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# Fast and robust estimation for positions and velocities from noisy accelerations using generalized modulating functions method

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## Abstract

The aim of this paper is to fast and robustly estimate the positions and velocities from the measured noisy accelerations for a class of dynamical systems modelled by a set of second order linear differential equations. The obtained estimators can be applied to a large number of on-line practical applications in noisy environment. For this purpose, the generalized modulating functions method recently developed for state and derivative estimation is adopted. First, the considered equations are transformed such that the matrix associated to the position vector is lower triangular. Then, the generalized modulating functions method is applied to obtain exact algebraic integral formulas for the positions and velocities using a recursive way. These formulas can give fast estimations using sliding integration windows. Second, computation schemes are provided to construct the required generalized modulating functions. These functions depend on design parameters such as the length of sliding windows, which can be chosen using an error bound to minimize the noise error contributions. Finally, the accuracy and robustness of the proposed method are illustrated in numerical simulations.

*Keywords:* Non-asymptotic fast estimation; Generalized modulating functions method; Design parameter selection; Noise error analysis.

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## 1. Introduction

In mechanical engineering, the vibration caused by the machine itself and the external environment excitation usually affects its reliability and service life. In order to overcome this problem, one solution is to design a closed-loop controller based on the position and velocity of the machine [1]. However, due to the limitation of sensor assembly, only the acceleration can be measured in most cases, and the measurement is usually corrupted by noise. Since many mechanical systems are modelled by a set of second order linear differential equations [1, 2, 3], one idea is to provide robust estimation for the position and velocity from the acceleration measurement based on the system model. Moreover, fast estimation is useful for on-line operation. Recently, some works were developed for such estimations in [4, 5, 6]. The estimators proposed in [4, 5] were obtained by applying some algebraic operations: multiplication by  $t^n$  and multiple integration from 0 to  $t$ . Hence, a sliding integration window was not considered. This can produce large estimation errors when  $t$  increases (Indeed, there exists an optimal value of the length of integration window). Moreover, undesired initial values were not eliminated in [4]. In [6], the estimator was proposed using a more general way by considering modulating functions and a sliding integration window. Indeed, the algebraic operations applied in [4, 5] correspond to a class of modulating functions of polynomial type. Consequently, the

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modulating functions method is considered in this paper, but with a different kind of modulating functions than [6].

Modulating functions method was originally introduced by Shinbrot [7] for identification of continuous-time systems [8, 9, 10]. Very recently, an innovative idea was presented in [11], which extended the modulating functions method to state estimation for a class of time-varying linear systems with unknown input. Then, on the one hand, the generalized modulating functions were later applied in a series of works to design fractional order differentiators for integer order linear systems in [12], and for fractional order linear systems in [13, 14, 15]. On the other hand, by relaxing the modulating function conditions, one-sided modulating functions were applied to estimate the initial and fractional derivative values for a class of linear systems in [16]. Independently, another kind of one-sided modulating functions were applied to estimate the state for a class of nonlinear systems in [17], and to estimate the velocity and position from the measurement of acceleration in [6]. Let us recall that the estimators proposed by the modulating functions method in previous works are non-asymptotic thus useful for fast estimation in on-line applications, and robust to corrupting noises, thanks to the obtained algebraic integral formulas [18, 19]. Thus, the modulating functions method is promising for solving more problems.

Having the previous ideas in mind, the aim of this paper is to extend the modulating functions method to design fast and robust estimators using measured noisy accelerations for the positions and velocities of a class of dynamical systems modelled by a set of second order linear differential equations. Exact algebraic integral formulas for the positions and velocities will be provided. The obtained estimators can be applied to a large number of on-line practical applications in noisy environment. Comparing to existing works, the contributions of this work can be given as follows:

1. Different from the previously mentioned estimators proposed in [11, 12, 13, 14, 15] by the modulating functions methods, the measurement of acceleration is considered, instead of position. Indeed, the previous ones were constructed based on the measurement of position, they are not applicable to the case where the measurement of acceleration is considered.
2. Different from the estimator proposed in [4, 5, 6]:
  - (a) The considered system is more general, which is modeled by a set of differential equations which can be coupled.
  - (b) Thanks to the properties of the generalized modulating functions, the estimators of positions and velocities are directly given. However, the ones proposed in [4, 5, 6] were obtained by solving a linear system depending on the system parameters. This can produce singularities when the parameters are not exactly known.
  - (c) Computation schemes are provided to construct the desired generalized modulating functions, which provide more choices on the required modulating functions.
  - (d) A strategy is given to show how to choose the modulating functions and the length of the sliding integration window in order to reduce estimation errors. Indeed, there was a lack of such analysis in [4, 5, 6].

The rest of the paper is organized as follows. Section 2 presents the problem statement and some introductions of modulating functions. The main results are given in Section 3. First, by applying the generalized modulating functions method, algebraic integral formulas are provided for positions and velocities. Then, it is shown how to construct the required modulating functions containing two design parameters. Moreover, some error analysis is given, which is useful to choose the design parameters. In Section 4, numerical results are given to illustrate the efficiency of the proposed method. Finally, conclusions are outlined in Section 5.

## 2. Preliminaries

Throughout this framework, the following notations are adopted:

- $\mathbb{R}_+$  denotes the set of positive real numbers,
- $\mathbb{R}^*$  denotes the set of non-zero real numbers,

- $\mathbb{N}^*$  denotes the set of positive integers,
- $I$  denotes an interval of  $\mathbb{R}_+ \cup \{0\}$  of length  $l_I$ ,
- $\mathcal{C}^L(I)$  ( $L \in \mathbb{N}$ ) denotes the set of functions  $L$  times continuously differentiable on  $I$ ,
- $I_T := \{t \in I \mid [t - T, t] \subset I\}$  with  $T \in \mathbb{R}_+$  and  $T < l_I$ .

### 2.1. Motivation and Problem statement

Before stating the problem to be solved in this paper, let us present a machine tool model which motivates this work. The dynamical model of a machine tool or its component can usually be described by a set of second order linear differential equations [1, 2, 3]. Recall that 30% – 50% of the rigidity of a machine tool is determined by the characteristics of contact stiffness, and more than 90% of the damping of a machine tool comes from joint interfaces [20, 21]. Hence, by considering the contact between machine tool and foundation, and the one between foundation and soil, which has a great influence on a machine tool [22, 23], the machine tool-foundation mechanical system can be described by Fig. 1, where  $m_1$  and  $m_2$  refer to the masses of the foundation and the machine tool respectively,  $k_1$ ,  $c_1$  are the stiffness and damping of the foundation and soil contact,  $k_2$ ,  $c_2$  are the ones of the machine tool and the foundation contact, and  $f_1$  and  $f_2$  are the forces acting on the foundation and the machine tool respectively. Moreover, the machine tool-foundation

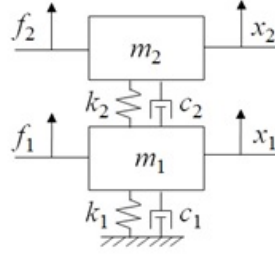


Figure 1: Machine tool-foundation mechanical system.

mechanical system can be modelled by the following equations:  $\forall t \in I$ ,

$$\begin{cases} m_1 \ddot{x}_1(t) + (c_1 + c_2) \dot{x}_1(t) - c_2 \dot{x}_2(t) + (k_1 + k_2) x_1(t) - k_2 x_2(t) = f_1(t), \\ m_2 \ddot{x}_2(t) - c_2 \dot{x}_1(t) + c_2 \dot{x}_2(t) - k_2 x_1(t) + k_2 x_2(t) = f_2(t), \\ y_1(t) = \dot{x}_1(t), \\ y_2(t) = \dot{x}_2(t). \end{cases} \quad (1)$$

Motivated by the previous example of machine tool model, the following linear dynamical system defined by a set of second order linear differential equations is considered in this work:  $\forall t \in I$ ,

$$\begin{cases} M \ddot{x}(t) + C \dot{x}(t) + K x(t) = f(t), \\ y(t) = \dot{x}(t), \end{cases} \quad (2)$$

where  $M, C, K \in \mathbb{R}^{n \times n}$  with components  $m_{ij}, c_{ij}, k_{ij}$ ,  $n \in \mathbb{N}^*$ ,  $x = (x_1, \dots, x_n)^\top$  is the vector of positions,  $f = (f_1, \dots, f_n)^\top$  is the input vector which is assumed to be known,  $y = (y_1, \dots, y_n)^\top$  is the measurable output vector.

The objective of this work is to estimate the position vector  $x$  and the velocity vector  $\dot{x}$  using the input vector  $f$  and the measured output vector of  $\dot{x}$  in noisy environment.

**Remark 1** If the matrices  $M$ ,  $C$  and  $K$  are all diagonal, then the equations of the system defined in (2) are independent, otherwise the system is coupled. If a system is coupled, it can be decoupled by applying classical modal analysis with a sufficient condition:  $C = \alpha M + \beta K$  for some constants  $\alpha$  and  $\beta$  [24], and a sufficient and necessary condition:  $CM^{-1}K = KM^{-1}C$  [25], where  $M$ ,  $C$  and  $K$  are assumed to be Symmetric Positive Definite (SPD). Without the assumption of SPD, a coupled system can be decoupled by applying phase synchronization where  $M$  is assumed to be invertible [26], or by applying isospectral transformation for a singular  $M$  [27]. However, the two later methods require the initial values  $x(0)$  and  $\dot{x}(0)$  which are usually unknown in practice. In this paper, the considered system can be coupled. Hence, neither the restrict conditions on  $M$ ,  $C$  and  $K$ , nor the initial values are required to decouple the system.

## 2.2. Modulating functions

Let us recall the definition of the modulating functions and the generalized ones.

**Definition 1** Let  $[a, b] \subset \mathbb{R}$ ,  $L \in \mathbb{N}^*$ ,  $n \in \mathbb{N}$  with  $n \leq L - 1$ ,  $g$  and  $g_n$  be two functions defined on  $[a, b]$  between which  $g_n$  depends on  $n$ .

1. If  $g$  satisfies the properties:  $\forall k \in \{0, \dots, L - 1\}$ ,

$$(P_1) : g \in C^L([a, b]),$$

$$(P_2) : g^{(k)}(a) = 0,$$

$$(P_3) : g^{(k)}(b) = 0.$$

Then,  $g$  is called  $L^{\text{th}}$  order modulating function on  $[a, b]$  [8].

