

A criterion for density of the isoperiodic leaves in rank 1 affine invariant orbifolds

Florent Ygouf

January 26, 2020

Abstract

We define on any affine invariant orbifold \mathcal{M} a foliation $\mathcal{F}^{\mathcal{M}}$ that generalises the isoperiodic foliation on strata of the moduli space of translation surfaces and study the dynamics of its leaves in the rank 1 case. We establish a criterion that ensures the density of the leaves and provide two applications of this criterion. The first one is a classification of the dynamical behavior of the leaves of $\mathcal{F}^{\mathcal{M}}$ when \mathcal{M} is Prym eigenform locus in genus 3 and the second provides the first examples of dense isoperiodic leaves in the stratum $\mathcal{H}(2, 1, 1)$.

1 Introduction

1.1 Context

Let $g \geq 1$ and let κ be a integer partition of $2g - 2$. A stratum $\mathcal{H}_g(\kappa)$ of the moduli space of translation surfaces is the set of isomorphism classes of pairs (X, ω) where X is a genus g Riemann surface and ω is a non vanishing holomorphic 1-form on X whose zeroes have multiplicities given by κ . This space admits a natural action by the group $GL_2^+(\mathbb{R})$, which is a generalization of the action of $GL_2^+(\mathbb{R})$ on the space of flat tori $GL_2^+(\mathbb{R})/SL(2, \mathbb{Z})$. The classification of the closed invariant sets is a central problem in Teichmüller dynamics. Recently, a deep theorem of Eskin, Mirzakhani and Mohammadi has shed light on the structure of such sets: they are immersed orbifolds cut out by linear equations with real coefficients in period coordinates. These objects are referred to as affine invariant orbifolds. See [EMM15].

Transverse to the $GL_2^+(\mathbb{R})$ -action, there is a local action by $\mathbb{C}^{|\kappa|-1}$ that fits into a natural holomorphic foliation of the strata of the moduli space. It is usually referred to as the isoperiodic foliation, the kernel foliation or the Rel foliation. The leaf \mathcal{F}_X of a translation surface (X, ω) is locally obtained as a level set of the absolute periods map. See [HW15] for a careful definition of this foliation. This foliation has been introduced about 25 years ago by Kontsevich and Eskin, and later by McMullen and Calta before it became a central

object in Teichmüller dynamics. See for example how it is involved in the classification of horocycle orbits of Prym eigenforms in $\mathcal{H}(1, 1)$ obtained by Bainbridge, Smillie and Weiss in [BSW16].

Several papers have been devoted to understanding the dynamics of its leaves. McMullen showed that the foliation is ergodic with respect to the Masur-Veech measure in the principal stratum in genus 2 and 3 using Ratner's theory in [McM14]. Hooper and Weiss gave the first examples of dense leaves outside the principal stratum in [HW15]. Shortly after Cal-samiglia, Deroin and Francaviglia have obtained a Ratner-like classification of the minimal sets in the principal stratum and obtained the ergodicity with respect to the Masur-Veech measure as a consequence of this classification. See [CDF15]. Simultaneously, Hamenstädt reproved this ergodicity statement using a different approach, still in the principal strata, in [Ham18]. Both this last two results used McMullen's result as a base case for an induction. Very surprisingly, apart from the examples of Hooper and Weiss, nothing is known for the dynamics of the isoperiodic foliation in strata where at least one zero is not simple.

1.2 Statement of the results

We define on any affine invariant orbifold \mathcal{M} a foliation $\mathcal{F}^{\mathcal{M}}$ that generalizes the isoperiodic foliation on the strata of the moduli space. Those foliations are particularly interesting in the light of Eskin, Mirzakhani and Mohammadi's result as any information on the geometry of affine invariant orbifolds directly translates into information on the dynamics of the $GL_2^+(\mathbb{R})$ -action. The dynamics of the foliation $\mathcal{F}^{\mathcal{M}}$ also contains information on the dynamics of the isoperiodic foliation itself as any leaf $\mathcal{F}_X^{\mathcal{M}}$ is a connected component of $\mathcal{F}_X \cap \mathcal{M}$.

The foliation $\mathcal{F}^{\mathcal{M}}$ will be referred to as the \mathcal{M} -isoperiodic foliation. For some affine invariant orbifolds, the leaves of $\mathcal{F}^{\mathcal{M}}$ have dimension 0. It is for instance the case for Teichmüller discs or for hyperelliptic loci of non hyperelliptic strata. More sophisticated examples are given in [EMMW18]. The other affine invariant orbifolds are called non absolute. In this text we study the dynamics of the foliation $\mathcal{F}^{\mathcal{M}}$ in the case where \mathcal{M} is a non absolute rank 1 affine invariant orbifold. We isolate a property \mathcal{P} that describes cylinder decompositions of a particular type for surfaces in \mathcal{M} and establish the following criterion, key to the rest of the paper:

Theorem A. *Let \mathcal{M} be a non absolute rank 1 affine invariant orbifold. If \mathcal{M} satisfies property \mathcal{P} , then all the leaves of $\mathcal{F}^{\mathcal{M}}$ are projectively dense.*

Here, projectively dense means that the leaf of any surface (X, ω) is dense in the locus of surfaces in \mathcal{M} that have the same area as (X, ω) . Equivalently, the projection of $\mathcal{F}_X^{\mathcal{M}}$ in $\mathbb{P}\mathcal{M}$ is dense. This has to do with the fact that the area of a translation surface can be computed only with its absolute periods and thus the area function is constant along the leaves of the \mathcal{M} -isoperiodic foliation. See definition 2.4 for more details. The proof of

theorem A relies on the fact that for a surface with a cylinder decomposition provided by property \mathcal{P} , the associated horocycle orbit is approximable by \mathcal{M} -isoperiodic deformations and thus is contained in the closure of its \mathcal{M} -isoperiodic leaf. We show that this can be done in a direction where the horocycle orbit closure coincide with the whole $SL_2(\mathbb{R})$ orbit closure, concluding the proof. We then give examples of affine invariant orbifolds that have property \mathcal{P} and others that do not. Special attention is dedicated to Prym eigenforms of genus 3 and we show the following:

Theorem B. *Let $\kappa = (2, 2)^{odd}, (2, 1, 1)$ or $(1, 1, 1, 1)$ and let \mathcal{M} be a connected component of $\Omega E_D(\kappa)$. Then either all the leaves of $\mathcal{F}^{\mathcal{M}}$ are all closed or all the leaves of $\mathcal{F}^{\mathcal{M}}$ are projectively dense. The last case occurs if, and only if, D is not a square.*

The remaining connected components $\mathcal{H}^{odd}(4)$, $\mathcal{H}^{hyp}(4)$ and $\mathcal{H}(2, 2)^{hyp}$ are not included in Theorem B as the corresponding Prym eigenform loci are union of Teichmüller discs and the leaves have dimension 0, hence closed even when D is not a square. This is clear when there is only a single singularity. See proposition 2.3 [LN14] for a proof of that claim in $\mathcal{H}^{hyp}(2, 2)$. The genus 2 case is an immediate consequence of Theorem A and had already been established by McMullen. It is known that Prym eigenforms of discriminant D are primitive if and only if D is not a square. Theorem B can thus be understood as a dynamical characterisation of primitivity for Prym eigenforms of genus 3.

Finally, we use Theorem A to investigate the geometric structure of the closure of isoperiodic leaves and deduce the following:

Theorem C. *Let $\kappa = (2, 1, 1)$ or $(1, 1, 1, 1)$ and let (X, ω) be a Prym eigenform in $\Omega E_D(\kappa)$. If D is not a square, then the leaf \mathcal{F}_X is projectively dense in $\mathcal{H}_3(\kappa)$.*

In the case of the Prym eigenforms in $\mathcal{H}^{odd}(2, 2)$, the leaves are not projectively dense in the stratum, and the computation of their closure is contained in Theorem A since in this situation the leaves of $\mathcal{F}^{\mathcal{M}}$ with \mathcal{M} being a connected component of $\Omega E_D(\kappa)$ are equal to the leaves of \mathcal{F} . It is interesting to note the dichotomy arising with the number of singularities. Theorem C gives, to our knowledge, the first examples of dense isoperiodic leaves in $\mathcal{H}(2, 1, 1)$ and gives a new proof, based on dynamics, of the projective density of the isoperiodic leaves of prym eigenforms in the case $\kappa = (1, 1, 1, 1)$. This was already known as part of the work of Calsamiglia, Deroin and Francaviglia.

1.3 Organisation of the paper

In section 2, we collect relevant definitions. We review the structures of the strata and of the closed $GL_2^+(\mathbb{R})$ -invariant sets, recall the definitions of the Prym eigenform loci, and define the foliation $\mathcal{F}^{\mathcal{M}}$. In section 3, we give a common framework for cylinder and isoperiodic deformations. In section 4, we establish the criterion presented in Theorem A and draw its first consequences on the structure of the closure of the leaves of $\mathcal{F}^{\mathcal{M}}$. In

section 5, we provide a list of affine invariant orbifolds that have property \mathcal{P} , and deduce Theorem B. In section 6, we use the structural result established in section 4 to prove theorem C. Appendix A is dedicated to enumerating all the cylinder decompositions of a certain type of Prym eigenforms in $\mathcal{H}(1,1,1,1)$. This list is used to show that Prym eigenforms in this stratum have property \mathcal{P} .

1.4 Acknowledgements

I am deeply indebted to Erwan Lanneau for introducing to the world of translation surfaces and suggesting the questions studied in this text, and many more. I would also like to thank Barak Weiss, Pascal Hubert and Duc Manh Nguyen for having discussed with me those results which have significantly benefited from their insightful comments. This work has been partially supported by the LabEx PERSYVAL-Lab (ANR-11-LABX-0025-01) funded by the French program Investissement d'avenir.

2 Framework

2.1 Period coordinates & the action of $GL_2^+(\mathbb{R})$

We denote by $\tilde{\mathcal{H}}(\kappa)$ the space of equivalence classes of marked translation surfaces (X, ω, f) where $f : S \rightarrow X$ is a homeomorphism from a fixed genus g surface S such that the preimage of the singularities of ω by f is a subset Σ , and the order of the singularities of ω are prescribed by κ . The following map is known as the period map :

$$\Phi : \begin{array}{ccc} \tilde{\mathcal{H}}(\kappa) & \rightarrow & H^1(S, \Sigma, \mathbb{C}) \\ (X, \omega, f) & \mapsto & (\gamma \mapsto \int_{f \circ \gamma} \omega) \end{array}$$

There is a complex structure on $\tilde{\mathcal{H}}(\kappa)$ that turns Φ into a local biholomorphism, and if $MCG(S, \Sigma)$ denotes the relative mapping class group of S that fixes Σ globally, then $MCG(S, \Sigma)$ acts almost freely on $\tilde{\mathcal{H}}(\kappa)$ by precomposition: $\varphi \cdot (X, \omega, f) = (X, \omega, f \circ \varphi^{-1})$. The quotient set is isomorphic to $\mathcal{H}(\kappa)$ and the latter is endowed with the complex orbifold structure that turns the canonical projection $\pi : \tilde{\mathcal{H}}(\kappa) \rightarrow \mathcal{H}(\kappa)$ into a local biholomorphism (in the orbifold sense, see [CJ19] for relevant definitions). The space $\tilde{\mathcal{H}}(\kappa)$ is endowed with a group action by $GL_2^+(\mathbb{R})$ defined by :

$$\forall g \in GL_2^+(\mathbb{R}) \quad \Phi(g \cdot (X, \omega, f)) = g \cdot \Phi(X, \omega, f)$$

That action descends to an action on $\mathcal{H}(\kappa)$ in a way that the canonical projection π is $GL_2^+(\mathbb{R})$ -equivariant. If $\theta \in S^1$, we denote by $R_\theta \in SL_2(\mathbb{R})$ the rotation of angle θ . We will use the following notation:

$$\forall \theta \in S^1 \quad \forall t \in \mathbb{R}, \quad g_t^\theta = R_\theta \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} R_{-\theta}, \quad h_t^\theta = R_\theta \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} R_{-\theta}$$

For any θ , the action of $A_\theta = \{g_t^\theta, t \geq 0\}$ and $H_\theta = \{h_t^\theta, t \geq 0\}$ are known as the geodesic flow and horocycle flow in direction θ . More details on the structures of these spaces and the action of $GL_2^+(\mathbb{R})$ can be found in [Zor06] or [FM13].

