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► To cite this version:

Cyril Bénézet, Jean-François Chassagneux, Adrien Richou. Switching problems with controlled randomisation and associated obliquely reflected BSDEs. *Stochastic Processes and their Applications*, 2022, 144, pp.Pages 23-71. 10.1016/j.spa.2021.10.010 . hal-02463626v2

HAL Id: hal-02463626

<https://hal.science/hal-02463626v2>

Submitted on 13 Dec 2021

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Switching problems with controlled randomisation and associated obliquely reflected BSDEs

Cyril Bénézet*, Jean-François Chassagneux[†], Adrien Richou[‡]

December 13, 2021

Abstract

We introduce and study a new class of optimal switching problems, namely *switching problem with controlled randomisation*, where some extra-randomness impacts the choice of switching modes and associated costs. We show that the optimal value of the switching problem is related to a new class of multidimensional obliquely reflected BSDEs. These BSDEs allow as well to construct an optimal strategy and thus to solve completely the initial problem. The other main contribution of our work is to prove new existence and uniqueness results for these obliquely reflected BSDEs. This is achieved by a careful study of the domain of reflection and the construction of an appropriate oblique reflection operator in order to invoke results from [6].

1 Introduction

In this work, we introduce and study a new class of optimal switching problems in stochastic control theory. The interest in switching problems comes mainly from their connections to financial and economic problems, like the pricing of *real options* [4]. In a celebrated article [14], Hamadène and Jeanblanc study the fair valuation of a company producing electricity. In their work, the company management can choose between two modes of production for their power plant – operating or close – and times of switching from one state to another, in order to maximise its expected return. Typically, the company will buy electricity on the market if the power station is not operating. The company receives a profit for delivering electricity in each regime. The main point here is that a fixed cost penalizes the profit upon switching. This *switching problem* has been generalized to more than two modes of production [10]. Let us now discuss

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this *switching problem* with $d \geq 2$ modes in more details. The costs to switch from one state to another are given by a matrix $(c_{i,j})_{1 \leq i,j \leq d}$. The management optimises the expected company profits by choosing switching strategies which are sequences of stopping times $(\tau_n)_{n \geq 0}$ and modes $(\zeta_n)_{n \geq 0}$. The current state of the strategy is given by $a_t = \sum_{k=0}^{+\infty} \zeta_k 1_{[\tau_k, \tau_{k+1})}(t)$, $t \in [0, T]$, where T is a terminal time. To formalise the problem, we assume that we are working on a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$ supporting a Brownian motion W . The stopping times are defined with respect to the filtration $(\mathcal{F}_t^0)_{t \geq 0}$ generated by this Brownian motion. Denoting by $f^i(t)$ the instantaneous profit received at time t in mode i , the time cumulated profit associated to a switching strategy is given by $\int_0^T f^{a_t}(t) dt - \sum_{k=0}^{+\infty} c_{\zeta_k, \zeta_{k+1}} 1_{\{\tau_{k+1} \leq T\}}$. The management solves then at the initial time the following control problem

$$\mathcal{V}_0 = \sup_{a \in \mathcal{A}} \mathbb{E} \left[\int_0^T f^{a_t}(t) dt - \sum_{k=0}^{+\infty} c_{\zeta_k, \zeta_{k+1}} 1_{\{\tau_{k+1} \leq T\}} \right], \quad (1.1)$$

where \mathcal{A} is a set of admissible strategies that will be precisely described below in our framework (see Section 2.1). We shall refer to problems of the form (1.1) under the name of *classical switching problems*. These problems have received a lot of interest and are now quite well understood [14, 10, 17, 5]. In our work, we introduce a new kind of switching problems, to model more realistic situations, by taking into account uncertainties that are encountered in practice. Coming back to the simple but enlightening example of an electricity producer described in [14], we introduce some extra-randomness in the production process. Namely, when switching to the operating mode, it may happen with – hopefully – a small probability that the station will have some dysfunction. This can be represented by a new mode of “production” with a greater switching cost than the business as usual one. To capture this phenomenon in our mathematical model, we introduce a randomisation procedure: the management decides the time of switching but the mode is chosen randomly according to some extra noise source. We shall refer to this kind of problems by *randomised switching problems*. However, we do not limit our study to this framework. Indeed, we allow some control by the agent on this randomisation. Namely, the agent can choose optimally a probability distribution P^u on the modes space given some parameter $u \in \mathcal{C}$, the control space. The new mode ζ_{k+1} is then drawn, independently of everything up to now, according to this distribution P^u , and a specific switching cost $c_{\zeta_k, \zeta_{k+1}}^u$ is applied. The management strategy is thus given now by the sequence $(\tau_k, u_k)_{k \geq 0}$ of switching times and controls. The maximisation problem is still given by (1.1). Let us observe however that $\mathbb{E} \left[c_{\zeta_k, \zeta_{k+1}}^{u_k} \right] = \mathbb{E} \left[\sum_{1 \leq j \leq d} P_{\zeta_k, j}^{u_k} c_{\zeta_k, j}^{u_k} \right]$, thanks to the tower property of conditional expectation. In particular, we will only work with the mean switching costs $\bar{c}_i^u := \sum_{1 \leq j \leq d} P_{i, j}^u c_{i, j}^u$ in (1.1). We name this kind of control problems *switching problems with controlled randomisation*. Although their apparent modeling power, this kind of control problems has not been considered in the literature before, to the best of our knowledge. In particular, we will show that *classical* and *randomised switching problems* are just special instances of this more generic problem. The *switching problem with controlled randomisation* is introduced rigorously in Section 2.1 below.

