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Connections and Frame Bundle Reductions

M. Lachièze-Rey

*APC-Astroparticule et Cosmologie (UMR-CNRS 7164),
Université Paris Diderot-Paris 7, 10, rue Alice Domon et Léonie Duquet
F-75205 Paris Cedex 13, France
mlr@apc.univ-paris7.fr*

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Abstract

In general relativity, the gravitational potential is represented by the Levi-Civita connection, the only symmetric connection preserving the metric. On a differentiable manifold, a metric identifies with an orthogonal structure, defined as a Lorentz reduction of the frame bundle. The Levi-Civita connection appears as the only symmetric connection preserving the reduction. This paper presents generalization of this process to other approaches of gravitation: Weyl structure with Weyl connections, teleparallel structures with Weitzenböck connections, unimodular structure, similarly appear as frame bundle reductions, with preserving connections.

To each subgroup H of the linear group GL correspond reduced structures, or H -structures. They are subbundles of the frame bundle (with GL as principal group), with H as principal group. A linear connection in a manifold M is a principal connection on the frame bundle. Given a reduction, the corresponding *preserving connections* on M are the linear connections which preserve it.

I also show that the *time gauge* used in the 3+1 formalism for general relativity similarly appears as the result of a bundle reduction.

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1 Introduction

Principal bundle reductions are the mathematical expression of a physical process of symmetry reduction, from a principal group to one of its subgroups. They play an important role in physics and are presented in many textbooks, mainly for the archetypal case of symmetry breaking in gauge theories, a variant of the *Higgs mechanism*.

It is maybe not so well known that general relativity may be seen as the result of a similar symmetry reduction, from the linear group to the Lorenz group: a metric on a manifold identifies with a reduction from its

frame bundle to an ON tetrad bundle. Then the metric connections are those (among the linear connections) which preserve this reduction.

This paper shows how some other approaches of gravitation, with their relevant connections, may be similarly considered as resulting from a reduction of the frame bundle. This applies to Weyl theory with Weyl connections; to teleparallel theories with Weitzenböck connections; to unimodular theories.

In physics, a symmetry corresponding to a Lie group G has a *local* expression under the form of a G -principal bundle \mathcal{P} (with principal group G), with the space-time manifold as basis. A bundle reduction leads to a H -reduced structure, with H a subgroup of G . This is a H -principal subbundle, and a preserving connection is a connection on \mathcal{P} which preserves the reduction.

The theory of general relativity may be formulated for instance as a particular case of orthogonal reduction of the frame bundle Fr , a GL -principal bundle (see, *e.g.*, [9, 10]). A solution of the theory is a Lorentzian structure, *i.e.*, a SO -principal subbundle of Fr , with the Lorentz group SO as principal group. This identifies with the orthogonal tetrad bundle, itself equivalent to the (Lorentzian) metric. The preserving connections are the metric connections. Among them, the unique symmetric one, the Levi-Civita connection, is identified to the gravitational potential.

The section 2 first introduces the *group reductions*; in particular for a linear group $\text{GL}(E)$ acting on its representation vector space E , as well as on the space $B(E)$ of its vector bases. A reduction of GL to a subgroup H is defined as a subspace of $B(E)$ which is an H -orbit. It may then be expressed under the form of some specific structure on E (like for instance an inner product in the case of orthogonal reduction).

Section 3 recalls the general process of *bundle reduction*, a generalization of group reduction. The section 4 specializes to the case of the frame bundle, and presents the generalizations (from global to local) of the reductions presented in section 2. This make appear different structures which have been proposed to express gravitation as results of such reductions: general relativity, but also Weyl theory, teleparallel theory, unimodular theory... The *time gauge* in the 3+1 formalism for general relativity (which is used, for instance, in Loop quantum gravity) appears similarly as the result of a bundle reduction.

Section 5 reminds some remarks about the link between Cartan connections and bundle reductions.

2 Group Quotients and Reductions

2.1 Group Quotients

Throughout the paper, G designs a Lie-group (later we will specialize to the linear group GL) of dimension D and H a subgroup of dimension d , with inclusion map $H \xrightarrow{i} G$.

We have a similar inclusion $\underline{\mathfrak{h}} \rightarrow \underline{\mathfrak{g}}$ of the corresponding Lie algebras; and $\underline{\mathfrak{g}}$ may be splitted as a direct sum of vector spaces, $\underline{\mathfrak{g}} = \underline{\mathfrak{h}} \oplus V$, with V a vector space (in general not an algebra) isomorphic to $\underline{\mathfrak{g}}/\underline{\mathfrak{h}}$. There are different ways to perform such splitting, and thus different isomorphisms¹.