Definition 2.1 (affine invariant orbifold). *An affine invariant orbifold is a closed connected subset \mathcal{M} of $\mathcal{H}(\kappa)$ obtained as the image of a complex orbifold \mathfrak{M} by a proper immersion ι that satisfies the following property: if $x \in \mathfrak{M}$, there is an open set \mathcal{U} around x , an orbifold chart (\mathcal{V}, φ) around $\iota(x)$ and a \mathbb{R} -linear vector subspace V of $H^1(S, \Sigma, \mathbb{R})$ such that:*

$$\iota(\mathcal{U}) \cap \varphi(\mathcal{V}) = \varphi(\mathcal{V} \cap V \otimes \mathbb{C}) \quad (1)$$

In the previous definition, immersion refers to an immersion of orbifold, See [CJ19] for relevant definitions. Affine invariant orbifolds are invariant under the action of $GL_2^+(\mathbb{R})$ and Eskin, Mirzakhani and Mohammadi proved in a celebrated result that the converse is true. More details can be found in [EMM15]. We will use the following useful characterisation.

Proposition 2.1. *Let \mathcal{M} be an affine invariant orbifold, and let $\iota : \mathfrak{M} \rightarrow \mathcal{H}(\kappa)$ as in definition 2.1. If $p : \hat{\mathfrak{M}} \rightarrow \mathfrak{M}$ is the orbifold universal cover of \mathfrak{M} , there is linear subspace $V \in H^1(S, \Sigma, \mathbb{R})$ and a map $\hat{\iota} : \hat{\mathfrak{M}} \rightarrow \tilde{\mathcal{H}}(\kappa)$ such that $\pi \circ \hat{\iota} = \iota \circ p$ and $\Phi \circ \hat{\iota}$ is a local biholomorphism from $\hat{\mathfrak{M}}$ to $V \otimes \mathbb{C}$.*

Proof. By construction, the orbifold fundamental group of $\hat{\mathfrak{M}}$ is trivial. The smooth map $\iota \circ p$ can thus be lifted to a smooth map $\hat{\iota} : \hat{\mathfrak{M}} \rightarrow \tilde{\mathcal{H}}(\kappa)$ such that $\pi \circ \hat{\iota} = \iota \circ p$. Now, equation (1) implies that for any $x \in \mathfrak{M}$ there are a neighborhood \mathcal{U}_1 of x , a neighborhood \mathcal{U}_2 of $\Phi \circ \hat{\iota}(x)$ and a subspace $V_x \subset H^1(S, \Sigma, \mathbb{R})$ so that:

$$\Phi \circ \hat{\iota}(\mathcal{U}_1) = \mathcal{U}_2 \cap V_x \otimes \mathbb{C}$$

By connectedness of \mathcal{M} , one deduces that all the V_x are equal, proving that the image of $\Phi \circ \hat{\iota}$ is indeed of the form $V \otimes \mathbb{C}$. □

For details on orbifold fundamental groups, orbifold coverings and lifts of maps, the reader is referred to part \mathcal{G} chapter 1 and 3 of [BH13]

An important numerical invariant associated to these loci is the rank, defined as follows: define $\rho : H^1(S, \Sigma, \mathbb{C}) \rightarrow H^1(S, \mathbb{C})$ to be the canonical restriction map, and chose a local model V of \mathcal{M} as in (1). Avila, Eskin and Möller proved in [AEM17] that $\rho(V \otimes \mathbb{C})$ is a symplectic subspace of $H^1(S, \mathbb{C})$. The rank of \mathcal{M} , denoted $rk(\mathcal{M})$, is then defined as half the complex dimension of this space (this notion is well defined as affine invariant orbifolds are required to be connected). More details can be found in [Wri15]. The following definition will be important for the remainder of this text :

Definition 2.2. Let \mathcal{M} be a affine invariant orbifold. The field of definition of \mathcal{M} is the smallest subfield $k(\mathcal{M})$ of \mathbb{R} such that any V as in (1) can be written as $V = V_0 \otimes_{k(\mathcal{M})} \mathbb{R}$, where V_0 is a $k(\mathcal{M})$ -linear subspace of $H^1(S, \Sigma, k(\mathcal{M}))$

By definition, affine invariant orbifolds are cut out by real linear equations and the field of definition of \mathcal{M} is thus the smallest subfield of \mathbb{R} in which those linear equations take their coefficients. Wright has proved that the field of definition can be computed using periodic translation surfaces: let (X, ω) be a translation surface. the quadratic form $\omega \otimes \bar{\omega}$ induces a flat metric on X , away from the singularities of ω . We say that (X, ω) is periodic in direction θ if the geodesic rays in direction θ are either periodic, or saddle connections (they start and end at singularities of ω). It is well known that if a translation surface is periodic in a given direction, it is decomposed as a union of cylinders (those are subsets isometric to $[0, h] \times \mathbb{R}/c\mathbb{Z}$) whose boundary components are union of saddle connections in direction θ). Wright proved the following in [Wri15]:

Proposition 2.2. Let \mathcal{M} be an affine invariant orbifold, and let (X, ω) be a periodic surface in \mathcal{M} with m cylinders. Then, $k(\mathcal{M})$ is contained in $\mathbb{Q}[c_2c_1^{-1}, \dots, c_m c_1^{-1}]$ where the c_i are the circumferences of the cylinders.

Wright actually proved a stronger result and proved the other inclusion, provided one considers only a subclass of cylinders. See [Wri15] for more details.

2.2 The \mathcal{M} -isoperiodic foliation

The aim of this subsection is to define a foliation on any affine invariant orbifold. In that perspective, let us first recall the definition of the usual isoperiodic foliation \mathcal{F} : the map $\rho \circ \Phi$ is a $MCG(S, \Sigma)$ -equivariant submersion. The isoperiodic foliation is defined as the quotient foliation of the foliation of $\mathcal{H}(\kappa)$ by connected components of the level sets of $\rho \circ \Phi$. We are going to adapt this construction. Let \mathcal{M} be an invariant affine orbifold and (X, ω) be a surface in \mathcal{M} . We denote by $\mathcal{F}_X^{\mathcal{M}}$ the connected component of $\mathcal{F}_X \cap \mathcal{M}$ that contains (X, ω) .

Proposition 2.3. Let $\iota : \mathfrak{M} \rightarrow \mathcal{H}(\kappa)$ be an immersion such that $\iota(\mathfrak{M}) = \mathcal{M}$ as in definition 2.1. There is a foliation \mathfrak{F} on \mathfrak{M} such that for any $(X, \omega) \in \mathcal{M}$ and $x \in \mathfrak{M}$ such that $\iota(x) = (X, \omega)$, there is a neighborhood \mathcal{U} around x in \mathfrak{M} such that:

$$\iota(\mathcal{U} \cap \mathfrak{F}_x) = \iota(\mathcal{U}) \cap \mathcal{F}_X^{\mathcal{M}}$$

Proof. Let $\hat{\iota} : \hat{\mathfrak{M}} \rightarrow \mathcal{H}(\kappa)$ and $V \in H^1(S, \Sigma, \mathbb{R})$ as in proposition 2.1. Since $\Phi \circ \hat{\iota}$ is a local biholomorphism and ρ is a linear surjection, the map $\rho \circ \Phi \circ \hat{\iota}$ is a submersion onto $\rho(V \otimes \mathbb{C})$. The connected components of its level sets induce a foliation on $\hat{\mathfrak{M}}$. The map ι induces a representation of the orbifold fundamental group of \mathcal{M} in $MCG(S, \Sigma)$ and thus an action on $\rho(V \otimes \mathbb{C})$. By construction the map $\rho \circ \Phi \circ \hat{\iota}$ is $\pi_1^{orb}(\mathfrak{M})$ -equivariant. This proves that

the aforementioned foliation descends to a foliation \mathfrak{F} on \mathfrak{M} .

Now, let (X, ω) and x be as in the statement and choose a neighborhood \mathcal{U} of x , a point $\hat{x} \in \hat{\mathfrak{M}}$ and a $\hat{\mathcal{U}}$ of \hat{x} in $\hat{\mathfrak{M}}$ such that $p(\hat{\mathcal{U}}) = \mathcal{U}$. Up to shrinking \mathcal{U} , one can assume that $\Phi \circ \hat{\iota}$ induces a diffeomorphism from $\hat{\mathcal{U}}$ to $\mathcal{V} \cap V \otimes \mathbb{C}$. By definition of \mathfrak{F} , we have:

$$\mathcal{U} \cap \mathfrak{F}_x = p((\Phi \circ \hat{\iota})^{-1}(\xi + \text{Ker} \rho \cap V \otimes \mathbb{C} \cap \mathcal{V}))$$

Consequently, using the relation $\pi \circ \hat{\iota} = \iota \circ p$, we obtain:

$$\begin{aligned} \iota(\mathcal{U} \cap \mathfrak{F}_x) &= \pi(\Phi^{-1}(\xi + \text{Ker} \rho \cap V \otimes \mathbb{C} \cap \mathcal{V})) \\ &= \pi(\Phi^{-1}(V \otimes \mathbb{C} \cap \mathcal{V})) \cap \pi(\Phi^{-1}(\xi + \text{Ker} \rho)) \\ &= \iota(\mathcal{U}) \cap \mathcal{F}_X \end{aligned}$$

To conclude, notice that $\iota(\mathcal{U}) = \iota(\mathcal{U}) \cap \mathcal{M}$ and thus $\iota(\mathcal{U}) \cap \mathcal{F}_X = \iota(\mathcal{U}) \cap \mathcal{F}_X^{\mathcal{M}}$ □

Remark. In the previous construction, the foliation \mathfrak{F} is independent of the choices made for $\hat{\mathcal{M}}$ and $\hat{\iota}$ as any other choice only differ by the action of an element of $\pi_1^{\text{orb}}(\mathfrak{M})$ and the leaves of \mathfrak{F} are obtained after taking quotient by $\pi_1^{\text{orb}}(\mathfrak{M})$.

Given the nature of proposition 2.3, we shall say that the $\mathcal{F}_X^{\mathcal{M}}$ are the leaves of an immersed foliation, which will be referred to as the **\mathcal{M} -isoperiodic foliation**

Examples.

1. If \mathcal{M} is the whole stratum $\mathcal{H}(\kappa)$, the foliation $\mathcal{F}^{\mathcal{M}}$ is the usual isoperiodic foliation (or Kernel foliation or Rel foliation).
2. Let \mathcal{M} and \mathcal{N} be two affine invariant orbifolds with $\mathcal{M} \subset \mathcal{N}$. Then the \mathcal{M} -isoperiodic leaf of any surface of \mathcal{M} is contained in its \mathcal{N} -isoperiodic leaf.
3. Let (X, ω) be a flat cover of a Veech surface (X_0, ω_0) with ramification over the singularities of ω_0 and a non periodic point. Let \mathcal{M} be the $GL_2^+(\mathbb{R})$ -orbit closure of (X, ω) . Then the \mathcal{M} -isoperiodic leaf of any surface in \mathcal{M} is isomorphic to the surface X_0 punctured at the singularities of ω_0 .
4. Let \mathcal{M} be a Teichmuller disc. Then the \mathcal{M} -isoperiodic leaf of any surface $(X, \omega) \in \mathcal{M}$ is the singleton $\{(X, \omega)\}$.

