



**HAL**  
open science

## On 2-Salem polynomials and 2-Salem numbers in positive characteristic

Mabrouk Ben Nasr, Hassen Kthiri, Jean-Louis Verger-Gaugry

► **To cite this version:**

Mabrouk Ben Nasr, Hassen Kthiri, Jean-Louis Verger-Gaugry. On 2-Salem polynomials and 2-Salem numbers in positive characteristic. 2020. hal-02464774

**HAL Id: hal-02464774**

**<https://hal.science/hal-02464774>**

Preprint submitted on 3 Feb 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On 2-Salem polynomials and 2-Salem numbers in positive characteristic

Mabrouk Ben Nasr<sup>1</sup>, Hassen Kthiri<sup>2</sup>, and Jean-Louis Verger-Gaugry<sup>3</sup>

<sup>1</sup>Department of Mathematics, Faculty of sciences, Sfax, Tunisia , e.mail:  
mabrouk\_bennasr@yahoo.fr

<sup>2</sup>Department of Mathematics, Faculty of sciences, Sfax, Tunisia, e.mail:  
hassenkthiri@gmail.com

<sup>3</sup>LAMA, CNRS UMR 5127, Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, F-73000  
Chambéry , France, e.mail: jean-louis.verger-gaugry@univ-smb.fr

## Abstract

Bateman and Duquette have characterized Salem numbers in positive characteristic. This work extends their results to 2-Salem numbers from 2-Salem minimal polynomials of the type  $Y^n + \lambda_{n-1}Y^{n-1} + \dots + \lambda_1Y + \lambda_0 \in \mathbb{F}_q[X][Y]$  where  $n \geq 2$ ,  $\lambda_0 \neq 0$  and  $\deg \lambda_{n-1} < \deg \lambda_{n-2} = \sup_{i \neq n-2} \deg(\lambda_i)$ . The existence of 2-Salem polynomials over  $\mathbb{F}_q$ , and related questions of irreducibility and algebraicity of 2-Salem numbers, are studied by means of Weiss's method of upper Newton polygons, for  $q \neq 2^r$ ,  $r \geq 1$ .

**2010 Mathematics Subject Classification.:** 11R04; 11R06, 11R09, 11R52, 12D10.

**Key words:** Finite field, Laurent series, 2-Salem series, irreducible polynomial, Newton polygon.

## Contents

1	Introduction	2
2	Salem series in $\mathbb{F}_q((X^{-1}))$	3
3	Multiplicative properties of 2-Salem series	4
4	A first characterization of 2-Salem series	6
5	Criteria of existence of roots and conjugates in $\mathbb{F}_q((X^{-1}))$	9
6	Proof of Theorem 1.1	12
7	A criterium of irreducibility	15

# 1 Introduction

A Salem number is a real algebraic integer  $\theta > 1$  of even degree at least 4, conjugated to  $\theta^{-1}$ , having all its conjugates  $\theta_i$  excluding  $\theta$  and  $\theta^{-1}$  of modulus exactly 1 [9]. The minimal polynomial  $\Lambda(z)$  of a Salem number  $\theta$  is reciprocal: it satisfies the equation  $z^{\deg \Lambda(z)} \Lambda(\frac{1}{z}) = \Lambda(z)$ . To put it simply, this means that its coefficients form a palindromic sequence: they read the same backwards as forwards. Therefore  $\theta + \theta^{-1}$  is a real algebraic integer  $\theta > 2$  such that its conjugates  $\neq \theta + \theta^{-1}$  lie in the real interval  $[-2, 2]$ . The Mahler measure  $M(\theta) := \prod_{i=1}^{\deg \theta} \max\{1, |\theta_i|\}$  of  $\theta$  satisfies  $M(\theta) = \theta$  and the trace  $tr(\theta) := \sum_{i=1}^{\deg \theta} \theta_i \in \mathbb{Z}$  is not bounded and can take arbitrary negative values [6].

In 1945, Salem [8] first defined and studied Salem numbers. They were called after his name. The set of Salem numbers is traditionally denoted by  $T$ , the letter just after  $S$ , where  $S$  denotes the set of Pisot numbers [2] [9]. The smallest element known of  $T$  is Lehmer's number  $\beta_0 = 1.1762\dots$  of degree 10, as dominant root of Lehmer's polynomial:

$$P(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1. \quad (1)$$

The problem of Lehmer for Salem numbers  $\theta \in T$  can be formulated as follows: does there exist an absolute constant  $c > 0$  such that: if  $M(\theta) > 1$  then  $M(\theta) > 1 + c$ ? If such a nontrivial minoration occurs, is the infimum of  $T$  a Salem number [3]? The problem of Lehmer stated for Mahler measures of nonzero algebraic numbers, not being a root of unity, became the Conjecture of Lehmer, stating that such an absolute constant exists. The Surveys [9] [10] take stock of the problem of minoration of the Mahler measure in all its forms.

Kerada [5] defined, as a generalization of a Salem number, a  $j$ -Salem number,  $j \geq 2$ . In particular, a 2-Salem number or a pair of Salem numbers is a pair  $(\beta_1, \beta_2)$  of conjugate algebraic integers of modulus  $> 1$  whose remaining conjugates have modulus at most 1, with at least one having modulus exactly 1. The set of 2-Salem numbers is denoted by  $T_2$ . It is partitioned as  $T_2 = T'_2 \cup T''_2$  where  $T'_2$  is the set of 2-Salem numbers with  $\beta_1, \beta_2 \in \mathbb{R}$  and  $T''_2$  the set of 2-Salem numbers for which  $\beta_1$  and  $\beta_2$  are complex non-real (and so complex conjugates of one another,  $\beta_1 = \bar{\beta}_2$ ).

In this work, instead of the classical setting of the real numbers, the analogues of 2-Salem numbers over the ring of formal Laurent series over finite fields are investigated. Bateman and Duquette [1] initiated the study of Salem numbers in positive characteristic. In this context 2-Salem numbers in positive characteristic will be called 2-Salem series, 2-Salem elements or 2-Salem numbers. The objectives of the present note consist in extending some of the results of Bateman and Duquette to 2-Salem series over  $\mathbb{F}_q[X]$ ,  $q \neq 2^r$ , and to study the analogues of the above-mentioned properties of 2-Salem series. More precisely, let  $\mathbb{F}_q$  denote the finite field having  $q$  elements,  $q \geq 3$ , and let  $p$  be the characteristic of  $\mathbb{F}_q$ ;  $q$  is a power of  $p$ . Let  $X$  be an indeterminate over  $\mathbb{F}_q$  and denote  $k := \mathbb{F}_q(X)$ . Let  $\infty$  be the unique place of  $k$  which is a pole of  $X$ , and denote  $k_\infty := \mathbb{F}_q((\frac{1}{X}))$ . Let  $\mathbb{C}_\infty$  be a completion of an algebraic closure of  $k_\infty$ . Then  $\mathbb{C}_\infty$  is algebraically closed and complete, and we denote by  $v_\infty$  the valuation on  $\mathbb{C}_\infty$  normalized by  $v_\infty(X) = -1$ . We fix an embedding of an algebraic closure of  $k$  in  $\mathbb{C}_\infty$  so that all the finite extensions of  $k$  mentioned in this work will be contained in  $\mathbb{C}_\infty$ . An explicit description of  $v_\infty$  is done in section 2. For simplicity's sake the algebraic closure of  $k_\infty$  will be often denoted by  $\overline{\mathbb{F}_q((X^{-1}))}$ .

2-Salem series over  $\mathbb{F}_q[X]$  may belong to  $k_\infty$  or to finite extensions of  $k_\infty$ . By analogy with Kerada's notations we denote by  $T_2^*$  the set of 2-Salem series. It can be partitioned as  $T_2^* = T_2'^* \cup T_2''^*$  where  $T_2'^*$  denotes those 2-Salem series  $(\omega_1, \omega_2)$  over  $\mathbb{F}_q[X]$  which (both) belong to  $\mathbb{F}_q((X^{-1}))$ , and  $T_2''^*$  those 2-Salem series, not in  $\mathbb{F}_q((X^{-1}))$ , such that  $(\omega_1^n, \omega_2^n) \in T_2'^*$  for some integer  $n \geq 2$ .