A key point in our work is to relate these control problems to a new class of obliquely reflected Backward Stochastic Differential Equations (BSDEs). In the first part, following the approach of [14, 10, 17], we completely solve the *switching problem with controlled randomisation* by providing an optimal strategy. The optimal strategy is built using the solution to a well chosen obliquely reflected BSDE. Although this approach is not new, the link between the obliquely reflected BSDE and the switching problem is more subtle than in the classical case due to the state uncertainty. In particular, some care must be taken when defining the adaptedness property of the strategy and associated quantities. Indeed, a tailor-made filtration, studied in details in Appendix A.2, is associated to each admissible strategy. The state and cumulative cost processes are adapted to this filtration, and the associated reward process is defined as the Y -component of the solution to some “switched” BSDE in this filtration. The classical estimates used to identify an optimal strategy have to be adapted to take into account the extra orthogonal martingale arising when solving this “switched” BSDE in a non Brownian filtration.

In the second part of our work, we study the auxiliary obliquely reflected BSDE, which is written in the Brownian filtration and represents the optimal value in all the possible starting modes. Reflected BSDEs were first considered by Gegout-Petit and Pardoux [13], in the multidimensional setting of normal reflections. In one dimension, they have also been studied in [11] in the so called simply reflected case, and in [7] in the doubly reflected case. The multidimensional RBSDE associated to the *classical switching problem* is reflected in a specific convex domain and involves oblique directions of reflection. Due to the controlled randomisation, the domain in which the Y -component of the auxiliary RBSDE is constrained is different from the *classical switching problem* domain and its shape varies a lot from one model specification to another. The existence of a solution to the obliquely reflected BSDE has thus to be studied carefully. We do so by relying on the article [6] that studies, in a generic way, the obliquely reflected BSDE in a fixed convex domain in both Markovian and non-Markovian setting. The main step for us here is to exhibit an oblique reflection operator, with the good properties to use the results in [6]. We are able to obtain new existence results for this class of obliquely reflected BSDEs. Because we are primarily interested in solving the control problem, we derive the uniqueness of the obliquely reflected BSDEs in the Hu and Tang specification for the driver [17], namely $f^i(t, y, z) := f^i(t, y^i, z^i)$ for $i \in \{1, \dots, d\}$. But our results could be easily generalized to the specification $f^i(t, y, z) := f^i(t, y, z^i)$ by using similar arguments as in [5].

The rest of the paper is organised as follows. In Section 2, we introduce the *switching problem with controlled randomisation*. We prove that, if the costs are positive and there exists a solution to the associated BSDE with oblique reflections, then its Y -component coincides with the value of the switching problem. A verification argument allows then to deduce uniqueness of the solution of the obliquely reflected BSDE, and we obtain an uniqueness result in the signed costs setting as a byproduct, assuming non-emptiness of the interior of the domain of reflection. In Section 3, we study the geometry of the domain of reflection in the signed costs case, and we provide explicit conditions on the coefficients which are equivalent to a domain of reflection with a non-empty interior. In Section 4, we show that there exists indeed a solution to the obliquely reflected BSDE

under the previous conditions in a Markovian setting, and in some particular instances in the non-Markovian case. We also prove uniqueness of the solution under some structural condition on the driver f . Finally, we gather in the Appendix section some technical results. In particular, in a positive costs setting, we prove a Martingale decomposition theorem in the filtration associated to an admissible strategy, which is a new result.

Notations If $n \geq 1$, we let \mathcal{B}^n be the Borelian sigma-algebra on \mathbb{R}^n . If E is a measurable subset of \mathbb{R}^n , we denote by $\overset{\circ}{E}$ its interior and by \overline{E} its closure. For any filtered probability space $(\Omega, \mathcal{A}, \mathbb{F} := (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and constants $T > 0$ and $p \geq 1$, we define the following spaces:

- $L_n^p(\mathcal{A})$ is the set of \mathcal{A} -measurable random variables X valued in \mathbb{R}^n satisfying $\mathbb{E}[|X|^p] < +\infty$,
- $\mathcal{P}(\mathbb{F})$ is the predictable sigma-algebra on $\Omega \times [0, T]$,
- $\mathbb{H}_n^p(\mathbb{F})$ is the set of predictable processes ϕ valued in \mathbb{R}^n such that

$$\|\phi\|_{\mathbb{H}_n^p(\mathbb{F})} := \mathbb{E} \left[\int_0^T |\phi_t|^p dt \right]^{\frac{1}{p}} < +\infty, \quad (1.2)$$

- $\mathbb{S}_n^p(\mathbb{F})$ is the set of càdlàg adapted processes ϕ valued in \mathbb{R}^n such that

$$\|\phi\|_{\mathbb{S}_n^p(\mathbb{F})} := \mathbb{E} \left[\sup_{0 \leq t \leq T} |\phi_t|^p \right]^{\frac{1}{p}} < +\infty, \quad (1.3)$$

- $\mathbb{A}_n^p(\mathbb{F})$ is the set of adapted continuous processes ϕ valued in \mathbb{R}^n such that $\phi_T \in L_n^p(\mathcal{F}_T)$ and ϕ^i is non-decreasing for all $i = 1, \dots, n$.

For X, Y two càdlàg processes, the quadratic covariation process is denoted $[X, Y]$ and the quadratic variation process $[X]$.

If $n = 1$, we omit the subscript n in previous notations.