The last example of this list leads us to the following definition:

Definition 2.3. *An affine invariant orbifold \mathcal{M} is said to be non absolute when leaves of $\mathcal{F}^{\mathcal{M}}$ have positive dimension. This is equivalent to the fact that any local model V of \mathcal{M} as in (1) intersects $\text{ker} \rho$ non trivially.*

Notice that all the local models of \mathcal{M} intersect $\ker \rho$ non trivially if, and only if, one of its model does. An affine invariant orbifold \mathcal{M} is non absolute whenever $\dim_{\mathbb{C}}(\mathcal{M}) > 2rk(\mathcal{M})$. The \mathcal{M} -isoperiodic foliation behave nicely with respect to the action of $GL_2^+(\mathbb{R})$, as shows the following:

Proposition 2.4. *Let \mathcal{M} be a non absolute affine invariant orbifold, let (X, ω) be a translation surface in \mathcal{M} and let $g \in GL_2^+(\mathbb{R})$. The following formulas holds :*

$$g \cdot \mathcal{F}_X^{\mathcal{M}} = \mathcal{F}_{g \cdot X}^{\mathcal{M}} \quad (2)$$

and

$$GL_2^+(\mathbb{R}) \cdot \mathcal{F}_X^{\mathcal{M}} = \mathcal{M} \quad (3)$$

Proof. Equation (2) is a consequence of the fact the Period map is $GL_2^+(\mathbb{R})$ -equivariant. For equation (3), let (X_1, ω_1) be a translation surface in \mathcal{M} . Notice that sets $GL_2^+(\mathbb{R}) \cdot \mathcal{F}_X^{\mathcal{M}}$ and $GL_2^+(\mathbb{R}) \cdot \mathcal{F}_{X_1}^{\mathcal{M}}$ both contain an open set and since the action of $GL_2^+(\mathbb{R})$ on \mathcal{M} is topologically transitive, we deduce that there is a matrix $g \in GL_2^+(\mathbb{R})$ such that $g \cdot \mathcal{F}_X^{\mathcal{M}} \cap \mathcal{F}_{X_1}^{\mathcal{M}}$ is non empty. But by (2) $g \cdot \mathcal{F}_X^{\mathcal{M}} = \mathcal{F}_{g \cdot X}^{\mathcal{M}}$ and thus $g \cdot \mathcal{F}_X^{\mathcal{M}} = \mathcal{F}_{X_1}^{\mathcal{M}}$. This concludes. \square

Recall that the area of a surface $(X, \omega) \in \mathcal{H}(\kappa)$ is defined as the integral of $\frac{i}{2} \cdot \omega \wedge \bar{\omega}$ over X . It is well known that the area of (X, ω) only depends upon the integral of ω over the absolute cycles, and thus the area function is constant along the leaves of the \mathcal{M} -isoperiodic foliation. This leads to the following definition.

Definition 2.4. *Let \mathcal{M} be a non absolute affine invariant orbifold, and let (X, ω) be a surface in \mathcal{M} . The leaf $\mathcal{F}_X^{\mathcal{M}}$ is said to be projectively dense if it dense in the set of surfaces in \mathcal{M} that have the same area as (X, ω) .*

2.3 Prym eigenform loci

In this section, we recall a construction giving an infinite family of rank one affine invariant orbifolds discovered by McMullen. Let (X, ω) be a translation surface endowed with a holomorphic involution τ . We denote by $\Omega(X)$ the set of holomorphic 1-forms, and by $\Omega^-(X)$ the set of τ -anti invariant holomorphic 1-forms. We say that (X, ω) is a Prym form if $\omega \in \Omega^-(X)$, that is $\tau^* \omega = -\omega$, and $\dim \Omega^-(X) = 2$. The Prym variety $Prym(X, \tau)$ is defined as the 2-dimensional abelian variety $(\Omega^-(X))^*/H_1^-(X, \mathbb{Z})$ endowed with the polarization coming from the intersection form on $H_1(X, \mathbb{Z})$. Finally, let D be a positive integer congruent to 0 or 1 mod 4 and let $\mathcal{O}_D \simeq \mathbb{Z}[X]/(X^2 + bX + c)$ be the real quadratic order of discriminant $D = b^2 - 4c$.

Definition 2.5. *A Prym eigenform is a Prym form (X, ω) corresponding to an involution τ such that there is an injective ring morphism $\mathfrak{i} : \mathcal{O}_D \rightarrow \text{End}(Prym(X, \tau))$ satisfying the following properties:*

1. $\mathfrak{i}(\mathcal{O}_D)$ is a proper self adjoint subring of $\text{End}(\text{Prym}(X), \tau)$.
2. ω is an eigenvector for the action of \mathcal{O}_D on $\Omega(X)^-$

We will denote by $\Omega E_D(\kappa)$ the set Prym eigenforms as in the previous definition contained in the stratum $\mathcal{H}(\kappa)$. For more details, see [McM07b]. McMullen proved the following:

Proposition 2.5. *The $GL_2^+(\mathbb{R})$ -orbit closure of any Prym eigenform is a rank one affine invariant orbifold.*

The Prym eigenforms play a crucial role in McMullen's classification of affine invariant orbifold in genus 2. It is proved in [McM07a] that if the orbit of a surface is neither closed nor dense in the stratum in which it belongs, then it is a Prym eigenform, the Prym involution being given by the hyperelliptic involution. In [McM07b], infinite families of Prym eigenforms are constructed in genus up to 5, and it is a consequence of Riemann-Hurwitz formula that Prym eigenforms can not exist for genus bigger than 5.

3 Modifying the twist parameters

In this section, we establish a number of useful results and tools to deform translation surfaces.

3.1 The twist map

A classic and useful construction to navigate inside the moduli space is as follows. Suppose (X, ω) is periodic in direction θ and decomposed into m cylinders, which we denote by $\mathcal{C}_1, \dots, \mathcal{C}_m$. We denote by γ_i the core curve of the cylinder \mathcal{C}_i . One can define a smooth map (in the orbifold sense) τ from \mathbb{R}^m to $\mathcal{H}(\kappa)$ in the following way: choose (x_1, \dots, x_m) in \mathbb{R}^m , cut open the surface along the core curves of its cylinders, rotate the top component of each cylinder \mathcal{C}_i by an amount of $x_i c_i$ and glue back. This construction yields a smooth embedding from the m -dimensional torus to $\mathcal{H}(\kappa)$. See figure 1. This embedding, called the twist map, is of particular interest as it allows to see specific deformations of surfaces as linear flows, of which we have a detailed understanding.

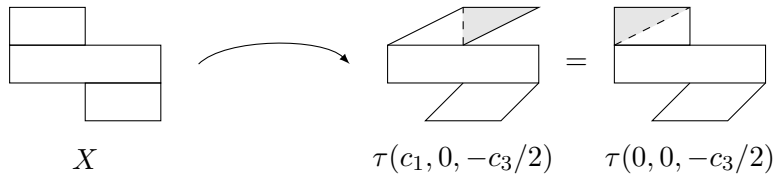


Figure 1: twisting cylinders

More precisely, let $f : S \rightarrow X$ be a marking on X , and define $\xi = \Phi(X, \omega, f)$. Recall that the geometric intersection induces a non degenerate form $I : H_1(S - \Sigma, -\mathbb{Z}) \times H_1(S, \Sigma, \mathbb{Z}) \rightarrow \mathbb{Z}$. The core curves of the cylinders can be seen as elements of $H_1(S - \Sigma, \mathbb{Z})$ and we denote by C_i^* the cocycle $I(f^{-1} \circ \gamma_i, \cdot)$. Define E to be the linear subspace of $H^1(S, \Sigma, \mathbb{R})$ spanned by the C_i^* and finally set \mathcal{C} to be the arc-wise component of $\{(Y, \theta, h) \in \tilde{\mathcal{H}}(\kappa) \mid \Phi(Y, \theta, h) \in \xi + E\}$ that contains (X, ω, f) .

Proposition 3.1. *The period map induces a diffeomorphism from \mathcal{C} to $\xi + E$.*

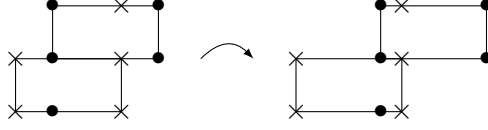
Now, define a map $\tilde{\tau}$ from \mathbb{R}^m to $\tilde{\mathcal{H}}(\kappa)$ that sends (x_1, \dots, x_m) to the surface of \mathcal{C} whose image by the period map is $\xi + \sum c_i x_i C_i^*$. The action of \mathbb{Z}^m on \mathbb{R}^m by translation is intertwined by $\tilde{\tau}$ with the action of the subgroup of Γ_g spanned the Dehn twists about the core curves of the cylinders. There is also a finite subgroup Δ , perhaps trivial, of automorphisms of (X, ω) that acts on X by permuting cylinders and on the m -dimensional torus \mathbb{T}^m by permuting the coordinates. Consequently, there is a map $\tau : \mathbb{T}^m / \Delta$ to $\mathcal{H}(\kappa)$ that fits into the commutative diagram depicted in figure 2. We shall refer to the map τ as the twists map associated to (X, ω) . Notice that by construction $\Phi \circ \tilde{\tau}$ is affine, and its constant part is ξ . For a similar construction and more details, the reader is referred to the section 6.2 of [HW15].

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{\tilde{\tau}} & \tilde{\mathcal{H}}(\kappa) \\ q \downarrow & & \downarrow \pi \\ \mathbb{T}^m / \Delta & \xrightarrow{\tau} & \mathcal{H}(\kappa) \end{array}$$

Figure 2: the twist map

3.2 Navigating inside affine invariant orbifolds with twists

Let \mathcal{M} is a non absolute affine invariant orbifold that contains the surface (X, ω) . We denote by k the rank of \mathcal{M} and by r the dimension of the foliation $\mathcal{F}^{\mathcal{M}}$. This last quantity is known as the Rel of \mathcal{M} . From now on, we assume that the cylinder decomposition of (X, ω) is $\mathcal{F}^{\mathcal{M}}$ -**stable**: this means that any every horizontal saddle connection of X vanishes as an element of $T_X^* \mathcal{F}_X^{\mathcal{M}}$. Equivalently, this means that there are no saddle connections joining different singularities along a boundary of a cylinder, or if there is one it has to remain so along any isoperiodic deformation inside \mathcal{M} that preserves the cylinder decomposition. The stable decomposition is relevant in our context as they allow to recognize some isoperiodic deformation as cylinder deformations. The following figure shows an example of isoperiodic deformation that is not expressed as a cylinder deformation:



Let $\hat{\iota} : \mathfrak{M} \rightarrow \mathcal{H}(\kappa)$ and $V \in H^1(S, \Sigma, \mathbb{C})$ as in proposition 2.1 such that (X, ω, f) belongs to the image of $\hat{\iota}$. Define $V_{\mathcal{M}}$ and $K_{\mathcal{M}}(X)$ to be the preimages of V and $\xi + \text{Ker}\rho \cap V$ by the map $\Phi \circ \tilde{\tau}$. Those spaces are linear as preimages of affine subspaces by an affine map that contain 0. Notice that by construction $\tau \circ q(V_{\mathcal{M}})$ is contained in \mathcal{M} and $\tau \circ q(K_{\mathcal{M}}(X))$ is contained in $\mathcal{F}_X^{\mathcal{M}}$. We emphasize that $V_{\mathcal{M}}$ and $K_{\mathcal{M}}(X)$ are defined up to the choice of the pair $(\hat{\iota}, V)$.