¹ Since $\underline{\mathfrak{h}}$ is a Lie algebra, we have $[\underline{\mathfrak{h}}, \underline{\mathfrak{h}}]_{\underline{\mathfrak{g}}} = [\underline{\mathfrak{h}}, \underline{\mathfrak{h}}]_{\underline{\mathfrak{h}}} \subseteq \underline{\mathfrak{h}}$. When in addition $[\underline{\mathfrak{h}}, V]_{\underline{\mathfrak{g}}} = \text{Ad}(H) \cdot V \subseteq V$, then V is Ad_H -invariant and the splitting is said reductive.

An action of G on some space M induces an action of H on the same space. Any point $m \in M$ has an H -orbit $[m] \stackrel{\text{def}}{=} \{h m; h \in H\}$, which is an H -equivalence class in M . A *group-reduction* is defined as the choice of such a class in M . The space M itself is G -invariant but each class is only H -invariant: this is a symmetry reduction.

The set of such classes (equivalently, of reductions) is the quotient space B/H , with dimension $D - d$.

2.2 Linear Groups and Frame Reductions

We specialize to the linear group $G = \text{GL}(\mathbb{R}^n) = \text{Aut}(\mathbb{R}^n)$, the group of linear transformations (automorphisms) of the vector space $E = \mathbb{R}^n$, with dimension $D = n^2$.

The linear group GL has also a natural action on the set $B(E) = \{b\}$ of vector bases (that I also call *frames*) of E . Each $g \in G$ identifies with a change of basis $b \rightarrow g b$. This action is free and transitive, so that any choice of a preferred basis (identified to the identity of G) provides $B(E)$ with a group structure isomorphic to GL ².

A subgroup $H \subset \text{GL}$ also acts on $B(E)$ and associates to any basis b its H -orbit $[b] \stackrel{\text{def}}{=} \{h b; h \in H\}$. Any choice of such an H -equivalence class $[b]$ in B is a symmetry reduction, from the linear group GL to H .

The set of such classes, or equivalently the set of reductions, is the quotient space B/H , with dimension $n^2 - d$.

In section 4, such group reductions are extended to frame bundle reductions, by performing them continuously in each fiber of the frame bundle of a manifold. This represents a reduction of the corresponding *local* symmetry. I give before some examples of frame-reductions.

2.2.1 Reduction to the orthogonal group

The typical example is $H = \text{O}$, an orthogonal group³, with dimension $d = \frac{n^2-n}{2}$. The space of possible reductions is the quotient space B/O , with dimension $D - d = \frac{n^2+n}{2} \approx 10$ (I indicate with the sign \approx the value for $n = 4$, relevant for physical space-time).

Each reduction selects an O -equivalence class $[b]$ in B : a class of vector bases linked together by orthogonal transformations (which are the elements of O). That class defines a unique inner product p in E (with the signature of O), such that all its frames are orthonormed (ON) w.r.t. p . It results that the $(D - d)$ -dimensional set of reductions identifies with the set of possible inner products in E (see table 1).

Note that an inner product is defined by the components of the symmetric $n \times n$ matrix representing it (in an arbitrary basis). Their number is $D - d = \frac{n^2+n}{2}$, the dimension of G/O .

2.2.2 Reduction to the Weyl group

We apply a similar procedure to the *Weyl group* $H = W \stackrel{\text{def}}{=} \mathfrak{R}^+ \times \text{O}$ (with $d = \frac{n^2-n}{2} + 1 \approx 7$). Here \mathfrak{R}^+ is the multiplicative group; each element, a positive real number, acts as a dilatation of the frame.

² This means that $B(E)$ is a torsor, *i.e.*, a group without unit element.

³ I do not specify now. This holds for $\text{O}(p, n - p)$; Later we will specialize to space-time, so that $n = 4$, and to the Lorentz group $\text{O}(1, 3)$. The procedure works for any dimension and with any signature.

Each reduction corresponds to a W -equivalence class $[b] \subset B$. Its elements are bases linked together by elements of W , *i.e.*, orthogonal transformations as above, and dilatations.

This defines not an inner product like above, but a family \mathcal{C} of inner products related by multiplication by a positive real number. This family exactly represents a *conformal structure* on E . The frames of the class are those which are ON w.r.t. one of the inner products of \mathcal{C} . The reduction is equivalent to this class, or to the conformal structure (see table 1).

A conformal structure (a reduction) is defined by $D-d = \frac{n^2+n}{2} - 1 \approx 9$ numbers, the dimension of G/W ; one number less than for a metric, corresponding to the dilatation freedom.