2. If  $g$  only satisfies  $(P_1)$ – $(P_2)$  (resp.  $(P_1)$ ,  $(P_3)$ ), then  $g$  is called left-sided (resp. right-sided)  $L^{\text{th}}$  order modulating function on  $[a, b]$  [16].

3. If  $g$  only satisfies  $(P_1)$ – $(P_2)$  (resp.  $(P_1)$ ,  $(P_3)$ ) and the following property:  $\forall k \in \{0, \dots, L - 1\}$ ,

$$(P_4) : g^{(k)}(b) \neq 0 \text{ (resp. } g^{(k)}(a) \neq 0),$$

then  $g$  is called strict left-sided (resp. strict right-sided)  $L^{\text{th}}$  order modulating function on  $[a, b]$  [17].

4. If  $g_n$  satisfies  $(P_1)$ – $(P_2)$  and the following property:  $\forall k \in \{0, \dots, L - 1\}$ ,

$$(P_5) : g_n^{(k)}(b) = 1, \text{ if } k = n, g_n^{(k)}(b) = 0, \text{ else.}$$

Then,  $g_n$  is called  $(L, n)^{\text{th}}$  order (generalized) modulating function on  $[a, b]$  [13].

Recall that the previous generalized modulating functions were applied to design fractional order differentiators in [15, 16] and the references therein. These functions will be applied in this paper. Independently, the strict left-sided modulating functions were applied to estimate the state for a class of nonlinear systems in [17], and to estimate the velocity and position from the measurement of acceleration in [6].

Based on Definition 1, we can get the following lemmas.

**Lemma 1** Under the conditions of Definition 1, let  $g_n$  be a  $(L, n)^{\text{th}}$  order modulating function on  $[a, b]$ , then  $g_{n,a}(\cdot) := g_n(\cdot + a)$  is a  $(L, n)^{\text{th}}$  order modulating function on  $[0, b - a]$ .

**Lemma 2** Under the conditions of Definition 1, let  $g_n$  be a  $(L, n)^{\text{th}}$  order modulating function on  $[a, b]$  and  $L' \in \mathbb{N}$  with  $L' < L$ . Then,  $g_n$  is a  $(L', n)^{\text{th}}$  order modulating function on  $[a, b]$  if  $n \leq L' - 1$ , and  $g_n$  is a  $(L')^{\text{th}}$  order modulating function on  $[a, b]$ , else.

One indispensable step of the modulating functions method is to apply the integration by parts formula, which is recalled in the following lemma.

**Lemma 3** Let  $f \in C^l(\mathbb{R})$  and  $g \in C^m(\mathbb{R})$ , where  $l, m \in \mathbb{N}^*$  with  $m \leq l$ . Then, for any interval  $[a, b] \subset \mathbb{R}$ , we have:

$$\int_a^b g(t) f^{(l)}(t) dt = (-1)^m \int_a^b g^{(m)}(t) f^{(l-m)}(t) dt + \sum_{k=0}^{m-1} (-1)^k \left[ g^{(k)}(t) f^{(l-1-k)}(t) \right]_{t=a}^{t=b}. \quad (3)$$

### 3. Main results

In this section, the main results of this paper will be given in the following subsections.

#### 3.1. Algebraic integral formulas for positions and velocities

The classical modulating functions was originally extended to design integer order differentiators by introducing generalized modulating functions in [11]. Then, it was developed to design fractional order differentiators in [12, 13, 14, 15]. Inspired by these works, the generalized modulating functions method will be extended in this subsection to provide algebraic integral formulas for the position and velocity vectors of the system defined in (2). Different from the previous works, the acceleration  $\ddot{x}$  is involved in the obtained formulas. Moreover, the studied model contains a set of differential equations.

Now, we are ready to introduce the first main result of this paper, by considering the following assumption.

**Assumption 1** *It is assumed that the matrix  $K$  in (2) is invertible.*

Remark that the machine tool-foundation mechanical system defined in (1) with  $k_1 \neq 0$  and  $k_2 \neq 0$  fulfills Assumption 1.

**Theorem 1** *For  $i = 1, \dots, n$ ,  $d = 0, 1$ , let  $T_{i,d} \in \mathbb{R}_+$  with  $T_{i,d} < l_I$ . For  $t \in \bigcap_{d=0}^1 \bigcap_{i=1}^n I_{T_{i,d}}$ , let  $g_{i,d}$  be  $(2, d)^{\text{th}}$  order modulating functions on  $[t - T_{i,d}, t]$ . Then, under Assumption 1, the velocity vector of the system defined in (2) can be given by algebraic integral formulas as follows: for  $i = 1, \dots, n$ ,*

$$\begin{aligned} \dot{x}_i(t) = & - \int_0^{T_{i,0}} \ddot{g}_{i,0}(t - T_{i,0} + \tau) \hat{f}_i(t - T_{i,0} + \tau) d\tau \\ & + \int_0^{T_{i,0}} \ddot{g}_{i,0}(t - T_{i,0} + \tau) \sum_{j=1}^n \hat{m}_{ij} y_j(t - T_{i,0} + \tau) d\tau \\ & - \int_0^{T_{i,0}} \dot{g}_{i,0}(t - T_{i,0} + \tau) \sum_{j=1}^n \hat{c}_{ij} y_j(t - T_{i,0} + \tau) d\tau \\ & + \int_0^{T_{i,0}} g_{i,0}(t - T_{i,0} + \tau) y_i(t - T_{i,0} + \tau) d\tau, \end{aligned} \quad (4)$$

where  $\hat{c}_{ij}$ ,  $\hat{m}_{ij}$  and  $\hat{f}_i$  are the components of  $K^{-1}C$ ,  $K^{-1}M$  and  $K^{-1}f$ , respectively. Moreover, the position vector of the system defined in (2) can be given by algebraic integral formulas as follows: for  $i = 1, \dots, n$ ,

$$\begin{aligned} x_i(t) = & \int_0^{T_{i,1}} \ddot{g}_{i,1}(t - T_{i,1} + \tau) \hat{f}_i(t - T_{i,1} + \tau) d\tau \\ & - \int_0^{T_{i,1}} \ddot{g}_{i,1}(t - T_{i,1} + \tau) \sum_{j=1}^n \hat{m}_{ij} y_j(t - T_{i,1} + \tau) d\tau \\ & + \int_0^{T_{i,1}} \dot{g}_{i,1}(t - T_{i,1} + \tau) \sum_{j=1}^n \hat{c}_{ij} y_j(t - T_{i,1} + \tau) d\tau \\ & - \int_0^{T_{i,1}} g_{i,1}(t - T_{i,1} + \tau) y_i(t - T_{i,1} + \tau) d\tau - \sum_{j=1}^n \hat{c}_{ij} \dot{x}_j(t). \end{aligned} \quad (5)$$

**Proof.** If  $K$  is invertible, then the system defined in (2) can be transformed as follows:  $\forall t \in I$ ,

$$\hat{M}\ddot{x}(t) + \hat{C}\dot{x}(t) + x(t) = \hat{f}(t), \quad (6)$$

where  $\hat{M} = K^{-1}M$ ,  $\hat{C} = K^{-1}C$  and  $\hat{f} = K^{-1}f$ . Then, this proof can be completed by the following steps:  
**Step 1. Multiplication and integration:** For  $i = 1, \dots, n$ , let  $T_i \in \mathbb{R}_+$  with  $T_i < l_I$ , and  $g_i \in \mathcal{C}^2([t - T_i, t])$  for  $t \in \bigcap_{i=1}^n I_{T_i}$ . Then, by multiplying each equation in (6) by  $\ddot{g}_i$  and integrating from  $t - T_i$  to  $t$ , we get: for  $i = 1, \dots, n$ ,

$$\int_{t-T_i}^t \ddot{g}_i(\tau) \left( \sum_{j=1}^n \hat{c}_{ij} \dot{x}_j(\tau) + x_i(\tau) \right) d\tau = \int_{t-T_i}^t \ddot{g}_i(\tau) \left( \hat{f}_i(\tau) - \sum_{j=1}^n \hat{m}_{ij} \ddot{x}_j(\tau) \right) d\tau. \quad (7)$$

**Step 2. Integration by parts:** In order to introduce  $\ddot{x}_j$  for  $j = 1, \dots, n$ , by applying the integration by parts formula given in (3) to (7), we obtain: for  $i = 1, \dots, n$ ,

$$\int_{t-T_i}^t \ddot{g}_i(\tau) \sum_{j=1}^n \hat{c}_{ij} \dot{x}_j(\tau) d\tau = - \int_{t-T_i}^t \dot{g}_i(\tau) \sum_{j=1}^n \hat{c}_{ij} \ddot{x}_j(\tau) d\tau + \left[ \dot{g}_i(\tau) \sum_{j=1}^n \hat{c}_{ij} \dot{x}_j(\tau) \right]_{t-T_i}^t, \quad (8)$$

and

$$\int_{t-T_i}^t \ddot{g}_i(\tau) x_i(\tau) d\tau = \int_{t-T_i}^t g_i(\tau) \ddot{x}_i(\tau) d\tau + [\dot{g}_i(\tau) x_i(\tau)]_{t-T_i}^t - [g_i(\tau) \dot{x}_i(\tau)]_{t-T_i}^t. \quad (9)$$

**Step 3. Eliminating unknown boundary values:** In order to eliminate the unknown boundary values of  $x_j$ ,  $\dot{x}_j$  on  $t - T_i$  for  $i, j = 1, \dots, n$  in (8)-(9), we assume  $g_i(t - T_i) = \dot{g}_i(t - T_i) = 0$ . Hence, using (7)-(9) we get: for  $i = 1, \dots, n$

$$\begin{aligned} & \dot{g}_i(t) \sum_{j=1}^n \hat{c}_{ij} \dot{x}_j(t) + \dot{g}_i(t) x_i(t) - g_i(t) \dot{x}_i(t) \\ &= \int_{t-T_i}^t \dot{g}_i(\tau) \sum_{j=1}^n \hat{c}_{ij} \ddot{x}_j(\tau) d\tau - \int_{t-T_i}^t g_i(\tau) \ddot{x}_i(\tau) d\tau \\ &+ \int_{t-T_i}^t \ddot{g}_i(\tau) \left( \hat{f}_i(\tau) - \sum_{j=1}^n \hat{m}_{ij} \ddot{x}_j(\tau) \right) d\tau. \end{aligned} \quad (10)$$

**Step 4. Algebraic integral formulas of  $\dot{x}_i(t)$ :** In order to get the formulas of  $\dot{x}_i(t)$  for  $i = 1, \dots, n$ , we replace  $g_i$  by  $g_{i,0}$  and  $T_i$  by  $T_{i,0}$  in (10), where  $g_{i,0}$  are  $(2,0)^{th}$  order modulating functions on  $[t - T_{i,0}, t]$ . Thus,  $\dot{x}_i(t)$  can be given by algebraic integral formulas as follows: for  $i = 1, \dots, n$ ,

$$\begin{aligned} \dot{x}_i(t) &= - \int_{t-T_{i,0}}^t \ddot{g}_{i,0}(\tau) \left( \hat{f}_i(\tau) - \sum_{j=1}^n \hat{m}_{ij} \ddot{x}_j(\tau) \right) d\tau \\ &- \int_{t-T_{i,0}}^t \dot{g}_{i,0}(\tau) \sum_{j=1}^n \hat{c}_{ij} \ddot{x}_j(\tau) d\tau + \int_{t-T_{i,0}}^t g_{i,0}(\tau) \ddot{x}_i(\tau) d\tau. \end{aligned} \quad (11)$$

**Step 5. Algebraic integral formulas of  $x_i(t)$ :** In order to get the formulas of  $x_i(t)$  for  $i = 1, \dots, n$ , we replace  $g_i$  by  $g_{i,1}$  and  $T_i$  by  $T_{i,1}$  in (10), where  $g_{i,1}$  are  $(2,1)^{th}$  order modulating functions on  $[t - T_{i,1}, t]$ .

