in F_\sigma} \left( \prod_{e \in E(\Gamma)} lk\left(z_{f(x(e))}^{\sigma(x(e))}, z_{f(y(e))}^{\sigma(y(e))}\right) \right) I(f)$$

with

$$I(f) = \int_{\prod_{i=1}^{2n} (A^{(\sigma(i))} \cup -A^{(\sigma(i))'})} \bigwedge_{e \in E(\Gamma)} p_{x(e)}^*\left(\eta\left(a_{f(x(e))}^{\sigma(x(e))}\right)\right) \wedge p_{y(e)}^*\left(\eta\left(a_{f(y(e))}^{\sigma(y(e))}\right)\right).$$

We also have

$$I(f) = \prod_{i=1}^{2n} \int_{(A^{(\sigma(i))} \cup -A^{(\sigma(i))'})} \bigwedge_{c \in H(\Gamma) : v(c)=i} p_i^* \left( \eta(a_{f(c)}^{\sigma(i)}) \right),$$

where we order the exterior product's factors over the half-edges of  $v^{-1}(i)$  according to the vertex-orientation of  $i$ . Observe

$$\int_{A^{(\sigma(i))} \cup (-A^{(\sigma(i))'})} \bigwedge_{c \in v^{-1}(i)} \eta(a_{f(c)}^{\sigma(i)}) = \mathcal{I}_{A^{(\sigma(i))} A^{(\sigma(i))'}} \left( \bigotimes_{c \in v^{-1}(i)} a_{f(c)}^{\sigma(i)} \right),$$

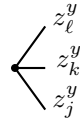
where the factors of the tensor product are ordered according to the vertex-orientation of  $i$ , again. Indeed, the closed form  $\eta(a_{f(c)}^{\sigma(i)})$  is dual to the homology class  $\partial_{MV}^{-1}(a_{f(c)}^{\sigma(i)})$  in  $A^{(\sigma(i))} \cup (-A^{(\sigma(i))'})$ , with the notation introduced before Theorem 18.6.

Summarizing, we get

$$\Delta_\sigma = \sum_{f \in F_\sigma} \left( \left( \prod_{e \in E(\Gamma)} lk(z_{f(x(e))}^{\sigma(x(e))}, z_{f(y(e))}^{\sigma(y(e))}) \right) \left( \prod_{i \in \underline{2n}} \mathcal{I}_{A^{(\sigma(i))} A^{(\sigma(i))'}} \left( \bigotimes_{c \in v^{-1}(i)} a_{f(c)}^{\sigma(i)} \right) \right) \right).$$

We may restrict the sum to the subset  $\tilde{F}_\sigma$  of  $F_\sigma$  consisting of the maps  $f$  of  $F_\sigma$  that restrict to  $v^{-1}(i)$  as injections for any  $i$ .

Finally,  $\Delta$  is a sum running over all the ways of renumbering the vertices of  $\Gamma$  by elements of  $\underline{x}$  (via  $\sigma$ ) and of coloring the half-edges  $c$  of  $v^{-1}(i)$  by three distinct curves  $z_{f(c)}^{\sigma(i)}$  via  $f$ . In particular, a pair  $(\sigma, f)$  provides a tripod



such that  $1 \leq j < k < \ell \leq g_y$  for any  $y \in \underline{x}$ , and it provides a pairing of the ends of the univalent vertices of the tripods, giving rise to the graph  $\Gamma$  with a possibly different vertex-orientation. The vertices of the obtained graph are furthermore numbered by the numbering of the vertices of  $\Gamma$ , and its edges are identified with the original edges of  $\Gamma$ .

Fix a set of tripods associated to the elements of  $\underline{x}$  as above and a pairing of their univalent vertices giving rise to  $\Gamma$  as a nonoriented graph. Then there are  $|\text{Aut}(\Gamma)|$  ways of numbering its vertices and edges to get a graph isomorphic to  $\Gamma$ . So the pairing occurs  $|\text{Aut}(\Gamma)|$  times.  $\square$

**PROOF OF THEOREM 18.6 UP TO THE UNPROVED ASSERTIONS OF THIS SECTION**

... RESTATED PRECISELY AT THE END OF THE PROOF: Lemma 18.34 and Proposition 7.25 easily imply

$$\begin{aligned} \sum_{I \subseteq \underline{x}} (-1)^{|I|} Z_n(\mathcal{C}_I, L, (\omega_I)) &= 0 && \text{if } 2n < x, \\ &= \left[ \langle \langle \bigsqcup_{i \in \underline{x}} T(\mathcal{I}_{A^{(i)} A^{(i)'}}) \rangle \rangle \right] && \text{if } 2n = x. \end{aligned}$$

Theorem 19.17 (or Theorem 12.7 if pseudo-parallelizations are not required) implies

$$\mathcal{Z}(\mathcal{C}, L) = \exp\left(-\frac{1}{4}p_1(\tau)\beta\right) \prod_{j=1}^k \left(\exp(-I_\theta(K_j, \tau)\alpha) \#_j\right) Z(\mathcal{C}, L, \tau).$$

Set

$$Y_n = \sum_{I \subseteq \underline{x}} (-1)^{|I|} \left(\exp\left(-\frac{1}{4}p_1(\tau_I)\beta\right) Z\right)_n(\mathcal{C}_I, L, (\omega_I)),$$

where  $(\cdot)_n$  stands for the degree  $n$  part. Note that the framing corrections  $\prod_{j=1}^k (\exp(-I_\theta(K_j, \tau_I)\alpha) \#_j)$  do not depend on  $I$ . Therefore, it suffices to prove

$$Y_n = \sum_{I \subseteq \underline{x}} (-1)^{|I|} Z_n(\mathcal{C}_I, L, (\omega_I))$$

when  $2n \leq x$ . We have

$$\begin{aligned} \left(\exp\left(-\frac{1}{4}p_1(\tau_I)\beta\right) Z\right)_n(\mathcal{C}_I, L, (\omega_I)) &= Z_n(\mathcal{C}_I, L, (\omega_I)) \\ &\quad + \sum_{j < n} Z_j(\mathcal{C}_I, L, (\omega_I)) P_{n-j}(I), \end{aligned}$$

with

$$P_{n-j}(I) = \left(\exp\left(-\frac{1}{4}(p_1(\tau) + \sum_{i \in I} p(i))(\beta_1 + \beta_3 + \dots))\right)\right)_{n-j},$$

thanks to Lemma 18.32.

This element  $P_{n-j}(I)$  of  $\mathcal{A}_{n-j}(\emptyset)$  can be expanded as a combination  $\sum m_{\Gamma, g, K} [\Gamma]$ , where

- the  $\Gamma$  are degree  $(n - j)$  Jacobi diagrams,
- the  $K$  are subsets of  $I$  with cardinality at most  $n - j$ ,
- the  $m_{\Gamma, g, K}$  are monomials of degree at most  $(n - j)$  in  $p_1(\tau)$  and in the  $p(i)$  for  $i \in K$ ,



- the  $m_{\Gamma,g,K}$  do not depend on  $I$ ,
- for every  $i \in K$ , the variable  $p(i)$  actually occurs in the monomial  $m_{\Gamma,g,K}$ , and
- for a subset  $J$  of  $\underline{x}$ ,  $m_{\Gamma,g,K}[\Gamma]$  appears in  $P_{n-j}(J)$  if and only if  $K$  is a subset of  $J$ .

Thus, we can write the sum of the undesired terms in  $Y_n$  by factoring out the  $m_{\Gamma,g,K}[\Gamma]$ . The factor of  $m_{\Gamma,g,K}[\Gamma]$  is

$$\sum_{I: K \subseteq I \subseteq \underline{x}} (-1)^{|I|} Z_j(\mathcal{C}_I, L, (\omega_I)).$$

This sum actually runs over the subsets of  $\underline{x} \setminus K$ . The cardinality of  $\underline{x} \setminus K$  is at least  $x + j - n$ . The inequalities  $2n \leq x$  and  $j < n$  imply  $j < n \leq x - n$ , and hence  $2j < x + j - n$ . Therefore, the beginning of the proof ensures that the above factor of  $m_{\Gamma,g,K}[\Gamma]$  is zero. Hence we have  $Y_n = \sum_{I \subseteq \underline{x}} (-1)^{|I|} Z_n(\mathcal{C}_I, L, (\omega_I))$ . This concludes the reduction of the proof of Theorem 18.6 to the proof of Proposition 18.33 given in Chapter 20, and to the proofs that pseudo-parallelizations and associated propagating forms exist and satisfy the announced properties, given in Chapter 19.  $\square$

## 18.8 Mixed universality statements

We can mix the statements of Theorem 17.30 and 18.6 to get the following statement, which covers both of them.

**Theorem 18.35.** *Let  $y, z \in \mathbb{N}$ . Recall  $\underline{y} = \{1, 2, \dots, y\}$ . Set  $(\underline{z} + y) = \{y + 1, y + 2, \dots, y + z\}$ . Let  $L$  be a singular  $q$ -tangle representative in a rational homology cylinder  $\mathcal{C}$ , whose double points are numbered by  $\underline{y}$  and sitting in balls  $B_b$  of desingularizations for  $b \in \underline{y}$ . For a subset  $I$  of  $\underline{y}$ , let  $L_I$  denote the  $q$ -tangle obtained from  $L$  by performing negative desingularizations on double points of  $I$  and positive ones on double points of  $\underline{y} \setminus I$  in the balls  $B_b$ . Let  $\sqcup_{i=y+1}^{y+z} A^{(i)}$  be a disjoint union of rational homology handlebodies embedded in  $\mathcal{C} \setminus (L \cup_{b=1}^y B_b)$ . Let  $(A^{(i)'}/A^{(i)})$  be rational LP-surgeries in  $\mathcal{C}$ .*

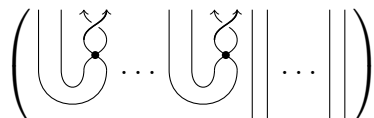
*Set  $X = [\mathcal{C}, L; (A^{(i)'}/A^{(i)})_{i \in \underline{z}+y}]$  and, using Notation 17.29,*

$$\overline{\mathcal{Z}}_n(X) = \sum_{I \subseteq \underline{y}+z} (-1)^{|I|} \overline{\mathcal{Z}}_n(\mathcal{C}((A^{(i)'}/A^{(i)})_{i \in I \cap (\underline{z}+y)}), L_{I \cap \underline{y}}).$$

If  $2n < 2y + z$ , then  $\overline{\mathcal{Z}}_n(X)$  vanishes. If  $2n = 2y + z$ , then we have

$$\overline{\mathcal{Z}}_n(X) = \left[ \left\langle \left\langle \bigsqcup_{i \in \underline{z+y}} T(\mathcal{I}_{A^{(i)}A^{(i)'}}) \right\rangle \right\rangle_{\tilde{R}(\mathcal{C})} \sqcup \Gamma_{\mathcal{C}}(L) \right].$$

PROOF ASSUMING THEOREM 18.6: Write  $(\mathcal{C}, L)$  as a product of a tangle  $L_1$  of the form



in the standard rational homology cylinder  $D_1 \times [0, 1]$ , by a nonsingular tangle  $L_2$  in  $\mathcal{C}$ , so that  $\sqcup_{i=y+1}^{y+z} A^{(i)}$  is in the latter factor. This can be achieved by moving the double points below. Then Proposition 17.32 and Theorem 18.6 imply that

$$\sum_{I \subseteq \underline{y+z}} (-1)^{|I|} \mathcal{Z}_n^f \left( \mathcal{C}((A^{(i)'}/A^{(i)})_{i \in I \cap (\underline{z+y})}), L_{I \cap \underline{y}} \right)$$

satisfies the conclusions of the statement with  $\mathcal{Z}^f$  instead of  $\overline{\mathcal{Z}}$ , by functoriality. Now, to prove Theorem 18.35 it suffices to “multiply the proof of Theorem 17.30 (at the end of Section 17.6) by  $[\langle \langle \bigsqcup_{i \in \underline{z+y}} T(\mathcal{I}_{A^{(i)}A^{(i)'}}) \rangle \rangle_{\tilde{R}(\mathcal{C})}]$ ”  $\square$ .

In order to prove more interesting mixed universality properties, we say more about the normalization of the propagating forms of Proposition 18.33.

Recall that  $[-4, 4] \times \partial A^{(i)}$  denotes a regular neighborhood of  $\partial A^{(i)}$  embedded in  $\mathcal{C}$ , that intersects  $A^{(i)}$  as  $[-4, 0] \times \partial A^{(i)}$ . All the neighborhoods  $[-4, 4] \times \partial A^{(i)}$  are disjoint from each other and from  $L$ . Throughout this paragraph, we use the corresponding coordinates on the image of this implicit embedding.

For  $t \in [-4, 4]$ , set

$$A_t^{(i)} = \begin{cases} A^{(i)} \cup ([0, t] \times \partial A^{(i)}) & \text{if } t \geq 0 \\ A^{(i)} \setminus ([t, 0] \times \partial A^{(i)}) & \text{if } t \leq 0. \end{cases}$$

We have  $\partial A_t^{(i)} = \{t\} \times \partial A^{(i)}$ .

For  $i \in \underline{x}$ , choose a basepoint  $p^i$  in  $\partial A^{(i)}$  outside the neighborhoods  $a_j^i \times [-1, 1]$  of the  $a_j^i$  and outside neighborhoods  $z_j^i \times [-1, 1]$  the  $z_j^i$ . Fix a path  $[p^i, q^i]$  from  $p^i$  to a point  $q^i$  of  $\partial \mathcal{C}$  in

$$\mathcal{C} \setminus \left( L \cup \text{Int}(A^{(i)}) \cup \cup_{k:k \neq i} A_4^{(k)} \right)$$

so that the paths  $[p^i, q^i]$  are pairwise disjoint. Choose a closed 2-form  $\omega(p^i)$  on  $(C_1(R_I) \setminus \text{Int}(A^{(i)}))$  such that

- the integral of  $\omega(p^i)$  along a closed surface of  $(\mathcal{C}_I \setminus \text{Int}(A^{(i)}))$  is its algebraic intersection with  $[p^i, q^i]$ ,
- the support of  $\omega(p^i)$  intersects  $(\mathcal{C}_I \setminus \text{Int}(A^{(i)}))$  inside a tubular neighborhood of  $[p^i, q^i]$  disjoint from

$$\left( \bigcup_{k:k \neq i} A_4^k \right) \cup L \cup \left( [0, 4] \times \left( \bigcup_{j=1}^{g_i} \left( (a_j^i \times [-1, 1]) \cup (z_j^i \times [-1, 1]) \right) \right) \right).$$

- $\omega(p^i)$  restricts as the usual volume form  $\omega_{S^2}$  on  $\partial C_1(R_I) = S^2$ .

For  $i \in \underline{x}$ , for  $j = 1, \dots, g_i$ , the curve  $\{4\} \times a_j^i$  bounds a rational chain  $\Sigma(a_j^i)$  in  $A_4^{(i)}$  and a rational chain  $\Sigma'(a_j^i)$  in  $A_4^{(i)'}$ . When viewed as a chain in  $\mathcal{C}_I$ , such a chain is denoted by  $\Sigma_I(a_j^i)$ . We have  $\Sigma_I(a_j^i) = \Sigma(a_j^i)$  if  $i \notin I$ , and  $\Sigma_I(a_j^i) = \Sigma'(a_j^i)$  if  $i \in I$ . The form  $\eta(a_j^i)$  supported on  $[-4, 4] \times a_j^i \times [-1, 1]$  in  $A_{I,4}^{(i)} \setminus A_{I,-4}^{(i)}$  may be expressed as  $\eta(a_j^i) = p_{[-1,1]}^* (\eta_{[-1,1]})$  there. Thus, it extends naturally to  $A_{I,4}^{(i)} = (A_I^{(i)})_4$ , as a closed form dual to the chain  $\Sigma_I(a_j^i)$ .

For  $i \in \underline{x}$ , for  $j = 1, \dots, g_i$ ,  $z_j^i$  bounds a rational chain in  $\mathcal{C}_I$ . Therefore, it cobounds a rational chain  $\Sigma_I(z_j^i)$  in  $(\mathcal{C}_I \setminus \mathring{A}_I^{(i)}) \setminus (\cup_{i=1}^x [p^i, q^i])$  with a combination of  $a_\ell^i$  with rational coefficients. We have

$$\partial \Sigma_I(z_j^i) = z_j^i - \sum_{\ell=1}^{g_i} lk_e(z_j^i, \{-1\} \times z_\ell^i) a_\ell^i = z_j^i.$$

Furthermore,  $\Sigma_I(z_j^i)$  may be assumed to intersect  $\mathring{A}_I^{(k)}$  as

$$\sum_{m=1}^{g_k} lk(z_j^i, z_m^k) \Sigma_I(a_m^k),$$

for  $k \neq i$ . There is a closed one-form  $\eta_I(z_j^i)$  dual to  $\Sigma_I(z_j^i)$  in  $(\mathcal{C}_I \setminus \mathring{A}_I^{(i)})$ , such that  $\eta_I(z_j^i)$  is supported near  $\Sigma_I(z_j^i)$  and outside the supports of  $\omega(p^i)$  and of the other  $\omega(p^k)$ , and we have

$$\eta_I(z_j^i) = \sum_{m=1}^{g_k} lk(z_j^i, z_m^k) \eta(a_m^k)$$

on  $\mathring{A}_I^{(k)}$  for  $k \neq i$ . The integral of  $\eta_I(z_j^i)$  along a closed curve of  $(\mathcal{C}_I \setminus \mathring{A}_I^{(i)})$  is its linking number with  $z_j^i$  in  $\mathcal{C}_I$ .

We will prove the following proposition in Chapter 20.

**Proposition 18.36.** *The antisymmetric propagating forms  $\omega_I$  of  $(C_2(R_I), \tau_I)$  of Proposition 18.33 can be chosen so that*

1. *for every  $i \in \underline{x}$ , the restriction of  $\omega_I$  to*

$$\left( A_I^{(i)} \times \left( C_1(R_I) \setminus A_{I,3}^{(i)} \right) \right) \subset C_2(R_I)$$

*is equal to*

$$\sum_{j \in g_i} p_1^* (\eta_I(a_j^i)) \wedge p_2^* (\eta_I(z_j^i)) + p_2^* (\omega(p^i)),$$

*where  $p_1$  and  $p_2$  respectively denote the first and second projection of  $A_I^{(i)} \times (C_1(R_I) \setminus A_{I,3}^{(i)})$  to  $C_1(R_I)$ , and*

2. *for every  $i$ , for any  $j \in \{1, 2, \dots, g_i\}$ , we have*

$$\int_{\Sigma_I(a_j^i) \times p^i} \omega_I = 0,$$

*where  $p^i \in \partial A_I^{(i)}$  and  $\partial \Sigma_I(a_j^i) \subset \{4\} \times \partial A_I^{(i)}$ .*

A *two-leg Jacobi diagram* is a uni-trivalent Jacobi diagram with two univalent vertices, called *legs*. When these legs are colored by possibly noncompact connected components  $K_j$  of a tangle  $L$ , a two-leg diagram gives rise to a diagram on the domain  $\mathcal{L}$  of the tangle by attaching the legs to the corresponding components. The class in  $\mathcal{A}(\mathcal{L})$  of this diagram is well-defined. Indeed, according to Lemma 6.26, Jacobi diagrams with one univalent vertex vanish in  $\mathcal{A}(\mathcal{L})$ . Therefore, the STU relation guarantees that if the same noncompact component colors the two legs, changing their order with respect to the orientation component does not change the diagram class. (See also Lemma 12.26.)

Generalize the contraction of trivalent graphs associated to LP-surgeries of Notation 18.5 to graphs with legs as follows.

**Notation 18.37.** Let  $L: \mathcal{L} \leftrightarrow \mathcal{C}$  be a long tangle representative in a rational homology cylinder  $\mathcal{C}$ .

Let  $G$  be a graph with oriented trivalent vertices and with two kinds of univalent vertices, the *decorated ones* and the *legs*, where the components of the legs of  $G$  are edges from a leg to a decorated univalent vertex. The *legs* are univalent vertices on  $\mathcal{L}$ . The decorated univalent vertex in a leg segment is decorated with the leg component. The other *decorated univalent vertices*

of  $G$  are decorated with disjoint curves of  $\check{R} = \check{R}(\mathcal{C})$  disjoint from the image of  $L$ . Such a curve  $c$  bounds a compact oriented surface  $\Sigma(c)$  in  $\mathcal{C}$ , and its linking number with a component  $K_u$  of  $L$  is the algebraic intersection  $\langle K_u, \Sigma(c) \rangle$ . Let  $\check{P}(G)$  be the set of partitions of the set of decorated univalent vertices of  $G$  in disjoint pairs.

For  $p \in \check{P}(G)$ , identifying the two decorated vertices of each pair provides a vertex-oriented Jacobi diagram  $\Gamma_p$  on  $\mathcal{L}$ . Multiplying it by the product  $\ell(p)$  over the pairs of  $p$  of the linking numbers of the curves that decorate the two vertices yields an element  $[\ell(p)\Gamma_p]$  of  $\mathcal{A}(\mathcal{L})$ .

Define

$$\langle\langle G \check{\rangle}\rangle = \check{p} \left( \sum_{p \in \check{P}(G)} [\ell(p)\Gamma_p] \right) \in \check{\mathcal{A}}(\mathcal{L})$$

Extend this contraction to linear combinations of graphs, linearly. Assume that the components of  $L$  are numbered in  $\underline{k}$ . To a pair  $(u, v)$  of  $\underline{k}$ , associate the univalent graph  $G_{uv} = \downarrow_{\bullet_u} \overset{v}{\leftarrow} \uparrow$  consisting of two distinct segments, the first one with its leg on  $K_u$ , and the second one with its leg on  $K_v$ . The legs are considered as distinct even if there is an automorphism of  $G_{uu} = \downarrow \downarrow_u$  that maps a leg to the other when  $K_u$  is a circle. We think of the leg of the first segment as the first leg of  $G_{uv}$ , and the other leg is the second leg of  $G_{uv}$ .

For a finite collection  $(A^{(i)'}/A^{(i)})_{i \in \underline{2x}}$  of disjoint rational LP-surgeries in  $\check{R}(\mathcal{C}) \setminus L$ , define the element

$$[\Gamma_{(2)}(\mathcal{C}, L; (A^{(i)'}/A^{(i)})_{i \in \underline{2x}})] = \frac{1}{2} \sum_{(u,v) \in \underline{k}^2} \left[ \left\langle \left\langle \bigsqcup_{i \in \underline{2x}} T(\mathcal{I}_{A^{(i)}A^{(i)'}) \sqcup G_{uv} \check{\rangle}\right\rangle \right]$$

of  $\check{\mathcal{A}}(\mathcal{L})$ .

**Examples 18.38.** Assume

$$T(\mathcal{I}_{A^{(1)}A^{(1)'}}) = \begin{matrix} & z_3^1 & \\ & \swarrow & \searrow \\ z_2^1 & & z_1^1 \\ & \nwarrow & \nearrow \end{matrix} \quad \text{and} \quad T(\mathcal{I}_{A^{(2)}A^{(2)'}}) = \begin{matrix} & z_3^2 & \\ & \swarrow & \searrow \\ z_2^2 & & z_1^2 \\ & \nwarrow & \nearrow \end{matrix},$$

where  $lk(z_i^1, z_j^2) = 0$  as soon as  $i \neq j$ , and  $lk(z_i^1, K_u) = lk(z_i^2, K_u) = 0$  if  $i \neq 3$ . If  $K_u$  is an interval, then we get

$$\left\langle \left\langle \bigsqcup_{i \in \underline{2}} T(\mathcal{I}_{A^{(i)}A^{(i)'}) \sqcup G_{uu} \check{\rangle}\right\rangle \right\rangle = -2lk(z_1^1, z_1^2) lk(z_2^1, z_2^2) lk(z_3^1, K_u) lk(z_3^2, K_u) \left[ \begin{matrix} K_u \\ \downarrow \\ \bullet \end{matrix} \right].$$

If  $K_u$  is a circle, then we get

$$\left\langle \left\langle \bigsqcup_{i \in \underline{2}} T(\mathcal{I}_{A^{(i)}A^{(i)'}) \sqcup G_{uu} \check{\rangle}\right\rangle \right\rangle = -2lk(z_1^1, z_1^2) lk(z_2^1, z_2^2) lk(z_3^1, K_u) lk(z_3^2, K_u) \left[ \begin{matrix} K_u \\ \circ \end{matrix} \right].$$

**Theorem 18.39.** *Let  $L: \mathcal{L} \hookrightarrow \mathcal{C}$  be a long tangle representative in a rational homology cylinder  $\mathcal{C}$ . Let  $\sqcup_{i=1}^{2x} A^{(i)}$  be a disjoint union of rational homology handlebodies embedded in  $\mathcal{C} \setminus L$ . Let  $(A^{(i)'}/A^{(i)})$  be rational LP-surgeries in  $\mathcal{C}$ . Set  $X = [\mathcal{C}, L; (A^{(i)'}/A^{(i)})_{i \in \underline{2x}}$ ] and*

$$\check{Z}_n(X) = \sum_{I \subseteq \underline{2x}} (-1)^{|I|} \check{Z}_n(\mathcal{C}_I, L),$$

where  $\mathcal{C}_I = \mathcal{C}((A^{(i)'}/A^{(i)})_{i \in I})$  is the rational homology cylinder obtained from  $\mathcal{C}$  by performing the LP-surgeries that replace  $A^{(i)}$  by  $A^{(i)'}$  for  $i \in I$ . Assume that  $x \neq 0$ . If  $n < x + 1$ , then  $\check{Z}_n(X)$  vanishes. If  $n = x + 1$ , then we have

$$\check{Z}_n(X) = \left[ \Gamma_{(2)}(\mathcal{C}, L; (A^{(i)'}/A^{(i)})_{i \in \underline{2x}}) \right],$$

where  $\check{Z}$  is defined in Proposition 17.28. See also Notation 6.19 and Notation 18.37.

**Remark 18.40.** This result holds modulo 1T when  $x = 0$ .

**PROOF OF THEOREM 18.39 ASSUMING PROPOSITION 18.36:** Recall that  $\check{Z}$  takes its values in  $\check{\mathcal{A}}(\mathcal{L})$ , where the diagrams with connected trivalent components vanish. Therefore, Theorem 18.6 implies the result when  $n < x + 1$ . Assume that  $n = x + 1$ . Since the framing correction terms involving  $\beta$  vanish in  $\check{\mathcal{A}}(\mathcal{L})$ , and since the other correction terms are the same for all the  $\check{Z}(\mathcal{C}_I, L)$ , we get

$$\check{Z}_{x+1}(X) = \sum_{I \subseteq \underline{2x}} (-1)^{|I|} \check{Z}_n(\mathcal{C}_I, L, (\omega_I)).$$

Let  $\Gamma$  be a Jacobi diagram of degree  $x + 1$  on  $\mathcal{L}$  that contributes to  $\check{Z}_{x+1}(X)$ . Since its class does not vanish in  $\check{\mathcal{A}}_{x+1}(\mathcal{L})$ , each component of  $\Gamma$  must contain at least two univalent vertices because one-leg diagrams vanish in  $\check{\mathcal{A}}(\mathcal{L})$ . Furthermore, as in the proof of Lemma 18.34, the configurations must involve at least one point in some  $A_I^{(i)}$ , for each  $i$  in  $\underline{2x}$ . Such a point must be a trivalent vertex position. Therefore, the graph  $\Gamma$  has at least  $2x$  trivalent vertices. Finally,  $\Gamma$  must be a connected two-leg Jacobi diagram of degree  $x + 1$  on  $\mathcal{L}$ . Assume that the univalent vertices of  $\Gamma$  are on components  $K_u$  and  $K_v$ . Order the set of univalent vertices of  $\Gamma$  so that the first is on  $K_u$  and the second is on  $K_v$ . Let  $\Gamma^U$  denote the graph  $\Gamma$  equipped with this order. Number the trivalent vertices of  $\Gamma$  in  $\underline{2x}$ . Orient the open configuration space  $\check{C}(\check{R}_I, L; \Gamma)$  as an open oriented submanifold of  $K_u \times K_v \times \check{R}_I^{2x}$  with respect to the order of  $V(\Gamma)$  induced by the numbering of its elements. Orient the edges

and the trivalent vertices of  $\Gamma$  so that the induced orientation of  $\check{C}(\check{R}_I, L; \Gamma)$  matches the previous one.

As in the proof of Lemma 18.34, we have

$$\Delta(\Gamma) = \sum_{I \subseteq \underline{2x}} (-1)^{|I|} I(R_I, L, \Gamma, o(\Gamma), (\omega_I)) [\Gamma] = \sum_{\sigma \in \mathfrak{S}_{2x}} \Delta_\sigma [\Gamma],$$

with

$$\Delta_\sigma = \sum_{I \subseteq \underline{2x}} (-1)^{|I|} \int_{D(I, \sigma)} \bigwedge_{e \in E(\Gamma)} p_e^*(\omega_I),$$

and

$$D(I, \sigma) = K_u \times K_v \times \prod_{i=1}^{2x} A_I^{(\sigma(i))},$$

when  $K_u \neq K_v$ , or when  $K_u$  is a closed component. When  $K_u = K_v$  and  $K_u$  is a long component, the factor  $K_u \times K_v$  must be replaced by one of its two subsets of pairs with a fixed order on  $K_u$ . When  $e$  is an oriented edge between two trivalent vertices, recall the expression of  $p_e^*(\omega_I)$  from Proposition 18.33. Without loss of generality, assume that the legs are the first half-edges of their edges. With the projections  $p_i : \check{C}(\check{R}_I, L; \Gamma) \rightarrow \check{R}_I$ , Proposition 18.36 implies

$$p_e^*(\omega_I) |_{D(I, \sigma)} = \sum_{j \in \underline{g_{\sigma(y(e))}}} p_{x(e)}^* \left( \eta(z_j^{\sigma(y(e))}) \right) \wedge p_{y(e)}^* \left( \eta(a_j^{\sigma(y(e))}) \right)$$

for an edge from a leg  $x(e)$  to a trivalent vertex  $y(e)$ . In this case, the edge  $e$  is associated to  $x(e)$  and denoted by  $e(x(e))$ . Let  $H_T(\Gamma)$  denote the subset of  $H(\Gamma)$  consisting of the half-edges that contain a trivalent vertex, and let  $E_T(\Gamma)$  denote the set of edges of  $\Gamma$  between trivalent vertices. Let  $F_\sigma$  denote the set of maps  $f$  from  $H_T(\Gamma)$  to  $\mathbb{N}$  such that  $f(c)$  belongs to  $\{1, 2, \dots, g_{\sigma(v(c))}\}$  for any  $c \in H_T(\Gamma)$ . We have

$$\Delta_\sigma = \sum_{f \in F_\sigma} \left( \prod_{e \in E_T(\Gamma)} lk \left( z_{f(x(e))}^{\sigma(x(e))}, z_{f(y(e))}^{\sigma(y(e))} \right) \right) I(f),$$

where  $I(f)$  is equal to

$$\int_{K_u} \eta \left( z_{f(y(u))}^{\sigma(y(u))} \right) \int_{K_v} \eta \left( z_{f(y(v))}^{\sigma(y(v))} \right) \times \prod_{i=1}^{2x} \int_{(A^{(\sigma(i))} \cup -A^{(\sigma(i))})'} \bigwedge_{c \in H(\Gamma) : v(c)=i} p_i^* \left( \eta(a_{f(c)}^{\sigma(i)}) \right),$$

when  $K_u \neq K_v$ , or when  $K_u$  is a closed component. Recall

$$\int_{K_u} \eta \left( z_{f(y(u))}^{\sigma(y(u))} \right) = lk \left( K_u, z_{f(y(u))}^{\sigma(y(u))} \right).$$

Summarizing, when  $K_u \neq K_v$  or when  $K_u$  is a closed component, we have

$$\Delta_\sigma = \sum_{f \in F_\sigma} \left( \left( \prod_{e \in E(\Gamma)} lk(e; f) \right) \left( \prod_{i \in \underline{2x}} \mathcal{I}_{A(\sigma(i))A(\sigma(i))} \left( \bigotimes_{c \in v^{-1}(i)} a_{f(c)}^{\sigma(i)} \right) \right) \right),$$

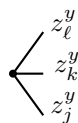
where  $lk(e; f) = lk \left( z_{f(x(e))}^{\sigma(x(e))}, z_{f(y(e))}^{\sigma(y(e))} \right)$  when  $e \in E_T(\Gamma)$ , and

$$lk(e; f) = lk \left( K_{x(e)}, z_{f(y(e))}^{\sigma(y(e))} \right)$$

when  $x(e)$  is univalent.

Finally,  $\Delta(\Gamma)$  is a sum, running over all the ways of renumbering the trivalent vertices of  $\Gamma$  by elements of  $\underline{2x}$  (via  $\sigma$ ), and of coloring the half-edges  $c$  of  $v^{-1}(i)$  by three distinct curves  $z_{f(c)}^{\sigma(i)}$  via  $f$ .

In particular, a pair  $(\sigma, f)$  provides a tripod



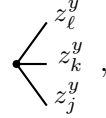
for any  $y \in \underline{2x}$  such that  $1 \leq j < k < \ell \leq g_y$ . It also provides a pairing of the ends of the univalent vertices of the tripods and of the legs on  $K_u$  and  $K_v$  (the first one and the second one when  $v = u$ ), which gives rise to the graph  $\Gamma$  with a possibly different vertex-orientation. The vertices of the obtained graph are furthermore numbered by the numbering of the vertices of  $\Gamma$ , and its edges are identified with the original edges of  $\Gamma$ . The order of  $U(\Gamma)$  is induced by the order on the legs of  $G_{uv}$ .

Let  $\text{Aut}(\Gamma^U)$  denote the set of automorphisms of  $\Gamma$  that fix the univalent vertices of  $\Gamma$  (pointwise). (The set  $\text{Aut}(\Gamma^U)$  is distinct from  $\text{Aut}(\Gamma)$  if and only if  $u=v$ ,  $K_u$  is a closed component, and there exists an automorphism of  $\Gamma$  that exchanges its two univalent vertices.)

Fix a set of tripods associated to the elements of  $\underline{2x}$  as above and a pairing of their univalent vertices and the legs on  $K_u$  and  $K_v$  (which are distinguished as the first one and the second one when  $v = u$ ). Then there are exactly  $|\text{Aut}(\Gamma^U)|$  ways of numbering its vertices and edges to get a graph isomorphic to  $\Gamma^U$  by an isomorphism that fixes the univalent vertices. So the pairing occurs  $|\text{Aut}(\Gamma^U)|$  times.



Let  $G$  be a family of  $2x$  tripods



one for each  $y \in \underline{2x}$ , such that  $1 \leq j < k < \ell \leq g_y$ . Define

$$[\langle\langle G \sqcup G_{uv} \rangle\rangle_{\Gamma^U}] = \sum_{p \in \check{P}(G \sqcup G_{uv}) : \Gamma_p \text{ isomorphic to } \Gamma^U} [\ell(p)\Gamma_p],$$

where the sum runs over the  $p$  such that  $\Gamma_p$  is isomorphic to  $\Gamma^U$ , as a nonoriented uni-trivalent graph on  $\mathcal{L}$ , equipped with a fixed order on  $U(\Gamma)$ , with the notation introduced before the statement of Theorem 18.39. Similarly define

$$[\langle\langle G \sqcup G_{uv} \rangle\rangle_{\Gamma}] = \sum_{p \in \check{P}(G \sqcup G_{uv}) : \Gamma_p \text{ isomorphic to } \Gamma} [\ell(p)\Gamma_p],$$

where the sum runs over the  $p$  such that  $\Gamma_p$  is isomorphic to  $\Gamma$ , as a nonoriented uni-trivalent graph on  $\mathcal{L}$ . (We forget the order on  $U(\Gamma)$ .)

We get

$$\Delta(\Gamma) = |\text{Aut}(\Gamma^U)| \left[ \left\langle \left\langle \bigsqcup_{i \in \underline{2x}} T(\mathcal{I}_{A^{(i)}A^{(i)'}}) \sqcup G_{uv} \right\rangle \right\rangle_{\Gamma^U} \right].$$

If  $u \neq v$ , then  $\Gamma^U$  and  $\Gamma$  coincide, and we have

$$\frac{\Delta(\Gamma)}{|\text{Aut}(\Gamma)|} = \frac{1}{2} \left( \left[ \left\langle \left\langle \bigsqcup_{i \in \underline{2x}} T(\mathcal{I}_{A^{(i)}A^{(i)'}}) \sqcup G_{uv} \right\rangle \right\rangle_{\Gamma} \right] + \left[ \left\langle \left\langle \bigsqcup_{i \in \underline{2x}} T(\mathcal{I}_{A^{(i)}A^{(i)'}}) \sqcup G_{vu} \right\rangle \right\rangle_{\Gamma} \right] \right).$$

Assume that  $u = v$  and that  $K_u$  is a circle. If there exists an automorphism of  $\Gamma$  that exchanges its univalent vertices, then we have  $|\text{Aut}(\Gamma)| = 2|\text{Aut}(\Gamma^U)|$  and  $\langle\langle \cdot \rangle\rangle_{\Gamma^U} = \langle\langle \cdot \rangle\rangle_{\Gamma}$ . Otherwise, we have  $|\text{Aut}(\Gamma)| = |\text{Aut}(\Gamma^U)|$  and  $[\langle\langle \cdot \rangle\rangle_{\Gamma}] = 2[\langle\langle \cdot \rangle\rangle_{\Gamma^U}]$ . So we obtain

$$\Delta(\Gamma) = \frac{1}{2} |\text{Aut}(\Gamma)| \left[ \left\langle \left\langle \bigsqcup_{i \in \underline{2x}} T(\mathcal{I}_{A^{(i)}A^{(i)'}}) \sqcup G_{uu} \right\rangle \right\rangle_{\Gamma} \right]$$

in any case.

It remains to study the case in which the univalent vertices of  $\Gamma$  belong to the same noncompact component  $K_u$ . In this case, we compute the sum

$\Delta(\Gamma) + \Delta(\Gamma^s)$ , where  $\Gamma^s$  is obtained from  $\Gamma$  by exchanging the order of its univalent vertices on  $K_u$ .

Again, we find

$$\Delta(\Gamma) + \Delta(\Gamma^s) = |\text{Aut}(\Gamma)| \left[ \left\langle \left\langle \bigsqcup_{i \in 2x} T(\mathcal{I}_{A^{(i)} A^{(i)'}}) \sqcup G_{uu} \right\rangle \right\rangle_{\Gamma} \right],$$

where the contraction  $\langle\langle \cdot \rangle\rangle_{\Gamma}$  keeps only the graphs that are isomorphic to  $\Gamma$  (as a graph with an ordered pair of free legs). (Recall  $[\Gamma^s] = [\Gamma]$  in  $\check{\mathcal{A}}(\mathcal{L})$ ). If  $\Gamma$  and  $\Gamma^s$  are isomorphic, then we have  $\Delta(\Gamma) = \Delta(\Gamma^s)$  and

$$\Delta(\Gamma) = \frac{1}{2} |\text{Aut}(\Gamma)| \left[ \left\langle \left\langle \bigsqcup_{i \in 2x} T(\mathcal{I}_{A^{(i)} A^{(i)'}}) \sqcup G_{uu} \right\rangle \right\rangle_{\Gamma} \right].$$

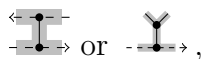
Otherwise, we have  $|\text{Aut}(\Gamma)| = |\text{Aut}(\Gamma^s)|$  and  $[\langle\langle \cdot \sqcup G_{uu} \rangle\rangle_{\Gamma}] = [\langle\langle \cdot \sqcup G_{uu} \rangle\rangle_{\Gamma^s}]$ . So we get

$$\sum_{\substack{\Gamma \text{ as above} \\ \text{with 2 univalent vertices on } K_u}} \frac{\Delta(\Gamma)}{|\text{Aut}(\Gamma)|} = \frac{1}{2} \left[ \left\langle \left\langle \bigsqcup_{i \in 2x} T(\mathcal{I}_{A^{(i)} A^{(i)'}}) \sqcup G_{uu} \right\rangle \right\rangle \right]$$

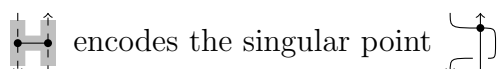
in any case. □

As in Section 18.4, we can associate an alternate sum of tangles to a framed embedding of a Jacobi diagram on a tangle  $L$  into a rational homology cylinder.

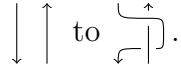
Let  $\Gamma$  be such a Jacobi diagram, whose trivalent vertices are numbered in  $\underline{x}$ . Let  $\Sigma(\Gamma)$  be an oriented surface containing  $\Gamma$  in its interior such that  $\Sigma(\Gamma)$  is a regular neighborhood of  $\Gamma$  in  $\Sigma(\Gamma)$ . Equip  $\Gamma$  with its vertex-orientation induced by the orientation of  $\Sigma(\Gamma)$ . Embed  $\Sigma(\Gamma)$  in  $\mathcal{C}$ , so that  $L$  intersects  $\Sigma(\Gamma)$  near univalent vertices as in



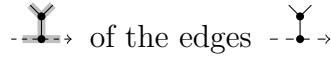
$L$  is tangent to  $\Sigma(\Gamma)$  at univalent vertices, and  $L$  does not meet  $\Sigma(\Gamma)$  outside such neighborhoods of the univalent vertices. Note that the embedding of  $\Sigma(\Gamma)$  induces a local orientation of  $L$  as in Definition 6.16. Replace a chord between two univalent vertices by a crossing change so that



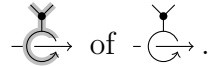
associated to the positive crossing change from



Replace neighborhoods



between a trivalent vertex and a univalent vertex by neighborhoods



Finally, replace neighborhoods of the edges between trivalent vertices as in Section 18.4.

Thus,  $\Sigma(\Gamma)$  transforms  $L$  into a singular tangle  $L^s$  whose double points are associated to the chords of  $\Gamma$  as above, equipped with a collection of disjoint oriented surfaces

$$\Sigma(Y) = \begin{array}{c} \circ \\ \diagdown \quad \diagup \\ \circ \quad \circ \end{array},$$

associated to trivalent vertices of  $\Gamma$ . The oriented surfaces  $\Sigma(Y)$  are next thickened to become framed genus 3 handlebodies.

Define  $\psi(\Sigma(\Gamma))$  to be  $[\mathcal{C}, L^s; (A^{(i)'}/A^{(i)})_{i \in \underline{x}}]$ , where the surgeries  $(A^{(i)'}/A^{(i)})$  associated to the trivalent vertices of  $\Gamma$  are defined as in Section 18.4. Set  $\check{Z}(\psi(\Sigma(\Gamma))) = \sum_{I \subseteq \underline{x}} (-1)^{x+|I|} \check{Z}_n(\mathcal{C}_I, L^s)$ , where  $\mathcal{C}_I = \mathcal{C}((A^{(i)'}/A^{(i)})_{i \in I})$ .