Lemma 3.1. *The space $V_{\mathcal{M}}(X)$ is rational and its dimension is at most $k+r$, the dimension of $K_{\mathcal{M}}(X)$ is exactly r .*

Proof. 1. Choose a neighborhood \mathcal{U} of (X, ω, f) in $\tilde{\mathcal{H}}(\kappa)$ such that any point point in \mathcal{U} represents a surface in \mathcal{M} if, and only if, its image by the period map is contained in V . Now, let $V_{\mathcal{M}}(X)^{\mathbb{Q}}$ be the smallest rational linear subspace that contains $V_{\mathcal{M}}(X)$. It is known that $q(V_{\mathcal{M}}(X))$ is dense in $q(V_{\mathcal{M}}(X)^{\mathbb{Q}})$ and since \mathcal{M} is closed, $\tau \circ q(V_{\mathcal{M}}(X)^{\mathbb{Q}})$ is contained in \mathcal{M} . As a consequence, $\Phi \circ \tilde{\tau}(V_{\mathcal{M}}(X)^{\mathbb{Q}}) \cap \Phi(\mathcal{U})$ is contained in V . By definition of $V_{\mathcal{M}}$, this implies that $V_{\mathcal{M}}^{\mathbb{Q}}$ is contained in $V_{\mathcal{M}}$, hence $V_{\mathcal{M}}$ is rational. Now, notice that $\rho(d_0\tau(\mathbb{R}^m))$ is isotropic as it is spanned by the C_i^* and the γ_i do not intersect. By [AEM17], we know that $\rho(V)$ is symplectic and has dimension $2k$ by definition. This implies that $\rho(d_0\tau(V_{\mathcal{M}}(X)))$ has dimension at most k and since the dimension of $\text{ker}\rho \cap V$ is r by definition, we deduce that V has dimension at most $k + r$.

2. We shall prove that the linear part of $\Phi \circ \tilde{\tau}$ is onto $\text{Ker}\rho \cap V$. Let u be in $\text{Ker}\rho \cap V$. Since the cylinder decomposition of X is stable, it implies that u vanishes on the cycle forming the boundaries of the cylinders. It also vanishes on the core curves of the cylinders as they are absolute cycles. Consequently, the only cycles on which it may not vanish are the cross curves of the cylinders (those are path inside a cylinder that connects the bottom component to the top component), which we denote by σ_i . Consequently, we get that $u = \sum u(\sigma_i)C_i^*$, and thus $u = \Phi \circ \tilde{\tau}(u(\sigma_1)c_1^{-1}, \dots, u(\sigma_m)c_m^{-1}) - \xi$. As the dimension of $\text{Ker}\rho \cap V$ is r by definition and $\Phi \circ \tilde{\tau}$ is injective, we get that the dimension of $K_{\mathcal{M}}(X)$ is r . \square

The case of equality for the dimension of $V_{\mathcal{M}}$ in the previous proposition is obtained precisely when the two subspaces $\text{Pres}(X, \mathcal{M})$ and $\text{Twist}(X, \mathcal{M})$ defined in [Wri15] are equal. Reciprocally, if we do not require the decomposition of X to be $\mathcal{F}^{\mathcal{M}}$ -stable, the dimension of $K_{\mathcal{M}}(X)$ is only bounded above by r . This expresses the fact that some isoperiodic deformations that preserve that cylinder decomposition of (X, ω) will not be recognized as

twist deformations.

Examples.

- If $\mathcal{M} = \mathcal{H}(\kappa)$ then $V_{\mathcal{M}}(X) = \mathbb{R}^m$ and is indeed rational.
- If \mathcal{M} is a connected component of $\Omega E_D(\kappa)$ then $v \in K_{\mathcal{M}}(X)$ if and only if $\tau(v)$ is in \mathcal{F}_X and the Prym involution of (X, ω) persits on $\tau(v)$. This comes from the fact that the condition defining Prym eigenforms among Prym forms does not involve relative periods.

Definition 3.1. *The support of $u \in K_{\mathcal{M}}(X)$ is the collection of indices i so that $u_i \neq 0$. If there are no other vector in $K_{\mathcal{M}}(X)$ whose support is stricly contained the one of u , we say that u is a minimal \mathcal{M} -isoperiodic deformation. Finally, two minimal \mathcal{M} -isoperiodic deformations are said to be transverse if their supports are disjoint.*

The support of u records the information of which cylinders have been twisted on (X, ω) to get to the surface $\tau(u)$, and u is a minimal \mathcal{M} -isoperiodic deformation if it is not possible to twist some cylinders in the support of u while staying inside $\mathcal{F}_X^{\mathcal{M}}$ without twisting the other cylinders of the support of u . Finally, we define the degree of $u \in \mathbb{R}^m$ as the $\dim \text{Span}_{\mathbb{Q}} \langle u_i \rangle - 1$

Definition 3.2 (Property \mathcal{P}). *The surface (X, ω) is said to have property \mathcal{P} if it has a horizontal $\mathcal{F}^{\mathcal{M}}$ -stable cylinder decomposition such that $K_{\mathcal{M}}(X)$ contains a minimal \mathcal{M} -isoperiodic deformation of degree at least 1. By extension, we say that an affine invariant orbifold \mathcal{M} has property \mathcal{P} if it contains a surface that has property \mathcal{P} .*

Remark. Having property \mathcal{P} does not depend on the choice of the pair (\hat{t}, V) even though the spaces $V_{\mathcal{M}}(X)$ and $K_{\mathcal{M}}(X)$ do, as we mentionned. This is due to the fact $MCG(S, \Sigma)$ acts on $H^1(S, \Sigma, \mathbb{C})$ by integer matrices.

4 Property \mathcal{P} and density of the leaves

4.1 Proof of Theorem A

Proposition 4.1. *Let \mathcal{M} be a non absolute rank one affine invariant orbifold and suppose (X, ω) is a surface in \mathcal{M} that has property \mathcal{P} . Then the closure of $\mathcal{F}^{\mathcal{M}}$ contains the horocycle orbit of (X, ω) .*

Proof. Let $u \in K_{\mathcal{M}}(X)$ be a minimal deformation of positive degree. There is a rational subspace $V_u \subset \mathbb{R}^m$ of dimension $1 + \deg(u)$ so that $q(V_u)$ is the closure of $\{q(t \cdot u), t \in \mathbb{R}\}$. By proposition 3.1, V_u is contained in $V_{\mathcal{M}}$. However, it is not contained in $K_{\mathcal{M}}(X)$. Indeed, the support of any vector in V_u is contained in the one of u , this is due to the fact that

vanishing of a coordinate is a rational linear relation that has to be satisfied by any vector in V_u . But since the degree of u is at least 2, there is a vector for which this inclusion is strict. By definition, such a vector can not be in $K_{\mathcal{M}}(X)$. Now, the subspace $K_{\mathcal{M}}(X) + V_u$ contains $K_{\mathcal{M}}(X)$ and this inclusion is strict. Consequently, its dimension is at least $k + 1$. But, since \mathcal{M} is rank 1, the dimension of $V_{\mathcal{M}}(X)$ is at most $k + 1$. Hence, $V_{\mathcal{M}}(X) = K + V_u$ and the dimension of $V_{\mathcal{M}}(X)$ is $k + 1$. As a consequence of this equality, we deduce that $\tau \circ q(V_{\mathcal{M}})$ is contained in the closure of $\mathcal{F}_X^{\mathcal{M}}$. Indeed, let $w = k + v \in V_{\mathcal{M}}$, with $v \in V_u$. Then there is a growing sequence $t_n > 0$ so that $q(t_n \cdot u)$ converges to v . But then, by continuity, $\tau \circ q(k + t_n \cdot v)$ is a sequence of points in $\mathcal{F}_X^{\mathcal{M}}$ converging to $\tau \circ q(w)$. To conclude, notice that $h_t \cdot (X, \omega) = \tau(t \cdot (h_1 c_1^{-1}, \dots, h_m c_m^{-1}))$ is contained in $\tau \circ q(V_{\mathcal{M}}(X))$. \square

Proposition 4.2. *Let \mathcal{M} be a non absolute rank 1 affine invariant orbifold and $(X, \omega) \in \mathcal{M}$. If the horocycle orbit of (X, ω) is contained in its isoperiodic leaf, then so is its $SL_2(\mathbb{R})$ -orbit.*

Proof. We will need the following lemma:

Lemma 4.1. *The \mathcal{M} -isoperiodic leaf of (X, ω) contains a surface fixed by a non trivial hyperbolic matrix.*

proof of the lemma. It is proved in, for instance, [Wri14] that any affine invariant orbifold contains a surface fixed by a non trivial hyperbolic matrix, using a closing lemma for the geodesic flow. Let (X_1, ω_1) be such a surface associated to a hyperbolic matrix g . Now corollary 3 applied to (X_1, ω_1) shows that there is a matrix u that sends (X_1, ω_1) to the \mathcal{M} -isoperiodic leaf of (X, ω) . The surface $u \cdot (X_1, \omega_1)$ is fixed by the hyperbolic matrix ugu^{-1} . \square

Now, pick a surface $(X_0, \omega_0) \in \mathcal{F}_X^{\mathcal{N}}$ as in the previous lemma, with $g \cdot (X_0, \omega_0) = (X_0, \omega_0)$ for some hyperbolic matrix g and denote by θ the angle between the horizontal line and the expanding line of this matrix. Using proposition (2) and the fact that the closure of $\mathcal{F}_X^{\mathcal{M}}$ is saturated by $\mathcal{F}^{\mathcal{M}}$ we show that the horocycle orbit of (X_0, ω_0) is also contained in the closure of the leaf of (X, ω) . Let θ_n be the angle between the horizontal line and the image under g^n of this horizontal line. Note that this sequence converges to θ . If $H_{\theta_n} \cdot (X, \omega)$ is contained in the closure of $\mathcal{F}_X^{\mathcal{M}}$, using proposition 2, one gets that $g \cdot H_{\theta_n} \cdot (X, \omega)$ is also contained in the closure of $\mathcal{F}_X^{\mathcal{M}}$. But since $g \cdot H_{\theta_n} = H_{\theta_{n+1}} \cdot g$, the latter is exactly $H_{\theta_{n+1}} \cdot g \cdot (X, \omega) = (X, \omega)$, this gives that for all $n > 0$ $H(\theta_n) \cdot (X, \omega)$ is contained in the closure of $\mathcal{F}_X^{\mathcal{M}}$. By continuity, this implies that the horocycle orbit in direction θ of (X, ω) also is contained in the closure of its \mathcal{M} -isoperiodic leaf. To conclude, use proposition 5.2 in [HW15] to deduce that the closure of $\mathcal{F}_X^{\mathcal{M}}$ contains the $SL_2(\mathbb{R})$ -orbit of (X_1, ω_1) . Once again, we use proposition 2 to deduce that the $SL_2(\mathbb{R})$ -orbit of (X, ω) is also contained in the closure of its \mathcal{M} -isoperiodic leaf. \square

We can now prove:

Theorem 4.1 (Theorem A). *Let \mathcal{M} be a non absolute rank one affine invariant orbifold having property \mathcal{P} . Then the \mathcal{M} -isoperiodic foliation is projectively minimal.*

Proof. Let (X, ω) be a surface in \mathcal{M} that has property \mathcal{P} . According to proposition 4.1, the $SL_2(\mathbb{R})$ -orbit of (X, ω) is contained in the closure of its \mathcal{M} -isoperiodic leaf. Using equation (2) and the fact the closure of $\mathcal{F}_X^{\mathcal{M}}$ saturated by $\mathcal{F}^{\mathcal{M}}$, we deduce that the closure $\mathcal{F}_X^{\mathcal{M}}$ is invariant under the action of $SL_2(\mathbb{R})$. Using equation (3), we conclude that the \mathcal{M} -isoperiodic leaf of (X, ω) is projectively dense. Finally, another utilisation of equation (3) together with 2 proves that there is an element of $GL_2^+(\mathbb{R})$ that sends the \mathcal{M} -isoperiodic leaf of (X, ω) to the \mathcal{M} -isoperiodic leaf of any other surface in \mathcal{M} . Since this action is continuous, all the leaves are projectively dense. \square

In section 5, we will give examples of rank 1 affine invariant orbifolds that have property \mathcal{P} . However, not all of them have it, as shows the following example.