2.2.3 Reduction to the identity

A trivial reduction corresponds to H the group with the identity for unique element. The H -class of any basis b reduces to that basis b alone, so that $B/H = B$. Each reduction identifies with the selection of a single basis $b_A \in B$ (a choice of $n^2 = 16$ numbers, the dimension of $G/H = G$). Note that this frame defines in turn an unique inner product w.r.t. which it is ON.

2.2.4 Reduction to the special linear group

Choosing the special linear group $H = \text{SL}$ ($d = n^2 - 1 \approx 15$), GL/H identifies with $\mathfrak{R} - \{0\}$. A reduction corresponds to a subspace $B^A \subset B$ of vector bases which share the same determinant $A \neq 0$ (as expressed in an arbitrary predetermined frame).

Each reduction is expressed by a number A and is equivalent to the class B^A of vector bases sharing the same determinant A (see table 1).

2.2.5 A double Reduction gives unimodular inner product

A first SL -reduction, as in the previous step, defines the class B^A of the bases which share the same determinant A (w.r.t. to a given fixed basis).

Then the group inclusion $\text{SO} \xrightarrow{i} \text{SL}$ defines the quotient space SL/SO with dimension $\frac{n^2+n}{2} - 1 \approx 9$. This allows further reductions by selecting in B^A a subclass B^{AS} of bases related by SO transformations (which preserve the determinant A).

As above these bases define an inner product p_{AS} . It is however constrained (by the first reduction) to have the determinant ϵA^2 (in the predetermined frame); the value of ϵ depends on the signature of SO ($\epsilon = 1$ for Riemannian; $\epsilon = -1$ for Lorentzian).

Finally, the double reduction is equivalent to first choosing a number A , and then an inner product p_{AS} having determinant ϵA^2 . It is called *unimodular* since the value of A is usually taken to be 1 (in fact the choice of A w.r.t. a predetermined reference frame is equivalent to the choice of a reference frame such that $A^2 = 1$).

Extended to the frame bundle, this is at the basis of unimodular theories of gravity (see 4.4).

Table 1: Reductions of the linear group $G = \text{GL}$

subgroup H	$\dim(\text{H})$ $= d$	$\dim(\text{G}/\text{H})$ $= D - d$	reduction	reduced frames
O	$\frac{n^2-n}{2}$	$\frac{n^2+n}{2}$	inner product p	ON w.r.t. w.r.t. p
W	$\frac{n^2-n}{2} + 1$	$\frac{n^2+n}{2} - 1$	conformal structure $[p]$	ON w.r.t. any $p \in [p]$
II	0	n^2	one frame	one frame
SL	$n^2 - 1$	1	$A \in \mathfrak{R} - \{0\}$	frames with det. = A
SL — → O	$n^2 - 1$ — $\frac{n^2-n}{2}$	1 — $\frac{n^2+n}{2} - 1$	A — unimodular inner product with det = ϵA^2	fr. with det= A — ON fr. w.r.t. p with det= A

3 Principal bundle reductions

A Lie group G expresses a *global* symmetry; a principal bundle represents its *local* counterpart. Its automorphisms form a group \mathcal{G} much larger than G , often called the *local* (not global) symmetry (or gauge) group. A group reduction, as in previous section, may be generalized to a principal bundle reduction, which corresponds to a reduction of the local (infinite) Lie group \mathcal{G} . In particular group reductions of GL generalize to reductions of the GL -principal frame bundle.

The next section recalls the standard construction of *reduced structures*; then we apply it to the frame bundle of a differentiable manifold. Hereafter M is a differentiable manifold of dimension n .

3.1 Reduced Structures

A G -principal bundle $\pi : \mathcal{P} \mapsto M$ has its typical fiber F isomorphic to its principal group G (dimension D). The action $y \rightarrow gy$ of G on \mathcal{P} defines that of its subgroup H . The latter assigns to each $y \in \mathcal{P}$ its H -orbit $[y] \stackrel{\text{def}}{=} \{hy; h \in H\}$. This is an H -equivalence class in \mathcal{P} ⁴ and we have the natural projection $\pi_H : y \rightarrow [y]$.

Physically, the G -principal bundle \mathcal{P} represents a system with *local* symmetry G and G is often called the *global* symmetry group. A reduction is performed like above, by the choice of an H -equivalence class *in each fiber*. This choice will appear equivalent to the construction of a H -principal bundle \mathcal{P}^σ .