**Corollary 18.41.** *Let  $n$  be a positive integer. Let  $\Gamma$  be a degree  $n$  Jacobi diagram with two univalent vertices. Let  $\Sigma(\Gamma)$  be a regular neighborhood of  $\Gamma$  embedded in  $\mathcal{C}$  as above. Then we have  $\check{Z}_{\leq n}(\psi(\Sigma(\Gamma))) = [\Gamma]$ .*

PROOF: This follows from Theorem 18.39 as in the proof of Lemma 18.24. □

**Remark 18.42.** This corollary could be true for more general Jacobi diagrams.

To finish this section, we apply Theorem 18.39 with the LP-surgery of Subsection 18.3.1 to compute the degree 2 part of  $\check{Z}$  for a null-homologous knot and prove Theorem 18.43 below. Recall the definition of  $a_2(K)$  from the statement of Proposition 18.12 and the lines that follow.

**Theorem 18.43.** *Let  $K$  be a null-homologous knot in a rational homology sphere  $R$ . Then we have*

$$\check{Z}_2(R, K) = \left( \frac{1}{24} - a_2(K) \right) \left[ \begin{array}{c} \circ \\ \diagdown \quad \diagup \\ \circ \quad \circ \end{array} \right].$$

PROOF: As in the proof of Proposition 18.12 in Subsection 18.3.2, let  $K$  bound a Seifert surface  $\Sigma$  equipped with a symplectic basis  $(x_i, y_i)_{i \in g}$  in  $R$ . Let  $\Sigma \times [-1, 2]$  be a collar of  $\Sigma$  in  $R$ , and let  $(A'/A) = (A'_F/A_F)$  and  $(B'/B) = (B'_F/B_F)$  be the LP-surgeries of Subsection 18.3.1. Let  $U$  be a meridian of  $K$  passing through  $d \times [-1, 2]$ . According to Proposition 18.16,  $(R(A'/A), U)$  is diffeomorphic to  $(R, K)$ . Similarly,  $(R(B'/B), U)$  is diffeomorphic to  $(R, -K)$ , while  $(R(A'/A, B'/B), U)$  is diffeomorphic to  $(R, U)$ . So we have

$$\begin{aligned} \check{Z}_2([R, U; A'/A, B'/B]) &= 2\check{Z}_2(R, U) - \check{Z}_2(R, K) - \check{Z}_2(R, -K) \\ &= [\Gamma_{(2)}(R, U; A'/A, B'/B)], \end{aligned}$$

with

$$[\Gamma_{(2)}(R, U; A'/A, B'/B)] = \frac{1}{2} \left[ \left\langle \left\langle T(\mathcal{I}_{AA'}) \sqcup T(\mathcal{I}_{BB'}) \sqcup \text{circle with arrow} \right\rangle \right\rangle \right] \in \mathcal{A}_2(\mathcal{L}),$$



$$T(\mathcal{I}_{AA'}) = \sum_{i=1}^g \left\langle \begin{array}{l} x_i \\ y_i \\ c \end{array} \right\rangle,$$

and

$$T(\mathcal{I}_{BB'}) = \sum_{j=1}^g \left\langle \begin{array}{l} x_j^+ \\ y_j^+ \\ c^+ \end{array} \right\rangle,$$

according to Lemma 18.18. We get

$$[\Gamma_{(2)}(R, U; A'/A, B'/B)] = a_2(K) \left[ \text{circle with arrow and chord} \right].$$

Since  $\check{\mathcal{A}}_2(S^1)$  is generated by the chord diagrams  and , which are symmetric with respect to the orientation change on  $S^1$ , we have  $\check{Z}_2(R, K) = \check{Z}_2(R, -K)$ . So we obtain

$$2\check{Z}_2(R, U) - 2\check{Z}_2(R, K) = a_2(K) \left[ \text{circle with arrow and chord} \right] = 2a_2(K) \left[ \text{circle with arrow and chord, rotated} \right].$$

According to Example 7.22 and the multiplicativity of  $\mathcal{Z}$  under connected sum of Theorem 10.26, we get  $\check{Z}_2(R, U) = \frac{1}{24} \left[ \text{circle with arrow and chord, rotated} \right]$ .  $\square$

**Remark 18.44.** This theorem generalizes a result of Enore Guadagnini, Maurizio Martellini, and Mihail Mintchev in [GMM90] for the case  $R = S^3$ , to any rational homology sphere  $R$ . In the case of  $S^3$ , the known proof relies on the facts that  $\check{Z}_2$  is of degree 2 and that the space of real-valued knot

invariants of degree at most 2 is generated by  $a_2$  and a constant nonzero invariant. This uses the fact that any knot can be unknotted by crossing changes. This is no longer true in general rational homology spheres since crossing changes do not change the homotopy class. Our proof is more direct, and our result holds for any null-homologous knot in a rational homology sphere.

Recall the Conway weight system  $w_C$  from Example 6.11, the Alexander polynomial of Definition 18.13, and Proposition 6.21. A *long knot*  $\check{K}$  of a  $\mathbb{Q}$ -sphere  $R(\mathcal{C})$  is an embedding of  $\mathbb{R}$  into  $\check{R}(\mathcal{C})$ , whose image intersects the complement of  $\mathcal{C}$ , as the vertical embedding ( $j_{\mathbb{R}}: t \mapsto (0, 0, t)$ ) does. Replace  $j_{\mathbb{R}}(\mathbb{R} \setminus ]0, 1[)$  by an arc of  $\check{R} \setminus \mathring{\mathcal{C}}$  from  $(0, 0, 1)$  to  $(0, 0, 0)$ , which cobounds an embedded topological disk in  $\check{R} \setminus \mathring{\mathcal{C}}$  with an arc of  $\partial\mathcal{C}$  with the same ends. This provides a knot  $\overline{\check{K}}$ , whose isotopy class is well-defined. In [Let22], David Leturcq proves the following theorem. Together with the properties of the functor  $\mathcal{Z}^f$  of the third part of the book, this Leturcq theorem implies Theorem 18.43.

**Theorem 18.45** (Leturcq). *For any long knot  $\check{K}$  in a rational homology sphere  $\check{R}$ , such that  $\overline{\check{K}}$  is null-homologous, we have the following equality in  $\mathbb{R}[[h]]$ :*

$$\sum_{n \in \mathbb{N}} w_C(\check{Z}_n(R, \check{K})) h^n = \Delta_{\overline{\check{K}}}(\exp(h)).$$

Leturcq's proof of this theorem relies on a direct computation with appropriate propagating forms. In [Let23], David Leturcq obtains a similar theorem for the Bott–Cattaneo–Rossi invariants of higher dimensional knots [CR05, Let21]. This Leturcq theorem in higher dimensions generalizes a theorem of Tadayuki Watanabe [Wat07], who proved it for ribbon knots.

# Chapter 19

## More flexible definitions of $\mathcal{Z}$ using pseudo-parallelizations

This chapter presents the *pseudo-parallelizations* introduced in Section 18.7 and defined in Section 19.2. Our proof of the universality theorem 18.6 uses these generalizations of parallelizations first introduced in [Les04b, Section 4.3 and 4.2] and furthermore studied in [Les10, Section 10] and [Les13, Sections 7 to 10].

Pseudo-parallelizations allow us to give more flexible definitions for our invariants  $\mathcal{Z}$  and  $\check{\mathcal{Z}}$ . In Theorem 19.17, we generalize the definition of the invariant  $\mathcal{Z}$  of Theorem 12.7, and thus the definition of the  $q$ -tangle invariant  $\mathcal{Z}^f$  of Definition 13.10, by allowing propagating forms associated with pseudo-parallelizations. We conclude the proof of Theorem 19.17 in Section 19.5. This proof involves all the previous sections.

In Chapter 21, we present variants of the definition of  $\mathcal{Z}^f$  involving nonhomogeneous propagating forms or propagating chains associated with pseudo-parallelizations. We will not use these variants in the proof of Theorem 18.6.

### 19.1 Why we need pseudo-parallelizations

This section explains why a parallelization of the exterior of a  $\mathbb{Q}$ -handlebody  $A$  does not necessarily extend to  $A'$  after a rational LP-surgery ( $A'/A$ ). It justifies why I could not avoid this chapter and some of its difficulties.

**Notation 19.1.** Let  $\alpha$  be a smooth map from  $[-1, 1]$  to  $[0, 2\pi]$  such that  $\alpha$  maps  $[-1, -1 + \varepsilon]$  to 0 for some  $\varepsilon \in ]0, \frac{1}{8}[$ , it increases from 0 to  $2\pi$  on  $[-1 + \varepsilon, 1 - \varepsilon]$ , and  $\alpha(-u) + \alpha(u) = 2\pi$  for any  $u \in [-1, 1]$ .

Let  $\Sigma$  be a surface, orientable or not. Let  $\gamma$  be a two-sided curve properly embedded in  $\Sigma$  and equipped with a collar  $\gamma \times [-1, 1]$  in  $\Sigma$ . Let  $\rho_{\alpha(u)} =$

$\rho(\alpha(u), (0, 0, 1))$  denote the rotation of  $\mathbb{R}^3$  of angle  $\alpha(u)$  whose axis is directed by  $(0, 0, 1)$ . Then  $\mathcal{R}_\gamma$  denotes the map from  $\Sigma$  to  $SO(3)$  sending  $\Sigma \setminus (\gamma \times [-1, 1])$  to  $\mathbf{1}$  and satisfying

$$\mathcal{R}_\gamma(c \in \gamma, u \in [-1, 1]) = \rho_{\alpha(u)}.$$

The homotopy class of  $\mathcal{R}_\gamma$  is well-defined.

**Lemma 19.2.** *Let  $\mathbb{R}P_\pi^2$  denote the nonorientable submanifold of  $SO(3)$  of the rotations of angle  $\pi$ . Let  $\Sigma$  be a connected surface, orientable or not. Let  $f: \partial\Sigma \rightarrow SO(3)$  be a map transverse to  $\mathbb{R}P_\pi^2$ . Then  $f$  extends to  $\Sigma$  if and only if the  $\mathbb{Z}/2\mathbb{Z}$ -valued algebraic intersection  $\langle f(\partial\Sigma), \mathbb{R}P_\pi^2 \rangle_{SO(3)}$  of  $f(\partial\Sigma)$  and  $\mathbb{R}P_\pi^2$  in  $SO(3)$  is zero.*

**Remark 19.3.** Recall  $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$  from Section 4.2. The homotopy class of a loop of  $SO(3)$  is determined by its  $\mathbb{Z}/2\mathbb{Z}$ -valued algebraic intersection with  $\mathbb{R}P_\pi^2$ . This proves Lemma 19.2 when  $\Sigma$  is a disk.

**PROOF OF LEMMA 19.2:** When  $f$  extends to  $\Sigma$ , we may choose an extension  $f_\Sigma$  transverse to  $\mathbb{R}P_\pi^2$ . Then  $f(\partial\Sigma) \cap \mathbb{R}P_\pi^2$  bounds the one-manifold  $f_\Sigma(\Sigma) \cap \mathbb{R}P_\pi^2$ . So the  $\mathbb{Z}/2\mathbb{Z}$ -valued intersection  $\langle f(\partial\Sigma), \mathbb{R}P_\pi^2 \rangle_{SO(3)}$  is zero.

Conversely, assume that  $\langle f(\partial\Sigma), \mathbb{R}P_\pi^2 \rangle_{SO(3)}$  is zero. Then there is a disjoint union  $\gamma$  of intervals embedded in  $\Sigma$  and transverse to  $\partial\Sigma$  with the following properties.

- The boundary  $\partial\gamma$  of  $\gamma$  is in  $\partial\Sigma$ .
- The connected components  $K$  of  $\partial\Sigma$  such that  $\langle f(K), \mathbb{R}P_\pi^2 \rangle = 0$  do not meet  $\gamma$ .
- The connected components  $K$  of  $\partial\Sigma$  such that  $\langle f(K), \mathbb{R}P_\pi^2 \rangle = 1$  meet  $\gamma$  at one point (of  $\partial\gamma$ ).

Then the restriction to  $\partial\Sigma$  of the map  $\mathcal{R}_\gamma$  associated to  $(\Sigma, \gamma)$  as in Notation 19.1 is homotopic to  $f$ . Therefore,  $f$  extends to  $\Sigma$ .  $\square$

Recall that a *framed knot* is a knot equipped with a parallel (up to homotopy). Equivalently, it is a knot equipped with a normal nonzero vector field  $\vec{n}_K$ . Let  $K$  be framed knot in an oriented 3-manifold  $M$ . Let  $\vec{t}_K$  be a tangent vector of  $K$  that induces the orientation of  $K$ . These data induce the direct trivialization  $\tau_K$  of  $TM|_K$  (up to homotopy) such that  $\tau_K(e_1) = \vec{t}_K$  and  $\tau_K(e_2) = \vec{n}_K$ . The homotopy class of the trivialization  $\tau_K$  is well-defined and does not depend on the orientation of  $K$ .

**Lemma 19.4.** *Let  $K$  be a framed knot bounding a possibly nonorientable compact surface  $\Sigma$  in an oriented 3-manifold  $M$ . Assume that  $\Sigma$  (or, more precisely, the parallel of  $K$  on  $\Sigma$ ) induces the framing of  $K$ . Let  $\tau$  be a trivialization of the tangent space of  $M$  over  $\Sigma$ . Then the restriction of  $\tau$  to  $K$  is not homotopic to  $\tau_K$ .*

PROOF: Let us prove that the homotopy class of the restriction of  $\tau$  to  $K$  is independent of the trivialization  $\tau$  of  $TM$  over  $\Sigma$ . Any other trivialization of  $TM$  over  $\Sigma$  may be written as  $\tau \circ \psi_{\mathbb{R}}(f)$  for a map  $f$  from  $\Sigma$  to  $SO(3)$ , with the notation of Section 4.2. Lemma 19.2 and Remark 19.3 imply that the restriction  $f|_K$  of such a map  $f$  is homotopically trivial. Thus, the homotopy class of the restriction of  $\tau$  to  $K$  is independent of the trivialization  $\tau$  of  $TM$  over  $\Sigma$ . It is also independent of the oriented 3-manifold  $M$  that contains  $\Sigma$ . Since the tangent bundle of an oriented 3-manifold over a possibly nonorientable closed surface is trivializable, the homotopy class of the restriction of  $\tau$  to  $K$  is also independent of  $\Sigma$ . Therefore, it is enough to prove the lemma when  $\Sigma$  is a disk, and it is obvious in this case.  $\square$

**Definition 19.5.** If  $(K, \tau_K)$  is a framed knot in an oriented 3-manifold  $M$  and if  $\tau$  is a trivialization of the restriction of  $TM$  to  $K$ , we say that  $K$  is  $\tau$ -*bounding* if  $\tau$  is not homotopic to  $\tau_K$ . (This notion is independent of the whole manifold  $M$ , depending only on what happens in a tubular neighborhood of  $K$ .)

**Definition 19.6.** Let  $\Sigma$  be a surface, orientable or not. Let  $\gamma$  be a two-sided curve properly embedded in  $\Sigma$ . Recall the map  $\mathcal{R}_\gamma$  from  $\Sigma$  to  $SO(3)$  from Notation 19.1. Define the *twist map*  $\mathcal{T}_\gamma$  across  $\gamma$  to be the map from  $\Sigma \times \mathbb{R}^3$  to itself such that

$$\mathcal{T}_\gamma(y \in \Sigma; X \in \mathbb{R}^3) = (y; \mathcal{R}_\gamma(y)(X)).$$

Assume that  $\Sigma$  is embedded in an oriented 3-manifold  $M$ . For a trivialization  $\tau$  of  $TM|_\Sigma$ , the *twist of  $\tau|_\Sigma$  across  $\gamma$*  is the trivialization  $\tau|_\Sigma \circ \mathcal{T}_\gamma$  (defined up to homotopy).

Let  $A$  be a compact oriented connected 3-manifold with boundary  $\partial A$ . Define the  $\mathbb{Z}/2\mathbb{Z}$ -Lagrangian  $\mathcal{L}_A^{\mathbb{Z}/2\mathbb{Z}}$  of  $A$  to be the kernel

$$\mathcal{L}_A^{\mathbb{Z}/2\mathbb{Z}} = \text{Ker}(H_1(\partial A; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H_1(A; \mathbb{Z}/2\mathbb{Z}))$$

of the map induced by the inclusion map. This is a Lagrangian subspace of  $(H_1(\partial A; \mathbb{Z}/2\mathbb{Z}); \langle \cdot, \cdot \rangle)$ .

The curves of an oriented compact surface  $F$  embedded in an oriented 3-manifold are naturally framed by the surface  $F$ : They are framed by a nonzero normal vector field tangent to  $F$ .



**Proposition 19.7.** *Let  $\partial A$  be a connected oriented compact surface. Let  $\tau$  be a trivialization of  $T(\partial A \times [-2, 2])$ . Then there exists a unique map*

$$\phi_\tau: H_1(\partial A; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \frac{\mathbb{Z}}{2\mathbb{Z}}$$

such that

1. *for a connected curve  $x$  of  $\partial A = \partial A \times \{0\}$ , we have  $\phi_\tau(x) = 0$  if and only if  $x$  (framed by  $\partial A$ ) is  $\tau$ -bounding and,*
2. *for any pair  $(x, y)$  of elements of  $H_1(\partial A; \mathbb{Z}/2\mathbb{Z})$ , we have*

$$\phi_\tau(x + y) = \phi_\tau(x) + \phi_\tau(y) + \langle x, y \rangle_{\partial A}.$$

The map  $\phi_\tau$  satisfies the following properties.

- *Let  $x$  be a disjoint union of curves in  $\partial A$ . Assume that  $\partial A \times [-2, 2]$  is embedded in an oriented 3-manifold  $M$ , where  $x$  bounds a connected surface  $\Sigma$ , orientable or not. If  $\Sigma$  and  $\partial A$  induce the same framing of  $x$ , then  $\tau|_x$  extends to  $\Sigma$  as a trivialization of  $TM|_\Sigma$  if and only if  $\phi_\tau(x)$  is zero.*
- *Let  $c$  be curve of  $\partial A$ . Let  $\mathcal{T}_c$  denote the twist across  $c$  of Definition 19.6. For any  $x \in H_1(\partial A; \mathbb{Z}/2\mathbb{Z})$ , we have*

$$\phi_{\tau \circ \mathcal{T}_c}(x) = \phi_\tau(x) + \langle x, c \rangle_{\partial A}.$$

- *When  $A$  is a compact oriented connected 3-manifold with boundary  $\partial A$ , the trivialization  $\tau$  extends as a trivialization over  $A$  if and only if  $\phi_\tau(\mathcal{L}_A^{\mathbb{Z}/2\mathbb{Z}}) = \{0\}$ .*

**PROOF:** For a disjoint union  $x = \sqcup_{i=1}^n x_i$  of connected curves  $x_i$  on  $\partial A$ , define  $\phi_\tau(x) = \sum_{i=1}^n \phi_\tau(x_i)$  in  $\mathbb{Z}/2\mathbb{Z}$ , with  $\phi_\tau(x_i) = 0$  if  $x_i$  (framed by  $\partial A$ ) is  $\tau$ -bounding, and  $\phi_\tau(x_i) = 1$  otherwise.

Assume that such a framed disjoint union  $x$  bounds a connected surface  $\Sigma$  in an oriented 3-manifold  $M$ . Also assume that  $\Sigma$  and  $\partial A$  induce the same framing of  $x$ . Let us prove that  $\tau|_x$  extends to  $\Sigma$  as a trivialization of  $TM|_\Sigma$  if and only if  $\phi_\tau(x)$  is zero.

When  $\phi_\tau(x) = 0$ , group all the curves  $x_i$  such that  $\phi_\tau(x_i) = 1$  by pairs. Make each such pair bound an annulus that induces the framing. Make each curve  $x_i$  such that  $\phi_\tau(x_i) = 0$  bound a disk which induces the framing. Let  $\hat{\Sigma}$  denote the union of  $\Sigma$  with the above disks and annuli. The restriction to  $\hat{\Sigma}$  of the tangent bundle of an oriented 3-manifold  $M$  in which  $\hat{\Sigma}$  embeds is

independent of  $M$ . It admits a trivialization  $\hat{\tau}$ . Extend  $\tau$  to  $\hat{\Sigma} \setminus \mathring{\Sigma}$ . Write the restriction of  $\hat{\tau}$  to  $(\hat{\Sigma} \setminus \mathring{\Sigma}) \times \mathbb{R}^3$  as  $\hat{\tau} = \tau \circ \psi_{\mathbb{R}}(f)$  for a map  $f$  from  $\hat{\Sigma} \setminus \mathring{\Sigma}$  to  $SO(3)$ . According to Lemma 19.2, the intersection of  $f(x)$  and  $\mathbb{R}P_{\pi}^2$  is zero, and the map  $f$  extends to  $\hat{\Sigma}$ . Thus  $\tau = \hat{\tau} \circ \psi_{\mathbb{R}}(f)^{-1}$  also extends to  $\Sigma$ .

When  $\phi_{\tau}(x) = 1$ , assume that  $\phi_{\tau}(x_1) = 1$  without loss of generality. Group the other curves  $x_i$  such that  $\phi_{\tau}(x_i) = 1$  pairwise, make them cobound a disjoint union of annuli, and make the curves  $x_i$  such that  $\phi_{\tau}(x_i) = 0$  bound disks, in a framed way as above. Let  $\hat{\Sigma}$  denote the union of  $\Sigma$  with these disks and these annuli. The boundary of  $\hat{\Sigma}$  is  $x_1$ . The trivialization  $\tau$  still extends to  $\hat{\Sigma} \setminus \mathring{\Sigma}$ . If  $\tau$  extends to  $\Sigma$ , then it extends to  $\hat{\Sigma}$ , and  $x_1$  is  $\tau$ -bounding. So  $\phi_{\tau}(x_1) = 0$ , which is absurd. Therefore,  $\tau$  does not extend to  $\Sigma$ .

Let us prove that our above definition of  $\phi_{\tau}(x)$  depends only on the class of  $x$  in  $H_1(\partial A; \mathbb{Z}/2\mathbb{Z})$ . Let  $x$  be an embedded (possibly nonconnected) curve in  $\partial A$ . Let  $y$  be another such curve in  $\partial A \times \{-1\}$  homologous to  $x$  modulo  $2H_1(\partial A; \mathbb{Z})$ . Then there exists a framed (possibly nonorientable) cobordism between  $x$  and  $y$  in  $\partial A \times [-1, 1]$ , and it is easy to see that  $\phi_{\tau}(x) = 0$  if and only if  $\phi_{\tau}(y) = 0$ .

Let us check that  $\phi_{\tau}$  behaves as predicted under addition. Because we are dealing with elements of  $H_1(\partial A; \mathbb{Z}/2\mathbb{Z})$ , we can consider representatives  $x$  and  $y$  of  $x$  and  $y$  such that  $x$  is connected and intersects  $y$  at most once. Next, the known additivity under disjoint union reduces the proof to the case in which  $x$  and  $y$  are connected and  $x$  and  $y$  intersect once. Note that both sides of the equality to be proved vary in the same way under trivialization changes. Consider the punctured torus neighborhood of  $x \cup y$  and a trivialization  $\tau$  that restricts to the punctured torus as the direct sum of a parallelization of the torus and the normal vector to  $\partial A$ . Then we have  $\phi_{\tau}(x + y) = \phi_{\tau}(x) = \phi_{\tau}(y) = 1$ . We leave the last two assertions to the reader.  $\square$

**Example 19.8.** For any  $\mathbb{Q}$ -handlebody  $A$ , there exists a Lagrangian subspace  $\mathcal{L}^{\mathbb{Z}}$  of  $(H_1(\partial A; \mathbb{Z}); \langle \cdot, \cdot \rangle)$ , such that  $\mathcal{L}_A = \mathcal{L}^{\mathbb{Z}} \otimes \mathbb{Q}$ . The following example shows that  $\mathcal{L}_A^{\mathbb{Z}/2\mathbb{Z}}$  may differ from  $\mathcal{L}^{\mathbb{Z}} \otimes \mathbb{Z}/2\mathbb{Z}$ .

Let  $\mathcal{M}$  be a Möbius band embedded in the interior of a solid torus  $D^2 \times S^1$  so that the core of the solid torus is the core of  $\mathcal{M}$ . Embed  $D^2 \times S^1$  into  $S^2 \times S^1 = D^2 \times S^1 \cup_{\partial D^2 \times S^1} (-D^2 \times S^1)$  as the first copy. Orient the knot  $\partial \mathcal{M}$  so that  $\partial \mathcal{M}$  pierces twice  $S^2 \times 1$  positively. Let  $m$  be the meridian of  $\partial \mathcal{M}$ . Let  $\ell$  be the parallel of  $\partial \mathcal{M}$  induced by  $\mathcal{M}$ . Let  $A$  be the exterior of the knot  $\partial \mathcal{M}$  in  $S^2 \times S^1$ . The reader can check that  $A$  is a  $\mathbb{Q}$ -handlebody, as an exercise. Observe  $\mathcal{L}_A^{\mathbb{Z}} = \mathbb{Z}[2m]$ ,  $\mathcal{L}_A = \mathbb{Q}[m]$ , and  $\mathcal{L}_A^{\mathbb{Z}/2\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z}[\ell]$ .

Let  $A_0$  be the solid torus with boundary  $\partial A = \partial A_0$ , where  $m$  bounds a disk. We have  $\mathcal{L}_{A_0} = \mathbb{Q}[m]$ . Equip  $A_0$  with a parallelization  $\tau$  such that

$\phi_\tau(\ell) = 1$ . According to Proposition 19.7, the restriction to  $\partial A$  of  $\tau$  does not extend to  $A$ .

## 19.2 Definition of pseudo-parallelizations

**Definition 19.9.** Recall  $\varepsilon$  and the maps  $\alpha$  and  $\rho_{\alpha(u)}$  from Notation 19.1. Set  $N(\partial[-1, 1]) = [-1, 1] \setminus ]-1 + \varepsilon, 1 - \varepsilon[$ . A *pseudo-parallelization*  $\tilde{\tau} = (N(\gamma); \tau_e, \tau_b)$  of an oriented 3-manifold  $A$  with possible boundary consists of

- a framed link  $\gamma$  of the interior of  $A$ , which will be called *the link of the pseudo-parallelization*  $\tilde{\tau}$ , equipped with a neighborhood  $N(\gamma) = [a, b] \times \gamma \times [-1, 1]$ , for real numbers  $a$  and  $b$  such that  $a < b$  and  $\varepsilon < \frac{b-a}{4}$ ,
- a parallelization  $\tau_e$  of  $A$  outside  $]a + \varepsilon, b - \varepsilon[ \times \gamma \times ]-1 + \varepsilon, 1 - \varepsilon[$ ,
- a parallelization  $\tau_b: N(\gamma) \times \mathbb{R}^3 \rightarrow TN(\gamma)$  of  $N(\gamma)$  such that

$$\tau_b = \begin{cases} \tau_e & \text{over } [b - \varepsilon, b] \times \gamma \times [-1, 1] \text{ and } [a, b] \times \gamma \times N(\partial[-1, 1]) \\ \tau_e \circ \mathcal{T}_\gamma & \text{over } [a, a + \varepsilon] \times \gamma \times [-1, 1], \end{cases}$$

where

$$\mathcal{T}_\gamma(t, c \in \gamma, u \in [-1, 1]; X \in \mathbb{R}^3) = (t, c, u; \rho_{\alpha(u)}(X)).$$

**Lemma 19.10.** *Let  $A$  be a compact oriented 3-manifold and let  $\tau$  be a trivialization of  $TA$  defined on a collar  $[-4, 0] \times \partial A$  of  $\partial A (= \{0\} \times \partial A)$ . Then there is a pseudo-parallelization of  $A$  that extends the restriction of  $\tau$  to  $[-1, 0] \times \partial A$ .*

PROOF: There exists a trivialization  $\tau'$  of  $TA$  on  $A$ . After a homotopy of  $\tau$  around  $\{-2\} \times \partial A$ , there exists a union  $\gamma \times [-1, 1]$  of annuli of  $\{-2\} \times \partial A$  such that  $\tau = \tau' \circ \mathcal{T}_\gamma$  on  $\{-2\} \times \partial A$ . (When  $\partial A$  is connected,  $\gamma$  can be assumed to be connected, too.) Consider the neighborhood  $N(\gamma) = [-2, -1] \times \gamma \times [-1, 1]$  of  $\gamma$ . Define  $\tau_e$  to coincide with  $\tau$  on  $([-2, 0] \times \partial A) \setminus \text{Int}(N(\gamma))$ , and with  $\tau'$  on  $A \setminus ([-2, 0] \times \partial A)$ . Define  $\tau_b$  to coincide with  $\tau$  on  $N(\gamma)$ .  $\square$

**Definition 19.11.** [Trivialization  $\tilde{\tau}_{\mathbb{C}}$  of  $TA \otimes_{\mathbb{R}} \mathbb{C}$ ] Define a smooth map

$$F_U: [a, b] \times [-1, 1] \longrightarrow SU(3)$$

such that

$$F_U(t, u) = \begin{cases} \mathbf{1} & \text{if } |u| > 1 - \varepsilon \\ \rho_{\alpha(u)} & \text{if } t < a + \varepsilon \\ \mathbf{1} & \text{if } t > b - \varepsilon. \end{cases}$$

Since  $\pi_1(SU(3))$  is trivial, it is possible to define such a smooth map. Define a trivialization  $\tilde{\tau}_{\mathbb{C}}$  of  $TA \otimes_{\mathbb{R}} \mathbb{C}$ , associated with the pseudo-parallelization  $\tilde{\tau}$  of Definition 19.9, as follows.

- On  $(A \setminus N(\gamma)) \times \mathbb{C}^3$ , we have  $\tilde{\tau}_{\mathbb{C}} = \tau_e \otimes 1_{\mathbb{C}}$ ,
- Over  $[a, b] \times \gamma \times [-1, 1]$ , we have  $\tilde{\tau}_{\mathbb{C}}(t, c, u; X) = \tau_b(t, c, u; F_U(t, u)^{-1}(X))$ .

Since  $\pi_2(SU(3))$  is trivial, the homotopy class of  $\tilde{\tau}_{\mathbb{C}}$  is well-defined.

**Definition 19.12.** Let  $M_0$  and  $M_1$  be two compact connected oriented 3-manifolds whose boundaries  $\partial M_0$  and  $\partial M_1$  have collars identified by a diffeomorphism. Let  $\tau_0$  be a pseudo-parallelization of  $M_0$ , which restricts to a collar neighborhood of  $\partial M_0$  as a genuine trivialization. Let  $\tau_1$  be a pseudo-parallelization of  $M_1$  that coincides with  $\tau_0$  on this collar neighborhood. We use the definition of Proposition 5.10 of relative Pontrjagin numbers and define  $p_1(\tau_0, \tau_1)$  to be  $p_1(\tau_{0,\mathbb{C}}, \tau_{1,\mathbb{C}})$ . Let  $\check{R}$  be a rational homology  $\mathbb{R}^3$ . A pseudo-parallelization  $\tilde{\tau}$  of  $\check{R}$  is *asymptotically standard* if it coincides with the standard parallelization  $\tau_s$  of  $\mathbb{R}^3$  outside  $B_R$ . (Recall Definition 3.6.) For such an asymptotically standard pseudo-parallelization, set  $p_1(\tilde{\tau}) = p_1((\tau_s)|_{B^3}, \tilde{\tau}|_{B_R})$ .

**Definition 19.13.** [Homogeneous boundary form associated with  $\tilde{\tau}$ ] Let  $\tilde{\tau} = (N(\gamma); \tau_e, \tau_b)$  be a pseudo-parallelization of a 3-manifold  $A$ . Recall  $\varepsilon$  and the map  $\alpha$  from Notation 19.1 and Definition 19.9. Define a smooth map

$$F: [a, b] \times [-1, 1] \longrightarrow SO(3)$$

such that

$$F(t, u) = \begin{cases} \mathbf{1} & \text{if } |u| > 1 - \varepsilon \\ \rho_{\alpha(u)} & \text{if } t < a + \varepsilon \\ \rho_{-\alpha(u)} & \text{if } t > b - \varepsilon. \end{cases}$$

Since the restriction of  $F$  to the boundary of  $[a, b] \times [-1, 1]$  is trivial in  $\pi_1(SO(3))$ , it is possible to define such a smooth map  $F$ .

Let  $p_{\tau_b} = p(\tau_b)$  denote the projection from  $UN(\gamma)$  to  $S^2$  induced by  $\tau_b$ . We have  $p_{\tau_b}(\tau_b(t, c, u; X \in S^2)) = X$ . Define

$$F(\gamma, \tau_b): \begin{array}{ccc} [a, b] \times \gamma \times [-1, 1] \times S^2 & \longrightarrow & [a, b] \times \gamma \times [-1, 1] \times S^2 \\ (t, c, u; Y) & \mapsto & (t, c, u; F(t, u)(Y)). \end{array}$$

Define the closed two-form  $\omega(\gamma, \tau_b)$  on  $U([a, b] \times \gamma \times [-1, 1])$  to be

$$\omega(\gamma, \tau_b) = \frac{p(\tau_b \circ \mathcal{T}_{\gamma}^{-1})^*(\omega_{S^2}) + p(\tau_b \circ F(\gamma, \tau_b)^{-1})^*(\omega_{S^2})}{2}.$$

The *homogeneous boundary form* associated to  $(\tilde{\tau}, F)$  is the following closed 2-form  $\omega(\tilde{\tau}, F)$  on  $UA$ .

$$\omega(\tilde{\tau}, F) = \begin{cases} p_{\tau_e}^*(\omega_{S^2}) & \text{on } U(A \setminus N(\gamma)) \\ \omega(\gamma, \tau_b) & \text{on } U(N(\gamma)). \end{cases}$$

A *homogeneous boundary form* of  $(UA, \tilde{\tau})$  is a homogeneous boundary form associated to  $(\tilde{\tau}, F)$  for some  $F$  as above.

We will justify the consistency of Definition 19.13 by applying the following lemma with the constant map  $\kappa$  with value one. We will use the general lemma in Lemma 21.3.

**Lemma 19.14.** *Let  $(e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1))$  denote the standard basis of  $\mathbb{R}^3$ . Let  $v_i : \mathbb{R}^3 \rightarrow \mathbb{R}$  denote the  $i$ th coordinate with respect to this basis. Let  $\rho_\theta = \rho_{\theta, e_3}$  denote the rotation of  $\mathbb{R}^3$  of angle  $\theta$  whose axis is directed by  $e_3$ . For  $k \in \mathbb{Z}$ , define*

$$\begin{aligned} \mathcal{T}_k : \mathbb{R} \times S^2 &\longrightarrow S^2 \\ (\theta, X) &\mapsto \rho_{k\theta}(X). \end{aligned}$$

Let  $\kappa : [-1, 1] \rightarrow \mathbb{R}$  be a smooth map. Consider the associated volume form  $(\kappa \circ v_3)\omega_{S^2}$  on  $S^2$ . It is invariant under the rotations  $\rho_\theta$ . Then we have

$$\mathcal{T}_k^*((\kappa \circ v_3)\omega_{S^2}) = \mathcal{T}_0^*((\kappa \circ v_3)\omega_{S^2}) + \frac{k(\kappa \circ v_3)}{4\pi} d\theta \wedge dv_3$$

PROOF: Recall the homogeneous two-form  $\omega_{S^2}$  on  $S^2$  with total area 1. When  $X \in S^2$ , and when  $v$  and  $w$  are two tangent vectors of  $S^2$  at  $X$ , we have

$$\omega_{S^2}(v \wedge w) = \frac{1}{4\pi} \det(X, v, w),$$

where  $X \wedge v \wedge w = \det(X, v, w)e_1 \wedge e_2 \wedge e_3$  in  $\bigwedge^3 \mathbb{R}^3$ .

Since  $\rho_\theta$  preserves the area in  $S^2$  and leaves  $v_3$  invariant, the restrictions of  $\mathcal{T}_k^*((\kappa \circ v_3)\omega_{S^2})$  and  $\mathcal{T}_0^*((\kappa \circ v_3)\omega_{S^2})$  coincide on  $\bigwedge^2 T_{(\theta, X)}(\{\theta\} \times S^2)$ . Therefore, we are left with the computation of

$$\left( \mathcal{T}_k^*((\kappa \circ v_3)\omega_{S^2}) - \mathcal{T}_0^*((\kappa \circ v_3)\omega_{S^2}) \right) (u \wedge v)$$

when  $u \in T_{(\theta, X)}(\mathbb{R} \times \{X\})$  and  $v \in T_{(\theta, X)}(\{\theta\} \times S^2)$ . We have

$$\mathcal{T}_0^*((\kappa \circ v_3)\omega_{S^2})(u \wedge v) = 0,$$

$$\mathcal{T}_k^*((\kappa \circ v_3)\omega_{S^2})_{(\theta, X)}(u \wedge v) = \frac{\kappa \circ v_3(X)}{4\pi} \det(\rho_{k\theta}(X), T_{(\theta, X)}\mathcal{T}_k(u), T_{(\theta, X)}\mathcal{T}_k(v)),$$

and  $T_{(\theta,X)}\mathcal{T}_k(v) = \rho_{k\theta}(v)$ . Since  $\rho_{k\theta}$  preserves the volume in  $\mathbb{R}^3$ , we get

$$\mathcal{T}_k^*((\kappa \circ v_3)\omega_{S^2})_{(\theta,X)}(u \wedge v) = \frac{\kappa \circ v_3(X)}{4\pi} \det(X, \rho_{-k\theta}(T_{(\theta,X)}\mathcal{T}_k(u)), v).$$

Let  $X_i$  stand for  $v_i(X)$ . We have

$$T_{(\theta,X)}\mathcal{T}_k(u) = kd\theta(u)\rho_{k\theta+\pi/2}(X_1e_1 + X_2e_2).$$

We obtain

$$\mathcal{T}_k^*(\omega_{S^2})_{(\theta,X)}(u \wedge v) = \frac{kd\theta(u)}{4\pi} \det(X, -X_2e_1 + X_1e_2, v),$$

and therefore

$$\begin{aligned} \mathcal{T}_k^*(\omega_{S^2})(u \wedge \cdot) &= \frac{kd\theta(u)}{4\pi} \det \begin{pmatrix} X_1 & -X_2 & dv_1 \\ X_2 & X_1 & dv_2 \\ X_3 & 0 & dv_3 \end{pmatrix} \\ &= \frac{kd\theta(u)}{4\pi} (-X_3X_1dv_1 - X_3X_2dv_2 + (1 - X_3^2)dv_3) \\ &= \frac{kd\theta(u)}{4\pi} dv_3. \end{aligned}$$

□

**PROOF OF THE CONSISTENCY OF DEFINITION 19.13:** It suffices to prove

$$p(\tau_b \circ \mathcal{T}_\gamma^{-1})^*(\omega_{S^2}) + p(\tau_b \circ \mathcal{T}_\gamma)^*(\omega_{S^2}) = 2p(\tau_b)^*(\omega_{S^2})$$

on  $U([b - \varepsilon, b] \times \gamma \times [-1, 1])$ , where we have

$$p_{\tau_b \circ \mathcal{T}_\gamma^{\pm 1}}(\tau_b(t, c, u; X)) = p_{\tau_b \circ \mathcal{T}_\gamma^{\pm 1}}(\tau_b \circ \mathcal{T}_\gamma^{\pm 1}(t, c, u; \rho_{\mp\alpha(u)}(X))) = \rho_{\mp\alpha(u)}(X).$$

Set  $\tilde{p}_{\tau_b} = p_{[-1,1]} \times p_{\tau_b} : U([b - \varepsilon, b] \times \gamma \times [-1, 1]) \rightarrow [-1, 1] \times S^2$ . We have

$$p(\tau_b) = \mathcal{T}_0 \circ (\alpha \times \mathbf{1}_{S^2}) \circ \tilde{p}_{\tau_b} \text{ and } p(\tau_b \circ \mathcal{T}_\gamma^{\pm 1}) = \mathcal{T}_{\mp 1} \circ (\alpha \times \mathbf{1}_{S^2}) \circ \tilde{p}_{\tau_b}.$$

We get  $p(\tau_b \circ \mathcal{T}_\gamma^{\pm 1})^*(\omega_{S^2}) = ((\alpha \times \mathbf{1}_{S^2}) \circ \tilde{p}_{\tau_b})^*(\mathcal{T}_{\mp 1}^*(\omega_{S^2}))$ .

Thus, Lemma 19.14 implies that Definition 19.13 is consistent. □

**Definition 19.15.** Let  $\check{R}$  be a rational homology  $\mathbb{R}^3$ , equipped with an asymptotically standard pseudo-parallelization  $\tilde{\tau}$ . A *homogeneous propagating form* of  $(C_2(R), \tilde{\tau})$  is a propagating form of  $C_2(R)$  (as in Definition 3.11) that coincides with a homogeneous boundary form of  $(U\check{R}, \tilde{\tau})$  as in Definition 19.13 on  $U\check{R}$ .

**Lemma 19.16.** *Such homogeneous propagating forms exist for any  $(R, \tilde{\tau})$ .*

PROOF: See Section 3.3. □

As in Definition 12.4, a *pseudo-parallelization* of a rational homology cylinder  $\mathcal{C}$  is a pseudo-parallelization of  $\check{R}(\mathcal{C})$  that agrees with the standard parallelization of  $\mathbb{R}^3$  outside  $\mathcal{C}$ .

The main result of this chapter is the following theorem.

**Theorem 19.17.** *Let  $\mathcal{C}$  be a rational homology cylinder. Let  $\tau = (N(\gamma); \tau_e, \tau_b)$  be a pseudo-parallelization of  $\mathcal{C}$ . Let  $L: \mathcal{L} \hookrightarrow \check{R}(\mathcal{C}) \setminus N(\gamma)$  be a long tangle representative in  $\check{R}(\mathcal{C}) \setminus N(\gamma)$ .*

*Definition 12.6 of  $I_\theta(K, \tau)$  naturally extends for such a pseudo-parallelization  $\tau$  when  $K$  is a component of  $L$ .*

*With this extended definition of  $I_\theta$  and with Definition 19.12 of  $p_1(\tau)$ , Theorem 7.20 and Theorem 12.7 also hold when  $\tau$  is a pseudo-parallelization  $\tau = (N(\gamma); \tau_e, \tau_b)$  of  $\mathcal{C}$  such that  $N(\gamma)$  does not meet the image of the long tangle representative (or the link)  $L: \mathcal{L} \hookrightarrow \check{R}(\mathcal{C})$ .*

In order to prove Theorem 19.17 in Section 19.5, we prove some preliminary lemmas in the next sections.

### 19.3 Integration of homogeneous propagating forms along surfaces

**Definition 19.18.** Let  $\Sigma$  be a compact oriented surface with boundary. Let  $\Sigma_0$  denote the image of the zero section in the tangent bundle  $T\Sigma$  of  $\Sigma$ . Let  $X$  be a nowhere vanishing section of  $T\Sigma$  along the boundary of  $\Sigma$ . Let  $\tilde{X}$  be an extension of  $X$  over  $\Sigma$  whose image  $\tilde{X}(\Sigma)$  in  $T\Sigma$  is transverse to  $\Sigma_0$ . The sections  $\Sigma_0$  and  $\tilde{X}(\Sigma)$  are naturally oriented by  $\Sigma$ . The *relative Euler number*  $\chi(X; \Sigma)$  is their algebraic intersection  $\langle \tilde{X}(\Sigma), \Sigma_0 \rangle_{T\Sigma}$  in  $T\Sigma$ .

