

# Non arithmetic affine invariant orbifolds in $\mathcal{H}^{odd}(2, 2)$ and $\mathcal{H}(3, 1)$

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## Abstract

We classify the non arithmetic rank one affine invariant orbifolds that do not arise from Veech surfaces in  $\mathcal{H}(3, 1)$  and  $\mathcal{H}^{odd}(2, 2)$ . We also give rigidity results on the isoperiodic leaf of non arithmetic Veech surfaces.

## 1 Introduction

### 1.1 Context and results

The moduli space of pairs  $(X, \omega)$  where  $X$  is a genus  $g \geq 1$  Riemann surface and  $\omega$  is a non vanishing holomorphic 1-form on  $X$  carries a natural action by  $GL_2^+(\mathbb{R})$  which is a generalization of the action of  $GL_2^+(\mathbb{R})$  on the space of flat tori  $GL_2^+(\mathbb{R})/SL(2, \mathbb{Z})$ . This action preserves the stratification of the moduli space induced by the combinatorics of the singularities, and the classification of the closed invariant sets of the strata is a central problem in Teichmüller dynamics. Such a classification in genus 2 has been initiated by Calta in [Cal04] and by McMullen in [McM07a]. In particular, the latter proved that the orbit of a surface  $(X, \omega)$  is either closed, dense, or contained in a locus of surfaces whose jacobian have a special property called "Real multiplication by a quadratic order", with  $\omega$  being an eigenform. See section 2 for more details. McMullen also discovered a generalization of those objects in higher genera and provided an infinite sequence of non trivial closed invariant sets  $\Omega E_D$ , parametrized by their discriminant  $D$ . Those loci, called the Prym eigenform loci, will play a central role in the remainder of this text. The question of the classification of closed orbits in genus 2 has been addressed by Hubert and Lelièvre in [HL04], then later by Duryev in [Dur18], but is still incomplete. Hubert, Lanneau and Möller have computed the orbit closure of many hyperelliptic surfaces in  $\mathcal{H}(2, 2)^{odd}$  in [HLM12]. Since, much effort has been made toward a classification in higher genera. In a very celebrated result, Eskin, Mirzakhani and Mohammadi have proved a structural result on the closed invariant sets: they are immersed orbifolds cut out by linear equations with real coefficients. See definition 2.1. Such objects are called affine invariant orbifolds. Wright

strengthened the conclusion of this result and proved that the coefficients of equations defining the orbifold belongs to a number field whose degree is bounded above by the genus. See [Wri14]. This number field will be referred to as the field of definition. These results opened the way to new powerful tools, and has been the starting point of every classification result obtained so far. Wright also introduced an important numerical invariant called the rank  $rk(\mathcal{M})$ . This is a modified version of the dimension and it measures the size of affine invariant orbifolds up to isoperiodic deformations. See [Wri15] for more details. Mirzakhani conjectured that arithmetic affine invariant orbifolds whose rank is bigger than 2 should arise from covering constructions over quadratic differentials. Arithmetic means here that the field of definition is  $\mathbb{Q}$ . This conjecture is now known to be false due to work of Eskin, McMullen, Mukamel and Wright in [EMMW18] but counterexamples are expected to be rare. Mirzakhani and Wright proved in [MW18] that the only affine invariant orbifolds of maximal rank are the strata themselves and the hyperelliptic locus of those strata. Then, Apisa proved in [Api17] and [Api18] that the orbits of translation surfaces in the hyperelliptic strata are either closed, dense, or contained in loci of branched covers. Finally, a classification of affine invariant orbifolds with  $rk(\mathcal{M}) \geq 2$  in genus 3 has been obtained by Nguyen and Aulicino in [AN16b] and [AN16a]. In this paper, we pursue the classification in genus 3, and prove :

**Theorem A.** *Let  $\mathcal{M}$  be a proper non arithmetic affine invariant orbifold in  $\mathcal{H}^{odd}(2, 2)$ . Then  $k(\mathcal{M})$  is a totally real quadratic number field of discriminant  $D$  and  $\mathcal{M}$  is a connected component of  $\Omega E_D^{odd}(2, 2)$ .*

Here, an affine invariant orbifold is said to be proper if it is either a closed orbit or the whole statum, and non arithmetic means that  $k(\mathcal{M})$  is strictly bigger than  $\mathbb{Q}$ . The sets  $\Omega E_D^{odd}(2, 2)$  are the intersection of the Prym eigenform loci of McMullen with the connected component  $\mathcal{H}^{odd}(2, 2)$ . The connected components of  $\Omega E_D^{odd}(2, 2)$  have been classified by Lanneau and Nguyen in [LN14]. In particular, this gives an alternative proof in this setting of the fact that  $k(\mathcal{M})$  is totally real quadratic extension of  $\mathbb{Q}$ . The fact that the field of definition is a totally real extension of  $\mathbb{Q}$  is due to Filip (Theorem 1.6 in [Fil16]). In the stratum  $\mathcal{H}(3, 1)$ , the situation is different and we prove :

**Theorem B.** *There are no proper non arithmetic affine invariant orbifold in  $\mathcal{H}(3, 1)$ .*

Notice that to establish theorem A and B, we only need to consider rank one affine invariant orbifolds as lemma 6.5 of [Wri14] implies that if  $\mathcal{M}$  is non arithmetic, then  $rk(\mathcal{M}) = 1$ .

## 1.2 Outline of the proof

The techniques we use rely on deformations, known as cylinder deformations, of the flat geometry of translation surfaces initiated by Wright in [Wri15]. For a horizontally periodic translation surface  $(X, \omega)$ , the horocycle flow (the action of the one parameter subgroup of unipotent matrices in  $SL_2(\mathbb{R})$ ) and the Rel flow (see section 3 for more details) read as

linear flows on a torus, and their orbit closures are thus given by rational subtori. When  $(X, \omega)$  is contained in a proper affine invariant orbifold  $\mathcal{M}$ , the equations defining those tori are linked to the equations defining the affine invariant orbifold  $\mathcal{M}$ . The non arithmeticity assumption implies that those tori have a non trivial intersection and thus their defining equations are not independent. The equations given by the Rel flow only involve the circumferences of the cylinders while the equations induced by the horocycle flow are expressed with the circumferences and the height of the cylinders. As a consequence one would expect equations on the heights of the cylinders. This is the object of proposition 3.2. To implement this strategy, we crucially need the existence of cylinder decompositions. This is obtained through the complete periodicity property that prevails in rank 1. More details can be found in section 3. However, this strategy fails if there are what we call non mixed cylinders: those are cylinders not affected by the isoperiodic deformations. The circumferences of such cylinders therefore do not intervene in the equations obtained. In appendix A, we give a list of the cylinder decompositions in the strata at stakes. This allows us to show that there is always a decomposition with only mixed cylinders in  $\mathcal{H}(2, 2)$ . We then look at how the equations on the heights given by proposition 3.2 transpose for the surfaces in this list: they rule out all the surfaces except one. We repeatedly apply the same argument in different direction keeping in mind that equations on the heights in one direction give equations on the circumferences in other directions. After having collected all the equations obtained, we then show that the surface has a prym involution. We conclude with a criterion given by McMullen to recognizes Prym eigenforms. For the stratum  $\mathcal{H}(3, 1)$ , the situation is different and we cannot rule out the existence of non mixed cylinders. We show that this implies the commensurability of the mixed cylinders in a way that is not compatible with non arithmeticity.

### 1.3 Organisation of the paper

We start by recalling basic definitions for the moduli space of translation surfaces in Section 2. In section 3 we collect the important results we will use in the course of our proofs. Section 4 is dedicated to the proof of Theorem A. We also draw a corollary on the leaf of Veech surfaces in  $\mathcal{H}^{odd}(2, 2)$ . Section 5 is dedicated to the proof Theorem B, and we draw the same corollary as in the previous section in the stratum  $\mathcal{H}(3, 1)$ . Finally, Appendix A provides a list of the possible stable cylinder decompositions in  $\mathcal{H}(2, 2)$  and  $\mathcal{H}(3, 1)$ .

### 1.4 Acknowledgements

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## 2 Framework

The stratum  $\tilde{\mathcal{H}}(\kappa)$  of the Teichmüller space of translation surfaces is the set of isomorphism classes of marked translation surfaces  $(X, \omega, f)$ , where  $f : S \rightarrow X$  is a homeomorphism from a fixed genus  $g$  surface  $S$  such that the preimage of the singularities of  $\omega$  by  $f$  is a subset  $\Sigma$ , and the order of the singularities of  $\omega$  are prescribed by  $\kappa$ . The following map is known as the period map :

$$\Phi : \begin{array}{ccc} \tilde{\mathcal{H}}(\kappa) & \rightarrow & H^1(S, \Sigma, \mathbb{C}) \\ (X, \omega, f) & \mapsto & (\gamma \mapsto \int_{f \circ \gamma} \omega) \end{array}$$

There is a complex structure on  $\tilde{\mathcal{H}}(\kappa)$  that turns  $\Phi$  into a local biholomorphism, and if  $MCG(S, \Sigma)$  denotes the relative mapping class group of  $S$  that fixes globally  $\Sigma$ , then  $MCG(S, \Sigma)$  acts almost freely on  $\tilde{\mathcal{H}}(\kappa)$  by precomposition:  $\varphi \cdot (X, \omega, f) = (X, \omega, f \circ \varphi^{-1})$ . The quotient set is isomorphic to  $\mathcal{H}(\kappa)$  and the latter is endowed with the complex orbifold structure that turns the canonical projection  $\pi : \tilde{\mathcal{H}}(\kappa) \rightarrow \mathcal{H}(\kappa)$  into a local biholomorphism (in the orbifold sense, see [CJ19] for relevant definitions). The space  $\tilde{\mathcal{H}}(\kappa)$  is endowed with a group action by  $GL_2^+(\mathbb{R})$  defined by :

$$\forall g \in GL_2^+(\mathbb{R}) \quad \Phi(g \cdot (X, \omega, f)) = g \cdot \Phi(X, \omega, f)$$

That action descends to an action on  $\mathcal{H}(\kappa)$  in a way that the canonical projection  $\pi$  is  $GL_2^+(\mathbb{R})$ -equivariant. The action of the subgroup of diagonal matrices with determinant 1 is known as the Teichmüller geodesic flow, while the action of the subgroup of upper triangular matrices with only 1 on the diagonal is known as the horocycle flow. More details on the structures of these spaces can be found in [Zor06] or [FM13].

**Definition 2.1** (affine invariant orbifold). *An affine invariant orbifold is a closed connected immersed orbifold  $\iota : \mathcal{M} \rightarrow \mathcal{H}(\kappa)$  that satisfies the following property: if  $x \in \mathcal{M}$ , there is an open set  $\mathcal{U}$  around  $x$ , an orbifold chart  $(\mathcal{V}, \varphi)$  around  $\iota(x)$  and a  $\mathbb{R}$ -linear vector subspace  $V$  of  $H^1(S, \Sigma, \mathbb{R})$  such that:*

$$\iota(\mathcal{U}) \cap \varphi(\mathcal{V}) = \varphi(\mathcal{V} \cap V \otimes \mathbb{C}) \tag{1}$$

*Remark.* In the previous definition, immersion refers to an immersion of orbifold, See [CJ19] for relevant definitions. Most of the time, we identify  $\mathcal{M}$  with its image in  $\mathcal{H}(\kappa)$ .