**Theorem 1.1.** *Suppose  $q \neq 2^r$  for any integer  $r \geq 1$ . Denote by  $\omega_1$  and  $\omega_2$  the dominant roots of the irreducible polynomial*

$$\Lambda(Y) = Y^n + \lambda_{n-1}Y^{n-1} + \lambda_{n-2}Y^{n-2} + \dots + \lambda_1Y + \lambda_0 \in \mathbb{F}_q[X][Y] \quad (2)$$

where  $n \geq 3$ ,  $\lambda_0 \neq 0$ ,  $\deg \lambda_{n-1} < \deg \lambda_{n-2} = \sup_{i \neq n-2} \deg(\lambda_i)$ . Then

- (i) *if  $\deg \lambda_{n-2} > 2 \deg \lambda_{n-1}$ , then the pair  $(\omega_1, \omega_2) \in T_2'^*$  if and only if  $\deg \lambda_{n-2}$  is even, the opposite  $-\alpha_{2s}$  of the dominant coefficient of  $\lambda_{n-2} = \alpha_{2s}X^{2s} + \dots + \alpha_0$  is a square in  $\mathbb{F}_q$ , and  $\deg \lambda_{n-3} < \deg \lambda_{n-2}$ .*
- (ii) *if  $\deg \lambda_{n-2} < 2 \deg \lambda_{n-1}$ , then  $(\omega_1, \omega_2) \in T_2'^*$ .*

The paper is organized as follows. In section 2 the fields of formal power series and the valuation used in the sequel are recalled. In this context the main result of Bateman and Duquette, characterizing Salem elements, is restated. Section 3 is devoted to arithmetical and topological properties of 2-Salem series in  $\overline{\mathbb{F}_q((X^{-1}))}$ . In section 4 Weiss's method of the upper Newton polygon is explicited to characterize 2-Salem series in  $\overline{\mathbb{F}_q((X^{-1}))}$ . In section 5 attention is focused on those 2-Salem series which lie in the field  $\mathbb{F}_q((X^{-1}))$ , by establishing criteria discriminating whether they belong to  $\mathbb{F}_q((X^{-1}))$  or to  $\overline{\mathbb{F}_q((X^{-1}))} \setminus \mathbb{F}_q((X^{-1}))$ . The proof of Theorem 1.1 is given in section 6. In Theorem 1.1 the polynomial given by (2) is assumed irreducible. More generally, the question of irreducibility of a polynomial of the general form (2) is discussed in section 7.

## 2 Salem series in $\mathbb{F}_q((X^{-1}))$

For  $p$  a prime and  $q$  a power of  $p$ , let  $\mathbb{F}_q((X^{-1}))$  be the set of Laurent series over  $\mathbb{F}_q$  which is defined as follows

$$\mathbb{F}_q((X^{-1})) = \left\{ \omega = \sum_{i \geq n_0} \omega_i X^{-i} : n_0 \in \mathbb{Z} \text{ and } \omega_i \in \mathbb{F}_q \right\}.$$

We know that every algebraic element over  $\mathbb{F}_q[X]$  can be writing explicitly as a formal series because  $\mathbb{F}_q[X] \subseteq \mathbb{F}_q((X^{-1}))$ . However, as  $\mathbb{F}_q((X^{-1}))$  is not algebraically closed, such an element is not necessarily expressed as a power series. We refer to Kedlaya [4], for a full characterization of the algebraic closure of  $\mathbb{F}_q[X]$ . We denote by  $\overline{\mathbb{F}_q((X^{-1}))}$  an algebraic closure of  $\mathbb{F}_q((X^{-1}))$ . Indifferently we will speak of 2-Salem elements, 2-Salem numbers or 2-Salem series in the present context.

Let  $\omega$  be an element of  $\mathbb{F}_q((X^{-1}))$ , its polynomial part is denoted by  $[\omega] \in \mathbb{F}_q[X]$  and  $\{\omega\}$  its fractional part. We can remark that  $\omega = [\omega] + \{\omega\}$ . If  $\omega \neq 0$ , then the polynomial degree of  $\omega$  is  $\gamma(\omega) = \sup\{-i : \omega_i \neq 0\}$ , the degree of the highest-degree nonzero monomial in  $\omega$ , and  $\gamma(0) = -\infty$ . Note that if  $[\omega] \neq 0$  then  $\gamma(\omega)$  is the degree of the polynomial  $[\omega]$ . Thus, we define the absolute value

$$|\omega| = \begin{cases} q^{\gamma(\omega)} & \text{for } \omega \neq 0; \\ 0 & \text{for } \omega = 0. \end{cases}$$

Since  $|\cdot|$  is not archimedean,  $|\cdot|$  fulfills the strict triangle inequality

$$\begin{aligned} |\omega + \nu| &\leq \max(|\omega|, |\nu|) && \text{and} \\ |\omega + \nu| &= \max(|\omega|, |\nu|) && \text{if } |\omega| \neq |\nu|. \end{aligned}$$

**Definition 2.1.** *A Salem ( resp. Pisot) element  $\omega$  in  $\mathbb{F}_q((X^{-1}))$  is an algebraic integer over  $\mathbb{F}_q[X]$  such that  $|\omega| > 1$ , whose remaining conjugates in  $\overline{\mathbb{F}_q((X^{-1}))}$  have an absolute value no greater than 1, and at least one has absolute value exactly 1. (resp. whose remaining conjugates in  $\overline{\mathbb{F}_q((X^{-1}))}$  have an absolute value strictly less than 1). The set of Salem (resp. Pisot) numbers is denoted  $T^*$  (resp.  $S^*$ ).*

**Theorem 2.2.** ([1]) An element  $\omega$  in  $\mathbb{F}_q((X^{-1}))$  is a Salem (resp. Pisot) element if and only if its minimal polynomial can be written as  $\Lambda(Y) = Y^s + \lambda_{s-1}Y^{s-1} + \dots + \lambda_0$ ,  $\lambda_i \in \mathbb{F}_q[X]$  for  $i = 0, \dots, s-1$  with  $|\lambda_{s-1}| = \sup_{0 \leq i \leq s-2} |\lambda_i|$ . (resp.  $|\lambda_{s-1}| > \sup_{0 \leq i \leq s-2} |\lambda_i|$ ).

Similarly to the real case, in the sequel we will focus on 2-Salem series in  $k_\infty$ : such a 2-Salem number or a pair of Salem series as a pair  $(\omega_1, \omega_2)$  in  $\mathbb{F}_q((X^{-1})) \times \mathbb{F}_q((X^{-1}))$  has an absolute value greater than 1, such that  $\omega_1$  is an algebraic integer over  $\mathbb{F}_q[X]$ , with the property that all of its conjugates lie on or within the unit circle, and at least one conjugate lies on the unit circle. This implies that all 2-Salem elements are necessarily separable over  $\mathbb{F}_q(X)$ . Also, for any  $(\omega_1, \omega_2) \in T_2'^*$ , then by definition,  $(\omega_1, \omega_2) \notin (\mathbb{F}_q((X^{-1})) \times \mathbb{F}_q((X^{-1})))$  such that  $(\omega_1^n, \omega_2^n) \in T_2'^*$ , for some integer  $n \geq 2$ . So we can check easily that any element in  $T_2^*$  is either in  $T_2'^*$  or in  $T_2''^*$ .

Let us remark that it is easy to construct a 2-Salem number over  $\mathbb{F}_q$  with  $q = 2$  and then to show that 2-Salem numbers do exist without the assumption  $q \neq 2^r, r \geq 1$ , taken in Theorem 1.1. The exclusion case  $q \neq 2^r$  of Theorem 1.1 will arise in a general setting from Lemma 5.3 and its consequences.

### 3 Multiplicative properties of 2-Salem series

**Lemma 3.1.** If  $\omega$  and  $\nu$  are algebraic integers, then  $\omega + \nu$  and  $\omega\nu$  are algebraic integers.

*Proof.* Let  $\omega$  and  $\nu$  be two algebraic integers and

$$P_\omega(Y) = \sum_{i=0}^d A_i Y^i \in \mathbb{F}_q[X][Y]$$

and  $P_\nu \in \mathbb{F}_q[X][Y]$  their minimal polynomials. Then,  $A_d \in \mathbb{F}_q^*$ .