For $d \geq 1$, we denote by $(e_i)_{i=1}^d$ the canonical basis of \mathbb{R}^d and $S_d(\mathbb{R})$ the set of symmetric matrices of size $d \times d$ with real coefficients. Moreover, we let $\mathbf{1} := \sum_{i=1}^d e_i$.

If \mathcal{D} is a convex subset of \mathbb{R}^d ($d \geq 1$) and $y \in \mathcal{D}$, we define the set $\mathcal{C}(y)$ by

$$\mathcal{C}(y) := \left\{ v \in \mathbb{R}^d \mid v^\top (z - y) \leq 0 \text{ for all } z \in \mathcal{D} \right\}. \quad (1.4)$$

This is the outward normal cone at y when y is a boundary point of \mathcal{D} , and it is obviously reduced to $\{0\}$ when y is an interior point. We also set $\mathbf{n}(y) := \mathcal{C}(y) \cap \{v \in \mathbb{R}^d \mid |v| = 1\}$. For any closed convex set \mathcal{O} , we denote by $\mathfrak{P}_{\mathcal{O}}(\cdot)$ the projection onto \mathcal{O} .

If X is a matrix of size $n \times m$, $\mathcal{I} \subset \{1, \dots, n\}$ and $\mathcal{J} \subset \{1, \dots, m\}$, we set $X^{(\mathcal{I}, \mathcal{J})}$ the

matrix of size $(n - |\mathcal{I}|) \times (m - |\mathcal{J}|)$ obtained from X by deleting rows with index $i \in \mathcal{I}$ and columns with index $j \in \mathcal{J}$. If $\mathcal{I} = \{i\}$ we set $X^{(i, \mathcal{J})} := X^{(\mathcal{I}, \mathcal{J})}$, and similarly if $\mathcal{J} = \{j\}$.

If v is a vector of size n and $1 \leq i \leq n$, we set $v^{(i)}$ the vector of size $n - 1$ obtained from v by deleting coefficient i .

For $(i, j) \in \{1, \dots, d\}$, we define $i^{(j)} := i - \mathbf{1}_{\{i > j\}} \in \{1, \dots, d - 1\}$, for $d \geq 2$.

We denote by \geq the component by component partial ordering relation on vectors and matrices.

2 Switching problems with controlled randomisation

We introduce here a new kind of stochastic control problems that we name *switching problems with controlled randomisation*. In contrast with the usual switching problems [14, 16, 17], the agent cannot choose directly the new state, but chooses a probability distribution under which the new state will be determined. In this section, we assume the existence of a solution to some auxiliary obliquely reflected BSDE to characterize the value process and an optimal strategy for the problem, see Assumption 2.2 below.

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space. We fix a finite time horizon $T > 0$ and $\kappa \geq 1, d \geq 2$ two integers. We assume that there exists a κ -dimensional Brownian motion W and a sequence $(\mathfrak{U}_n)_{n \geq 1}$ of independent random variables, independent of W , uniformly distributed on $[0, 1]$. We also assume that \mathcal{G} is generated by the Brownian motion W and the family $(\mathfrak{U}_n)_{n \geq 1}$. We define $\mathbb{F}^0 = (\mathcal{F}_t^0)_{t \geq 0}$ as the augmented Brownian filtration, which satisfies the usual conditions.

Let \mathcal{C} be an ordered compact metric space and $F : \mathcal{C} \times \{1, \dots, d\} \times [0, 1] \rightarrow \{1, \dots, d\}$ a measurable map. To each $u \in \mathcal{C}$ is associated a transition probability function on the state space $\{1, \dots, d\}$, given by $P_{i,j}^u := \mathbb{P}(F(u, i, \mathfrak{U}) = j)$ for \mathfrak{U} uniformly distributed on $[0, 1]$. We assume that for all $(i, j) \in \{1, \dots, d\}^2$, the map $u \mapsto P_{i,j}^u$ is continuous.

Let $\bar{c} : \{1, \dots, d\} \times \mathcal{C} \rightarrow \mathbb{R}, (i, u) \mapsto \bar{c}_i^u$ a map such that $u \mapsto \bar{c}_i^u$ is continuous for all $i = 1, \dots, d$. We denote $\sup_{i \in \{1, \dots, d\}, u \in \mathcal{C}} \bar{c}_i^u := \check{c}$ and $\inf_{i \in \{1, \dots, d\}, u \in \mathcal{C}} \bar{c}_i^u := \hat{c}$.

Let $\xi = (\xi^1, \dots, \xi^d) \in L_d^2(\mathcal{F}_T^0)$ and $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times \kappa} \rightarrow \mathbb{R}^d$ a map satisfying

- f is $\mathcal{P}(\mathbb{F}^0) \otimes \mathcal{B}^d \otimes \mathcal{B}^{d \times \kappa}$ -measurable and $f(\cdot, 0, 0) \in \mathbb{H}_d^2(\mathbb{F}^0)$.
- There exists $L \geq 0$ such that, for all $(t, y, y', z, z') \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d \times \kappa} \times \mathbb{R}^{d \times \kappa}$,

$$|f(t, y, z) - f(t, y', z')| \leq L(|y - y'| + |z - z'|).$$

The above assumptions will be in force throughout our work. We shall also use, in this section only, the following additional assumptions.

Assumption 2.1. *i) Switching costs are positive, i.e. $\hat{c} > 0$.*

ii) For all $(t, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times \kappa}$, it holds almost-surely,

$$f(t, y, z) = (f^i(t, y^i, z^i))_{1 \leq i \leq d}. \quad (2.1)$$

iii) For all $u \in \mathcal{C}$ and $i \in \{1, \dots, d\}$, we have $P_{i,i}^u \neq 1$.