Affine invariant orbifolds are invariant under the action of  $GL_2^+(\mathbb{R})$  and Eskin, Mirzakhani and Mohammadi proved in a celebrated result that the converse is true. More details can be found in [EMM15]. An important numerical invariant associated to these loci is the rank, defined as follows: define  $\rho : H^1(S, \Sigma, \mathbb{C}) \rightarrow H^1(S, \mathbb{C})$  to be the canonical restriction map, and for any  $V$  as in (1) of Avila, Eskin and Möller proved in [AEM17] that  $\rho(V)$  is a symplectic subspace of  $H_1(S)$ . The rank of  $\mathcal{M}$ , denoted  $rk(\mathcal{M})$ , is then defined as

half the dimension of this space. Notice that this dimension is constant by connectedness. More details can be found in [Wri15]. The following definition will be important for the remainder of this text :

**Definition 2.2.** *Let  $\mathcal{M}$  be a affine invariant orbifold. The field of definition of  $\mathcal{M}$  is the smallest subfield  $k(\mathcal{M})$  of  $\mathbb{R}$  such that any  $\mathcal{M}$  as in (1) can be written as  $V = V_0 \otimes_{k(\mathcal{M})} \mathbb{R}$ , where  $V_0$  is a  $k(\mathcal{M})$ -linear subspace of  $H^1(S, \Sigma, k(\mathcal{M}))$ .*

A particularly interesting family of rank 1 affine invariant orbifolds has been discovered by McMullen in [McM07b]. We recall here the definition. Let  $(X, \omega)$  be a translation surface endowed with a holomorphic involution  $\tau$ . We denote by  $\Omega(X)$  the set of holomorphic 1-forms, and by  $\Omega^-(X)$  the set of  $\tau$ -anti invariant holomorphic 1-forms. We say that  $(X, \omega)$  is a Prym form if  $\omega \in \Omega^-(X)$ , that is  $\tau^*\omega = -\omega$ , and  $\dim \Omega^-(X) = 2$ . The Prym variety  $\text{Prym}(X, \tau)$  is defined as the 2-dimensional abelian variety  $(\Omega^-(X))^*/H_1^-(X, \mathbb{Z})$  endowed with the polarization coming from the intersection form on  $H_1(X, \mathbb{Z})$ . Finally, let  $D$  be a positive integer congruent to 0 or 1 mod 4 and let  $\mathcal{O}_D \simeq \mathbb{Z}[X]/(X^2 + bX + c)$  be the real quadratic order of discriminant  $D = b^2 - 4c$ .

**Definition 2.3.** *A Prym eigenform is a Prym form  $(X, \omega)$  corresponding to an involution  $\tau$  such that there is an injective ring morphism  $\mathfrak{i} : \mathcal{O}_D \rightarrow \text{End}(\text{Prym}(X, \tau))$  satisfying the following properties:*

1.  $\mathfrak{i}(\mathcal{O}_D)$  is a proper self adjoint subring of  $\text{End}(\text{Prym}(X, \tau))$ .
2.  $\omega$  is an eigenvector for the action of  $\mathcal{O}_D$  on  $\Omega(X)^-$

We will denote by  $\Omega E_D(\kappa)$  the set Prym eigenforms as in the previous definition contained in the stratum  $\mathcal{H}(\kappa)$ . For more details, see [McM07b]. The following important result is due to McMullen

**Theorem 2.1.** *Any connected component of  $\Omega E_D(\kappa)$  is a rank 1 invariant orbifold.*

In the component  $\mathcal{H}^{odd}(2, 2)$ , those loci are proper and Theorem A states that those are the only rank 1 invariant orbifolds. Finally, note that  $\Omega E_D(3, 1)$  is empty. Indeed if  $(X, \omega)$  is Prym form in  $\mathcal{H}(3, 1)$  corresponding to a involution  $\tau$ , then  $\omega^2$  is invariant by  $\tau$  and thus descends to a quadratic form on the torus whose singularities counted with multiplicity do not add up to 0, as they should according to Riemann-Roch theorem.

### 3 Preparation of a toolkit

In this section, we collect the tools we will use in the proof of the theorems A and B.

### 3.1 Isoperiodic foliation and Rel flow

The stratum  $\mathcal{H}(\kappa)$  is endowed with a foliation which we describe in this section. It is usually referred to as the isoperiodic foliation or the kernel foliation. Since  $\Phi$  is a local biholomorphism and  $\rho$  is a surjective linear map, the composition  $\rho \circ \Phi$  is a submersion. The connected components of the level sets is thus a foliation of  $\tilde{\mathcal{H}}(\kappa)$  whose plaques are modeled on  $\text{Ker}\rho$ . The group  $MCG(S, \Sigma)$  acts on both  $\tilde{\mathcal{H}}(\kappa)$  by precomposition on the marking and on  $H^1(S, \mathbb{C})$  by the Torelli representation. With respect to those actions, the map  $\rho \circ \Phi$  is  $MCG(S, \Sigma)$ -equivariant. This means that the aforementioned foliation on  $\tilde{\mathcal{H}}(\kappa)$  descend to a foliation on  $\mathcal{H}(\kappa)$ . This foliation is usually referred to as the isoperiodic foliation, the kernel foliation or the Rel foliation.

Let us define a local flow on the leaves of the isoperiodic foliation. Let  $\sigma_i$  be a family of paths in  $S$  that link a fixed singularity to all the others and define the following positive quadratic form:

$$\mathcal{Q} : \begin{array}{ccc} H^1(S, \Sigma, \mathbb{C}) & \rightarrow & \mathbb{C} \\ \xi & \mapsto & \sum \xi(\sigma_i) \overline{\xi(\sigma_i)} \end{array}$$

Its restriction to  $\text{Ker}(\rho)$  is definite and preserved by the action of  $MCG(S, \Sigma)$ . This last claim is due to the fact that  $MCG(S, \Sigma)$  acts on  $\Sigma$  by permutation and thus, up to absolute cycles,  $MCG(S, \Sigma)$  only permutes the  $\sigma_i$  while, perhaps, reversing their orientation. All this means that there is a  $MCG(S, \Sigma)$ -invariant hermitian inner product on  $\text{Ker}\rho$  and thus its real part induces a flat metric on the leaves of  $\mathcal{F}$ . Denote by  $p$  the canonical projection  $T\mathcal{H}(\kappa) \rightarrow \mathcal{H}(\kappa)$  and if  $u$  is an element of  $T_{(X, \omega)}\mathcal{F}$  and  $t \in \mathbb{R}$  is small enough, then define  $\text{Rel}_u^t(X, \omega)$  to be the image by  $p$  of the point obtained in  $T\mathcal{H}(\kappa)$  after flowing along the geodesic flow for a time  $t$  starting at  $((X, \omega); u)$ . Note that this might not be defined for all  $t$  as singularities might collide along the trajectories. This construction is usually referred to as the Rel flow. See [HW15] for more discussions on the isoperiodic foliation and the rel flow where everything is defined using charts.

### 3.2 Modifying the twist parameters

Let  $(X, \omega)$  be a translation surface in  $\mathcal{H}(\kappa)$  that has a cylinder decomposition into  $m$  cylinders, which we denote by  $\mathcal{C}_1, \dots, \mathcal{C}_m$ . One can define a smooth map (in the orbifold sense)  $\tau$  from  $\mathbb{R}^m$  to  $\mathcal{H}(\kappa)$  in the following way: choose  $(x_1, \dots, x_m)$ , cut open the surface along the core curves of its cylinders, rotate the top component of each cylinder  $\mathcal{C}_i$  by an amount of  $x_i c_i$  and glue back. The action of  $\mathbb{Z}^m$  on  $\mathbb{R}^m$  by translation is intertwined by  $\tau$  with the action of the subgroup of  $MCG(S, \Sigma)$  spanned by the Dehn twists about the core curves of the cylinders. There is also a finite subgroup  $\Delta$ , perhaps trivial, of automorphisms of  $(X, \omega)$  that acts on  $X$  by permuting some cylinders and on the  $m$ -dimensional torus  $\mathbb{T}^m$  by permuting the coordinates. See section 6.2 of [HW15] for more details. Consequently, there is an induced map  $\tau : \mathbb{T}^m / \Delta \rightarrow \mathcal{H}(\kappa)$ , and it is a diffeomorphism onto its image.

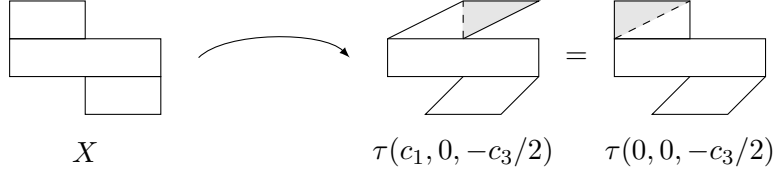


Figure 1: twisting cylinders

From now on and until the end of this section we assume that  $\mathcal{H}(\kappa)$  is a stratum of surfaces with exactly two singularities, and that the decomposition of  $X$  is stable: we mean by that that any component of the boundaries of cylinders contain only one singularity. A cylinder is said to be mixed if the singularity on the top component of the cylinder is not the same as the one on the bottom component. Denote by  $\mathfrak{C}_{mix}$  the family of mixed cylinders of  $(X, \omega)$ . There is an equivalence relation on  $\mathfrak{C}_{mix}$ , defined as follows. Denote by  $\partial$  the boundary map from  $H_1(X, \Sigma)$  to  $H_1(\Sigma)$  and choose for each cylinder  $\mathcal{C}_i$  a cross section  $\beta_i$ , that is a path inside  $\mathcal{C}_i$  connecting the singularity on the bottom component of the cylinder to the singularity on the top component. Since we assumed that there are only two singularities, the image of  $\partial$  is one dimensional. Thus  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are equivalent if  $\partial(\beta_1)$  is positively colinear to  $\partial(\beta_2)$ . There are two equivalence classes which we denote by  $\mathfrak{C}^+$  and  $\mathfrak{C}^-$ . Note that the cylinders that are not mixed are exactly the ones whose cross sections are sent to 0 by the map  $\partial$ . Let  $\delta_i$  be 1 if  $i \in \mathfrak{C}^+$ ,  $-1$  if  $i \in \mathfrak{C}^-$  and 0 if  $i$  does not correspond to a mixed cylinder. For a given cylinder  $\mathcal{C}_i$ , we denote by  $h_i$  its height,  $c_i$  its circumference,  $\gamma_i = c_i^{-1}$ , and  $\mu_i = h_i \gamma_i$ , its modulus. Finally, let  $\mu := (\mu_1, \dots, \mu_m)$ ,  $u = (\delta_1 \gamma_1, \dots, \delta_m \gamma_m)$ ,  $V(X) = \text{span}\langle \mu, u \rangle$  and let  $d$  denote the algebraic degree of the  $\delta_i \gamma_i$  (we mean the dimension of the  $\mathbb{Q}$ -vector space spanned by these numbers). The following lemma is fundamental for the remainder of this text.