Let  $\nu_1 = \nu, \nu_2, \dots, \nu_s$  the roots of  $P_\nu$ . Therefore  $Q(Y) = \prod_{j=1}^s P_\omega(Y - \nu_j)$  is a polynomial with coefficients in  $\mathbb{F}_q[X]$ , a dominant coefficient is  $(A_d)^s \in \mathbb{F}_q^*$  and  $Q(\omega + \nu) = 0$ .

For the second one, we take the polynomial

$$R(Y) = \prod_{j=1}^s \nu_j^d P_\omega\left(\frac{Y}{\nu_j}\right)$$

is a polynomial with coefficients in  $\mathbb{F}_q[X]$ , whose dominant coefficient is  $(A_d)^s \in \mathbb{F}_q^*$  and  $R(\omega\nu) = 0$ .  $\square$

**Proposition 3.2.**

Let  $(\omega_1, \omega_2) \in T_2'^*$ , then  $(\omega_1^n, \omega_2^n) \in T_2'^*$ , for all  $n \in \mathbb{N}^*$ .

*Proof.* Let  $M \in \mathbb{F}_q[X][Y]$  the minimal polynomial of integer algebraic  $\omega$  of degree  $d$  and  $\omega_2, \dots, \omega_d$  the conjugates of  $\omega$ . Then there exists exactly one conjugate  $\omega_2$  of  $\omega_1$  that lie outside the unit disc and for  $3 \leq i \leq d$ , the conjugates  $\omega_i$  lie on or within the unit circle, and at least one conjugate  $\omega_j$  lies on the unit circle for  $3 \leq j \leq d$ . Let  $\omega_3, \dots, \omega_d$  denote the other roots of  $M$ . Since  $\omega$  is an algebraic integer, by Lemma 3.1,  $\forall n \in \mathbb{N}$   $\omega_1^n$  is also. Let  $\Lambda \in \mathbb{F}_q[X][Y]$  be the minimal polynomial of  $\omega_1^n$ .

Now, we consider the embedding  $\sigma_i$  of  $\mathbb{F}_q(X)(\omega_1)$  into  $\overline{\mathbb{F}_q((X^{-1}))}$ , which fixes  $\mathbb{F}_q(X)$  and maps  $\omega_1$  to  $\omega_i$ .

$$\Lambda(\omega_i^n) = \Lambda((\sigma_i(\omega_1))^n) = \Lambda(\sigma_i(\omega_1^n)) = \sigma_i(\Lambda(\omega_1^n)) = \sigma_i(0) = 0$$

So for all  $i \leq d$ ,  $\omega_i^n$  is a root of the equation  $\Lambda(Y) = 0$ . We have,

$$[\mathbb{F}_q(X)(\omega_1^n) : \mathbb{F}_q(X)] \leq [\mathbb{F}_q(X)(\omega_1) : \mathbb{F}_q(X)].$$

This shows that  $\deg(\Lambda) \leq \deg(M)$ . So  $\omega_1^n, \omega_2^n, \dots, \omega_d^n$  are all the roots of  $\Lambda$ .

If  $3 \leq i \leq d$ , then  $|\omega_i^n| = |\omega_i|^n \leq 1$  and there exists at least  $3 \leq j \leq n$  such that  $|\omega_j^n| = |\omega_j|^n = 1$ . Therefore  $(\omega_1^n, \omega_2^n) \in T_2'^*$ , for all  $n \in \mathbb{N}^*$ .  $\square$

Note that the converse is false in general. For instance, take  $q = 3, d = 4$  and  $n = 2$ . Then, the polynomial

$$Y^4 - 2X^2Y^2 + 2X^2$$

over  $\mathbb{F}_3$  is irreducible and its two roots of absolute value  $> 1$  defined by

$$(\omega_1, \omega_2) = ((\sqrt{2}(X - \frac{1}{X^3} + \dots), -(\sqrt{2}(X - \frac{1}{X^3} + \dots))),$$

not lie in  $\mathbb{F}_3((X^{-1}))$ . The other conjugates defined by

$$(\omega_3, \omega_4) = (1 - \frac{1}{X^2} + \dots, -(1 + \frac{1}{X^2} + \dots)).$$

We can see that  $(\omega_1^2, \omega_2^2)$  lie in  $\mathbb{F}_3((X^{-1}))$ .

The 2- Salem series have the following basic property, as is easily seen by considering the trace.

**Proposition 3.3.** *Let  $(\omega_1, \omega_2) \in T_2'^*$ , then  $\{\omega_1^n + \omega_2^n\}$  is bounded.*

*Proof.* Let  $N$  an integer and  $(\omega_1, \omega_2)$  be a 2-Salem series and  $\omega_3, \dots, \omega_d$  the other conjugates of  $\omega_1$  and  $\omega_2$ . From the preceding proposition result, for all  $n \geq N$  integer,  $\omega_1^n$  and  $\omega_2^n$  are the roots of the same degree  $d$  irreducible polynomial,  $\Lambda_n$  in  $\mathbb{F}_q[X]$ . Also,

$$\text{tr}(\Lambda_n) = \sum_{i=1}^d \omega_i^n \in \mathbb{F}_q[X].$$

So  $\{\text{tr}(\Lambda_n)\} = 0$ . The above can be rewritten as

$$\{\text{tr}(\Lambda_n) = \sum_{i=1}^d \omega_i^n\} = \{\omega_1^n + \omega_2^n + \sum_{i=3}^d \omega_i^n\}.$$

Since, for  $3 \leq i \leq d$ , by definition  $|\omega_i| \leq 1$ , and there exists at least  $3 \leq j \leq n$  such that  $|\omega_j^n| = |\omega_j|^n = 1$ . Further, passing to absolute value, it is easy to show that

$$\lim_{n \rightarrow +\infty} |\{\sum_{i=3}^d \omega_i^n\}| \leq \max_{i=3, \dots, d} |\{\omega_i^n\}| \leq C \in \mathbb{F}_q.$$

Therefore  $\{\omega_1^n + \omega_2^n\}$  is bounded.  $\square$

The following remark is a particular case of Proposition 3.3.

**Remark 3.4.**

Let  $(\omega_1, \omega_2) \in T_2'^*$  such that it has only one root  $\omega_3$  with absolute value equal to 1 and the other conjugates have an absolute value strictly less than 1. Then  $\lim_{n \rightarrow +\infty} \{\omega_1^n + \omega_2^n\} = 0$ .

*Proof.* By Proposition 4.1, we can see that  $\omega_3 \in \mathbb{F}_q((X^{-1}))$ . Thus

$$\lim_{n \rightarrow +\infty} \{\omega_3^n\} = 0. \tag{3}$$

On the other hand, by the proof of Proposition 3.3, we have

$$\omega_1^n + \omega_2^n = \text{Tr}(\omega_1^n) - \omega_3^n - \omega_4^n - \dots - \omega_d^n$$

which implies for  $n \geq N$ ,

$$\begin{aligned} |\{\omega_1^n + \omega_2^n\}| &= |\{\omega_3^n\} + \{\omega_4^n\} + \dots + \{\omega_d^n\}| \\ &\leq |\{\omega_3^n\} + \omega_4^n + \dots + \omega_d^n| \\ &\leq \max_{i=4, \dots, d} (|\{\omega_3^n\}|, |\omega_i^n|). \end{aligned}$$

Since  $|\omega_i| < 1$  for  $i = 4, \dots, d$  and by (3), the assertion of the Remark follows.  $\square$

**Proposition 3.5.** *Let  $(\omega_1, \omega_2) \in T_2'^*$  with a minimal polynomial  $\Lambda \in \mathbb{F}_q[X][Y]$  of degree 4 and  $\omega_1, \omega_2, \omega_3$  and  $\omega_4$  its roots such that  $\deg \omega_3 = 0$ . If  $\Lambda(0) \in \mathbb{F}_q^*$ , then  $\omega_1\omega_2\omega_3 \in T^*$ .*