Proposition 4.3. *Let (X_0, ω_0) be a Veech surface, that is a surface whose $GL_2^+(\mathbb{R})$ -orbit is closed, and let $p : (X, \omega) \rightarrow (X_0, \omega_0)$ be a flat cover: a holomorphic map such that $p^*\omega_0 = \omega$. Finally, let \mathcal{M} be the $GL_2^+(\mathbb{R})$ -orbit closure of (X, ω) . Then, \mathcal{M} is a rank 1 affine invariant orbifold such that $\mathcal{F}_X^{\mathcal{M}}$ is closed. In particular, if \mathcal{N} is an arithmetic rank 1 affine invariant orbifold, then the leaves of $\mathcal{F}^{\mathcal{N}}$ are closed.*

Proof. Let us prove the first assertion first. Let \mathcal{M}_0 be the $GL_2^+(\mathbb{R})$ -orbit of (X_0, ω_0) . It is easy to show that any surface in \mathcal{M} is also a flat cover over a surface in \mathcal{M}_0 . Thus there is a $GL_2^+(\mathbb{R})$ -equivariant map Ψ from \mathcal{M} to \mathcal{M}_0 that maps a surface to the unique surface it covers, and this map is continuous. To conclude, it remains to notice that the leaf $\mathcal{F}_X^{\mathcal{M}}$ is a connected component of the preimage of (X_0, ω_0) by Ψ , which is closed.

For the second assertion, notice that in this case, \mathcal{N} contains a surface whose period are rational and thus is a flat cover over a square torus, which is a Veech surface. Up to flowing along $\mathcal{F}^{\mathcal{N}}$, we get a new surface that still is a flat cover of a torus and whose $GL_2^+(\mathbb{R})$ -orbit closure is \mathcal{M} . Then the second assertion follows from the first one. \square

Remark. The affine invariant orbifold \mathcal{M} in proposition 4.3 is non absolute if and only if the set of ramification points of p contains a non periodic point. See [GHS03] for more details.

4.2 Structure of the closure of the leaves

Theorem A allows us to compute the closure of the leaves of the full isoperiodic foliation \mathcal{F} in $\mathcal{H}(\kappa)$.

Theorem 4.2. *If (X, ω) is a surface in a rank 1 affine invariant orbifold \mathcal{M} with property \mathcal{P} , then the isoperiodic leaf \mathcal{F}_X is projectively dense in an affine invariant orbifold*

\mathcal{N} . Furthermore, if $K_{\mathcal{H}(\kappa)}(X)$ contains m transverse minimal isoperiodic deformations of degree $(d_i)_{i \leq m}$, then :

$$rk(\mathcal{N}) \geq \sum d_i$$

Proof. We show as in the proof of Theorem A that the closure of \mathcal{F}_X is invariant under the action of $SL_2(\mathbb{R})$. Therefore the leaf \mathcal{F}_X is projectively dense in a affine invariant orbifold \mathcal{N} . Now, notice that $K_{\mathcal{H}(\kappa)}(X) = K_{\mathcal{N}}(X)$, and denote by u_i be the family of transverse minimal deformations of the statement. Denote also by V_i the associated subspace of $V_{\mathcal{N}}$ that gives the closure of $\{q(t \cdot u_i), t > 0\}$, and by $p_{\mathcal{N}}$ the canonical projection from $V_{\mathcal{N}}$ to $V_{\mathcal{N}}/K_{\mathcal{N}}$. We claim that the $p_{\mathcal{N}}(V_i)$ form a direct sum. Indeed, let $v_i \in V_i$ so that $\sum p_{\mathcal{N}}(v_i) = 0$. Then, there is a $k \in K_{\mathcal{N}}$ so that $\sum v_i = k$. Let $k_i \in K_{\mathcal{N}} \cap V_i$ so that $k = \sum k_i$. We then have $\sum (v_i - k_i) = 0$. Notice that since the u_i are transverse, if two vectors belong to different V_i , their support are disjoint. This implies that $v_i = k_i$, and thus $p_{\mathcal{N}}(v_i) = 0$. We are thus left with proving that the dimension of $p_{\mathcal{N}}(V_i)$ is m_i . Proposition 1.4.1 of [KH95] implies that for any i , there is an independant family $v_{i,j}$ of $m_i + 1$ deformations such that $v_{i,0} = v_i$ and the support of $v_{i,j+1}$ is strictly contained in the one of $v_{i,j}$. By definition of being minimal, this implies that the $v_{i,j}$ are not contained in $K_{\mathcal{N}}$, for every $j \geq 1$. This gives that the dimension of $K_{\mathcal{N}}/V_{\mathcal{N}}$ is bounded below by $\sum d_i$, and since by lemma 3.1 the dimension of $K_{\mathcal{N}}/V_{\mathcal{N}}$ is bounded above by $rk(\mathcal{N})$. \square

5 Examples of rank 1 affine invariant orbifold with property \mathcal{P}

5.1 The genus 2 case

Proposition 5.1. *Let \mathcal{M} be a non arithmetic rank one affine invariant orbifold contained in $\mathcal{H}(1,1)$. Then \mathcal{M} has property \mathcal{P} .*

Proof. Let (X, ω) be a horizontally periodic surface in \mathcal{M} , and up to flowing along the isoperiodic foliation, we can assume that this decomposition is $\mathcal{F}^{\mathcal{M}}$ -stable. That means that (X, ω) is cylinder equivalent (meaning up to adjusting height and twist parameter of the cylinders) to the following surface:

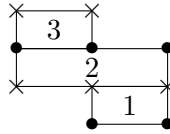


fig. The only stable cylinder decomposition in genus 2

This affirmation can be checked very easily as there are only two possibilities for the diagram of separatrices in this case, and they are isomorphic. See Appendix A for more details. Then the vector $(\gamma_1, -\gamma_2, \gamma_3)$ belongs to $K_{\mathcal{M}}(X)$, and it is necessarily of positive degree or else proposition 2.2 would imply that \mathcal{M} is arithmetic. \square

We recall that the non absolute rank one affine invariant orbifolds in genus 2 have been classified by McMullen in [McM07a], and they are connected components of prym eigenform loci. In this case \mathcal{F} and $\mathcal{F}^{\mathcal{M}}$ are the same as the $\Omega E_D(1, 1)$ are saturated by the isoperiodic foliation. These remarks, together with Theorem A and proposition 4.3 then implies:

Theorem 5.1. *Let (X, ω) be a Prym eigenform in $\Omega E_D(1, 1)$, then the leaf \mathcal{F}_X is either closed or projectively dense in $\Omega E_D(1, 1)$. The latter case occurs if, and only if, D is not a square.*

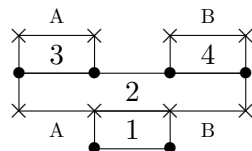
This result was already known prior to that work. It is for instance part of the classification established in [CDF15]. However it was interesting to obtain an elementary proof of this result that does not make use of degeneration of translation surfaces as it is the case in the aforementioned paper.

5.2 Prym eigenform loci in genus 3

Proposition 5.2. *Let $\kappa = (2, 2)^{odd}, (2, 1, 1)$ or $(1, 1, 1, 1)$ and let \mathcal{M} be a connected component of $\Omega E_D(\kappa)$. Then \mathcal{M} has property \mathcal{P} if and only if D is not a square.*

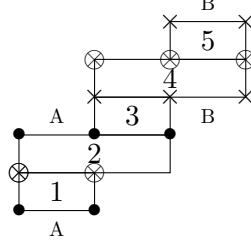
Proof. First, notice that D is a square if and only if \mathcal{M} is arithmetic as $k(\mathcal{M}) = \mathbb{Q}(\sqrt{D})$. The case where D is a square has already been proved in proposition 4.3. We assume from now on that D is not a square. We treat separately the different strata:

- If \mathcal{M} is contained in $\Omega E_D^{odd}(2, 2)$, theorem B of [LN14] states that \mathcal{M} contains a translation surface that is cylinder equivalent to the following surface:



Clearly, the vector $v = (c_1^{-1}, -c_2^{-1}, c_3^{-1}, c_4^{-1}) \in K_{\mathcal{M}}(X)$ as the Prym involution persists on $\tau(v)$, and it is a minimal \mathcal{M} -isoperiodic deformation. If its degree is 0, that means that all the cylinders have a commensurable circumferences and thus that \mathcal{M} is arithmetic by proposition 2.2, which is not. Then the degree of v is at least one, and \mathcal{M} has property \mathcal{P} .

- If \mathcal{M} is contained in $\Omega E_D(2, 1, 1)$, then once again theorem *B* of [LN14] states that \mathcal{M} contains a translation surface that is cylinder equivalent to the following surface:

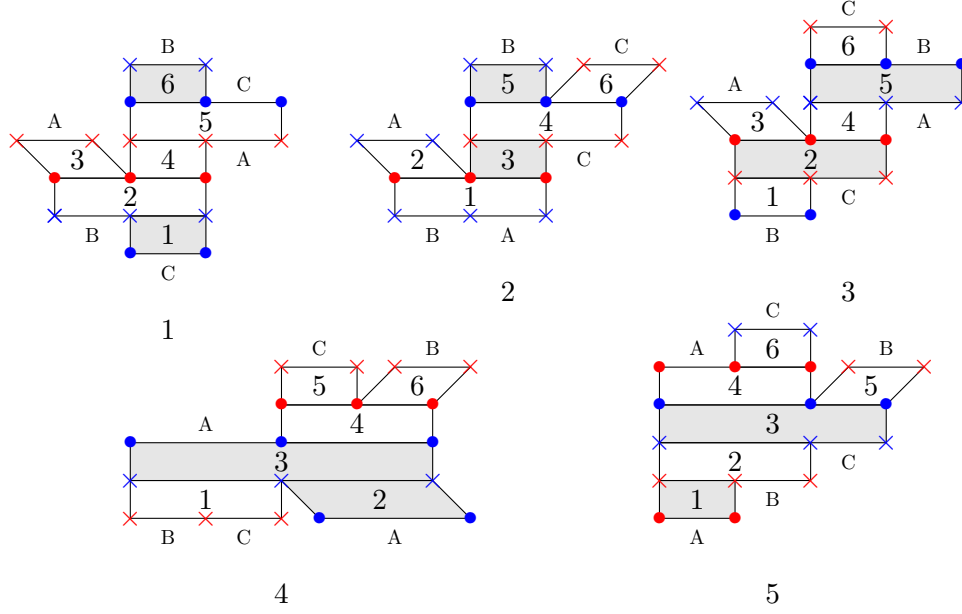


In this, the vector $v = (c_1^{-1}, -c_2^{-1}, 2c_3^{-1}, -c_4^{-1}, c_5^{-1})$ lies in $K_{\mathcal{M}}(X)$, and it is minimal. Note that the Prym involution exchanges 1 with 5 and 2 with 4. For the same reason as in the previous case, the degree of this deformation is at least one.

- If \mathcal{M} is contained in $\Omega E_D(1^4)$, let (X, ω) be a horizontally periodic surface in \mathcal{M} . It is well known that those surfaces always exist in affine invariant orbifolds, without any assumption on the rank. See for instance corollary 6 of [SW04]. Up to flowing along $\mathcal{F}^{\mathcal{M}}$, we can assume that the corresponding cylinder decomposition is $\mathcal{F}^{\mathcal{M}}$ -stable. We claim that we can even assume it is stable: it is not the case that there are two different singularities on the same component of a cylinder. Indeed, if there are two singularities on the same component, then these singularities are not swapped by the Prym involution, or else their cone angle would be 2π . They can thus be disaligned by an \mathcal{M} -isoperiodic deformation. This implies that there are 6 cylinders, as can be seen when computing the Euler characteristic.