Thus,  $x_i(t)$  can be given by algebraic integral formulas by: for  $i = 1, \dots, n$ ,

$$\begin{aligned} x_i(t) &= \int_{t-T_{i,1}}^t \ddot{g}_{i,1}(\tau) \left( \hat{f}_i(\tau) - \sum_{j=1}^n \hat{m}_{ij} \ddot{x}_j(\tau) \right) d\tau \\ &+ \int_{t-T_{i,1}}^t \dot{g}_{i,1}(\tau) \sum_{j=1}^n \hat{c}_{ij} \ddot{x}_j(\tau) d\tau \\ &- \int_{t-T_{i,1}}^t g_{i,1}(\tau) \ddot{x}_i(\tau) d\tau - \sum_{j=1}^n \hat{c}_{ij} \dot{x}_j(t). \end{aligned} \quad (12)$$

**Step 6. Substitution and changes of variables:** Finally, this proof can be finished by substituting  $\ddot{x}_i$  by  $y_i$  in (11)-(12) and by applying the changes of variables  $\tau \rightarrow \tau + t - T_{i,d}$ , for  $i = 1, \dots, n$ ,  $d = 0, 1$ . ■

The previous result is based on Assumption 1. If the matrix  $K$  in (2) is singular, let us introduce the following lemma.

**Lemma 4** *If the matrix  $K$  in (2) is singular with  $\text{rank}(K) = r < n$ , there exist  $P, Q \in \mathbb{R}^{n \times n}$  such that  $\bar{K} = PKQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$  where  $I_r \in \mathbb{R}^{r \times r}$  is the identity matrix, and  $Q$  is invertible. Then, the system model given in (2) can be transformed as follows:  $\forall t \in I$ ,*

$$\begin{cases} \bar{M}_h \ddot{z}(t) + \bar{C}_h \dot{z}(t) + z_h(t) = \bar{f}_h(t), \\ \bar{M}_l \ddot{z}(t) + \bar{C}_l \dot{z}(t) = \bar{f}_l(t), \\ \bar{y}(t) = \ddot{z}(t), \end{cases} \quad (13)$$

where  $z = Q^{-1}x$  with  $z_h = (z_1, \dots, z_r)^\top$ ,  $z_l = (z_{r+1}, \dots, z_n)^\top$ ,  $\bar{y} = Q^{-1}y$ ,  $\bar{M} = PMQ = \begin{pmatrix} \bar{M}_h \\ \bar{M}_l \end{pmatrix}$ ,  $\bar{C} = PCQ = \begin{pmatrix} \bar{C}_h \\ \bar{C}_l \end{pmatrix}$ , and  $\bar{f} = Pf = \begin{pmatrix} \bar{f}_h \\ \bar{f}_l \end{pmatrix}$ .

**Proof.** If  $\text{rank}(K) = r < n$ , then there exist two elementary matrices  $P$  and  $Q$  such that  $\bar{K} = PKQ$  [28]. Hence, by defining  $z = Q^{-1}x$ , (2) is transformed as follows:  $\forall t \in I$ ,

$$\begin{cases} \bar{M} \ddot{z}(t) + \bar{C} \dot{z}(t) + \bar{K} z(t) = \bar{f}(t), \\ y(t) = Q \ddot{z}(t). \end{cases} \quad (14)$$

Thus, this proof is completed. ■

Based on the previous lemma, let us give the second main result of this paper, by considering the following assumption.

**Assumption 2** *If the matrix  $K$  in (2) is singular with  $\text{rank}(K) = r < n$ , then it is assumed that  $\bar{C}_{l,r} \in \mathbb{R}^{(n-r) \times (n-r)}$  is invertible, where  $\bar{C}_l = \begin{pmatrix} \bar{C}_{l,l} & \bar{C}_{l,r} \end{pmatrix}$  is given in Lemma 4.*

**Theorem 2** *For  $i = 1, \dots, n$ , let  $T_{i,0} \in \mathbb{R}_+$  with  $T_{i,0} < l_I$ . For  $t \in \bigcap_{i=1}^n I_{T_{i,0}}$ , let  $g_{i,0}$  be  $(2, 0)^{\text{th}}$  order modulating functions on  $[t - T_{i,0}, t]$  for  $i = 1, \dots, r$ , and  $g_{i,0}$  be  $(1, 0)^{\text{th}}$  order modulating functions on  $[t - T_{i,0}, t]$  for  $i = r + 1, \dots, n$ . Then, under Assumption 2, the velocity vector of the system defined in (13) can be given by algebraic integral formulas using a recursive way as follows:*



- for  $i = 1, \dots, r$ ,

$$\begin{aligned}
\dot{z}_i(t) = & - \int_0^{T_{i,0}} \ddot{g}_{i,0}(t - T_{i,0} + \tau) \bar{f}_i(t - T_{i,0} + \tau) d\tau \\
& + \int_0^{T_{i,0}} \ddot{g}_{i,0}(t - T_{i,0} + \tau) \sum_{j=1}^n \bar{m}_{ij} \bar{y}_j(t - T_{i,0} + \tau) d\tau \\
& - \int_0^{T_{i,0}} \dot{g}_{i,0}(t - T_{i,0} + \tau) \sum_{j=1}^n \bar{c}_{ij} \bar{y}_j(t - T_{i,0} + \tau) d\tau \\
& + \int_0^{T_{i,0}} g_{i,0}(t - T_{i,0} + \tau) \bar{y}_i(t - T_{i,0} + \tau) d\tau,
\end{aligned} \tag{15}$$

where  $\bar{m}_{ij}$ ,  $\bar{c}_{ij}$ , and  $\bar{f}_i$  are the components of  $\bar{M}_h$ ,  $\bar{C}_h$ , and  $\bar{f}_h$ , respectively,

- for  $i = r + 1, \dots, n$ ,

$$\begin{aligned}
\dot{z}_i(t) = & \int_0^{T_{i,0}} \dot{g}_{i,0}(t - T_{i,0} + \tau) \tilde{f}_{i-r}(t - T_{i,0} + \tau) d\tau \\
& - \int_0^{T_{i,0}} \dot{g}_{i,0}(t - T_{i,0} + \tau) \sum_{j=1}^n \tilde{m}_{(i-r)j} \tilde{y}_j(t - T_{i,0} + \tau) d\tau \\
& + \int_0^{T_{i,0}} g_{i,0}(t - T_{i,0} + \tau) \sum_{j=1}^r \tilde{c}_{(i-r)j} \tilde{y}_j(t - T_{i,0} + \tau) d\tau \\
& + \int_0^{T_{i,0}} g_{i,0}(t - T_{i,0} + \tau) \tilde{y}_i(t - T_{i,0} + \tau) d\tau - \sum_{j=1}^r \tilde{c}_{(i-r)j} \dot{z}_j(t),
\end{aligned} \tag{16}$$

where  $\tilde{m}_{(i-r)j}$ ,  $\tilde{c}_{(i-r)j}$ , and  $\tilde{f}_{i-r}(t)$  are the components of  $\bar{C}_{l,r}^{-1} \bar{M}_l$ ,  $\bar{C}_{l,r}^{-1} \bar{C}_{l,l}$  and  $\bar{C}_{l,r}^{-1} \bar{f}_l$ , respectively. Moreover, the velocity vector of the system defined in (2) can be given by  $\dot{x} = Q\dot{z}$ , respectively, where  $\dot{z}$  is given by (15)-(16).

**Proof.** This proof can be completed using a similar way as done in the proof of Theorem 1.

On the one hand, by applying **Step 1** - **Step 4** and **Step 6** in the proof of Theorem 1 to the first  $r$  equations in (13), similar to (4), (15) can be obtained.

On the other hand, let us consider the  $(r + 1)^{th}$  to the  $n^{th}$  equations in (13), which can be given as follows:

$$\bar{M}_l \ddot{z}(t) + \bar{C}_{l,r} \dot{z}_l(t) = \bar{f}_l(t) - \bar{C}_{l,l} \dot{z}_h(t). \tag{17}$$

Then, under Assumption 2, we get:

$$\tilde{M}_l \ddot{z}(t) + \dot{z}_l(t) = \tilde{f}_l(t) - \tilde{C}_{l,l} \dot{z}_h(t), \tag{18}$$

where  $\tilde{M}_l = \bar{C}_{l,r}^{-1} \bar{M}_l$ ,  $\tilde{f}_l = \bar{C}_{l,r}^{-1} \bar{f}_l$  and  $\tilde{C}_{l,l} = \bar{C}_{l,r}^{-1} \bar{C}_{l,l}$ . Hence, we have: for  $i = r + 1, \dots, n$ ,

$$\sum_{j=1}^n \tilde{m}_{(i-r)j} \ddot{z}_j(t) + \dot{z}_i(t) = \tilde{f}_{i-r}(t) - \sum_{j=1}^r \tilde{c}_{(i-r)j} \dot{z}_j(t). \tag{19}$$

Hence, by multiplying  $\dot{g}_{i,0}$  to (19) and applying **Step 1** - **Step 4** and **Step 6** in the proof of Theorem 1, (16) can be obtained. Thus, this proof is completed using  $\dot{x} = Q\dot{z}$ .  $\blacksquare$

**Remark 2** It can be seen in Theorem 1 that the essential idea of the proposed method is to remove the derivative operations from the modulating functions to  $x$  and  $\dot{x}$  such that the values of position and velocity

can be produced. It does not depend on the invertibility of  $M$  and  $C$ . However, without Assumption 1, the position vector  $x$  of the system defined in (2) cannot be obtained by the proposed method. In fact, the system given in (13) is not algebraically observable [29] (see [4] for a simple example), hence neither the one defined in (2). This is why only the first  $r$  components of  $\dot{z}$  and  $z$  can be obtained in Theorem 2. Then, under Assumption 2, the last  $n - r$  components of  $\dot{z}$  can be obtained such that the velocity vector  $\dot{x}$  can be also.