Remark 2.1. i) It is usual to assume positive costs in the literature on switching problem. In particular, it implies that the cumulative cost process, see (2.2), is non-decreasing. Introducing signed costs adds extra technical difficulties in the proof of the representation theorem (see e.g. [19] and references therein). We postpone the adaptation of our results in this more general framework to future works.

ii) The structural condition on f is also standard: it allows to get a comparison result for BSDEs, which is a key point to obtain the representation theorem. Note however that our results can be generalized to the case $f^i(t, y, z) = f^i(t, y, z^i)$ for $i \in \{1, \dots, d\}$ by using similar arguments as in [5].

2.1 Solving the control problem using obliquely reflected BSDEs

We define in this section the stochastic optimal control problem. We first introduce the strategies available to the agent and related processes. The definition of the strategy is more involved than in the usual switching problem setting since its adaptedness property is understood with respect to a filtration built recursively.

A strategy is thus given by $\phi = (\zeta_0, (\tau_n)_{n \geq 0}, (\alpha_n)_{n \geq 1})$ where $\zeta_0 \in \{1, \dots, d\}$, $(\tau_n)_{n \geq 0}$ is a non-decreasing sequence of random times and $(\alpha_n)_{n \geq 1}$ is a sequence of \mathcal{C} -valued random variables, which satisfy:

- $\tau_0 \in [0, T]$ and $\zeta_0 \in \{1, \dots, d\}$ are deterministic.
- For all $n \geq 0$, τ_{n+1} is a \mathbb{F}^n -stopping time and α_{n+1} is $\mathcal{F}_{\tau_{n+1}}^n$ -measurable (recall that \mathbb{F}^0 is the augmented Brownian filtration). We then set $\mathbb{F}^{n+1} = (\mathcal{F}_t^{n+1})_{t \geq 0}$ with $\mathcal{F}_t^{n+1} := \mathcal{F}_t^n \vee \sigma(\mathbf{U}_{n+1} 1_{\{\tau_{n+1} \leq t\}})$ for all $t \geq 0$.

Lastly, we define $\mathbb{F}^\infty = (\mathcal{F}_t^\infty)_{t \geq 0}$ with $\mathcal{F}_t^\infty := \bigvee_{n \geq 0} \mathcal{F}_t^n$, $t \geq 0$.

For a strategy $\phi = (\zeta_0, (\tau_n)_{n \geq 0}, (\alpha_n)_{n \geq 1})$, we set, for $n \geq 0$ and $t \geq 0$,

$$\zeta_{n+1} := F(\alpha_{n+1}, \zeta_n, \mathbf{U}_{n+1}) \text{ and } a_t := \sum_{k=0}^{+\infty} \zeta_k 1_{[\tau_k, \tau_{k+1})}(t),$$

which represent the state after a switch and the state process, respectively. We also introduce two processes, for $t \geq 0$,

$$A_t^\phi = \sum_{k=0}^{+\infty} \bar{c}_{\zeta_k}^{\alpha_{k+1}} 1_{\{\tau_{k+1} \leq t\}} \text{ and } N_t^\phi := \sum_{k \geq 0} 1_{\{\tau_{k+1} \leq t\}}. \quad (2.2)$$

The random variable A_t^ϕ is the cumulative cost up to time t and N_t^ϕ is the number of switches before time t . Notice that the processes (a, A^ϕ, N^ϕ) are adapted to \mathbb{F}^∞ and that A^ϕ is a non-decreasing process.

We say that a strategy $\phi = (\zeta_0, (\tau_n)_{n \geq 0}, (\alpha_n)_{n \geq 1})$ is an *admissible strategy* if the cumulative cost process satisfies

$$A_T^\phi - A_{\tau_0}^\phi \in L^2(\mathcal{F}_T^\infty) \quad \text{and} \quad \mathbb{E} \left[\left(A_{\tau_0}^\phi \right)^2 \middle| \mathcal{F}_{\tau_0}^0 \right] < +\infty \text{ a.s.} \quad (2.3)$$

We denote by \mathcal{A} the set of admissible strategies, and for $t \in [0, T]$ and $i \in \{1, \dots, d\}$, we denote by \mathcal{A}_t^i the subset of admissible strategies satisfying $\zeta_0 = i$ and $\tau_0 = t$.

Remark 2.2. *i) The recursive definition of an admissible strategy allows to construct strategies for which decisions are taken using all the information available at that time: the Brownian trajectory up to that time and the values of the noise factors \mathfrak{U}_k associated to earlier decisions. When switching costs are positive, the integrability conditions ensure that the total cost and the total number of switches are almost-surely finite and are square integrable.*

ii) The definition of an admissible strategy is slightly weaker than usual [17], which requires the stronger property $A_T^\phi \in L^2(\mathcal{F}_T^\infty)$. But, importantly, the above definition is enough to define the switched BSDE associated to an admissible control, see below. Moreover, we observe in the next section that optimal strategies are admissible with respect to our definition, but not necessarily with the usual one, due to possible simultaneous jumps at the initial time.

iii) For technical reasons involving possible simultaneous jumps, we cannot consider the generated filtration associated to the state process a , which is contained in \mathbb{F}^∞ .