Note that this definition makes sense since all the extensions of  $X$  are homotopic relatively to  $\partial\Sigma$ . This Euler number is an obstruction to extending  $X$  over  $\Sigma$  as a nowhere vanishing section of  $T\Sigma$ . Here are some other well-known properties of this number.

**Lemma 19.19.** *Let  $\Sigma$  be a compact oriented surface with boundary, and let  $X$  be a section of  $U\Sigma$  along the boundary of  $\Sigma$ .*

- *If  $\Sigma$  is connected and if  $\chi(X; \Sigma) = 0$ , then  $X$  extends as a nowhere vanishing section of  $T\Sigma$ .*

- If  $X$  is tangent to the boundary of  $\Sigma$ , then  $\chi(X; \Sigma)$  is the Euler characteristic  $\chi(\Sigma)$  of  $\Sigma$ .
- More generally, let  $a^{(1)}, \dots, a^{(k)}$  denote the  $k$  connected components of the boundary  $\partial\Sigma$  of  $\Sigma$ . For  $i = 1, \dots, k$ , the unit bundle  $U\Sigma|_{a^{(i)}}$  of  $T\Sigma|_{a^{(i)}}$  is an  $S^1$ -bundle over  $a^{(i)}$  with a canonical trivialization induced by  $Ta^{(i)}$ . Let  $d(X, a^{(i)})$  be the degree of the projection on the fiber  $S^1$  of this bundle of the section  $X$ , with respect to this canonical trivialization. Then we have

$$\chi(X; \Sigma) = \sum_{i=1}^k d(X, a^{(i)}) + \chi(\Sigma).$$

PROOF: First observe all these properties when  $\Sigma$  is a disk. When  $\Sigma$  is connected, there is a disk  $D$  that contains all the zeros of an arbitrary generic extension of  $\tilde{X}$  of  $X$ . If  $\chi(X; \Sigma) = 0$ , then  $\chi(\tilde{X}|_{\partial D}; D) = 0$ , and  $\tilde{X}|_{\partial D}$  extends to  $D$  as a nowhere vanishing section. So  $X$  extends to  $\Sigma$  as a nowhere vanishing section.

Let  $U^+\partial\Sigma$  denote the unit vector field of  $\partial\Sigma$  that is tangent to  $\partial\Sigma$  and induces its orientation. Let us prove that the following equality  $(*(\Sigma))$

$$\chi(U^+\partial\Sigma; \Sigma) = \chi(\Sigma)$$

holds for a general  $\Sigma$ .

For  $i = 1, 2$ , let  $\Sigma_i$  be a compact oriented surface, and let  $c_i$  be a connected component of  $\partial\Sigma_i$ . Set  $\Sigma = \Sigma_1 \cup_{c_1 \sim -c_2} \Sigma_2$ . Since the section  $U^+c_1$  is homotopic to  $(-U^+c_1)$  as a section of  $U\Sigma$ , we have

$$\chi(U^+\partial\Sigma; \Sigma) = \chi(U^+\partial\Sigma_1; \Sigma_1) + \chi(U^+\partial\Sigma_2; \Sigma_2).$$

Assume that  $(*(\Sigma_1))$  holds. Since  $\chi(\Sigma) = \chi(\Sigma_1) + \chi(\Sigma_2)$ , the equalities  $(*(\Sigma_2))$  and  $(*(\Sigma))$  are equivalent. Since  $S^1 \times S^1$  is parallelizable, the equality  $(*(S^1 \times S^1))$  holds. So  $(*(S^1 \times S^1 \setminus \mathring{D}^2))$  holds. The general case follows easily.

The third property of  $\chi(X; \Sigma)$  is an easy consequence of the previous one. □

**Lemma 19.20.** *Recall the vectors  $e_2 = (0, 1, 0)$  and  $e_3 = (0, 0, 1)$  of  $\mathbb{R}^3$ . Let  $\Sigma$  be a compact oriented surface immersed in a 3-manifold  $M$  equipped with a parallelization  $\tau$ . Assume that  $\tau(\cdot \times e_3)$  is a positive normal to  $\Sigma$  along  $\partial\Sigma$ . Let  $s_+(\Sigma) \subset UM$  (resp.  $s_-(\Sigma) \subset UM$ ) be the graph of the section of  $UM|_\Sigma$  in  $UM$  associated to the positive (resp. negative) normal to  $\Sigma$ . Let  $s_\tau(\Sigma; e_3)$  be the graph of the section  $\tau(\Sigma \times \{e_3\})$ . Then the cycles*

$$2(s_+(\Sigma) - s_\tau(\Sigma; e_3)) - \chi(\tau(\cdot \times e_2)|_{\partial\Sigma}; \Sigma)UM|_*$$



and

$$2(s_-(\Sigma) - s_\tau(\Sigma; -e_3)) + \chi(\tau(\cdot \times e_2)|_{\partial\Sigma}; \Sigma)UM|_*$$

of  $UM|_\Sigma$  are null-homologous in  $UM|_\Sigma$ .

PROOF: Consider the involution  $\iota_\Sigma$  of  $UM|_\Sigma$  that sends a vector to its opposite. This involution reverses the orientation of a fiber  $UM|_*$ . Therefore, it sends the second cycle (with  $s_-(\Sigma)$ ) to the first one (with  $s_+(\Sigma)$ ). Thus, it suffices to prove that

$$2(s_+(\Sigma) - s_\tau(\Sigma; e_3)) - \chi(\tau(\cdot \times e_2)|_{\partial\Sigma}; \Sigma)UM|_*$$

is a null-homologous cycle in  $UM|_\Sigma$ . The trivialization  $\tau$  can be homotoped so that  $\tau(\cdot \times e_3)$  is a positive normal to  $\Sigma$  over a one-skeleton of  $\Sigma$ . Therefore, we can assume that  $\tau(\cdot \times e_3)$  is a positive normal to  $\Sigma$  over the complement of a disjoint union of disks embedded in the interior of  $\Sigma$ . So it suffices to prove that  $2(s_+(D) - s_\tau(D; e_3)) - \chi(\tau(\cdot \times e_2)|_{\partial D}; D)UM|_*$  is null-homologous, for such a disk  $D$ . Let  $\tau_0$  be a trivialization of  $UM|_D$  such that  $\tau_0(\cdot, e_3)$  is the positive normal to  $D$  and  $\tau_0(\cdot, e_2)$  is tangent to  $\Sigma$ . The trivialization  $\tau_0$  identifies  $UM|_D$  with  $D \times S^2$ , and we have

$$[s_+(D)] - [s_\tau(D; e_3)] = -\langle s_\tau(D; e_3), s_{\tau_0}(D; -e_3) \rangle_{UM|_D} [UM|_*].$$

Let  $[e_2, e_3]$  denote the shortest arc of great circle from  $e_2$  to  $e_3$  on  $S^2$ . We have  $\langle s_\tau(D; e_3), s_{\tau_0}(D; -e_3) \rangle_{UM|_D} = \langle s_\tau(D; e_2), s_{\tau_0}(D; -e_3) \rangle_{UM|_D}$  because  $s_\tau(D; e_3) \cap s_{\tau_0}(D; -e_3)$  and  $s_\tau(D; e_2) \cap s_{\tau_0}(D; -e_3)$  cobound  $\tau(D \times [e_2, e_3]) \cap s_{\tau_0}(D; -e_3)$  in the interior of  $UM|_D$ . We similarly get

$$\langle s_\tau(D; e_3), s_{\tau_0}(D; -e_3) \rangle_{UM|_D} = \langle s_\tau(D; -e_2), s_{\tau_0}(D; -e_3) \rangle_{UM|_D}.$$

Applying the involution  $\iota_\Sigma$ , we obtain

$$2\left([s_+(D)] - [s_\tau(D; e_3)]\right) = \langle s_\tau(D; e_2), s_{\tau_0}(D; e_3) - s_{\tau_0}(D; -e_3) \rangle_{UM|_D} [UM|_*].$$

The projection to  $\mathbb{R}^2 \times \{0\}$  of  $\tau(\cdot \times e_2)$  is an extension of the section  $\tau(\cdot \times e_2)$  of  $TD$  to  $D$ . Its intersection with the zero section of  $TD$  is the above intersection number.  $\square$

**Definition 19.21.** A homotopy from a pseudo-parallelization  $(N(\gamma); \tau_e, \tau_b)$  to another such  $(N(\gamma); \tau'_e, \tau'_b)$  is a homotopy from the pair  $(\tau_e, \tau_b)$  to the pair  $(\tau'_e, \tau'_b)$  such that

$$\tau_b = \begin{cases} \tau_e & \text{over } \partial([a, b] \times \gamma \times [-1, 1]) \setminus (\{a\} \times \gamma \times [-1, 1]) \\ \tau_e \circ \mathcal{T}_\gamma & \text{over } \{a\} \times \gamma \times [-1, 1] \end{cases}$$

at any time.

**Proposition 19.22.** *Let  $\Sigma$ ,  $e_3$ ,  $s_+(\Sigma)$ , and  $s_-(\Sigma)$  be as in Lemma 19.20. Let  $\tilde{\tau}$  be a pseudo-parallelization restricting to a neighborhood of  $\partial\Sigma$  as a genuine parallelization of  $M$  such that  $\tilde{\tau}(\cdot \times e_3)$  is the positive normal to  $\Sigma$  along  $\partial\Sigma$ . Let  $\omega(\tilde{\tau})$  be a homogeneous boundary form associated with  $\tilde{\tau}$  (as in Definition 19.13). Then we have*

$$\int_{s_+(\Sigma)} \omega(\tilde{\tau}) = \frac{1}{2} \chi(\tilde{\tau}(\cdot \times e_2)|_{\partial\Sigma}; \Sigma)$$

and

$$\int_{s_-(\Sigma)} \omega(\tilde{\tau}) = -\frac{1}{2} \chi(\tilde{\tau}(\cdot \times e_2)|_{\partial\Sigma}; \Sigma).$$

PROOF: Observe that  $\omega(\tilde{\tau})$  is a closed form, which satisfies  $\int_{UM|_*} \omega(\tilde{\tau}) = 1$ . So Lemma 19.20 implies

$$\int_{s_{\pm}(\Sigma)} \omega(\tilde{\tau}) = \mp \frac{1}{2} \chi(\tilde{\tau}(\cdot \times e_2)|_{\partial\Sigma}; \Sigma) + \int_{s_{\tilde{\tau}}(\Sigma; \pm e_3)} \omega(\tilde{\tau}).$$

When  $\tilde{\tau}$  is a genuine parallelization, the term  $\int_{s_{\tilde{\tau}}(\Sigma; \pm e_3)} \omega(\tilde{\tau})$  is zero, and the proposition follows.

In general, for  $\tilde{\tau} = (N(\gamma) = [a, b] \times \gamma \times [-1, 1]; \tau_e, \tau_b)$ , the integral  $\int_{s_+(\Sigma)} \omega(\tilde{\tau})$  is invariant under an isotopy of  $\Sigma$  that fixes  $\partial\Sigma$  since  $\omega(\tilde{\tau})$  is closed. It is also invariant under a homotopy of  $(\tau_e, \tau_b)$  as in Definition 19.21 that is fixed in a neighborhood of  $\partial\Sigma$ . (See Lemma B.2.)

In particular, there is no loss of generality in assuming that  $\Sigma$  meets  $N(\gamma)$  along disks  $D_c = [a, b] \times \{c\} \times [-1, 1]$ , and that  $\tau_b(\cdot \times e_3)$  is the positive normal to  $D_c$  along  $\partial D_c$  for these disks. Thus, thanks to the good behavior of the two sides of the equality to be proved under gluings along circles that satisfy the boundary conditions, it suffices to prove the proposition when  $\Sigma$  is a meridian disk  $D_c$  of  $\gamma$  (with its corners smoothed) such that  $\tau_b(\cdot \times e_3)$  is the positive normal to  $\Sigma$  along  $\partial\Sigma$ . On  $UM|_{D_c}$ , the form  $\omega(\tilde{\tau})$  is then equal to

$$\omega(\gamma, \tau_b) = \frac{p(\tau_b \circ \mathcal{T}_\gamma^{-1})^*(\omega_{S^2}) + p(\tau_b \circ F(\gamma, \tau_b)^{-1})^*(\omega_{S^2})}{2},$$

where  $p(\tau_b \circ \mathcal{T}_\gamma^{-1})^*(\omega_{S^2})$  and  $p(\tau_b \circ F(\gamma, \tau_b)^{-1})^*(\omega_{S^2})$  are propagating forms respectively associated with the parallelizations  $\tau_b \circ \mathcal{T}_\gamma^{-1}$  and  $(\tau_b \circ F(\gamma, \tau_b)^{-1})$ , and  $\chi(\tau_b \circ \mathcal{T}_\gamma^{-1}(\cdot \times e_2)|_{\partial\Sigma}; \Sigma) = \chi(\tau_b(\cdot \times e_2)|_{\partial\Sigma}; \Sigma)$ .

Therefore, we have

$$\int_{s_+(\Sigma)} \omega(\tilde{\tau}) = \frac{1}{4} \left( \chi(\tau_b(\cdot \times e_2)|_{\partial\Sigma}; \Sigma) + \chi((\tau_b \circ F(\gamma, \tau_b)^{-1})(\cdot \times e_2)|_{\partial\Sigma}; \Sigma) \right).$$

Thanks to Lemma 19.19, this average is  $\frac{1}{2}(\chi(\tau_e(\cdot \times e_2)|_{\partial\Sigma}; \Sigma))$ . This concludes the computation of  $\int_{s_+(\Sigma)} \omega(\tilde{\tau})$ . The computation of  $\int_{s_-(\Sigma)} \omega(\tilde{\tau})$  is similar.  $\square$

## 19.4 Anomalous terms for pseudo-parallelizations

**Proposition 19.23.** *Let  $A$  be a compact 3-manifold equipped with two pseudo-parallelizations  $\tau_0$  and  $\tau_1$  that coincide with a common genuine parallelization along a regular neighborhood of  $\partial A$ . There exists a closed 2-form  $\omega$  on  $[0, 1] \times UA$  that restricts*

- to  $\{0\} \times UA$  as a homogeneous boundary form  $\omega(\tau_0)$  of  $(UA, \tau_0)$ ,
- to  $\{1\} \times UA$  as a homogeneous boundary form  $\omega(\tau_1)$  of  $(UA, \tau_1)$ ,
- to  $[0, 1] \times UA|_{\partial A}$  as  $p_{UA}^*(\omega(\tau_0))$  with respect to the natural projection  $p_{UA}: [0, 1] \times UA \rightarrow UA$ .

**PROOF:** Without loss of generality, assume that  $A$  is connected. Set  $X = [0, 1] \times UA$ . Then  $X$  is diffeomorphic to  $[0, 1] \times A \times S^2$  by a diffeomorphism induced by a parallelization  $\tau$ . The closed two-form  $\omega$  is defined consistently on  $\partial X$ . It suffices to prove that the coboundary map  $\partial$  of the long exact cohomology sequence associated to the pair  $(X, \partial X)$  maps the class of  $\omega|_{\partial X}$  to 0 in  $H^3(X, \partial X)$ . Since  $H_3(X, \partial X)$  is Poincaré dual to

$$H_3(X) \cong (H_1(A) \otimes H_2(S^2)) \oplus (H_3(A) \otimes H_0(S^2)),$$

it is generated

- by classes of the form  $[0, 1] \times s_+(\Sigma)$  for surfaces  $\Sigma$  of  $A$  such that  $\partial\Sigma \subset \partial A$  and for graphs  $s_+(\Sigma)$  of sections in  $UA$  associated to positive normals of the  $\Sigma$ , and
- by  $[0, 1] \times \{a\} \times S^2$  for some  $a \in A$ , when  $\partial A = \emptyset$ .

The evaluation of  $\partial[\omega|_{\partial X}]$  on these classes is the evaluation of  $[\omega|_{\partial X}]$  on their boundary. It is clearly zero for  $\partial[0, 1] \times \{a\} \times S^2$  since

$$\int_{\{(1,a)\} \times S^2} \omega = \int_{\{(0,a)\} \times S^2} \omega = 1.$$

Let us conclude the proof by proving

$$\int_{\partial([0,1] \times s_+(\Sigma))} \omega = 0$$

for a surface  $\Sigma$  as above transverse to  $\{-1\} \times \partial A$ . This integral is invariant under the homotopies of  $(\tau_0, \tau_1)$  that fix  $(\tau_0, \tau_1)$  near  $\partial A$ . (See Lemma B.2.) Therefore, we can assume that  $\tau_0 = \tau_1$  in a neighborhood  $[-2, 0] \times \partial A$  of  $\partial A$  in  $A$ , that  $\tau_0$  is a genuine parallelization in this neighborhood, and that the positive normal to  $\Sigma$  is  $\tau_0(\cdot \times e_3)$  along  $\Sigma \cap (\{-1\} \times \partial A)$ . Set  $A_{-1} = A \setminus ([-1, 0] \times \partial A)$  and  $\Sigma_{-1} = \Sigma \cap A_{-1}$ . Extend  $\omega$  so that  $\omega = p_{UA}^*(\omega(\tau_0))$  over  $[0, 1] \times UA|_{[-2,0] \times \partial A}$ . We have  $\int_{\partial([0,1] \times s_+(\Sigma))} \omega = \int_{\partial([0,1] \times s_+(\Sigma_{-1}))} \omega$ . Proposition 19.22 ensures  $\int_{\{0\} \times s_+(\Sigma_{-1})} \omega = \int_{\{1\} \times s_+(\Sigma_{-1})} \omega$ . Since  $\int_{[0,1] \times \partial s_+(\Sigma_{-1})} \omega = 0$ , we get  $\int_{\partial([0,1] \times s_+(\Sigma))} \omega = 0$ .  $\square$

**Proposition 19.24.** *Let  $A$  be a compact oriented 3-manifold equipped with three pseudo-parallelizations  $\tau_0, \tau_1,$  and  $\tau_2$  that coincide with a common genuine parallelization along a regular neighborhood of  $\partial A$ . Let  $n \in \mathbb{N}$ . Let  $\omega(\tau_0)$  and  $\omega(\tau_1)$  be homogeneous boundary forms respectively associated with  $(UA, \tau_0)$  and  $(UA, \tau_1)$ .*

*Under the assumptions of Proposition 19.23, as in Corollary 9.4, set*

$$z_n([0, 1] \times UA; \omega) = \sum_{\Gamma \in \mathcal{D}_n^c} \zeta_\Gamma \int_{[0,1] \times \bar{S}_V(\Gamma)(TA)} \bigwedge_{e \in E(\Gamma)} p_e^*(\omega) [\Gamma].$$

*If  $A$  embeds in a rational homology 3-ball, then  $z_n([0, 1] \times UA; \omega)$  depends only on the pseudo-parallelizations  $\tau_0$  and  $\tau_1$ . It is denoted by  $z_n(A; \tau_0, \tau_1)$ , and the following properties are satisfied.*

- *If  $n$  is even, then  $z_n(A; \tau_0, \tau_1) = 0$ .*
- *If  $B$  is a compact oriented 3-manifold embedded in the interior of  $A$ , if  $\tau_0$  and  $\tau_1$  coincide on a neighborhood of  $A \setminus B$ , and if  $\tau_0$  restricts to a neighborhood of  $\partial B$  as a genuine parallelization, then  $z_n(B; \tau_0, \tau_1) = z_n(A; \tau_0, \tau_1)$ .*
- *If  $\tau_0$  and  $\tau_1$  are actual parallelizations, then we have*

$$z_n(A; \tau_0, \tau_1) = \frac{p_1(\tau_0, \tau_1)}{4} \beta_n.$$

- *We have  $z_n(A; \tau_0, \tau_2) = z_n(A; \tau_0, \tau_1) + z_n(A; \tau_1, \tau_2)$ . (In particular, we have  $z_n(A; \tau_0, \tau_1) = -z_n(A; \tau_1, \tau_0)$ .)*

- For any orientation-preserving bundle isomorphism  $\Psi$  of  $UA$  over the identity map of  $A$ , we have  $z_n(A; \Psi \circ \tau_0, \Psi \circ \tau_1) = z_n(A; \tau_0, \tau_1)$ .
- For any orientation-preserving diffeomorphism  $\psi$  from  $A$  to another compact oriented 3-manifold  $B$ , we have

$$z_n\left(B; T\psi \circ \tau_0 \circ (\psi^{-1} \times 1_{\mathbb{R}^3}), T\psi \circ \tau_1 \circ (\psi^{-1} \times 1_{\mathbb{R}^3})\right) = z_n(A; \tau_0, \tau_1).$$

- If  $\tau'_1$  is homotopic to  $\tau_1$  relatively to  $\partial A$  in the sense of Definition 19.21, then we have

$$z_n(A; \tau_0, \tau'_1) = z_n(A; \tau_0, \tau_1).$$

- For any orientation-preserving diffeomorphism  $\psi_1$  of  $A$  isotopic to the identity map of  $A$  relatively to  $\partial A$ , we have

$$z_n\left(A; \tau_0, T\psi_1 \circ \tau_1 \circ (\psi_1^{-1} \times 1_{\mathbb{R}^3})\right) = z_n(A; \tau_0, \tau_1),$$

where  $\psi_1$  is used to carry the required parametrization of  $N(\gamma)$ .

PROOF: In this proof, the manifold  $A$  embeds in a rational homology 3-ball. Lemma 9.12 implies that  $z_n(A; \tau_0, \tau_1) = 0$  when  $n$  is even. Assume that  $n$  is odd from now on. Let us first prove that  $z_n([0, 1] \times UA; \omega)$  does not depend on the closed extension  $\omega$  when  $A$  is a rational homology 3-ball and when  $\tau_0$  is standard near  $\partial A$ . According to Lemma 19.16,  $\omega(\tau_0)$  (resp.  $\omega(\tau_1)$ ) extends to a homogeneous propagating form of  $(C_2(S^3(A/B_{S^3})), \tau_0)$  (resp. of  $(C_2(S^3(A/B_{S^3})), \tau_1)$ ). Set  $X = [0, 1] \times C_2(S^3(A/B_{S^3}))$ . The above extensions of  $\omega(\tau_0)$  and  $\omega(\tau_1)$  together with  $\omega$  (extended as  $p_{\tau_s}^*(\omega_{S^2})$  on  $[0, 1] \times (\partial C_2(S^3(A/B_{S^3})) \setminus UA)$ ) determine a closed 2-form of  $\partial X$ . This form extends as a closed form on  $X$  by Lemma 9.1. Then Corollary 9.4 implies that  $z_n([0, 1] \times UA; \omega)$  does not depend on  $\omega$  when  $A$  is a rational homology 3-ball and when  $\tau_0$  is standard near  $\partial A$ . In general, embed  $A$  in the interior of such a space  $B_R$ . The pseudo-parallelization  $\tau_0$  extends to  $B_R$  as a pseudo-parallelization standard near  $\partial B_R$  according to Lemma 19.10. The form  $\omega$  of Proposition 19.23 extends to  $[0, 1] \times UB_R$  as  $p_{UB_R}^*(\omega(\tau_0))$  on  $[0, 1] \times U(B_R \setminus A)$ . Then  $z_n([0, 1] \times U(B_R \setminus A); \omega) = 0$  since  $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega)$  pulls back through  $\check{S}_{V(\Gamma)}(T(B_R \setminus A))$  for any involved  $\Gamma$ . So  $z_n([0, 1] \times UA; \omega) = z_n([0, 1] \times UB_R; \omega)$  is independent of  $\omega|_{[0, 1] \times UA}$ .

Set  $z_n(A; \omega(\tau_0), \omega(\tau_1)) = z_n([0, 1] \times UA; \omega)$ . We easily observe

$$z_n(A; \omega(\tau_0), \omega(\tau_2)) = z_n(A; \omega(\tau_0), \omega(\tau_1)) + z_n(A; \omega(\tau_1), \omega(\tau_2)).$$

Assume  $\tau_0 = \tau_1 = (N(\gamma); \tau_e, \tau_b)$ . Assume that the forms  $\omega(\tau_0) = \omega(\tau_0, F_0)$  and  $\omega(\tau_1) = \omega(\tau_0, F_1)$  of Definition 19.13 are obtained from one another by

changing the map  $F = F_0: [a, b] \times [-1, 1] \rightarrow SO(3)$  to another one  $F_1$ . There exists a homotopy  $F_t$  from  $F_0$  to  $F_1$ . Such a homotopy induces a homotopy

$$F_t(\gamma, \tau_b): \begin{array}{ccc} [a, b] \times \gamma \times [-1, 1] \times S^2 & \longrightarrow & [a, b] \times \gamma \times [-1, 1] \times S^2 \\ (s, c, u; Y) & \mapsto & (s, c, u; F_t(s, u)(Y)). \end{array}$$

Then we have  $z_n(A; \omega(\tau_0), \omega(\tau_1)) = z_n(N(\gamma); \omega(\tau_0, F_0), \omega(\tau_0, F_1))$ . Use  $\tau_b$  to identify  $UN(\gamma)$  with  $[a, b] \times \gamma \times [-1, 1] \times S^2$ , and define  $\omega(\gamma, \tau_b)$  on  $[0, 1] \times [a, b] \times \gamma \times [-1, 1] \times S^2$  with respect to the formula for  $\omega(\gamma, \tau_b)$  in Definition 19.13 by

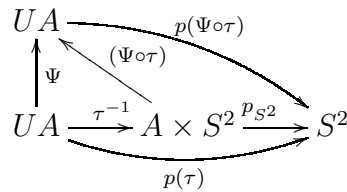
$$\omega(\gamma, \tau_b) = \frac{p(\tau_b \circ \mathcal{T}_\gamma^{-1})^*(\omega_{S^2}) + p(\tau_b \circ F_t(\gamma, \tau_b)^{-1})^*(\omega_{S^2})}{2}.$$

This formula does not depend on the coordinate along  $\gamma$ . So  $\omega$  pulls back through a projection from  $[0, 1] \times UN(\gamma)$  to  $[0, 1] \times [a, b] \times [-1, 1] \times S^2$ . The  $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega)$  pull back through  $[0, 1] \times \mathcal{S}_{V(\Gamma)}(TN(\gamma)|_{[a,b] \times \{c\} \times [-1,1]})$ . Hence  $z_n([0, 1] \times UN(\gamma); \omega)$  vanishes. This proves that  $z_n(N(\gamma); \omega(\tau_0, F_0), \omega(\tau_0, F_1))$  vanishes. We conclude that  $z_n(A; \omega(\tau_0), \omega(\tau_1))$  depends only on  $\tau_0$  and  $\tau_1$ .

For an orientation-preserving bundle isomorphism  $\Psi$  of  $UA$  over the identity map of  $A$ , the pseudo-parallelization

$$\Psi \circ (\tau_0 = (N(\gamma); \tau_e, \tau_b)) = (N(\gamma); \Psi \circ \tau_e, \Psi \circ \tau_b)$$

makes unambiguous sense. The following commutative diagram



shows that  $p(\tau_b) = p(\Psi \circ \tau_b) \circ \Psi$ . So we have

$$\omega(\Psi \circ \tau_0, F) = (\Psi^{-1})^* \omega(\tau_0, F).$$

The form  $\omega$  on  $[0, 1] \times UA$  can be pulled back by the orientation-preserving  $1_{[0,1]} \times \Psi^{-1}$ , similarly. We get

$$z_n(A; \Psi \circ \tau_0, \Psi \circ \tau_1) = z_n(A; \tau_0, \tau_1).$$

Proposition 10.7 implies that  $z_n(A; \tau_0, \tau_1) = \frac{p_1(\tau_0, \tau_1)}{4} \beta_n$  as soon as  $\tau_0$  and  $\tau_1$  are actual parallelizations and  $A$  embeds in a rational homology ball to which  $\tau_0$  extends as a genuine parallelization. Embed  $A$  in a rational homology ball  $B_R$ . Let  $\tau$  be a parallelization of  $B_R$ . Then  $\tau|_A =$

$\Psi \circ \tau_0$  for an orientation-preserving bundle isomorphism  $\Psi$  of  $UA$ . We have  $z_n(A; \Psi \circ \tau_0, \Psi \circ \tau_1) = \frac{p_1(\Psi \circ \tau_0, \Psi \circ \tau_1)}{4} \beta_n$ . The above behavior of  $z_n$  under an orientation-preserving bundle isomorphism of  $UA$  allows us to conclude that  $z_n(A; \tau_0, \tau_1) = \frac{p_1(\tau_0, \tau_1)}{4} \beta_n$  as soon as  $\tau_0$  and  $\tau_1$  are actual parallelizations.

For any orientation-preserving diffeomorphism  $\psi$  from  $A$  to  $B$ , the reader can check

$$z_n(B; T\psi \circ \tau_0 \circ (\psi^{-1} \times 1_{\mathbb{R}^3}), T\psi \circ \tau_1 \circ (\psi^{-1} \times 1_{\mathbb{R}^3})) = z_n(A; \tau_0, \tau_1)$$

as above.

If  $\tau_1$  is homotopic to  $\tau_0$  in the sense of Definition 19.21, then there exists a map  $\Psi: [0, 1] \times UA \rightarrow UA$  such that  $(t \mapsto \Psi(t, \cdot) \circ \tau_0)$  is a homotopy of pseudo-parallelizations from  $\tau_0$  to  $\tau_1$ . So we have

$$z_n(A; \tau_0, \tau_1) = z_n([0, 1] \times UA; \omega = (\Psi^{-1})^*(\omega(\tau_0, F))).$$

The form  $\omega$  pulls back through a map from  $[0, 1] \times UA$  to  $UA$ , and the forms  $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega)$  pull back through  $\tilde{S}_{V(\Gamma)}(TA)$ . So  $z_n([0, 1] \times UA; \omega)$  vanishes.

Let  $\psi: [0, 1] \times A \rightarrow A$  be an isotopy, which maps  $(t, u)$  to  $\psi_t(u)$ , such that  $\psi_0 = \mathbf{1}$ . It induces the homotopy

$$\begin{aligned} \Psi: [0, 1] \times UA &\rightarrow UA \\ (t, u) &\mapsto \tau_1 \circ (\psi_t \times 1_{S^2}) \circ \tau_1^{-1} \circ T\psi_t^{-1}(u), \end{aligned}$$

which satisfies

$$\begin{aligned} p(T\psi_t \circ \tau_1 \circ (\psi_t^{-1} \times 1_{S^2})) &= p_{S^2} \circ \tau_1^{-1} \circ \tau_1 \circ (T\psi_t \circ \tau_1 \circ (\psi_t^{-1} \times 1_{S^2}))^{-1} \\ &= p(\tau_1) \circ \Psi(t, \cdot). \end{aligned}$$

We get

$$\begin{aligned} z_n(A; \tau_1 = T\psi_0 \circ \tau_1 \circ (\psi_0^{-1} \times 1_{\mathbb{R}^3}), T\psi_1 \circ \tau_1 \circ (\psi_1^{-1} \times 1_{\mathbb{R}^3})) \\ = z_n([0, 1] \times UA; (\Psi)^*(\omega(\tau_1))) = 0. \end{aligned}$$

□

The main result of this section is the following theorem.

**Theorem 19.25.** *Let  $A$  be a compact 3-manifold equipped with two pseudo-parallelizations  $\tau_0$  and  $\tau_1$  that coincide with a common genuine parallelization along a regular neighborhood of  $\partial A$ . Assume that  $A$  embeds in a rational homology 3-ball. With the notation of Proposition 19.24, we have*

$$z_n(A; \tau_0, \tau_1) = \frac{p_1(\tau_0, \tau_1)}{4} \beta_n$$

for any natural integer  $n$ .

To prove this theorem, we will prove it in special cases and show that these special cases are sufficient to get a complete proof.

**Lemma 19.26.** *Assume  $A = [2, 9] \times \gamma \times [-2, 2]$ . Equip  $A$  with a pseudo-parallelization  $\tau_0 = (N(\tilde{\gamma}); \tau_e, \tau_b)$  such that*

$$N(\tilde{\gamma}) = [3, 5] \times \gamma \times [-1, 1] \sqcup [6, 8] \times \gamma \times [-1, 1].$$

*Then there exists a parallelization  $\tau_1$  of  $A$  that coincides with  $\tau_e$  in a neighborhood of  $\partial A$ . Furthermore, we have*

$$z_n(A; \tau_0, \tau_1) = \frac{p_1(\tau_0, \tau_1)}{4} \beta_n$$

*for any such parallelization.*

**PROOF:** We first prove the lemma for some chosen pseudo-parallelizations  $\tilde{\tau}_0$  and  $\tilde{\tau}_1$  satisfying the assumptions and behaving as “products by  $\gamma$ ”. For these pseudo-parallelizations, this product behavior will imply  $p_1(\tilde{\tau}_0, \tilde{\tau}_1) = 0$  and  $z_n(A; \tilde{\tau}_0, \tilde{\tau}_1) = 0$ . Define the parallelization

$$\begin{aligned} \tau_A: \quad A \times \mathbb{R}^3 &\quad \rightarrow \quad UA \\ (s_0, c_0, u_0; e_1) &\mapsto \frac{d}{ds}(s, c_0, u_0)(s_0, c_0, u_0) \\ (s_0, c_0, u_0; e_2) &\mapsto \frac{d}{dc}(s_0, c, u_0)(s_0, c_0, u_0) \\ (s_0, c_0, u_0; e_3) &\mapsto \frac{d}{du}(s_0, c_0, u)(s_0, c_0, u_0) \end{aligned}$$

of  $A$ . Define  $\tilde{\tau}_e: (A \setminus \mathring{N}(\tilde{\gamma})) \times \mathbb{R}^3 \rightarrow U(A \setminus \mathring{N}(\tilde{\gamma}))$  by

$$\tilde{\tau}_e = \begin{cases} \tau_A & \text{on } A \setminus ([2, 8[ \times \gamma \times ]-1, 1[) \times \mathbb{R}^3 \\ \tau_A \circ \mathcal{T}_\gamma^{-1} & \text{on } [5, 6] \times \gamma \times [-1, 1] \\ \tau_A \circ \mathcal{T}_\gamma^{-2} & \text{on } [2, 3] \times \gamma \times [-1, 1], \end{cases}$$

where  $\mathcal{T}_\gamma(t, c \in \gamma, u \in [-1, 1]; X \in \mathbb{R}^3) = (t, c, u; \rho_{\alpha(u)}(X))$  as in Definition 19.9. Define  $\tilde{\tau}_b: N(\tilde{\gamma}) \times \mathbb{R}^3 \rightarrow UN(\tilde{\gamma})$  by

$$\tilde{\tau}_b = \begin{cases} \tau_A & \text{on } [6, 8] \times \gamma \times [-1, 1] \\ \tau_A \circ \mathcal{T}_\gamma^{-1} & \text{on } [3, 5] \times \gamma \times [-1, 1]. \end{cases}$$

Set  $\tilde{\tau}_0 = (N(\tilde{\gamma}); \tilde{\tau}_e, \tilde{\tau}_b)$ . Define a map

$$\begin{aligned} \tilde{F}: \quad [3, 8] \times [-1, 1] &\quad \rightarrow \quad SO(3) \\ (3, u) &\mapsto \rho_{-2\alpha(u)} \\ (8, u) &\mapsto 1_{SO(3)} \\ (t, \pm 1) &\mapsto 1_{SO(3)}. \end{aligned}$$



Finally define  $\tilde{\tau}_1$  such that

$$\tilde{\tau}_1 = \begin{cases} \tau_A & \text{on } A \setminus ([2, 8[ \times \gamma \times ]-1, 1[) \times \mathbb{R}^3 \\ \tau_A \circ \mathcal{T}_\gamma^{-2} & \text{on } [2, 3] \times \gamma \times [-1, 1] \end{cases}$$

and

$$\tilde{\tau}_1(s, c, u; X) = \left( \tau_A(s, c, u; \tilde{F}(s, u)(X)) \right)$$

when  $(s, u) \in [3, 8] \times [-1, 1]$ .

Let us prove  $p_1(\tilde{\tau}_0, \tilde{\tau}_1) = 0$ . The involved trivializations of  $T([0, 1] \times A) \otimes C$  on  $\partial[0, 1] \times A$  are obtained from the natural parallelization  $T[0, 1] \oplus \tau_A$  by composition by a map from  $\partial([0, 1] \times A) = \gamma \times \partial([0, 1] \times [2, 9] \times [-2, 2])$  to  $SU(4)$ , which does not depend on the coordinate along  $\gamma$ . Since  $\pi_2(SU(4)) = \{0\}$ , this map extends to  $SU(4)$ .

Let us similarly prove  $z_n(A; \tilde{\tau}_0, \tilde{\tau}_1) = 0$ . Set  $Y = [0, 1] \times S^2 \times [2, 9] \times [-2, 2]$ . The parallelization  $\tau_A$  identifies  $[0, 1] \times UA$  with  $\gamma \times Y$ . We have  $z_n(A; \tilde{\tau}_0, \tilde{\tau}_1) = z_n([0, 1] \times UA; \omega)$  for a closed two-form  $\omega$  whose restriction  $\omega|_\partial$  to  $\partial([0, 1] \times UA)$  factors through the projection of  $\gamma \times \partial Y$  onto  $\partial Y$ . The involved closed 2-form on  $\partial Y$  extends to  $Y$  as a closed form  $\omega_Y$  since  $\omega|_\partial$  extends to the whole  $[0, 1] \times UA$ , according to Proposition 19.23. Then  $\omega$  can be chosen as the pull-back of  $\omega_X$  under the projection of  $\gamma \times X$  onto  $X$ . Thus, the forms  $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega)$  again pull back through a projection onto a space of dimension smaller than the degree of the forms. So  $z_n([0, 1] \times UA; \omega)$  vanishes.

There exists an orientation-preserving bundle isomorphism  $\Psi$  of  $UA$  over the identity map of  $A$ , such that  $\tau_0 = \Psi \circ \tilde{\tau}_0$ . The parallelization  $\tau_1 = \Psi \circ \tilde{\tau}_1$  satisfies the assumptions of the lemma. We have  $z_n(A; \Psi \circ \tilde{\tau}_0, \Psi \circ \tilde{\tau}_1) = z_n(A; \tilde{\tau}_0, \tilde{\tau}_1) = 0$  and  $p_1(\Psi \circ \tilde{\tau}_0, \Psi \circ \tilde{\tau}_1) = p_1(\tilde{\tau}_0, \tilde{\tau}_1) = 0$ . So we have  $z_n(A; \tau_0, \tau_1) = \frac{p_1(\tau_0, \tau_1)}{4} \beta_n$ . We conclude for any another parallelization  $\tau'_1$  that coincides with  $\tau_e$  near  $\partial A$ , because  $z_n(A; \tau_1, \tau'_1) = \frac{p_1(\tau_1, \tau'_1)}{4} \beta_n$ .  $\square$

**Lemma 19.27.** *Let  $A$  be a compact oriented 3-manifold that embeds in a rational homology 3-ball. Let  $[-7, 0] \times \partial A$  be a collar neighborhood of  $A$ . Let  $\gamma \times [-2, 2]$  be a disjoint union of annuli in  $\partial A$ . Set  $N(\gamma) = [-2, -1] \times \gamma \times [-1, 1]$ . Let  $\tau_0 = (N(\gamma); \tau_e, \tau_b)$  be a pseudo-parallelization of  $A$  that coincides with the restriction of a parallelization  $\tau_1$  of  $A$  in a neighborhood of  $\partial A$ . Then we have*

$$z_n(A; \tau_0, \tau_1) = \frac{p_1(\tau_0, \tau_1)}{4} \beta_n.$$

PROOF: Recall  $A_{-2} = A \setminus (]-2, 0] \times \partial A)$ . Figure 19.1 shows the schema of the proof.

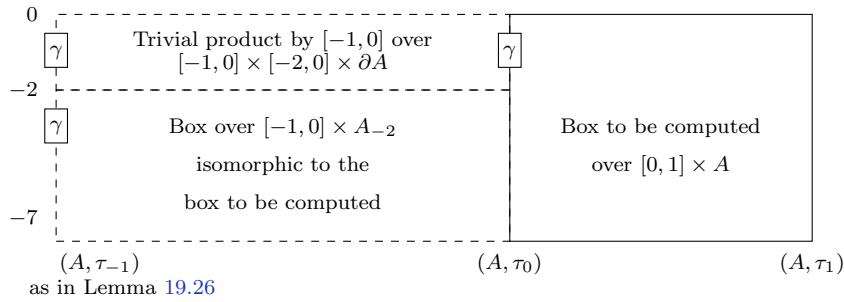


Figure 19.1: Schema of proof for Lemma 19.27

Let  $f: [-7, 0] \rightarrow [-7, -2]$  be a diffeomorphism such that  $f(t) = t - 2$  when  $t \geq -3$ , and  $f(t) = t$  when  $t \leq -6$ . Let  $\psi: A \rightarrow A_{-2}$  be a diffeomorphism restricting to  $A \setminus ([-7, 0] \times \partial A)$  as the identity map, and mapping  $(t, x) \in [-7, 0] \times \partial A$  to  $(f(t), x)$ .