**Lemma 3.1.** *If  $\mathcal{M}$  is a proper rank 1 affine invariant orbifold and  $(X, \omega)$  is a horizontally periodic surface in  $\mathcal{M}$ , then  $V(X)$  is rational. In particular,  $d \leq 2$ .*

*Proof.* First, note that the assumption that  $\mathcal{M}$  is a proper rank 1 affine invariant orbifold together with the fact that there are only two singularities implies that  $\mathcal{M}$  is saturated by the isoperiodic foliation. Indeed, let  $(X, \omega)$  be a surface in  $\mathcal{M}$  and chose a chart  $\varphi$  defined on a neighborhood  $\mathcal{U}$  of  $\xi$  in  $H^1(S, \Sigma, \mathbb{C})$  such that  $\varphi(\xi) = (X, \omega)$ . By definition 2.1, there is a linear subspace  $V \in H^1(S, \Sigma, \mathbb{R})$  such that  $\varphi(V \otimes \mathbb{C} \cap \mathcal{U})$  is contained in  $\mathcal{M}$ . The assumption that  $\mathcal{M}$  is a proper orbifold means that the dimension of  $V$  is at least 3, and the dimension on the rank implies that  $\rho(V)$  has dimension 2. Since there are only two singularities, the kernel of  $\rho$  has dimension 1, and thus the dimension of  $V$  is exactly 3 and  $\text{Ker} \rho$  is contained in  $V \otimes \mathbb{C}$ . But by definition,  $\varphi(\xi + \text{Ker} \rho \cap \mathcal{U}) = \varphi(\mathcal{U}) \cap \mathcal{F}_X$ . This proves our claim.

Now, denote by  $q : \mathbb{R}^m \rightarrow \mathbb{T}^m/\Delta$  the canonical projection and let  $v = t_1\mu + t_2u \in V(X)$ . Notice that by construction,  $t \mapsto \tau \circ q(t \cdot u)$  is a path in the isoperiodic leaf of  $(X, \omega)$ . Denoting  $u_0 = d_0\tau(u) \in T_{(X, \omega)}\mathcal{F}$ , one gets:

$$\tau(v) = \begin{pmatrix} 1 & t_1 \\ 0 & 1 \end{pmatrix} \cdot \text{Rel}_u^{t_2}(X, \omega)$$

Thus  $\tau \circ q(v)$  is contained in  $\mathcal{M}$  as the latter is invariant under the action of  $SL_2(\mathbb{R})$  and saturated by the isoperiodic foliation. Let  $V_0$  be the smallest rational linear subspace that contains  $V(X)$ . It is a classical result that  $q(V(X))$  is dense in  $q(V_0)$ . see for instance page 33 of [KH95]. Since  $\mathcal{M}$  is closed, we deduce that  $\tau \circ q(V_0)$  is contained in  $\mathcal{M}$ . As a consequence  $d_0\tau(V_0)$  is contained in  $V$  and its image by  $\rho$  is an isotropic subspace of  $\rho(V)$  and hence has dimension 1. The isotropic claim is a consequence of the fact that the twist map does not change the period of the core curves of the cylinders. Since  $\tau$  is a diffeomorphism onto its image  $d_0\tau(V_0)$  has dimension at least 2 and then exactly 2 by what precedes. We conclude that  $V_0 = V(X)$ . For the second assertion, it is known that  $d$  is the dimension of the smallest rational linear subspace that contains  $u$ . By the first claim, we deduce that  $d \leq 2$ .  $\square$

**Proposition 3.1.** *If  $d = 2$ , any rational relation satisfied by the  $\delta_i\gamma_i$  are also satisfied by the coordinates of the elements of  $V(X)$ .*

*Proof.* By lemma 3.1, the smallest rational linear space that contains  $u$  is  $V(X)$ . In particular, any rational relation satisfied by the coordinates of  $u$  has to be satisfied by the coordinates of any vector in  $V(X)$ .  $\square$

**Corollary 3.1.** *If the horizontal decomposition of  $X$  is not mixed, then all the mixed cylinders have commensurable circumferences.*

*Proof.* Let  $\mathcal{C}_i$  be a non mixed cylinder and suppose there are mixed cylinders with non commensurable circumferences. Then  $d = 2$  and  $\delta_i\gamma_i = 0$  is a rational relation satisfied by the coordinates of  $u$ . By proposition 3.1, this equation must be satisfied by the coordinates of  $\mu$ . This is a contradiction as moduli do not vanish.  $\square$

**Proposition 3.2.** *If  $d = 2$ , two non equivalent cylinders can not have commensurable circumferences.*

*Proof.* Suppose to a contradiction that two non equivalent cylinders  $\mathcal{C}_1$  and  $\mathcal{C}_2$  have commensurable circumferences, and say  $\mathcal{C}_1$  belongs to  $\mathfrak{C}^+$ . There is thus a positive rational number  $q$  such that  $\gamma_1 = q\gamma_2 = -q\delta_2\gamma_2$ . By proposition 3.1, it implies that  $\mu_1 = -q\mu_2$ . This is a contradiction as moduli are positive numbers.  $\square$

**Corollary 3.2.** *If  $d = 2$ , any pair of mixed cylinders with commensurable circumferences have same height. Reciprocally, if two equivalent cylinders have same height, then their circumferences are commensurable.*



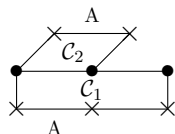
*Proof.* For the first assertion, suppose  $d = 2$  and let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be two mixed cylinders with commensurable circumferences. Proposition 3.2 implies that these cylinders are equivalent. Take a rational number  $q$  such that  $\gamma_1 = q\gamma_2$ . By proposition 3.1, it implies that  $\mu_1 = q\mu_2 = \frac{\gamma_1}{\gamma_2}\mu_2$ . This equation is exactly  $h_1 = h_2$ .

For the second assertion, notice that the projection of  $V(X)$  on the 2-dimensional torus corresponding to the coordinates of  $\mathcal{C}_1$  and  $\mathcal{C}_2$  is a line. This is due to the fact that  $V(X)$  is spanned by  $u$  and  $\mu$  and that the coordinates corresponding to  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are proportional, with ratio given by the common height.  $\square$

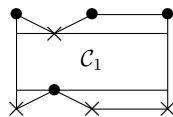
Now, we say that two cylinders are adjacent if they share a saddle connection on their boundaries, and that they are 2-adjacent if they share two saddle connections, one on each boundary. Finally, denote by  $\mathfrak{C}_0^\pm$  the collection of cylinders whose height is  $\min\{h_C \mid C \in \mathfrak{C}^\pm\}$ . Denote by  $\mathfrak{C}_1^\pm$  the complementary of  $\mathfrak{C}_0^\pm$  in  $\mathfrak{C}^\pm$ . The following proposition appeared first in [Apl17].

**Proposition 3.3.** *Suppose  $d = 2$ . If there is a cylinder in  $\mathfrak{C}_0^+$  that is 2-adjacent to a cylinder in  $\mathfrak{C}_0^-$ , but not adjacent to any other cylinder in  $\mathfrak{C}_1^+$ , then the circumferences of the cylinders in  $\mathfrak{C}_1^+$  are commensurable.*

*Proof.* Suppose  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are as in the statement with  $\mathcal{C}_1$  in  $\mathfrak{C}^+$  and denote by  $h$  the height of the cylinders in  $\mathfrak{C}_0^+$ . Define  $X' = Rel_{i,u}^{h+\varepsilon}(X)$ . Notice that on  $X'$ , the cylinders  $\mathcal{C}_1$  and the cylinders in  $\mathfrak{C}_1^+$  have persisted, but  $\mathcal{C}_1$  is now in  $\mathfrak{C}^-$ , while the cylinders in  $\mathfrak{C}_1^+$  in  $X$  stay in  $\mathfrak{C}^+$  on  $X'$ . This comes from the adjacency conditions. The following picture depicts the deformation of the cylinders 1 and 2.



On the surface  $X$



On the surface  $X'$

Suppose that the circumferences of the cylinders in  $\mathfrak{C}_1^+$  are not commensurable. Then, there are rational numbers  $p_i$  so that  $\gamma_1 = \sum_{i \in \mathfrak{C}_1^+} p_i \gamma_i$ . Now, proposition 3.1 applied to  $X$  and  $X'$  implies the following two equations:

$$\begin{aligned} \mu_1 &= \sum_{i \in \mathfrak{C}_1^+} p_i \mu_i \\ -\mu'_1 &= \sum_{i \in \mathfrak{C}_1^+} p_i \mu'_i \end{aligned}$$

Adding this two equations yields, using the fact  $\mu'_i = \mu_i - \frac{h+\varepsilon}{c_i}$  if  $i \geq 2$ , and  $\mu'_1 = \mu_1 + \frac{h-\varepsilon}{c_1}$  :

$$(\varepsilon - h)\gamma_1 = \mu_1 - \mu'_1 = \sum_{i \in \mathcal{C}_1^+} p_i \delta_i (\mu_i + \mu'_i) = \sum_{i \in \mathcal{C}_1^+} p_i \delta_i (2\mu_i - (h + \varepsilon)\gamma_i) = 2\mu_1 - (h + \varepsilon)\gamma_1$$

This equation simplifies to  $\varepsilon = h_1$ , which is a contradiction as  $\varepsilon$  can be chosen arbitrarily small.  $\square$

### 3.3 The field of definition

In this section we recall a very useful formula for the field of definition of an affine invariant orbifold  $\mathcal{M}$ . It has been proved by Wright in [Wri15].

**Proposition 3.4.** *Let  $(X, \omega)$  be a translation surface in  $\mathcal{M}$  that is decomposed into  $m$  cylinders whose circumferences are denoted by  $c_i$ . Then, the following formula holds :*

$$k(\mathcal{M}) \subseteq \mathbb{Q}[c_2 c_1^{-1}, \dots, c_m c_1^{-1}]$$

Wright actually proved a stronger version and established the other inclusion if one consider only a subclass of cylinders, but we will not need such generality.

### 3.4 Complete periodicity

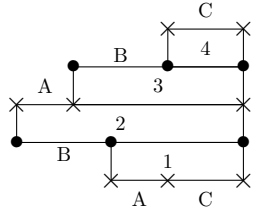
The last tool we will need is the complete periodicity property. See [Wri15] for more details.