Consequently, algebraic integral formulas are obtained in Theorem 1 and Theorem 2, which do not contain any sources of errors. Moreover, it can be seen in Theorem 1 that  $\dot{x}_i$  for  $i = 1, \dots, n$ , are given by the sum of integrals involving  $y$  and  $f$ , then  $x_i$  for  $i = 1, \dots, n$ , are given using  $\dot{x}_j$  for  $j = 1, \dots, n$ . Hence, they are also given by the sum of integrals involving  $y$  and  $f$ . Moreover, when we calculate  $x_i(t)$  using (4), the lengths of the integration windows for  $\dot{x}_j(t)$  for  $j = 1, \dots, n$  are also taken to  $T_{i,1}$ . Consequently,  $x_i^{(d)}$  for  $i = 1, \dots, n$ ,  $d = 0, 1$ , are calculated by the sum of integrals involving  $y$  and  $f$  of length  $T_{i,d}$ . Thus, for each  $t \in \bigcap_{d=0}^1 \bigcap_{i=1}^n I_{T_{i,d}}$ , the desired position and velocity values are calculated using sliding integration windows on  $[t - T_{i,d}, t]$ . Similarly, for each  $t \in \bigcap_{i=1}^n I_{T_{i,0}}$ , the values of  $\dot{z}_i(t)$  for  $i = 1, \dots, n$ , are calculated using sliding integration windows on  $[t - T_{i,0}, t]$  in Theorem 2. Then,  $\dot{x}(t)$  is given by  $Q\dot{z}(t)$ . This advantage of using a sliding integration window can provide fast estimations for on-line applications. The length  $T_{i,d}$  is a design parameter for  $x_i^{(d)}$ , the selection of which will be studied later.

### 3.2. Constructions of modulating functions

There exist several kinds of modulating functions in the literature [7, 9, 13, 30], such as sinusoidal functions, Hermite functions, Poisson moment functionals, and polynomials, etc. However, only generalized modulating functions of polynomial type were constructed in the literature [11, 12, 13, 15]. In following proposition, a computation scheme is provided to show how to construct generalized modulating functions in a more general case based on left-sided modulating functions.

**Proposition 1** *Let  $T \in \mathbb{R}_+$ ,  $L \in \mathbb{N}^*$ ,  $n \in \mathbb{N}$  with  $n \leq L - 1$ . For  $i = 0, \dots, L - 1$ , let  $g_i$  be a left-sided  $L^{\text{th}}$  order modulating function on  $[0, T]$ . Then, let us consider the following function:  $\forall \tau \in [0, t]$ ,*

$$g(\tau) = \sum_{i=0}^{L-1} \lambda_i g_i(\tau). \quad (20)$$

*If the coefficient vector  $\lambda = (\lambda_0, \dots, \lambda_{L-1})^\top$  is the unique solution of the following linear system:*

$$B\lambda = b_n, \quad (21)$$

*where  $B \in \mathbb{R}^{L \times L}$  assumed to be invertible, and  $b_n \in \mathbb{R}^L$  are defined as follows: for  $k, i = 0, \dots, L - 1$ ,*

$$B(k, i) = g_i^{(k)}(T), \quad (22)$$

*and for  $k = 0, \dots, L - 1$ ,*

$$b_n(k) = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{else,} \end{cases} \quad (23)$$

*then  $g$  is a  $(L, n)^{\text{th}}$  order modulating function on  $[0, T]$ .*

**Proof.** Since  $g$  is a linear combination of functions satisfying  $(P_1)$ – $(P_2)$ ,  $g$  also satisfies  $(P_1)$ – $(P_2)$ . Hence, it is necessary to verify  $g$  satisfies  $(P_4)$ . By calculating the  $k^{\text{th}}$  derivative of  $g$  at  $\tau = T$ , we obtain: for  $k = 0, \dots, L - 1$ ,

$$g^{(k)}(T) = \sum_{i=0}^{L-1} \lambda_i g_i^{(k)}(T). \quad (24)$$

Thus, if the coefficient vector  $\lambda$  is the solution of the system in (21) where the matrix  $B$  is invertible, then for  $k = 0, \dots, L - 1$ ,  $g^{(k)}(T) = 1$  if  $k = n$ ,  $g^{(k)}(T) = 0$  else, *i.e.*  $(P_4)$  is fulfilled. Consequently, this proof is completed.  $\blacksquare$

Consequently, by applying Proposition 1, the left-sided modulating functions involving exponential functions considered in [17, 31] can also be used to construct generalized modulating functions. Now, if modulating functions of polynomial type are considered in Theorem 1 and Theorem 2, then the following corollary can be deduced.

**Corollary 1** *Let  $T \in \mathbb{R}_+$ ,  $m \in \mathbb{N}^*$ . Then, we have:*

1. *if  $m \geq 1$ ,  $q_{0,m}$  is a  $(1, 0)^{th}$  order modulating function on  $[0, T]: \forall \tau \in [0, t]$ ,*

$$q_{0,m}(\tau) = \frac{\tau^m}{T^m}, \quad (25)$$

2. *if  $m \geq 2$ ,  $p_{0,m}$  is a  $(2, 0)^{th}$  order modulating function on  $[0, T]: \forall \tau \in [0, t]$ ,*

$$p_{0,m}(\tau) = (m+1) \frac{\tau^m}{T^m} - m \frac{\tau^{m+1}}{T^{m+1}}, \quad (26)$$

3. *if  $m \geq 2$ ,  $p_{1,m}$  is a  $(2, 1)^{th}$  order modulating function on  $[0, T]: \forall \tau \in [0, t]$ ,*

$$p_{1,m}(\tau) = \frac{\tau^{m+1}}{T^m} - \frac{\tau^m}{T^{m-1}}. \quad (27)$$

**Proof.** First, it can be easily verified that  $q_{0,m}$  is a  $(1, 0)^{th}$  order modulating function on  $[0, T]$ . Second, for  $n = 0, 1$ , let us consider the following functions:  $\forall \tau \in [0, T]$ ,

$$p_{n,m}(\tau) = \sum_{i=0}^1 \lambda_i g_i(\tau), \quad (28)$$

where  $g_i(\tau) = T^{n-m-i} \tau^{m+i}$ . Then, let us define the following matrix:

$$B_m = \begin{pmatrix} g_0(T) & g_1(T) \\ \dot{g}_0(T) & \dot{g}_1(T) \end{pmatrix} = T^{n-1} \begin{pmatrix} T & T \\ m & m+1 \end{pmatrix}. \quad (29)$$

Hence, we obtain:  $\det(B_m) = T^{2n-1} \neq 0$ , and the inverse of  $B_m$  is given by:

$$B_m^{-1} = \frac{1}{T^n} \begin{pmatrix} m+1 & -T \\ -m & T \end{pmatrix}. \quad (30)$$

It can be verified that  $g_i$  for  $i = 0, 1$ , satisfy  $(P_1)$ - $(P_2)$ . Hence, according to Proposition 1,  $p_{n,m}$  is a  $(2, n)^{th}$  order modulating function on  $[0, T]$  with  $\lambda = B_m^{-1} b_n$ . Thus, this proof is completed.  $\blacksquare$

According to Lemma 2, on the one hand,  $p_{0,m}$  is also a  $(1, 0)^{th}$  order modulating function on  $[0, T]$ . On the other hand, a  $(L, 0)^{th}$  (with  $2 \leq L$ ) order modulating function on  $[0, T]$  can also be used in Theorem 1 and Theorem 2. Consequently, by applying Proposition 1, more polynomials can be constructed for Theorem 1 and Theorem 2.

It can be seen that the generalized modulating functions constructed in Corollary 1 depend on two parameters  $m$  and  $T$ . The way of the parameter selection will be presented in numerical simulations using the error bound provided in the next subsection.

### 3.3. Error analysis in discrete noisy cases

From now on, the discrete noisy observation  $y^\varpi$  of  $y$  on  $I$  is considered: for  $j = 1, \dots, n$ ,

$$y_j^\varpi(t_i) = y_j(t_i) + \varpi_j(t_i), \quad (31)$$

where  $t_i = iT_s$ , for  $i = 0, 1, \dots, S$ , with an equidistant sampling period  $T_s = \frac{L}{S}$ , and  $\varpi = (\varpi_1, \dots, \varpi_n)^\top$  is the corrupting noise vector. Then, the algebraic integral formulas provided in Theorem 1 and Theorem 2 will be applied in the discrete noisy case, where  $y$  is replaced by its observation  $y^\varpi$ . For this, a numerical integration method needs to be used to approximate the integrals. Consequently, it can be deduced that the estimators proposed in Theorem 1 and Theorem 2 contain two sources of errors:

1. the numerical errors due to the used numerical integration method;
2. the noise error contributions due to the noise vector  $\varpi$ .