There exists a bundle isomorphism  $\Phi$  of  $UA_{-2}$  over the identity map of  $A_{-2}$  such that  $\tau_e|_{A_{-2}} = \Phi \circ T\psi \circ \tau_1 \circ (\psi^{-1} \times 1_{\mathbb{R}^3})$ . Let  $\tau_{-1}$  be the pseudo-parallelization of  $A$  that coincides with  $\tau_0$  over  $[-2, 0] \times \Sigma$ , and with  $\Phi \circ T\psi \circ \tau_0 \circ (\psi^{-1} \times 1_{\mathbb{R}^3})$  over  $A_{-2}$ . Then  $\tau_{-1}$  is a parallelization outside  $[-7, 0] \times \gamma \times [-2, 2]$ . Since  $[-7, 0] \times \gamma \times [-2, 2]$  satisfies the hypotheses of Lemma 19.26 up to reparametrization, Lemma 19.26 and Proposition 19.24 ensure that

$$z_n(A; \tau_{-1}, \tau_1) = \frac{p_1(\tau_{-1}, \tau_1)}{4} \beta_n.$$

To conclude, we prove that  $z_n(A; \tau_0, \tau_1) = \frac{1}{2}z_n(A; \tau_{-1}, \tau_1)$  and  $p_1(\tau_0, \tau_1) = \frac{1}{2}p_1(\tau_{-1}, \tau_1)$ . The element  $z_n(A; \tau_{-1}, \tau_0)$  of  $\mathcal{A}_n(\emptyset)$  can be written as

$$z_n(A_{-2}; \Phi \circ T\psi \circ \tau_0 \circ (\psi^{-1} \times 1_{\mathbb{R}^3}), \Phi \circ T\psi \circ \tau_1 \circ (\psi^{-1} \times 1_{\mathbb{R}^3})) = z_n(A; \tau_0, \tau_1).$$

We similarly have  $p_1(\tau_{-1}, \tau_0) = p_1(\tau_0, \tau_1)$ . □

**Lemma 19.28.** *Let  $\Sigma$  be a compact oriented surface. Let  $\gamma_0$  and  $\gamma_1$  be two disjoint unions of curves of  $\Sigma$  with respective tubular neighborhoods  $\gamma_0 \times [-1, 1]$  and  $\gamma_1 \times [-1, 1]$ . Set  $A = [0, 3] \times \Sigma$ ,  $N(\gamma_0) = [1, 2] \times \gamma_0 \times [-1, 1]$  and  $N(\gamma_1) = [1, 2] \times \gamma_1 \times [-1, 1]$ .*

*There exist two pseudo-parallelizations  $\tau_0 = (N(\gamma_0); \tau_{e,0}, \tau_{b,0})$  and  $\tau_1 = (N(\gamma_1); \tau_{e,1}, \tau_{b,1})$  that coincide near  $\partial A$  if and only if  $\gamma_0$  and  $\gamma_1$  have the same class in  $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ . In this case, we have*

$$z_n(A; \tau_0, \tau_1) = \frac{p_1(\tau_0, \tau_1)}{4} \beta_n.$$

PROOF: Without loss of generality, assume that  $\Sigma$  is connected and that the complement of  $(\gamma_0 \times [-1, 1]) \cup (\gamma_1 \times [-1, 1])$  in  $\Sigma$  is not empty. Let  $\tau$  be a parallelization of  $A = [0, 3] \times \Sigma$ .

Assume that  $\tau_0$  and  $\tau_1$  are two pseudo-parallelizations as in the statement, which coincide near  $\partial A$ . Let us prove that  $\gamma_0$  and  $\gamma_1$  are homologous modulo  $2H_1(\Sigma; \mathbb{Z})$ .

If the boundary of  $\Sigma$  is empty, choose a disk  $D$  of  $\Sigma$  outside  $\gamma_0 \times [-1, 1] \cup \gamma_1 \times [-1, 1]$  and assume that  $\tau_{e,0}$  and  $\tau_{e,1}$  coincide on  $[0, 3] \times D$ , without loss of generality. This allows us to assume  $\partial\Sigma \neq \emptyset$ , without loss of generality, by possibly removing the interior of  $D$  from  $\Sigma$ . For  $i \in \{0, 1\}$  and for the pseudo-parallelization  $\tau_i$  of the statement, write  $\tau_{e,i}$  as  $\tau \circ \psi_{\mathbb{R}}(g_i)$  for some  $g_i: A \setminus \overset{\circ}{N}(\gamma_i) \rightarrow SO(3)$ , with the notation of Section 4.2. Then the restriction of  $g_i$  to a meridian curve of  $\gamma_i$  is not homotopic to a constant loop. The maps  $g_0$  and  $g_1$  coincide near  $\partial A$ . Let  $c: [0, 1] \rightarrow \Sigma$  be a path such that  $c(0)$  and  $c(1)$  are in  $\partial\Sigma$ . Then the restriction of  $g_i$  to

$$\left( \{3\} \times c([0, 1]) \right) \cup \left( -[0, 3] \times c(1) \right) \cup \left( -\{0\} \times c([0, 1]) \right) \cup \left( [0, 3] \times c(0) \right)$$

is null-homotopic if and only if the mod 2 intersection of  $c$  with  $\gamma_i$  is trivial. So  $\gamma_0$  and  $\gamma_1$  must be homologous modulo  $2H_1(\partial A; \mathbb{Z})$ .

Conversely, assume that  $\gamma_0$  and  $\gamma_1$  are homologous modulo  $2H_1(\Sigma; \mathbb{Z})$ . Define  $g_0: A \setminus \overset{\circ}{N}(\gamma_0) \rightarrow SO(3)$  to be the map sending  $A \setminus ([0, 2] \times \gamma_0 \times ]-1, 1])$  to the identity of  $SO(3)$ , and mapping  $(t, c, u) \in ([0, 1] \times \gamma_0 \times [-1, 1])$  to  $\rho_{-\alpha(u)}$ . First define  $g_1: A \setminus \overset{\circ}{N}(\gamma_1) \rightarrow SO(3)$  on  $(A \setminus ([0, 1[ \times \Sigma)) \setminus \overset{\circ}{N}(\gamma_1)$  so that  $g_1$  sends  $(A \setminus ([0, 1] \times \Sigma)) \setminus \overset{\circ}{N}(\gamma_1)$  to the identity of  $SO(3)$  and  $g_1$  maps  $(t, c, u) \in (\{1\} \times \gamma_1 \times [-1, 1])$  to  $\rho_{-\alpha(u)}$ . Since the classes of  $\gamma_0$  and  $\gamma_1$  coincide in  $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ , the restrictions to  $\{1\} \times \Sigma$  of  $g_1$  and  $g_0$  are homotopic on the one-skeleton of  $\Sigma$ , and hence on  $\Sigma$ . This allows us to extend  $g_1$  to  $[0, 1] \times \Sigma$  so that  $g_0$  and  $g_1$  coincide in a neighborhood of  $\partial A$ . Then, for  $i \in \{0, 1\}$ , there exists  $\tau_{b,i}$  such that  $\tau_i = (N(\gamma_i); \tau_{e,i} = \tau \circ \psi_{\mathbb{R}}(g_i), \tau_{b,i})$  is a pseudo-parallelization. The pseudo-parallelizations  $\tau_0$  and  $\tau_1$  coincide near  $\partial A$ .

Set  $B = [0, 6] \times \Sigma$  and  $N(\gamma'_0) = [4, 5] \times \gamma_0 \times [-1, 1]$ . Extend  $\tau_0$  to  $B$  as a pseudo-parallelization  $\tau_{0,B} = (N(\gamma_0) \sqcup N(\gamma'_0); \tau_{e,B,0}, \tau_{b,B,0})$ . Extend  $\tau_1$  to  $B$  as a pseudo-parallelization  $\tau_{1,B}$  that coincides with  $\tau_{0,B}$  on  $[3, 6] \times \Sigma$ . We have  $z_n(A; \tau_0, \tau_1) = z_n(B; \tau_{0,B}, \tau_{1,B})$  and  $p_1(\tau_0, \tau_1) = p_1(\tau_{0,B}, \tau_{1,B})$ .

According to Lemma 19.26, there is a parallelization  $\tau_2$  of  $B$  that coincides with  $\tau_{0,B}$  near  $\partial B$ , and we have  $z_n(B; \tau_{0,B}, \tau_2) = \frac{p_1(\tau_{0,B}, \tau_2)}{4} \beta_n$ .

Apply Lemma 19.27 to prove that  $z_n(B; \tau_{1,B}, \tau_2) = \frac{p_1(\tau_{1,B}, \tau_2)}{4} \beta_n$ . (To apply Lemma 19.27 as it is stated in  $B$ , first rotate  $N(\gamma_0) = [1, 2] \times \gamma_0 \times [-1, 1]$

around  $\gamma_0$  by an isotopy that sends  $1 \times [-1, 1]$  to  $2 \times [1, -1]$  and apply Proposition 19.24.) This implies that  $z_n(B; \tau_{0,B}, \tau_{1,B}) = \frac{p_1(\tau_{0,B}, \tau_{1,B})}{4}$ .  $\square$

**Lemma 19.29.** *Let  $\Sigma$  be a compact connected oriented surface with boundary. Let  $\gamma, \gamma_0,$  and  $\gamma_1$  be three disjoint unions of curves of  $\Sigma$  with respective tubular neighborhoods  $\gamma \times [-1, 1], \gamma_0 \times [-1, 1],$  and  $\gamma_1 \times [-1, 1]$ .<sup>1</sup> Assume that  $[\gamma_1] = [\gamma_0] + [\gamma]$  in  $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ . Set  $A = [0, 6] \times \Sigma, N(\gamma) = [4, 5] \times \gamma \times [-1, 1], N(\gamma_0) = [1, 2] \times \gamma_0 \times [-1, 1],$  and  $N(\gamma_1) = [1, 2] \times \gamma_1 \times [-1, 1].$  Let  $\tau_0 = (N(\gamma) \sqcup N(\gamma_0); \tau_{e,0}, \tau_{b,0})$  and  $\tau_1 = (N(\gamma_1); \tau_{e,1}, \tau_{b,1})$  be two pseudo-parallelizations which coincide near  $\partial A$ . Then we have*

$$z_n(A; \tau_0, \tau_1) = \frac{p_1(\tau_0, \tau_1)}{4} \beta_n.$$

PROOF: Lemma 19.28 allows us to choose arbitrary representatives of  $[\gamma]$  and  $[\gamma_0]$  for the proof, without loss of generality. In particular, there is no loss of generality in assuming that  $\gamma_0$  is connected and that the intersection of  $\gamma$  and  $\gamma_0$  has no more than one point.

If  $\gamma$  and  $\gamma_0$  are disjoint, then we can perform an isotopy in  $A$  to lower  $\gamma,$  and the result is a direct consequence of Lemma 19.28.

Assume that the intersection of  $\gamma$  and  $\gamma_0$  has one transverse point. Attach two copies  $\Sigma$  and  $\Sigma'$  of  $\Sigma$  to a disk  $D$  along intervals  $I$  and  $I'$ . Let  $\gamma', \gamma'_0,$  and  $\gamma'_1$  be the respective copies of  $\gamma, \gamma_0,$  and  $\gamma_1$  in  $\Sigma'$ . Let  $\tilde{\Sigma} = \Sigma \cup D \cup \Sigma'$  as in Figure 19.2.

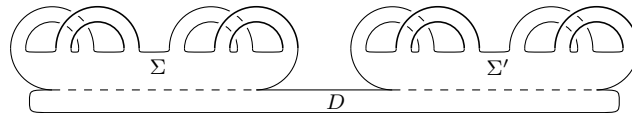


Figure 19.2:  $\tilde{\Sigma}$

Let  $B = [0, 6] \times \tilde{\Sigma}$ . Let  $\tau_{B,0}$  be a pseudo-parallelization of  $B$  extending the pseudo-parallelization  $\tau_0$  used both for  $A$  and for  $A' = [0, 6] \times \Sigma'$ . Let  $\tau_{B,1}$  be a pseudo-parallelization that coincides with  $\tau_1$  on  $A$  and on  $A',$  and with  $\tau_{B,0}$  on  $[0, 6] \times D$ . Then we have  $z_n(A; \tau_0, \tau_1) = \frac{1}{2} z_n(B; \tau_{B,0}, \tau_{B,1})$  and  $p_1(\tau_0, \tau_1) = \frac{1}{2} p_1(\tau_{B,0}, \tau_{B,1})$ . Since the intersection of  $(\gamma \cup \gamma')$  and  $(\gamma_0 \cup \gamma'_0)$  is zero modulo  $2H_1(\partial A; \mathbb{Z}),$  the homology classes of these curves can be represented by curves that do not intersect. So we have  $z_n(B; \tau_{B,0}, \tau_{B,1}) = \frac{p_1(\tau_{B,0}, \tau_{B,1})}{4} \beta_n.$   $\square$

PROOF OF THEOREM 19.25: Let us first prove the theorem when  $A$  is a rational homology ball, according to the schema of Figure 19.3. Then there ex-

<sup>1</sup>The disjoint union of curves  $\gamma$  may intersect  $\gamma_0$  and  $\gamma_1,$  and  $\gamma_0$  may intersect  $\gamma_1.$

ists a parallelization  $\tau_2$  of  $A$  that coincides with  $\tau_0 = (N(\gamma); \tau_e, \tau_b)$  in a neighborhood of  $\partial A$ . Thicken the neighborhood  $N(\gamma)$  to  $[a - 7, b + 7] \times \gamma \times [-2, 2]$ . Add bands to  $\gamma \times [-2, 2]$  so that the disjoint union  $\gamma \times [-2, 2]$  is embedded in a connected oriented surface  $\Sigma$  of  $A$  with one boundary component. Let  $[a - 7, b + 7] \times \Sigma$  be embedded in  $A$  so that this parametrization matches the previous one.

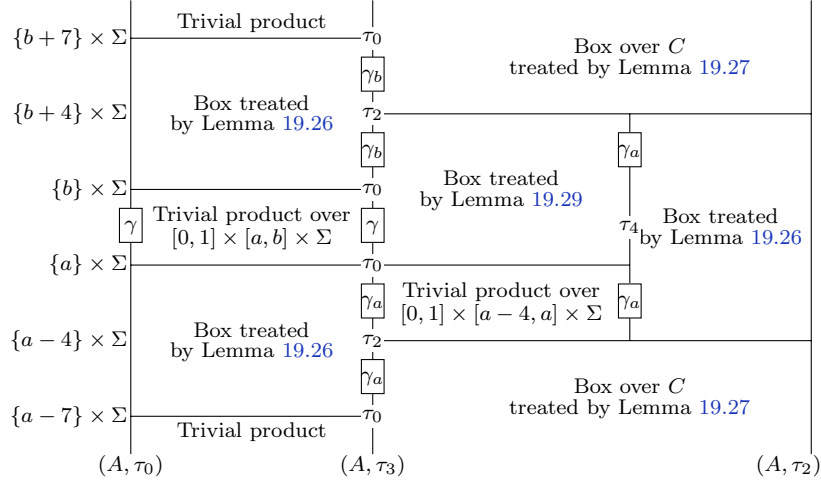


Figure 19.3: Schema of proof for Theorem 19.25

After a possible homotopy of  $\tau_2$ , there exist annuli  $\gamma_a \times [-1, 1]$  and  $\gamma_b \times [-1, 1]$  in  $\Sigma$  such that

- $\tau_0$  coincides with  $\tau_2$  in a neighborhood of  $[a - 7, b + 7] \times \partial\Sigma$ ,
- $\tau_0|_{\{a-4\} \times \Sigma} = \tau_2 \circ \mathcal{T}_{\gamma_a}$ , and
- $\tau_0|_{\{b+4\} \times \Sigma} = \tau_2 \circ \mathcal{T}_{\gamma_b}^{-1}$ .

Set

$$N(\gamma_3) = N(\gamma) \sqcup ([b + 2, b + 3] \sqcup [b + 5, b + 6]) \times \gamma_b \times [-1, 1] \sqcup ([a - 6, a - 5] \sqcup [a - 3, a - 2]) \times \gamma_a \times [-1, 1].$$

Let  $\tau_3 = (N(\gamma_3); \tau_{3,e}, \tau_{3,b})$  be a pseudo-parallelization, which coincides with  $\tau_0$  outside  $([b + 1, b + 7] \times \gamma_b \times [-1, 1] \sqcup ([a - 7, a - 1] \times \gamma_a \times [-1, 1])$ , and which coincides with  $\tau_2$  on  $\{a - 4\} \times \Sigma$  and on  $\{b + 4\} \times \Sigma$ . According to Lemma 19.26 and to Proposition 19.24, we have

$$z_n(A; \tau_0, \tau_3) = \frac{p_1(\tau_0, \tau_3)}{4} \beta_n,$$

as the left part of Figure 19.3 shows.

Set  $B = [a - 4, b + 4] \times \Sigma$  and  $C = A \setminus \left( ]a - 4, b + 4[ \times \overset{\circ}{\Sigma} \right)$ . We have

$$z_n(A; \tau_2, \tau_3) = z_n(B; \tau_2, \tau_3) + z_n(C; \tau_2, \tau_3),$$

and  $p_1$  decomposes similarly.

Let us now prove

$$z_n(C; \tau_2, \tau_3) = \frac{1}{4}p_1(\tau_2|_C, \tau_3|_C)\beta_n.$$

To do this, we apply Lemma 19.27, after an isotopy of  $[b + 4, b + 7] \times \Sigma$  which sends  $[b + 5, b + 6] \times \gamma_b \times [-1, 1]$  to itself (at the end) so that  $\{b + 5\} \times \gamma_b \times [-1, 1]$  is sent to  $\{b + 6\} \times \gamma_b \times (-[-1, 1])$ , and  $\{b + 6\} \times \gamma_b \times [-1, 1]$  is sent to  $\{b + 5\} \times \gamma_b \times (-[-1, 1])$ .

Let  $\tau_4 = (([a - 3, a - 2] \sqcup [b + 2, b + 3]) \times \gamma_a \times [-1, 1]; \tau_{4,e}, \tau_{4,b})$  be a pseudo-parallelization of  $B$  that coincides with  $\tau_2$  in a neighborhood of  $\partial B$ . Lemma 19.26 implies

$$z_n(B; \tau_4, \tau_2) = \frac{1}{4}p_1(\tau_4, \tau_2|_B)\beta_n.$$

Since  $[\gamma_a] + [\gamma] + [\gamma_b] = 0$  in  $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ , Lemma 19.29 and Proposition 19.24 imply  $z_n(B; \tau_3, \tau_4) = \frac{1}{4}p_1(\tau_3|_B, \tau_4)\beta_n$ . We get  $z_n(B; \tau_2, \tau_3) = \frac{1}{4}p_1(\tau_2|_B, \tau_3|_B)\beta_n$ , and hence

$$z_n(A; \tau_2, \tau_3) = \frac{1}{4}p_1(\tau_2, \tau_3)\beta_n.$$

So we have  $z_n(A; \tau_0, \tau_2) = \frac{1}{4}p_1(\tau_0, \tau_2)\beta_n$ . For the same reasons, we have  $z_n(A; \tau_1, \tau_2) = \frac{1}{4}p_1(\tau_1, \tau_2)\beta_n$ . Hence the lemma is proved when  $A$  is a rational homology ball.

In general, the manifold  $A$  is assumed to embed into a rational homology ball  $B$ , the pseudo-parallelization  $\tau_0$  on  $A$  extends to a pseudo-parallelization  $\tilde{\tau}_0$  of  $B$ , and the pseudo-parallelization  $\tau_1$  over  $A$  extends to a pseudo-parallelization  $\tilde{\tau}_1$  of  $B$ , which coincides with  $\tilde{\tau}_0$  over  $B \setminus \overset{\circ}{A}$ . We have

$$z_n(A; \tau_0, \tau_1) = z_n(B; \tilde{\tau}_0, \tilde{\tau}_1) = \frac{p_1(\tilde{\tau}_0, \tilde{\tau}_1)}{4}\beta_n = \frac{p_1(\tau_0, \tau_1)}{4}\beta_n.$$

□

## 19.5 Proof of Theorem 19.17

**Proposition 19.30.** *Let  $\check{R}$  be a rational homology  $\mathbb{R}^3$  equipped with an asymptotically standard pseudo-parallelization  $\tau$ . Let  $\omega(\tau)$  be a homogeneous propagating form of  $(C_2(R), \tau)$ . Let  $n \in \mathbb{N}$ . With the notation of Corollary 10.9 and Notation 7.16, we have*

$$z_n(\check{R}, \omega(\tau)) = \mathfrak{z}_n(R) + \frac{1}{4}p_1(\tau)\beta_n$$

and

$$\mathcal{Z}(R) = Z(\check{R}, \emptyset, \omega(\tau)) \exp\left(-\frac{p_1(\tau)}{4}\beta\right).$$

PROOF: Let  $\tau_0$  and  $\tau_1$  be two pseudo-parallelizations of  $\check{R}$  standard outside  $B_R$ . Let  $\omega$  be a 2-form on  $[0, 1] \times UB_R$  as in Lemma 9.1. Corollary 9.4 implies

$$z_n(\check{R}, \omega(\tau_1)) - z_n(\check{R}, \omega(\tau_0)) = z_n([0, 1] \times UB_R; \omega),$$

while Theorem 19.25 implies

$$z_n([0, 1] \times UB_R; \omega) = \frac{1}{4}(p_1(\tau_1) - p_1(\tau_0))\beta_n.$$

Conclude with Corollary 10.9. □

PROOF OF THEOREM 19.17: Theorem 12.7 implies

$$Z(\mathcal{C}, L, \tau) = \exp\left(\frac{1}{4}p_1(\tau)\beta\right) \prod_{j=1}^k \left(\exp(I_\theta(K_j, \tau)\alpha) \#_j\right) \check{\mathcal{Z}}(\mathcal{C}, L)$$

for any actual parallelization  $\tau$  of  $\mathcal{C}$ . We want to prove the same equality when  $\tau$  is a pseudo-parallelization that is an actual parallelization over a tubular neighborhood  $N(L)$  of  $L$ . Recall Lemma 7.27. Proposition 19.30 leaves us with the proof that

$$\check{Z}(\mathcal{C}, L, \tau') = \prod_{j=1}^k \left(\exp(I_\theta(K_j, \tau')\alpha) \#_j\right) \check{\mathcal{Z}}(\mathcal{C}, L)$$

for any pseudo-parallelization  $\tau'$  of  $\mathcal{C}$  that is an actual parallelization over a tubular neighborhood  $N(L)$  of  $L$ .

First assume that the restriction of  $\tau'$  to the tubular neighborhood  $N(L)$  of  $L$  extends to an actual parallelization  $\tau$  of  $\mathcal{C}$ . In this case, we apply

Proposition 14.49 with a closed 2-form  $\tilde{\omega}$  on  $[0, 1] \times UC$  as in its statement. Thus, we get

$$\check{Z}(\mathcal{C}, L, \tau') = \left( \prod_{j=1}^k \exp(I_j) \#_j \right) \check{Z}(\mathcal{C}, L, \tau),$$

where  $I_j$  is defined from  $\tilde{\omega}$  for any component  $K_j$  of  $L = \sqcup_{j=1}^k K_j$  in Proposition 14.49. Let  $p_{UN(L)}: [0, 1] \times UN(L) \rightarrow UN(L)$  denote the projection to the second factor. According to Proposition Proposition 19.23, the form  $\tilde{\omega}$  may be expressed as  $p_{UN(L)}^* p_{\tau}^*(\omega_{S^2})$  over  $[0, 1] \times UN(L)$ . This factorization implies that the  $I_j$  vanish. So, we have  $\check{Z}(\mathcal{C}, L, \tau') = \check{Z}(\mathcal{C}, L, \tau)$ . The degree one part of this equality implies  $I_{\theta}(K_j, \tau') = I_{\theta}(K_j, \tau)$ . Hence the theorem is proved when the restriction of  $\tau'$  to the tubular neighborhood  $N(L)$  of  $L$  extends to an actual parallelization  $\tau$  of  $\mathcal{C}$ .

Recall that  $D_r$  is the disk of the complex numbers of module less than or equal to  $r$ . Let  $\tau_s$  denote both the standard parallelization of  $\mathbb{R}^3$  and its restriction to  $D_4 \times [-2, 2]$ . Consider the neighborhood

$$N(\gamma_2) = (D_3 \setminus \dot{D}_1) \times [-1, 1]$$

of  $(\gamma_2 = \partial D_2 \times \{0\})$ . Let  $\tau_2 = (N(\gamma_2); \tau_e, \tau_b)$  be a pseudo-parallelization of  $D_4 \times [-2, 2]$  that coincides with  $\tau_s$  in a neighborhood of  $\partial(D_4 \times [-2, 2])$  and that maps  $e_3$  to the vertical direction of  $\{0\} \times [-2, 2]$  along  $\{0\} \times [-2, 2]$ .

Let  $\tau_s^{\mathcal{C}}$  and  $\tau_2^{\mathcal{C}}$  be two pseudo-parallelizations of  $\mathcal{C}$  that satisfy the following set  $(*)$   $(\tau_s, \tau_2, \mathcal{C}, L, K_j)$  of assumptions: They are actual parallelizations over a tubular neighborhood  $N(L)$  of  $L$ . There is an embedding of  $D_4 \times [-2, 2]$  in the rational homology cylinder  $\mathcal{C}$  equipped with the long tangle representative  $L$  so that (the image of)  $D_4 \times [-2, 2]$  intersects (the image of)  $L$  along  $\{0\} \times [-2, 2]$  and the orientations of  $\{0\} \times [-2, 2]$  and  $L$  match. With respect to this embedding,  $\tau_s^{\mathcal{C}}$  and  $\tau_2^{\mathcal{C}}$  respectively coincide with  $\tau_s$  and  $\tau_2$  on the image of  $D_4 \times [-2, 2]$ , and they coincide with each other elsewhere. The component of  $L$  that intersects  $D_4 \times [-2, 2]$  is denoted by  $K_j$ .

According to Proposition 19.23, there exists a closed 2-form  $\tilde{\omega}$  on  $[0, 1] \times U(D_4 \times [-2, 2])$  that restricts to

$$\left( \{0\} \times U(D_4 \times [-2, 2]) \right) \cup \left( [0, 1] \times U(D_4 \times [-2, 2])|_{\partial(D_4 \times [-2, 2])} \right)$$

as  $p_{\tau_s}^*(\omega_{S^2})$ , and to  $\{1\} \times U(D_4 \times [-2, 2])$  as a homogeneous propagating form of  $C_2(R(\mathcal{C}), \tau_2^{\mathcal{C}})$  does. This closed 2-form is actually independent of  $(\mathcal{C}, L)$ , so is the induced quantity  $I(\tau_2)$  of Proposition 14.49 such that

$$\check{Z}(\mathcal{C}, L, \tau_2^{\mathcal{C}}) = \exp(I(\tau_2)) \#_j \check{Z}(\mathcal{C}, L, \tau_s^{\mathcal{C}}).$$



In particular, the difference  $(I_\theta(K_j, \tau_2^{\mathcal{C}}) - I_\theta(K_j, \tau_s^{\mathcal{C}}))$  is a constant  $\ell(\tau_2)$ . It can be obtained from the degree one part of  $I(\tau_2)$ .

Apply this computation when  $R(\mathcal{C})$  is  $SO(3)$ ,  $(L = K)$  is a knot, the homology class of  $K$  represents the generator of  $H_1(SO(3); \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ , and  $\tau_s^{\mathcal{C}}$  is an actual parallelization. The two homotopy classes of parallelizations of  $N(K)$  are obtained from one another by composition by the restriction to  $N(K) \times \mathbb{R}^3$  of the map

$$\begin{aligned} SO(3) \times \mathbb{R}^3 &\rightarrow SO(3) \times \mathbb{R}^3 \\ (\rho, x) &\mapsto (\rho, \rho(x)). \end{aligned}$$

So, all parallelizations of  $N(K)$  extend to  $\mathcal{C}$ . In particular,  $\tau_2^{\mathcal{C}}|_{N(K)}$  extends to  $\mathcal{C}$  as a parallelization standard near  $\partial\mathcal{C}$ .

Then the first studied case implies  $I(\tau_2) = (I_\theta(K, \tau_2^{\mathcal{C}}) - I_\theta(K, \tau_s^{\mathcal{C}})) \alpha = \ell(\tau_2)\alpha$ . So we get  $I(\tau_2) = (I_\theta(K_j, \tau_2^{\mathcal{C}}) - I_\theta(K_j, \tau_s^{\mathcal{C}})) \alpha$ , and hence

$$\begin{aligned} \prod_{\ell=1}^k \left( \exp(-I_\theta(K_\ell, \tau_2^{\mathcal{C}})\alpha) \#_\ell \right) \check{Z}(\mathcal{C}, L, \tau_2^{\mathcal{C}}) \\ = \prod_{\ell=1}^k \left( \exp(-I_\theta(K_\ell, \tau_s^{\mathcal{C}})\alpha) \#_\ell \right) \check{Z}(\mathcal{C}, L, \tau_s^{\mathcal{C}}) \end{aligned}$$

for any two pseudo-parallelizations  $\tau_s^{\mathcal{C}}$  and  $\tau_2^{\mathcal{C}}$  that satisfy  $(*)(\tau_s, \tau_2, \mathcal{C}, L, K_j)$ .

Let  $\tau'$  be a pseudo-parallelization of  $\mathcal{C}$  that coincides with an actual parallelization  $\tau_N$  of  $\mathcal{C}$  on  $N(L)$ . Let  $\tau$  be a parallelization of  $\mathcal{C}$ . The restrictions of  $\tau_N$  and  $\tau$  to  $\partial N(L)$  are homotopic along the meridians of  $L$ . They differ by the action of the generator of  $\pi_1(SO(2))$  along parallels on components  $K_j$  for  $K_j$  in some finite set  $A$ . If  $A = \emptyset$ , the first studied case implies

$$\prod_{j=1}^k \left( \exp(-I_\theta(K_j, \tau')\alpha) \#_j \right) \check{Z}(\mathcal{C}, L, \tau') = \check{Z}(\mathcal{C}, L).$$

Otherwise, equip each component  $K_j$  of  $A$  with one embedding of  $D_4 \times [-2, 2]$  whose image meets  $L$  in  $K_j$  along  $\{0\} \times [-2, 2]$ , so that the orientations of  $\{0\} \times [-2, 2]$  and  $L$  match. Perform a homotopy of  $\tau'$  as in Definition 19.21 to transform  $\tau'$  to a pseudo-parallelization  $\tau''$  such that  $\tau''$  is induced by  $\tau_s$  on the image of the above embeddings. For  $B \subseteq A$ , let  $\tau''_B$  be obtained from  $\tau''$  by changing  $\tau_s$  to  $\tau_2$  on the images of the embeddings of  $D_4 \times [-2, 2]$  that meet an element of  $B$ . Then for  $B \subseteq A$  and for  $K_\ell \in A \setminus B$ , the pseudo-parallelizations  $\tau''_B$  and  $\tau''_{B \cup \{K_\ell\}}$  satisfy  $(*)(\tau''_B, \tau''_{B \cup \{K_\ell\}}, \mathcal{C}, L, K_\ell)$ . Therefore,

we have

$$\begin{aligned} \prod_{j=1}^k \left( \exp(-I_\theta(K_j, \tau''_{B \cup \{K_\ell\}})\alpha) \#_j \right) \check{Z}(\mathcal{C}, L, \tau''_{B \cup \{K_\ell\}}) \\ = \prod_{j=1}^k \left( \exp(-I_\theta(K_j, \tau''_B)\alpha) \#_j \right) \check{Z}(\mathcal{C}, L, \tau''_B). \end{aligned}$$

We get

$$\prod_{j=1}^k \left( \exp(-I_\theta(K_j, \tau''_A)\alpha) \#_j \right) \check{Z}(\mathcal{C}, L, \tau''_A) = \prod_{j=1}^k \left( \exp(-I_\theta(K_j, \tau'')\alpha) \#_j \right) \check{Z}(\mathcal{C}, L, \tau'')$$

by induction on  $|A|$ . Since  $\tau''_A$  and  $\tau$  are homotopic on  $N(L)$ , the first studied case implies

$$\prod_{j=1}^k \left( \exp(-I_\theta(K_j, \tau''_A)\alpha) \#_j \right) \check{Z}(\mathcal{C}, L, \tau''_A) = \check{Z}(\mathcal{C}, L).$$

So it suffices to prove that

$$\prod_{j=1}^k \left( \exp(-I_\theta(K_j, \tau'')\alpha) \#_j \right) \check{Z}(\mathcal{C}, L, \tau'') = \prod_{j=1}^k \left( \exp(-I_\theta(K_j, \tau')\alpha) \#_j \right) \check{Z}(\mathcal{C}, L, \tau')$$

or that  $\prod_{j=1}^k (\exp(-I_\theta(K_j, \tau')\alpha) \#_j) \check{Z}(\mathcal{C}, L, \tau')$  is invariant by a homotopy of  $\tau'$ , as in Definition 19.21, supported in a ball where  $\tau'$  is a genuine parallelization (namely, around an image of  $D_4 \times [-2, 2]$ ).

Again, the effect on  $\check{Z}(\mathcal{C}, L, \tau')$  of such a homotopy depends only on the homotopy inside the ball, according to Propositions 14.49 and 19.23. Since such a ball equipped with the homotopy may be inserted in a tangle equipped with a genuine trivialization  $\tau$ , we conclude that

$$\prod_{j=1}^k (\exp(-I_\theta(K_j, \tau')\alpha) \#_j) \check{Z}(\mathcal{C}, L, \tau')$$

is indeed invariant under the above homotopies. This is sufficient to conclude the proof of Theorem 19.17.  $\square$

We give more definitions of  $Z$  involving non-necessarily homogeneous propagating forms and pseudo-parallelizations in Chapter 21.



# Chapter 20

## Simultaneous normalization of propagating forms

This chapter is devoted to the proof of Propositions 18.33 and 18.36. As shown in Section 18.7, this is sufficient to prove Theorem 18.6. In this chapter, we use real coefficients for homology and cohomology unless otherwise mentioned.

### 20.1 Sketch

First note that the homogeneous boundary form of Definition 19.13 defined on  $\partial C_2(R)$  (or the form  $p_\tau^*(\omega_{S^2})$  as in Definition 3.9 when pseudo-parallelizations are not involved) is antisymmetric on  $\partial C_2(R)$  as in Definition 3.14. So it extends as a closed antisymmetric 2-form  $\omega = \omega_\emptyset$  on  $C_2(R)$  as in Lemma 3.16. Also note that if the restriction of  $\omega_I$  to

$$A_I^{(i)} \times (C_1(R_I) \setminus A_{I,3}^{(i)}) \subset C_2(R_I)$$

equals

$$\sum_{j \in g_i} p_1^*(\eta_I(a_j^i)) \wedge p_2^*(\eta_I(z_j^i)) + p_2^*(\omega(p^i))$$

as stated in Proposition 18.36, then the restriction of  $\omega_I$  to  $A_I^{(i)} \times A_I^{(k)}$  equals

$$\sum_{(j,\ell) \in \underline{g_i} \times \underline{g_k}} lk(z_j^i, z_\ell^k) p_{A_I^{(i)}}^*(\eta(a_j^i)) \wedge p_{A_I^{(k)}}^*(\eta(a_\ell^k)),$$

for  $k \neq i$ , as required in Proposition 18.33.

To arrange the propagating forms  $\omega_I$  as in Propositions 18.33 and 18.36, we will first show how to make  $\omega$  satisfy the conditions of Proposition 18.36,

with respect to the notation before Proposition 18.36. More precisely, we will prove the following proposition in Subsection 20.2.

**Proposition 20.1.** *Let  $\tilde{\omega}$  be a propagating form of  $C_2(R)$  as in Definition 3.11. Its restriction to  $\partial C_2(R) \setminus UB_R$  may be expressed as  $p_\tau^*(\tilde{\omega}_{S^2})$ , for some volume-one form  $\tilde{\omega}_{S^2}$  of  $S^2$ . Let  $\tilde{\omega}(p^i)$  (resp.  $\tilde{\omega}(p^i)_\iota$ ) be a degree-two form on  $(C_1(R) \setminus \text{Int}(A^{(i)}))$  that satisfies the same properties as the form  $\omega(p^i)$  (introduced before Proposition 18.36) except that it restricts to  $\partial C_1(R) = S^2$  as  $\tilde{\omega}_{S^2}$  (resp. as  $-\iota_{S^2}^* \tilde{\omega}_{S^2}$ ) instead of the usual volume form  $\omega_{S^2}$ .<sup>1</sup> If  $\tilde{\omega}$  is antisymmetric, then assume that  $\tilde{\omega}(p^i)_\iota = \tilde{\omega}(p^i)$ .*

*Then there exists a propagating form  $\omega$  of  $C_2(R)$  such that*

1. *the form  $\omega$  coincides with  $\tilde{\omega}$  on  $\partial C_2(R)$ ,*
2. *for every  $i \in \underline{x}$ , the restriction of  $\omega$  to*

$$A^{(i)} \times (C_1(R) \setminus A_3^{(i)}) \subset C_2(R)$$

*is equal to*

$$\sum_{j=1}^{g_i} p_1^*(\eta(a_j^i)) \wedge p_2^*(\eta(z_j^i)) + p_2^*(\tilde{\omega}(p^i)),$$

*where  $p_1$  and  $p_2$  respectively denote the first and the second projection of  $A^{(i)} \times (C_1(R) \setminus A_3^{(i)})$  to  $C_1(R)$ , and the restriction of  $\omega$  to*

$$(C_1(R) \setminus A_3^{(i)}) \times A^{(i)} \subset C_2(R)$$

*is equal to*

$$\sum_{j \in \underline{g}_i} p_1^*(\eta(z_j^i)) \wedge p_2^*(\eta(a_j^i)) - p_1^*(\tilde{\omega}(p^i)_\iota),$$

3. *for every  $i \in \underline{x}$ , for any  $j \in \{1, 2, \dots, g_i\}$ , we have*

$$\int_{\Sigma(a_j^i) \times p^i} \omega = 0 \quad \text{and} \quad \int_{p^i \times \Sigma(a_j^i)} \omega = 0,$$

*where  $p^i \in \partial A^{(i)}$  and  $\partial \Sigma(a_j^i) \subset \{4\} \times \partial A^{(i)}$ , and*

4.  *$\omega$  is antisymmetric if  $\tilde{\omega}$  is.*

---

<sup>1</sup>In our proof of Propositions 18.33 and 18.36, we use Proposition 20.1 only when  $\tilde{\omega}_{S^2} = \omega_{S^2}$  and  $\tilde{\omega}(p^i) = \tilde{\omega}(p^i)_\iota = \omega(p^i)$ , but the general statement is useful in other related work.

Assume that Proposition 20.1 is proved. This is the goal of Subsection 20.2. Recall that we will use Proposition 20.1 only when  $\tilde{\omega}_{S^2} = \omega_{S^2}$  and  $\tilde{\omega}(p^i) = \tilde{\omega}(p^i)_\iota = \omega(p^i)$ . Also recall that the  $\eta(a_j^i)$  are defined both in  $A^{(i)}$  and  $A^{(i)'}$  and they are identical near  $\partial A^{(i)}$  and  $\partial A^{(i)'}$ . Finally recall that  $\omega(p^i)$  is supported outside  $\cup_{k \in \underline{x}} \text{Int}(A^{(k)})$  and the  $\eta(z_j^i)$  restrict to the  $A^{(k)}$  as a combination of  $\eta(a_\ell^k)$  (fixed by the linking numbers). When changing some  $A^{(i)}$  into some  $A^{(i)'}$  with the same Lagrangian, it is easy to change the restrictions of  $\omega$  inside the parts  $(A^{(r)} \times (C_1(R) \setminus A_3^{(r)}))$  or  $(C_1(R) \setminus A_3^{(r)}) \times A^{(r)} \subset C_2(R)$  mentioned in the statement of Proposition 20.1. Indeed, all the forms  $\eta(a_j^i)$ ,  $\eta(z_j^i)$ , and  $\omega(p^i)$  can be defined on the parts of the  $R_I$  where they are needed so that these forms coincide with each other whenever it makes sense, and so that they have the properties that were required for  $R$ . Define  $\omega_0(R_I)$  on

$$D(\omega_0(R_I)) = \left( C_2(R_I) \setminus \left( \cup_{i \in I} p_b^{-1}((A_{-1}^{(i)'}) \times A_3^{(i)'}) \cup (A_3^{(i)'}) \times A_{-1}^{(i)'} \right) \right) \cup p_b^{-1}(\text{diag}(\check{R}_I))$$

so that

1. we have  $\omega_0(R_I) = \omega$  on  $C_2(R \setminus (\cup_{i \in I} A_{-1}^{(i)'}))$ ,
2. when  $i \in I$ , we have

$$\omega_0(R_I) = \sum_{i=1}^{g_i} p_1^* (\eta(a_j^i)) \wedge p_2^* (\eta(z_j^i)) + p_2^* (\omega(p^i))$$

on  $A^{(i)'}) \times (\check{R}_I \setminus A_3^{(i)'})$ ,

3. when  $i \in I$ , we have  $\omega_0(R_I) = -\iota^* (\omega_0(R_I))$  on  $(\check{R}_I \setminus A_3^{(i)'}) \times A^{(i)'}$ , and
4. on  $\partial C_2(R_I)$ , the form  $\omega_0(R_I)$  coincides with the homogeneous boundary form  $\omega(\tau_I, F)$  of Definition 19.13, for a map  $F$  that is the same for all  $I \subseteq \underline{x}$ .

Note that this definition is consistent.

Set  $R_i = R_{\{i\}}$  and  $D_A(\omega_0(R_i)) = C_2(A_4^{(i)'}) \cap D(\omega_0(R_i))$ .

**Lemma 20.2.** *With the above notation, for any  $i \in \underline{x}$ , the cohomology class of  $\omega_0(R_i)$  vanishes on the kernel of the map*

$$H_2\left(D_A(\omega_0(R_i))\right) \longrightarrow H_2\left(C_2(A_4^{(i)'})\right)$$

*induced by the inclusion.*

This lemma was surprisingly difficult to prove for me. It will be proved in Subsection 20.3. Assume it for the moment. Then (the cohomology class of)  $\omega_0(R_i)$  is in the image of the natural map

$$H^2\left(C_2(A_4^{(i)'})\right) \longrightarrow H^2\left(D_A(\omega_0(R_i))\right).$$

So  $\omega_0(R_i)$  extends to a closed form  $\omega_1(R_i)$  on  $C_2(A_4^{(i)'})$ . Change this form to  $\omega_{\{i\}} = \frac{1}{2}(\omega_1(R_i) - \iota^*(\omega_1(R_i)))$  to get an antisymmetric homogeneous propagating form of  $(C_2(R_i), \tau_{\{i\}})$ . For any  $I \subseteq \underline{x}$ , define

$$\omega_I = \begin{cases} \omega_0(R_I) & \text{on } C_2(R_I) \setminus \left(\cup_{i \in I} p_b^{-1}((A_{-1}^{(i)'} \times A_4^{(i)'}) \cup (A_4^{(i)'} \times A_{-1}^{(i)}))\right) \\ \omega_{\{i\}} & \text{on } C_2(A_4^{(i)'}) \text{ for } i \in I. \end{cases}$$

This definition is consistent since the  $C_2(A_4^{(i)'})$  do not intersect. The forms  $\omega_I$  satisfy the properties of Proposition 18.33. In order to finish the proof of Proposition 18.36, up to Proposition 20.1 and Lemma 20.2, let us prove that  $\int_{\Sigma'(a_j^i) \times p^i} \omega_{\{i\}} = 0$ . Note

$$\int_{\Sigma'(a_j^i) \times p^i} \omega_{\{i\}} = \int_{\Sigma'(a_j^i) \times (p^i \times \{4\})} \omega_{\{i\}} + \int_{\partial \Sigma'(a_j^i) \times (p^i \times [0,4])} \omega_{\{i\}}.$$

The same formula applied to  $\Sigma(a_j^i)$  instead of  $\Sigma'(a_j^i)$  yields

$$0 = \int_{\Sigma(a_j^i) \times (p^i \times \{4\})} \omega + \int_{\partial \Sigma(a_j^i) \times (p^i \times [0,4])} \omega.$$

The prescribed behavior of the forms on  $A_I^{(i)} \times (\check{R}_I \setminus A_{I3}^{(i)})$  implies

$$\int_{(\Sigma'(a_j^i) \cap A_0^{(i)'}) \times (p^i \times \{4\})} \omega_{\{i\}} = \int_{(\Sigma(a_j^i) \cap A_0^{(i)}) \times (p^i \times \{4\})} \omega = 0.$$

Since  $\omega$  and  $\omega_{\{i\}}$  coincide on  $C_2\left(A_4^{(i)} \setminus A_{-1}^{(i)}\right)$ , we also have

$$\int_{((-\partial \Sigma(a_j^i) \times [0,4]) \times (p^i \times \{4\})) \cup (\partial \Sigma(a_j^i) \times (p^i \times [0,4]))} (\omega_{\{i\}} - \omega) = 0.$$

This shows  $\int_{\Sigma'(a_j^i) \times p^i} \omega_{\{i\}} = 0$ .