*Proof.* Let  $(\omega_1, \omega_2) \in T_2'^*$  and

$$\Lambda(Y) = Y^4 + \lambda_3 Y^3 + \lambda_2 Y^2 + \lambda_1 Y + \lambda_0.$$

the minimal polynomial of  $\omega_1$  and  $\omega_2$ . Let  $\omega_3$  and  $\omega_4$  the other conjugates. Consider

$$Q(Y) = Y^4 \Lambda\left(\frac{1}{Y}\right).$$

Clearly  $Q$  is an irreducible monic over  $\mathbb{F}_q[X]$ , and has four roots

$$\frac{1}{\omega_1}, \frac{1}{\omega_2}, \frac{1}{\omega_3} = \omega_1\omega_2\omega_4 \quad \text{and} \quad \frac{1}{\omega_4} = \omega_1\omega_2\omega_3.$$

We have

$$\begin{aligned} \left|\frac{1}{\omega_3}\right| &= |\omega_1\omega_2\omega_4| = 1 \\ \left|\frac{1}{\omega_4}\right| &= |\omega_1\omega_2\omega_3| > 1 \end{aligned}$$

and  $\left|\frac{1}{\omega_i}\right| < 1$ , for  $i = 1, 2$ . Therefore  $\omega_1\omega_2\omega_3$  is a Salem series.  $\square$

## 4 A first characterization of 2-Salem series

We will now take up the following definition which we need. Recall the definition of upper Newton polygon of a polynomial

$$\Lambda(X, Y) = \lambda_n Y^n + \lambda_{n-1} Y^{n-1} + \dots + \lambda_1 Y + \lambda_0 \in \mathbb{F}_q[X, Y] \quad (4)$$

to each monomial  $\lambda_i Y^i \neq 0$ , we assign the point  $(i, \deg(\lambda_i)) \in \mathbb{Z}^2$ . For  $\lambda_i = 0$ , we ignore the corresponding point  $(i, -\infty)$ . If we consider the upper convex hull of the set of points

$$\{(0, \deg(\lambda_0)), \dots, (n, \deg(\lambda_n))\},$$

we obtain the so-called upper Newton polygon of  $\Lambda(X, Y)$  with respect to  $Y$ . The polygon is a sequence of line segments  $E_1, E_2, \dots, E_t$ , with monotonous decreasing slopes.

The following Proposition of Weiss in [11] is main tool for our purposes. The Newton polygon is an important tool in the study of 2-Pisot series. Recall the definition of upper Newton polygon of a polynomial

$$f(X, Y) = A_m Y^m + A_{m-1} Y^{m-1} + \dots + A_1 Y + A_0 \in \mathbb{F}_q[X, Y]. \quad (5)$$

To each monomial  $A_i Y^i \neq 0$ , we assign the point  $(i, \deg(A_i)) \in \mathbb{Z}^2$ . For  $A_i = 0$ , we ignore the corresponding point  $(i, -\infty)$ . If we consider the upper convex hull of the set of points

$$\{(0, \deg(A_0)), \dots, (m, \deg(A_m))\},$$

we obtain the so-called upper Newton polygon of  $f(X, Y)$  with respect to  $Y$ . The Newton polygon of  $f(X, Y)$  is a sequence of line segments with monotonous decreasing pairwise distinct slopes. The slope of a segment of the Newton polygon of  $f(X, Y)$  joins, for instance, the point  $(r, \deg(A_r))$  to  $(r + s, \deg(A_{r+s}))$  for some  $0 \leq r < r + s \leq m$ . The slope is

$$k = \frac{\deg(A_{r+s}) - \deg(A_r)}{s}.$$

Denote by  $K_f$  the set of the slopes.

**Proposition 4.1** (Weiss). *Let*

$$\Lambda(X, Y) = Y^n + \lambda_{n-1} Y^{n-1} + \dots + \lambda_1 Y + \lambda_0 \in \mathbb{F}_q[X, Y]$$

and  $K_\Lambda$  the set of the slopes of its Newton polygon. Then, for every  $k \in K_\Lambda$ ,

i)  $\Lambda(X, Y)$ , as a polynomial in  $Y$ , has  $s$  roots  $\alpha_1, \dots, \alpha_s$  with the same degree  $-k$  and

$$|\alpha_1| = \dots = |\alpha_s| = q^{-k}$$

.

ii) The polynomial

$$\Lambda_k(X, Y) = \prod_{i=1}^s (Y - \alpha_i) \in \mathbb{F}_q((X^{-1}))[Y]$$

divides  $\Lambda(X, Y)$ , with

$$\Lambda(X, Y) = \prod_{k \in K_\Lambda} \Lambda_k(X, Y).$$

**Corollary 4.2.** [7] *Let*

$$\Lambda(X, Y) = \lambda_n Y^n + \lambda_{n-1} Y^{n-1} + \dots + \lambda_1 Y + \lambda_0 \in \mathbb{F}_q[X][Y]. \quad (6)$$

and  $\omega$  a root of  $\Lambda$ . If  $|\lambda_n| = \sup_{0 \leq k \leq n} |\lambda_k|$ , then  $|\omega| \leq 1$ .

**Corollary 4.3.** *Let*

$$\Lambda(X, Y) = Y^n + \lambda_{n-1} Y^{n-1} + \dots + \lambda_1 Y + \lambda_0 \in \mathbb{F}_q[X][Y] \quad (7)$$

with  $\lambda_0 \neq 0$  and  $\sup_{0 \leq k < n-2} \deg \lambda_k = \deg \lambda_{n-2} < 2 \deg \lambda_{n-1}$ . Then,  $\Lambda$  has only two roots  $\omega_1, \omega_2 \in \mathbb{F}_q((X^{-1}))$  satisfying  $|\omega_1| > 1$  and  $|\omega_2| > 1$  and at least one conjugate lies on the unit circle.



*Proof.* First notice that the stated condition implies that  $\deg \lambda_{n-1} > 0$ . Moreover the Newton polygon of  $\Lambda$  contains the line with a slope  $k_1$  joining  $(n-1, \deg \lambda_{n-1})$  and  $(n, 0)$ , the line with a slope  $k_2$  joining  $(n-2, \deg \lambda_{n-2})$  and  $(n-1, \deg \lambda_{n-1})$  and the line with a slope  $k_3$  joining  $(n-2, \deg \lambda_{n-2})$  and  $(n-k, \deg \lambda_{n-k})$ . By Proposition 4.1 (i),  $\Lambda$  has exactly two roots  $\omega_1, \omega_2$

$$\begin{cases} |\omega_1| = & = q^{\deg \lambda_{n-1}} = q^{-k_1} > 1 \\ |\omega_2| = & \omega_2| = |q^{\deg \lambda_{n-2} - \deg \lambda_{n-1}} = q^{-k_2} > 1. \end{cases}$$

and  $j = n - 2 - k$  roots  $\omega_j$  such that

$$|\omega_j| = |q^{\frac{-\deg \lambda_{n-2} + \deg \lambda_k}{n-2-k}}| = q^{-k_3} = 1.$$

By Proposition 4.1 (ii), there are two factors  $\Lambda_{k_1}(X, Y) = (Y - \omega_1) \in \mathbb{F}_q((X^{-1}))[Y]$  and  $\Lambda_{k_2}(X, Y) = (Y - \omega_2) \in \mathbb{F}_q((X^{-1}))[Y]$  of  $\Lambda$ . Hence  $\omega_1$  and  $\omega_2 \in \mathbb{F}_q((X^{-1}))$ .  $\square$

**Theorem 4.4.** *Let  $\Lambda$  the polynomial of degree  $n \geq 3$  defined by*

$$\Lambda(Y) = Y^n + \lambda_{n-1}Y^{n-1} + \lambda_{n-2}Y^{n-2} + \dots + \lambda_1Y + \lambda_0 \in \mathbb{F}_q[X][Y]$$

*with  $\lambda_0 \neq 0$ . Then,  $\Lambda$  has exactly 2 roots in  $\overline{\mathbb{F}_q((X^{-1}))}$  of an absolute strictly greater than 1 and all remaining roots in  $\overline{\mathbb{F}_q((X^{-1}))}$  have an absolute value less or equal to 1 and at least one conjugate lies on the unit circle if and only if  $|\lambda_{n-1}| < |\lambda_{n-2}| = \sup_{i < n-2} |\lambda_i|$ .*

*Proof.* Let  $w = \omega_1, \omega_2, \dots, \omega_n$  be the roots of  $\Lambda$ . Suppose that  $|\omega_1| \geq |\omega_2| > 1 \geq |\omega_3| \geq \dots \geq |\omega_n|$  and there exists at least one  $3 \leq j \leq n$  such that  $|\omega_j| = 1$ . For  $k \in \{1, \dots, n\}; k \neq 2$ , we have

$$|\lambda_{n-k}| = \left| \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \omega_{i_1} \omega_{i_2} \dots \omega_{i_k} \right| \leq |\omega_1 \omega_2 \dots \omega_k| \leq |\omega_1 \omega_2| = |\lambda_{n-2}|$$

and

$$|\lambda_{n-j}| = \left| \sum_{1 \leq i_1 < i_2 < \dots < i_j \leq n} \omega_{i_1} \omega_{i_2} \dots \omega_{i_j} \right| = |\omega_1 \omega_2 \dots \omega_j| = |\omega_1 \omega_2| = |\lambda_{n-2}|$$

Then

$$|\lambda_{n-2}| = \sup_{i \neq n-2} |\lambda_i|.$$

For the converse, it follows easily from Proposition 4.1.