We are now in position to introduce the reward associated to an admissible strategy. If $\phi = (\zeta_0, (\tau_n)_{n \geq 0}, (\alpha_n)_{n \geq 1}) \in \mathcal{A}$, the reward is defined as the value $\mathbb{E} \left[U_{\tau_0}^\phi - A_{\tau_0}^\phi \middle| \mathcal{F}_{\tau_0}^0 \right]$, where $(U^\phi, V^\phi, M^\phi) \in \mathbb{S}^2(\mathbb{F}^\infty) \times \mathbb{H}_\kappa^2(\mathbb{F}^\infty) \times \mathbb{H}^2(\mathbb{F}^\infty)$ is the solution of the following switched BSDE (see e.g. [17] for the definition of a switched BSDE) on the filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}^\infty, \mathbb{P})$:

$$U_t = \xi^{a_T} + \int_t^T f^{a_s}(s, U_s, V_s) ds - \int_t^T V_s dW_s - \int_t^T dM_s - \int_t^T dA_s^\phi, \quad t \in [\tau_0, T], \quad (2.4)$$

where $\int_t^T dX_s = \int_{(t, T]} dX_s = X_T - X_t$ for $X = M$ and A^ϕ , noticing that these processes are discontinuous.

Remark 2.3. *This switched BSDE rewrites as a classical BSDE in \mathbb{F}^∞ , see Remark A.3, hence existence, uniqueness and a comparison theorem are available in this setting. We refer to Section A.2.2 for more details.*

For $t \in [0, T]$ and $i \in \{1, \dots, d\}$, the agent aims thus to solve the following maximisation problem:

$$\mathcal{V}_t^i = \text{ess sup}_{\phi \in \mathcal{A}_t^i} \mathbb{E} \left[U_t^\phi - A_t^\phi \middle| \mathcal{F}_t^0 \right]. \quad (2.5)$$

We first remark that this control problem corresponds to (1.1) as soon as f does not depend on y and z . Moreover, the term $\mathbb{E}\left[A_t^\phi \middle| \mathcal{F}_t^0\right]$ is non-zero if and only if we have at least one instantaneous switch at initial time t . Notice that in the context of usual switching problems, this control problem reduces to the classical one studied in [17]. We also remark that, if we study a problem (1.1) where the randomness of the running profit comes from a diffusion X and where the manager is allowed to control the drift of this diffusion (continuously in time), then we get a generator f that depends on z (see [17] for some precise results), which justify the level of generality of (2.4).

The main result of this section is the Theorem 2.1 below that relates the value process \mathcal{V} to the solution of an obliquely reflected BSDE, introduced in the following assumption:

Assumption 2.2. *There exists a solution $(Y, Z, K) \in \mathbb{S}_d^2(\mathbb{F}^0) \times \mathbb{H}_{d \times \kappa}^2(\mathbb{F}^0) \times \mathbb{A}_d^2(\mathbb{F}^0)$ to the following obliquely reflected BSDE:*

$$Y_t^i = \xi^i + \int_t^T f^i(s, Y_s^i, Z_s^i) ds - \int_t^T Z_s^i dW_s + \int_t^T dK_s^i, \quad t \in [0, T], \quad i \in \mathcal{I}, \quad (2.6)$$

$$Y_t \in \mathcal{D}, \quad t \in [0, T], \quad (2.7)$$

$$\int_0^T \left(Y_t^i - \sup_{u \in \mathcal{C}} \left\{ \sum_{j=1}^d P_{i,j}^u Y_t^j - \bar{c}_i^u \right\} \right) dK_t^i = 0, \quad i \in \mathcal{I}, \quad (2.8)$$

where $\mathcal{I} := \{1, \dots, d\}$ and \mathcal{D} is the following convex subset of \mathbb{R}^d :

$$\mathcal{D} := \left\{ y \in \mathbb{R}^d \middle| y_i \geq \sup_{u \in \mathcal{C}} \left\{ \sum_{j=1}^d P_{i,j}^u y_j - \bar{c}_i^u \right\}, i \in \mathcal{I} \right\}. \quad (2.9)$$

Let us observe that the positive costs assumption, Assumption 2.1-i), implies that \mathcal{D} has a non-empty interior, since $y = 0$ is an interior point. Except for Section 2.2, this is the main setting for this part, recall Remark 2.1. In Section 4, the system (2.6)-(2.7)-(2.8) is studied in details in a general costs setting: Then, an important step will be to understand when \mathcal{D} has a non-empty interior, see Section 3.

We also introduce a strategy which turns out to be optimal for the control problem. This strategy is the natural extension to our setting of the optimal one for *classical switching problem*, see e.g. [17]. A key step is to prove that this strategy is admissible, which is more involved than in the classical case due to the randomisation, see Section 2.3.2.

For $(t, i) \in [0, T] \times \mathcal{I}$, let $\phi^\star = (\zeta_0^\star, (\tau_n^\star)_{n \geq 0}, (\alpha_n^\star)_{n \geq 1})$ defined by $\tau_0^\star = t$ and $\zeta_0^\star = i$ and inductively by:

$$\tau_{k+1}^\star = \inf \left\{ \tau_k^\star \leq s \leq T \middle| Y_s^{\zeta_k^\star} = \max_{u \in \mathcal{C}} \left\{ \sum_{j=1}^d P_{\zeta_k^\star, j}^u Y_s^j - \bar{c}_{\zeta_k^\star}^u \right\} \right\} \wedge (T+1), \quad (2.10)$$

$$\alpha_{k+1}^\star = \min \arg \max_{u \in \mathcal{C}} \left\{ \sum_{j=1}^d P_{\zeta_k^\star, j}^u Y_{\tau_{k+1}^\star}^j - \bar{c}_{\zeta_k^\star}^u \right\}, \quad (2.11)$$

recall that \mathcal{C} is ordered.