Now, an application of Riemann-Hurwitz formula reveals that the corresponding Prym form have 4 fixed points. They are either on core curves of cylinders or on their boundary components. Notice that whenever a cylinder is fixed, there are 2 fixed points of the involution on the core curve of this cylinder. For parity reason, there are either 0 or 2 fixed cylinders. Consider first the case where no cylinders are fixed by the Prym form. There are thus 3 pairs of cylinders exchanged by the Prym involution and 4 fixed points. This means that there is a pair of exchanged cylinders that contain two fixed points on the union of their boundary components. But when a saddle connection contains such a point, this saddle connection is shared by the two cylinders and thus there is a cylinder in X that has a fixed point on its two boundary components. Chose a geodesic segment α inside this cylinder that joins those fixed points. The concatenation of α with its image by the prym involution is a geodesic loop fixed by the involution. By the complete periodicity of rank 1 affine invariant orbifolds (Theorem 1.5 of [Wri15]), the direction of α is a new periodic

direction for which there is at least one cylinder fixed by the prym involution. This direction can be assumed to be stable by the same argument that we used before. We can thus consider only the second case where two cylinders are fixed by the involution. Appendix A, provides a list of such decompositions and thus (X, ω) is cylinder equivalent to one of the following surfaces:



We provide a minimal \mathcal{M} -isoperiodic deformation of positive degree for each of those decompositions.

1. $(c_1^{-1}, 0, -c_3^{-1}, -c_4^{-1}, 0, c_6^{-1})$
2. $(c_1^{-1}, -c_2^{-1}, 0, c_4^{-1}, -2c_5^{-1}, -c_6^{-1})$
3. $(c_1^{-1}, 0, c_3^{-1}, c_4^{-1}, -2c_5^{-1}, c_6^{-1})$
4. $(0, c_2^{-1}, -c_3^{-1}, 0, c_5^{-1}, c_6^{-1})$
5. $(c_1^{-1}, 0, -c_3^{-1}, 0, c_5^{-1}, c_6^{-1})$

□

Proposition 5.2 together with Theorem A and proposition 4.3 implies the following:

Corollary 5.1 (Theorem B). *Let $\kappa = (2, 2)^{odd}, (2, 1, 1)$ or $(1, 1, 1, 1)$ and let \mathcal{M} be a connected component of $\Omega E_D(\kappa)$. Then either all the leaves of $\mathcal{F}^{\mathcal{M}}$ are all closed or all the leaves of $\mathcal{F}^{\mathcal{M}}$ are projectively dense. The last case occurs if, and only if, D is not a square.*

5.3 The hyperelliptic case

Proposition 5.3. *Let \mathcal{M} be a non absolute rank one affine invariant orbifold in $\mathcal{H}(g - 1^2)^{hyp}$. Then \mathcal{M} has property \mathcal{P} if and only if it is not arithmetic.*

Proof. Let (X, ω) be a translation surface in \mathcal{M} , and choose a periodic direction on X . Up to a small isoperiodic deformation, we can ensure that the direction is $\mathcal{F}^{\mathcal{M}}$ -stable (which is equivalent to \mathcal{F} -stable here as there are only two singularities). Note that the singularity present on a boundary component of a cylinder is different from the singularity present on the other boundary component. This is due to the fact that the hyperelliptic involution exchanges the singularities of ω while fixing all the cylinders. A consequence of that is that a minimal \mathcal{M} -isoperiodic deformation is supported on all the cylinders and has the form $(\delta_1 c_1^{-1}, \dots, \delta_m c_m^{-1})$ with the δ_i belonging to $\{1, -1\}$. The fact that \mathcal{M} is non arithmetic is equivalent to the fact that this deformation has non vanishing degree by 2.2. Therefore \mathcal{M} has property \mathcal{P} if and only if \mathcal{M} is non arithmetic. \square

Theorem 5.2. *Let \mathcal{M} be a non absolute rank 1 affine invariant orbifold contained in $\mathcal{H}(g - 1^2)^{hyp}$, and let (X, ω) be a surface in \mathcal{M} . Then the leaf of $\mathcal{F}_X^{\mathcal{M}}$ is either closed or projectively dense in \mathcal{M} . The latter case occurs if, and only if \mathcal{M} is non arithmetic.*

Paul Apisa has classified the rank one affine invariant orbifolds in the hyperelliptic strata and showed that if \mathcal{M} is a non arithmetic rank 1 affine invariant orbifold, then it is a translation cover of a surface in $\Omega E_D(1, 1)$. See [Api17].

6 Isoperiodic leaves of genus 3 Prym eigenforms

In this section we compute the closure of the isoperiodic leaves or Prym eigenforms of genus 3. We emphasize the fact that Theorem A dealt with the subfoliation $\mathcal{F}^{\mathcal{M}}$ induced on $\Omega E_D(\kappa)$ while in this section we deal with the full isoperiodic foliation.

6.1 The case $\mathcal{H}^{odd}(2, 2)$

Let \mathcal{M} be a connected component of $\Omega E_D^{odd}(2, 2)$. In the case where there are only two singularities, the dimension of the leaves of $\mathcal{F}^{\mathcal{M}}$ is the same as the ones of \mathcal{F} . This observation readily implies that for any surface $(X, \omega) \in \mathcal{M}$, there is equality between $\mathcal{F}_X^{\mathcal{M}}$ and \mathcal{F}_X . Thus the following is only a reformulation of Theorem A:

Proposition 6.1. *Let (X, ω) be a Prym eigenform in $\mathcal{H}^{odd}(2, 2)$, then the leaf \mathcal{F}_X is either closed or projectively dense in the connected component of $\Omega E_D^{odd}(2, 2)$ in which it belongs. The latter case occurs if, and only if, D is not a square.*

6.2 The case $\mathcal{H}_3(\kappa)$ with $|\kappa| > 2$

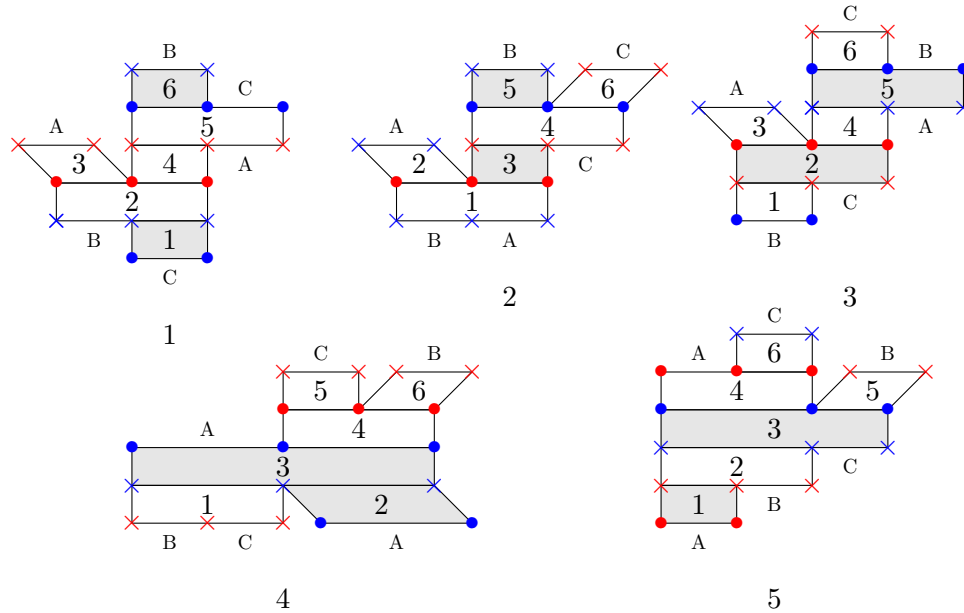
This section is dedicated to proving Theorem C.

Theorem 6.1 (Theorem C). *Let (X, ω) be a Prym eigenform in $\mathcal{H}_3(\kappa)$ with $|\kappa| > 2$, then the leaf \mathcal{F}_X is either closed or projectively dense. The last case occurs if, and only if, D is not a square.*

Proof. To prove this result, we are going to use theorem 4.2 and additional computations to prove that if the leaf of (X, ω) is not closed, it has to be projectively dense in an affine invariant orbifold \mathcal{M} of rank at least 2. However, the classification of Aulicino and Nguyen of rank 2 affine invariant orbifolds ([AN16]) shows that if an affine invariant orbifold is \mathcal{F} -saturated, as is \mathcal{M} , it cannot be rank 2. Thus the rank is 3, and by [MW18], it has to be the whole stratum, since the hyperelliptic locus is not saturated by \mathcal{F} neither.

If \mathcal{M} is contained in $\mathcal{H}(2, 1, 1)$, then we saw in the course of the proof of proposition 5.2, that we can assume that (X, ω) is horizontally periodic with 5 cylinders, and that $u = (1, -c_2^{-1}, c_3^{-1}, 0, 0)$ and $v = (0, 0, c_3^{-1}, -c_4^{-1}, c_5^{-1})$ belong to $K_{\mathcal{M}}(X)$ and have a positive degree, with $c_1^{-1} = c_5^{-1}$, and $c_2^{-1} = c_4^{-1}$. The projection of $K_{\mathcal{M}}(X)$ on the two first coordinates is a 2-dimensional space, or c_1 and c_2 would be \mathbb{Q} -dependant, and the relation $c_2 = c_1 + c_3$ would imply that the degree of u vanishes. Similarly, the projection of $K_{\mathcal{M}}(X)$ on the two last factors is also a 2-dimensional space. In particular, the dimension of $K_{\mathcal{M}}(X)$ is at least 4. By lemma 3.1, we have $r + 2 \geq 4$, and thus the rank of \mathcal{M} is at least 2.

If \mathcal{M} is contained in $\mathcal{H}(1^4)$, we saw that (X, ω) can be assumed to be cylinder equivalent to one of the following surfaces:



We provide a pair of minimal transverse \mathcal{M} -isoperiodic deformations for each of the first four decompositions:

1. $(c_1^{-1}, 0, 0, 0, -c_5^{-1}, c_6^{-1})$ and $(0, c_2^{-1}, -c_3^{-1}, -c_4^{-1}, 0, 0)$
2. $(c_1^{-1}, -c_2^{-1}, 0, 0, -c_5^{-1}, 0)$ and $(0, 0, c_3^{-1}, -c_4^{-1}, 0, c_6^{-1})$
3. $(c_1^{-1}, 0, 0, 0, -c_5^{-1}, c_6^{-1})$ and $(0, c_2^{-1}, -c_3^{-1}, -c_4^{-1}, 0, 0)$
4. $(c_1^{-1}, 0, 0, 0, -c_5^{-1}, -c_6^{-1})$ and $(0, c_2^{-1}, -c_3^{-1}, c_4^{-1}, 0, 0)$

Those minimal deformations have to be a positive degree, and we conclude in these cases using proposition 4.2. Now, if (X, ω) is cylinder equivalent to the fifth surface, then $K_{\mathcal{M}}(X)$ contains the following three vectors $(c_1^{-1}, 0, 0, -c_4^{-1}, 0, c_6^{-1})$, $(0, c_2^{-1}, 0, -c_4^{-1}, -c_5^{-1}, c_6^{-1})$ and $(0, 0, c_3^{-1}, -c_4^{-1}, -c_5^{-1}, 0)$, and each of them is of positive degree. Using the same argument as in the case where $\mathcal{M} \subset \mathcal{H}(2, 1, 1)$, we conclude that $k + 3 \geq 5$, and thus the rank of \mathcal{M} has to be at least 2.

□

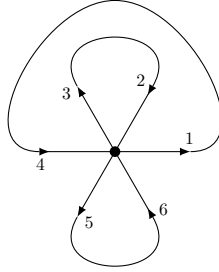
A Stable cylinder decompositions in $\text{Prym}(1, 1, 1, 1)$

A.1 Diagram of separatrices

We recall from [?] the common framework for enumeration of cylinder decompositions.

Definition A.1. A *prediagram of separatrices* is a quadruplet $\Gamma = (E, \sigma, \tau, \theta)$, where τ is a fixed point free involution of E , σ is a permutation of E and θ is a map from $E/\tau \rightarrow E$ such that $p \circ \theta = \text{id}$, where $p : E \rightarrow E/\tau$ is the canonical projection.