**Proposition 3.5.** *Let  $\mathcal{M}$  be a rank one affine invariant orbifold, and let  $(X, \omega)$  be a translation surface in  $\mathcal{M}$ . If there is a saddle connection on  $X$  in direction  $\theta$  that joins a singularity to itself, then  $X$  is periodic in direction  $\theta$ .*

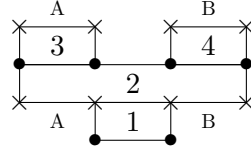
## 4 The stratum $\mathcal{H}^{odd}(2, 2)$

**Theorem 4.1** (Theorem A). *Let  $\mathcal{M}$  be a non absolute rank one affine invariant orbifold in  $\mathcal{H}(2, 2)^{odd}$ . If  $\mathcal{M}$  is non arithmetic, then  $k(\mathcal{M})$  is a totally real quadratic field, and  $\mathcal{M}$  is a component of  $\Omega E_D^{odd}(2, 2)$ , where  $D$  is the discriminant of  $k(\mathcal{M})$ .*

*Proof.* Let  $(X, \omega)$  be a horizontally periodic surface in  $\mathcal{M}$ . Up to flowing along the Rel flow, it can be assumed that the cylinder decomposition is stable. Since  $(X, -\omega) = -id \cdot (X, \omega)$  and  $\mathcal{M}$  is invariant under the action of  $GL_2^+(\mathbb{R})$ , we can thus assume that  $(X, \omega)$  is cylinder equivalent to one of the surface given in A.1. Applying  $Rel_{iu_0}$  (we use the notation of the proof of lemma 3.1) takes one cylinder decomposition to another and using this trick it can even be assumed that  $(X, \omega)$  is cylinder equivalent to one of the following two surfaces that have a mixed decomposition:

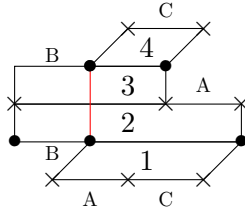


Decomposition A

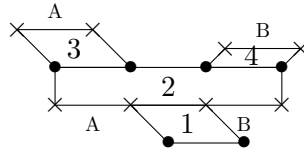


Decomposition B

We shall denote by  $\tau_i$  the twist of the cylinder  $i$ . It is the period of a cross section and is defined only mod  $c_i$ . Suppose  $(X, \omega)$  has a cylinder decomposition as in A. Then, with the notation of section 3,  $u = (\gamma_1, -\gamma_2, \gamma_3, -\gamma_4)$ . Proposition 3.2 shows that the circumferences of the cylinders 1 and 2 are not commensurable. Consequently, the twists of the cylinders 2 and 3 can be chosen independently. More precisely, there is a vector  $v \in V(X)$  so that  $\tau(v)$  is the following surface :

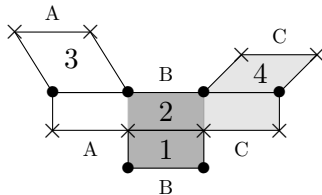


On that surface, the saddle connection drawn in red starts and ends at the same singularity. By proposition 3.5 that implies that the vertical direction is completely periodic. There is a small  $\varepsilon$  such that on  $Rel_{u_0}^\varepsilon(X)$  the vertical decomposition is stable, and the decomposition must appear in the list provided in appendix A. Recall that  $u_0$  was defined in the proof of lemma 3.1 as the image of  $u$  by the derivative at 0 of  $\tau$ . On this surface the red saddle connection borders a cylinders, and notice that the total angle along this cylinder is at least  $3\pi$ . The only cylinder decomposition that presents this feature is the decomposition B. That means we can only consider this decomposition. From now on,  $(X, \omega)$  is the following surface:



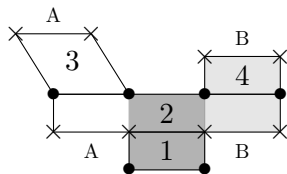
The surface  $(X, \omega)$

We shall start by proving that two of the cylinders among  $\{1, 3, 4\}$  have same height and commensurable circumferences. Here,  $u = (\gamma_1, -\gamma_2, \gamma_3, \gamma_4)$  and its algebraic degree is 2 or else proposition 3.4 would imply that  $\mathcal{M}$  is arithmetic. If  $\mathfrak{C}_0^+$  has at least two elements, the claim follows from proposition 3.2. If not, the circumferences of the two remaining cylinders are commensurable by proposition 3.3 and corollary 3.2 implies that they have same height. We can assume that those two cylinders are 1 and 4. Now, for the same reason as before,  $\gamma_1$  and  $\gamma_2$  are not commensurable, and thus there is a  $v \in V(X)$  such that on  $\tau(v)$  the twists of the cylinder 1 and 2 equal to zero.



The surface  $\tau(v)$

If the twist of  $\mathcal{C}_1$  on the surface  $\tau(v)$  equals 0, that means that  $v_1 = -\frac{\tau_1}{c_1}$ . Since  $\gamma_1$  and  $\gamma_4$  are rationally dependent, proposition 3.1 implies that  $v_4 = \frac{\gamma_4}{\gamma_1}v_1 = \frac{c_1}{c_4}v_1 = -\frac{\tau_1}{c_4}$ , and the twist of  $\mathcal{C}_4$  is now  $\tau_4 - \tau_1$ . Notice that the dark grey part of the surface is crossed by a vertical geodesic. By proposition 3.5, that implies that the vertical direction is periodic. This is possible only if  $\tau_4 - \tau_1$  is a rational multiple of  $c_4$ : let  $p, q$  two coprime integers such that  $\tau_4 - \tau_1 = \frac{p}{q}c_4$ . The vertical cylinder decomposition is thus made out of three cylinders: the white one, and the two grey ones. The circumference of the dark grey cylinder is  $h_1 + h_2$  and its height is  $c_1$ , while the circumference of the light grey cylinder is  $q(h_2 + h_4)$  and its height is  $\frac{c_4}{q}$ . Applying proposition 3.2 to  $Rel_u^\varepsilon(X)$  gives  $c_1 = \frac{c_4}{q}$ , that is to say  $\tau_4 - \tau_1 = pc_1$ . Up to applying powers of the Dehn twist about the core curve of the cylinder  $\mathcal{C}_1$ , the twist of the cylinders 1 and 4 are equal. We can replace  $(X, \omega)$  with the following surface:



The surface  $(X, \omega)$  now has a vertical cylinder decomposition into 3 cylinders that we have colored once more in white, dark grey and light grey. The two greys cylinders have the same circumference given by  $h_1 + h_2 = h_2 + h_4$ , and their heights are  $c_1$  and  $c_4$ . Applying proposition 3.2 to  $Rel_u^\varepsilon(X)$  gives that  $c_1 = c_4$ . Consequently, there is an involution that takes the light grey cylinder to the dark grey one, while fixing the white one and such that  $\tau^*\omega = -\omega$ . Notice that the genus of  $X/\tau$  is 1. That means that  $\tau$  is a Prym involution.

Now, we invoke theorem 3.5 in [McM07b]. This theorem states that if the Veech group of a Prym form contains a hyperbolic element, then it is a Prym eigenform. Indeed, the involution that we constructed persist under small deformations along the isoperiodic foliation (small enough so that the cylinder decomposition persists) and under the action of  $GL_2^+(\mathbb{R})$ . This means that any surface in a neighborhood of  $(X, \omega)$  in  $\mathcal{M}$  is also a Prym form. Finally, an application of the Poincaré recurrence theorem together with a closing lemma for the Teichmüller flow that can be found, for instance, in [Wri14] shows that one of the surfaces in this neighborhood is fixed by an hyperbolic matrix.  $\square$

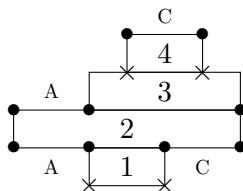
**Proposition 4.1.** *If  $X$  is a non arithmetic Veech surface in  $\mathcal{H}^{odd}(2, 2)$  that is not contained in the Prym locus, then the subset  $G \cdot \mathcal{F}_X$  is dense in  $\mathcal{H}^{odd}(2, 2)$ .*

*Proof.* Let  $\mathcal{M}$  be the closure of  $G \cdot \mathcal{F}_X$ . This is an affine invariant orbifold as it is connected, closed, and  $GL_2^+(\mathbb{R})$  invariant, and it is saturated by the isoperiodic foliation. Its rank is at least two, as otherwise theorem A would imply that  $X$  is contained in a Prym locus. The work of Aulicino and Nguyen (Theorem 1.1 in [AN16b]) implies it can not be rank two either as the Prym locus is the only rank two affine invariant orbifold that is saturated by the isoperiodic foliation. Its rank is thus 3, and as it is saturated by the isoperiodic foliation, by [MW18] this is actually the whole stratum  $\mathcal{H}^{odd}(2, 2)$ .  $\square$

## 5 The stratum $\mathcal{H}(3, 1)$

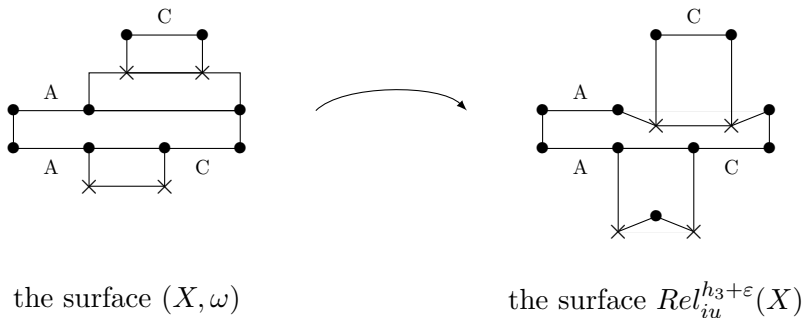
**Theorem 5.1.** *Any non absolute rank one affine invariant orbifold in  $\mathcal{H}(3, 1)$  is arithmetic.*

*Proof.* Let  $\mathcal{M}$  be a non absolute rank 1 affine invariant orbifold, and let  $(X, \omega)$  be a horizontally periodic surface in  $\mathcal{M}$ . Up to flowing along the rel flow, it can be assumed that the corresponding cylinder decomposition is stable. Since  $(X, -\omega) = -id \cdot (X, \omega)$  and  $\mathcal{M}$  is invariant under the action of  $GL_2^+(\mathbb{R})$ , we can thus assume that  $(X, \omega)$  is cylinder equivalent to one of the surface given in A.3. Consider the first decomposition of the list. Up to modifying the twists and heights of the cylinders, the surface  $(X, \omega)$  is as follows :



With the same notation as previously  $u = (\gamma_1, 0, -\gamma_3, \gamma_4)$ . If the algebraic degree of  $u$  were to be 2, then by proposition 3.1, the circumferences of the cylinders 1, 3 and 4 are pairwise rationally dependent. Note that proposition 3.4 is of no help at this stage to prove

that  $\mathcal{M}$  is arithmetic as we do not know anything yet on the circumference of the second cylinder. The remaining of the argument is to prove that  $c_2$  is indeed commensurable to the circumference of the other cylinders. To do so, consider the surface  $Y = Rel_{iu_0}^{h_3+\epsilon}(X)$ , where  $h_3$  is the height of the cylinder 3 and  $\epsilon$  is small enough. The deformation is depicted in the following picture:



For the same reason as previously, on the surface  $Y$  the cylinder 1, 2 and 4 are pairwise commensurable. But notice that the circumferences of the cylinder 2 or 3 are the same on  $X$  and  $Y$ . This concludes that all the cylinders of  $(X, \omega)$  are commensurable. Proposition 3.4 shows that  $\mathcal{M}$  is thus arithmetic. The scheme of proof can be used for the decomposition 1, 2, 3, 4, 5 and 6. To deal with the other two cases, note that the surface can be deformed through the horocycle flow and the Rel flow so that the vertical direction is periodic with one non mixed cylinder. It means that the vertical decomposition falls in the previous list. This concludes the proof.  $\square$