Thus, Proposition 20.1 and Lemma 20.2 imply Propositions 18.33 and 18.36. Their proofs occupy the next two subsections.

## 20.2 Proof of Proposition 20.1

The homology classes of the  $(z_j^i \times (4 \times a_k^i))_{(j,k) \in \{1, \dots, g_i\}^2}$  and  $(p^i \times \partial C_1(R))$  form a basis of

$$H_2\left(A^{(i)} \times (C_1(R) \setminus A_3^{(i)})\right) = \left(H_1(A^{(i)}) \otimes H_1(R \setminus A^{(i)})\right) \oplus H_2\left(C_1(R) \setminus A^{(i)}\right).$$

According to Lemma 3.12, the evaluation of the cohomology class of any propagating form of  $C_2(R)$  at these classes is  $lk(z_j^i, (4 \times a_k^i)) = \delta_{kj}$  for the first ones and 1 for the last one. In particular, the form of the statement integrates correctly on this basis.

Let us first prove Proposition 20.1 when  $\underline{x} = \{1\}$ . Set  $A^1 = A$ , and forget about the superfluous superscripts 1. Let  $\omega_0$  be a propagating two-form of  $C_2(R)$  that restricts to  $\partial C_2(R) \setminus UB_R$  as  $p_\tau^*(\tilde{\omega}_{S^2})$ , and let  $\omega_b$  be the closed 2-form defined on  $(A_1 \times (C_1(R) \setminus \text{Int}(A_2)))$  by the statement (extended naturally). Since this form  $\omega_b$  integrates correctly on

$$H_2\left(A_1 \times (C_1(R) \setminus \text{Int}(A_2))\right),$$

there exists a one-form  $\eta$  on  $(A_1 \times (C_1(R) \setminus \text{Int}(A_2)))$  such that  $\omega_b = \omega_0 + d\eta$ .

This form  $\eta$  is closed on  $A_1 \times \partial C_1(R)$ . Since  $H^1(A_1 \times (C_1(R) \setminus \text{Int}(A_2)))$  maps surjectively to  $H^1(A_1 \times \partial C_1(R))$ , we may extend  $\eta$  as a closed one-form  $\tilde{\eta}$  on  $(A_1 \times (C_1(R) \setminus \text{Int}(A_2)))$ . Changing  $\eta$  into  $(\eta - \tilde{\eta})$  turns  $\eta$  to a primitive of  $(\omega_b - \omega_0)$  that vanishes on  $A_1 \times \partial C_1(R)$ . Let  $\chi: C_2(R) \rightarrow [0, 1]$  be a smooth function supported in  $(A_1 \times (C_1(R) \setminus \text{Int}(A_2)))$  and constant with the value 1 on  $(A \times (C_1(R) \setminus A_3))$ . Set

$$\omega_a = \omega_0 + d\chi\eta.$$

Then  $\omega_a$  is a closed form that has the required form on  $(A \times (C_1(R) \setminus A_3))$ . Furthermore, the restrictions of  $\omega_a$  and  $\omega_0$  agree on  $\partial C_2(R)$  since  $d\chi\eta$  vanishes there (because  $\eta$  vanishes on  $A_1 \times \partial C_1(R)$ ).

Adding to  $\eta$  a combination  $\eta_c$  of the closed forms  $p_2^*(\eta(z_j))$ , which vanish on  $A_1 \times \partial C_1(R)$ , does not change the above properties, but adds

$$\int_{p \times ([2,4] \times a_j)} d(\chi\eta_c) = \int_{p \times (4 \times a_j)} \eta_c$$

to  $\int_{p \times \Sigma(a_j)} \omega_a$ . Therefore, since the  $p_2^*(\eta(z_j))$  generate the dual of  $\mathcal{L}_A$ , we may choose  $\eta_c$  so that all the  $\int_{p \times \Sigma(a_j)} \omega_a$  vanish. After this step,  $\omega_a$  is a closed form, which takes the prescribed values on

$$PS_a = \partial C_2(R) \cup (A \times (C_1(R) \setminus A_3)),$$



and such that all the  $\int_{p \times \Sigma(a_j)} \omega_a$  vanish. To make  $\omega_a$  take the prescribed values on  $\iota(PS_a)$  and integrate as required on  $\Sigma(a_j) \times p$ , we apply similar modifications to  $\omega_a$  on the symmetric part  $(C_1(R) \setminus \text{Int}(A_2)) \times A_1$ . The support of these modifications is disjoint from the support of the previous ones. Therefore, they do not interfere and transform  $\omega_a$  into a closed form  $\omega_b$  with the following additional properties:

- the form  $\omega_b$  has the prescribed form on  $(C_1(R) \setminus A_3) \times A$ ,
- we have  $\int_{\Sigma(a_j) \times p} \omega_b = 0$ , for all  $j = 1, \dots, g_1$ .

Thus, the form  $\omega = \omega_b$  (resp.  $\omega = \frac{\omega_b - \iota^*(\omega_b)}{2}$  if antisymmetry is desired) has all the required properties, and Proposition 20.1 is proved for  $\underline{x} = \{1\}$ .

We now proceed by induction on  $x$ . We start with a 2-form  $\omega_0$  that satisfies all the hypotheses with  $\underline{x} - 1$  instead of  $\underline{x}$ . By the first step, we also assume that we have a 2-form  $\omega_b$  satisfying all the hypotheses with  $\{x\}$  instead of  $\underline{x}$ , with the enlarged  $A_1^{(x)}$  replacing  $A^{(x)}$ .

Now, we proceed similarly. There exists a one-form  $\eta$  on  $C_2(R)$  such that  $\omega_b = \omega_0 + d\eta$ . The exact sequence

$$0 = H^1(C_2(R)) \longrightarrow H^1(\partial C_2(R)) \longrightarrow H^2(C_2(R), \partial C_2(R)) \cong H_4(C_2(R)) = 0$$

implies that  $H^1(\partial C_2(R))$  is trivial. Therefore, the form  $\eta$  is exact on  $\partial C_2(R)$ . Thus, we may assume that  $\eta$  vanishes on  $\partial C_2(R)$ , which we do. Let

$$\chi: C_2(R) \rightarrow [0, 1]$$

be a smooth function supported in  $(A_1^{(x)} \times (C_1(R) \setminus \text{Int}(A_2^{(x)})))$  and constant with the value 1 on  $(A^{(x)} \times (C_1(R) \setminus A_3^{(x)}))$ . Again, we are going to modify  $\eta$  by some closed forms so that

$$\omega_a = \omega_0 + d\chi\eta$$

has the prescribed value on

$$PS_a = \partial C_2(R) \cup \left( \bigcup_{k=1}^x (A^{(k)} \times (C_1(R) \setminus A_3^{(k)})) \right) \cup \left( \bigcup_{k=1}^{x-1} ((C_1(R) \setminus A_3^{(k)}) \times A^{(k)}) \right).$$

Our form  $\omega_a$  is as required everywhere except possibly on

$$\left( A_1^{(x)} \times (C_1(R) \setminus \text{Int}(A_2^{(x)})) \right) \setminus \left( A^{(x)} \times (C_1(R) \setminus A_3^{(x)}) \right)$$

and in particular on the intersection of this domain with the domains where it was normalized previously, which is included in

$$A_1^{(x)} \times (\partial C_1(R) \cup (\cup_{k=1}^{x-1} A^{(k)})).$$

Recall that  $\eta$  vanishes on  $A_1^{(x)} \times \partial C_1(R)$ . Our assumptions also imply that  $\eta$  is closed on  $A_1^{(x)} \times A^{(k)}$ , for any  $k < x$ . Let us prove that they imply that  $\eta$  is exact on  $A_1^{(x)} \times A^{(k)}$ , for any  $k < x$ . To do that, it suffices to check the following two assertions.

1. For any  $j = 1, \dots, g_x$ , we have  $\int_{z_j^x \times p^k} \eta = 0$ .
2. For any  $j = 1, \dots, g_k$ , we have  $\int_{p^x \times z_j^k} \eta = 0$ .

Let us prove the first assertion. Let  $\infty(v) \in \partial C_1(R)$ , and let  $[p^k, \infty(v)]$  be a path from  $p^k$  to  $\infty(v)$  in  $C_1(R)$  that intersects  $\mathcal{C}$  like the path  $[p^k, q^k]$  introduced before Proposition 18.36. Since  $\int_{z_j^x \times \infty(v)} \eta = 0$ , we have

$$\int_{z_j^x \times p^k} \eta = \int_{\partial(z_j^x \times [p^k, \infty(v)])} \eta = \int_{z_j^x \times [p^k, \infty(v)]} (\omega_b - \omega_0),$$

where  $\int_{z_j^x \times [p^k, \infty(v)]} \omega_b = 0$  because the supports of the  $\eta(z_\ell^x)$  do not intersect  $[p^k, \infty(v)]$ . Let us compute

$$\int_{z_j^x \times [p^k, \infty(v)]} \omega_0 = - \int_{\Sigma(z_j^x) \times \partial[p^k, \infty(v)]} \omega_0 = \int_{\Sigma(z_j^x) \times \{p^k\}} \omega_0.$$

The last integral vanishes because

1. the surface  $\Sigma(z_j^x)$  intersects  $A_4^{(k)}$  as copies of  $\Sigma(a_\ell^k)$ ,
2. we have  $\int_{\Sigma(a_\ell^k) \times p^k} \omega_0 = 0$ , thanks to the third condition of Proposition 20.1, and
3. the integral of  $\omega_0$  also vanishes on the remaining part of  $\Sigma(z_j^x) \times p^k$  since  $\omega_0$  is determined on  $((C_1(R) \setminus A_4^{(k)}) \times A^{(k)})$  and since the support of  $\omega(p^k)$  is disjoint from  $\Sigma(z_j^x)$ .

Let us prove the second assertion, namely  $\int_{p^x \times z_j^k} \eta = 0$  for  $j \in \underline{g}_k$ . Again, we have  $\int_{\infty(v) \times z_j^k} \eta = 0$  since  $\eta$  vanishes on  $\partial C_2(R)$ . We get

$$\int_{p^x \times z_j^k} \eta = - \int_{[p^x, \infty(v)] \times z_j^k} (\omega_b - \omega_0).$$

The integral  $\int_{[p^x, \infty(v)] \times z_j^k} \omega_0$  is zero because of the behavior of  $\omega_0$  on  $(C_1(R) \setminus A_4^{(k)}) \times A^{(k)}$ . We are left with the computation of

$$\int_{[p^x, \infty(v)] \times z_j^k} \omega_b = \int_{(\partial[p^x, \infty(v)]) \times \Sigma(z_j^k)} \omega_b = - \int_{\{p^x\} \times \Sigma(z_j^k)} \omega_b.$$

Again, we know that this integral is zero along the intersection of  $\{p^x\} \times \Sigma(z_j^k)$  with  $A^{(x)} \times (C_1(R) \setminus A_4^{(x)})$  because  $\Sigma(z_j^k)$  does not meet the support of  $\omega(p^x)$ . We conclude because  $\int_{\{p^x\} \times \Sigma(a_\ell^x)} \omega_b = 0$  and because  $\Sigma(z_j^k)$  intersects  $A_4^{(x)}$  along copies of  $\Sigma(a_\ell^x)$ .

Since  $\eta$  is exact on  $A_1^{(x)} \times (\cup_{k=1}^{x-1} A^{(k)})$ , we can assume that it vanishes identically there.

Thus, the form  $\omega_a$  takes the prescribed values on  $A^{(x)} \times (C_1(R) \setminus A_4^{(x)})$ , it coincides with  $\omega_0$  where  $\omega_0$  was prescribed, and it integrates correctly along the  $\Sigma(a_\ell^k) \times p^k$  and their symmetric with respect to  $\iota$ , for  $k \neq x$ . Let us now modify  $\eta$  so that the integrals of  $\omega_a$  along the  $\{p^x\} \times \Sigma(a_\ell^x)$  vanish for  $\ell = 1, \dots, g_x$ , too. We do this by adding to  $\eta$  a linear combination  $(-\eta_c)$  of  $p_2^*(\eta(z_j^x))$  that vanishes on the  $A_1^{(x)} \times A^{(k)}$ , for  $k < x$ , so that we do not change the properties obtained above. Let  $f: H_1(R \setminus \text{Int}(A^{(x)})) \rightarrow \mathbb{R}$  be the linear map defined by

$$f(a_\ell^x) = \int_{\{p^x\} \times \Sigma(a_\ell^x)} \omega_a.$$

Set  $\eta_c = \sum_{\ell=1}^{g_x} f(a_\ell^x) p_2^*(\eta(z_\ell^x))$ . We have  $f(y) = \int_{p^x \times y} \eta_c$  for any  $y \in \mathcal{L}_{A^{(x)}}$ .

Fix  $k < x$ . For any  $j \in \underline{g}_k$ , we have

$$\begin{aligned} f(z_j^k) &= \sum_{\ell=1}^{g_x} lk(z_j^k, z_\ell^x) f(a_\ell^x) = \int_{\{p^x\} \times \Sigma(z_j^k)} \omega_a \\ &= \int_{\{\infty(v)\} \times \Sigma(z_j^k)} \omega_a - \int_{[p^x, \infty(v)] \times z_j^k} \omega_a = 0. \end{aligned}$$

The restriction of  $\eta_c$  to  $A_1^{(x)} \times A_k$  may be expressed as

$$\sum_{\ell=1}^{g_x} f(a_\ell^x) \sum_{j=1}^{g_k} lk(z_j^k, z_\ell^x) p_2^*(\eta(a_j^k)) = \sum_{j=1}^{g_k} f(z_j^k) p_2^*(\eta(a_j^k)) = 0.$$

So  $\eta_c$  vanishes on the  $A_1^{(x)} \times A^{(k)}$  for any  $k < x$ . Changing  $\eta$  into  $(\eta - \eta_c)$  does not change  $\omega_a$  on the prescribed set but removes  $\int_{\{p^x\} \times \Sigma(a_\ell^x)} d\chi \eta_c = \int_{\{p^x\} \times (4 \times a_\ell^x)} \eta_c = f(a_\ell^x)$  from  $\int_{\{p^x\} \times \Sigma(a_\ell^x)} \omega_a$ , which becomes 0.

After this step,  $\omega_a$  is a closed form taking the prescribed values on  $PS_a$  such that the integrals of  $\omega_a$  along the  $(\{p^x\} \times \Sigma(a_\ell^x))$  vanish for  $\ell = 1, \dots, g_i$ . To make  $\omega_a$  take the prescribed values on  $\iota(PS_a)$ , we apply similar modifications to  $\omega_a$  on the symmetric part  $(C_1(R) \setminus \mathring{A}_2^{(x)}) \times A_1^{(x)}$ . Again, the support of these modifications is disjoint from the support of the previous ones. Thus, they do not interfere. They transform  $\omega_a$  to a closed form  $\omega_c$  with the following additional properties.

- The form  $\omega_c$  has the prescribed form on  $(C_1(R) \setminus A_3^{(x)}) \times A^{(x)}$ .
- We have  $\int_{\Sigma(a_j^x) \times p^x} \omega_c = 0$  for all  $j = 1, \dots, g_x$ .

Now, the form  $\omega = \omega_c$  (resp.  $\omega = \frac{\omega_c - \iota^*(\omega_c)}{2}$  if antisymmetry is desired) has all the required properties, and Proposition 20.1 is proved. □

### 20.3 Proof of Lemma 20.2

In this section, we conclude the proofs of Propositions 18.33 and 18.36 by proving Lemma 20.2. We first state some homological lemmas.

**Lemma 20.3.** *Let  $S$  be a closed (oriented) surface. Let  $S$  and  $S^+$  be two copies of  $S$ . Let  $(c_i)_{i \in \underline{2g}}$  and  $(c_i^*)_{i \in \underline{2g}}$  be two dual bases of  $H_1(S; \mathbb{Z})$  such that  $\langle c_i, c_j^* \rangle_S = \delta_{ij}$ . Let  $*$  be a point of  $S$ . Set  $\text{diag}(S \times S^+) = \{(x, x^+) : x \in S\}$ . We have the following equality in  $H_2(S \times S^+; \mathbb{Z})$ :*

$$[\text{diag}(S \times S^+)] = [* \times S^+] + [S \times *^+] + \sum_{i=1}^{2g} [c_i \times c_i^{*+}].$$

PROOF: We have

$$H_2(S \times S^+; \mathbb{Z}) = \mathbb{Z} [* \times S^+] \oplus \mathbb{Z} [S \times *^+] \oplus \bigoplus_{(i,j) \in \underline{2g}^2} \mathbb{Z} [c_i \times c_j^{*+}].$$

The dual basis of the above basis with respect to the intersection form is

$$\left( [S \times *^+], [* \times S^+], ([c_i^* \times c_j^+])_{(i,j) \in \underline{2g}^2} \right).$$

To get the coordinates of  $[\text{diag}(S \times S^+)]$  with respect to the first basis, we compute its intersection numbers with the second one. We have

$$\langle [\text{diag}(S \times S^+)], [c_i^* \times c_i^+] \rangle = \pm 1.$$

The tangent space to  $\text{diag}(S \times S^+)$  is naturally parametrized by  $(u_i, v_i^*, u_i, v_i^*)$ , and the tangent space to  $[c_i^* \times c_i^+]$  is naturally parametrized by  $(0, w_i^*, x_i, 0)$ . So the intersection sign is the sign of the permutation

$$(u, v, w, x) \mapsto (u, w, x, v),$$

which is  $+1$ . □

**Lemma 20.4.** *Let  $\Sigma$  be a connected compact oriented surface with one boundary component  $J(S^1)$  equipped with a basepoint  $* = J(1)$ . Let  $(c_i)_{i \in \underline{2g}}$  and  $(c_i^*)_{i \in \underline{2g}}$  be two dual bases of  $H_1(\Sigma; \mathbb{Z})$  such that  $\langle c_i, c_j^* \rangle = \delta_{ij}$ . Let  $\Sigma$  and  $\Sigma^+$  be two copies of  $\Sigma$ . Set  $J^+ = J^+(S^1) = \partial\Sigma^+$ . Define the subspaces  $J \times_{*, \leq} J^+$  and  $J \times_{*, \geq} J^+$  of  $J \times J^+$  to be*

$$J \times_{*, \leq} J^+ = \left\{ (J(\exp(2i\pi t)), J(\exp(2i\pi u))) : (t, u) \in [0, 1]^2, t \leq u \right\}$$

and

$$J \times_{*, \geq} J^+ = \left\{ (J(\exp(2i\pi t)), J(\exp(2i\pi u))) : (t, u) \in [0, 1]^2, t \geq u \right\}.$$

Let  $\text{diag}(\Sigma \times \Sigma^+)$  be the subspace  $\{(x, x) : x \in \Sigma\}$  of  $\Sigma \times \Sigma^+$ . Then the chains

$$C_{*, \leq}(\Sigma, \Sigma^+) = \text{diag}(\Sigma \times \Sigma^+) - * \times \Sigma^+ - \Sigma \times *^+ - J \times_{*, \leq} J^+$$

and

$$C_{*, \geq}(\Sigma, \Sigma^+) = \text{diag}(\Sigma \times \Sigma^+) - * \times \Sigma^+ - \Sigma \times *^+ + J \times_{*, \geq} J^+$$

are cycles, and we have

$$[C_{*, \leq}(\Sigma, \Sigma^+)] = [C_{*, \geq}(\Sigma, \Sigma^+)] = \sum_{i=1}^{2g} [c_i \times c_i^{*+}]$$

in  $H_2(\Sigma \times \Sigma^+; \mathbb{Z})$ .

**PROOF:** Since  $\partial(J \times_{*, \leq} J^+) = \text{diag}(J \times J^+) - * \times J^+ - J \times *^+$ , the chain  $C_{*, \leq}(\Sigma, \Sigma^+)$  is a cycle. Consider the closed surface  $S$  obtained from  $\Sigma$  by gluing a disk  $D$  along  $J$ . According to Lemma 20.3, we have

$$[\text{diag}(S \times S^+)] = [* \times S^+] + [S \times *^+] + \sum_{i=1}^{2g} [c_i \times c_i^{*+}]$$

in  $H_2(S \times S^+; \mathbb{Z})$ . This implies

$$[C_{*,\leq}(\Sigma, \Sigma^+) - C_{*,\leq}(-D, (-D)^+)] = \sum_{i=1}^{2g} [c_i \times c_i^{*+}]$$

in  $H_2(S \times S^+; \mathbb{Z})$ . Since the cycle  $C_{*,\leq}(-D, (-D)^+)$  lies in  $D \times D^+$ , it is null-homologous there. Since  $H_2(\Sigma \times \Sigma^+; \mathbb{Z})$  injects naturally into  $H_2(S \times S^+; \mathbb{Z})$ , we deduce that  $[C_{*,\leq}(\Sigma, \Sigma^+)] = \sum_{i=1}^{2g} [c_i \times c_i^{*+}]$  in  $H_2(\Sigma \times \Sigma^+; \mathbb{Z})$ . The proof for  $C_{*,\geq}(\Sigma, \Sigma^+)$  is similar.  $\square$

Consider a rational homology handlebody  $A$  with a collar  $[-4, 0] \times \partial A$  of its boundary. For  $s \in [-4, 0]$ , recall  $A_s = A \setminus ([s, 0] \times \partial A)$  and  $\partial A_s = \{s\} \times \partial A$ . Fix pairwise disjoint simple closed curves  $(a_i)_{i=1, \dots, g_A}$  and pairwise disjoint simple closed curves  $(z_i)_{i=1, \dots, g_A}$  on  $\partial A$  such that  $\mathcal{L}_A = \bigoplus_{i=1}^{g_A} [a_i]$  and  $\langle a_i, z_j \rangle_{\partial A} = \delta_{ij}$ .

Consider a curve  $a$  of  $\partial A$  disjoint from the curves  $a_i$ . The class of  $a$  is in  $\mathcal{L}_A$ . Let  $k \in \mathbb{N} \setminus \{0\}$  be its order  $H_1(A; \mathbb{Z})$ . Let  $\Sigma = k\Sigma(a)$  be a surface of  $A$  immersed in  $A$  bounded by  $ka$ . Assume that  $\Sigma$  intersects  $[-1, 0] \times \partial A$  as  $k$  copies of  $[-1, 0] \times a$ , the interior  $\text{Int}(A_{-1})$  of  $A_{-1}$  as an embedded surface, and  $\text{Int}(A_{-1}) \setminus A_{-2}$  as  $k$  disjoint annuli. See the thick part of Figure 20.1. For  $s \in [-2, 0]$ , set  $\Sigma_s = \Sigma \cap A_s$ .

**Lemma 20.5.** *With the above notation, let  $(c_i)_{i=1, \dots, 2g}$  and  $(c_i^*)_{i=1, \dots, 2g}$  be two dual bases of  $H_1(\Sigma_{-2}; \mathbb{Z})/H_1(\partial\Sigma_{-2}; \mathbb{Z})$  such that  $\langle c_i, c_j^* \rangle = \delta_{ij}$ . Represent  $(c_i)_{i=1, \dots, 2g}$  and  $(c_i^*)_{i=1, \dots, 2g}$  by curves  $(c_i)_{i=1, \dots, 2g}$  and  $(c_i^*)_{i=1, \dots, 2g}$  of  $\Sigma_{-2}$ . Let  $\Sigma_{-2} \times [-1, 1]$  denote a tubular neighborhood of  $\Sigma_{-2} = \Sigma_{-2} \times \{0\}$  in  $A_{-2}$ . For a curve  $\sigma$  of  $\Sigma_{-2}$ , the curve  $\sigma \times \{1\}$  is denoted by  $\sigma^+$ .*

*Then  $\sum_{i=1}^{2g} c_i \times c_i^*$  is homologous to  $\sum_{(j,\ell) \in g_A^2 \setminus \text{diag}} \langle \Sigma, \Sigma(a_j), \Sigma(a_\ell) \rangle_A z_j \times z_\ell$  in  $A^2$ . Furthermore, the sum  $\sum_{i=1}^{2g} c_i \times c_i^{*+}$  is homologous to*

$$\sum_{(j,\ell) \in g_A^2 \setminus \text{diag}} \langle \Sigma, \Sigma(a_j), \Sigma(a_\ell) \rangle_A z_j \times z_\ell - gUA|_*$$

in  $C_2(A)$ .

**PROOF:** Assume that  $\Sigma$  and the  $\Sigma(a_j)$  are transverse to each other. For  $(j, \ell) \in \{1, \dots, g_A\}^2$ , set  $\gamma_{\Sigma j} = \Sigma \cap \Sigma(a_j)$  and  $\gamma_{\Sigma \ell} = \Sigma \cap \Sigma(a_\ell)$ . If  $j \neq \ell$ , also set  $\gamma_{j\ell} = \Sigma(a_j) \cap \Sigma(a_\ell)$ . We have

$$c_i = \sum_{j=1}^{g_A} \langle c_i, \Sigma(a_j) \rangle_A z_j = \sum_{j=1}^{g_A} \langle c_i, \gamma_{\Sigma j} \rangle_\Sigma z_j \quad \text{and} \quad c_i^* = \sum_{\ell=1}^{g_A} \langle c_i^*, \gamma_{\Sigma \ell} \rangle_\Sigma z_\ell$$

in  $H_1(A)$ . This implies

$$c_i \times c_i^* = \sum_{(j,\ell) \in g_A^2} \langle c_i, \gamma_{\Sigma j} \rangle_{\Sigma} \langle c_i^*, \gamma_{\Sigma \ell} \rangle_{\Sigma} z_j \times z_{\ell}$$

in  $H_2(A^2)$ . On the other hand, we have

$$\gamma_{\Sigma j} = \sum_{i=1}^{2g} \langle c_i, \gamma_{\Sigma j} \rangle_{\Sigma} c_i^* \text{ and } \gamma_{\Sigma \ell} = - \sum_{i=1}^{2g} \langle c_i^*, \gamma_{\Sigma \ell} \rangle_{\Sigma} c_i$$

in  $H_1(\Sigma_{-2})/H_1(\partial\Sigma_{-2})$ . This implies

$$\langle \gamma_{\Sigma j}, \gamma_{\Sigma \ell} \rangle_{\Sigma} = \sum_{i=1}^{2g} \langle c_i, \gamma_{\Sigma j} \rangle_{\Sigma} \langle c_i^*, \gamma_{\Sigma \ell} \rangle_{\Sigma}.$$

In particular, we have

$$\sum_{i=1}^{2g} \langle c_i, \gamma_{\Sigma j} \rangle_{\Sigma} \langle c_i^*, \gamma_{\Sigma j} \rangle_{\Sigma} = 0$$

for any  $j \in \{1, \dots, g_A\}$ . If  $j \neq \ell$ , then  $\langle \Sigma, \Sigma(a_j), \Sigma(a_{\ell}) \rangle_A = \langle \gamma_{\Sigma j}, \gamma_{\Sigma \ell} \rangle_{\Sigma}$ . So  $\alpha = \sum_{i=1}^{2g} c_i \times c_i^{*+}$  is homologous to  $\sum_{(j,\ell) \in g_A^2 \setminus \text{diag}} \langle \Sigma, \Sigma(a_j), \Sigma(a_{\ell}) \rangle_A z_j \times z_{\ell}$  as announced.

Let us now compute the homology class  $[\alpha]$  of  $\alpha$  in  $H_2(C_2(A))$ . For a curve  $\sigma$  of  $\Sigma_{-2}$ , both  $a \times \sigma^+$  and  $\sigma \times a^+$  are null-homologous in  $C_2(A)$ . Hence, the class  $[\alpha]$  depends only on the class of  $\sum_{i=1}^{2g} c_i \otimes c_i^{*+}$  in  $H_1(\Sigma_{-2})/H_1(\partial\Sigma_{-2}) \otimes H_1(\Sigma_{-2}^+)/H_1(\partial\Sigma_{-2}^+)$ . This class is determined by the property that for any two closed curves  $e$  and  $f$  of  $\Sigma$ , we have  $\langle e \times f^+, \alpha \rangle_{\Sigma \times \Sigma^+} = -\langle e, f \rangle_{\Sigma}$ . So the class  $[\alpha] \in H_2(C_2(A))$  is independent of the dual bases  $(c_i)$  and  $(c_i^*)$ . In particular, we have  $[\alpha] = [\sum_{i=1}^{2g} c_i^* \times (-c_i^+)]$ . Set

$$\beta = \sum_{(j,\ell) \in g_A^2 \setminus \text{diag}} \langle \Sigma, \Sigma(a_j), \Sigma(a_{\ell}) \rangle_A z_j \times z_{\ell} - gUA|_*.$$

The previous computation tells us that the difference  $[\alpha - \beta]$  is a rational multiple of  $[UA|_*]$  in  $H_2(C_2(A))$ . To evaluate this multiple, we embed  $A$  in a rational homology ball obtained from  $A$  by adding thickened disks along neighborhoods of the  $z_i$ . Embed this rational homology ball in a rational homology sphere  $R$ . We get

$$[\alpha - \beta] = \frac{1}{2} \sum_{i=1}^{2g} (lk_R(c_i, c_i^{*+}) - lk_R(c_i^{*-}, c_i)) [UA|_*] - \sum_{(j,\ell) \in \{1, \dots, g_A\}^2 \setminus \text{diag}} \langle \Sigma, \Sigma(a_j), \Sigma(a_{\ell}) \rangle_A lk_R(z_j, z_{\ell}) [UA|_*]$$

in  $H_2(C_2(R); \mathbb{R})$ . Since  $lk_R(z_j, z_\ell) = 0$ , the second row vanishes and we obtain  $[\alpha - \beta] = -g[UA|_*]$  in  $H_2(C_2(R); \mathbb{R})$ . Since  $[UA|_*] \neq 0$  in  $H_2(C_2(R); \mathbb{R})$ , this equality also holds in  $H_2(C_2(A); \mathbb{R})$ .  $\square$

We now define a cycle  $F^2(\Sigma(a))$  of  $\partial C_2(A)$  associated to the surface  $\Sigma = k\Sigma(a)$  introduced before Lemma 20.5. Let  $(a \times [-1, 1])$  be a tubular neighborhood of  $a$  in  $\partial A$ . Let  $p(a) \in a$ . View  $a$  as the image of a map  $a: [0, 1] \rightarrow a$  such that  $a(0) = a(1) = p(a)$ . Set

$$\Sigma^+ = \Sigma_{-1} \cup k \left\{ (t - 1, a(\alpha), t) : (t, \alpha) \in [0, 1]^2 \right\}.$$

So  $\partial \Sigma^+ = ka^+$ , where  $a^+ = a \times \{1\}$ . Set

$$p(a)^+ = (p(a), 1) = (0, p(a), 1) \in a \times [-1, 1] \subset (\partial A = \{0\} \times \partial A).$$

Recall  $a \times_{p(a), \geq} a^+ = \{((a(v), 0), (a(w), +1)) : (v, w) \in [0, 1]^2, v \geq w\}$ . Let  $T(a)$  be the closure of  $\{((a(v), 0), (a(v), t)) : (t, v) \in ]0, 1] \times [0, 1]\}$  (oriented by  $(t, v)$ ) in  $\partial C_2(A)$ . Let  $s_+(\Sigma)$  be the positive normal section of  $U(A)|_\Sigma$ . Let  $g$  be the genus of  $\Sigma$ . Set  $e(\Sigma(a)) = \frac{1}{k}\Sigma = \frac{1}{k}(g + k - 1)$ . We have

$$e(\Sigma(a)) = \frac{-\chi(\Sigma)}{2k} + \frac{1}{2}.$$

**Lemma 20.6.** *With the above notation, the chain*

$$\begin{aligned} F^2(\Sigma(a)) &= \frac{1}{k}s_+(\Sigma) + T(a) - \left(p(a) \times \frac{1}{k}\Sigma^+\right) - \left(\frac{1}{k}\Sigma \times p(a)^+\right) + \left(a \times_{p(a), \geq} a^+\right) \\ &\quad + e(\Sigma(a))[UA|_*] - \sum_{(j, \ell) \in \{1, \dots, gA\}^2 \setminus \text{diag}} \langle \Sigma(a), \Sigma(a_j), \Sigma(a_\ell) \rangle z_j \times z_\ell \end{aligned}$$

is a cycle, and it is null-homologous in  $C_2(A)$ .

**PROOF:** For  $k = 1$  (when we are dealing with integral homology handlebodies, for example), it is a direct consequence of Lemma 20.4 and Lemma 20.5 above. Let us now focus on the case  $k > 1$ . Observe that  $F^2(\Sigma(a))$  is a cycle. Without loss of generality, assume

$$\Sigma \cap ([-2, -1] \times \partial A) = \left\{ \left(t - 2, a(\alpha), (1 - t)\frac{j - 1}{k}\right) : (t, \alpha) \in [0, 1]^2, j \in \underline{k} \right\}.$$

For the proof, we change  $\Sigma^+$  to a surface (still denoted by  $\Sigma^+$ ) with the same boundary as follows.



Recall  $\Sigma_{-2} = \Sigma \cap A_{-2}$ . We have  $\partial\Sigma_{-2} = \cup_{j=1}^k (\{-2\} \times a \times \{\frac{j-1}{k}\})$ . Let  $\Sigma_{-2}^+$  be a parallel copy of  $\Sigma$  on its positive side with boundary  $\partial\Sigma_{-2}^+ = \cup_{j=1}^k (\{-2\} \times a \times \{\frac{j-1/2}{k}\})$ . Redefine  $\Sigma^+$  so that we have  $\Sigma^+ \cap A_{-2} = \Sigma_{-2}^+$ ,

$$\Sigma^+ \cap ([-1, 0] \times \partial A) = k([-1, 0] \times a \times \{1\}),$$

and

$$\Sigma^+ \cap ([-2, -1] \times \partial A) = \left\{ (t-2, a(\alpha), (1-t)\frac{j-1/2}{k} + t) : (t, \alpha) \in [0, 1]^2, j \in \underline{k} \right\}$$

as in Figure 20.1, which represents  $\Sigma \cap ([-2, 0] \times p(a) \times [-1, 1])$  as the thick lines and  $\Sigma^+ \cap ([-2, 0] \times p(a) \times [-1, 1])$  as the thin lines when  $k = 3$ . Observe that this modification of  $\Sigma^+$  changes neither the boundary of  $\Sigma^+$  nor the class of the resulting cycle  $F^2(\Sigma(a))$  in  $H_2(C_2(A))$ .

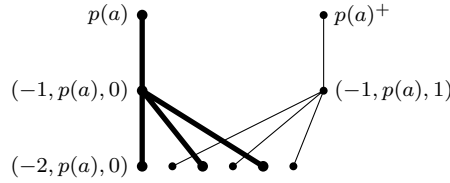


Figure 20.1: How  $\Sigma$  and  $\Sigma^+$  intersect  $([-2, 0] \times p(a) \times [-1, 1])$  in the proof of Lemma 20.6.

Let  $S$  be the closed surface obtained from  $\Sigma_{-2}$  by gluing abstract disks  $D_j$  with respective boundaries  $\{-2\} \times (-a) \times \{\frac{j-1}{k}\}$  on  $\partial\Sigma_{-2}$ . Let  $S^+$  be similarly obtained from  $\Sigma_{-2}^+$  by gluing abstract disks  $D_j^+$  with boundaries  $\{-2\} \times (-a) \times \{\frac{j-1/2}{k}\}$  on  $\partial\Sigma_{-2}^+$ . For  $j = 1, \dots, k$ , set  $p_j = (-2, p(a), \frac{j-1}{k}) \in \partial A_{-2}$  and  $p_j^+ = (-2, p(a), \frac{j-1/2}{k})$ . Recall the dual bases  $(c_i)$  and  $(c_i^*)$  of Lemma 20.5. Lemma 20.3 implies that

$$C(S) = \text{diag}(S \times S^+) - p_1 \times S^+ - S \times p_k^+ - \sum_{i=1}^{2g} c_i \times c_i^{*+}$$

is null-homologous in  $H_2(S \times S^+)$ . Choose closed representatives  $c_i$  of the classes  $c_i$  in the interior of  $\Sigma_{-2}$  so that  $(\Sigma_{-2} \setminus \cup_{i=1}^{2g} c_i)$  is connected. Let  $[p_1, p_j]$  denote a path from  $p_1$  to  $p_j$  in  $(\Sigma_{-2} \setminus \cup_{i=1}^{2g} c_i)$ . Let  $[p_j^+, p_k^+]$  be a path from  $p_j^+$  to  $p_k^+$  in  $(\Sigma_{-2}^+ \setminus \cup_{i=1}^{2g} c_i^+)$ . Add to  $C(S)$  the null-homologous cycles

$$\begin{aligned} \partial(-[p_1, p_j] \times D_j^+) &= p_1 \times D_j^+ - p_j \times D_j^+ + [p_1, p_j] \times \partial D_j^+, \\ \partial(D_j \times [p_j^+, p_k^+]) &= D_j \times p_k^+ - D_j \times p_j^+ + \partial D_j \times [p_j^+, p_k^+], \end{aligned}$$

and the null-homologous cycles  $(-C_{*,\leq}(D_j, D_j^+))$  of Lemma 20.4, for  $j = 1, \dots, k$ . This addition transforms  $C(S)$  to the null-homologous cycle

$$C(\Sigma_{-2}) = \text{diag}(\Sigma_{-2} \times \Sigma_{-2}^+) - p_1 \times \Sigma_{-2}^+ - \Sigma_{-2} \times p_k^+ - \sum_{i=1}^{2g} c_i \times c_i^{*+} + \sum_{j=1}^k (\partial D_j \times [p_j^+, p_k^+] + [p_1, p_j] \times \partial D_j^+ + \partial D_j \times_{p(a), \leq} \partial D_j^+).$$

Let us deform the cycle  $C(\Sigma_{-2})$  continuously in  $\Sigma \times \Sigma^+$  to move the level  $\{-2\} \times \partial A$  to the level  $\{0\} \times \partial A$ . During such a deformation, when  $s$  tends to  $(-1)$ , the path  $[\{s\} \times p_j^+, \{s\} \times p_k^+]$  becomes a loop  $[p_j^+, p_k^+]_{-1}$  on  $\Sigma_{-1}^+ = \Sigma^+ \cap A_{-1}$ , and the path  $[\{s\} \times p_1, \{s\} \times p_j]$  becomes a loop  $[p_1, p_j]_{-1}$  on  $\Sigma_{-1}$ . Thus the cycle  $C(\Sigma_{-2})$  can be naturally deformed in  $\Sigma \times \Sigma^+$  to the following still null-homologous cycle:

$$C(\Sigma) = \text{diag}(\Sigma \times \Sigma^+) - p(a) \times \Sigma^+ - \Sigma \times p(a)^+ - \sum_{i=1}^{2g} c_i \times c_i^{*+} + \sum_{j=1}^k ((-a) \times [p_j^+, p_k^+]_{-1} + [p_1, p_j]_{-1} \times (-a^+)) + k(a \times_{p(a), \geq} a^+).$$

Since  $a$  bounds  $\frac{1}{k}\Sigma$ , the cycle  $(-a) \times [p_j^+, p_k^+]_{-1}$  is homologous to the cycle  $\langle \frac{-1}{k}\Sigma, [p_j^+, p_k^+]_{-1} \rangle_A UA|_*$ , in  $C_2(A)$ . Similarly,  $[p_1, p_j]_{-1} \times (-a^+)$  is homologous to  $\langle \frac{-1}{k}\Sigma^+, [p_1, p_j]_{-1} \rangle_A UA|_*$ , in  $C_2(A)$ . Intersections occur where  $\Sigma$  and  $\Sigma^+$  intersect, in  $([-2, -1] \times \partial A)$ , as shown in Figure 20.2, where the positive normal to  $\Sigma$  goes from left to right.

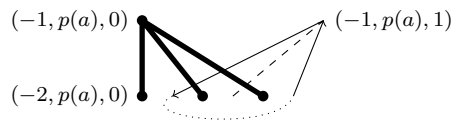


Figure 20.2: The intersection  $\Sigma \cap \Sigma^+$  and the loop  $[p_j^+, p_k^+]_{-1}$  ( $j = 1, k = 3$ )

For any  $j \in \underline{k}$ , we have

$$\left\langle \frac{-1}{k}\Sigma, [p_j^+, p_k^+]_{-1} \right\rangle_A = \frac{k-j}{k} \text{ and } \left\langle \frac{-1}{k}\Sigma^+, [p_1, p_j]_{-1} \right\rangle_A = \frac{j-1}{k}.$$

Therefore, Lemma 20.5 implies that the null-homologous cycle  $C(\Sigma)$  is homologous to

$$\text{diag}(\Sigma \times \Sigma^+) - p(a) \times \Sigma^+ - \Sigma \times p(a)^+ - \sum_{(j,\ell) \in \{1, \dots, g_A\}^2 \setminus \text{diag}} \langle \Sigma, \Sigma(a_j), \Sigma(a_\ell) \rangle z_j \times z_\ell + gUA|_* + (k-1)UA|_* + k(a \times_{p(a), \geq} a^+).$$

This cycle is naturally homologous to  $kF^2(\Sigma(a))$ , which is therefore homologous to zero.  $\square$

**Lemma 20.7.** *If  $A$  is a rational homology handlebody such that  $H_1(A) = \bigoplus_{j=1}^{g(A)} \mathbb{R}[z_j]$ , then we have*

$$H_3(C_2(A)) = \bigoplus_{j=1}^{g(A)} \mathbb{R} [UA|_{z_j}].$$

PROOF: The configuration spaces  $C_2(A)$  and  $C_2(\mathring{A})$  have the same homotopy type, which is the homotopy type of  $\mathring{A}^2 \setminus \text{diag}$ . We have  $H_3(\mathring{A}^2) = H_4(\mathring{A}^2) = 0$  and  $H_4(\mathring{A}^2, \mathring{A}^2 \setminus \text{diag}) \cong H_4(\mathring{A}^2 \times B^3, \mathring{A}^2 \times S^2) = \bigoplus_{j=1}^{g(A)} \mathbb{R}[z_j \times B^3]$ .  $\square$

**Lemma 20.8.** *Let  $i \in \underline{x}$ . For any  $j \in \underline{g}_i$ , assume that the chains  $\Sigma(a_j^i)$  and  $\Sigma'(a_j^i)$  defined before Proposition 18.36 intersect  $[-1, 4] \times \partial A^{(i)}$  as  $[-1, 4] \times a_j^i$ . Also assume that they may be respectively expressed as  $\frac{1}{k'}\Sigma'$  and  $\frac{1}{k}\Sigma$ , for immersed surfaces  $\Sigma$  and  $\Sigma'$ , which respectively intersect  $\text{Int}(A_{-1}^{(i)})$  and  $\text{Int}(A_{-1}^{(i)'})$  as embedded surfaces (as before Lemma 20.5). Fix  $p(a_j^i)$  on  $\{4\} \times a_j^i$ . Then the classes of the cycles  $F^2(\Sigma'(a_j^i))$  of  $\partial C_2(A_4^{(i)'})$  defined in Lemma 20.6, for  $j \in \underline{g}_i$  generate the kernel of the map induced by the inclusion*

$$H_2\left(D_A(\omega_0(R_i))\right) \longrightarrow H_2\left(C_2(A_4^{(i)'})\right)$$

for the domain  $D_A(\omega_0(R_i))$  defined before Lemma 20.2.