Let us give some error analysis for the estimators proposed in Theorem 1. Similar results can be deduced for the ones proposed in Theorem 2. It is shown in Theorem 1 that  $x_i^{(d)}$  for  $i = 1, \dots, n$ ,  $d = 0, 1$ , are in fact given by the sum of integrals involving  $y$  and  $f$  of length  $T_{i,d}$ . In order to simplify the presentation, let us denote the integral formulas of  $x_i^{(d)}$  by a unified expression as follows:

$$x_i^{(d)}(t) = \mathcal{I}_{T_{i,d}}\{y\}(t) + \int_0^{T_{i,d}} \sum_{j=1}^n p_{i,d,j}(\tau) f_j(t - T_{i,d} + \tau) d\tau, \quad (32)$$

where

$$\mathcal{I}_{T_{i,d}}\{y\}(t) := \int_0^{T_{i,d}} \sum_{j=1}^n q_{i,d,j}(\tau) y_j(t - T_{i,d} + \tau) d\tau, \quad (33)$$

$p_{i,d,j}$  and  $q_{i,d,j}$  are the associated functions to  $f_j$  and  $y_j$  respectively, which are linear combinations of the adopted modulating functions and their derivatives. Then, by taking  $T_{i,d} = s_{i,d}T_s$  with  $s_{i,d} \in \mathbb{N}^*$ ,  $\mathcal{I}_{T_{i,d}}\{y\}(t)$  can be approximated in the discrete noisy case by:

$$\mathcal{S}_{T_{i,d}}\{y^\varpi\}(t) := \mathcal{S}_{T_{i,d}}\{y\}(t) + \mathcal{S}_{T_{i,d}}\{\varpi\}(t), \quad (34)$$

where

$$\mathcal{S}_{T_{i,d}}\{y\}(t) := T_s \sum_{k=0}^{s_{i,d}} w_k \sum_{j=1}^n q_{i,d,j}(kT_s) y_j(t - T_{i,d} + kT_s), \quad (35)$$

$$\mathcal{S}_{T_{i,d}}\{\varpi\}(t) := T_s \sum_{k=0}^{s_{i,d}} w_k \sum_{j=1}^n q_{i,d,j}(kT_s) \varpi_j(t - T_{i,d} + kT_s), \quad (36)$$

where  $w_k \in \mathbb{R}_+$  are the weights of a used numerical integration method. It is well known that the numerical errors between  $\mathcal{I}_{T_{i,d}}\{y\}(t)$  and  $\mathcal{S}_{T_{i,d}}\{y\}(t)$  converge to zero when  $T_s \rightarrow 0$  (see, e.g. [32]).

Let us assume that the noise  $\{\varpi(t), t \in I\}$  is a vector of continuous stochastic processes, which satisfies the following conditions:

- (C<sub>1</sub>) : for any  $s, t \in I$ ,  $s \neq t$ ,  $\varpi_j(s)$  and  $\varpi_j(t)$  are independent;
- (C<sub>2</sub>) : for any  $i, j \in \{1, \dots, n\}$ ,  $i \neq j$ ,  $\varpi_i(t)$  and  $\varpi_j(t)$  are independent;
- (C<sub>3</sub>) : for any  $s, t \in I$ ,  $s \neq t$ , for any  $i, j \in \{1, \dots, n\}$ ,  $i \neq j$ ,  $\varpi_i(s)$  and  $\varpi_j(t)$  are independent;
- (C<sub>4</sub>) : the mean value function of  $\{\varpi_j(t), t \in I\}$  denoted by  $\mathbb{E}[\cdot]$  is equal to zero;
- (C<sub>5</sub>) : the variance function of  $\{\varpi_j(t), t \in I\}$  denoted by  $\text{Var}[\cdot]$  is bounded on  $I$ , i.e.  $\exists \delta_j \in \mathbb{R}_+, \forall t \in I, \text{Var}[\varpi_j(t)] \leq \delta_j$ .

Then, it can be verified that:

$$\mathbb{E} [\mathcal{S}_{T_{i,d}}\{\varpi\}(t)] = 0, \quad (37)$$

$$\text{Var} [\mathcal{S}_{T_{i,d}}\{\varpi\}(t)] \leq T_s^2 \sum_{k=0}^{s_{i,d}} w_k^2 \sum_{j=1}^n \delta_j q_{i,d,j}^2(kT_s). \quad (38)$$

Hence, using a similar technique as done in [12, 13], the convergence in mean square of the noise error contributions  $\mathcal{S}_{T_{i,d}}\{\varpi\}(t)$  can be deduced:

$$\mathbb{E} \left[ (\mathcal{S}_{T_{i,d}}\{\varpi\}(t))^2 \right] \xrightarrow{T_s \rightarrow 0} 0. \quad (39)$$

Consequently, both the numerical errors and the noise error contributions in the proposed estimators can be reduced by decreasing the sampling period  $T_s$ .

When the sampling period is set, using (37)  $\mathcal{S}_{T_{i,d}}\{\varpi\}(t)$  can be bounded using the Bienaymé-Chebyshev inequality as follows:  $\forall \gamma \in \mathbb{R}_+^*$ ,

$$\Pr \left( \left| \mathcal{S}_{T_{i,d}}\{\varpi\}(t) \right| < \gamma \left( \text{Var}[\mathcal{S}_{T_{i,d}}\{\varpi\}(t)] \right)^{\frac{1}{2}} \right) > 1 - \frac{1}{\gamma^2}, \quad (40)$$

*i.e.* the probability for  $\left| \mathcal{S}_{T_{i,d}}\{\varpi\}(t) \right|$  to be smaller than  $\gamma \left( \text{Var}[\mathcal{S}_{T_{i,d}}\{\varpi\}(t)] \right)^{\frac{1}{2}}$  is larger than  $1 - \frac{1}{\gamma^2}$ . Thus, the following error bound is deduced from (38) and (40):

$$\left| \mathcal{S}_{T_{i,d}}\{\varpi\}(t) \right| \stackrel{p_\gamma}{\leq} \delta^{\frac{1}{2}} \gamma T_s \left( \sum_{k=0}^{s_{i,d}} w_k^2 \sum_{j=1}^n q_{i,d,j}^2(kT_s) \right)^{\frac{1}{2}}, \quad (41)$$

where  $\delta = \max_{j=1,\dots,n} (\delta_j)$ ,  $a \stackrel{p_\gamma}{<} b$  means that the probability for a real number  $b$  to be larger than another real number  $a$  is equal to  $p_\gamma$  with  $p_\gamma > 1 - \frac{1}{\gamma^2}$ .

Thanks to the probability property, the noise error bound obtained in (41) is sharp. Once the required modulating functions are constructed, for example using Corollary 1, their derivatives can be analytically calculated, then as well as can  $q_{i,d,j}$ . Since the constructed modulating functions depend on the design parameters, so do  $q_{i,d,j}$  and the error bound. Consequently, in order to reduce the noise error contribution, the design parameters can be chosen in the way to minimize the error bound. Thus, the property of robustness of the proposed method can be guaranteed. Moreover, since the error bound is linear with respect to  $\delta^{\frac{1}{2}} \gamma$ , the location of its minimum does not depend on the value of  $\delta^{\frac{1}{2}} \gamma$ . Consequently, this analysis can be done in an off-line work without knowing the noise levels.

Once the design parameters are selected,  $q_{i,d,j}$  are calculated in an off-line work, then they are presented by vectors in discrete case. Thus,  $\mathcal{S}_{T_{i,d}}\{y^\varpi\}(t)$  can be calculated just by dot products of vectors of  $q_{i,d,j}$  and  $y^\varpi$ . On the other hand, since the input vector  $f$  is known, the integrals involving  $f$  can also be approximated using dot products of vectors in discrete case. Consequently, the proposed estimators can be realized by dot products of the vectors of  $y^\varpi$ ,  $f$ , and the ones depending on the modulating functions designed in an off-line work. Thus, the computation effect is significantly reduced for on-line applications (such idea was also used in a similar context in [33]).

Let us summarize how to apply the proposed method using the matrices  $M$ ,  $C$ ,  $K$ , the input vector  $f$ , the output vector  $y$  and the sampling period  $T_s$ :

**Step 1:** verify if  $K$  is invertible. If  $K$  is invertible, calculate  $K^{-1}$  and follow **Step 1 – Step 5**. Otherwise, follow **Step 6-Step 9**.

**Step 2:** apply Proposition 1 to construct the required modulating functions  $g_{i,d}$  for  $i = 1, \dots, n$ ,  $d = 0, 1$ .

**Step 3:** apply Theorem 1 to find  $p_{i,d,j}$  and  $q_{i,d,j}$  for  $x_i^{(d)}$  for  $i, j = 1, \dots, n$ ,  $d = 0, 1$ , using  $K^{-1}$  and  $g_{i,d}$ .

**Step 4:** apply the error bound given in (41) to select design parameters  $m$  and  $T_{i,d}$  for  $x_i^{(d)}$  for  $i = 1, \dots, n$ ,  $d = 0, 1$ , using  $T_s$ ,  $q_{i,d,j}$  and a numerical integration method, where we can take  $\delta = 1$  and  $\gamma = 3$ .

**Step 5:** use the selected design parameters to estimate  $x_i^{(d)}$  by calculating the sum of the dot products of the vectors  $p_{i,d,j}$  and  $f$  and the ones of  $q_{i,d,j}$  and  $y$ .

**Step 6:** calculate the rank of  $K$ , and find  $P$ ,  $Q$ . Then, calculate  $Q^{-1}$ ,  $\bar{y}$ ,  $\bar{f}$ ,  $\bar{M}$  and  $\bar{C}$  using Lemma 4.

**Step 7:** under Assumption 2, calculate  $\bar{C}_{l,r}^{-1}$ .

**Step 8:** similar to **Step 2 – Step 4**, find the required vectors by applying Theorem 2, which involve the modulating functions constructed by applying Proposition 1 with the parameters selected by using (41).

**Step 9:** estimate  $\dot{z}$  by calculating the dot products of  $\bar{y}$  and  $\bar{f}$  with the vectors found in **Step 8**, respectively. Then, estimate  $\dot{x} = Q\dot{z}$ .

Recall that **Step 1 – Step 4** and **Step 6 – Step 8** can be done in off-line work, and **Step 5** and **Step 9** are done in on-line work.

Finally, let us remark that on the one hand the proposed algorithm works in the case where some components of  $f$  are identical to zero. Indeed, the algebraic integral formulas obtained in Theorem 1 and Theorem 2 always hold if some components of  $f$  are identical to zero. On the other hand, the proposed algorithm also works if the components of  $f$  contain some kind of  $PD$ -controls using the estimated values of position and velocity as done in [34]. Indeed, the controller value at time instant  $t$  is designed based on the estimated values of position and velocity at  $t - T_s$ . The previous two cases will be considered in the numerical tests in the following section.

#### 4. Numerical tests

In this section, the efficiency and robustness of the proposed method will be shown in numerical simulations. For this purpose, two examples are considered.