The elements of E are called oriented edges. An edge γ in E is said to be positively oriented if $\theta \circ p(\gamma) = \gamma$ and negatively oriented if $\theta \circ p(\gamma) = \tau(\gamma)$. The set of positively oriented edges will be denoted by E_+ and the set of negatively oriented edges will be denoted by E_- . A prediagram of separatrices is called alternating if $\sigma(E_+) = \sigma(E_-)$. A cylinder component is defined as an orbit of $\sigma_\infty := \sigma \circ \tau$. Such a cylinder component is said to be positively oriented if it corresponds to a positively oriented edge, and negatively oriented otherwise. We denote by \mathcal{C}_+ and \mathcal{C}_- the set of positively and negatively oriented cylinder components. A pairing of cylinder components is a bijection from \mathcal{C}_+ to \mathcal{C}_- . Finally, we define a metric on Γ as a strictly positive τ -invariant function l on E , and we consider its natural extension \hat{l} to the set of cylinder components defined by $\hat{l}(c) : \sum_n l(\sigma_\infty^n(\gamma))$, where c is the cylinder component associated to γ . Such an object can be encoded by a directed graph with additional information. Its set of vertices is the set of orbits of σ , and its set of edges is E/σ . Set $p_0 : E \rightarrow E/\sigma$ to be the canonical projection. The beginning of an edge is $p_0(e)$, and its end is $p_0 \circ \tau(e)$. There is an cyclic ordering on the star of each vertex define by σ . In the following figure is depicted the graph associated to the alternating prediagram of separatrices $(\{1, \dots, 6\}, \text{id}, (14)(23)(56), \theta_0)$, where $\theta_0(i) = i$ for any $i \in \{1, 2, 5\}$.



The graph associated to $(\{1, \dots, 6\}, \text{id}, (14)(23)(56), \theta_0)$

Definition A.2. A *diagram of separatrices* (Γ, m, l) is the data of an alternating prediagram of separatrices Γ , together with a matching of its cylinder components and a metric l on Γ that is invariant by m , that is $\hat{l} \circ m = \hat{l}$.

To any translation surface $(X, \omega) \in \mathcal{H}(\kappa)$, there is a canonical diagram of separatrices $\Gamma(X, \omega)$ associated. It is defined as follows: the set E is the collection of all geodesics rays γ :

$[0, 1] \rightarrow X$, such that $\gamma^{-1}(\Sigma) = \{0, 1\}$, and $\gamma^*(Im(\omega)) = 0$, taken up to reparametrization. We define τ to be the orientation reversing map : $\tau(\gamma)(t) : \gamma(1-t)$. For any $\gamma \in E$, there is a chart around $\gamma(0)$ that takes ω to $z^k dz$; where k is the order of the singularity. The set of germs of geodesic rays that begin at $\gamma(0)$ is thus invariant by multiplication by $e^{\frac{2i\pi}{k+1}}$, and the germ of a geodesic ray at its beginning completely determines it. We thus define $\sigma(\gamma)$ to be the geodesic ray whose germ at its beginning is the one of $e^{\frac{i\pi}{k+1}}\gamma$. Finally, to define θ we need to define a map that is τ -invariant. If γ is in E , we define $\theta(\gamma) = \gamma$ if $\gamma^*(Re(\omega)) > 0$, and $\theta(\gamma) = \tau(\gamma)$ otherwise. The metric on $\Gamma(X, \omega)$ is given by $l(\gamma) = |\int_{\gamma} \omega|$. Note that the orbits of σ_{∞} are in correspondence with oriented core curves of cylinders. The matching m is then defined to map the orbit that corresponds to the positively oriented core curve to the one that corresponds to the negatively oriented core curves of the same cylinder. In the reverse direction, to any diagram of separatrices Γ there is horizontally periodic translation surface $(X_{\Gamma}, \omega_{\Gamma})$. It is defined in the following way: replace any $\gamma \in \theta(E/\tau)$ by a strip of length $l(\gamma)$, and replace any element in E/σ by a disk. The permutation σ defines a cyclic ordering on any orbit. Using this ordering, one can glue the strips to the disks. We get a topological surface with boundary, and the boundary components correspond to orbits of σ_{∞} . Since by requirement, the paired components have same length (defined by \hat{l}) we can glue those components using the pairing, and we get a horizontally periodic flat surface in $\mathcal{H}(\kappa)$, where $|\kappa|$ is the cardinal of E/σ and k_i is half the cardinal of the orbit i minus 1. The fact that the cardinal of an orbit is even comes from the alternating condition.

Two prediagrams of separatrices $\Gamma_1 = (E^1, \tau_1, \sigma_1, \theta_1)$ and $\Gamma_2 = (E^2, \tau_2, \sigma_2, \theta_2)$ are isomorphic if there is a map $\varphi : E_1 \rightarrow E_2$ such that $\varphi \circ \sigma_1 = \sigma_2 \circ \varphi$, $\varphi \circ \tau_1 = \tau_2 \circ \varphi$, and $\varphi(E_+^1) = E_+^2$, or equivalently $\varphi \circ \theta_1 = \theta_2$. Finally, two diagrams of separatrices (Γ_1, l_1, m_1) and (Γ_2, l_2, m_2) are isomorphic if there is an isomorphism of prediagram of separatrices φ between the two such that $l_2 \circ \varphi = l_1$, and if two cylinders components on Γ_1 associated to the orbits of γ_1 and γ_2 are paired by m_1 , then the cylinder components of Γ_2 associated to $\varphi(\gamma_1)$ and $\varphi(\gamma_2)$ are paired by m_2 .

Proposition A.1. *Two horizontally periodic translation surfaces (X_1, ω_1) and (X_2, ω_2) in $\mathcal{H}(\kappa)$ are isomorphic to cylinder equivalents surfaces if, and only if, the associated diagrams of separatrices are isomorphic.*

We refer to [KZ03] for a proof of that result.

Definition A.3. *A connected component of a prediagram of separatrices of $\Gamma = (E, \sigma, \tau, \theta)$ is an orbit of the group $\langle \sigma, \tau \rangle$. A prediagram of separatrices is said to be connected when it is reduced to a single connected component.*

If E' is a connected component of Γ , there is an induced prediagram of separatrices $(E', \sigma|_{E'}, \tau|_{E'}, \theta|_{p(E')})$.

Definition A.4. *A prediagram of separatrices $(E, \tau, \sigma, \theta)$ is said to be stable if τ preserves the orbits of σ .*

The notion of stable for prediagram coincide with the notion of stable for cylinder decomposition for surfaces. It precisely means that the geodesic rays start and end at the same singularity, without going through any other singularity. Note that when a prediagram of separatrices $\Gamma = (E, \sigma, \tau, \theta)$ is both stable and connected, then E is reduced to a single orbit of σ . This remark allows to define the type of Γ as follows. Let x be a positively oriented edge. Any positively oriented edge can be written as $\sigma^{2k}(x)$, and any negatively oriented edge can be written as $\sigma^{c_n(2l)}(x)$. Since τ reverses the orientation, denoting by n the number of positively oriented edges, there is a map $f \in \mathfrak{S}_n$ such that for any k :

$$\tau \circ \sigma^{2k}(x) = \sigma^{2f(k)+1}(x)$$

Denote by c_n the element of \mathfrak{S}_n that sends i to $i+1$ and n to 1. The group $H = \langle c_n \rangle$ acts by conjugation on \mathfrak{S}_n and the type of the component is defined as the orbit of f under this action. The fact that we defined f up to conjugation by H comes from the fact we could have chosen any other even iteration of x as a generator of the orbit. More generally, we say that Γ is of type (f_i) if the type of the minimal connected components are given by the f_i .

Proposition A.2. *Two stable alternating prediagrams are isomorphic if, and only if, they have the same type.*

Proof. It is enough to prove the result for minimal connected components : Let $\Gamma_i = (E_i, \sigma_i, \tau_i, \theta_i)$ for $i \in \{1, 2\}$ be two minimal prediagrams of same type. Pick x_i in E_i that is positively oriented and define f_i such that $\tau_i(\sigma_i^{2k}(x_i)) = \sigma_i^{2(f_i(k)+1)}(x_i)$. Saying that the diagrams have the same type means there is l such that $f_2 = c_n^l \circ f_1 \circ c_n^{-l}$. Define φ such that for all j $\varphi(\sigma_1^j(x_1)) = \sigma_2^{2l+j}(x_2)$. We claim that φ is an isomorphism of prediagram. Indeed, let $\zeta \in E_1$, and pick j such that $\sigma_1^j(x_1) = \zeta$. Thus:

$$\begin{aligned} \varphi \circ \sigma_1(\zeta) &= \varphi \circ \sigma_1^{j+1}(x_1) \\ &= \sigma_2^{2l+j+1}(x_2) \\ &= \sigma_2(\sigma_2^{2l+j}(x_2)) \\ &= \sigma_2 \circ \varphi(\sigma_1^j(x_1)) \\ &= \sigma_2 \circ \varphi(\zeta) \end{aligned}$$

We also have:

$$\begin{aligned}
\tau_2 \circ \varphi(\sigma_1^{2k}(x_1)) &= \tau_2 \circ \sigma_2^{2(k+l)}(x_2) \\
&= \sigma_2^{2f_2(k+l)+1}(x_2) \\
&= \sigma_2^{2c_n^l \circ f_1 \circ c_n^{-l}(k+l)+1}(x_2) \\
&= \sigma_2^{2(f_1(k)+l)+1}(x_2) \\
&= \varphi \circ \sigma_1^{2f_1(k)+1} \\
&= \varphi \circ \tau_1(\sigma^{2k}(x_1)).
\end{aligned}$$

By construction, $\varphi(E_+^1) = E_+^2$. Reciprocally, if there is an isomorphism of prediagrams between Γ_1 and Γ_2 , then $\varphi(x_1) = \sigma_2^{2l}(x_1)$, and then:

$$f_2 = c_n^l \circ f_1 \circ c_n^{-l}$$

□

We denote by $\bar{\Gamma}$ the prediagram $(E, \sigma, \tau, \tau \circ \theta)$. if Γ corresponds to a surface (X, ω) , then $\bar{\Gamma}$ corresponds to the surface $(X, -\omega)$. Note that if (X, ω) is represented by a polygon P , the surface $(X, -\omega)$ is represented by the image of the polygon by the rotation of angle π .

Proposition A.3. *Let Γ be a minimal stable alternating prediagram of type f . The type of $\bar{\Gamma}$ is given by $(f \circ c_n)^{-1}$.*

Proof. Let $x \in E$ be positively oriented, and such that $\tau \circ \sigma^{2k}(x) = \sigma^{2f(k)+1}(x)$. Then $\sigma(x)$ is positively oriented on $\bar{\Gamma}$. Then:

$$\begin{aligned}
\tau \circ \sigma^{2j}(\sigma(x)) &= \sigma^{2f^{-1}(j)}(x) \\
&= \sigma^{2c_n^{-1}(f^{-1}(j))}(\sigma^2(x))
\end{aligned}$$

□

These propositions enable us to enumerate all the possible stable alternating prediagrams up to isomorphism. We shall represent only one of the surfaces associated to Γ or $\bar{\Gamma}$ as one is obtained from the other by a rotation of angle π .

A.2 Combinatorial Prym involution

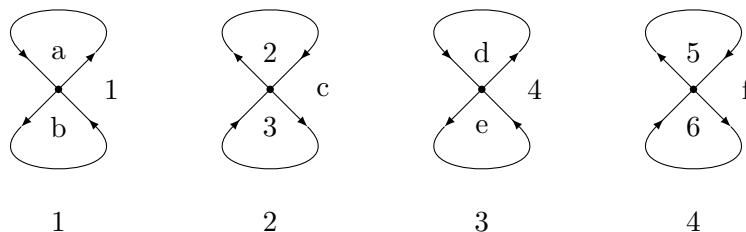
In this appendix we enumerate the possible cylinder decompositions of a certain type for a surface in the Prym locus $\text{Prym}(1, 1, 1, 1)$. The following definition is the combinatorial version of the Prym involution.