This represents a local symmetry breaking from G to H . For instance, a choice of this kind is at the basis of the Higgs mechanism in gauge theories, where the reduced structure identifies with the Higgs field. Here we will consider different approaches to gravitation as similar reductions of

⁴ Since the action of G , and thus of H , is vertical, the orbit $[y]$ is contained in F_y , the fibre over y .

the frame bundle, from its original local GL symmetry to that of a subgroup. This generalizes the well-known result that a Riemannian structure (a metric) identifies with an orthogonal structure, *i.e.*, a reduction from the frame bundle to the ON frame bundle (as we recall below).

3.1.1 General Construction of the Reduced Structure

Each section s of $\mathcal{P} \mapsto M$ defines the map

$$m \rightarrow \sigma(m) \stackrel{\text{def}}{=} (\pi_H \circ s)(m) \stackrel{\text{def}}{=} [s(m)].$$

It sends any basis point to the H -orbit of its image through the section s . An important theorem (see, *e.g.*, [3] p.119) states that $m \rightarrow \sigma(m)$ is a section of the fiber bundle

$$\Sigma \stackrel{\text{def}}{=} \pi_\Sigma : \mathcal{P}/H \rightarrow M,$$

with $\pi = \pi_H \circ \pi_\Sigma$. This quotient bundle has typical fiber F/H isomorphic to G/H (dimension $D - d$).

Then each section $\sigma : M \mapsto \Sigma : m \rightarrow \sigma(m)$ of this quotient bundle defines a reduction, under the form of the subbundle of \mathcal{P} ,

$$\pi_\sigma : \mathcal{P}^\sigma \rightarrow M.$$

This subbundle is defined as containing only those elements in \mathcal{P} which belong to the image of the section σ :

$$\mathcal{P}^\sigma = \{y \in \mathcal{P}, \exists m : y \in \sigma(m)\},$$

with inclusion $\mathcal{P}^\sigma \xrightarrow{i} \mathcal{P}$. Its fiber over m is $(\pi_\sigma)^{-1}(m) \stackrel{\text{def}}{=} \sigma(m)$ ⁵.

The H -principal bundle \mathcal{P}^σ is the [reduced] H -structure associated to the section σ . The construction shows the one-to-one correspondence between such reductions and global sections σ of the quotient bundle $\Sigma \stackrel{\text{def}}{=} \mathcal{P}/H \mapsto M$.

Degrees of Freedom

One may count the degrees of freedom: each fiber of \mathcal{P} has the dimension D of G : we have D degrees of freedom for choosing a section. The fiber of \mathcal{P}/H has dimension $D - d$: a section represents $(D - d)$ degree of freedoms. The fiber of \mathcal{P}^σ has dimension d : a section has d degrees of freedom.

In the language of gauge theories, a reduction represents a *symmetry breaking*: from the whole (un broken) symmetry group G to the group H of broken symmetries. A section σ can be interpreted as a classical Higgs field [9]. Matter fields are sections of some vector bundle associated to \mathcal{P}^σ , whose fibers are representations of H .

3.2 Preserving connections

Now we assume a bundle reduction $\mathcal{P}^H \xrightarrow{i} \mathcal{P}$ linked to the choice of a section σ of $\mathcal{P}/H \rightarrow M$, as above. Such a choice fixes d local (for each point) degrees of freedom among D , so that $D - d$ remain.

On the other hand an (Ehresmann) principal G-connection on \mathcal{P} is defined by a connection form ω . This is a one-form in \mathcal{P} ⁶ with values

⁵ Locally, above an open U_M of M , we have $\mathcal{P}^\sigma \simeq U_M \times H$.

⁶ It does not depend on a trivialization, in contrary to *local* connection forms, which are defined in the basis manifold M .

Table 2: The “ localization ” of Table 1: reductions of the Frame Bundle; “ frames ” hold for “ moving frames ”

subgroup H	structure	reduction	reduced bundle	connection
O	Orthogonal structure	metric g	ON frames w.r.t. g	metric connection
W	conformal structure	Conformal structure $[g]$	ON frames w.r.t. any $g \in [g]$	Weyl c.
\mathbb{I}	Teleparallel	one frame e	unique frame e	Weitzenböck connection
SL		$A \in \mathfrak{K} - \{0\}$	frames with determinant A	
SL \rightarrow O	Unimodular structure	“ unimodular ” metric g with $\det = \epsilon A^2$	ON frames w.r.t. g with $\det = A$	connection metric w.r.t. g

in \underline{g} . We may demand that the connection preserves the reduced structure. This is equivalent to demanding that ω takes its value in $\underline{\mathfrak{h}} \subset \underline{\mathfrak{g}}$ only. We refer to this case as a *preserving connection*. We show below how specific types of linear connections in manifolds, in relation to some gravitation theories, identify with particular preserving connections.