##### Example 1.

Let us take the following parameters in (1):  $m_1 = 0.1$ ,  $m_2 = 0.6$ ,  $c_1 = 0.5$ ,  $c_2 = 0.4$ ,  $k_1 = 0.3$  and  $k_2 = 0.2$ . Then, we assume the system is excited by  $f(t) = (\sin(2t), \cos(t))^T$  on  $I = [0, 50]$  with the following initial conditions  $x_1(0) = 0$ ,  $x_2(0) = 0$ ,  $\dot{x}_1(0) = 0$  and  $\dot{x}_2(0) = 0$ . Moreover, we assume the accelerations  $\ddot{x}_1$  and  $\ddot{x}_2$  are measured by discrete noisy observations:  $y_j^\varpi(t_i) = y_j(t_i) + \sigma_j \varpi_j(t_i)$  for  $j = 1, 2$ , with  $t_i = iT_s$  and  $T_s = 0.001$ . The noise  $\varpi = (\varpi_1, \varpi_2)^T$  is a vector of continuous stochastic process and satisfies  $(C_1)$ - $(C_5)$ , where each  $\varpi_j$  is a zero-mean white Gaussian noise, and the values of  $\sigma_j$  is taken to 0.5. Then, the output vector  $y$  and its noisy observation  $y^\varpi$  on  $[0, 20]$  are shown in Fig. 2, where  $y$  is obtained by the well-known finite difference method [36]. Hence, the algebraic integral formulas obtained in (4)-(5) are applied to estimate  $x_1$ ,  $x_2$ ,  $\dot{x}_1$  and  $\dot{x}_2$  from the discrete noisy output observation  $y^\varpi$ , where the trapezoidal numerical integration method is used.

First, the required modulating functions are constructed using Corollary 1. Then, the error bound given in (41) is used to select the design parameter  $m$  and  $T_{i,d}$  for  $i = 1, 2$  and  $d = 0, 1$ . For this, we take  $\gamma = 3$ ,  $\delta = 1$  which is unknown in practice, then we study the variations of the error bounds with respect to  $m$  and  $T_{i,d}$ , and we take the optimal values which minimize the error bounds. The variations are shown in Fig. 3, where  $m = 2, \dots, 6$  and  $T_{i,d} = 1, 1 + T_s, \dots, 10$ . Consequently, the following parameter values are selected:

1. for  $x_1$ , when  $t \in ]T_s, 5]$ ,  $m = 2$  and  $T_{1,1} = iT_s$ , then  $m = 3$  and  $T_{1,1} = 5$ ,
2. for  $x_2$ , when  $t \in ]T_s, 5.5]$ ,  $m = 2$  and  $T_{2,1} = iT_s$ , then  $m = 2$  and  $T_{2,1} = 5.5$ ,
3. for  $\dot{x}_1$ , when  $t \in ]T_s, 5.5]$ ,  $m = 2$  and  $T_{1,0} = iT_s$ , then  $m = 4$  and  $T_{1,0} = 5.5$ ,
4. for  $\dot{x}_2$ , when  $t \in ]T_s, 6.5]$ ,  $m = 2$  and  $T_{2,0} = iT_s$ , when  $t \in ]6.5, 8.5]$ ,  $m = 3$  and  $T_{2,0} = iT_s$ , then  $m = 4$  and  $T_{2,0} = 8.5$ .

In fact, when the time  $t$  is smaller than the optimal length of the sliding integration window, the length is taken equal to  $t$  with optimal value of  $m$ .

Second, using the selected design parameter values, the estimated positions and velocities on  $[0, 20]$  are shown in Fig. 4, where the ones calculated by the finite difference method in noise free case are used to verify the accuracy of the proposed method. Consequently, it can be seen that the proposed method provides reliable estimation results against the corrupting noises. Moreover, in order to verify the efficiency of the proposed error bounds, the associated noise errors and the corresponding error bounds obtained by taking  $\delta = 0.5$  and  $\gamma = 3$  according to the empirical rule [37] are shown in Fig. 5.

Third, a  $PD$ -controller is designed to the system on [20, 50] in order to stabilize the system by eliminating the influence of  $f$ . For this purpose, we take: for  $i = 1, 2$ ,  $u_i(t) = K_{p_i}e_i(t) + K_{d_i}\dot{e}_i(t)$ , where  $e_i = r_i(t) - \tilde{x}_i(t)$ ,  $\dot{e}_i = \dot{r}_i(t) - \dot{\tilde{x}}_i(t)$ , the reference is  $r_i(t) = 0$ ,  $\tilde{x}_i(t)$  and  $\dot{\tilde{x}}_i(t)$  are estimated with the previous design parameters. Finally, the position trajectories are shown in Fig. 6, which are obtained by taking  $K_{p_1} = 50$ ,  $K_{d_1} = 10$ ,  $K_{p_2} = 50$  and  $K_{d_2} = 10$ .

**Example 2.**

Let us take the following parameter and matrices in (2):  $n = 3$ ,  $M = \begin{pmatrix} 1.9259 & 1.7539 & 2.2229 \\ 1.3083 & 1.2223 & 1.4568 \\ 2.1476 & 2.2102 & 2.4447 \end{pmatrix}$ ,

$C = \begin{pmatrix} 2.1342 & 3.0594 & 3.0594 \\ 1.6716 & 2.1342 & 2.1342 \\ 3.3431 & 4.2684 & 4.2684 \end{pmatrix}$ ,  $K = \begin{pmatrix} 7.0000 & 4.0000 & 7.0000 \\ 4.0000 & 2.5000 & 4.0000 \\ 5.0000 & 3.5000 & 5.0000 \end{pmatrix}$ . Then, we assume the system

is excited by  $f(t) = (\cos(\sqrt{t}), e^{-\sin(t)}, 0)^\top$  on  $I = [0, 20]$  with the following initial conditions  $x_1(0) = 0.1$ ,  $x_2(0) = -0.2$ ,  $x_3(0) = 0.3$ ,  $\dot{x}_1(0) = -0.4$ ,  $\dot{x}_2(0) = 0.7$  and  $\dot{x}_3(0) = 0.6$ . Moreover, **in order to better show the robustness of the proposed method**, we assume there exist perturbations in the system:  $M\ddot{x}(t) + C\dot{x}(t) + Kx(t) + d(t) = f(t)$  with  $d(t) = (0.3(d_1(t) + 1), -0.23(d_2(t) + 1), 0.42(d_3(t) + 1))^\top$ , and the accelerations  $\ddot{x}_1, \ddot{x}_2, \ddot{x}_3$  are measured by discrete noisy observations:  $y_j^\varpi(t_i) = y_j(t_i) + \sigma_j\varpi_j(t_i)$  for  $j = 1, 2, 3$ , with  $t_i = iT_s$  and  $T_s = 0.0005$ , where  $y$  is the output of the system with perturbations. **The perturbations  $d_j$  and the noises  $\varpi_j$  for  $j = 1, 2, 3$  are simulated using zero-mean white Gaussian noises**, and the value of  $\sigma_j$  is adjusted such that the Signal-to-Noise Ratio (SNR) [35] is equal to 20 dB, 15 dB, and 20 dB, respectively. **Remark that a constant term is considered in each perturbation in  $d(t)$ , which can not be estimated only using the acceleration measurements. Moreover, there is no bias term considered in the acceleration measurements in this paper, the reader can see [6] for this study.**

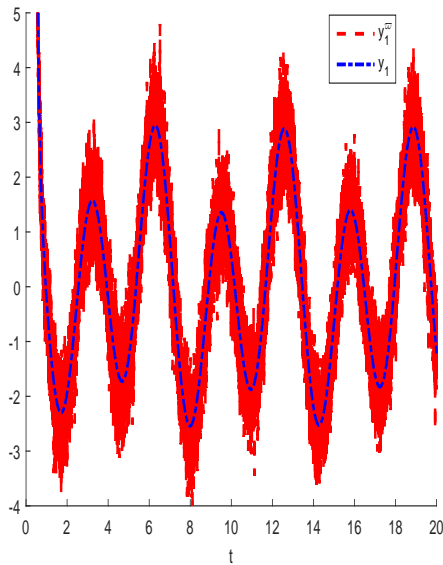
It can be calculated that  $\text{rank}(K) = 2$  with  $P = \begin{pmatrix} 1 & 0 & -1 \\ -4 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$ ,  $Q = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$  and  $\bar{C}_{l,r}$  is invertible. Hence, the algebraic integral formulas obtained in (15)-(16) are applied to estimate  $\dot{x}_1, \dot{x}_2$  and  $\dot{x}_3$  from the discrete noisy output observation  $y^\varpi$ , where the trapezoidal numerical integration method is used. Then, using a similar way as done in Example 1, the required modulating functions are constructed using Corollary 1, and by using the error bound given in (41), the following parameter values are selected:

1. for  $\dot{x}_1$ , when  $t \in ]T_s, 5]$ ,  $m = 2$  and  $T_{1,0} = iT_s$ , then  $m = 3$  and  $T_{1,0} = 5$ ,
2. for  $\dot{x}_2$ , when  $t \in ]T_s, 5]$ ,  $m = 2$  and  $T_{2,0} = iT_s$ , then  $m = 2$  and  $T_{2,0} = 5$ ,
3. for  $\dot{x}_3$ , when  $t \in ]T_s, 5]$ ,  $m = 2$  and  $T_{2,0} = iT_s$ , then  $m = 2$  and  $T_{2,0} = 5$ .

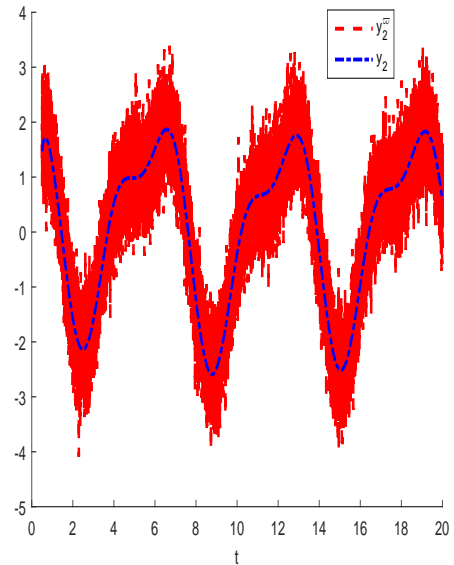
Finally, the noisy outputs and the estimated velocities are shown in Fig. 7 and Fig. 8, respectively.