**Example 4.5.**

Let

$$\Lambda(Y) = Y^3 + (X+1)Y^2 + (X^4 + X^3)Y + X^4 + X^3 + X^2 + X + 1 \in \mathbb{F}_2[X][Y].$$

By Theorem 4.4,  $\Lambda(Y)$  has two roots  $\omega_1$  and  $\omega_2$  of an absolute value strictly greater than 1 and one root  $\omega_3$  of an absolute value exactly equal to 1. Using the fact that

- $[\omega_1 + \omega_2 + \omega_3] = X + 1$
- $[\omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3] = X^4 + X^3$
- $[\omega_1 \omega_2 \omega_3] = X^4 + X^3 + X^2 + X + 1.$

Then  $\omega_1$ ,  $\omega_2$  and  $\omega_3$  are defined by:

$$\begin{cases} \omega_1 = X^2 + 1 + \frac{1}{Z_1} \text{ such that } |Z_1| > 1; \\ \omega_2 = X^2 + X + \frac{1}{Z_2} \text{ such that } |Z_2| > 1; \end{cases}$$

and  $\omega_3 = 1 + \frac{1}{Z_3}$  such that  $|Z_3| > 1$ . For  $j = 1, 2$   $\Lambda(\omega_j) = 0$  implies  $Z_1$  (rep.  $Z_2$ ) is a root of the polynomial  $H_1$  (resp.  $H_2$ ) defined by

$$H_1 = Z^3 + (X^3 + 1)Z^2 + (X^2 + X)Z + 1. \quad (8)$$

and

$$H_2 = (X^2 + X + 1)Z^3 + (X^3 + X^2)Z^2 + (X^2 + 1)Z + 1. \quad (9)$$

Applying Proposition 4.1 to the equations (9) and (8), we obtain  $Z_1, Z_2 \in \mathbb{F}_2((X^{-1}))$ .

Therefore  $\omega_1, \omega_2 \in \mathbb{F}_2((X^{-1}))$ . Since  $\Lambda$  is monic and irreducible over  $\mathbb{F}_2[X]$ , we deduce that  $(\omega_1, \omega_2)$  is 2-Salem series and  $\Lambda$  is the minimal polynomial of  $\omega_1$ . □

## 5 Criteria of existence of roots and conjugates in $\mathbb{F}_q((X^{-1}))$

Before giving the proof of our results, we will present some lemmas that we will need:

**Lemma 5.1.** *Let  $n \geq 3$  and  $\Lambda$  defined by*

$$\Lambda(Y) = Y^n + \lambda_{n-1}Y^{n-1} + \lambda_{n-2}Y^{n-2} + \dots + \lambda_1Y + \lambda_0 \in \mathbb{F}_q[X][Y].$$

*Suppose  $\lambda_0 \neq 0$  and  $\sup_{i \neq n-2} \deg \lambda_i = \deg \lambda_{n-2} \geq 2 \deg \lambda_{n-1}$ . If  $\deg(\lambda_{n-2})$  is odd, then  $\Lambda$  has no roots in  $\mathbb{F}_q((X^{-1}))$  with absolute value  $> 1$ .*

*Proof.* By Theorem 4.4,  $\Lambda$  has two roots  $\omega_1$  and  $\omega_2$  such that  $|\omega_1| > 1$  and  $|\omega_2| > 1$ . The remaining roots  $\omega_3, \dots, \omega_n$  have an absolute value less or equal to 1 and at least one conjugate lies on the unit circle for  $3 \leq j \leq n$ . As  $\deg \lambda_{n-2} \geq 2 \deg \lambda_{n-1}$ , then the Newton polygon of  $\Lambda$  contains the line connecting the points  $(n-2, \deg \lambda_{n-2})$  and  $(n, 0)$ . The slope of this line is  $k = -\frac{\deg \lambda_{n-2}}{2}$ . By Proposition 4.1 (i),  $\Lambda$  has  $n - (n-2) = 2$  roots  $\omega_1$  and  $\omega_2$ . Since they have the same absolute value  $q^{-k} > 1$ . Then

$$\deg \omega_1 = \deg \omega_2 = -k = -\frac{\deg \lambda_{n-2}}{2} \notin \mathbb{Z}. \quad (10)$$

Therefore  $\omega_1, \omega_2 \notin \mathbb{F}_q((X^{-1}))$ . □

**Lemma 5.2.** *Let  $\Lambda$  defined by*

$$\Lambda(Y) = Y^n + \lambda_{n-1}Y^{n-1} + \lambda_{n-2}Y^{n-2} + \dots + \lambda_1Y + \lambda_0 \in \mathbb{F}_q[X][Y].$$

*Suppose  $\lambda_0 \neq 0$  and  $\sup_{i \neq n-2} \deg \lambda_i < \deg \lambda_{n-3} = \deg \lambda_{n-2} \geq 2 \deg \lambda_{n-1}$ . If  $\deg(\lambda_{n-2})$  is odd, then  $\Lambda$  is irreducible over  $\mathbb{F}_q[X]$ .*

*Proof.* By Considering the Newton polygon of  $\Lambda$ , then it has exactly two roots  $\omega_1$  and  $\omega_2$  such that  $|\omega_1| > 1$  and  $|\omega_2| > 1$ , one root  $\omega_3$  such that  $|\omega_3| = 1$  and remaining roots  $\omega_4, \dots, \omega_n$  have an absolute value strictly less than 1. Suppose that

$$\Lambda(Y) = \Lambda_1(Y) \cdot \Lambda_2(Y) \quad (11)$$

$$= (Y^s + A_{s-1}Y^{s-1} + \dots + A_1Y + A_0)(Y^m + B_{m-1}Y^{m-1} + \dots + B_1Y + B_0). \quad (12)$$

such that  $\Lambda_1, \Lambda_2 \in \mathbb{F}_q[X][Y]$  and  $s > 0, m > 0$ .

If  $\Lambda_1(\omega_i) = 0$  for  $i = 1, 2, 3$ ; then all roots of  $\Lambda_2$  have an absolute value strictly less 1, which is a contradiction, because  $|B_0| > 1$ .

If  $\Lambda_1(\omega_i) = 0$  for  $i = 1, 2$ ; then the roots of  $\Lambda_2$  contains one root of absolute value equal to 1 and the other roots have an absolute value strictly less 1, which is a contradiction,  $|B_0| > 1$ .