The space \mathcal{A} of (smooth) connections in \mathcal{P} admits $n D \approx 64$ local degrees of freedom. For a given reduction σ , the space \mathcal{A}_σ of preserving connections (also in \mathcal{P}) has only $n d$ local degrees of freedom ⁷.

A preserving connection is a particular case of *reductive connection*. For the later, the connection form admits an Ad_H -invariant splitting $\omega = \omega^h + \omega^v$, where ω^h takes its values in $\underline{\mathfrak{h}}$. This implies that $i^* \omega^h$ is a principal H-connection in the H-principal bundle \mathcal{P}^H : it takes its values in $\underline{\mathfrak{h}}$ and is H-equivariant. The case of a reducing connection here corresponds to $\omega^v = 0$.

We apply below to linear connections in a manifold, which are in fact GL-principal connections in the frame bundle.

4 Reductions of the Frame Bundle

4.1 The Frame Bundle

The frame bundle $\text{Fr}(M) \mapsto M$ of a manifold M is a GL-principal fiber bundle with $G = \text{GL}$ the linear group, and $D = n^2$. Now I write Fr for $\text{Fr}(M)$. Its fiber over m is the set $\text{Fr}_m \stackrel{\text{def}}{=} \{f_m\}$ of possible vector bases of $T_m M$, the tangent space at m ; Fr_m is (non canonically) isomorphic to GL and we recover in each fiber the situation of the first section.

A section of Fr is a *moving frame* $f : m \rightarrow f_m \in \text{Fr}_m$. It assigns to each m a basis f_m of $T_m M$. A smooth *local* action of $G = \text{GL}$ (the

⁷ This may be seen from the fact that the reduced structure corresponds to $D - d$ degrees of freedom. The preservation conditions represent $n (D - d)$ equations in the $(n D)$ -dimensional space of connections, leaving a space of preserving connections with dimension $n d$.

action of an element of \mathbf{GL} at each point of M) identifies with a change of moving frame. It is free and transitive. Such local actions form the infinite-dimensional group \mathcal{G} ⁸.

4.1.1 Linear Connections

A linear connection on a manifold M is a principal connection on the frame bundle, with $G = \mathbf{GL}$. The space \mathcal{A} of [smooth] linear connections has $n^2 D = n^3 \approx 64$ local degrees of freedom.

One may describe a linear connection by its local connection form (in a given fixed moving frame), a \mathbf{GL} -equivariant one-form ω taking its value in \mathfrak{g} . Since G is the linear group, the components of ω are usually written with manifold indices (in the same frame) as $\Gamma_{\beta}^{\alpha} = \Gamma_{\mu\beta}^{\alpha} \theta^{\mu}$. The connection is usually represented by its connection coefficients $\Gamma_{\mu\beta}^{\alpha}$. Its n^3 manifold indices represent the n^3 degrees of freedom of \mathcal{A} .

4.2 Orthogonal reduction

The well known case (see, *e.g.*, [9, 10]) of *orthogonal reduction* corresponds to the choice of an orthogonal group $H = \mathbf{O}$, with $d = n(n-1)/2$ degrees of freedom (the procedure works for any dimension and with any signature). Each reduction defines an orthogonal structure.

We apply a reduction like in 2.2.1, but fiberwise. A fiber of $\text{Fr}/\mathbf{O} \rightarrow M$ is the quotient Fr_m/\mathbf{O} , the set of \mathbf{O} -equivalence classes of local bases; or equivalently, of \mathbf{O} -orbits in Fr_m . A section σ of $\text{Fr}/\mathbf{O} \rightarrow M$ is a continuous choice $\sigma(m)$ of such a class for each m . Like above, all the frames in the class $\sigma(m)$ are linked together by \mathbf{O} -transformations, and they appear as ON w.r.t. an inner product p_m in $T_m M$. The reunion of these inner products (one for each m) builds a metric g over M . Finally, the reduction, and equivalently the section of Fr/\mathbf{O} , identifies with the metric g . For this reason, Fr/\mathbf{O} is called the *metric bundle*. We have the equivalences **reduction \simeq section of $\text{Fr}/\mathbf{O} \simeq$ metric $g \simeq$ class of ON moving frames (tetrads) w.r.t. g .**

Each reduction (choice of g) defines the related *orthogonal structure* $\text{Fr}^{\mathbf{O}}(M) \subset \text{Fr}(M)$, called the tetrad bundle, or ON frame bundle, corresponding to g . This sub-bundle of Fr contains, as sections, only the moving frames taken in the class, *i.e.*, ON tetrads w.r.t. g . It admits the structure group \mathbf{O} .