## 5. Conclusions

Motivated by a machine tool-foundation mechanical system, the modulating functions method recently generalized for state and derivative estimation was adopted in this paper to fast and robustly estimate the positions and velocities from the measured noisy accelerations for a class of dynamical systems modelled by a set of second order linear differential equations. First, after transforming the considered equations, the generalized modulating functions method was applied to obtain algebraic integral formulas for the positions and velocities using a recursive way. These formulas can give fast estimations using sliding integration windows. Second, a general computation scheme was provided to construct the required generalized modulating functions. In particular, modulating functions of polynomial type were constructed, which depend on two design parameters including the length of the sliding integration window. Third, an error analysis was given to show the property of robustness of the proposed method. Moreover, an error bound was introduced to guide the parameter selection in order to minimize the noise error contributions. Finally, the accuracy and robustness of the proposed method was illustrated in numerical simulations. In this paper, a linear system model was considered to introduce the proposed method by giving the strategy of constructing the required



(a)  $y_1$  and its discrete noisy observation.



(b)  $y_2$  and its discrete noisy observation.

Figure 2: Example 1 : output  $y$  in discrete noisy case.

modulating functions and selecting the design parameters. The non-linear case will be studied in a future work.

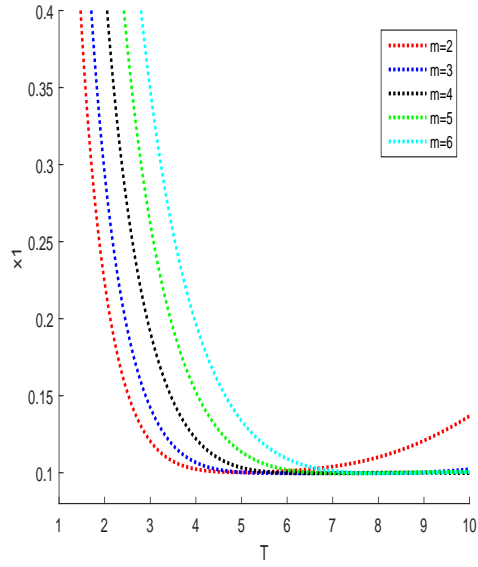
## Acknowledgements

This work was supported by the Natural Science Foundation of Liaoning Province, China (20170540431). The authors gratefully acknowledge the anonymous reviewers for their careful reading and useful comments that allowed to improve the quality of this paper.

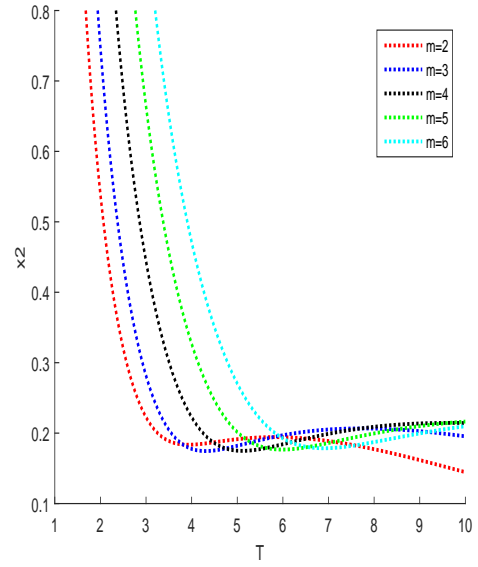
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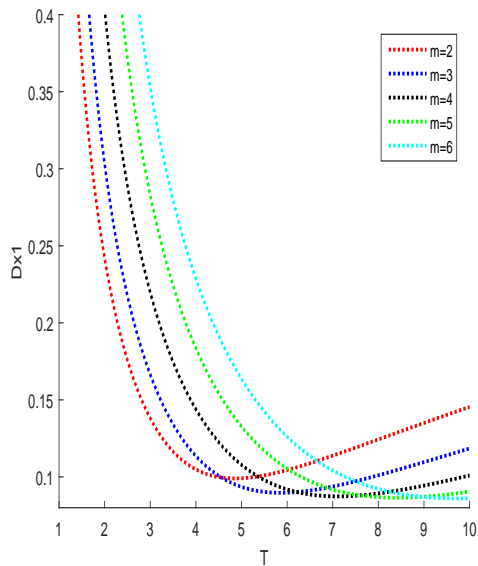




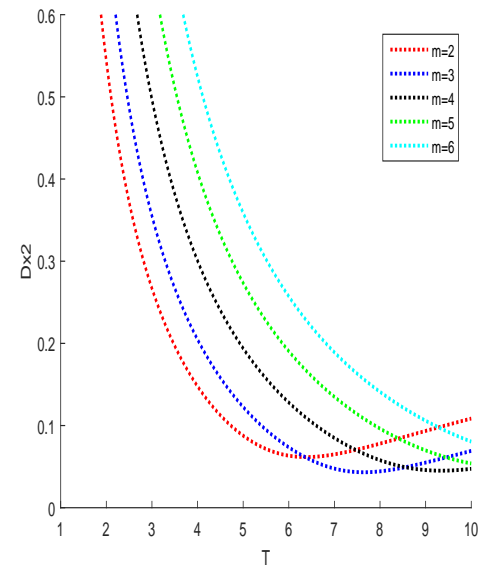
(a) Parameter selection for  $x_1$ .



(b) Parameter selection for  $x_2$ .

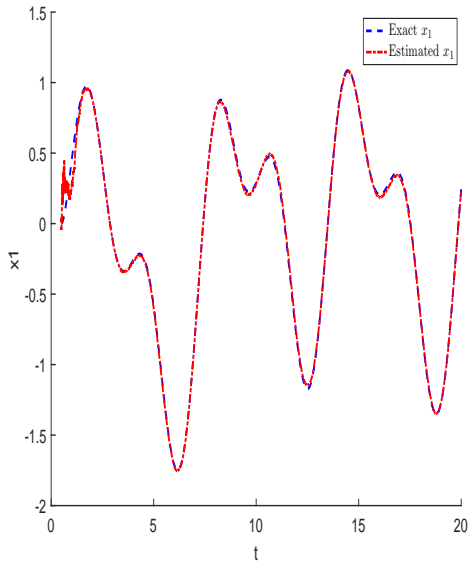


(c) Parameter selection for  $\hat{x}_1$ .

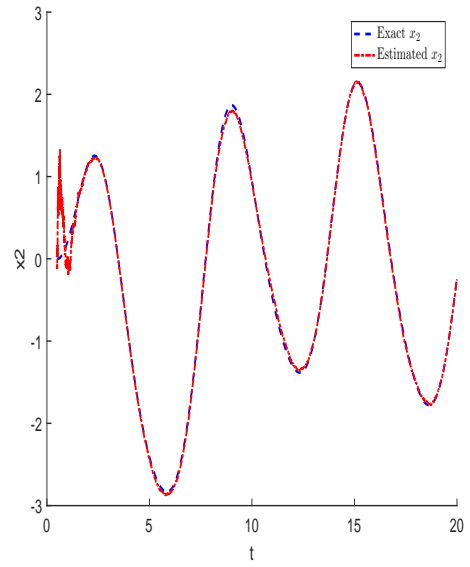


(d) Parameter selection for  $\hat{x}_2$ .

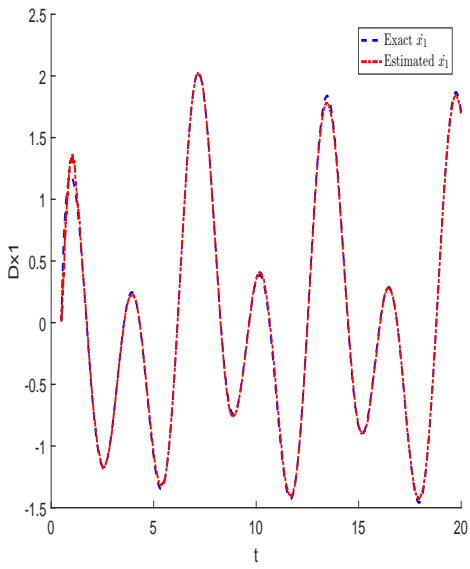
Figure 3: Noise error bounds with respect to different values of  $m$  and  $T_{i,d}$ .



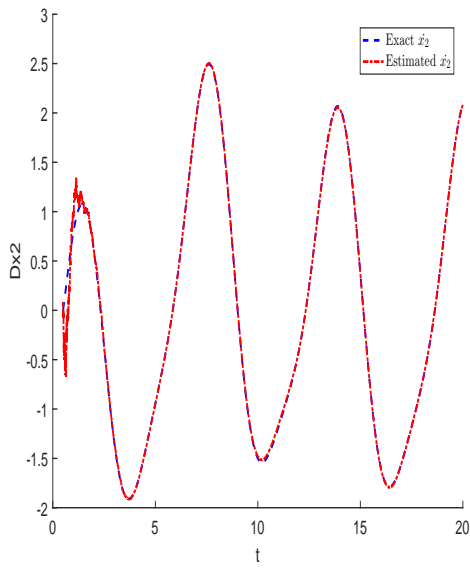
(a)  $x_1$  and its estimation.



(b)  $x_2$  and its estimation.

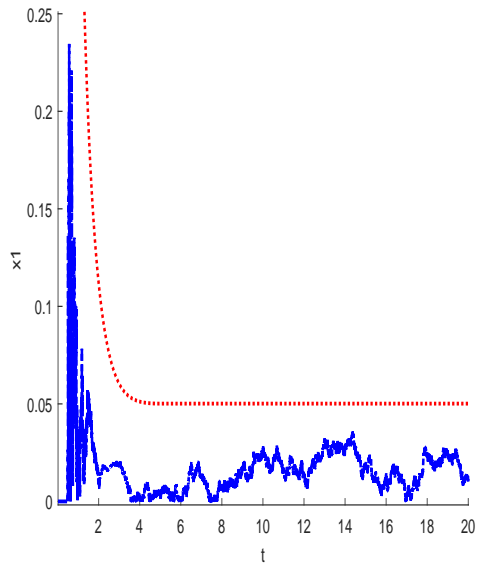


(c)  $\dot{x}_1$  and its estimation.