If  $\Lambda_2(\omega_3) = 0$  and all the other conjugates are the root of  $\Lambda_1$ . Then we can write  $\Lambda$  of the form (11) by

$$\Lambda(Y) = (Y^{n-1} + A_{n-2}Y^{n-2} + \dots + A_1Y + A_0)(Y + B_0) \in \mathbb{F}_q[X][Y].$$

such that  $\deg B_0 = \deg(\omega_3) = 0$ , so  $B_0 = b_0 \in \mathbb{F}_q \setminus \{0\}$ . Thus

$$\deg \lambda_{n-2} = \deg A_{n-3} > \sup_{j \neq n-3} \deg A_j = \sup_{j \neq n-2} \deg \lambda_j.$$

contradicting the hypothesis of the lemma. Then we may conclude that  $\Lambda_1(\omega_1) = 0$  and  $\Lambda_2(\omega_2) = 0$ . The remaining roots of  $\Lambda_1$  and  $\Lambda_2$  have an absolute value  $\leq 1$ . (\*\*\*)

Since  $-A_{s-1}$  (resp.,  $-B_{m-1}$ ) is the sum of the roots of  $\Lambda_1$  (resp.,  $\Lambda_2$ ) and by the symmetric function of the roots, it follows that

$$\deg A_{s-1} = \deg \omega_1 = \sup_{i \neq s-1} \deg A_i \quad \text{and} \quad \deg B_{m-1} = \deg \omega_2 > \sup_{j \neq m-1} \deg B_j.$$

By Lemma 5.1, it follows that

$$\deg A_{s-1} = \deg B_{m-1}.$$

On the other hand  $|A_{s-2}| \leq |\omega_1|$  and  $|B_{m-2}| < |\omega_2|$ . Then

$$\deg \lambda_{n-2} = \deg(A_{s-2} + A_{s-1}B_{m-1} + B_{m-2}) = \deg A_{s-1} + \deg B_{m-1} = 2 \deg A_{s-1},$$

a contradiction. □

**Lemma 5.3.** *Let  $q \neq 2^r$  and  $\Lambda$  the polynomial defined by*

$$\Lambda(Y) = Y^n + \lambda_{n-1}Y^{n-1} + \lambda_{n-2}Y^{n-2} + \dots + \lambda_1Y + \lambda_0 \in \mathbb{F}_q[X][Y]$$

where  $\lambda_0 \neq 0$ ,  $\deg \lambda_{n-2} \geq \sup_{i \neq n-2} \deg(\lambda_i)$  and  $\deg \lambda_{n-2} > 2 \deg \lambda_{n-1}$ . Let  $\omega_1$  be a root of  $\Lambda$  such that  $|\omega_1| > 1$ .

If  $\deg \lambda_{n-3} = \deg \lambda_{n-2}$ , then  $\omega_1 \in \overline{\mathbb{F}_q((X^{-1}))} \setminus \mathbb{F}_q((X^{-1}))$

*Proof.* According to Lemma 5.1, we conclude that  $\deg \lambda_{n-2}$  is even. Set  $\deg \lambda_{n-2} = 2s$ , then  $\deg \omega_1 = \deg \omega_2 = s$ . Consider

$$\omega_1 = \sum_{i=0}^s a_i X^i + \frac{1}{Z_1}$$

such that  $a_s \neq 0$  and  $|Z_1| > 1$ . Let  $\lambda_n = 1$ ,

$$\lambda_i = \sum_{k_i=0}^{m_i} \alpha_{(k_i, i)} X^{k_i}$$

with  $m_i \leq s$  for  $i = 0, \dots, n-1$  and

$$\lambda_{n-2} = \sum_{j=0}^{2s} \alpha_{(j, n-2)} X^j$$

such that  $\alpha_{(2s, n-2)} \neq 0$ .  $\Lambda(\omega_1) = 0$  implies

$$([\omega_1] + \frac{1}{Z_1})^n + \lambda_{n-1}([\omega_1] + \frac{1}{Z_1})^{n-1} + \lambda_{n-2}([\omega_1] + \frac{1}{Z_1})^{n-2} + \dots + \lambda_1([\omega_1] + \frac{1}{Z_1}) + \lambda_0 = 0.$$

Multiplying by  $Z_1^n$ , we obtain

$$\begin{aligned} & Z_1^n \left( \sum_{k=0}^n \lambda_k [\omega_1]^k \right) + Z_1^{n-1} \left( \sum_{k=1}^n k \lambda_k [\omega_1]^k \right) + Z_1^{n-2} \left( \sum_{k=2}^n \frac{k(k-1)}{2} \lambda_k [\omega_1]^k \right) \\ & + \dots + Z_1^{n-k} \left( \frac{(n-k) \dots (n-(k+1))}{k!} \right) \lambda_{n-k} [\omega_1]^{n-k} + \dots + 1 = 0. \end{aligned}$$

Whence  $Z_1$  is the root of polynomials  $H$  defined by

$$H(Z) = A_n Z^n + A_{n-1} Z^{n-1} + \dots + 1 \in \mathbb{F}_q[X, Y]$$

with

$$A_i = \sum_{k=0}^i \binom{n-k}{i-k} \lambda_{n-k} [\omega_1]^{i-k}, \quad 0 \leq i \leq n. \quad (13)$$

Moreover

$$-\lambda_{n-1} = [\omega_1] + [\omega_2] \quad (14)$$

and

$$\lambda_{n-2} = \omega_1 \omega_2 + \omega_1 \omega_3 + \dots + \omega_{n-1} \omega_n \quad (15)$$

$$= ([\omega_1][\omega_2]) + Q \quad (16)$$

with  $Q \in \mathbb{F}_q[X, Y]$  and  $\deg Q \leq s-1$ . Notice that  $\deg \lambda_{n-2} > 2 \deg \lambda_{n-1}$  implies

$$\deg \lambda_{n-1} = \deg([\omega_1] + [\omega_2]) < s. \quad (17)$$

It follows from (14) and (15) that

$$\begin{cases} \deg(\lambda_{n-k} [\omega_1]^{i-k}) = is & \forall k = 0, 2 \\ \deg(\lambda_{n-k} [\omega_1]^{i-k}) < is & \forall k \neq 0, 2. \end{cases}$$

Then

$$\deg A_i \leq is, \quad 0 \leq i \leq n.$$

In view of (13), we can write

$$\begin{aligned} A_n &= [\omega_1]^n + \lambda_{n-1} [\omega_1]^{n-1} + \dots + \lambda_0 \\ &= [\omega_1]^{n-2} Q + \lambda_{n-3} [\omega_1]^{n-3} + \dots + \lambda_0. \end{aligned}$$

Thus

$$\deg A_n = (n-1)s.$$

Again, by (13), it is easy to show that

$$\deg A_i \leq (n-1)s, \quad \text{quand } 0 \leq i \leq n-1.$$

As a result, by applying of Corollary 4.2, we obtain  $|Z_1| \leq 1$ , a contradiction.  $\square$

## 6 Proof of Theorem 1.1

The next result, interesting in its own right, will play a dominant role in the characterization of 2-Salem numbers.

**Theorem 6.1.** *Let  $n \geq 3$  and  $\Lambda$  the polynomial defined by*

$$\Lambda(Y) = Y^n + \lambda_{n-1}Y^{n-1} + \lambda_{n-2}Y^{n-2} + \dots + \lambda_1Y + \lambda_0 \in \mathbb{F}_q[X][Y] \quad (18)$$

where  $\lambda_0 \neq 0$  and  $\deg \lambda_{n-1} < \deg \lambda_{n-2} = \sup_{i \neq n-2} \deg(\lambda_i)$ . Suppose  $q \neq 2^r$ ,  $\deg \lambda_{n-2} > 2 \deg \lambda_{n-1}$  and let  $\omega_1$  be a root of  $\Lambda$  such that  $|\omega_1| > 1$ . Then  $\omega_1 \in \mathbb{F}_q((X^{-1}))$  if and only if  $[\omega_1] \in \mathbb{F}_q[X]$ ,  $\deg \lambda_{n-2}$  is even and  $\deg \lambda_{n-3} < \deg \lambda_{n-2}$ .