PROOF: First note that  $D_A(\omega_0(R_i))$  is homotopically equivalent by retraction to  $\partial C_2(A_4^{(i)'})$ . The cycles  $F^2(\Sigma'(a_j^i))$  lie in  $\partial C_2(A_4^{(i)'})$  and bound chains  $G^3(a_j^i)$  in  $C_2(A_4^{(i)'})$  according to Lemma 20.6. These chains can be assumed to be transverse to the boundary. So they satisfy

$$\left\langle [G^3(a_j^i)], [UA^{(i)'}|_{z_k^i}] \right\rangle_{C_2(A_4^{(i)'})} = \pm \left\langle F^2(\Sigma'(a_j^i)), UA^{(i)'}|_{\{0\} \times z_k^i} \right\rangle_{\partial C_2(A_4^{(i)'})} = \pm \delta_{jk}.$$

Therefore, Poincaré duality and Lemma 20.7 imply

$$H_3(C_2(A_4^{(i)'}), \partial C_2(A_4^{(i)'})) = \bigoplus_{j=1}^{g_i} \mathbb{R} [G^3(a_j^i)].$$

The boundary map of the long exact sequence associated to the pair  $(C_2(A_4^{(i)'}), \partial C_2(A_4^{(i)'}))$  sends the cycle  $[G^3(a_j^i)]$  of  $(C_2(A_4^{(i)'}), \partial C_2(A_4^{(i)'}))$  to  $[F^2(\Sigma'(a_j^i))]$ .  $\square$

PROOF OF LEMMA 20.2: According to Lemma 20.8, it suffices to prove

$$\int_{F^2(\Sigma'(a_j^i))} \omega_0(R_i) = 0$$

for any  $i \in \underline{x}$ , and for any  $j \in \underline{g}_i$ . Fix  $i \in \underline{x}$  and  $j \in \underline{g}_i$ . Set  $a = \{4\} \times a_j^i$ . Let  $F'$  denote the cycle  $F^2(\Sigma'(a))$  of  $\partial C_2(A_4^{(i)'})$  associated to  $\Sigma' = k'\Sigma'(a)$  and to  $p(a) = p(a_j^i)$ , and let  $F$  denote the cycle  $F^2(\Sigma(a))$  of  $\partial C_2(A_4^{(i)})$  similarly associated to  $\Sigma(a)$  and to  $p(a)$ . We have

$$F' = T(a) + a \times_{p(a), \geq} a^+ - p(a) \times \frac{1}{k'} \Sigma'^+ - \frac{1}{k'} \Sigma' \times p(a)^+ + \frac{1}{k'} s_+(\Sigma') + e(\Sigma'(a)) [UA|_*] - \sum_{(p,q) \in \{1, \dots, g_i\}^2 \setminus \text{diag}} \langle \Sigma'(a), \Sigma'(a_p^i), \Sigma'(a_q^i) \rangle (\{4\} \times z_p^i) \times (\{4\} \times z_q^i)$$

Set  $\Sigma'_{-1}^+ = \Sigma'^+ \cap A_{-1}^{(i)'}$ . The integral of  $\omega_0(R_i)$  along

$$-p(a) \times \frac{1}{k'} \Sigma'_{-1}^+ - \frac{1}{k'} \Sigma'_{-1} \times p(a)^+$$

is zero because of the prescribed form of  $\omega_0(R_i)$  on

$$((\check{R} \setminus A_3^{(i)'}) \times A^{(i)'}) \cup (A^{(i)' \prime} \times (\check{R} \setminus A_3^{(i)' \prime})).$$

For the form  $\omega$  of Proposition 20.1, the same argument implies

$$\int_{-p(a) \times \Sigma'_{-1}^+(a_j^i) - \Sigma_{-1}(a_j^i) \times p(a)^+} \omega = 0.$$

The part

$$C = T(a) + a \times_{p(a), \geq} a^+ - (p(a) \times \frac{1}{k'} (\Sigma'^+ \setminus \Sigma'_{-1}^+)) - (\frac{1}{k'} (\Sigma' \setminus \Sigma'_{-1}) \times p(a)^+) + \frac{1}{k'} s_+(\Sigma' \setminus \Sigma'_0)$$

of  $F'$  (or  $F$ ) lies in the intersection of  $C_2(A_4^{(i)})$  and  $C_2(A_4^{(i)'})$ , inside which  $\omega = \omega_0(R_i)$ . So we have

$$\int_C \omega = \int_C \omega_0(R_i).$$

Recall

$$\int_{(\{4\} \times z_p^i) \times (\{4\} \times z_q^i)} \omega = lk(z_p^i, z_q^i)$$

for any  $(p, q) \in \{1, \dots, g_i\}^2 \setminus \text{diag}$ . Since  $lk(z_p^i, z_q^i) = lk(z_q^i, z_p^i)$ , and since

$$\langle \Sigma(a), \Sigma(a_p^i), \Sigma(a_q^i) \rangle = -\langle \Sigma(a), \Sigma(a_q^i), \Sigma(a_p^i) \rangle,$$

the integral of  $\omega$  along

$$\sum_{(p,q) \in \{1, \dots, g_i\}^2 \setminus \text{diag}} \langle \Sigma(a), \Sigma(a_p^i), \Sigma(a_q^i) \rangle (\{4\} \times z_p^i) \times (\{4\} \times z_q^i)$$

is equal to zero. The integral of  $\omega_0(R_i)$  along

$$\sum_{(p,q) \in \{1, \dots, g_i\}^2 \setminus \text{diag}} \langle \Sigma'(a), \Sigma'(a_p^i), \Sigma'(a_q^i) \rangle (\{4\} \times z_p^i) \times (\{4\} \times z_q^i)$$

vanishes similarly.

Since  $\int_F \omega = 0$ , we have

$$\begin{aligned} \int_{F'} \omega_0(R_i) &= \int_{F'} \omega_0(R_i) - \int_F \omega \\ &= \int_{\frac{1}{k'} s_+(\Sigma'_0)} \omega_0(R_i) - \int_{s_+(\Sigma_0(a_j^i))} \omega + e(\Sigma'(a)) - e(\Sigma(a)). \end{aligned}$$

Recall that  $\tau$  coincides with  $\tau_i$  on  $[0, 4] \times \partial A^{(i)}$ . When  $\tau$  maps  $e_3$  to the positive normal to  $\Sigma_0^+$  along  $\partial \Sigma_0^+$ , Proposition 19.22 and Lemma 19.19 imply

$$\int_{\frac{1}{k'} s_+(\Sigma'_0)} \omega_0(R_i) = \frac{1}{2} d(\tau(\cdot \times e_2), \{0\} \times a) + \frac{1}{2k'} \chi(\Sigma'_{-2}).$$

We get

$$\begin{aligned} \int_{\frac{1}{k'} s_+(\Sigma'_0)} \omega_0(R_i) + e(\Sigma'(a)) &= \frac{1}{2} d(\tau(\cdot \times e_2), \{0\} \times a) + \frac{1}{2} \\ &= \int_{s_+(\Sigma_0(a_j^i))} \omega + e(\Sigma(a)) \end{aligned}$$

and  $\int_{F'} \omega_0(R_i) = 0$ . When  $\tau$  does not map  $e_3$  to the positive normal to  $\Sigma_0^+$  along  $\partial \Sigma_0^+$ , perform a simultaneous homotopy on  $\tau$  and  $\tau_i$  to make this happen without changing

$$\int_{\frac{1}{k'} s_+(\Sigma'_0)} \omega_0(R_i) - \int_{s_+(\Sigma_0(a_j^i))} \omega.$$

Thus, the above proof still implies  $\int_{F'} \omega_0(R_i) = 0$ . □

# Chapter 21

## Much more flexible definitions of $\mathcal{Z}$

### 21.1 More propagating forms associated to pseudo-parallelizations

In this section, we define nonhomogeneous propagating forms associated with pseudo-parallelizations, and we give more flexible definitions of  $Z$  involving these forms. In Section 21.2, we define propagating chains associated with pseudo-parallelizations, which allow discrete computations of  $Z$  associated with pseudo-parallelizations. As in Chapter 11 and Section 17.1, the corresponding discrete definition of  $Z$  are justified by using nonhomogeneous propagating forms  $\varepsilon$ -dual (as in Definition 11.6) to these propagating chains.

Let  $A$  be an oriented 3-manifold with possible boundary, equipped with a pseudo-parallelization  $\tilde{\tau} = (N(\gamma); \tau_e, \tau_b)$  as in Definition 19.9. Let  $\omega_s$  be a 2-form of  $S^2$  invariant under the rotations around the vertical axis such that  $\int_{S^2} \omega_s = 1$ . Let  $\omega_i$  be a 2-form of  $S^2$  such that  $\int_{S^2} \omega_i = 1$ . Let  $\eta_{i,s,1}$  be a 1-form of  $S^2$  such that  $\omega_i = \omega_s + d\eta_{i,s,1}$ . Let  $\varepsilon \in ]0, 1/2[$  be the small positive number of Definitions 19.9 and 19.13. Let  $\varepsilon_i \in ]0, \varepsilon/2[$ , and let  $k$  be a large integer greater than 3. Let  $p(\tau_b)$  denote the projection from  $UN(\gamma)$  to  $S^2$  induced by  $\tau_b$ . We have

$$p(\tau_b)(\tau_b(t, c, u; X \in S^2)) = X.$$

**Lemma 21.1.** *Under the hypotheses above, there exists a one-form  $\eta_{i,s}$  on  $U(A)$  such that*

- $\eta_{i,s} = p(\tau_e)^*(\eta_{i,s,1})$  on  $U\left(A \setminus ([a, b - \varepsilon_i + \varepsilon_i^k] \times \gamma \times [-1, 1])\right)$ ,

- $\eta_{i,s} = p(\tau_b)^*(\eta_{i,s,1})$  on  $[a, b] \times \gamma \times N(\partial[-1, 1]) \times S^2$ ,
- $\eta_{i,s} = \frac{1}{2} \left( p(\tau_b \circ \mathcal{T}_\gamma^{-1})^*(\eta_{i,s,1}) + p(\tau_b \circ F(\gamma, \tau_b)^{-1})^*(\eta_{i,s,1}) \right)$  on  $[a, b - \varepsilon_i - \varepsilon_i^k] \times \gamma \times [-1, 1] \times S^2$ , and
- $\eta_{i,s}$  pulls back through  $p_{[a,b]} \times p_{[-1,1]} \times p(\tau_b)$  on  $U(N(\gamma) = [a, b] \times \gamma \times [-1, 1])$ .

The following addition is useless in this book but useful for studying equivariant invariants as in [Les13]. Let  $c$  and  $d$  be two elements of  $] -1, 1[$  such that  $c < d$ . Let  $v_3$  denote the projection on the third coordinate in  $\mathbb{R}^3$ , and let  $S^2_{]c,d[} = \{X \in S^2 : v_3(X) \in ]c, d[ \}$ . If  $\omega_s$  and  $\omega_i$  are compactly supported in  $S^2_{]c,d[}$ , then  $\eta_{i,s,1}$  can be chosen so that it is also compactly supported in  $S^2_{]c,d[}$ , and  $\eta_{i,s}$  can be chosen so that it is compactly supported in  $[a, b] \times \gamma \times [-1, 1] \times S^2_{]c,d[}$ .

PROOF: The existence of  $\eta_{i,s,1}$  compactly supported in  $S^2_{]c,d[}$ , when  $\omega_s$  and  $\omega_i$  are compactly supported in  $S^2_{]c,d[}$ , comes from the fact that  $H_c^2(S^2_{]c,d[}) = \mathbb{R}$ , according to Theorem B.3.

Extend  $\eta_{i,s}$  to

$$C(\times \gamma) \times S^2 = [b - \varepsilon_i - \varepsilon_i^k, b - \varepsilon_i + \varepsilon_i^k] (\times \gamma) \times [-1 + \frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}] \times S^2$$

as follows. Cover  $S^2$  by three open spaces  $S^2_{]c,d[}$ ,  $N(N) = \{X \in S^2 : v_3(X) > \frac{c+d}{2}\}$ , and  $N(S) = \{X \in S^2 : v_3(X) < \frac{c+d}{2}\}$ . Pick a corresponding partition of unity  $(\chi_{S^2_{]c,d[}}, \chi_{N(S)}, \chi_{N(N)})$  of functions respectively compactly supported on  $S^2_{]c,d[}$ ,  $N(N)$ , and  $N(S)$  whose sum is one. Let  $S$  stand for  $S^2_{]c,d[}$ ,  $N(N)$ , or  $N(S)$ . On the product  $C(\times \gamma) \times S$  the form  $\eta_{i,s}$  is a combination of basic standard one-forms. Smoothly extend the  $\mathbb{R}$ -valued coordinate functions of the forms from  $(\partial(C(\times \gamma))) \times S$  to  $C(\times \gamma) \times S$  in order to obtain  $\eta_{i,s}$  with the required properties, so that  $\eta_{i,s}$  is compactly supported in  $[a, b] \times \gamma \times [-1, 1] \times S^2_{]c,d[}$  when  $\eta_{i,s,1}$  is compactly supported in  $S^2_{]c,d[}$ .  $\square$

Recall the notation of Definition 19.13 and extend  $\tau_e$  to

$$U([b - \varepsilon_i + \varepsilon_i^k, b] \times \gamma \times [-1, 1]),$$

so that it coincides with  $\tau_b$ , there.

**Definition 21.2.** Under the hypotheses of the beginning of this section, define  $\omega = \omega(\tilde{\tau}, \omega_i, k, \varepsilon_i, \eta_{i,s})$  on  $UA$  to satisfy

$$\omega = p(\tau_e)^*(\omega_i) \text{ on } U\left(A \setminus ([a, b - \varepsilon_i + \varepsilon_i^k] \times \gamma \times [-1, 1])\right),$$

$$\omega = \frac{1}{2} \left( p(\tau_b \circ \mathcal{T}_\gamma^{-1})^* (\omega_i) + p(\tau_b \circ F(\gamma, \tau_b)^{-1})^* (\omega_i) \right)$$

on  $U([a, b - \varepsilon_i - \varepsilon_i^k] \times \gamma \times [-1, 1])$ ,

and

$$\omega = p(\tau_b)^* (\omega_s) + d\eta_{i,s} \text{ on } U([b - \varepsilon_i - 2\varepsilon_i^k, b - \varepsilon_i + 2\varepsilon_i^k] \times \gamma \times ]-1, 1[).$$

**Lemma 21.3.** *Definition 21.2 of  $\omega(\tilde{\tau}, \omega_i, k, \varepsilon_i, \eta_{i,s})$  is consistent. Furthermore, we have  $\omega(\tilde{\tau}, \omega_i, k, \varepsilon_i, \eta_{i,s}) = \omega(\tilde{\tau}, \omega_s, k, \varepsilon_i, 0) + d\eta_{i,s}$  on  $UA$ .*

PROOF: Any 2-form  $\omega_s$  invariant under the rotations around the vertical axis may be expressed as  $(\kappa \circ \nu_3)\omega_{S^2}$ . Therefore, when  $\eta_{i,s}$  is zero, Lemma 19.14 ensures that the definitions of  $\omega(\tilde{\tau}, \omega_s, k, \varepsilon_i, 0)$  match on  $[b - \varepsilon_i - 2\varepsilon_i^k, b - \varepsilon_i - \varepsilon_i^k] \times \gamma \times ]-1, 1[$ . Conclude by observing that  $\omega(\tilde{\tau}, \omega_i, k, \varepsilon_i, \eta_{i,s}) = \omega(\tilde{\tau}, \omega_s, k, \varepsilon_i, 0) + d\eta_{i,s}$  on  $UN(\gamma)$ .  $\square$

**Definition 21.4** (General boundary form associated with a pseudo-parallelization  $\tilde{\tau}$ ). Let  $A$  be an oriented 3-manifold equipped with a pseudo-parallelization  $\tilde{\tau}$ . A *boundary form* of  $(A, \tilde{\tau})$  is a 2-form on  $UA$  that may be expressed as  $\omega(\tilde{\tau}, \omega_i, k, \varepsilon_i, \eta_{i,s})$  for some  $(\omega_i, k, \varepsilon_i, \eta_{i,s})$  as in Definition 21.2. When  $\eta_{i,s,1}$ ,  $\omega_s$ , and  $\omega_i$  are compactly supported in  $S_{]c,d[}^2$ , and when the intersection of the support of  $\eta_{i,s}$  with  $[b - \varepsilon_i - 2\varepsilon_i^k, b - \varepsilon_i + 2\varepsilon_i^k] \times \gamma \times [-1, 1] \times S^2$  is a compact subspace of  $[b - \varepsilon_i - 2\varepsilon_i^k, b - \varepsilon_i + 2\varepsilon_i^k] \times \gamma \times [-1, 1] \times S_{]c,d[}^2$ , the boundary form is said to be *adapted to  $S_{]c,d[}^2$* .

Note that Definition 21.4 coincides with Definition 19.13 of a homogeneous boundary form when  $\omega_i = \omega_s = \omega_{S^2}$  and  $\eta_{i,s} = 0$ .

**Lemma 21.5.** *Under the assumptions of Proposition 19.22, for any boundary form  $\omega$  of  $(M, \tilde{\tau})$  as in Definition 21.4, we have*

$$\int_{s_+(\Sigma)} \omega = \frac{1}{2} \chi(\tilde{\tau}(\cdot \times e_2)|_{\partial\Sigma}; \Sigma)$$

and

$$\int_{s_-(\Sigma)} \omega = -\frac{1}{2} \chi(\tilde{\tau}(\cdot \times e_2)|_{\partial\Sigma}; \Sigma).$$

PROOF: Thanks to Lemma 19.20, Lemma 21.5 holds when  $\Sigma$  does not meet  $N(\gamma)$ . Therefore, as in the proof of Proposition 19.22, it suffices to prove Lemma 21.5 when  $\Sigma$  is a meridian of the link  $\gamma$  of the pseudo-parallelization  $\tilde{\tau}$ . Let us treat this case. When  $\omega_i = \omega_s$  and  $\eta_{i,s} = 0$ , the proof of Proposition 19.22 applies. In general, let  $\tilde{\omega}$  be the form obtained from  $\omega$  by changing  $\omega_i$  to  $\omega_s$  and  $\eta_{i,s}$  to 0. The form  $\omega$  may be written as  $\tilde{\omega} + d\eta_{i,s}$  on



$UN(\gamma)$ , where  $\eta_{i,s}$  is expressed as  $p(\tau_e)^*(\eta_{i,s,1})$  along  $s_+(\Sigma)|_{\partial\Sigma}$ . We have  $\int_{s_+(\Sigma)} \omega - \int_{s_+(\Sigma)} \tilde{\omega} = \int_{s_+(\Sigma)|_{\partial\Sigma}} p(\tau_e)^*(\eta_{i,s,1})$ . This is zero since  $p(\tau_e)$  maps  $s_+(\Sigma)|_{\partial\Sigma}$  to a point. We similarly get  $\int_{s_-(\Sigma)} \omega = \int_{s_-(\Sigma)} \tilde{\omega}$ .  $\square$

Theorem 19.25 generalizes as follows to these boundary forms.

**Theorem 21.6.** *Let  $A$  be a compact 3-manifold that embeds in a rational homology 3-ball. Assume that  $A$  is equipped with two pseudo-parallelizations  $\tau_0$  and  $\tau_1$  that coincide with a common genuine parallelization along a regular neighborhood of  $\partial A$ . For  $i \in \underline{3n}$ , let  $\omega_{0,i}(\tau_0)$  be a boundary form of  $(A, \tau_0)$  and let  $\omega_{1,i}(\tau_1)$  be a boundary form of  $(A, \tau_1)$ . There exists a closed 2-form  $\omega(i)$  on  $[0, 1] \times UA$  that restricts*

- to  $\{0\} \times UA$  as  $\omega_{0,i}(\tau_0)$
- to  $\{1\} \times UA$  as  $\omega_{1,i}(\tau_1)$ ,
- to  $[0, 1] \times UA|_{\partial A}$  as  $(\mathbf{1}_{[0,1]} \times p_{\tau_0})^*(\omega_{S,i})$  with respect to a closed 2-form  $\omega_{S,i}$  on  $[0, 1] \times S^2$  and to the projection  $\mathbf{1}_{[0,1]} \times p_{\tau_0} : [0, 1] \times UA|_{\partial A} \rightarrow [0, 1] \times S^2$ .

Let  $n$  be a natural integer. As in Corollary 9.4, set

$$z_n\left([0, 1] \times UA; (\omega(i))_{i \in \underline{3n}}\right) = \sum_{\Gamma \in \mathcal{D}_n^c} \zeta_{\Gamma} \int_{[0,1] \times \check{S}_{V(\Gamma)}(TA)} \bigwedge_{e \in E(\Gamma)} p_e^*\left(\omega(j_E(e))\right) [\Gamma].$$

Then we have

$$z_n\left([0, 1] \times UA; (\omega(i))_{i \in \underline{3n}}\right) = \frac{p_1(\tau_0, \tau_1)}{4} \beta_n.$$

PROOF: The existence of  $\omega_{S,i}$  is proved in Lemma 9.1. The proof of the existence of the form  $\omega(i)$  with its prescribed properties is obtained from the proof of Proposition 19.23 by replacing Proposition 19.22 with Lemma 21.5.

Let  $t$  stand for the coordinate in  $[0, 1]$ . Assume that  $\omega_{0,i}(\tau_0)$  is a homogeneous boundary form and that we have  $\tau_0 = \tau_1$ . Choose  $\omega_{S,i} = \omega_{S^2} + d(t\eta_{i,s,1})$  with the notation before Lemma 21.1. Then the form  $\omega(i)$  can be chosen so that it is equal to

$$p(\tau_e)^*(\omega_{S,i}) \text{ on } [0, 1] \times U\left(A \setminus ([a, b - \varepsilon_i + \varepsilon_i^k] \times \gamma \times [-1, 1])\right),$$

$p(\tau_b)^*(\omega_{S^2}) + d(t\eta_{i,s})$  on  $[0, 1] \times U([b - \varepsilon_i - \varepsilon_i^k, b - \varepsilon_i + \varepsilon_i^k] \times \gamma \times ]-1, 1[)$ , and

$$\frac{1}{2}\left(p(\tau_b \circ \mathcal{T}_{\gamma}^{-1})^*(\omega_{S,i}) + p(\tau_b \circ F(\gamma, \tau_b)^{-1})^*(\omega_{S,i})\right)$$

on  $[0, 1] \times U([a, b - \varepsilon_i - \varepsilon_i^k] \times \gamma \times [-1, 1])$ . In this case, the parts over  $(A \setminus ([a, b] \times \gamma \times [-1, 1]))$  in  $z_n([0, 1] \times UA; (\omega(i))_{i \in \underline{3n}})$  cancel because the forms do not depend on the factor  $A$  (which is factored out via the parallelization  $\tau_e$ ), and the parts over  $([a, b] \times \gamma \times [-1, 1])$  also cancel because the forms do not depend on the factor  $\gamma$ . So we simply find  $z_n([0, 1] \times UA; (\omega(i))_{i \in \underline{3n}}) = 0$  as announced.

In general, as in the beginning of the proof of Proposition 19.24, and as in Corollary 9.4, the sum  $z_n([0, 1] \times UA; (\omega(i)))$  depends only on the restriction of the  $\omega(i)$  to  $\partial([0, 1] \times UA)$ . The above arguments can also be used to prove that  $z_n([0, 1] \times UA; (\omega(i))_{i \in \underline{3n}})$  is independent of the forms  $\omega_{S,i}$ . So  $z_n([0, 1] \times UA; (\omega(i))_{i \in \underline{3n}})$  depends only on the pairs  $(\omega_{0,i}(\tau_0), \omega_{1,i}(\tau_1))$ . Denote it by  $z_n((\omega_{0,i}(\tau_0), \omega_{1,i}(\tau_1))_{i \in \underline{3n}})$ . Observe

$$z_n\left((\omega_{0,i}(\tau_0), \omega_{2,i}(\tau_2))_{i \in \underline{3n}}\right) = z_n\left((\omega_{0,i}(\tau_0), \omega_{1,i}(\tau_1))_{i \in \underline{3n}}\right) + z_n\left((\omega_{1,i}(\tau_1), \omega_{2,i}(\tau_2))_{i \in \underline{3n}}\right)$$

again. Conclude with the study of the special case and with Theorem 19.25. □

**Definition 21.7.** Let  $\check{R}$  be a rational homology  $\mathbb{R}^3$  equipped with an asymptotically standard pseudo-parallelization  $\tilde{\tau}$ . A *propagating form* of  $(C_2(R), \tilde{\tau})$  is a propagating form of  $C_2(R)$  (as in Definition 3.11) that coincides with a boundary form (of Definition 21.4) of  $(\check{R}, \tilde{\tau})$  on  $U\check{R}$ .

**Lemma 21.8.** Let  $\check{R}$  be a rational homology  $\mathbb{R}^3$  equipped with an asymptotically standard pseudo-parallelization  $\tau = (N(\gamma); \tau_e, \tau_b)$ . Let  $K: \mathcal{L} \hookrightarrow \check{R} \setminus N(\gamma)$  be a knot embedding, which is straight with respect to  $\tau|_{\check{R} \setminus N(\gamma)}$ , as in Definition 7.37. Let  $K_{\parallel, \tau}$  denote the parallel of  $K$  induced by  $\tau$ , as in Lemma 7.35. For any propagating form  $\omega_p$  of  $(C_2(R), \tau)$ , we have

$$\int_{C(K; \overset{\tau}{\curvearrowright} S^1)} \omega_p = I_\theta(K, \tau) = lk(K, K_{\parallel, \tau, Y}).$$

PROOF: Lemma 7.35 extends to this case. See also Lemmas 7.13 and 21.3. □

Theorem 7.40 generalizes as follows to pseudo-parallelizations:

**Theorem 21.9.** Let  $\check{R}$  be a rational homology  $\mathbb{R}^3$  equipped with an asymptotically standard pseudo-parallelization  $\tau = (N(\gamma); \tau_e, \tau_b)$ . Let  $L: \mathcal{L} \hookrightarrow \check{R} \setminus N(\gamma)$  be a link embedding, which is straight with respect to  $\tau|_{\check{R} \setminus N(\gamma)}$ . Let  $L_{\parallel, \tau}$  denote

the parallel of  $L$  induced by  $\tau$ . For any  $i \in \underline{3n}$ , let  $\omega(i)$  be a propagating form of  $(C_2(R), \tau)$ . Then

$$Z_n^s(\check{R}, L, (\omega(i))_{i \in \underline{3n}}) = \sum_{\Gamma \in \mathcal{D}_n^s(\mathcal{L})} \zeta_\Gamma I(R, L, \Gamma, (\omega(i))_{i \in \underline{3n}}) [\Gamma] \in \mathcal{A}_n(\mathcal{L}).$$

is independent of the chosen  $\omega(i)$ . Set

$$Z_n^s(\check{R}, L, \tau) = Z_n^s(\check{R}, L, (\omega(i))_{i \in \underline{3n}})$$

and  $Z^s(\check{R}, L, \tau) = (Z_n^s(\check{R}, L, \tau))_{n \in \mathbb{N}}$ . Then we have

$$Z^s(\check{R}, L, \tau) = Z(\check{R}, L, \tau) = \mathcal{Z}^f(\check{R}, L, L_{\parallel, \tau}) \exp\left(\frac{p_1(\tau)}{4} \beta\right),$$

where  $Z(\check{R}, L, \tau)$  is defined in Theorem 19.17 (as in Theorem 7.20), with Definition 19.12 for  $p_1(\tau)$ , and Definition 7.41 for  $\mathcal{Z}^f(\check{R}, L, L_{\parallel, \tau})$ .

Theorem 21.9 is a consequence of the following generalization of Theorem 16.9 to pseudo-parallelizations:

**Theorem 21.10.** *Let  $\mathcal{C}$  be a rational homology cylinder equipped with a pseudo-parallelization  $\tau = (N(\gamma); \tau_e, \tau_b)$ . Let  $L: \mathcal{L} \hookrightarrow R(\mathcal{C}) \setminus N(\gamma)$  be a long tangle representative associated to a tangle in  $\mathcal{C}$ . Let  $\{K_j\}_{j \in I}$  be the set of components of  $L$ . Assume that the bottom (resp. top) configuration of  $L$  is represented by a map  $y^-: B^- \rightarrow D_1$  (resp.  $y^+: B^+ \rightarrow D_1$ ).*

*Let  $N \in \mathbb{N}$ . For  $i \in \underline{3N}$ , let  $\tilde{\omega}(i, S^2) = (\tilde{\omega}(i, t, S^2))_{t \in [0, 1]}$  be a closed 2-form on  $[0, 1] \times S^2$  such that  $\int_{S^2} \tilde{\omega}(i, 0, S^2) = 1$ . There exists a closed 2-form  $\tilde{\omega}(i) = (\tilde{\omega}(i, t))_{t \in [0, 1]}$  on  $[0, 1] \times C_2(R(\mathcal{C}))$  such that*

$$\tilde{\omega}(i) = (\mathbf{1}_{[0, 1]} \times p_{\tau_e})^* (\tilde{\omega}(i, S^2))$$

*on  $[0, 1] \times (\partial C_2(R(\mathcal{C})) \setminus UN(\gamma))$  and the restriction of  $\tilde{\omega}(i, t)$  to  $U\check{R}(\mathcal{C})$  is a boundary form of  $(\check{R}(\mathcal{C}), \tau)$  as in Definition 21.4 for all  $t \in [0, 1]$ .*

*For such a family  $(\tilde{\omega}(i))_{i \in \underline{3N}}$ , and for a subset  $A$  of  $\underline{3N}$  with cardinality  $3k$ , set*

$$Z(\mathcal{C}, L, \tau, A, (\tilde{\omega}(i, t))_{i \in A}) = \sum_{\Gamma \in \mathcal{D}_{k, A}^e(\mathcal{L})} \zeta_\Gamma I(\mathcal{C}, L, \Gamma, (\tilde{\omega}(i, t))_{i \in A}) [\Gamma] \in \mathcal{A}_k(\mathcal{L}).$$

Then

$$Z(\mathcal{C}, L, \tau, A)(t) = Z(\mathcal{C}, L, \tau, A, (\tilde{\omega}(i, t))_{i \in A})$$

depends only on  $(\tilde{\omega}(i, t, S^2))_{i \in A}$  for any  $t$  (and on  $(\mathcal{C}, L, \tau)$ ).

It will be denoted by  $Z(\mathcal{C}, L, \tau, A, (\tilde{\omega}(i, t, S^2))_{i \in A})$ . When  $\tilde{\omega}(i, t, S^2)$  is the standard homogeneous form  $\omega_{S^2}$  on  $S^2$  for any  $i$  and for all  $t \in [0, 1]$ , the map  $Z(\mathcal{C}, L, \tau, \cdot)(t)$  sends any subset of  $\underline{3N}$  with cardinality  $3k$  to  $Z_k(\mathcal{C}, L, \tau)$ .

Furthermore, with the notation of Definition 16.10, for any given orientation of  $L$ , we have

$$Z(\mathcal{C}, L, \tau, \cdot)(t) = \left( \left( \prod_{j \in I} \widetilde{\text{hol}}_{[0,t]}(\eta(\cdot, U_j^+)) \#_j \right) \widetilde{\text{hol}}_{[t,0] \times y^-}(\eta_{B^-, \cdot}) Z(\mathcal{C}, L, \tau, \cdot)(0) \widetilde{\text{hol}}_{[0,t] \times y^+}(\eta_{B^+, \cdot}) \right)_{\square},$$

where  $U_j^+ = p_{\tau}(U^+ K_j)$ .

**PROOF:** Fix  $i \in \underline{3N}$ . Pick a one-form  $\tilde{\eta}_{i,s,1}$  on  $[0, 1] \times S^2$  such that  $\tilde{\omega}(i, S^2) = p_{S^2}^*(\omega_{S^2}) + d\tilde{\eta}_{i,s,1}$ . Use this form to construct a one-form  $\tilde{\eta}_{i,s}(t)$  on  $[0, 1] \times U([a, b] \times \gamma \times [-1, 1])$  that pulls back through  $p_{[0,1]} \times p_{[a,b]} \times p_{[-1,1]} \times p(\tau_b)$  and such that its restriction to  $\{t\} \times U(N(\gamma))$  satisfies the conditions of Lemma 21.1, with respect to  $\tilde{\eta}_{i,s,1}|_{\{t\} \times U(N(\gamma))}$ .

Next use  $\tilde{\eta}_{i,s}$  to construct the restriction of  $\tilde{\omega}(i)$  to  $[0, 1] \times \partial C_2(R(\mathcal{C}))$ , such that  $\tilde{\omega}(i, t)|_{\{t\} \times U\check{R}(\mathcal{C})} = \omega(\tau, \tilde{\omega}(i, t, S^2), k, \varepsilon_i, \tilde{\eta}_{i,s}(t))$ . The existence of  $\tilde{\omega}(i)$  follows, as in Lemma 9.1.

Use the proof of Theorem 16.9 to prove the variation formula that expresses  $Z(\mathcal{C}, L, \tau, A, (\tilde{\omega}(i, t))_{i \in A})$  as a function of  $Z(\mathcal{C}, L, \tau, A, (\tilde{\omega}(i, 0))_{i \in A})$  for forms  $\tilde{\omega}(i, t)$  as in the statement.

Then note that if  $\tilde{\omega}(i, 0, S^2)$  is the standard homogeneous form  $\omega_{S^2}$  on  $S^2$  for any  $i$ , then  $Z(\mathcal{C}, L, \tau, \cdot)(0)$  maps any subset of  $\underline{3N}$  with cardinality  $3k$  to  $Z_k(\mathcal{C}, L, \tau)$ , by Theorem 19.17. Thanks to Lemma 9.1, this shows that  $Z(\mathcal{C}, L, \tau, A)(t)$  depends only on the forms  $\tilde{\omega}(i, S^2)$ , and therefore only on the forms  $\tilde{\omega}(i, t, S^2)$ .  $\square$

**PROOF OF THEOREM 21.9:** Apply the formula of Theorem 21.10. There are no factors  $\widetilde{\text{hol}}_{[t,0] \times y^-}(\eta_{B^-, \cdot})$  and  $\widetilde{\text{hol}}_{[0,t] \times y^+}(\eta_{B^+, \cdot})$  because  $L$  is a link. Since  $L$  is a straight link with respect to  $\tau$ , the factors  $\widetilde{\text{hol}}_{[0,t]}(\eta(\cdot, p_{\tau}(U^+ K_j)))$  also vanish, even though the notion of straight link of Definition 7.37 is less restrictive than the notion of straight tangle of Definition 16.11. Theorem 21.9 follows from Theorems 21.10 and 19.17, and Lemma 21.8.  $\square$

**Remark 21.11.** When  $\tilde{\omega}(i, 1, S^2)$  is the standard homogeneous form  $\omega_{S^2}$  on  $S^2$  for any  $i$ , the variation formula of Theorem 21.10 yields alternative expressions of  $Z$ .

## 21.2 Pseudo-sections associated with pseudo-parallelizations

For an asymptotically standard parallelization  $\tau$  of a punctured rational homology 3-sphere  $\check{R}$ , a propagating chain of  $(C_2(R), \tau)$  is defined in Definition 3.9 to be a 4-chain  $P$  of  $C_2(R)$  such that  $\partial P = p_\tau^{-1}(X)$  for some  $X \in S^2$ , where  $p_\tau^{-1}(X) \subset \partial C_2(R)$ , the intersection  $p_\tau^{-1}(X) \cap (\partial C_2(R) \setminus U\check{R})$  is independent of  $\tau$ , and

$$p_\tau^{-1}(X) \cap U\check{R} = \tau(\check{R} \times \{X\}).$$

The image in  $UA$  of the restriction of a section  $\tau(\check{R} \times X) = \tau(\check{R} \times \{X\})$  to a part  $A$  of  $\check{R}$  is denoted by  $s_\tau(A; X)$ .

In this section, we define pseudo-sections  $s_{\tilde{\tau}}(\check{R}; X)$  associated with pseudo-parallelizations  $\tilde{\tau}$ . A *propagating chain* of  $(C_2(R), \tilde{\tau})$  is a 4-chain  $P$  of  $C_2(R)$  such that

$$\partial P = (p_\tau|_{\partial C_2(R) \setminus U\check{R}})^{-1}(X) \cup s_{\tilde{\tau}}(\check{R}; X)$$

for some  $X \in S^2$ . Thus, the pseudo-sections  $s_{\tilde{\tau}}(\check{R}; X)$  will play the same role as the sections  $s_\tau(\check{R}; X)$  in the more flexible definition of  $\mathcal{Z}$  presented in Section 21.3.

**Definition 21.12** (Pseudo-sections  $s_{\tilde{\tau}}(\cdot; X)$ ). Recall the map  $F(\gamma, \tau_b)$  of Definition 19.13 and the notation of Definition 19.9.

Let  $X \in S^2$ . Let  $S^1(X)$  be the intersection of  $S^2$  with the plane orthogonal to the axis generated by  $(0, 0, 1)$  that contains  $X$ . So  $S^1(X)$  is a circle or a point. Let  $G_2(X)$  be a 2-dimensional chain in  $[-1, 1] \times S^1(X)$  whose boundary is  $\{(u, \rho_{-\alpha(u)}(X)), u \in [-1, 1]\} + \{(u, \rho_{\alpha(u)}(X)), u \in [-1, 1]\} - 2[-1, 1] \times \{X\}$ , as in Figure 21.1.

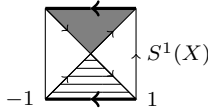


Figure 21.1: The chain  $G_2(X)$  in  $[-1, 1] \times S^1(X)$

Then  $s_{\tilde{\tau}}(A; X)$  is the following 3-cycle of  $(UA, UA|_{\partial A})$

$$s_{\tilde{\tau}}(A; X) = s_{\tau_e}(A \setminus \mathring{N}(\gamma); X) + \frac{1}{2} \left( s_{\tau_b \circ \tau_\gamma^{-1}}(N(\gamma); X) + s_{\tau_b \circ F(\gamma, \tau_b)^{-1}}(N(\gamma); X) + \{b\} \times \gamma \times G_2(X) \right),$$

where  $UA|_{\{b\} \times \gamma \times [-1, 1]}$  is identified with  $\{b\} \times \gamma \times [-1, 1] \times S^2$  by  $\tau_b$  (or in the same way by  $\tau_e$ ).

We also introduce small deformations of these sections, associated with  $\varepsilon_i$  such that  $0 \leq \varepsilon_i < \varepsilon$  (with respect to the  $\varepsilon$  of Definition 19.9) as follows. Let  $N(\gamma, \varepsilon_i) = [a, b - \varepsilon_i] \times \gamma \times [-1, 1] \times S^2$ . Extend  $\tau_e$  over  $[b - \varepsilon, b] \times \gamma \times [-1, 1]$  so that it coincides with  $\tau_b$ , there. Then  $s_{\tilde{\tau}}(A; X, \varepsilon_i)$  is the following 3-cycle of  $(UA, UA|_{\partial A})$

$$s_{\tilde{\tau}}(A; X, \varepsilon_i) = s_{\tau_e}(A \setminus \mathring{N}(\gamma, \varepsilon_i); X) + \frac{1}{2} \left( s_{\tau_b \circ \mathcal{T}_\gamma^{-1}}(N(\gamma, \varepsilon_i); X) + s_{\tau_b \circ F(\gamma, \tau_b)^{-1}}(N(\gamma, \varepsilon_i); X) \right) + \frac{1}{2} \{b - \varepsilon_i\} \times \gamma \times G_2(X).$$

When  $\Sigma$  is a 2-chain that intersects  $N(\gamma)$  along sections  $N_c(\gamma) = [a, b] \times \{c\} \times [-1, 1]$  (which are oriented as meridian disks of  $(-\gamma)$ ), set

$$s_{\tilde{\tau}}(\Sigma; X) = s_{\tilde{\tau}}(A; X) \cap UA|_{\Sigma}.$$

So  $s_{\tilde{\tau}}(N_c(\gamma); X)$  equals

$$\frac{1}{2} \left( s_{\tau_b \circ \mathcal{T}_\gamma^{-1}}(N_c(\gamma); X) + s_{\tau_b \circ F(\gamma, \tau_b)^{-1}}(N_c(\gamma); X) - \{b\} \times \{c\} \times G_2(X) \right).$$

Note that  $G_2(e_3)$  lies in  $[-1, 1] \times \{e_3\}$ . So  $s_{\tau}(A; \pm e_3)$  is simply given by

$$s_{\tau}(A; \pm e_3) = s_{\tau_e}(A \setminus \mathring{N}(\gamma); \pm e_3) + \frac{1}{2} \left( s_{\tau_b \circ \mathcal{T}_\gamma^{-1}}(N(\gamma); \pm e_3) + s_{\tau_b \circ F(\gamma, \tau_b)^{-1}}(N(\gamma); \pm e_3) \right).$$

Below, we discuss common properties of homology classes of sections and pseudo-sections.

**Lemma 21.13.** *Let  $W \in S^2$ . Let  $\Phi$  be a map from the unit disk  $D^2$  of  $\mathbb{C}$  to  $SO(3)$  such that  $\Phi(\exp(i\beta))$  is the rotation  $\rho_{2\beta, W}$  of angle  $2\beta$  whose axis is directed by  $W$  for  $\beta \in [0, 2\pi]$ . Then the map*

$$\Phi_W = \Phi(\cdot)(W): \quad \begin{array}{ccc} D^2 & \rightarrow & S^2 \\ z & \mapsto & \Phi(z)(W) \end{array}$$

*sends  $\partial D^2$  to  $W$ , and the degree of the induced map from  $D^2/\partial D^2$  to  $S^2$  is  $(-1)$ .*

**PROOF:** First note that the above degree does not depend on  $\Phi$  on the interior of  $D^2$  since  $\pi_2(SO(3)) = 0$ . View the restriction of  $\Phi_W$  to  $\partial D^2$ , as the path composition of the maps  $(\beta \in [0, 2\pi] \mapsto \rho_{\beta, W})$  and the inverse of  $(\beta \in [0, 2\pi] \mapsto \rho_{\beta, -W})$  (which is homotopic to the first map). Consider an arc  $\xi$  of a great circle of  $S^2$  from  $-W$  to  $W$ . Then  $\Phi$  can be regarded as the map

from  $\xi \times [0, 2\pi]$  to  $SO(3)$  that maps  $(X, \beta)$  to  $\rho_{\beta, X}$ . So the only preimage of  $-W$  under  $\Phi_W$  is  $(X_0, \pi)$ , where  $X_0 \in \xi$  and  $X_0 \perp W$ . The local degree is easily seen to be  $(-1)$ .  $\square$

Lemma 19.20 generalizes as follows to pseudo-parallelizations, with the notation of Definition 19.18.