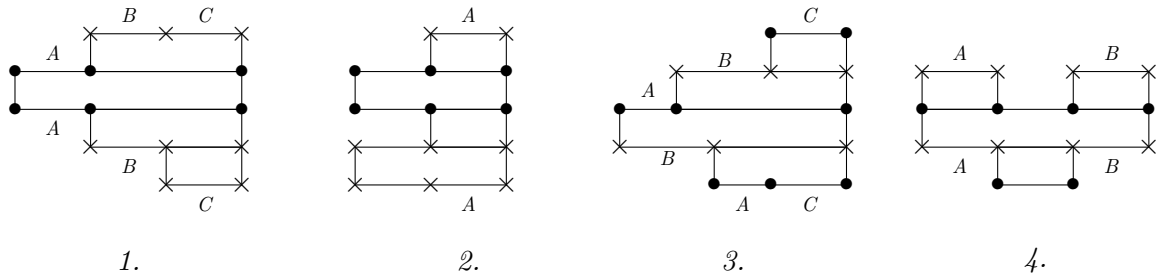
**Corollary 5.1.** *If  $(X, \omega)$  is a non arithmetic Veech surface in  $\mathcal{H}(3, 1)$ , then the subset  $GL_2^+(\mathbb{R}) \cdot \mathcal{F}_X$  is dense in  $\mathcal{H}(3, 1)$ .*

*Proof.* Let  $\mathcal{M}$  be the closure of  $GL_2^+(\mathbb{R}) \cdot \mathcal{F}_X$ . It is an affine invariant orbifold. If its rank were one, then  $\mathcal{M}$  would be non arithmetic and this is a contradiction with Theorem B. The rank of  $\mathcal{M}$  is then at least 2, but Nguyen and Aulicino proved that there are no rank 2 affine invariant orbifold in  $\mathcal{H}(3, 1)$ . Therefore the rank of  $\mathcal{M}$  is 3, and  $\mathcal{M}$  is the whole stratum  $\mathcal{H}(3, 1)$ .  $\square$

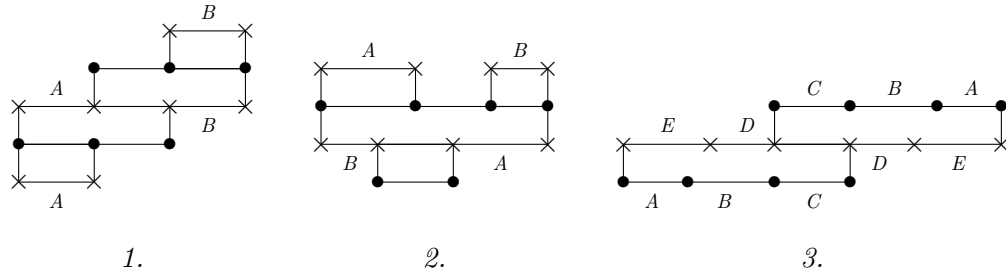
## A Stable Cylinder decompositions in $\mathcal{H}(2, 2)$ and $\mathcal{H}(3, 1)$

In this appendix, we give the possible stable cylinder decompositions in  $\mathcal{H}(2, 2)$ , and  $\mathcal{H}(3, 1)$ . More formally, two periodic translation surfaces are said to be cylinder equivalent if one is obtained from the other by modifying the height and twist parameter of the cylinders. We prove the following three propositions :

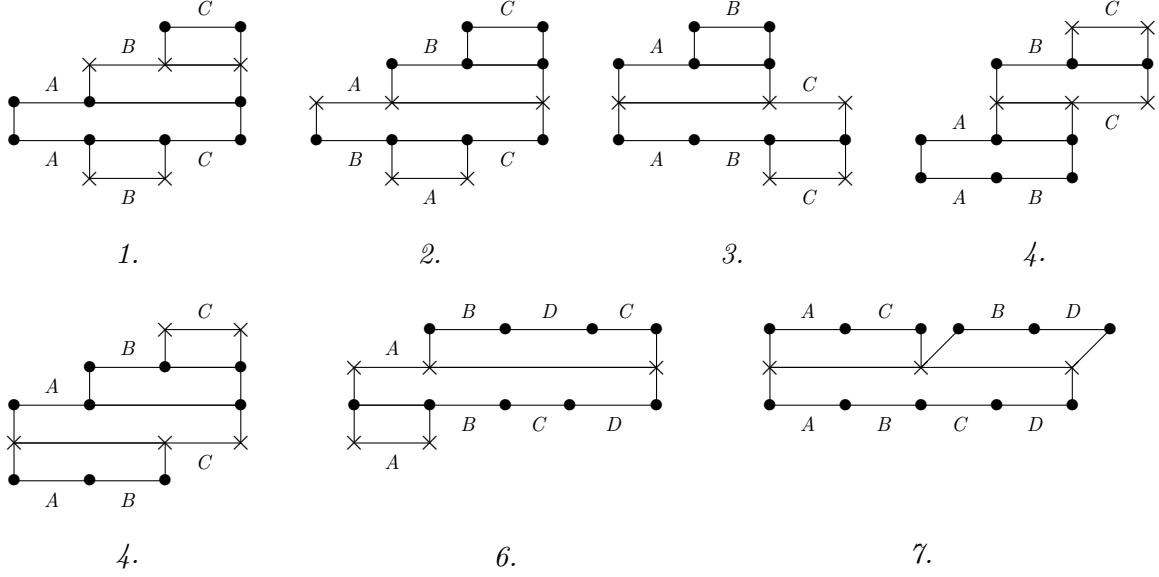
**Proposition A.1.** *If  $(X, \omega)$  is a horizontally periodic translation surface with mixed cylinders in  $\mathcal{H}^{odd}(2, 2)$ , then, maybe after replacing  $\omega$  by  $-\omega$ , it is cylinder equivalent to one of the following translation surfaces:*



**Proposition A.2.** *If  $(X, \omega)$  is a horizontally periodic translation surface with mixed cylinders in  $\mathcal{H}^{hyp}(2, 2)$ , then, maybe after replacing  $\omega$  by  $-\omega$ , it is cylinder equivalent to one of the following translation surfaces:*



**Proposition A.3.** *If  $(X, \omega)$  is a horizontally periodic translation surface with mixed cylinders in  $\mathcal{H}(3, 1)$ , then, maybe after replacing  $\omega$  by  $-\omega$ , it is cylinder equivalent to one of the following translation surfaces:*



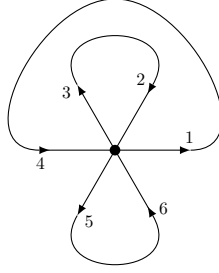
## A.1 Diagram of separatrices

We start by giving a common framework to study cylinder decompositions of translation surfaces.

**Definition A.1.** A *prediagram of separatrices* is a quadruplet  $\Gamma = (E, \sigma, \tau, \theta)$ , where  $\tau$  is a fixed point free involution of  $E$ ,  $\sigma$  is a permutation of  $E$  and  $\theta$  is a map from  $E/\tau \rightarrow E$  such that  $p \circ \theta = id$ , where  $p : E \rightarrow E/\tau$  is the canonical projection.

The elements of  $E$  are called oriented edges. An edge  $\gamma$  in  $E$  is said to be positively oriented if  $\theta \circ p(\gamma) = \gamma$  and negatively oriented if  $\theta \circ p(\gamma) = \tau(\gamma)$ . The set of positively oriented edges will be denoted by  $E_+$  and the set of negatively oriented edges will be denoted by  $E_-$ . A prediagram of separatrices is called alternating if  $\sigma(E_+) = \sigma(E_-)$ . A cylinder component is defined as an orbit of  $\sigma_\infty := \sigma \circ \tau$ . Such a cylinder component is said to be positively oriented if it corresponds to a positively oriented edge, and negatively oriented otherwise. We denote by  $\mathcal{C}_+$  and  $\mathcal{C}_-$  the set of positively and negatively oriented cylinder components. A pairing of cylinder components is a bijection from  $\mathcal{C}_+$  to  $\mathcal{C}_-$ . Finally, we define a metric on  $\Gamma$  as a strictly positive  $\tau$ -invariant function  $l$  on  $E$ , and we consider its natural extension  $\hat{l}$  to the set of cylinder components defined by  $\hat{l}(c) : \sum_n l(\sigma_\infty^n(\gamma))$ , where  $c$  is the cylinder component associated to  $\gamma$ . Such an object can be encoded by a directed graph with additional information. Its set of vertices is the set of orbits of  $\sigma$ , and its set of edges is  $E/\sigma$ . Set  $p_0 : E \rightarrow E/\sigma$  to be the canonical projection. The beginning of an edge is  $p_0(e)$ , and its end is  $p_0 \circ \tau(e)$ . There is an cyclic ordering on the star of each vertex define by  $\sigma$ . In the following figure is depicted the graph associated to the alternating prediagram of separatrices  $(\{1, \dots, 6\}, id, (14)(23)(56), \theta_0)$ , where  $\theta_0(i) = i$  for any  $i \in \{1, 2, 5\}$ .





The graph associated to  $(\{1, \dots, 6\}, id, (14)(23)(56), \theta_0)$

**Definition A.2.** A diagram of separatrices  $(\Gamma, m, l)$  is the data of an alternating prediagram of separatrices  $\Gamma$ , together with a matching of its cylinder components and a metric  $l$  on  $\Gamma$  that is invariant by  $m$ , that is  $\hat{l} \circ m = \hat{l}$ .

To any translation surface  $(X, \omega) \in \mathcal{H}(\kappa)$ , there is a canonical diagram of separatrices  $\Gamma(X, \omega)$  associated. It is defined as follows: the set  $E$  is the collection of all geodesics rays  $\gamma : [0, 1] \rightarrow X$ , such that  $\gamma^{-1}(\Sigma) = \{0, 1\}$ , and  $\gamma^*(Im(\omega)) = 0$ , taken up to reparametrization. We define  $\tau$  to be the orientation reversing map :  $\tau(\gamma)(t) : \gamma(1-t)$ . For any  $\gamma \in E$ , there is a chart around  $\gamma(0)$  that takes  $\omega$  to  $z^k dz$ ; where  $k$  is the order of the singularity. The set of germs of geodesic rays that begin at  $\gamma(0)$  is thus invariant by multiplication by  $e^{\frac{2i\pi}{k+1}}$ , and the germ of a geodesic ray at its beginning completely determines it. We thus define  $\sigma(\gamma)$  to be the geodesic ray whose germ at its beginning is the one of  $e^{\frac{i\pi}{k+1}}\gamma$ . Finally, to define  $\theta$  we need to define a map that is  $\tau$ -invariant. If  $\gamma$  is in  $E$ , we define  $\theta(\gamma) = \gamma$  if  $\gamma^*(Re(\omega)) > 0$ , and  $\theta(\gamma) = \tau(\gamma)$  otherwise. The metric on  $\Gamma(X, \omega)$  is given by  $l(\gamma) = |\int_{\gamma} \omega|$ . Note that the orbits of  $\sigma_{\infty}$  are in correspondence with oriented core curves of cylinders. The matching  $m$  is then defined to map the orbit that corresponds to the positively oriented core curve to the one that corresponds to the negatively oriented core curves of the same cylinder. In the reverse direction, to any diagram of separatrices  $\Gamma$  there is horizontally periodic translation surface  $(X_{\Gamma}, \omega_{\Gamma})$ . It is defined in the following way: replace any  $\gamma \in \theta(E/\tau)$  by a strip of length  $l(\gamma)$ , and replace any element in  $E/\sigma$  by a disk. The permutation  $\sigma$  defines a cyclic ordering on any orbit. Using this ordering, one can glue the strips to the disks. We get a topological surface with boundary, and the boundary components correspond to orbits of  $\sigma_{\infty}$ . Since by requirement, the paired components have same length (defined by  $\hat{l}$ ) we can glue those components using the pairing, and we get a horizontally periodic flat surface in  $\mathcal{H}(\kappa)$ , where  $|\kappa|$  is the cardinal of  $E/\sigma$  and  $k_i$  is half the cardinal of the orbit  $i$  minus 1. The fact that the cardinal of an orbit is even comes from the alternating condition.