*Proof.* Using Lemma 5.3, it follows that the condition is necessary. For sufficiency, we keep the same steps of the proof of Lemma 5.3 until the inequality (15). Since  $\deg \lambda_{n-3} < 2s - 1$ , then  $\deg A_n$  is at most  $(n-1)s - 1$ . From (17) and the condition  $q \neq 2^r$ , it follows that  $a_s \neq b_s$  and  $[\omega_1] \neq [\omega_2]$ . Hence

$$[\omega_1] - [\omega_2] = 2a_sX^s + (a_{s-1} - b_{s-1})X^{s-1} + \dots + (a_0 - b_0).$$

Thus  $\deg([\omega_1] - [\omega_2]) = s$ . Just as above, we see that

$$\begin{aligned} A_{n-1} &= [\omega_1]^{n-2}([\omega_1] - [\omega_2]) + (n-2)(Q[\omega_1]^{n-3} + \lambda_{n-3}[\omega_1]^{n-4}) \\ &\quad - \lambda_{n-3}[\omega_1]^{n-4} + (n-4)\lambda_{n-4}[\omega_1]^{n-5} + \dots + \lambda_1. \end{aligned}$$

and

$$\begin{aligned} A_{n-2} &= (n-1)[\omega_1]^{n-3}([\omega_1] - [\omega_2]) + [\omega_1]^{n-3}[\omega_2] + \\ &\quad + \frac{(n-2)(n-3)}{2}[\omega_1]^{n-4}Q + \frac{(n-3)(n-4)}{2}\lambda_{n-3}[\omega_1]^{n-5} + \dots + \lambda_1. \end{aligned}$$

Therefore

$$\deg A_{n-1} = (n-2)s + \deg([\omega_1] - [\omega_2]) = (n-1)s$$

and

$$\deg A_{n-2} = (n-2)s.$$

Notice that  $A_n \neq 0$ , if not, by Corollary 4.2, we get  $|Z_1| \leq 1$ , a contradiction. We conclude that

$$\deg A_{n-1} > \sup_{i \neq n-1} \deg A_i.$$

Finally, by (i) and (ii) of Proposition 4.1, the only root of  $H$  with an absolute value  $> 1$  is  $Z_1$  and there is a factor  $(Z - Z_1) \in \mathbb{F}_q((X^{-1}))[Z]$  of  $H$ . Then  $Z_1 \in \mathbb{F}_q((X^{-1}))$  and  $\omega_1 = [\omega_1] + \frac{1}{Z_1} \in \mathbb{F}_q((X^{-1}))$ , completes the proof.  $\square$

**Remark 6.2.**

(i) We mention that Theorem 6.1 is not always true in characteristic 3 in the case  $\deg \lambda_{n-2} = 2 \deg \lambda_{n-1}$ . (see Example 6.3).

(ii) We note also that this theorem is not always true for any field of characteristic  $p = 2$ . (see Example 4.5).

**Example 6.3.**

Let

$$\Lambda(Y) = Y^3 + (X+1)Y^2 + X^2Y - X^2 + 2 \in \mathbb{F}_3[X][Y]. \quad (19)$$

By Theorem 4.4,  $\Lambda(Y)$  has two roots  $\omega_1$  and  $\omega_2$  of an absolute value strictly greater than 1 and one root  $\omega_3$  of an absolute value equal to 1. Set  $\omega_1 = X + \frac{1}{Z_1} \in \mathbb{F}_3((X^{-1}))$  such that  $|Z_1| > 1$ .  $Z_1$  is the root of the polynomial defined by

$$2Z^3 + 2XZ^2 + (X+1)Z + 1 = 0. \quad (20)$$

By Proposition 4.1, we deduce that  $Z_1 \in \mathbb{F}_4((X^{-1}))$  and  $\omega_1 \in \mathbb{F}_3((X^{-1}))$ .

Set now  $\omega_2 = X + 1 + \frac{1}{Z_2} \in \mathbb{F}_3((X^{-1}))$  such that  $|Z_2| > 1$ . We obtain  $Z_2$  is the root of the polynomial defined by

$$Z^3 + (X^2 + X + 1)Z^2 + (2X^2 + X + 2)Z + 1 = 0. \quad (21)$$

Again by Proposition 4.1, we deduce that  $Z_2 \in \mathbb{F}_3((X^{-1}))$  and  $\omega_2 \in \mathbb{F}_3((X^{-1}))$ .

As  $\Lambda$  is monic and irreducible over  $\mathbb{F}_3[X]$ , it follows that  $(\omega_1, \omega_2)$  is 2-Salem series and  $\Lambda$  is the minimal polynomial of  $\omega_1$ .

*We can now give the proof of our main result Theorem 1.1 which characterizes 2-Salem series in the case  $q \neq 2^r$ :*

(i) Assume that  $\omega_1 \in \mathbb{F}_q((X^{-1}))$ . By Theorem 6.1, we get  $\deg \lambda_{n-2}$  is even and  $\deg \lambda_{n-3} < \deg \lambda_{n-2}$ . Set

$$\lambda_{n-2} = \alpha_{2s}X^{2s} + \alpha_{2s-1}X^{2s-1} + \dots + \alpha_0 \quad (22)$$

$$= (a_sX^s + a_{s-1}X^{s-1} + \dots + a_0)(b_sX^s + b_{s-1}X^{s-1} + \dots + b_0) + Q \quad (23)$$

As  $\deg \lambda_{n-1} < s$ , we obtain  $a_s = -b_s$ , which implies

$$-\alpha_{2s} = -a_s b_s = a_s^2.$$

For the converse, suppose  $-\alpha_{2s} = a^2$ . Putting  $a_s = a$ .

By induction, suppose we have  $a_s, a_{s-1}, \dots, a_{i+1}$  ( $0 \leq i \leq s-1$ ). From (22), we deduce that

$$\alpha_{s+i} = a_s b_i + a_{s-1} b_{i+1} + \dots + a_i b_s \quad (24)$$

$$= a(b_i - a_i) + d \quad (25)$$

where

$$d = a_{s-1} b_{i+1} + \dots + b_{s-1} a_{i+1}.$$

On the other hand, there is at least one conjugate  $\omega_j$  lies on the unit circle for  $3 \leq j \leq n$ . Let

$$\omega_j = c_0 + c_{-1}X^{-1} + \dots$$

set

$$\lambda_{n-1} = \beta_s X^s + \beta_{s-1} X^{s-1} + \dots + \beta_0$$

$$= -([\omega_1] + [\omega_2] + c_0).$$

Thus

$$-\beta_i = a_i + b_i \quad 1 \leq i \leq s. \quad (26)$$

and

$$-\beta_0 = a_0 + b_0 + c_0.$$

Combining (24) and (26) we get, for  $1 \leq i \leq s-1$ , that

$$a_i = -2^{-1}(\beta_i + a^{-1}(\alpha_{s+i} - d))$$

and

$$a_0 = -\beta_0 - b_0 - c_0.$$

Therefore,  $[\omega_1] \in \mathbb{F}_q[X]$  and from Theorem 6.1, we obtain  $\omega_1 \in \mathbb{F}_q((X^{-1}))$ . By the same way, we can show that  $\omega_2 \in \mathbb{F}_q((X^{-1}))$ . As  $\Lambda$  is monic and irreducible over  $\mathbb{F}_q[X]$ , then  $\omega_1$  is an algebraic integer. Therefore  $(\omega_1, \omega_2)$  is a 2-Salem pair.

(ii) This assertion follows immediately by Corollary 4.3.

**Remark 6.4.**

Note that Theorem 1.1 (i) is not always true in the case  $\deg \lambda_{n-2} = 2 \deg \lambda_{n-1}$ . To show this, we construct two counterexamples.

**Example 6.5.**

Let  $\Lambda$  the polynomial over  $\mathbb{F}_3[X]$  which is defined in (19). Then, in view of the above,  $\Lambda$  satisfies the conditions  $\deg \lambda_{n-2} = 2 \deg \lambda_{n-1}$  and  $-1$  is a non square in  $\mathbb{F}_3$ . In contrast,  $\Lambda$  has two dominant roots  $\omega_1, \omega_2 \in \mathbb{F}_3((X^{-1}))$ .