Metric connections

Preserving connections preserve the orthogonal structure: they are the *metric connections*. Given a metric, the space of metric connections admits $n^2(n-1)/2 \approx 24$ local degrees of freedom.⁹

Asking in addition for *symmetry* (no torsion) fixes all the remaining degrees of freedom, resulting in the unique Levi-Civita connection for that metric.¹⁰

⁸ It admits [the pullback of] the infinite Lie group Diff as a subgroup. Note that the reduction of linear moving frames to *holonomic* frames (or coordinate frames) may be seen as a reduction of \mathcal{G} to Diff .

⁹ Preserving the $D-d$ degrees of freedom of the metric correspond to $n \times (D-d)$ equations that the connection must obey. Since the space of connections has n^3 local degrees of freedom, this leaves d a space of metric connections.

¹⁰ The space of symmetric connections has $n(n^2+n)/2 \approx 40$ local degrees of freedom.

4.2.1 Lorentz Structure and Time Gauge

The case $n = 4$ and the choice of the Lorentz group $\text{SO}(1, 3)$ corresponds to space-time and general relativity. A *Lorentz structure* is an other name for a Lorentzian metric g , with Fr^{SO} identified with the corresponding ON tetrad bundle¹¹ on space-time, with $D = \frac{n^2-n}{2} \approx 6$ degrees of freedom. Each section is a tetrad. “The configuration space of general relativity can thus be seen as the space of sections of the bundle Fr/SO ” [7].

We may extend to a new reduction to $\text{SO}(3) \subset \text{SO}(1, 3)$. This defines an $\text{SO}(3)$ -structure on space-time, under the form of an $\text{SO}(3)$ -principal subbundle of Fr^{SO} .

We have $d = \frac{(n-1)^2-(n-1)}{2} \approx 3$. The space of such reductions is the quotient bundle $\text{Fr}^{\text{SO}}/\text{SO}(3)$, with dimension $D - d \approx 3$ per fibre. This is the number of degrees of freedom of the space of reductions.

A section of the reduced structure is a class Fr^u of ON tetrads which are linked together by the (local) orthogonal transformations of $\text{O}(3)$, *i.e.*, the spatial rotations in each fiber. In other words all the tetrads of this class share the same timelike unit vector-field u . And the reduction identifies with the choice of this vector-field u . This process is called a *time gauge* (see also [4]). The $\text{SO}(3)$ -structure is the corresponding *triad bundle*¹².

4.3 Weyl group and Weyl geometry

The choice of the Weyl group $H = W$ (with $d = n(n-1)/2 + 1 \approx 7$) defines the reduction to a Weyl structure, or conformal structure. This is a class $[g]$ of metrics related by [Weyl] scalings (*i.e.*, multiplications by a scalar positive function). Any moving frame in the reduced structure is ON w.r.t. one of the metrics of the class $[g]$. Such a *conformal structure* represents $\frac{n^2+n}{2} - 1 \approx 9$ degrees of freedom.

A connection preserving the conformal structure $[g]$ is called a *Weyl connection*. For a given reduction, the space of Weyl connections admits $n^2(n-1)/2 + n \approx 28$ local degrees of freedom. Imposing symmetry (no torsion), as usual, fixes $n^2(n-1)/2 \approx 24$ degrees of freedom among them. The $n \approx 4$ remaining correspond to the choice of a *Weyl form* A such that, for any member g of the class $[g]$, $\nabla g = A \otimes g$ (the form A depends on the representative of the class). This corresponds exactly to a *Weyl geometry*. It is *integrable* when A is an exact form.

4.4 Unimodular Theory

The frame bundle is a principal GL -bundle. The reduction of GL to SL (generalizing 2.2.5) fixes 1 degree of freedom. It is well expressed by the choice of a volume form Vol ¹³. A section of the SL structure is the class of moving frames whose associated volume element identifies with Vol . Such frames are linked together by the local SL group.

¹¹ For any signature, the group inclusions $\text{SO}^\uparrow \subset \text{SO} \subset \text{O} \subset \text{GL}$ allows us to replace O (reduced O structure Fr^{O} of ON frames) by SO or SO^\uparrow . One obtains respectively:

- the reduced SO structure Fr^{SO} of oriented ON frames;
- the reduced SO^\uparrow structure of oriented and time-oriented ON frames.

¹² Each tetrad is decomposed as u plus the (spacelike) triad.

¹³ This is equivalent to the choice of a scalar function in an arbitrarily chosen moving frame; but the formulation with a volume form offers the advantage of remaining covariant. Also, it is always possible to chose the reference frame such that f is a constant function equal to unity; hence the appellation *unimodular* [5].