Two prediagrams of separatrices  $\Gamma_1 = (E^1, \tau_1, \sigma_1, \theta_1)$  and  $\Gamma_2 = (E^2, \tau_2, \sigma_2, \theta_2)$  are isomorphic if there is a map  $\varphi : E_1 \rightarrow E_2$  such that  $\varphi \circ \sigma_1 = \sigma_2 \circ \varphi$ ,  $\varphi \circ \tau_1 = \tau_2 \circ \varphi$ , and  $\varphi(E_1^+) = E_2^+$ , or equivalently  $\varphi \circ \theta_1 = \theta_2$ . Finally, two diagrams of separatrices  $(\Gamma_1, l_1, m_1)$  and  $(\Gamma_2, l_2, m_2)$  are isomorphic if there is an isomorphism of prediagram of separatrices  $\varphi$  between the two such that  $l_2 \circ \varphi = l_1$ , and if two cylinders components on  $\Gamma_1$  associated to

the orbits of  $\gamma_1$  and  $\gamma_2$  are paired by  $m_1$ , then the cylinder components of  $\Gamma_2$  associated to  $\varphi(\gamma_1)$  and  $\varphi(\gamma_2)$  are paired by  $m_2$ .

**Proposition A.4.** *Two horizontally periodic translation surfaces  $(X_1, \omega_1)$  and  $(X_2, \omega_2)$  in  $\mathcal{H}(\kappa)$  are isomorphic to cylinder equivalents surfaces if, and only if, the associated diagrams of separatrices are isomorphic.*

We refer to [KZ03] for a proof of that result.

**Definition A.3.** *A connected component of a prediagram of separatrices of  $\Gamma = (E, \sigma, \tau, \theta)$  is an orbit of the group  $\langle \sigma, \tau \rangle$ . A prediagram of separatrices is said to be connected when it is reduced to a single connected component.*

If  $E'$  is a connected component of  $\Gamma$ , there is an induced prediagram of separatrices  $(E', \sigma|_{E'}, \tau|_{E'}, \theta|_{p(E')})$ .

**Definition A.4.** *A prediagram of separatrices  $(E, \tau, \sigma, \theta)$  is said to be stable if  $\tau$  preserves the orbits of  $\sigma$ .*

The notion of stable for prediagram coincide with the notion of stable for cylinder decomposition for surfaces. It precisely means that the geodesic rays start and end at the same singularity, without going through any other singularity. Note that when a prediagram of separatrices  $\Gamma = (E, \sigma, \tau, \theta)$  is both stable and connected, then  $E$  is reduced to a single orbit of  $\sigma$ . This remark allows to define the type of  $\Gamma$  as follows. Let  $x$  be a positively oriented edge. Any positively oriented edge can be written as  $\sigma^{2k}(x)$ , and any negatively oriented edge can be written as  $\sigma^{c_n(2l)}(x)$ . Since  $\tau$  reverses the orientation, denoting by  $n$  the number of positively oriented edges, there is a map  $f \in \mathfrak{S}_n$  such that for any  $k$ :

$$\tau \circ \sigma^{2k}(x) = \sigma^{2f(k)+1}(x)$$

Denote by  $c_n$  the element of  $\mathfrak{S}_n$  that sends  $i$  to  $i+1$  and  $n$  to 1. The group  $H = \langle c_n \rangle$  acts by conjugation on  $\mathfrak{S}_n$  and the type of the component is defined as the orbit of  $f$  under this action. The fact that we defined  $f$  up to conjugation by  $H$  comes from the fact we could have chosen any other even iteration of  $x$  as a generator of the orbit. More generally, we say that  $\Gamma$  is of type  $(f_i)$  if the type of the minimal connected components are given by the  $f_i$ .

**Proposition A.5.** *Two stable alternating prediagrams are isomorphic if, and only if, they have the same type.*

*Proof.* It is enough to prove the result for minimal connected components : Let  $\Gamma_i = (E_i, \sigma_i, \tau_i, \theta_i)$  for  $i \in \{1, 2\}$  be two minimal prediagrams of same type. Pick  $x_i$  in  $E_i$  that is positively oriented and define  $f_i$  such that  $\tau_i(\sigma_i^{2k}(x_i)) = \sigma_i^{2(f_i(k)+1)}$ . Saying that the diagrams have the same type means there is  $l$  such that  $f_2 = c_n^l \circ f_1 \circ c_n^{-l}$ . Define  $\varphi$  such

that for all  $j$   $\varphi(\sigma_1^j(x_1)) = \sigma_2^{2l+j}(x_2)$ . We claim that  $\varphi$  is an isomorphism of prediagram. Indeed, let  $\zeta \in E_1$ , and pick  $j$  such that  $\sigma^j(x_1) = \zeta$ . Thus:

$$\begin{aligned}\varphi \circ \sigma_1(\zeta) &= \varphi \circ \sigma^{j+1}(x_1) \\ &= \sigma_2^{2l+j+1}(x_2) \\ &= \sigma_2(\sigma_2^{2l+j}(x_2)) \\ &= \sigma_2 \circ \varphi(\sigma_1^j(x_1)) \\ &= \sigma_2 \circ \varphi(\zeta)\end{aligned}$$

We also have:

$$\begin{aligned}\tau_2 \circ \varphi(\sigma_1^{2k}(x_1)) &= \tau_2 \circ \sigma_2^{2(k+l)}(x_2) \\ &= \sigma_2^{2f_2(k+l)+1}(x_2) \\ &= \sigma_2^{2c_n^l \circ f_1 \circ c_n^{-l}(k+l)+1}(x_2) \\ &= \sigma_2^{2(f_1(k)+l)+1}(x_2) \\ &= \varphi \circ \sigma_1^{2f_1(k)+1} \\ &= \varphi \circ \tau_1(\sigma^{2k}(x_1)).\end{aligned}$$

By construction,  $\varphi(E_+^1) = E_+^2$ . Reciprocally, if there is an isomorphism of prediagrams between  $\Gamma_1$  and  $\Gamma_2$ , then  $\varphi(x_1) = \sigma_2^{2l}(x_1)$ , and then:

$$f_2 = c_n^l \circ f_1 \circ c_n^{-l}$$

□

We denote by  $\bar{\Gamma}$  the prediagram  $(E, \sigma, \tau, \tau \circ \theta)$ . if  $\Gamma$  corresponds to a surface  $(X, \omega)$ , then  $\bar{\Gamma}$  corresponds to the surface  $(X, -\omega)$ . Note that if  $(X, \omega)$  is represented by a polygon P, the surface  $(X, -\omega)$  is represented by the image of the polygon by the rotation of angle  $\pi$ .

**Proposition A.6.** *Let  $\Gamma$  be a minimal stable alternating prediagram of type  $f$ . The type of  $\bar{\Gamma}$  is given by  $(f \circ c_n)^{-1}$ .*

*Proof.* Let  $x \in E$  be positively oriented, and such that  $\tau \circ \sigma^{2k}(x) = \sigma^{2f(k)+1}$ . Then  $\sigma(x)$  is positively oriented on  $\bar{\Gamma}$ . Then:

$$\begin{aligned}\tau \circ \sigma^{2j}(\sigma(x)) &= \sigma^{2f^{-1}(j)}(x) \\ &= \sigma^{2c_n^{-1}(f^{-1}(j))}(\sigma^2(x))\end{aligned}$$

□

These propositions enable us to enumerate all the possible stable alternating prediagrams up to isomorphism. We shall represent only one of the surfaces associated to  $\Gamma$  or  $\bar{\Gamma}$  as one is obtained from the other by a rotation of angle  $\pi$ .

## A.2 Stable cylinder decompositions in $\mathcal{H}^{odd}(2, 2)$

Let  $(X, \omega)$  be a stable horizontally periodic translation surface in  $\mathcal{H}(2, 2)$ , and denote by  $\Gamma(X, \omega)$  the prediagram associated. The types of the minimal connected components of  $\Gamma(X, \omega)$  are, up to reversing the orientation, either  $id_{\mathfrak{S}_3}, (123)$  or  $(12)(3)$ .

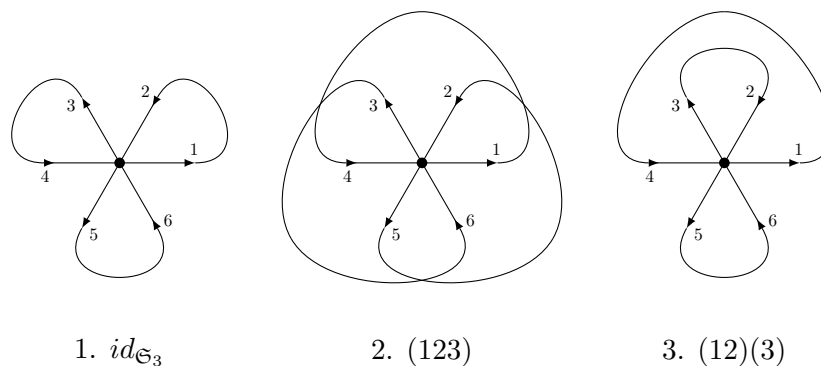
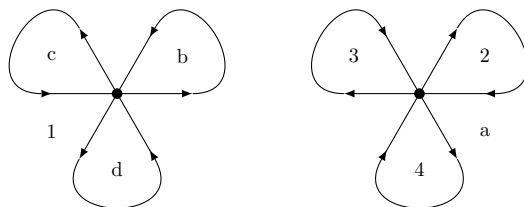


Fig. The possible types for the components of  $\Gamma(X, \omega)$ , up to orientation.