**Proposition 21.14.** *Recall  $e_3 = (0, 0, 1) \in \mathbb{R}^3$ . Let  $\Sigma$  be a compact oriented surface immersed in a 3-manifold  $M$  equipped with a pseudo-parallelization  $\tau = (N(\gamma); \tau_e, \tau_b)$ . Assume that  $\tau$  is an actual parallelization around the boundary  $\partial\Sigma$  of  $\Sigma$  and that  $\tau(\cdot \times e_3)$  is a positive normal to  $\Sigma$  along  $\partial\Sigma$ . Let  $s_+(\Sigma) \subset UM$  be the section of  $UM|_\Sigma$  in  $UM$  associated to the positive normal field  $n$  to  $\Sigma$ , which coincides with  $\tau(\partial\Sigma \times e_3)$  on  $\partial\Sigma$ . Then*

$$2(s_+(\Sigma) - s_\tau(\Sigma; e_3)) - \chi(\tau(\cdot \times e_2)|_{\partial\Sigma}; \Sigma)UM|_*$$

and

$$2(s_-(\Sigma) - s_\tau(\Sigma; -e_3)) + \chi(\tau(\cdot \times e_2)|_{\partial\Sigma}; \Sigma)UM|_*$$

are null-homologous cycles in  $UM|_\Sigma$ .

PROOF: Lemma 19.20 gives the result for actual parallelizations. Let us prove it for pseudo-parallelizations. Since  $s_\tau(\Sigma; \pm e_3) = s_\tau(M; \pm e_3) \cap UM|_\Sigma$  is well-defined for any surface  $\Sigma$ , as soon as the above intersection is transverse, there is no loss of generality in assuming that  $\Sigma$  meets  $N(\gamma)$  along sections  $N_c(\gamma)$ , for some  $c \in \gamma$ .

Since the cycles of the statement behave well under gluing (or cutting) surfaces along curves that satisfy the boundary assumptions, and since these assumptions are easily satisfied after a homotopy as in Definition 19.21 (when the cutting process is concerned), it suffices to prove the lemma for a meridian disk  $\Sigma$  of  $\gamma$ . For such a disk,  $s_\tau(\Sigma; e_3)$  is the average of two genuine sections corresponding to trivializations  $\tau_1$  and  $\tau_2$ . So, with the notation of Lemma 19.19, the chain

$$2(s_+(\Sigma) - s_\tau(\Sigma; e_3)) - \left( \frac{d(\tau_1(\cdot \times e_2), \partial\Sigma) + d(\tau_2(\cdot \times e_2), \partial\Sigma)}{2} + \chi(\Sigma) \right) UM|_*$$

is a null-homologous cycle. We conclude that the first cycle is null-homologous because the exterior trivialization  $\tau_e$  satisfies

$$d(\tau_e(\cdot \times e_2), \partial\Sigma) = \frac{d(\tau_1(\cdot \times e_2), \partial\Sigma) + d(\tau_2(\cdot \times e_2), \partial\Sigma)}{2}.$$

The second cycle can be treated similarly.  $\square$

The obvious property that *genuine sections  $s_\tau(\Sigma; X)$  and  $s_\tau(\Sigma; Y)$  corresponding to distinct  $X$  and  $Y$  of  $S^2$  are disjoint* generalizes as follows for pseudo-sections.

**Lemma 21.15.** *Let  $\Sigma$  be an oriented surface embedded in a 3-manifold  $A$  equipped with a pseudo-parallelization  $\tilde{\tau} = (N(\gamma); \tau_e, \tau_b)$  such that  $\Sigma$  intersects  $N(\gamma)$  only along sections  $N_{c_i}(\gamma) = [a, b] \times \{c_i\} \times [-1, 1]$  in the interior of  $\Sigma$ . If  $Y \in S^2$  and if  $Z \in S^2 \setminus S^1(Y)$ , then the algebraic intersection  $\langle s_{\tilde{\tau}}(\Sigma; Y), s_{\tilde{\tau}}(\Sigma; Z) \rangle_{UA|_{\Sigma}}$  of  $s_{\tilde{\tau}}(\Sigma; Y)$  and  $s_{\tilde{\tau}}(\Sigma; Z)$  in  $UA|_{\Sigma}$  is zero.*

PROOF: Recall that  $S^1(X)$  is the intersection circle of  $S^2$  with the plane orthogonal to the axis generated by  $e_3 = (0, 0, 1)$  that contains  $X$ . Consider the contribution to  $\langle s_{\tilde{\tau}}(\Sigma; Y), s_{\tilde{\tau}}(\Sigma; Z) \rangle_{UA|_{\Sigma}}$  of an intersection point  $c$  of  $\gamma$  with  $\Sigma$ . Such a contribution may be expressed as  $\pm \frac{1}{4}(d_Y(Z) + d_Z(Y))$  with

$$d_Y(Z) = \left\langle s_{\tau_b \circ F(\gamma, \tau_b)^{-1}}(N_c(\gamma); Z), s_{\tau_b \circ \mathcal{T}_\gamma^{-1}}(N_c(\gamma); Y) \right\rangle$$

and  $d_Z(Y) = \left\langle s_{\tau_b \circ F(\gamma, \tau_b)^{-1}}(N_c(\gamma); Y), s_{\tau_b \circ \mathcal{T}_\gamma^{-1}}(N_c(\gamma); Z) \right\rangle$ . We prove that both intersection numbers  $d_Y(Z)$  and  $d_Z(Y)$  are  $\pm 1$  and that their signs are opposite. Consider an arc  $\xi$  of great circle from  $e_3$  to  $-e_3$ . Identify  $\xi$  with  $[a, b]$  via an orientation-preserving diffeomorphism, so that the map  $F$  of Definition 19.13 can be regarded as the map that maps  $(V \in \xi, u \in [-1, 1])$  to the rotation  $\rho_{\alpha(u), V}$  with axis  $V$  and angle  $\alpha(u)$ . Then the intersection

$$d_Y(Z) = \left\langle s_{\tau_b \circ F(\gamma, \tau_b)^{-1}}(N_c(\gamma); Z), s_{\tau_b \circ \mathcal{T}_\gamma^{-1}}(N_c(\gamma); Y) \right\rangle$$

in

$$UA|_{N_c(\gamma)} =_{\tau_b \circ F(\gamma, \tau_b)^{-1}} N_c(\gamma) \times S^2$$

is the algebraic intersection

$$\left\langle N_c(\gamma) \times \{Z\}, F(\gamma, \tau_b) \circ \mathcal{T}_\gamma^{-1}(N_c(\gamma) \times \{Y\}) \right\rangle_{N_c(\gamma) \times S^2}.$$

It is the degree of the map

$$f_Y: \begin{array}{ccc} \xi \times [-1, 1] & \rightarrow & S^2 \\ (V, u) & \mapsto & \rho_{\alpha(u), V} \circ \rho_{\alpha(-u), e_3}(Y) \end{array}$$

at  $Z$ . The other algebraic intersection  $d_Z(Y)$  is the degree of the map  $f_Z$  at  $Y$ . The boundary of the image of  $f_Y$  is  $-2S^1(Y)$ , where  $S^1(Y)$  is oriented as the boundary of the closure of the connected component of  $S^2 \setminus S^1(Y)$  that contains  $e_3$ . Therefore, the degree increases by 2 from the component of  $S^2 \setminus S^1(Y)$  that contains  $e_3$  to the component of  $(-e_3)$ . Furthermore, the degree of  $f_Y$  on the component of  $e_3$  is independent of  $Y \neq e_3$ . Lemma 21.13 implies  $\deg(f_{-e_3}) = \deg(\Phi_{-e_3}) = -1$ . So the degree of  $f_Y$  at  $e_3$  is  $-1$  when  $Y \neq e_3$ . Therefore, the degree of  $f_Y$  on the component of  $-e_3$ , which is independent of  $Y \neq (-e_3)$ , is 1. Thus, the degree of  $f_Y$  at  $Z$  and the degree of  $f_Z$  at  $Y$  are opposite.  $\square$



**Lemma 21.16.** *Let  $A$  be a rational homology handlebody equipped with two pseudo-parallelizations  $\tilde{\tau}_0$  and  $\tilde{\tau}_1$  that coincide with the same genuine parallelization near the boundary of  $A$ . Let  $X \in S^2$ . Let  $\eta \in [0, \varepsilon[$ . There exists a rational 4-chain  $H(\tilde{\tau}_0, \tilde{\tau}_1, X, \eta)$  in  $UA$  such that*

$$\partial H(\tilde{\tau}_0, \tilde{\tau}_1, X, \eta) = s_{\tilde{\tau}_1}(A; X, \eta) - s_{\tilde{\tau}_0}(A; X, \eta).$$

PROOF: Let  $Y \in S^2 \setminus S^1(X)$ . Let us show that  $C = s_{\tilde{\tau}_1}(A; X, \eta) - s_{\tilde{\tau}_0}(A; X, \eta)$  vanishes in

$$H_3(UA; \mathbb{Q}) = H_1(A; \mathbb{Q}) \otimes H_2(S^2; \mathbb{Q}).$$

To do so, we prove that the algebraic intersections of  $C$  with  $s_{\tilde{\tau}_0}(S; Y)$  vanish for 2-cycles  $S$  of  $(A, \partial A)$  that generate  $H_2(A, \partial A)$ . We assume  $\eta = 0$  without loss. Of course  $s_{\tilde{\tau}_0}(\partial A; X)$  does not intersect  $s_{\tilde{\tau}_0}(S; Y)$ . According to Lemma 21.15, the chain  $s_{\tilde{\tau}_0}(S; X)$  does not intersect  $s_{\tilde{\tau}_0}(S; Y)$  in  $UA|_S$  algebraically. So  $s_{\tilde{\tau}_0}(A; X)$  does not intersect  $s_{\tilde{\tau}_0}(S; Y)$  algebraically, either. Since Proposition 21.14 guarantees that  $(s_{\tilde{\tau}_0}(S; X) - s_{\tilde{\tau}_1}(S; X))$  bounds in  $UA|_S$ , the chain  $s_{\tilde{\tau}_1}(S; X)$  does not intersect  $s_{\tilde{\tau}_0}(S; Y)$  algebraically in  $UA|_S$ . So  $s_{\tilde{\tau}_1}(A; X)$  does not intersect  $s_{\tilde{\tau}_0}(S; Y)$  algebraically. This proves the existence of the desired 4-chain  $H(\tilde{\tau}_0, \tilde{\tau}_1, X, \eta)$ .  $\square$

### 21.3 Definition of $\mathcal{Z}$ with respect to pseudo-sections

**Definition 21.17.** Let  $\check{R}$  be a rational homology  $\mathbb{R}^3$  equipped with an asymptotically standard pseudo-parallelization  $\tilde{\tau}$ . A *propagating chain* of  $(C_2(R), \tilde{\tau})$  is a propagating chain of  $C_2(R)$  (as in Definition 3.11) whose boundary intersects  $U\check{R}$  as a chain  $s_{\tilde{\tau}}(\check{R}; X, \varepsilon_i)$  as in Definition 21.12.

Again, for a given  $s_{\tilde{\tau}}(\check{R}; X, \varepsilon_i)$ , a propagating chain whose boundary intersects  $U\check{R}$  as  $s_{\tilde{\tau}}(\check{R}; X, \varepsilon_i)$  exists because  $H_3(C_2(R); \mathbb{Q}) = 0$ .

**Lemma 21.18.** *Let  $\check{R}$  be a rational homology  $\mathbb{R}^3$  equipped with an asymptotically standard pseudo-parallelization  $\tilde{\tau}$ . For any positive number  $\alpha$ , and for any propagating chain  $P$  of  $(C_2(R), \tilde{\tau})$ , transverse to  $\partial C_2(R)$ , there exists a propagating form of  $(C_2(R), \tilde{\tau})$  (as in Definition 21.7) that is  $\alpha$ -dual to  $P$  (as in Definition 11.6).*

PROOF: There is no loss of generality in assuming that  $P$  intersects a collar  $[-1, 0] \times \partial C_2(R)$  as  $[-1, 0] \times \partial P$ .

Assume that the boundary of  $P$  intersects  $U\check{R}$  as a chain  $s_\tau(\check{R}; X, \varepsilon_i)$ . Let  $\omega_i$  be volume-one form of  $S^2$ , which is  $\alpha_X$ -dual to  $X$  for some small  $\alpha_X \in ]0, \alpha[$ . Set

$$I(\varepsilon_i) = [b - \varepsilon_i - \varepsilon_i^k, b - \varepsilon_i + \varepsilon_i^k]$$

and  $I_2(\varepsilon_i) = [b - \varepsilon_i - 2\varepsilon_i^k, b - \varepsilon_i + 2\varepsilon_i^k] \times [-1, 1] \times S^2$ .

Forms  $\alpha$ -dual to a given chain can be constructed as in Lemma B.4. We use this process to construct a form  $\omega_1$   $\alpha$ -dual to  $P$ , which pulls back through  $\partial C_2(R)$  on  $[-1, 0] \times \partial C_2(R)$ , which may be expressed as in Definition 21.2 outside  $[-1, 0] \times U(I(\varepsilon_i) \times \gamma \times [-1, 1])$ , and which factors through  $I(\varepsilon_i) \times [-1, 1] \times S^2$  on  $[-1, 0] \times U(I(\varepsilon_i) \times \gamma \times [-1, 1])$ .

It remains to see that  $\omega_1$  can be assumed to have the form prescribed by Definition 21.7 on  $[-1, 0] \times U(I(\varepsilon_i) \times \gamma \times [-1, 1])$ .

We also have a form  $\omega_2$  as in Definition 21.7, which also pulls back through  $\partial C_2(R)$  on  $[-1, 0] \times \partial C_2(R)$  and which coincides with  $\omega_1$  on  $[-1, 0] \times U(\check{R} \setminus (I(\varepsilon_i) \times \gamma \times [-1 + \varepsilon, 1 - \varepsilon]))$ .

Both forms factor through  $I_2(\varepsilon_i) \times [-1, 1] \times S^2$ , and they are cohomologous there. So there exists a one-form  $\eta$  on  $I_2(\varepsilon_i) \times [-1, 1] \times S^2$  such that  $\omega_2 - \omega_1 = p^*(d\eta)$ , where  $p$  is the natural projection that forgets the  $[-1, 0]$  factor and the  $\gamma$  factor, and  $d\eta$  vanishes outside  $I(\varepsilon_i) \times [-1 + \varepsilon, 1 - \varepsilon] \times S^2$ .

To conclude the proof, it suffices to prove that such  $\eta$  can be chosen so that  $\eta$  is zero outside  $I(\varepsilon_i) \times [-1 + \varepsilon, 1 - \varepsilon] \times S^2$ , too. Therefore, it suffices to prove that the cohomology class of  $\eta$  is zero on

$$\left( (I_2(\varepsilon_i) \times [-1, 1]) \setminus (I(\varepsilon_i) \times [-1 + \varepsilon, 1 - \varepsilon]) \right) \times S^2.$$

Thus, it suffices to prove that the integral of  $\eta$  along

$$\partial(I_2(\varepsilon_i) \times [-1, 1] \times \{Y\})$$

is zero for some  $Y \in S^2$ . This integral is also the integral of  $(\omega_2 - \omega_1)$  along  $I_2(\varepsilon_i) \times [-1, 1] \times \{Y\}$  or along  $(N_c(\gamma) = [a, b] \times \{c\} \times [-1, 1]) \times \{Y\}$ , where the integral of  $\omega_1$  is the algebraic intersection of  $s_{\tau_b}([a, b] \times \{c\} \times [-1, 1]; Y)$  with  $P$ .

Note that nothing depends on the trivialization  $\tau_b$ , which identifies

$$U([a, b] \times \gamma \times [-1, 1])$$

with  $([a, b] \times \gamma \times [-1, 1]) \times S^2$ . Therefore, there is no loss of generality in assuming that  $\tau_b$  maps  $e_3$  to the direction of  $(-\gamma)$ , which is the positive normal to the meridian disks, so that  $(N_c(\gamma) \times \{e_3\}) = s_+(N_c(\gamma))$ .

Assume  $X \neq e_3$  and choose  $Y = e_3$ . Then it suffices to prove  $\int_{s_+(N_c(\gamma))} \omega_1 = \int_{s_+(N_c(\gamma))} \omega_2$ . Lemma 21.5 implies

$$\int_{s_+(N_c(\gamma))} \omega_2 = \frac{1}{2} \chi(\tau_e(\cdot \times e_2)|_{\partial N_c(\gamma)}; N_c(\gamma)).$$

Let us compute  $\int_{s_+(N_c(\gamma))} \omega_1 = \langle s_+(N_c(\gamma)), P \rangle$ . Thanks to Proposition 21.14,

$$2\left(s_+(N_c(\gamma)) - s_{\tilde{\tau}}(N_c(\gamma); e_3)\right) - \chi(\tau_e(\cdot \times e_2)|_{\partial N_c(\gamma)}; N_c(\gamma)) U\check{R}|_*$$

is null-homologous in  $U\check{R}|_*$ . So its algebraic intersection with  $P$  is zero. The algebraic intersection of  $s_{\tilde{\tau}}(N_c(\gamma); e_3)$  with  $P$  is also zero according to Lemma 21.15. We thus get  $\int_{s_+(N_c(\gamma))} \omega_1 = \int_{s_+(N_c(\gamma))} \omega_2$ , and the lemma is proved when  $e_3 \neq X$ .

When  $e_3 = X$ , choose  $Y = -e_3$  and conclude by computing the integrals along  $s_-(N_c(\gamma))$ , similarly.  $\square$

For a family of propagating chains  $P_i$  of  $(C_2(R), \tilde{\tau})$  in general  $3n$ -position with respect to  $L$  as in Definition 11.3, and for propagating forms  $\omega(i)$ , which are  $\alpha$ -dual to them for a sufficiently small  $\alpha$  as in Lemma 21.18, the integrals involved in  $Z_n^s(\check{R}, L, \tilde{\tau})$  in Theorem 21.9 can be computed as algebraic intersections of preimages of transverse propagating chains of  $(C_2(R), \tilde{\tau})$  as in Sections 11.1 and 17.1.

This provides the announced discrete definition of  $Z_n^s(\check{R}, L, \tilde{\tau})$  with respect to pseudo-parallelizations.

# Appendix A

## Some basic algebraic topology

This appendix reviews the main well-known results in algebraic topology used throughout the book. The proofs of these results can be found in several books about basic algebraic topology, such as [Gre67, Hat02, Spa81], by Marvin Greenberg, Allen Hatcher, and Edwin Spanier. Here we state only the weak versions used in the book, with some sketches of proofs and some hints to show how things work.

### A.1 Homology

We first review some properties of homology.

A *topological pair*  $(X, Y)$  consists of a topological space  $X$  and a subspace  $Y$  of  $X$ . A *map*  $f: (X, Y) \rightarrow (A, B)$  between such pairs is a continuous map  $f$  from  $X$  to  $A$  sending  $Y$  to  $B$ . A topological space  $X$  is identified with the pair  $(X, \emptyset)$ .

The coefficient ring  $\Lambda$  of the homology theories  $H(\cdot) = H(\cdot; \Lambda)$  considered in this book is  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ , or  $\mathbb{R}$ .

A *covariant functor*  $H$  from the category of topological pairs  $(X, Y)$  to the category of graded  $\Lambda$ -modules and homomorphisms of degree 0 maps any topological pair  $(X, Y)$  to a sequence  $H(X, Y; \Lambda) = H(X, Y) = (H_k(X, Y))_{k \in \mathbb{Z}}$  of  $\Lambda$ -modules. It associates to any map  $f: (X, Y) \rightarrow (A, B)$  between pairs a sequence of  $\Lambda$ -linear morphisms  $H(f) = (H_k(f): H_k(X, Y) \rightarrow H_k(A, B))_{k \in \mathbb{Z}}$  so that if  $g: (A, B) \rightarrow (V, W)$  is another map between pairs  $H_k(g \circ f) = H_k(g) \circ H_k(f)$ , and  $H_k$  maps the identity map of  $(X, Y)$  to the identity map of  $H_k(X, Y)$ .

The homology theories  $(H, \partial)$  that we consider consist of

- a *covariant functor*  $H$  from the category of topological pairs  $(X, Y)$  to the category of graded  $\Lambda$ -modules and homomorphisms of degree 0,

and

- sequences  $\partial_k(X, Y)$  of linear maps  $\partial_k(X, Y): H_k(X, Y) \rightarrow H_{k-1}(Y)$  associated to topological pairs  $(X, Y)$  such that for any map  $f: (X, Y) \rightarrow (A, B)$ , we have

$$\partial_k(A, B) \circ H_k(f) = H_{k-1}(f|_Y) \circ \partial_k(X, Y).$$

The considered homology theories satisfy the following Eilenberg–Steenrod axioms [Spa81, Chap. 4, Sec. 8, p.199].

- **Homotopy axiom:** For two topological pairs  $(X, Y)$  and  $(A, B)$ , if

$$\begin{aligned} f: [0, 1] \times X &\rightarrow A \\ (t, x) &\mapsto f_t(x) = f(t, x) \end{aligned}$$

is a continuous map sending  $[0, 1] \times Y$  to  $B$ , then  $H(f_0)$  is equal to  $H(f_1)$ .

- **Exactness axiom:** For any topological pair  $(X, Y)$  with inclusion maps  $i: Y \hookrightarrow X$  and  $j: (X, \emptyset) \hookrightarrow (X, Y)$ , there is a long exact sequence:

$$\dots \xrightarrow{\partial_{k+1}(X, Y)} H_k(Y) \xrightarrow{H_k(i)} H_k(X) \xrightarrow{H_k(j)} H_k(X, Y) \xrightarrow{\partial_k(X, Y)} H_{k-1}(Y) \xrightarrow{H_{k-1}(i)} \dots^1$$

- **Excision axiom:** For any topological pair  $(X, Y)$ , if  $U$  is an open subset of  $X$  whose closure lies in the interior of  $Y$ , then the inclusion map  $e: (X \setminus U, Y \setminus U) \rightarrow (X, Y)$  induces isomorphisms  $H_k(e): H_k(X \setminus U, Y \setminus U) \rightarrow H_k(X, Y)$  for any  $k \in \mathbb{Z}$ .
- **Dimension axiom:** If  $X$  has one element  $x$ , then  $H_k(X) \cong \{0\}$  for any  $k \neq 0$ . Furthermore,  $H_0(X)$  is isomorphic to  $\Lambda$  and is equipped with a canonical generator denoted by  $[x]$ . So we write  $H_0(\{x\}) = \Lambda[x]$ .

An example of such a homology theory is the *singular homology* described in [Gre67, Chap. 10 and 13] and in [Hat02, Section 2.1].

The spaces considered in this book are homotopy equivalent to finite CW-complexes. The above Eilenberg–Steenrod axioms characterize the homology of these spaces. Let us show a few examples of properties of homology and computations from these axioms. First note that the functoriality implies that  $H$  maps a homeomorphism  $f: (X, Y) \rightarrow (A, B)$  between topological

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<sup>1</sup>Such a sequence of homomorphisms is *exact* if and only if the image of a homomorphism is equal to the kernel of the following one.

pairs to an isomorphism  $H(f)$ . Also note that the homotopy axiom implies that the homologies of  $\mathbb{R}^n$  and its unit ball  $B^n$  are isomorphic to the homology of a point, which is determined by the dimension axiom.

Here are other easy consequences of the Eilenberg–Steenrod axioms.

**Proposition A.1.** *Let  $X$  and  $Y$  be two topological spaces. Let  $i_X$  and  $i_Y$  denote their respective inclusion maps into their disjoint union  $X \sqcup Y$ . Then for any  $k \in \mathbb{Z}$ ,  $H_k(i_X): H_k(X) \rightarrow H_k(X \sqcup Y)$  and  $H_k(i_Y): H_k(Y) \rightarrow H_k(X \sqcup Y)$  are injective, and we have*

$$H_k(X \sqcup Y) = H_k(i_X)(H_k(X)) \oplus H_k(i_Y)(H_k(Y)).$$

PROOF: The excision isomorphism  $H_k(e_X): H_k(X) = H_k(X, \emptyset) \rightarrow H_k(X \sqcup Y, Y)$  may be expressed as

$$H_k(e_X) = H_k(j: (X \sqcup Y, \emptyset) \hookrightarrow (X \sqcup Y, Y)) \circ H_k(i_X: X \hookrightarrow X \sqcup Y).$$

So  $H_k(j)$  is surjective and  $H_k(i_X)$  is injective. Similarly,  $H_k(i_Y)$  is injective. Thus, the long exact sequence of  $(X \sqcup Y, Y)$  yields short exact sequences

$$0 \rightarrow H_k(Y) \xrightarrow{H_k(i_Y)} H_k(X \sqcup Y) \xrightarrow{H_k(e_X)^{-1} \circ H_k(j)} H_k(X) \rightarrow 0$$

for any integer  $k$ . Since  $(H_k(e_X)^{-1} \circ H_k(j)) \circ H_k(i_X)$  is the identity map, the exact sequence splits, and we get the result.  $\square$

**Proposition A.2.** *Let  $X$  be a topological space. Let  $Y$  and  $Z$  be two subspaces of  $X$  such that  $Z \subseteq Y$ . Let*

$$(\partial_k = \partial_k(X, Y, Z)): H_k(X, Y) \rightarrow H_{k-1}(Y, Z)$$

be the composition of  $\partial_k(X, Y): H_k(X, Y) \rightarrow H_{k-1}(Y)$  and the map from  $H_{k-1}(Y)$  to  $H_{k-1}(Y, Z)$  induced by the inclusion. Then the long sequence

$$\begin{array}{ccccccc} H_{k+1}(X, Y) & \xrightarrow{\partial_{k+1}} & H_k(Y, Z) & \xrightarrow{H_k(i)} & H_k(X, Z) & \xrightarrow{H_k(j)} & H_k(X, Y) \\ & & & & \searrow \partial_k & & \\ & & H_{k-1}(Y, Z) & \xrightarrow{H_{k-1}(i)} & H_{k-1}(X, Z) & \xrightarrow{H_{k-1}(j)} & H_{k-1}(X, Y) & \xrightarrow{\partial_{k-1}} & H_{k-2}(\quad) \end{array}$$

where the  $H_k(i)$  and the  $H_k(j)$  are induced by inclusion, is exact. It is called the long exact sequence of homology of the triple  $(X, Y, Z)$ .

SKETCH OF PROOF: To prove that  $H_k(j) \circ H_k(i) = 0$ , note that the inclusion map from  $(Y, Z)$  to  $(X, Y)$  factors through  $(Y, Y)$ . Next chase in commutative diagrams using functoriality and the exactness axiom, which is this exact sequence when  $Z = \emptyset$ .  $\square$

Let  $X$  be a topological space with  $k$  connected components. Choose a basepoint  $x_i$  in each connected component  $X_i$ . Let  $Y$  denote the set of these basepoints equipped with the natural discrete topology. Let  $i: Y \hookrightarrow X$  be the inclusion map, and let  $p$  be the map sending each connected component to its basepoint. By functoriality  $H(p) \circ H(i)$  is the identity map. So  $H(i)$  is injective. In particular  $H_0(i)$  injects  $\bigoplus_{x_i \in Y} \Lambda[x_i]$  into  $H_0(X)$ . The element  $H_0(i)([x_i])$  is also denoted by  $[x_i]$ . The homotopy axiom implies that the element  $[x_i]$  of  $H_0(X_i)$  is independent of the basepoint  $x_i$  of  $X_i$  if  $X_i$  is path-connected.

**Lemma A.3.** *We have*

$$H_k(\mathbb{R}, \mathbb{R} \setminus \{0\}) = \begin{cases} \Lambda \partial_1^{-1}(\mathbb{R}, \mathbb{R} \setminus \{0\})([1] - [-1]) \cong \Lambda & \text{if } k = 1 \\ \{0\} & \text{otherwise.} \end{cases}$$

PROOF: Proposition A.1 and the observations before imply

$$H_0(\mathbb{R} \setminus \{0\}) = \Lambda[-1] \oplus \Lambda[+1].$$

The morphism  $H_0(i)$  induced by inclusion from  $H_0(\mathbb{R} \setminus \{0\})$  to  $H_0(\mathbb{R}) = \Lambda[1]$  maps  $[-1]$  and  $[+1]$  to the preferred generator  $[-1] = [0] = [1]$  of  $H_0(\mathbb{R})$ . The long exact sequence associated to  $(\mathbb{R}, \mathbb{R} \setminus \{0\}, \emptyset)$  implies that  $\partial_1(\mathbb{R}, \mathbb{R} \setminus \{0\}): H_1(\mathbb{R}, \mathbb{R} \setminus \{0\}) \rightarrow H_0(\mathbb{R} \setminus \{0\})$  is an isomorphism onto its image, which is the kernel  $\Lambda([+1] - [-1])$  of  $H_0(i)$ . The long exact sequence also implies  $H_q(\mathbb{R}, \mathbb{R} \setminus \{0\}) = \{0\}$  when  $q \neq 1$ .  $\square$

The reader can also apply the Eilenberg–Steenrod axioms and the above observations to prove the following proposition.

**Proposition A.4.** *Let  $n \in \mathbb{N} \setminus \{0\}$  and let  $k \in \mathbb{Z}$ . Recall*

$$B^n = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : \|x\|^2 = \sum_{i=1}^n x_i^2 \leq 1 \right\}$$

and  $S^{n-1} = \partial B^n$ .

Assume that  $n \geq 2$ . Set  $B_+^{n-1} = \{x \in S^{n-1} : x_1 \geq 0\}$ ,  $B_-^{n-1} = \{x \in S^{n-1} : x_1 \leq 0\}$ , and  $S_+^{n-2} = \{x \in S^{n-1} : x_1 = 0\}$ . The morphisms of the sequence

$$\begin{aligned} H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) &\xleftarrow{H_k(i)} H_k(B^n, S^{n-1}) \xrightarrow{\partial} H_{k-1}(S^{n-1}, B_-^{n-1}) \\ &\longleftarrow H_{k-1}(B_+^{n-1}, S_+^{n-2}) \xrightarrow{H_{k-1}(p)} H_{k-1}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1} \setminus \{0\}), \end{aligned}$$

where

- the unlabeled morphism and  $H_k(i)$  are induced by inclusions,
- $\partial$  is the morphism  $\partial_k(B^n, S^{n-1}, B_-^{n-1})$  of Proposition A.2, and
- $p : B_+^{n-1} \rightarrow \mathbb{R}^{n-1}$  forgets the first coordinate and shifts the numbering of the remaining ones by  $(-1)$ ,

are isomorphisms. In particular, when  $k = n$ , their composition

$$D_n : H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \rightarrow H_{n-1}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1} \setminus \{0\})$$

is an isomorphism. Let  $[\mathbb{R}, \mathbb{R} \setminus \{0\}] = \partial_1^{-1}(\mathbb{R}, \mathbb{R} \setminus \{0\})([1] - [-1])$  be the preferred generator of  $H_1(\mathbb{R}, \mathbb{R} \setminus \{0\})$  of Lemma A.3. Inductively define

$$[\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}] = D_n^{-1}[\mathbb{R}^{n-1}, \mathbb{R}^{n-1} \setminus \{0\}].$$

Then we have

$$H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) = \begin{cases} \Lambda[\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}] \cong \Lambda & \text{if } k = n \\ \{0\} & \text{otherwise.} \end{cases}$$

for any positive integer  $n$ . For  $n \geq 1$ , set

$$[B^n, S^{n-1}] = H_n(i)^{-1}([\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}]).$$

Then we have

$$H_k(B^n, S^{n-1}) = \begin{cases} \Lambda[B^n, S^{n-1}] \cong \Lambda & \text{if } k = n \\ \{0\} & \text{otherwise.} \end{cases}$$

□

As a corollary, we get the following proposition:

**Proposition A.5.** *Let  $n \in \mathbb{N}$ . Assume that  $n \geq 1$ . Set*

$$[S^n] = \partial_{n+1}(B^{n+1}, S^n)([B^{n+1}, S^n]).$$

We have

$$H_k(S^n) = \begin{cases} \Lambda[S^n] \cong \Lambda & \text{if } k = n \\ \Lambda[(1, 0, \dots, 0)] \cong \Lambda & \text{if } k = 0 \\ \{0\} & \text{otherwise.} \end{cases}$$

PROOF: Use the exact sequence associated to  $(B^{n+1}, S^n)$  and the previous proposition. □



**Proposition A.6.** *Let  $n \in \mathbb{N}$ . Let  $\phi$  be a diffeomorphism from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  sending  $0$  to  $\phi(0) = 0$ . If  $\phi$  preserves the orientation, then  $\phi$  induces the identity map on  $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ . Otherwise, we have*

$$H_n(\phi)\left([\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}]\right) = -[\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}].$$

PROOF: When  $n = 1$ , this is easy to see with the generator

$$[\mathbb{R}, \mathbb{R} \setminus \{0\}] = \partial_1(\mathbb{R}, \mathbb{R} \setminus \{0\})^{-1}([+1] - [-1]).$$

Let  $\iota_n: \mathbb{R}^n \rightarrow \mathbb{R}^n$  map  $(x_1, x_2, \dots, x_{n-1}, x_n)$  to  $(x_1, x_2, \dots, x_{n-1}, -x_n)$ . With the notation of Proposition A.4, we have  $H_{n-1}(\iota_{n-1}) \circ D_n = D_n \circ H_n(\iota_n)$ . We inductively deduce  $H_n(\iota_n)([\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}]) = -[\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}]$ . Any orientation-reversing linear isomorphism of  $\mathbb{R}^n$  is homotopic to  $\iota_n$  through linear isomorphisms. Therefore, we have  $H_n(\phi)([\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}]) = -[\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}]$  for any such isomorphism  $\phi$ . We similarly observe that  $H_n(\phi)$  sends  $[\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}]$  to itself for an orientation-preserving linear isomorphism  $\phi$ . Diffeomorphisms preserving  $0$  are homotopic to their linear derivative at  $0$  through maps preserving  $\mathbb{R}^n \setminus \{0\}$ , near  $0$ . So the result follows for general diffeomorphisms thanks to the excision axiom.  $\square$

Let  $X$  be a topological space equipped with a triangulation  $T$  as in Subsection 2.1.2. The Eilenberg–Steenrod axioms also imply that the homology  $H(X)$  of  $X$  can be computed as follows.<sup>2</sup> Equip each simplex of the triangulation with an arbitrary orientation, and let  $C_k(T) = C_k(T; \Lambda)$  denote the  $\Lambda$ -module freely generated by the simplices of dimension  $k$  of  $T$ . The *algebraic boundary* of such a  $k$ -dimensional simplex  $\Delta$  is the sum

$$\partial_k(\Delta) = \sum_{\delta} \varepsilon(\Delta, \delta)\delta,$$

running over all the  $(k - 1)$ -dimensional simplices  $\delta$  in the boundary of  $\Delta$ , where  $\varepsilon(\Delta, \delta)$  is  $1$  if  $\delta$  is oriented as (part of) the boundary of  $\Delta$  (with the outward normal first convention as usual), and  $(-1)$  otherwise. Define the *boundary map*  $\partial_k: C_k(T) \rightarrow C_{k-1}(T)$  to be the linear map sending a  $k$ -dimensional simplex  $\Delta$  to its algebraic boundary  $\partial_k(\Delta)$ . Then we have  $\partial_k \circ \partial_{k+1} = 0$ . So  $(C(T), \partial) = (C_k(T), \partial_k)_{k \in \mathbb{Z}}$  is a *chain complex*. Its homology

$$H(C(T), \partial) = \left( H_k(C(T), \partial) = \frac{\text{Ker } \partial_k}{\text{Im } \partial_{k+1}} \right)_{k \in \mathbb{Z}}$$

is canonically isomorphic to the homology of  $X$ .

<sup>2</sup>The reader can prove it, but it takes more space.

The elements of  $\text{Ker } \partial_k$  are the  $k$ -dimensional cycles of  $C(T)$ . The elements of  $\text{Im } \partial_{k+1}$  are the  $k$ -dimensional boundaries of  $C(T)$ . The elements of  $C_k(T) = C_k(T; \Lambda)$  are called the *simplicial chains* of  $T$  of dimension  $k$  with coefficients in  $\Lambda$ .

Thus, the homology of an  $n$ -dimensional manifold  $M$  that can be equipped with a triangulation  $T$  vanishes in degrees higher than  $n$  and in negative degrees. (The existence of a homology theory satisfying the axioms and Proposition A.4 imply that the notion of dimension is well-defined for topological manifolds.) Let  $C$  be an  $n$ -dimensional cycle in such a connected manifold  $M$ , and let  $\Delta_1$  and  $\Delta_2$  be two  $n$ -simplices of  $T$  intersecting along an  $(n-1)$ -simplex. Then the coefficients of  $\Delta_1$  and  $\Delta_2$  in  $C$  must coincide if the orientations of  $\Delta_1$  and  $\Delta_2$  are consistent along  $\Delta_1 \cap \Delta_2$ . They must be opposite to each other if the orientations are not consistent.

In particular, when  $\Lambda = \mathbb{Z}/2\mathbb{Z}$ , if  $M$  is connected, then the existence of a nonzero  $n$ -dimensional cycle implies that the boundary of  $M$  is empty and that  $M$  is compact. When  $\Lambda = \mathbb{Z}$ ,  $\mathbb{Q}$  or  $\mathbb{R}$ , if  $M$  is connected, then the existence of a nonzero  $n$ -dimensional cycle furthermore implies that  $M$  is orientable.

Assume that  $M$  is a compact,  $n$ -dimensional, connected, triangulable, oriented manifold with empty boundary, and assume that the  $n$ -simplices of  $T$  are equipped with the induced orientation. Then the sum of all these simplices is a cycle. Its homology class is called the *fundamental class* of  $M$ . It is denoted by  $[M]$ . (When  $\Lambda = \mathbb{Z}/2\mathbb{Z}$ , this definition does not require an orientation of  $M$ .) Let  $B^n$  be an  $n$ -dimensional closed ball embedded in  $M$  by an orientation-preserving embedding. Let  $0$  be the center of this ball. Thanks to the excision axiom, the inclusions induce isomorphisms from  $H_n(B^n, B^n \setminus \{0\})$  to  $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ , and from  $H_n(B^n, B^n \setminus \{0\})$  to  $H_n(M, M \setminus \{0\})$ . Let  $[B^n, B^n \setminus \{0\}]$  denote the generator of  $H_n(B^n, B^n \setminus \{0\})$  mapped to  $[\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}]$  by the first isomorphism and let  $[M, M \setminus \{0\}]$  denote the image of  $[B^n, B^n \setminus \{0\}]$  under the second isomorphism. Then the inclusion induces an isomorphism from  $H_n(M)$  to  $H_n(M, M \setminus \{0\})$ , which maps the generator  $[M]$  of  $H_n(M)$  to the generator  $[M, M \setminus \{0\}]$  of  $H_n(M, M \setminus \{0\})$ . This defines the generator  $[M]$  of  $H_n(M)$  independently of a triangulation.

Let  $X$  be a topological space equipped with a triangulation  $T$  as above. Let  $Y$  be a closed subspace of  $X$  that is a union of simplices of  $T$ . Let  $T_Y$  be the corresponding triangulation of  $Y$ . Set  $C_k(T, T_Y) = C_k(T)/C_k(T_Y)$  and define  $\partial_k(T, T_Y): C_k(T, T_Y) \rightarrow C_{k-1}(T, T_Y)$  to be the map induced by the previous boundary map  $\partial_k$ . Then the homology  $H(X, Y)$  is canonically isomorphic to the homology of the chain complex  $(C(T, T_Y), \partial(T, T_Y))$ .

When  $M$  is a connected, compact, oriented  $n$ -dimensional manifold with boundary, equipped with a triangulation  $T$  whose  $n$ -dimensional simplices

are oriented by the orientation of  $M$ , the sum of all  $n$ -dimensional simplices of  $T$  is a cycle of  $C_n(M, \partial M)$ . Its homology class is called the *fundamental class* of  $(M, \partial M)$ . It is denoted by  $[M, \partial M]$ . Again, if  $B^n$  is an  $n$ -dimensional closed ball embedded in  $M$  by an orientation-preserving embedding, then the inclusions induce isomorphisms from  $H_n(B^n, B^n \setminus \{0\})$  to  $H_n(M, M \setminus \{0\})$  and from  $H_n(M, \partial M)$  to  $H_n(M, M \setminus \{0\})$ , and the image of  $[B^n, B^n \setminus \{0\}]$  under the first isomorphism coincides with the image of  $[M, \partial M]$  under the second one. It is denoted by  $[M, M \setminus \{0\}]$ .

These considerations allow us to talk about *the homology class of a compact oriented  $p$ -dimensional submanifold  $P$  of a manifold  $M$* . It is the image of  $([P] \in H_p(P))$  in  $H_p(M)$  under the map induced by the inclusion, and it is often still denoted by  $[P]$ . When  $P$  is a compact oriented  $p$ -dimensional manifold with boundary embedded in a topological space  $X$  so that  $\partial P$  is embedded in a subspace  $Y$  of  $X$ , we define the class  $[P, \partial P]$  of  $(P, \partial P)$  in  $H_p(X, Y)$ , similarly.

With these conventions, when  $M$  is a connected, compact, oriented,  $n$ -dimensional manifold with boundary, such that  $M$  can be equipped with a triangulation, the boundary map  $\partial_n$  in the homology sequence of the pair  $(M, \partial M)$  maps  $[M, \partial M]$  to the class  $[\partial M]$ , where  $[\partial M]$  is the sum of the classes of the connected components of  $\partial M$  equipped with the orientation induced by the orientation of  $M$  with respect to the outward normal first convention.

All the manifolds considered in this book can be equipped with triangulations. If  $M$  is a manifold equipped with a triangulation  $T$ , and if  $P$  is a  $p$ -dimensional closed oriented manifold embedded in  $M$  that is a union of simplices of  $T$ , then the homology class of  $P$  in  $M$  vanishes if and only if the cycle that is the sum of the simplices of dimension  $p$  of  $P$  (equipped with the orientation of  $P$ ) is the (algebraic) boundary of a simplicial chain of  $T$  of dimension  $p + 1$ .

Homologies with various coefficients are related by the universal coefficient theorem. See [Gre67, 29.12], [Hat02, Thm. 3A.3], or [Spa81, Chap. 5, Sec.2], for example. Here is an excerpt of this theorem.

**Theorem A.7.** *When  $\Lambda = \mathbb{Q}$  or  $\mathbb{R}$ , we have*

$$H_k(X; \Lambda) = H_k(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda$$

*for any topological space  $X$ .*

In this book, we mostly use cohomology with coefficients in  $\mathbb{Q}$ ,  $\mathbb{Z}/2\mathbb{Z}$  or  $\mathbb{R}$ . In these cases, it can be defined by the following excerpt of the universal coefficient theorem for cohomology, which can be found in [Spa81, Chap.5, Sec.5, Thm. 3, page 243] and in [Hat02, Thm. 3.2], for example.