Theorem 2.1. *Let Assumptions 2.1 and 2.2 hold.*

1. For all $i \in \{1, \dots, d\}$, $t \in [0, T]$ and $\phi \in \mathcal{A}_t^i$, we have $Y_t^i \geq \mathbb{E}[U_t^\phi - A_t^\phi | \mathcal{F}_t^0]$.
2. We have $Y_t^i = \mathbb{E}[U_t^{\phi^*} - A_t^{\phi^*} | \mathcal{F}_t^0]$, where $\phi^* = (i, (\tau_n^*)_{n \geq 0}, (\alpha_n^*)_{n \geq 1}) \in \mathcal{A}_t^i$ is defined in (2.10)-(2.11).

The proof is given in Section 2.3. We first remark that, as an immediate consequence, we obtain the uniqueness for the first BSDE component used to characterize the value process of the control problem. Uniqueness of Z follows as usual by applying Itô's formula to $|Y - Y'|^2$ where (Y, Z, K) and (Y', Z', K') are two solutions since necessarily $Y = Y'$. Finally, K is uniquely defined in (2.6) as soon as Y and Z are unique.

Corollary 2.1. *Under Assumptions 2.1 and 2.2, there exists a unique solution $(Y, Z, K) \in \mathbb{S}_d^2(\mathbb{F}^0) \times \mathbb{H}_{d \times \kappa}^2(\mathbb{F}^0) \times \mathbb{A}_d^2(\mathbb{F}^0)$ to the obliquely reflected BSDE (2.6)-(2.7)-(2.8).*

Remark 2.4. *The classical switching problem is an example of switching problem with controlled randomisation. Indeed, we just have to consider $\mathcal{C} = \{1, \dots, d-1\}$,*

$$P_{i,j}^u = \begin{cases} 1 & \text{if } j - i = u \bmod d, \\ 0 & \text{otherwise,} \end{cases} \quad \forall u \in \mathcal{C}, 1 \leq i, j \leq d,$$

and

$$c_i^u = \begin{cases} c_{i,i+u} & \text{if } i + u \leq d, \\ c_{i,i+u-d} & \text{if } i + u > d, \end{cases} \quad \forall u \in \mathcal{C}, 1 \leq i \leq d,$$

where $(c_{i,j})_{1 \leq i,j \leq d}$ are the switching costs from the classical switching problem. We observe that, in this specific case, there is no extra-randomness introduced at each switching time and so there is no need to consider an enlarged filtration. In this setting, if Assumption 2.1-i)&ii) are satisfied, then Theorem 2.1 is already known and Assumption 2.2 is fulfilled, see e.g. [16, 17].

2.2 Uniqueness of solutions to reflected BSDEs with general costs

In this section, we extend the uniqueness result of Corollary 2.1. Namely, we consider the case where $\inf_{1 \leq i \leq d, u \in \mathcal{C}} \bar{c}_i^u = \hat{c}$ can be non-positive, meaning that only Assumption 2.1-ii)&iii) hold here. Assuming in addition that \mathcal{D} has a non-empty interior, we are then able to show uniqueness to (2.6)-(2.7)-(2.8) in Proposition 2.1 below.

Fix y^0 in the interior of \mathcal{D} . It is clear that for all $1 \leq i \leq d$,

$$y_i^0 > \sup_{u \in \mathcal{C}} \left\{ \sum_{j=1}^d P_{i,j}^u y_j^0 - \bar{c}_i^u \right\}.$$

We set, for all $1 \leq i \leq d$ and $u \in \mathcal{C}$,

$$\tilde{c}_i^u := y_i^0 - \sum_{j=1}^d P_{i,j}^u y_j^0 + \bar{c}_i^u > 0,$$

so that $\hat{c} := \inf_{1 \leq i \leq d, u \in \mathcal{C}} \tilde{c}_i^u > 0$ by compactness of \mathcal{C} and continuity of $u \mapsto (\bar{c}_i^u, P_{i,j}^u)$ for all $1 \leq i, j \leq d$. We also consider the following set

$$\tilde{\mathcal{D}} := \left\{ \tilde{y} \in \mathbb{R}^d \left| \tilde{y}_i \geq \sup_{u \in \mathcal{C}} \left\{ \sum_{j=1}^d P_{i,j}^u \tilde{y}_j - \tilde{c}_i^u \right\}, 1 \leq i \leq d \right. \right\}.$$

Lemma 2.1. *Assume that \mathcal{D} has a non-empty interior and let y^0 in its interior. Then,*

$$\tilde{\mathcal{D}} = \{y - y^0 \mid y \in \mathcal{D}\}.$$

Proof. If $y \in \mathcal{D}$, let $\tilde{y} := y - y^0$. For $1 \leq i \leq d$ and $u \in \mathcal{C}$, we have

$$\begin{aligned} \tilde{y}_i &= y_i - y_i^0 \geq \sum_{j=1}^d P_{i,j}^u y_j - \bar{c}_i^u - y_i^0 = \sum_{j=1}^d P_{i,j}^u (y_j - y_j^0) - (\bar{c}_i^u + y_i^0 - \sum_{j=1}^d P_{i,j}^u y_j^0) \\ &= \sum_{j=1}^d P_{i,j}^u \tilde{y}_j - \tilde{c}_i^u, \end{aligned}$$

hence $\tilde{y} \in \tilde{\mathcal{D}}$. Conversely, let $\tilde{y} \in \tilde{\mathcal{D}}$ and let $y := \tilde{y} + y^0$. We can show by the same kind of calculation that $y \in \mathcal{D}$. \square