Indeed, those permutations together the inverse of their composition with  $c_3$  are the only permutations of the group  $\mathfrak{S}_3$  up to conjugation by elements of  $H_3$ . The isomorphism class of the prediagram  $\Gamma(X, \omega)$  is thus completely determined by the choice of one of the three types for its two connected components, together with an orientation on those components. This orientation cannot be determined arbitrarily as there is a pairing on the orbits of  $\sigma_\infty$ . The components of type 1 and 3 both have 4 cylinder components, while the type 2 has 2 components. A pairing is possible only if there is an even number of cylinder components, half of them being positively oriented, and the other half being negatively oriented. Hence the type of  $\Gamma(X, \omega)$  is, up to reversing the orientation on the components, either  $(1, 1), (2, 2)$  or  $(2, 3)$ . The reason  $\Gamma(X, \omega)$  can not be of type  $(1, 2)$  is the fact the type 1 components has the same number of positively and negatively cylinder components, while the type 2 one has 3 cylinder components with the same orientation and only one with the opposite orientation. Hence there is no matching of the cylinder components possible.

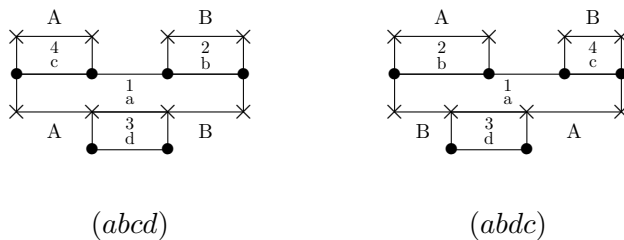
### A.2.1 Stable cylinder decompositions of type (1, 1)

The two minimal components need to be oppositely oriented so that a pairing is possible. The associated graph is planar and we can identify cylinder components to connected components of the plane once the graphs are removed. We have named those orbits by letters when they are positively oriented, and with a digit when they are negatively oriented.



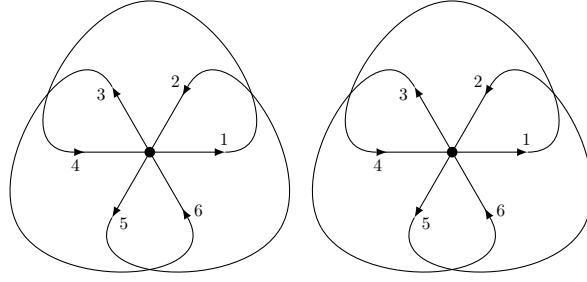
The prediagram of  $\Gamma(X, \omega)$  in the type (2, 2) case

We will denote a pairing by an ordered quadruple of letters. The cylinder component whose label appears first is matched with the cylinder component labeled by 1 etc. For instance, in the pairing labeled by  $(abcd)$ , the first components  $a$  is paired with the component 1, the component  $c$  with the component 2, etc. Here, for metric reason, the cylinder component denoted by 1 is necessarily paired with  $a$ . We can also assume that 2 is paired with  $b$  as any other choice produces isomorphic diagrams of separatrices. We depict the two corresponding surfaces associated:



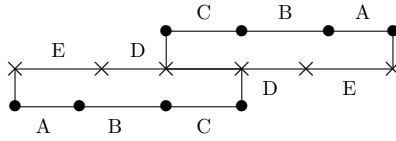
### A.2.2 Stable cylinder decomposition of type (2, 2)

Here, any choice of orientation produces isomorphic prediagrams, so we can chose an arbitrary orientation on each component.



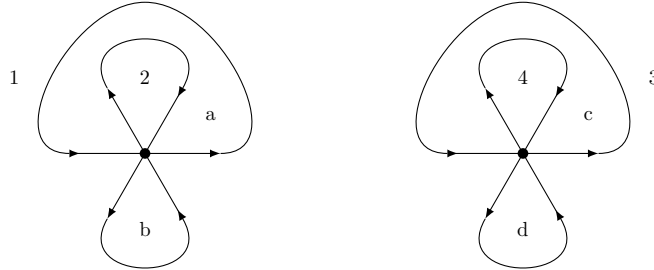
The prediagram of  $\Gamma(X, \omega)$  in the type (2, 2) case

In this case there is only one possible pairing that produces a connected translation surface. The associated surface is represented in the following picture.



### A.2.3 Stable cylinder decomposition of type (3, 3)

Here, any choice of orientation produces isomorphic prediagrams, so we can choose an arbitrary orientation on each connected component. Here again, the graph associated to the connected components is planar. We use the same notation as in the previous section.



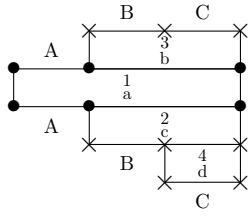
The prediagram of  $\Gamma(X, \omega)$  in the type (1, 1) case

There are a priori 24 possible pairings. However, the following 4 represent non connected surfaces :  $(abcd)$ ,  $(abdc)$ ,  $(bacd)$ ,  $(badc)$ , and on the following 13 there are compatible metric :

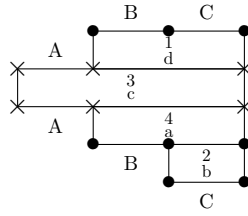
1.  $(acdb)$ . If  $l_a = l_1$ , then  $l_2 = l_b$ . But  $l_b$  must be equal to  $l_4$ , while  $l_2$  must be equal to  $l_c$ , and thus  $l_c = l_4$ . However,  $l_c$  should be strictly greater than  $l_4$ .

2.  $(adbc)$ . The pairing would give  $l_c = l_4$ , while we should get that  $l_2$  is strictly greater than  $l_4$ .
3.  $(bdca)$ . The pairing would give  $l_1 = l_b$ , while we should get that  $l_1$  is strictly greater than  $l_b$ .
4.  $(bcad)$ . The pairing would give  $l_1 = l_b$ , while we should get that  $l_1$  is strictly greater than  $l_b$ .
5.  $(cbda)$ . The pairing would give  $l_d = l_3$ , while we should get that  $l_3$  is strictly greater than  $l_d$ .
6.  $(cabd)$ . The pairing would give  $l_2 = l_a$ , while we should get that  $l_a$  is strictly greater than  $l_2$ .
7.  $(bcda)$ . The pairing would give  $l_1 = l_b$ , while we should get that  $l_1$  is strictly greater than  $l_b$ .
8.  $(dabc)$ . The pairing would give  $l_c = l_4$ , while we should get that  $l_2$  is strictly greater than  $l_4$ .
9.  $(dbac)$ . The pairing would give  $l_c = l_4$ , while we should get that  $l_2$  is strictly greater than  $l_4$ .
10.  $(bdac)$ . The pairing would give  $l_1 = l_b$ , while we should get that  $l_1$  is strictly greater than  $l_b$ .
11.  $(cadb)$ . The pairing would give  $l_d = l_3$ , while we should get that  $l_3$  is strictly greater than  $l_d$ .
12.  $(dacb)$ . The pairing would give  $l_2 = l_a$ , while we should get that  $l_a$  is strictly greater than  $l_2$ .
13.  $(dcba)$  For metric reason, we have  $l_a$  is greater than  $l_2$ , and likewise  $l_c$  greater than  $l_4$ . However the pairing requires  $l_2 = l_c$ , thus we get that  $l_4$  is greater than  $l_a$ . But the pairing would imply that  $l_4$  equals  $l_a$ .

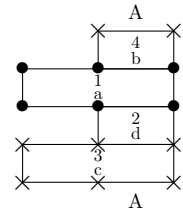
We depict the surfaces associated to the other diagrams:



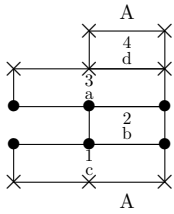
(acbd)



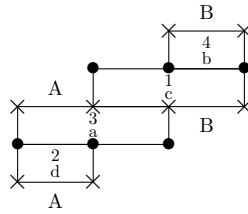
(dbca)



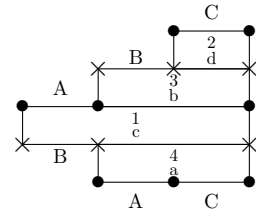
(adcb)



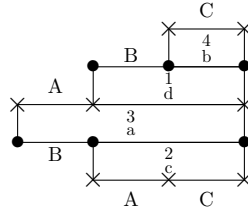
(cbad)



(cdab)



(cdba)



(dcab)

Finally, notice that the first two diagrams are isomorphic, as well as the last two. The isomorphism is, in both case the one that exchanges the two components while commuting with both  $\sigma$  and  $\tau$ . Intuitively, the corresponding isomorphism at the level of the surfaces swap the singularities.

### A.3 Stable cylinder decompositions in $\mathcal{H}(3, 1)$

The types of the connected components of order 3 of  $\Gamma(X, \omega)$  are, up to reversing the orientation, either  $(1)(243)$ ,  $(1)(3)(24)$ ,  $(1)(234)$ ,  $(13)(24)$  or  $id_{\mathfrak{S}_4}$ , and the type of the component corresponding to the singularity of order 1 is, up to reversing the orientation,  $id_{\mathfrak{S}_2}$ .



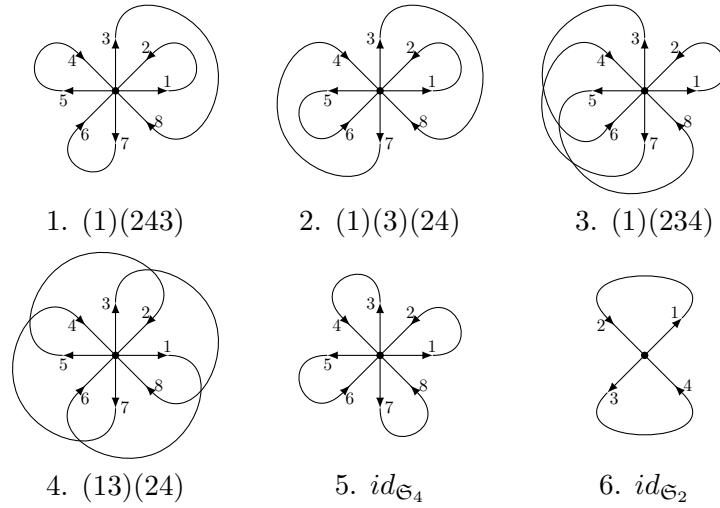
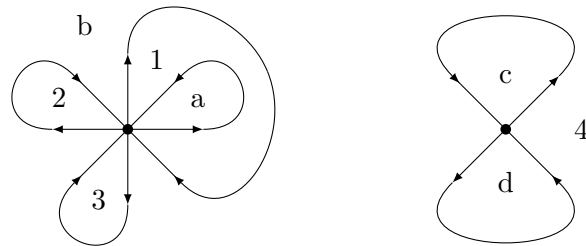


Fig. The possible types for the components of  $\Gamma(X, \omega)$

The types 1 and 3 both have 3 cylinder components with the same orientation and 2 with the opposite orientation, while types 3 and 4 have both 2 cylinder components with the same orientation and only one with the reverse orientation. The type 5 has 4 cylinder components with the same orientation and 1 with the opposite orientation. The type 6 has two cylinder components with the same orientation, and one with the reverse orientation. With the pairing condition, we deduce that  $\Gamma(X, \omega)$  can be type (1, 6), (2, 6), (3, 6) or (4, 6).