**Theorem A.8.** *When  $\Lambda$  is a field, we have*

$$H^k(X, Y; \Lambda) = \text{Hom}_\Lambda(H_k(X, Y; \Lambda), \Lambda)$$

for any  $k \in \mathbb{Z}$  and for any topological space  $X$ .

Note the sign  $=$  in the above theorems, meaning that the identifications are canonical. For a continuous map  $f: (X, Y) \rightarrow (A, B)$  and an integer  $k \in \mathbb{Z}$ , the morphism

$$H^k(f; \Lambda): H^k(A, B; \Lambda) \rightarrow H^k(X, Y; \Lambda)$$

maps a linear map  $g$  of  $H^k(A, B; \Lambda)$  to  $g \circ H_k(f; \Lambda)$ .

For a general  $\Lambda$ , and for a pair  $(X, Y)$  of topological spaces such that  $X$  is equipped with a triangulation  $T$  as above, and  $Y$  is equipped with a subtriangulation  $T_Y$  of  $T$  as before, the cohomology  $H^*(X, Y; \Lambda)$  of  $(X, Y)$  is the cohomology of the complex  $(C^*(T, T_Y; \Lambda), \partial^*(T, T_Y; \Lambda))$ , where  $C^k(T, T_Y; \Lambda)$  is equal to  $\text{Hom}(C_k(T, T_Y; \Lambda); \Lambda)$  and  $\partial^k: C^k(T, T_Y; \Lambda) \rightarrow C^{k+1}(T, T_Y; \Lambda)$  maps a linear form  $g$  to  $g \circ \partial_{k+1}$ . We have

$$H^k(X, Y; \Lambda) = \frac{\text{Ker}(\partial^k)}{\text{Im}(\partial^{k-1})}.$$

The elements of  $\text{Ker}(\partial^k)$  are the  $k$ -dimensional cocycles, and the elements of  $\text{Im}(\partial^{k-1})$  are the  $k$ -dimensional coboundaries.

Here is a weak version of the Poincaré duality theorem. See [Gre67, Chap. 26 to 28, in particular (28.18)] or [Hat02, Thm. 3.43].

**Theorem A.9.** *Let  $M$  be a compact,  $n$ -dimensional manifold with a possible boundary. Let  $H$  denote the singular homology. Then there are canonical Poincaré duality isomorphisms from  $H^k(M, \partial M; \mathbb{Z}/2\mathbb{Z})$  to  $H_{n-k}(M; \mathbb{Z}/2\mathbb{Z})$ , and from  $H^k(M; \mathbb{Z}/2\mathbb{Z})$  to  $H_{n-k}(M, \partial M; \mathbb{Z}/2\mathbb{Z})$ , for any  $k \in \mathbb{Z}$ .*

*If  $M$  is oriented, then for any  $\Lambda \in \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ , there are canonical Poincaré duality isomorphisms from  $H^k(M, \partial M; \Lambda)$  to  $H_{n-k}(M; \Lambda)$ , and from  $H^k(M; \Lambda)$  to  $H_{n-k}(M, \partial M; \Lambda)$ .*

*Let  $A$  be an oriented smooth submanifold of  $M$  of dimension  $n - k$  such that  $\partial A \subset \partial M$ . Let  $P^{-1}([A, \partial A]) \in H^k(M; \Lambda)$  denote the image of its class under the inverse of such a Poincaré duality isomorphism. Let  $B$  be the class of a closed oriented  $k$ -dimensional submanifold or a simplicial  $k$ -dimensional cycle transverse to  $A$ . Then the evaluation  $P^{-1}([A, \partial A])(B)$  of  $P^{-1}([A, \partial A])$  at  $B$  is the algebraic intersection  $\langle B, A \rangle_M$ .*

Here is a weak version of the Künneth theorem. See [Spa81, Chap.5, Sec.3, Thm. 10 (and Chap.5, Sec.2, Lem. 5)] or [Hat02, Section 3.B].

**Theorem A.10.** *Let  $H$  denote the singular homology with coefficients in a commutative field  $\Lambda$ . Then for any two topological spaces  $X$  and  $Y$ , for any  $k \in \mathbb{N}$ , we have*

$$H_k(X \times Y) = \bigoplus_{i=0}^k H_i(X) \otimes_{\Lambda} H_{k-i}(Y).$$

Again, the sign  $=$  means that the identification is canonical. For embeddings of oriented closed manifolds  $P$  into  $X$  and  $Q$  into  $Y$ , the tensor product  $[P] \otimes [Q]$  of the homology classes of their images, represents the homology class of  $P \times Q$ .

We end this section by stating a weak version of the following Mayer–Vietoris exact sequence, which can be recovered from the Eilenberg–Steenrod axioms. See [Gre67, Chap. 17, Thm. 17.6], for example.

**Theorem A.11.** *Let  $X$  be a topological space. Let  $A$  and  $B$  be subspaces of  $X$  whose union  $A \cup B$  equals  $X$ . Let  $i_A: A \cap B \hookrightarrow A$ ,  $i_B: A \cap B \hookrightarrow B$ ,  $j_A: A \hookrightarrow X$ ,  $j_B: B \hookrightarrow X$  denote the inclusion maps. Assume that  $j_A$  and  $j_B$  induce isomorphisms from  $H(A, A \cap B)$  to  $H(X, B)$  and from  $H(B, A \cap B)$  to  $H(X, A)$ . Let  $\partial_{MV,k+1}: H_{k+1}(X) \rightarrow H_k(A \cap B)$  be the composition of the map from  $H_{k+1}(X)$  to  $H_{k+1}(X, B)$  induced by inclusion, the inverse of the isomorphism from  $H_{k+1}(A, A \cap B)$  to  $H_{k+1}(X, B)$ , and the boundary map  $\partial_{k+1}(A, A \cap B): H_{k+1}(A, A \cap B) \rightarrow H_k(A \cap B)$  of the long exact sequence of  $(A, A \cap B)$ . Then there is a long exact sequence*

$$\dots \rightarrow H_{k+1}(X) \xrightarrow{\partial_{MV,k+1}} H_k(A \cap B) \xrightarrow{i_{MV,k}} H_k(A) \oplus H_k(B) \xrightarrow{j_{MV,k}} H_k(X) \xrightarrow{\partial_{MV,k}} \dots$$

such that  $j_{MV,k}(\alpha \in H_k(A), \beta \in H_k(B)) = H_k(j_A)(\alpha) + H_k(j_B)(\beta)$  and  $i_{MV,k}(\gamma) = (H_k(i_A)(\gamma), -H_k(i_B)(\gamma))$ .

## A.2 Homotopy groups

Let  $X$  be a topological space equipped with a basepoint  $x$ . Let  $n$  be a positive integer. The set of homotopy classes of maps from  $[0, 1]^n$  to  $X$  mapping  $\partial([0, 1]^n)$  to  $x$  is denoted by  $\pi_n(X, x)$ . It is a *homotopy group*. Its product sends a pair  $([f], [g])$  of homotopy classes of maps  $f$  and  $g$  such that  $f$  sends  $[1/2, 1] \times [0, 1]^{n-1}$  to  $x$  and  $g$  sends  $[0, 1/2] \times [0, 1]^{n-1}$  to  $x$  to the class  $[f][g]$  of the map that coincides with  $f$  on  $[0, 1/2] \times [0, 1]^{n-1}$  and with  $g$  on  $[1/2, 1] \times [0, 1]^{n-1}$ . This product is commutative when  $n \geq 2$ . The set of path-connected components of  $X$  is denoted by  $\pi_0(X)$ .

**Remark A.12.** Classically, the set  $\pi_n(X, x)$  is defined as the set of homotopy classes of maps from  $(S^n, (1, 0, \dots, 0))$  to  $(X, x)$ . The two definitions coincide since  $S^n$  is homeomorphic to the quotient  $[0, 1]^n / (\partial[0, 1]^n)$  for  $n \geq 1$ .

**Examples A.13.** Let  $k$  and  $n$  be positive integers, such that  $1 \leq k \leq n$ . A standard approximation theorem [Hir94, Chapter 2, Theorem 2.6, p. 49] implies that any continuous map from  $S^k$  to  $S^n$  is homotopic to a smooth map. The Morse–Sard theorem 1.4 ensures that if  $k < n$ , any smooth map is valued in the complement of a point in  $S^n$ , which is contractible. Therefore,  $\pi_k(S^n, *) = \{1\}$  if  $1 \leq k < n$ . The reader can develop these arguments to prove that there is a canonical isomorphism from  $\pi_n(S^n, *)$  to  $\mathbb{Z}$ , which maps the homotopy class of a smooth map from  $(S^n, *)$  to itself to its degree (see Definition 1.5).

A weak version of the *Hurewicz theorem* relating homotopy groups to homology groups, ensures that for any path-connected topological space  $X$  equipped with a basepoint  $x$ ,  $H_1(X; \mathbb{Z})$  is the abelianization of  $\pi_1(X, x)$ . See [Gre67, (12.1)], for example.

A map  $p: E \rightarrow B$  is called a *weak fibration* if it satisfies the following *homotopy lifting property with respect to cubes*:

For any integer  $n \in \mathbb{N}$ , for any pair  $(h_0: [0, 1]^n \times \{0\} \rightarrow E, H: [0, 1]^{n+1} \rightarrow B)$  of continuous maps such that  $H|_{[0, 1]^n \times \{0\}} = p \circ h_0$ , there exists a continuous extension  $h$  of  $h_0$  to  $[0, 1]^{n+1}$  such that  $H = p \circ h$ .

To such a weak fibration, we associate the following long exact sequence in homotopy [Spa81, Chap. 7, Sec. 2, Thm. 10].

**Theorem A.14.** *Let  $p: E \rightarrow B$  be a weak fibration. Let  $e \in E$  be a basepoint of  $E$ . Let  $b = p(e)$  denote its image under  $p$ , and let  $F = p^{-1}(b)$  denote the fiber over  $b$ . Then we have the long exact sequence*

$$\begin{aligned} \dots \pi_{n+1}(B, b) \rightarrow \pi_n(F, e) \rightarrow \pi_n(E, e) \rightarrow \pi_n(B, b) \rightarrow \pi_{n-1}(F, e) \dots \\ \dots \rightarrow \pi_1(B, b) \rightarrow \pi_0(F) \rightarrow \pi_0(E) \rightarrow \pi_0(B), \end{aligned}$$

where the maps between the  $\pi_n$  are respectively induced by the inclusion  $F \hookrightarrow E$  and by  $p$ , and the map from  $\pi_n(B, b) \rightarrow \pi_{n-1}(F, e)$  is constructed as follows. An element of  $\pi_n(B, b)$  is represented by a map  $H: [0, 1]^n \rightarrow B$ , which has a lift  $h: [0, 1]^n \rightarrow E$  that maps  $[0, 1]^{n-1} \times \{0\} \cup (\partial[0, 1]^{n-1} \times [0, 1])$  to  $e$ . Such an element is mapped to the homotopy class of the restriction of  $h$  to  $[0, 1]^{n-1} \times \{1\}$ . The last three maps of the exact sequence are just maps between sets. Exactness means that the preimage of the component of the basepoint is the image of the previous map.

**Remark A.15.** To define the map from  $\pi_n(B, b)$  to  $\pi_{n-1}(F, e)$  in the above theorem, we implicitly used the fact that the pair

$$\left( [0, 1]^n, [0, 1]^{n-1} \times \{0\} \cup (\partial[0, 1]^{n-1} \times [0, 1]) \right)$$

is homeomorphic to  $([0, 1]^n, [0, 1]^{n-1} \times \{0\})$  for all  $n \in \mathbb{N} \setminus \{0\}$ .

A path from a point  $x$  to another point  $y$  of  $X$  induces an isomorphism from  $\pi_n(X, x)$  to  $\pi_n(X, y)$  for any integer  $n$ . So the basepoint is frequently omitted from the notation  $\pi_n(X, x)$  when  $X$  is path-connected.

A map  $p: E \rightarrow B$  is called a *covering map* if every  $b \in B$  has a neighborhood  $U$  such that  $p^{-1}(U)$  is a disjoint union of subsets of  $E$ , each of which is mapped homeomorphically onto  $U$  by  $p$ . Such a covering map is an example of a weak fibration, for which  $p$  induces isomorphisms from  $\pi_n(E, e)$  to  $\pi_n(B, p(e))$  for any  $e \in E$  and any  $n \geq 2$ .

# Appendix B

## Differential forms and de Rham cohomology

Here are a few well-known results about differential forms used throughout the book. More detailed accounts can be found in the books [God71, Chapters XI and XII] by Claude Godbillon and [BT82] by Raoul Bott and Loring Tu.

### B.1 Differential forms

Let  $M$  be a smooth manifold with possible boundary and ridges. Degree 0 forms of  $M$  are smooth functions on  $M$ . The differential  $df = Tf: TM \rightarrow \mathbb{R}$  of a smooth map  $f$  from  $M$  to  $\mathbb{R}$  is an example of a degree-one form of  $M$ . In general, a *degree  $k$  differential form* on  $M$  is a smooth section of the bundle  $\bigwedge^k(TM)^* = \text{Hom}(\bigwedge^k TM; \mathbb{R})$  over  $M$ . So, such a form  $\omega$  maps any  $m \in M$  to an element  $\omega(m)$  of  $\text{Hom}(\bigwedge^k T_m M; \mathbb{R})$ , smoothly, in the sense below. On an open part  $U$  of a manifold identified with an open subspace of  $\mathbb{R}^n$  by a chart  $\phi: U \rightarrow \mathbb{R}^n$  mapping  $u \in U$  to  $\phi(u) = (\phi_1(u), \dots, \phi_n(u))$ , the degree  $k$  forms are uniquely expressed as

$$\sum_{(i_1, \dots, i_k) \in \mathbb{N}^k : 1 \leq i_1 < \dots < i_k \leq n} f_{i_1 \dots i_k} d\phi_{i_1} \wedge \dots \wedge d\phi_{i_k}.$$

for smooth maps  $f_{i_1 \dots i_k}: U \rightarrow \mathbb{R}$ . The vector space of degree  $k$  differential forms on  $M$  is denoted by  $\Omega^k(M)$ . The differential  $T\psi: TM \rightarrow TN$  of a smooth map  $\psi$  from a manifold  $M$  to a manifold  $N$  induces the *pull-back* of differential forms

$$\begin{aligned} \psi^*: \Omega^k(N) &\rightarrow \Omega^k(M) \\ \omega &\mapsto (x \mapsto \omega(\psi(x)) \circ \bigwedge^k T_x \psi). \end{aligned}$$



The antisymmetric bilinear exterior product  $\wedge$  equips  $\bigoplus_{k \in \mathbb{N}} \Omega^k(M)$  with a structure of graded algebra, such that  $\psi^*(\omega \wedge \omega') = \psi^*(\omega) \wedge \psi^*(\omega')$  for any two forms  $\omega$  and  $\omega'$  on  $N$ . This graded algebra is equipped with a unique operator  $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$  such that

- $(df = Tf)$  is the differential of  $f$  for any  $f \in \Omega^0(M)$ ,
- we have  $d \circ d(f) = 0$  for any  $f \in \Omega^0(M)$ ,
- for any  $\alpha \in \Omega^{|\alpha|}(M)$  and  $\beta \in \Omega^{|\beta|}(M)$ , we have

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta,$$

where  $|\alpha|$  and  $|\beta|$  denote the respective integral degrees of  $\alpha$  and  $\beta$ , and

- the derivation operator  $d$  commutes with the above pull-backs, we have  $d\psi^*(\omega) = \psi^*(d\omega)$ .

Then we have  $d \circ d = 0$ .

The *support* of a differentiable form is the closure of the set where it does not vanish.

Let  $x_i: \mathbb{R}^k \rightarrow \mathbb{R}$  be the usual coordinate functions. The *integral* of a degree  $k$  differential form  $\omega = f dx_1 \wedge \cdots \wedge dx_k$  over a  $k$ -dimensional compact submanifold  $C$  of  $\mathbb{R}^k$  with boundary and ridges (like  $[0, 1]^k$ ) is  $\int_C \omega = \int_C f dx_1 \dots dx_k$ . For any smooth map  $\psi$  of  $\mathbb{R}^k$  that restricts to an orientation-preserving diffeomorphism from  $C$  to its image, we have

$$\int_C \psi^*(\omega) = \int_{\psi(C)} \omega.$$

Thanks to this property, we unambiguously define the integral of a  $k$ -form  $\omega$  over any  $k$ -dimensional compact submanifold  $C$  of a manifold  $M$ , identified to a subspace of  $\mathbb{R}^k$  by a diffeomorphism  $\phi: C \rightarrow \mathbb{R}^k$  onto its image, to be

$$\int_C \omega = \int_{\phi(C)} \phi^{-1*}(\omega).$$

This definition extends naturally to general compact manifolds with boundaries and ridges.

One of the most useful theorems in this book is the following Stokes theorem. See [BT82, Theorem 3.5, Page 31].

**Theorem B.1** (Stokes' theorem). *Let  $\omega$  be a degree  $d$  form on an oriented smooth compact  $(d + 1)$ -manifold  $M$  with boundary  $\partial M$ , then we have*

$$\int_{\partial M} \omega = \int_M d\omega.$$

This theorem applies to manifolds with ridges, and  $\int_{\partial M} \omega$  is the sum of the  $\int_C \omega$ , over the codimension zero faces  $C$  of  $\partial M$ . (The closures of these faces are  $d$ -manifolds with boundaries.)

## B.2 De Rham cohomology

A degree  $k$  differential form  $\omega$  on  $M$  is *closed* if  $d\omega = 0$ . It is *exact* if  $\omega \in d\Omega^{k-1}(M)$ . Define the degree  $k$  de Rham cohomology module of  $M$  to be

$$H_{dR}^k(M) = \frac{\text{Ker}(d: \Omega^k(M) \rightarrow \Omega^{k+1}(M))}{d(\Omega^{k-1}(M))}.$$

For a compact submanifold  $N$  (without boundary) of the interior of  $M$ , or for a connected component  $N$  of  $\partial M$ , as in [God71, Chapter XII], let  $\Omega^k(M, N)$  be the kernel of the restriction map from  $\Omega^k(M)$  to  $\Omega^k(N)$ . The relative degree  $k$  de Rham cohomology module of  $(M, N)$  is

$$H_{dR}^k(M, N) = \frac{\text{Ker}(d: \Omega^k(M, N) \rightarrow \Omega^{k+1}(M, N))}{d(\Omega^{k-1}(M, N))}.$$

We have a natural short exact sequence of chain complexes

$$0 \rightarrow \Omega^k(M, N) \rightarrow \Omega^k(M) \rightarrow \Omega^k(N) \rightarrow 0.$$

This sequence induces a natural long exact cohomology sequence

$$\rightarrow H_{dR}^{k-1}(N) \rightarrow H_{dR}^k(M, N) \rightarrow H_{dR}^k(M) \rightarrow H_{dR}^k(N) \rightarrow$$

where the maps from  $H_{dR}^k(M, N)$  to  $H_{dR}^k(M)$  and from  $H_{dR}^k(M)$  to  $H_{dR}^k(N)$  are induced by the restrictions and the cohomology class  $[\eta]$  in  $H_{dR}^{k-1}(N)$  of a closed form  $\eta$  of  $N$  is mapped to  $([d\tilde{\eta}] \in H_{dR}^k(M, N))$  for an extension  $\tilde{\eta}$  of  $\eta$  to  $M$ .

The degree  $k$  forms with compact support in  $M$  also form a subspace  $\Omega_c^k(M)$  of  $\Omega^k(M)$ , and the degree  $k$  de Rham cohomology module with compact support of  $M$  is

$$H_{dR,c}^k(M) = \frac{\text{Ker}(d: \Omega_c^k(M) \rightarrow \Omega_c^{k+1}(M))}{d(\Omega_c^{k-1}(M))}.$$

For any smooth map  $\psi$  from  $M$  to another manifold  $M'$  that maps  $N$  to a submanifold  $N'$ , the pull-back  $\psi^*$  induces maps still denoted by  $\psi^*$  from  $H_{dR}^k(N')$  to  $H_{dR}^k(N)$ , and from  $H_{dR}^k(M', N')$  to  $H_{dR}^k(M, N)$ . If  $\phi$  is another such smooth map from the pair  $(M', N')$  to another such  $(M'', N'')$ , then we have

$$\psi^* \circ \phi^* = (\phi \circ \psi)^*.$$

When such a map  $\psi$  is *proper* (i.e., when the preimage of a compact is compact), the map  $\psi$  also induces  $\psi^*: H_{dR,c}^k(M') \rightarrow H_{dR,c}^k(M)$ .

The following standard lemma implies that  $\psi^*: H_{dR}^k(M') \rightarrow H_{dR}^k(M)$  depends only on the homotopy class of  $\psi: M \rightarrow M'$ .

**Lemma B.2.** *Let  $V$  and  $W$  be two smooth manifolds, and let*

$$\begin{aligned} h: [0, 1] \times V &\rightarrow W \\ (t, v) &\mapsto h_t(v) \end{aligned}$$

*be a smooth homotopy. Let  $\omega$  be a degree  $d$  closed form on  $W$ . Then we have*

$$h_t^*(\omega) - h_0^*(\omega) = d\eta_t(h, \omega)$$

*for any  $t \in [0, 1]$ , where  $\eta_t(h, \omega)$  is the following degree  $(d-1)$  form on  $V$ . For  $u \in [0, 1]$ , let  $i_u: V \rightarrow [0, 1] \times V$  map  $v \in V$  to  $i_u(v) = (u, v)$ . Let  $h^*(\omega)((u, v))(\frac{\partial}{\partial t} \wedge \cdot)$  be obtained from  $h^*(\omega)$  by evaluating it at the tangent vector  $\frac{\partial}{\partial t}$  to  $[0, 1] \times \{v\}$  at  $(u, v)$ . (Thus,  $h^*(\omega)((u, v))(\frac{\partial}{\partial t} \wedge \cdot)$  is the value at  $(u, v) \in [0, 1] \times V$  of a degree  $(d-1)$  form of  $[0, 1] \times V$ .) Then we define*

$$\eta_t(h, \omega)(v) = \int_0^t i_u^* \left( h^*(\omega)((u, v)) \left( \frac{\partial}{\partial t} \wedge \cdot \right) \right) du.$$

**PROOF:** Observe  $h_u^*(\omega) = i_u^*(h^*(\omega))$  and write

$$h^*(\omega) = \omega_1 + dt \wedge h^*(\omega) \left( \frac{\partial}{\partial t} \wedge \cdot \right),$$

where we have  $h_u^*(\omega) = i_u^*(\omega_1)$ . Since  $\omega$  is closed,  $dh^*(\omega)$  vanishes. Therefore, we obtain

$$0 = d\omega_1 - dt \wedge d \left( h^*(\omega) \left( \frac{\partial}{\partial t} \wedge \cdot \right) \right).$$

On the other hand, with the natural projection  $p_V: [0, 1] \times V \rightarrow V$ , we can write

$$(d\omega_1)((u, v)) = \left( dt \wedge \frac{\partial}{\partial t}(\omega_1) + p_V^*(dh_u^*(\omega)) \right)((u, v)).$$

We get

$$i_u^* \left( \frac{\partial}{\partial t}(\omega_1) \right) = i_u^* \left( d \left( h^*(\omega) \left( \frac{\partial}{\partial t} \wedge . \right) \right) \right)$$

and

$$\frac{\partial}{\partial u} h_u^*(\omega) = \frac{\partial}{\partial u} i_u^*(\omega_1) = i_u^* \left( \frac{\partial}{\partial t}(\omega_1) \right) = i_u^* \left( d \left( h^*(\omega) \left( \frac{\partial}{\partial t} \wedge . \right) \right) \right).$$

Since we have  $h_t^*(\omega)(v) - h_0^*(\omega)(v) = \int_0^t \frac{\partial}{\partial u} h_u^*(\omega)(v) du$ , the lemma follows.  $\square$

In particular, if there exist smooth maps  $f: M \rightarrow N$  and  $g: N \rightarrow M$  such that  $f \circ g$  is smoothly homotopic to the identity map of  $N$ , and  $g \circ f$  is smoothly homotopic to the identity map of  $M$ , then  $f^*$  is an isomorphism from  $H_{dR}^k(N)$  to  $H_{dR}^k(M)$  for any  $k$ . In particular, all the smoothly contractible manifolds, such as  $\mathbb{R}^n$ , have the same de Rham cohomology as the point. So, for such a manifold  $C$ ,  $H_{dR}^k(C) = \{0\}$  for any  $k \neq 0$ , and  $H_{dR}^0(C) = \mathbb{R}$ .

More generally, in 1931, Georges de Rham identified the de Rham cohomology of a smooth  $n$ -dimensional manifold  $M$  to its singular cohomology with coefficients in  $\mathbb{R}$ . See [War83, Pages 205-207], or [BT82, Theorems 8.9 Page 98, 15.8 page 191].

The de Rham isomorphism sends the cohomology class  $[\omega]$  of a closed degree  $k$  form to a linear map  $[\omega]_{dR}$  of  $\text{Hom}(H_k(M; \mathbb{R}); \mathbb{R}) = H^k(M; \mathbb{R})$ . The linear map  $[\omega]_{dR}$  sends the homology class of a closed oriented  $k$ -dimensional submanifold  $N$  of  $M$  to  $[\omega]_{dR}([N]) = \int_N \omega$ .

If  $M$  has a smooth triangulation  $T$ , any  $k$ -form  $\omega$  similarly defines a simplicial cochain, i.e., an element of  $C^k(T; \mathbb{R}) = \text{Hom}(C_k(T; \mathbb{R}); \mathbb{R})$ , by integration over the simplices. Stokes' theorem guarantees that the induced map from  $\Omega^k(M)$  to  $C^k(T; \mathbb{R})$  commutes with differentials, and induces a morphism from  $H_{dR}^k(M)$  to  $H^k(M; \mathbb{R}) = \text{Hom}(H_k(M; \mathbb{R}); \mathbb{R})$ .

We also have the following theorem [GHV72, page 197] or [BT82, (5.4) and Remark 5.7].

**Theorem B.3.** *For any oriented manifold  $M$  without boundary of dimension  $n$  (whose cohomology is not necessarily finite-dimensional), the morphism from  $\Omega^k(M; \mathbb{R})$  to  $\text{Hom}(\Omega_c^{n-k}(M); \mathbb{R})$*

$$\left( \omega \mapsto \left( \omega_2 \mapsto \int_M \omega \wedge \omega_2 \right) \right)$$

*induces an isomorphism from  $H^k(M; \mathbb{R})$  to  $\text{Hom}(H_c^{n-k}(M); \mathbb{R})$ , for any integer  $k$ .*

In particular, when the real cohomology of  $M$  is finite-dimensional, so is its homology, and  $H_c^{n-k}(M)$  is isomorphic to  $H_k(M; \mathbb{R})$ . Below, we exhibit the image  $[\omega_A]$  of the homology class of an oriented compact submanifold  $A$  with trivial normal bundle in  $H_k(M; \mathbb{R}) = \text{Hom}(H^k(M); \mathbb{R})$  under a canonical isomorphism from  $H_k(M; \mathbb{R})$  to  $H_c^{n-k}(M)$ .

Let  $A$  be such a  $k$ -dimensional submanifold without boundary of the manifold  $M$ , and let  $N(A) = B^{n-k} \times A$  be a tubular neighborhood of  $A$  in  $M$ . Let  $\omega_B$  be an  $(n-k)$ -form of  $B^{n-k}$  supported in the interior of  $B^{n-k}$  such that  $\int_{B^{n-k}} \omega_B = 1$ . Use the natural projection  $p_B: B^{n-k} \times A \rightarrow B^{n-k}$  to pull back  $\omega_B$  on  $N(A)$  and define a form  $\omega_A$  to coincide with  $p_B^*(\omega_B)$  on  $N(A)$  and to be zero outside  $N(A)$ . The cohomology class  $[\omega_A]$  of  $\omega_A$  is independent of the choice of  $\omega_B$ . The integral of the closed form  $\omega_A$  over a compact  $(n-k)$  submanifold  $B$  of  $M$  transverse to  $A$  is the algebraic intersection  $\langle B, A \rangle_M$  of  $B$  and  $A$ . Note that the support of  $\omega_A$  may be chosen in an arbitrarily small neighborhood of  $A$ . In the words of Definition 11.6, the form  $\omega_A$  may be chosen to be  $\alpha$ -dual to  $A$  for an arbitrarily small positive number  $\alpha$ .

**Lemma B.4.** *Let  $M$  be a compact manifold of dimension  $n$  with possible boundary. Assume that  $M$  is equipped with a smooth triangulation. The above process can be extended to produce canonical Poincaré duality isomorphisms from  $H_a(M, \partial M; \mathbb{R})$  to  $H_{dR}^{n-a}(M)$ , where such a Poincaré duality isomorphism maps the class of an  $a$ -dimensional cycle  $A$  of  $(M, \partial M)$  to the class of a closed  $(n-a)$ -form  $\omega_A$   $\alpha$ -dual to  $A$  for an arbitrarily small positive number  $\alpha$ , as follows.*

**PROOF:** Let  $A$  be a linear combination of  $a$ -dimensional simplices of the triangulation  $T$  of  $M$  such that the algebraic boundary of  $A$  is in  $C(T_{\partial M})$ . Such an  $A$  is a *simplicial  $a$ -cycle* of  $(M, \partial M)$ . Its *support*  $\bar{A}$  is the union in  $M$  of the closed simplices with a nonzero coefficient of  $A$ . We construct a form  $\omega_A$   $\alpha$ -dual to  $A$  as follows. Let  $A^{(k)}$  be the intersection of  $\bar{A}$  with the  $k$ -skeleton of  $T$ . ( $A^{(k)} = \emptyset$  if  $k < 0$ .) Let  $N(A^{(a-1)})$  be a small neighborhood of  $A^{(a-1)}$ .

First construct a closed form  $\omega_A$   $\alpha$ -dual to  $A$  outside  $N(A^{(a-1)})$  so that its support is in an arbitrarily small neighborhood of  $A$ , as explained above in the case of closed manifolds with trivial normal bundles.

Then extend  $\omega_A$  around each  $(a-1)$ -simplex  $\Delta$  of  $A$ , outside a small neighborhood  $N(A^{(a-2)})$  of  $A^{(a-2)}$  as follows.

Without loss of generality, assume that the intersection of a neighborhood of  $\Delta$  with the complement of  $N(A^{(a-2)})$  is diffeomorphic to  $\Delta' \times D^{n-a+1}$ , for some interior  $\Delta'$  of an  $(a-1)$ -simplex in the interior of  $\Delta$  (some *truncation* of  $\Delta$ ). See Figure B.1. The  $(n-a)$ -form  $\omega_A$  is defined and closed on a neighborhood of  $\Delta' \times \partial D^{n-a+1}$ . Its integral along  $\{x\} \times \partial D^{n-a+1}$  is the

algebraic intersection  $\langle \partial D^{n-a+1}, A \rangle_M$ . It is the coefficient of  $\Delta$  in  $\partial A$ , up to sign. Since this coefficient is zero, the form  $\omega_A$  is exact in a neighborhood of  $\Delta' \times \partial D^{n-a+1}$ . So  $\omega_A$  may be written as  $d\eta$  on this neighborhood. Choose a map  $\chi$  on  $\mathbb{R}^{n-a+1}$  that takes the value 1 outside a small neighborhood of 0 and that vanishes in a smaller neighborhood of 0 so that  $d\chi\eta$  makes sense on  $\Delta' \times D^{n-a+1}$  and extends  $\omega_A$  as a closed form.

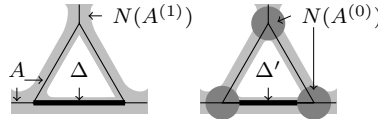


Figure B.1: The neighborhoods  $N(A^{(1)})$  and  $N(A^{(0)})$ , a simplex  $\Delta$  and its truncation  $\Delta'$ , when  $A$  is one-dimensional

This process allows us to define a closed form  $\omega_A$   $\alpha$ -dual to  $A$  outside a small neighborhood  $N(A^{(a-2)})$  of  $A^{(a-2)}$ . Iterate the process in order to extend such a form outside  $N(A^{(a-3)})$ , outside  $N(A^{(a-4)})$ ,  $\dots$ , and on the whole  $M$ . Note that the forms will be automatically exact around the truncated smaller cells since we have  $H^{n-a}(\partial D^{n-a+k}) = 0$  when  $k > 1$ .  $\square$

In this setting, the correspondence between chain boundaries and the differentiation operator  $d$  can be roughly seen as follows. Let  $A$  be a compact oriented  $a$ -dimensional submanifold with boundary of a manifold  $M$ . Assume that the normal bundle of  $A$  is trivial. Let  $[-1, 1] \times \partial A$  be embedded in the interior of  $M$  so that  $\{0\} \times \partial A = \partial A$  and  $[-1, 0] \times \partial A$  is a neighborhood of  $\partial A$  in  $A$ . Set  $A^+ = A \cup_{[-1,0] \times \partial A} [-1, 1] \times \partial A$ . Choose a map  $\chi: A^+ \rightarrow [0, 1]$  sending  $A$  to 1 and a neighborhood of  $\partial A^+$  to 0 such that  $\chi$  factors through the projection onto  $[0, 1]$  on  $[0, 1] \times \partial A$ . Let  $N(A^+) = B^{n-a} \times A^+$  be the total space of the normal bundle to  $A^+ = A \cup_{\partial A} ([0, 1] \times \partial A)$  embedded in  $M$ . Let  $p_{A^+}: N(A^+) \rightarrow A^+$  and  $p_B: N(A^+) \rightarrow B^{n-a}$  denote the natural projections. Let  $\omega_B$  be an  $(n - a)$ -form of  $B^{n-a}$  supported in the interior of  $B^{n-a}$  such that  $\int_{B^{n-a}} \omega_B = 1$ . The form

$$\omega_A = (\chi \circ p_{A^+})p_B^*(\omega_B)$$

extends as a smooth form that vanishes outside  $N(A^+)$ . The form  $d\omega_A$  is supported in  $B^{n-a} \times [0, 1] \times \partial A$ . Its integral along a chain  $C$  whose boundary does not meet  $B^{n-a} \times [0, 1] \times \partial A$  is  $\pm \langle C, \partial A \rangle_M$ .



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# Terminology

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# Index of notation

- $A_I^{(i)}$  3-handlebody, 483
- $A_t^{(i)}$  3-handlebody, 489
- $\mathcal{A}_n(\cdot)$  target quotiented diagram
  - space of  $\mathcal{Z}_n$ , 128, 129
- $\check{\mathcal{A}}_n(\cdot)$  no trivalent component, 130
- $\mathcal{A}_n^c(\emptyset)$  connected, 154
- $\overline{\mathcal{A}}_n$  chord diagrams mod 4T, 124
- $\alpha$  anomaly, 157
  
- $b(A)$  basepoint, 186
- $\mathcal{B}(A, C)$  blow-up, 66
- $\mathring{B}_{1,\infty}$  ball around  $\infty$ , 70
- $B_R$  rational homology ball, 73
- $\beta$  anomaly, 223
  
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  - $\check{C}_V[M]$  open, 191
  - $\check{C}(R, L; \Gamma)$  open, 145
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  - $D(V)$ , 172
  - $D(V, K(V))$ , 172
- $D_\varepsilon$  disk of radius  $\varepsilon$ , 258
- $D(\omega_0(R_I))$  domain in  $C_2$ , 533
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  - $\mathcal{D}_{n,m}^e(\mathcal{L})$ , 160
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- $\partial_j$  codimension  $j$  boundary, 56
  
- $E(\Gamma) = \{\text{edges of } \Gamma\}$ , 42
- $\mathbf{e}(\mathcal{E}, \ell)$  edge in a tree, 308
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- $\eta(a_j^i)$  one-form on  $A_I^{(i)}$ , 483
  
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  - $F_\infty(A, L, \Gamma)$ , 182
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- $\Gamma$  Jacobi diagram, 42
- $\Gamma_A$  subgraph of  $\Gamma$ , 202
  
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- $\widetilde{\text{hol}}_{[a,b]}(\cdot)$   $A$ -holonomy, 382
  
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  - $I_\theta(K, \tau)$  for a long knot, 274
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## Lagrangians

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Open configuration sets

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## 3-manifolds

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177 $\overline{S}_V(T)$  configuration space, 176 $\check{S}(T_\infty R, A)$  open configuration  
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72 $U(\Gamma) = \{\text{univalent vert. of } \Gamma\}$ , 42 $U^-K$  unit negative tangent  
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# Summarizing the main definitions of $Z$

In the informal summary below, we first recall the various definitions of  $Z$  and its variants  $(\check{Z}, z, \mathcal{Z}^f)$  for a given link embedding  $L = \sqcup_{j=1}^k K_j$  into a  $\mathbb{Q}$ -sphere  $R$ . The punctured  $\check{R} = R \setminus \{\infty\}$  is equipped with a parallelization  $\tau$  that makes  $(\check{R}, \tau)$  an asymptotic rational homology  $\mathbb{R}^3$  as in Definition 3.8.

For  $n \in \mathbb{N}$ , for a family  $(\omega(i))_{i \in \underline{3n}}$  of propagating forms of  $C_2(R)$  as in Section 3.3, we have

$$Z_n(\check{R}, L, (\omega(i))_{i \in \underline{3n}}) \stackrel{\text{Not. 7.16}}{=} \sum_{\Gamma \in \mathcal{D}_n^c(\mathcal{L})} \zeta_\Gamma I(R, L, \Gamma, (\omega(i))_{i \in \underline{3n}}) [\Gamma] \in \mathcal{A}_n(\mathcal{L}).^1$$

If the  $\omega(i)$  are propagating forms of  $(C_2(R), \tau)$  as in Definition 3.9, and if they are homogeneous as in Definition 3.13, then we have

$$Z_n(\check{R}, L, \tau) \stackrel{\text{Thm. 7.20}}{=} Z_n(\check{R}, L, (\omega(i))_{i \in \underline{3n}})$$

and

$$Z(\check{R}, L, \tau) = (Z_n(\check{R}, L, \tau))_{n \in \mathbb{N}} = \sum_{n \in \mathbb{N}} Z_n(\check{R}, L, \tau).$$

With the anomalies  $\alpha$  of Section 10.3 and  $\beta$  of Section 10.2, the Pontragin number  $p_1(\tau)$  of Definition 5.13, and the definition of  $I_\theta$  in Lemma 7.15, we have

$$\mathcal{Z}(R, L) \stackrel{\text{Thm. 7.20}}{=} \exp\left(-\frac{1}{4}p_1(\tau)\beta\right) \prod_{j=1}^k \left(\exp(-I_\theta(K_j, \tau)\alpha) \#_j\right) Z(\check{R}, L, \tau),^2$$

<sup>1</sup>When  $\omega(i) = \omega$  for all  $i$ , we set  $Z_n(\check{R}, L, \omega) = Z_n(\check{R}, L, (\omega(i))_{i \in \underline{3n}})$  as in Proposition 7.25. The use of a family of distinct  $\omega(i)$  allows us to discretize the definition of  $\mathcal{Z}$  in Chapter 11, thanks to Version 7.40 of Theorem 7.20.

<sup>2</sup>The similar letters  $Z$  and  $\mathcal{Z}$  respectively denote a function involving auxiliary data and the induced invariant.

and

$$\mathcal{Z}(R) = \mathcal{Z}(R, \emptyset).$$

With the projection  $p^c: \mathcal{A}(\emptyset) \rightarrow \mathcal{A}^c(\emptyset)$  on the connected part of Notation 7.16, we have

$$z(R) = p^c(\mathcal{Z}(R)) \text{ and } \mathcal{Z}(R) = \exp(z(R)).$$

With the projection  $\check{p}: \mathcal{A}(\mathcal{L}) \rightarrow \check{\mathcal{A}}(\mathcal{L})$  of Notation 7.16, which forgets diagrams without univalent vertices, we have

$$\check{\mathcal{Z}} = \check{p} \circ \mathcal{Z} \text{ and } \mathcal{Z}(R, L) = \mathcal{Z}(R)\check{\mathcal{Z}}(R, L).$$

When  $L = \sqcup_{j=1}^k K_j$  has a parallel  $L_{\parallel} = \sqcup_{j=1}^k K_{j\parallel}$ , the *framed* version  $\mathcal{Z}^f$  of  $\mathcal{Z}$  satisfies

$$\mathcal{Z}^f(\check{R}, L, L_{\parallel}) \stackrel{\text{Def. 7.41}}{=} \prod_{j=1}^k \left( \exp(\text{lk}(K_j, K_{j\parallel})\alpha) \#_j \right) \mathcal{Z}(R, L).$$

Now, we give a similar summary for the extension of  $\mathcal{Z}$  to tangles in  $\mathbb{Q}$ -cylinders of the book's third part. We fix such a  $\mathbb{Q}$ -cylinder  $\mathcal{C}$ , the associated  $\mathbb{Q}$ -sphere  $R(\mathcal{C})$ , a tangle  $L$  in  $\mathcal{C}$ , and the associated long tangle also denoted by  $L$  in  $\check{R}(\mathcal{C})$  as in Definition 12.1. Theorem 12.7 defines  $Z(\mathcal{C}, L, \tau)$  (resp.  $Z_n(\mathcal{C}, L, (\omega(i))_{i \in 3n})$ ) to be a natural extension of  $Z(R(\mathcal{C}), L, \tau)$  (resp.  $Z_n(R(\mathcal{C}), L, (\omega(i))_{i \in 3n})$ ) from the case where  $L$  is a link in  $\mathcal{C}$  to the case where  $L$  is any tangle of  $\mathcal{C}$  canonically extended to a long tangle as in Definition 12.1. We have

$$\mathcal{Z}(\mathcal{C}, L) = \exp\left(-\frac{1}{4}p_1(\tau)\beta\right) \prod_{j=1}^k \left( \exp(-I_{\theta}(K_j, \tau)\alpha) \#_j \right) Z(\mathcal{C}, L, \tau),$$

with respect to Definition 12.6 for  $I_{\theta}(K_j, \tau)$  when  $K_j$  is a long component. Definition 12.12 of

$$\mathcal{Z}^f(\mathcal{C}, (L, L_{\parallel})) = \prod_{j=1}^k \left( \exp(\text{lk}(K_j, K_{j\parallel})\alpha) \#_j \right) \mathcal{Z}(\mathcal{C}, L).$$

extends from tangles to  $q$ -tangles, which are cobordisms between limit configurations, by a natural limit process as in Remark 13.11.

Again, discrete computations and proofs of properties involve *straight* tangles as in Section 16.3 and distinct propagating forms *dual* to propagating chains as in Chapter 11. Theorems 16.9 and 16.16, and Definition 16.44 describe variants of  $\mathcal{Z}$  associated to such data. These variants depend on 2-forms over  $S^2$  in a way described in Theorem 16.45.