Proposition 2.1. *Let Assumption 2.1-ii)&iii) hold and assume furthermore that \mathcal{D} has a non-empty interior. Then there exists at most one solution to (2.6)-(2.7)-(2.8) in $\mathbb{S}_d^2(\mathbb{F}^0) \times \mathbb{H}_{d \times \kappa}^2(\mathbb{F}^0) \times \mathbb{A}_d^2(\mathbb{F}^0)$.*

Proof. Let us assume that (Y^1, Z^1, K^1) and (Y^2, Z^2, K^2) are two solutions to (2.6)-(2.7)-(2.8). We set $\tilde{Y}^1 := Y^1 - y^0$ and $\tilde{Y}^2 := Y^2 - y^0$. Then one checks easily that (\tilde{Y}^1, Z^1, K^1) and (\tilde{Y}^2, Z^2, K^2) are solutions to (2.6)-(2.7)-(2.8) with terminal condition $\xi = \xi - y^0$, driver \tilde{f} given by

$$\tilde{f}^i(t, \tilde{y}_i, z_i) := f^i(t, \tilde{y}_i + y_i^0, z_i), \quad 1 \leq i \leq d, t \in [0, T], \tilde{y} \in \mathbb{R}^d, z \in \mathbb{R}^{d \times \kappa},$$

and domain $\tilde{\mathcal{D}}$. This domain is associated to a switching problem with controlled randomisation with $\hat{c} > 0$, hence Corollary 2.1 gives that $(\tilde{Y}^1, Z^1, K^1) = (\tilde{Y}^2, Z^2, K^2)$ which implies the uniqueness. \square

2.3 Proof of the representation result

We prove here our main result for this part, namely Theorem 2.1. The proof is divided in several steps.

2.3.1 Preliminary estimates

We first introduce auxiliary processes associated to an admissible strategy and prove some key integrability properties.

Suppose that Assumption 2.2 is in force, and let (Y, Z, K) be a solution to (2.6)-(2.7)-(2.8). Let $i \in \{1, \dots, d\}$ and $t \in [0, T]$. We set, for $\phi \in \mathcal{A}_t^i$ and $t \leq s \leq T$,

$$\mathcal{Y}_s^\phi := \sum_{k \geq 0} Y_s^{\zeta_k} 1_{[\tau_k, \tau_{k+1})}(s), \quad (2.12)$$

$$\mathcal{Z}_s^\phi := \sum_{k \geq 0} Z_s^{\zeta_k} 1_{[\tau_k, \tau_{k+1})}(s), \quad (2.13)$$

$$\mathcal{K}_s^\phi := \sum_{k \geq 0} \int_{\tau_k \wedge s}^{\tau_{k+1} \wedge s} dK_u^{\zeta_k}, \quad (2.14)$$

$$\mathcal{M}_s^\phi := \sum_{k \geq 0} \left(Y_{\tau_{k+1}}^{\zeta_{k+1}} - \mathbb{E} \left[Y_{\tau_{k+1}}^{\zeta_{k+1}} \middle| \mathcal{F}_{\tau_{k+1}}^k \right] \right) 1_{\{t < \tau_{k+1} \leq s\}}, \quad (2.15)$$

$$\mathcal{A}_s^\phi = \sum_{k \geq 0} \left(Y_{\tau_{k+1}}^{\zeta_k} - \mathbb{E} \left[Y_{\tau_{k+1}}^{\zeta_{k+1}} \middle| \mathcal{F}_{\tau_{k+1}}^k \right] + \bar{c}_{\zeta_k}^{\alpha_{k+1}} \right) 1_{\{t < \tau_{k+1} \leq s\}}. \quad (2.16)$$

Remark 2.5. For all $k \geq 0$, since α_{k+1} is $\mathcal{F}_{\tau_{k+1}}^k$ -measurable, we have

$$\mathbb{E} \left[Y_{\tau_{k+1}}^{\zeta_{k+1}} \middle| \mathcal{F}_{\tau_{k+1}}^k \right] = \sum_{j=1}^d \mathbb{P}(\zeta_{k+1} = j \mid \mathcal{F}_{\tau_{k+1}}^k) Y_{\tau_{k+1}}^j = \sum_{j=1}^d P_{\zeta_k, j}^{\alpha_{k+1}} Y_{\tau_{k+1}}^j. \quad (2.17)$$

Lemma 2.2. Suppose that Assumption 2.1-i) and Assumption 2.2 are satisfied. For any admissible strategy $\phi \in \mathcal{A}_t^i$, \mathcal{M}^ϕ is a square integrable \mathbb{F}^∞ -martingale with $\mathcal{M}_t^\phi = 0$, \mathcal{K}^ϕ is a non-decreasing continuous process with $\mathcal{K}_t^\phi = 0$, \mathcal{A}^ϕ is non-decreasing and satisfies $\mathcal{A}_t^\phi = 0$ and $\mathcal{A}_T^\phi \in L^2(\mathcal{F}_T^\infty)$. In addition,

$$\mathbb{E} \left[\left(\sum_{k \geq 0} \left(Y_{\tau_{k+1}}^{\zeta_{k+1}} - \mathbb{E} \left[Y_{\tau_{k+1}}^{\zeta_{k+1}} \middle| \mathcal{F}_{\tau_{k+1}}^k \right] \right) 1_{\{\tau_{k+1} \leq t\}} \right)^2 \middle| \mathcal{F}_t^0 \right] < +\infty \quad a.s. \quad (2.18)$$