### A.3.1 Stable cylinder decomposition of type (1, 6)

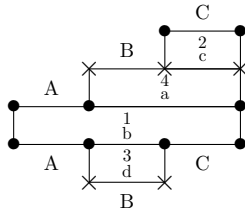
The following picture depicts the only possible orientations.



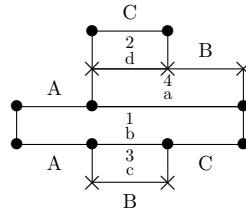
Here again, the associated graphs are planar, and we label the cylinder components with letters and digits, and a pairing by an ordered quadruple of letters. There are a priori 24 possible pairings. However, the following 18 cannot be endowed with a positive metric :

1.  $(abcd)$ . does not have a metric solution since  $\lambda_a > \lambda_1$
2.  $(abdc)$ . does not have a metric solution since  $\lambda_a > \lambda_1$
3.  $(acbd)$ . does not have a metric solution since  $\lambda_a > \lambda_1$
4.  $(acbd)$ . does not have a metric solution since  $\lambda_a > \lambda_1$
5.  $(adbc)$ . does not have a metric solution since  $\lambda_a > \lambda_1$
6.  $(adcb)$ . does not have a metric solution since  $\lambda_a > \lambda_1$
7.  $(bacd)$ . does not have a metric solution since  $\lambda_b - \lambda_2 - \lambda_3 = \lambda_1 - \lambda_a$  and  $\lambda_2 = \lambda_a$  together with  $\lambda_1 = \lambda_b$  imply  $\lambda_3 = 0$ .
8.  $(badc)$ . does not have a metric solution since  $\lambda_b - \lambda_2 - \lambda_3 = \lambda_1 - \lambda_a$  and  $\lambda_2 = \lambda_a$  together with  $\lambda_1 = \lambda_b$  imply  $\lambda_3 = 0$ .
9.  $(bcad)$ . does not have a metric solution since  $\lambda_b - \lambda_2 - \lambda_3 = \lambda_1 - \lambda_a$  and  $\lambda_3 = \lambda_a$  together with  $\lambda_1 = \lambda_b$  imply  $\lambda_2 = 0$ .
10.  $(bdac)$ . does not have a metric solution since  $\lambda_b - \lambda_2 - \lambda_3 = \lambda_1 - \lambda_a$  and  $\lambda_2 = \lambda_a$  together with  $\lambda_1 = \lambda_b$  imply  $\lambda_3 = 0$ .
11.  $(cabd)$ . does not have a metric solution since  $\lambda_3 < \lambda_b$
12.  $(cbad)$ . does not have a metric solution since  $\lambda_2 < \lambda_b$ .
13.  $(cbda)$ . does not have a metric solution since  $\lambda_2 < \lambda_b$ .
14.  $(cdba)$ . does not have a metric solution since  $\lambda_3 < \lambda_b$ .
15.  $(dabc)$ . does not have a metric solution since  $\lambda_2 < \lambda_b$ .
16.  $(dbac)$ . does not have a metric solution since  $\lambda_2 < \lambda_b$ .
17.  $(dbca)$ . does not have a metric solution since  $\lambda_2 < \lambda_b$ .
18.  $(dcba)$ . does not have a metric solution since  $\lambda_3 < \lambda_b$ .

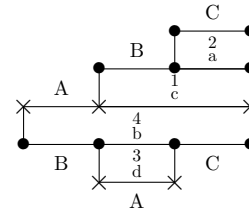
We depict the surfaces associated to the other diagrams:



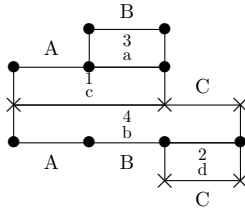
1.  $(bcda)$



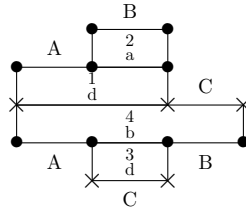
2.  $(bdca)$



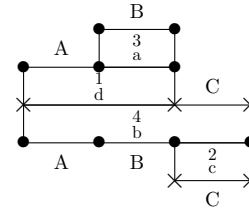
3.  $(cadb)$



4.  $(cdab)$



5.  $(dacb)$

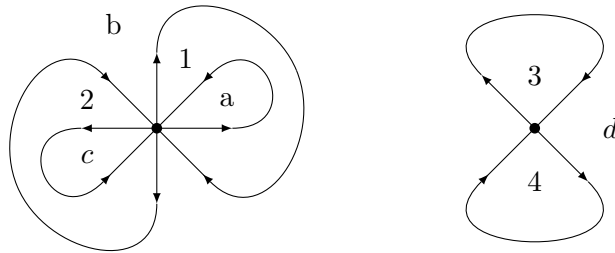


6.  $(dcab)$

To conclude, notice that the diagrams 1 and 2 are isomorphic, as well as 3 and 5, and 4 and 6.

### A.3.2 Stable cylinder decompositions of type $(2, 6)$

The following picture depicts the only possible orientation. Here again, the associated graphs are planar, and we label the components with letters and digits.



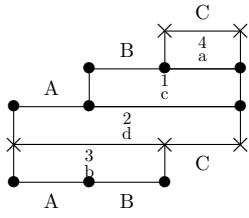
The prediagram of  $\Gamma(X, \omega)$  in the type  $(2, 6)$  case

We continue to denote a pairing by an ordered quadruple of letters. There are a priori 24 possible pairings. However, the following 18 cannot be endowed with a positive metric :

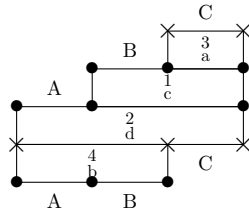
1.  $(abcd)$ . Does not have a metric solution as  $\lambda_1 > \lambda_a$ .
2.  $(abdc)$ . Does not have a metric solution as  $\lambda_1 > \lambda_a$ .

3.  $(acbd)$ . Does not have a metric solution as  $\lambda_1 > \lambda_a$ .
4.  $(acbd)$ . Does not have a metric solution as  $\lambda_1 > \lambda_a$ .
5.  $(adbcb)$ . Does not have a metric solution as  $\lambda_1 > \lambda_a$ .
6.  $(adcdb)$ . Does not have a metric solution as  $\lambda_1 > \lambda_a$ .
7.  $(bacd)$ . Does not have a metric solution as  $\lambda_d > \lambda_4$ .
8.  $(badc)$ . Does not have a metric solution as  $\lambda_d > \lambda_4$ .
9.  $(bcad)$ . Does not have a metric solution as  $\lambda_d > \lambda_4$ .
10.  $(bcda)$ . Does not have a metric solution as  $\lambda_2 > \lambda_c$ .
11.  $(bdac)$ . Does not have a metric solution as  $\lambda_1 - \lambda_a = \lambda_b - \lambda_2$  together with  $\lambda_b = \lambda_1$  imply  $\lambda_a = \lambda_2$ . But  $\lambda_a = \lambda_3 + \lambda_4$ ; and thus if  $\lambda_a = \lambda_3$  then  $\lambda_4 = 0$ .
12.  $(bdca)$ . Does not have a metric solution as  $\lambda_1 - \lambda_a = \lambda_b - \lambda_2$  together with  $\lambda_b = \lambda_1$  imply  $\lambda_a = \lambda_2$ . But if  $\lambda_2 = \lambda_d$  then  $\lambda_a = \lambda_3 + \lambda_4$ ; and thus if  $\lambda_a = \lambda_4$  then  $\lambda_3 = 0$ .
13.  $(cabd)$ . Does not have a metric solution as  $\lambda_d > \lambda_4$ .
14.  $(cadb)$ . Does not have a metric solution as  $\lambda_d > \lambda_4$ .
15.  $(cbad)$ . Does not have a metric solution as  $\lambda_d > \lambda_4$ .
16.  $(cbda)$ . Does not have a metric solution as  $\lambda_d > \lambda_3$ .
17.  $(dcab)$ . Does not have a metric solution as  $\lambda_d > \lambda_4$ .
18.  $(dcba)$ . Does not have a metric solution as  $\lambda_d > \lambda_4$ .

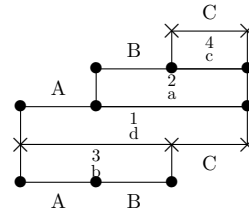
We depict the surfaces associated to the other diagrams:



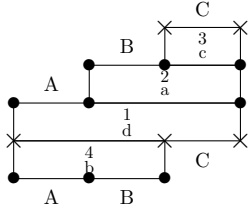
1. *cdab*



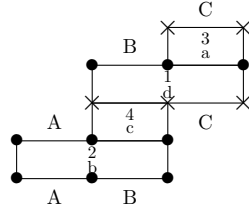
2. *cdba*



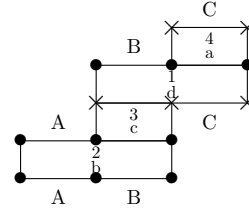
3. *dabc*



4. *dacb*



5. *dbac*

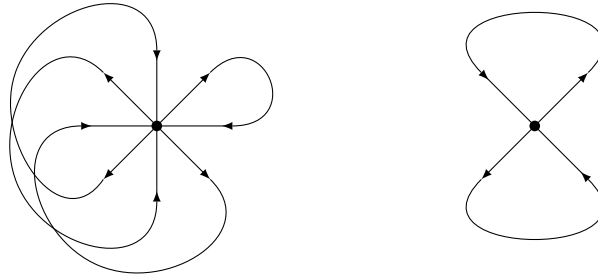


6. *dbca*

To conclude, notice that the four first diagrams are isomorphic, as well as the last two.

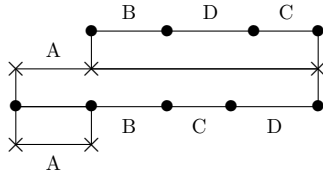
### A.3.3 Stable cylinder decomposition of type (3, 6)

In this case, only one choice of orientation is valid. There are two possible diagrams but one is obtain from the other by rotating the order one singularity and thus the diagrams are isomorphic.



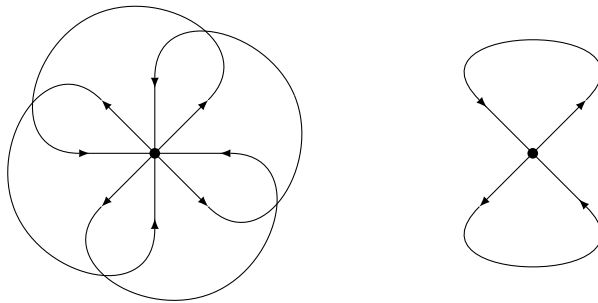
The prediagram of  $\Gamma(X, \omega)$  in the type (2, 6) case

We depict the only surface associated :



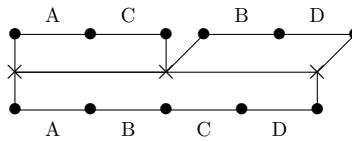
**A.3.4 Stable cylinder decomposition of type (4, 6)**

In this case, only one choice of orientation is valid. There are two diagrams but one is obtained from the other by rotating the order one singularity.



The prediagram of  $\Gamma(X, \omega)$  in the type (2, 6) case

We depict the only surface associated:



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