**Example 6.6.**

The polynomial

$$\Lambda_2 = Y^4 - XY^3 + X^2Y^2 + XY + X^2 + 1 \in \mathbb{F}_5[X][Y]$$

satisfies the conditions  $\deg \lambda_{n-2} = 2 \deg \lambda_{n-1}$  and  $-1$  is a square in  $\mathbb{F}_5$ . By Proposition 4.1 (i),  $\Lambda_2$  has exactly two dominate roots  $\omega_1$  and  $\omega_2$  with

$$\deg \omega_1 = \deg \omega_2 = 1.$$

The other conjugate roots  $\omega_3$  and  $\omega_4$  have the same degree equal to 0. Suppose  $[\omega_1] \in \mathbb{F}_5[X]$ , using the fact that

$$[\omega_1] + [\omega_2] + [\omega_3] + [\omega_4] = X,$$

this yields that  $[\omega_2] \in \mathbb{F}_5[X]$ . Let

$$[\omega_1] = a_1X + a_0, \quad [\omega_2] = b_1X + b_0$$

and

$$[\omega_3] = c_0, \quad [\omega_4] = d_0$$

where  $a_1, b_1, c_0$  and  $d_0$  are four integers in  $\mathbb{F}_5 \setminus \{0\}$ . It follows that

$$a_1 + b_1 = a_1b_1 = 1.$$

These equations have no solutions in  $\mathbb{F}_5$ .

## 7 A criterium of irreducibility

**Theorem 7.1.** *Let  $\Lambda$  the polynomial of the form (18) where  $n \geq 4$  and  $q \neq 2^r$  such that*

$$\sup_{i \in \{1, 2, \dots, n-4\} \cup \{n-1\}} \deg \lambda_i < \deg \lambda_{n-3} = \deg \lambda_{n-2} < 2 \deg \lambda_{n-1}.$$

*If, for  $1 \leq i \leq n-4$ ,  $\frac{\deg \lambda_{i+1} + \deg \lambda_{i-1}}{2} < \deg \lambda_i$  and  $\deg \lambda_{n-2} - \deg \lambda_{n-1} < \deg \lambda_{n-4} < \deg \lambda_{n-1}$ , then  $(\omega_1, \omega_2)$  is a 2-Salem series and  $\Lambda$  is its minimal polynomial.*

*Proof.* By Corollary 4.3,  $\Lambda(Y)$  has two roots  $\omega_1$  and  $\omega_2$  in  $\mathbb{F}_q((X^{-1}))$  of degree  $s$  and  $m$  respectively such that

$$1 < |\omega_2| = q^{\deg \lambda_{n-2} - \deg \lambda_{n-1}} = q^m < |\omega_1| = q^{\deg \lambda_{n-1}} = q^s$$

and there is exactly one conjugate  $\omega_3$  lies on the unit circle. The other conjugates  $\omega_4, \dots, \omega_n \in \overline{\mathbb{F}_q((X^{-1}))}$  have an absolute value strictly less than 1. Since for  $1 \leq i \leq n-4$ , we have

$$\frac{\deg \lambda_{i+1} + \deg \lambda_{i-1}}{2} < \deg \lambda_i, \text{ then}$$

$$|\omega_i| = q^{-k_j} < 1$$

where

$$-k_j = \deg \omega_i = \deg \lambda_{n-j} - \deg \lambda_{n-j+1}$$

for  $4 \leq j \leq n$ .

We keep the same steps of the proof of Lemma 5.2 until (\*\*). We may conclude that  $\Lambda_1(\omega_1) = 0$  and  $\Lambda_2(\omega_2) = 0$ . Suppose first that  $\Lambda_2(\omega_3) = 0$ , so we obtain  $\omega_1 \in S^*$  and  $\omega_2 \in T^*$ . Applying Theorem 2.2, we get

$$\deg A_{s-1} = \deg \omega_1 > \sup_{i < s-2} \deg A_i \text{ and } \deg B_{m-1} = \deg B_{m-2} = \deg \omega_2 > \sup_{j \neq m-1} \deg B_j. \quad (27)$$

From (11), we get, for  $A_s = B_m = 1$ , that

$$\lambda_{n-4} = \sum_{i+j=4, 0 \leq i \leq 4, 0 \leq j \leq 4} A_{s-i} B_{m-j}. \quad (28)$$

We can see that  $\Lambda_1(\omega_4) = 0$ . Indeed, if not then, by the symmetric function of the roots of  $\Lambda_2$  we obtain

$$\deg B_{m-3} = \deg(\omega_2 \omega_3 \omega_4) = \deg \lambda_{n-4} - \deg \lambda_{n-1} < 0,$$

a contradiction. In the other hand,  $\deg \lambda_{n-2} - \deg \lambda_{n-1} < \deg \lambda_{n-4} < \deg \lambda_{n-1}$  implies

$$m < \sup \deg((A_{s-2} B_{m-2} + A_{s-1} B_{m-3})) < s. \quad (29)$$

Therefore

$$\deg(A_{s-2} B_{m-2}) = \deg \lambda_{n-4} < \deg(A_{s-1} B_{m-3}) = s + \deg B_{m-3}. \quad (30)$$

It then follows from (28) and (30) that

$$\deg \lambda_{n-4} = \deg A_{s-1} + \deg B_{m-3} \geq s = \deg \lambda_{n-1},$$

a contradiction.

If now  $\Lambda_1(\omega_3) = 0$ , then  $\omega_1 \in T^*$  and  $\omega_2 \in S^*$ . The arguments of the proof are the same. Therefore  $\Lambda(Y)$  is irreducible over  $\mathbb{F}_q[X]$ . Finally, as  $\Lambda(Y)$  is monic, then  $(\omega_1, \omega_2)$  is a 2-Salem series and  $\Lambda$  is its minimal polynomial.  $\square$



**Example 7.2.** *Pair of 2-Salem series of degree 5 in  $\mathbb{F}_3((X^{-1}))$ .*

Let

$$\Lambda(Y) = Y^5 + X^4Y^4 + X^5Y^3 + X^5Y^2 + X^3Y + 1 \in \mathbb{F}_3[X][Y].$$

We deduce from Theorem 7.1, that  $\Lambda$  is irreducible over  $\mathbb{F}_3[X]$  and has 5 roots defined by

$$\left\{ \begin{array}{ll} \omega_1 = X^5 + 2X + \frac{1}{X^2} + \dots = X^5 + 2X + \frac{1}{Z_1} & \text{such that } |Z_1| > 1 \\ \omega_2 = X + 1 + \frac{1}{Z_2} & \text{such that } |Z_2| > 1 \\ \omega_3 = 2 + \frac{1}{Z_3} & \text{such that } |Z_3| > 1 \\ \omega_4 = \frac{1}{X^2} + \dots & \\ \omega_5 = \frac{1}{X^3} + \dots & \end{array} \right.$$

These roots correspond to the facets of the upper Newton polygon associated with the 2-Salem minimal polynomial  $\Lambda$ . As  $\Lambda$  is monic, then  $w_1$  is an algebraic integer. Therefore  $(\omega_1, \omega_2)$  is a 2-Salem series.

## References

- [1] P.T. Bateman, A.L. Duquette, *The analogue of the Pisot-Vijayaraghavan numbers in fields of formal power series*, Illinois J. Math. **6** (1962), 594–606.
- [2] M-J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Deleosse, J-P. Schreiber, *Pisot and Salem Numbers*. Basel: Birkhauser Verlag (1992).
- [3] D.W. Boyd, *Small Salem numbers*, Duke Math. J. **44**, no. 2 (1977), 315–328.
- [4] K.S. Kedlaya, *The algebraic closure of the power series field in positive characteristic*, Proc. Amer. Math. Soc **129**, no. 12 (2001), 3461–3470.
- [5] M. Kerada, *Une caractérisation de certaines classes d’entiers algébriques généralisant les nombres de Salem*, Acta Arith. **72** (1995), 55–65.
- [6] J. McKee and C. Smyth, *There are Salem Numbers of Every Trace*, Bull. London Math. Soc. **37** (2005), 25–36.
- [7] M. Ben Nasr and H. Kthiri, *Characterization of 2-Pisot elements in the field of Laurent series over a finite field*, Math. Notes (2019), to appear.
- [8] R. Salem, *Power series with integral coefficients*, Duke Math. J. **12** (1945), 153–172 .
- [9] C. Smyth, *Seventy Years of Salem Numbers: a Survey*, Bull. London Math. Soc. **47** (2015), 379–395.
- [10] J-L. Verger-Gaugry, *A Survey on the Conjecture of Lehmer and the Conjecture of Schinzel-Zassenhaus*, 2019, from <https://hal.archives-ouvertes.fr/hal-02315014/document>.
- [11] E. Weiss, *Algebraic number theory. Reprint of the 1963 original,* Mineola, NY: Dover Publications, 275 pp. (1998).