Definition A.5. A combinatorial Prym involution on a stable diagram of separatrices $\Gamma = (E, \sigma, \tau, \theta)$ is an isomorphism ρ of diagram between Γ and $\bar{\Gamma}$ whose square is the identity and such that $\#Fix(\rho_0) + \#Fix(\tau \circ \rho) + 2\#Fix(\rho \circ m) = 10 - 2g$, where g is the genus of the surface associated to Γ and ρ_0 is the map induced by ρ on E/σ .

The formula translates that the fixed points of the Prym involution are located either on the singularities, on the saddle connections or on the core curves of the cylinders (and in that case they come by pair).

Proposition A.4. If a horizontally periodic translation surface with a \mathcal{F} -stable decomposition has a Prym involution, then the morphism of diagram of separatrices it induces is a combinatorial Prym involution. Reciprocally, If a stable diagram of separatrices has a Prym involution, then the associated surface has a Prym involution.

This proposition enables us to enumerate the cylinder decomposition in Prym loci. In $\text{Prym}(1^4)$, the Prym involution does not fix any singularity and has 4 fixed points. It fixes either 0 or 2 cylinders. In this appendix, we will enumerate all the cylinder decompositions corresponding to the case where there are two cylinders fixed. We will say that the diagram associated is of the first kind if both the cylinders are bordered by the same pair of exchanged singularities, and of the second kind in the other case. Here, the type of the diagram is necessarily $(id_{\mathfrak{S}_2}, id_{\mathfrak{S}_2}, id_{\mathfrak{S}_2}, id_{\mathfrak{S}_2})$, and the Prym involution forces the following orientation:

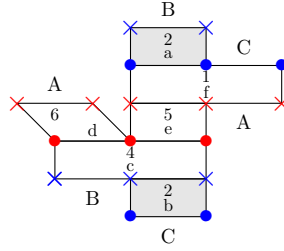


The prediagram of separatrices in $\text{Prym}(1^4)$

Since the connected components of the graph are planar, the cylinder components are in correspondence with the components of the plane after removing the edges. We label the positively oriented component by a letter and the negatively oriented components by a digit. A matching will be denoted by a 6-tuple of letters. For instance $(abcdef)$ means that the cylinder component 1 is matched with a, the cylinder component 2 is matched with b etc. We can assume that the involution τ exchanges the two first singularities and the two last, that is $\tau(1) = 2$ and $\tau(3) = 4$, and that it maps 2 to a and 4 to f. Other choices would produce isomorphic diagrams.

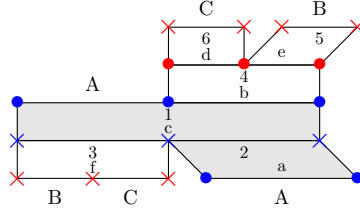
A.3 Stable decompositions of the first kind in $\text{Prym}(1, 1, 1, 1)$

- The first possible outcome is $m(a) = 2$ and $m(b) = 3$. Then $m(c)$ can be 4, 5 or 6. It cannot be 1 as this would give an extra fixed cylinder in contradiction with the Prym condition. The case 5 and 6 are isomorphic. So we only need to consider the case where $m(c)$ is 4 or 5. If it is 4, then, after applying τ_0 we get $m(f) = 1$. $m(d)$ is either 5 or 6, but the case $m(d) = 5$ produces one extra fixed cylinder, thus and then we get $m(d) = 6$, and then $m(e) = 5$. This gives that $m = (fabced)$. The other case is $m(c) = 5$. Then $m(d) = 1$. Now $m(e) = 4$ or 6 but only $m(e) = 6$ gives a metric solution, and this gives an extra fixed cylinder so this case is not possible.



the surface $(fabced)$

- The second possible outcome is $m(c) = 1$ and $m(a) = 2$ (or $m(b) = 3$ but this case is isomorphic). Then $m(f)$ cannot be 5 or 6 for metric reason, and it cannot be 4 either as this would produce an extra fixed cylinder. This means that $m(f) = 3$. Thus by the Prym involution, $m(b) = 4$. Now, $m(d)$ is either 5 or 6 but 5 gives an extra cylinder. Thus $m(d) = 6$, and $m(e) = 5$. The only possibility here is $(cafbed)$.

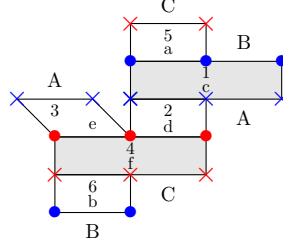


the surface $(cafbed)$

A.4 Stable decompositions of the second kind in $\text{Prym}(1, 1, 1, 1)$

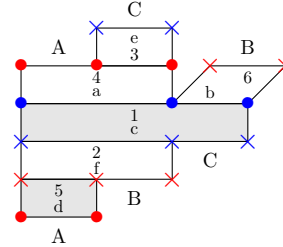
- The first case is $m(c) = 1$ and $m(f) = 4$. Then $m(a)$ is either 3, 5 or 6. The last two cases are isomorphic. Let us consider first the case where $m(a) = 3$, then $m(2) = b$, and $m(d) = 6$ and $m(e) = 5$. So we get $m = (cbafed)$, but this does not give a

connected surface, so this case does not exist. In the case where $m(a) = 5$. Then $m(d) = 2$, and then since there are not more than 2 fixed cylinders, $m(e) = 3$ and $m(b) = 6$. Thus we get $m = (cdefab)$.



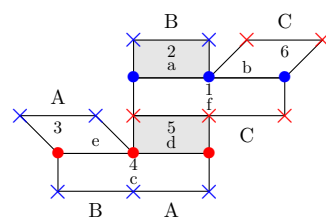
the surface $(cdefab)$

- Then $m(c) = 1$ and $m(d) = 5$ (the case where it $m(e) = 6$ instead of $m(d) = 5$ is isomorphic. If $m(b) = 3$ there is one extra fixed cylinder. If $m(b) = 2$ then $m(a) = 3$ and the surface is disconnected. Thus $m(b)$ is 4 or 6. In the first case, by applying the involution $m(f) = 3$. Likewise $m(a)$ cannot be 2 or there would be too many fixed cylinders thus $m(a) = 6$ and thus $m(e) = 2$. We get $m = (cefbda)$. If $m(b) = 6$, then $m(e) = 3$. $m(a)$ cannot be 2 as this would produce another fixed cylinder. thus $m(a) = 4$, and $m(f) = 2$. We get $m = (cfeadb)$. But this one is isomorphic to $(cefbda)$ by exchanging 2 with 3 and a with b.



the surface $(cfeadb)$

- Then $m(a) = 2$ and $m(d) = 5$. The other cases, like $m(f) = 4$ and $m(a) = 2$, are isomorphic, so we do not consider them. $m(c)$ cannot be neither 1, 2 nor 3. it is thus 4 or 6. If it is 4, then after applying the involution $m(f) = 1$. $m(b)$ cannot be 3, thus $m(b) = 6$ and $m(e) = 3$, and $m = (faecdb)$. If $m(c) = 6$, then $m(e) = 1$. $m(b)$ cannot be 3 thus $m(b) = 4$ and $m(f) = 3$. We get $m = (eafbdc)$. But this one do not have a metric solution. Indeed, we should have $l(e) = l(1)$ and $l(b) = l(4)$, but $l(1) > l(b) = l(4) > l(e)$.



the surface $(faecdb)$

References

- [AEM17] Artur Avila, Alex Eskin, and Martin Möller. Symplectic and isometric $\mathrm{SL}(2, \mathbb{R})$ -invariant subbundles of the hodge bundle. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2017(732):1–20, 2017.
- [AN16] David Auricino and Duc-Manh Nguyen. Rank two affine submanifolds in $\mathcal{H}(2, 2)$ and $\mathcal{H}(3, 1)$. *Geometry & Topology*, 20(5):2837–2904, 2016.
- [Api17] Paul Apisa. Rank one orbit closures in $\mathcal{H}^{hyp}(g-1, g-1)$. *arXiv preprint arXiv:1710.05507*, 2017.
- [BH13] Martin R Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319. Springer Science & Business Media, 2013.
- [BSW16] Matt Bainbridge, John Smillie, and Barak Weiss. Horocycle dynamics: new invariants and eigenform loci in the stratum $\mathcal{H}(1, 1)$. *arXiv preprint arXiv:1603.00808*, 2016.
- [CDF15] Gabriel Calsamiglia, Bertrand Deroin, and Stefano Francaviglia. A transfer principle: from periods to isoperiodic foliations. *arXiv preprint arXiv:1511.07635*, 2015.
- [CJ19] Francisco C Caramello Jr. Introduction to orbifolds. *arXiv preprint arXiv:1909.08699*, 2019.
- [EMM15] Alex Eskin, Maryam Mirzakhani, and Amir Mohammadi. Isolation, equidistribution, and orbit closures for the $\mathrm{SL}(2, \mathbb{R})$ action on moduli space. *Annals of Mathematics*, pages 673–721, 2015.
- [EMMW18] Alex Eskin, Curtis McMullen, Ronen Mukamel, and Alex Wright. Billiards, quadrilaterals and moduli spaces. *preprint*, 2018.
- [FM13] Giovanni Forni and Carlos Matheus. Introduction to teichmüller theory and its applications to dynamics of interval exchange transformations, flows on surfaces and billiards. *arXiv preprint arXiv:1311.2758*, 2013.
- [GHS03] Eugene Gutkin, Pascal Hubert, and Thomas A. Schmidt. Affine diffeomorphisms of translation surfaces : periodic points, fuchsian groups, and arithmeticity. *Annales scientifiques de l’École Normale Supérieure*, Ser. 4, 36(6):847–866, 2003.
- [Ham18] Ursula Hamenstädt. Ergodicity of the absolute period foliation. *Israel Journal of Mathematics*, 225:661–680, 2018.

- [HW15] W Patrick Hooper and Barak Weiss. Rel leaves of the arnoux–yoccoz surfaces. *Selecta Mathematica*, pages 1–60, 2015.
- [KH95] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54. Cambridge university press, 1995.
- [KZ03] Maxim Kontsevich and Anton Zorich. Connected components of the moduli spaces of abelian differentials with prescribed singularities. *Inventiones mathematicae*, 153(3):631–678, 2003.
- [LN14] Erwan Lanneau and Duc-Manh Nguyen. Connected components of prym eigenform loci in genus three. *Mathematische Annalen*, pages 1–41, 2014.
- [McM07a] Curtis T McMullen. Dynamics of $SL_2(\mathbb{R})$ over moduli space in genus two. *Annals of mathematics*, 165(2):397–456, 2007.
- [McM07b] Curtis T McMullen. Prym varieties and teichmüller curves. 2007.
- [McM14] Curtis McMullen. Moduli spaces of isoperiodic forms on riemann surfaces. *Duke Mathematical Journal*, 163(12):2271–2323, 2014.
- [MW18] Maryam Mirzakhani and Alex Wright. Full-rank affine invariant submanifolds. *Duke Mathematical Journal*, 167(1):1–40, 2018.
- [SW04] John Smillie and Barak Weiss. Minimal sets for flows on moduli space. *Israel Journal of Mathematics*, 142(1):249–260, 2004.
- [Wri14] Alex Wright. The field of definition of affine invariant submanifolds of the moduli space of abelian differentials. *Geometry & Topology*, 18(3):1323–1341, 2014.
- [Wri15] Alex Wright. Cylinder deformations in orbit closures of translation surfaces. *Geometry & Topology*, 19(1):413–438, 2015.
- [Zor06] Anton Zorich. Flat surfaces. *Frontiers in number theory, physics, and geometry I*, pages 439–585, 2006.