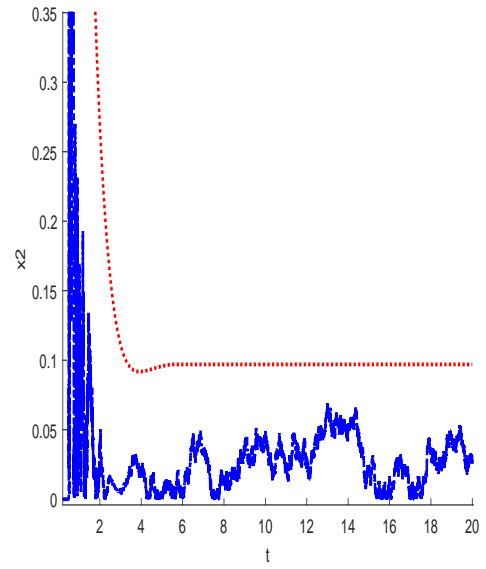


(d)  $\dot{x}_2$  and its estimation.

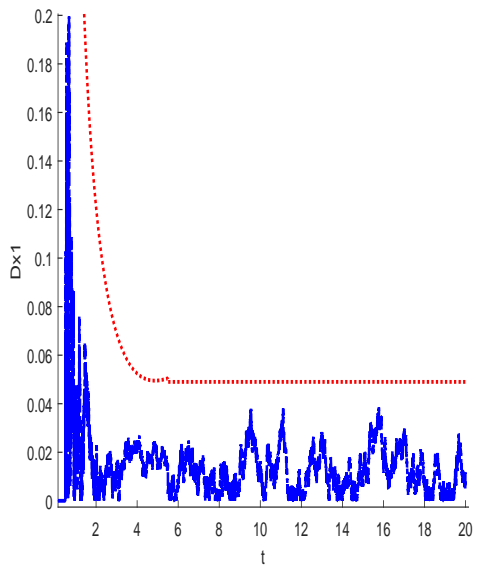
Figure 4: Positions and velocities estimation.



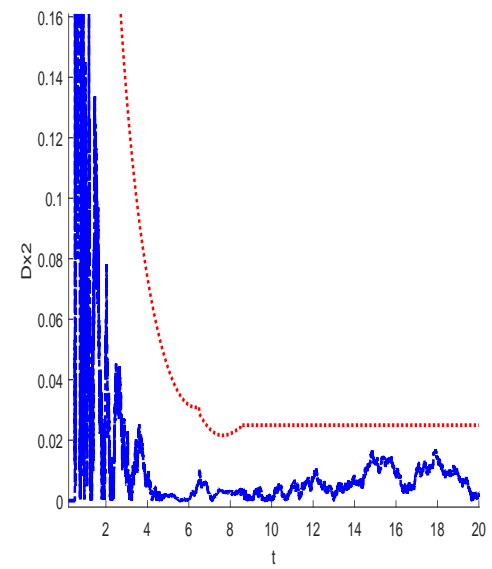
(a) In the case of  $x_1$ .



(b) In the case of  $x_2$ .



(c) In the case of  $\dot{x}_1$ .



(d) In the case of  $\dot{x}_2$ .

Figure 5: Noise error contributions and the corresponding noise errors bounds.

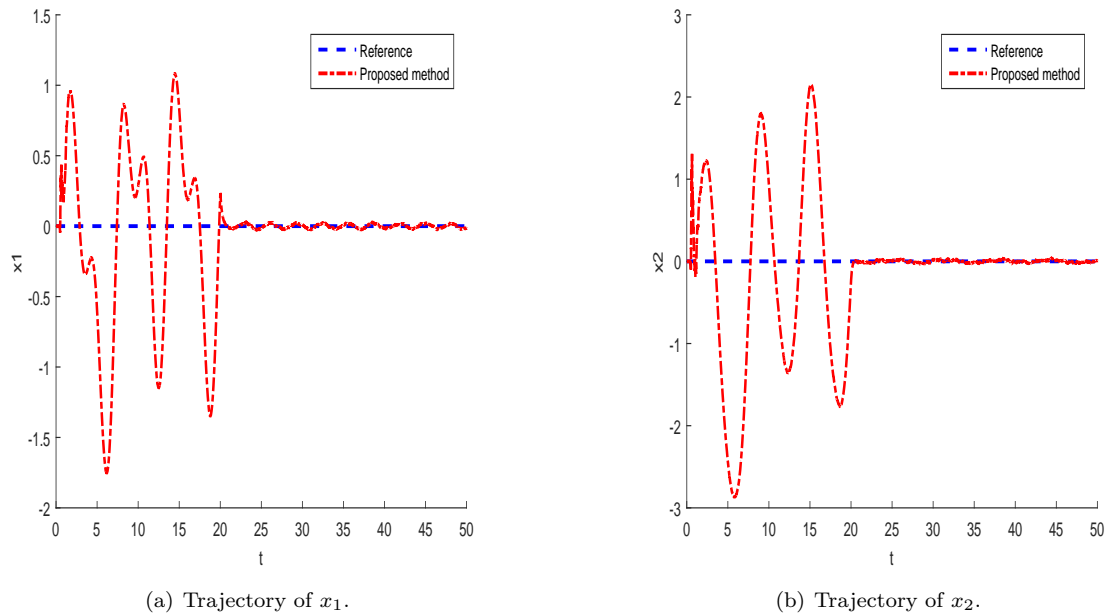
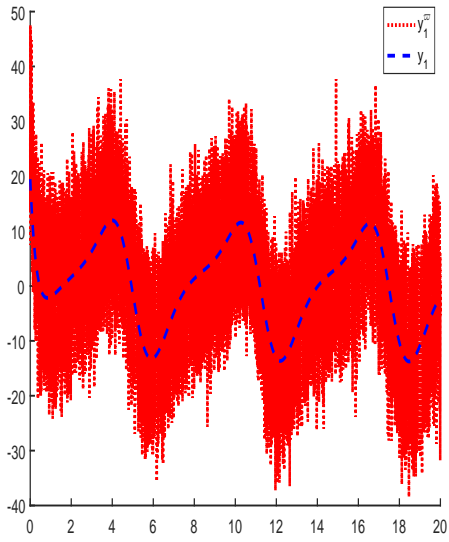
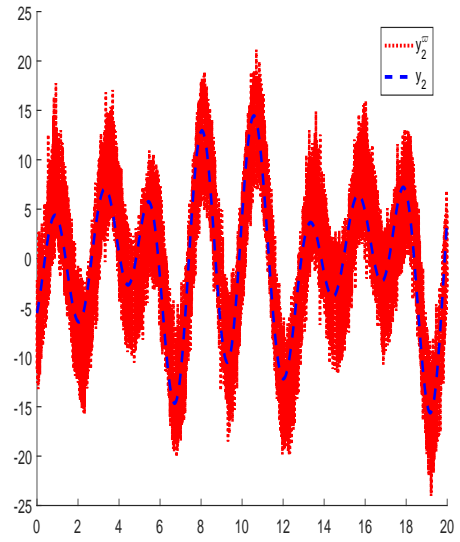


Figure 6: Position trajectories before and after adding a controller.

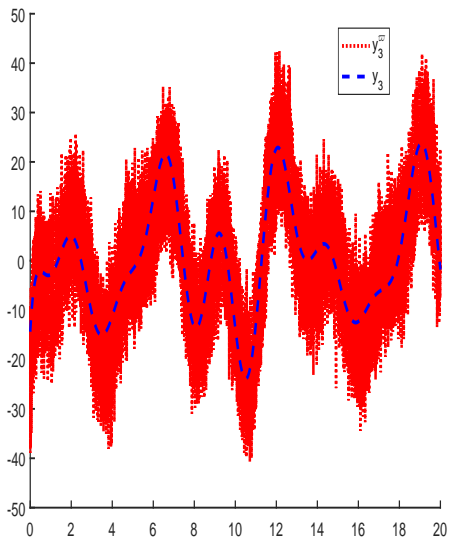
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(a)  $y_1$  and its discrete noisy observation.

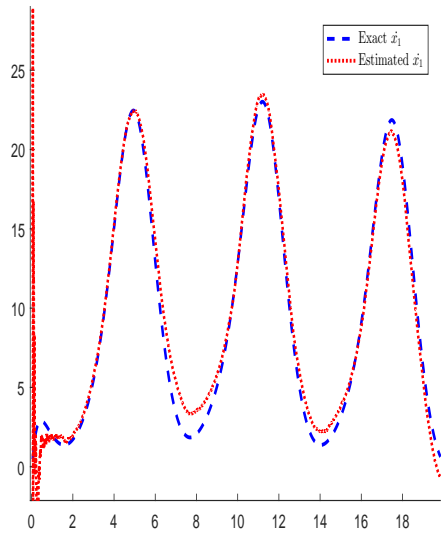


(b)  $y_2$  and its discrete noisy observation.

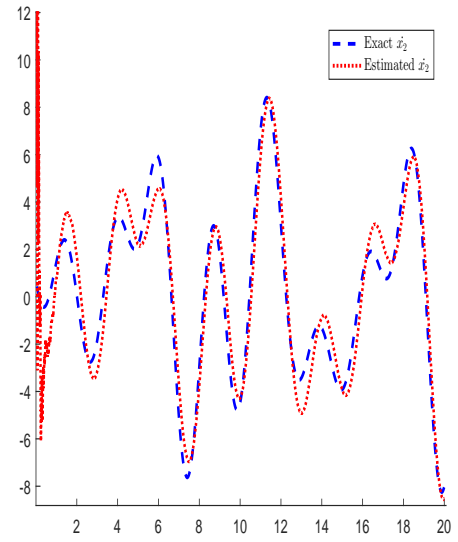


(c)  $y_3$  and its discrete noisy observation.

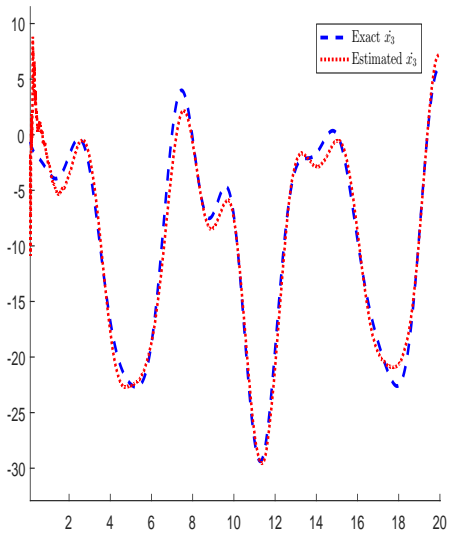
Figure 7: Example 2 : output  $y$  in discrete noisy case.



(a)  $\dot{x}_1$  and its estimation.



(b)  $\dot{x}_2$  and its estimation.



(c)  $\dot{x}_3$  and its estimation.

Figure 8: Velocities estimations.

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