Then a subsequent reduction of SL to an orthogonal group SO (like the Lorentz group) fixes $n(n-1)/2 \approx 6$ additional local degrees of freedom. A section is a family of moving frames linked by orthogonal transformations and sharing the same volume form Vol . Like in 2.2.5 this defines a metric g , which is constrained to obey $Vol_g = Vol$. The space of such unimodular metrics admits $n(n+1)/2 - 1 \approx 9$ local degrees of freedom.

A metric connection (like the Levi-Civita connection) w.r.t. an unimodular metric preserves both structures.

4.4.1 Weitzenböck connections

A trivial reduction of GL to identity — a generalization of 2.2.3 — selects an unique basis f_m in each fiber Fr_m ; thus a moving frame f corresponding to a section of $Fr = Fr/\mathbb{L}$. Note that such a moving frame, and thus the reduction, only exists (when continuity is required) for a parallelizable manifold. The reduced structure identifies with that moving frame, sometimes called the AP-frame, for “Absolute Parallelism”. In other words, the reduction fixes the totality of the degrees of freedom in Fr ¹⁴ and breaks totally the linear symmetry.

It turns out that there is an unique¹⁵ connection preserving the AP-frame (*i.e.*, the reduced structure): $\nabla e = 0$, which means $\nabla(e_I) = 0, \forall I$. It is called the *Weitzenböck connection* associated to the selected AP-frame e . This implies zero connection coefficients in the AP-frame, from which follows — as it is well known — that this connection is flat (with zero curvature)¹⁶. This connection has however torsion, described by the local torsion form $T^I = de^I$ in the AP-frame.

Like any coframe, the reciprocal coframe θ of the AP-frame defines in turn an unique metric $g_{\mu\nu} \stackrel{\text{def}}{=} \eta_{IJ} \theta^I \theta^J$, w.r.t. which it is ON. This AP-metric is also preserved by the Weitzenböck connection which is thus metric w.r.t. g . Of course, the later differs from the Levi-Civita connection associated to g but, as it has been emphasized a long time ago by Einstein himself, it is possible to construct from the torsion of the Weitzenböck connection an action which leads to equations for the metric similar to that of general relativity (obtained from the *Einstein-Hilbert action* constructed with the curvature scalar of the Levi-Civita connection). This analogy between this approach, named *teleparallel*, and general relativity may however hold only when a Weitzenböck connection exists, and thus for parallelizable manifolds.

The main difference is that, by construction, the teleparallel view breaks explicitly the Lorentz symmetry through the choice of the AP-tetrad: it breaks totally (not partially) the linear symmetry, although general relativity breaks it to Lorentz symmetry only¹⁷. Thus, a physical equivalence can only associate teleparallelism with general relativity plus an additional physical entity corresponding to the breaking of the Lorentz symmetry.

¹⁴ Preserving the 16 degrees of freedom of a tetrad correspond to $4 \times 16 = 64$ equations; which fix the 64 degrees of freedom of a connection.

¹⁵ In dimension 4, preserving the 16 degrees of freedom of a moving frame correspond to $4 \times 16 = 64$ equations, which fix the 64 degrees of freedom of a the possible connections, thus leading to the same conclusion.

¹⁶ Flat connection do not exist in any differentiable manifold.

¹⁷ One may obtain a teleparallel structure, with associated Weitzenböck connection, from a double reduction: first an orthogonal reduction, from the linear group to the orthogonal (Lorentz) group; then a second reduction entirely breaking the remaining orthogonal (Lorentz) symmetry, to preserve the AP-frame.

A link has also been suggested with a gauging of translations (see [8, 6] and references therein). A moving coframe θ is a one form taking its value in \mathfrak{R}^4 . The identification of the latter with the Lie algebra of the group of translations [1] may allow to consider θ as a principal connection for this group. This is however not a *linear* connection on the manifold and, in particular, torsion is not defined for it. It turns out however that the curvature form, $d\theta$, of this connection identifies with the torsion form of the Weitzenböck connection expressed in the same tetrad (considered as AP-tetrad). This mathematical analogy lacks however geometrical support since the involved “ translations ” have absolutely no geometrical action, on space-time or on its tangent space. This suggest that one should rather refer to an *internal \mathfrak{R}^4 gauging* than a *translation gauging*.

5 Some Remarks about Cartan Connections

Let me finish with some remarks concerning *Cartan Connections*, which are also linked to bundle reductions, although in a different way.

I assume a bundle reduction $\mathcal{P}^H \xrightarrow{i} \mathcal{P}$, with \mathcal{P} a G -principal bundle and \mathcal{P}^H a H -principal bundle. To fix the ideas, H may be an orthogonal group O and \mathcal{P}^H the orthogonal frame bundle like the usual *tetrad bundle* in Riemannian geometry; and G a larger group, including O as a subgroup, typically Poincaré, de Sitter or anti-de Sitter. Thus the resulting geometry is often called the gauge theory of these groups [11].

A principal connection form $\tilde{\omega}$ on \mathcal{P} footnote We precise that the G -principal bundle \mathcal{P} is only used as an auxiliary for the derivations and has not necessarily a physical interpretation: the physics is described in the H -principal bundle \mathcal{P}^H . The derivation of the Cartan geometry may be entirely formulated in it without implying the bundle \mathcal{P} . It involves a \mathfrak{g} -valued connection (not a principal connection) ω , which takes its values in \mathfrak{g} (not in \mathfrak{h}). ω takes values in \mathfrak{g} . Like above we split $\tilde{\omega} = \tilde{\omega}^H + \tilde{\omega}^V$, with $\tilde{\omega}^V$ taking its values in a vector space V (in general not an algebra) isomorphic to $\mathfrak{g}/\mathfrak{h}$ ¹⁸.

Then $\omega \stackrel{\text{def}}{=} i_*\tilde{\omega}$ is by definition a Cartan connection in \mathcal{P}^H ¹⁹ iff it verifies the *Cartan condition*:

for all $p \in \mathcal{P}^H$, ω defines a linear isomorphism $T_p\mathcal{P}^H \xrightarrow{\sim} \mathfrak{g}$.

This requires $D = d + n$. Taking its values in \mathfrak{g} , ω cannot be a principal connection since the principal group of \mathcal{P}^H is H . It is however H -equivariant [2].²⁰

When in addition the splitting $\omega = \omega^H + \omega^V$ is *reductive* (*i.e.*, also H -equivariant), then $\omega^H \stackrel{\text{def}}{=} i^*\tilde{\omega}^H$ is a H -principal connection on \mathcal{P}^H ,

¹⁸ Since \mathfrak{h} is a Lie algebra, we have $[\mathfrak{h}, \mathfrak{h}]_{\mathfrak{g}} = [\mathfrak{h}, \mathfrak{h}]_{\mathfrak{h}} \subseteq \mathfrak{h}$. When in addition $[\mathfrak{h}, V]_{\mathfrak{g}} = \text{Ad}(H) \cdot V \subseteq V$, then V is Ad_H -invariant and the splitting is said reductive.

¹⁹ Some authors call ω a Cartan connection on \mathcal{P} .

²⁰ Note that a Cartan connection ω can also be defined intrinsically in a H -principal bundle \mathcal{P}^H as

- taking values in \mathfrak{g} and being H -equivariant;
- verifying the Cartan condition above;
- and giving the value $\omega(\zeta) = \underline{\zeta}$ for the fundamental vector-field ζ (Killing vector-field) in \mathcal{P}^H corresponding to $\underline{\zeta} \in \mathfrak{h}$.

called a *reduced connection* and $e \stackrel{\text{def}}{=} i^* \omega^V$ a H-value one form on \mathcal{P}^H called the *solder form*.

5.1 Gauging of the Poincaré and de Sitter Groups

This applies in particular to G the Poincaré group, and $H = \text{SO}(1,3)$ the Lorentz group. We assume $n = 4$.

\mathcal{P} is a principal G -bundle over space-time M . The reduced Lorentz structure is Fr^O , the tetrad bundle over M , equivalent to a Lorentzian manifold with metric g . A principal connection $\tilde{\omega}$ over \mathcal{P} takes its values in $\underline{\mathfrak{g}}$. It does not generate a linear connection over M since this is not a linear connection on its frame bundle.

The commutator of a translation with a rotation being a translation, the Poincaré algebra admits a reductive splitting $\underline{\mathfrak{g}} = \underline{\mathfrak{h}} + V$. The connection $\tilde{\omega}$ splits as $\tilde{\omega} = \tilde{\omega}^H + \tilde{\omega}^V$, with the isomorphisms $T_m M \xrightarrow{\tilde{\omega}^V} \underline{\mathfrak{g}\mathfrak{h}}$, thus obeying the Cartan condition.

We obtain the Lorentz-invariant splitting $i_* \tilde{\omega} \stackrel{\text{def}}{=} \omega = \omega^{SO} + \omega^V$. Here ω is the Cartan connection on M (not a linear connection), and the principal Lorentz connection ω^{SO} defines a linear metric connection on M . The Minkowski vector space-valued one-form ω^V in Fr^O is the solder form, the bundle expression of the cotetrad. Note that the present construction also holds with the Poincaré group replaced by de Sitter or anti-de Sitter groups.

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