Proof. Let $\phi \in \mathcal{A}_t^i$. It is clear by (2.14) that \mathcal{K}^ϕ is non-decreasing, continuous and satisfies $\mathcal{K}_t^\phi = 0$. Moreover, by (2.15) and (2.16), we have that $\mathcal{M}_t^\phi = 0$ and $\mathcal{A}_t^\phi = 0$. Using Theorem A.3, we obtain that each summand in (2.15) is a \mathbb{F}^∞ -martingale, hence \mathcal{M}^ϕ is a \mathbb{F}^∞ -local martingale. We are going to show that $\mathbb{E} \left[\int_0^T d[\mathcal{M}^\phi]_s \right] < +\infty$, which implies that \mathcal{M}^ϕ is a \mathbb{F}^∞ -martingale, and we show at the same time that $\mathcal{A}_T^\phi \in L^2(\mathcal{F}_T^\infty)$. Eventually, (2.18) will be proved.

Using (2.16) and (2.17), we have, for all $s \in [t, T]$,

$$\mathcal{A}_s^\phi = \sum_{k \geq 0} \left(Y_{\tau_{k+1}}^{\zeta_k} - \sum_{j=1}^d P_{\zeta_k, j}^{\alpha_{k+1}} Y_{\tau_{k+1}}^j + \bar{c}_{\zeta_k}^{\alpha_{k+1}} \right) 1_{\{t < \tau_{k+1} \leq s\}}, \quad (2.19)$$

hence \mathcal{A}^ϕ is non-decreasing since each summand is positive as $Y \in \mathcal{D}$.

For $t \leq s \leq T$, distinguishing for each $k \geq 0$ between what happens on $[\tau_k, \tau_{k+1})$ (i.e. following the dynamics of Y^{ζ_k}) and at time τ_{k+1} (i.e. switching from Y^{ζ_k} to $Y^{\zeta_{k+1}}$), we have

$$\mathcal{Y}_s^\phi - \mathcal{Y}_t^\phi = \sum_{k \geq 0} \left(Y_{\tau_{k+1} \wedge s}^{\zeta_k} - Y_{\tau_k \wedge s}^{\zeta_k} \right) + \sum_{k \geq 0} \left(Y_{\tau_{k+1}}^{\zeta_{k+1}} - Y_{\tau_{k+1}}^{\zeta_k} \right) 1_{\{t < \tau_{k+1} \leq s\}}. \quad (2.20)$$

Using (2.6), we get, for all $k \geq 0$,

$$\begin{aligned} & Y_{\tau_{k+1} \wedge s}^{\zeta_k} - Y_{\tau_k \wedge s}^{\zeta_k} \\ &= - \int_{\tau_k \wedge s}^{\tau_{k+1} \wedge s} f^{\zeta_k}(u, Y_u^{\zeta_k}, Z_u^{\zeta_k}) du + \int_{\tau_k \wedge s}^{\tau_{k+1} \wedge s} Z_u^{\zeta_k} dW_u - \int_{\tau_k \wedge s}^{\tau_{k+1} \wedge s} dK_u^{\zeta_k}, \end{aligned}$$

recalling ζ_k is \mathcal{F}_{τ_k} -measurable. We also have, using (2.17), for all $k \geq 0$,

$$\begin{aligned} & Y_{\tau_{k+1}}^{\zeta_{k+1}} - Y_{\tau_{k+1}}^{\zeta_k} \\ &= \left(Y_{\tau_{k+1}}^{\zeta_{k+1}} - \mathbb{E} \left[Y_{\tau_{k+1}}^{\zeta_{k+1}} \middle| \mathcal{F}_{\tau_{k+1}}^k \right] \right) - \left(Y_{\tau_{k+1}}^{\zeta_k} - \sum_{j=1}^d P_{\zeta_k, j}^{\alpha_{k+1}} Y_{\tau_{k+1}}^j + \bar{c}_{\zeta_k}^{\alpha_{k+1}} \right) + \bar{c}_{\zeta_k}^{\alpha_{k+1}}. \end{aligned}$$

Inserting the two previous equalities into (2.20), we get:

$$\begin{aligned} \mathcal{Y}_s^\phi - \mathcal{Y}_t^\phi &= \sum_{k \geq 0} \left(- \int_{\tau_k \wedge s}^{\tau_{k+1} \wedge s} f^{\zeta_k}(u, Y_u^{\zeta_k}, Z_u^{\zeta_k}) du + \int_{\tau_k \wedge s}^{\tau_{k+1} \wedge s} Z_u^{\zeta_k} dW_u - \int_{\tau_k \wedge s}^{\tau_{k+1} \wedge s} dK_u^{\zeta_k} \right) \\ &\quad + \sum_{k \geq 0} \left(Y_{\tau_{k+1}}^{\zeta_{k+1}} - \mathbb{E} \left[Y_{\tau_{k+1}}^{\zeta_{k+1}} \middle| \mathcal{F}_{\tau_{k+1}}^k \right] \right) 1_{\{t < \tau_{k+1} \leq s\}} + A_s^\phi - A_t^\phi \\ &\quad - \sum_{k \geq 0} \left(Y_{\tau_{k+1}}^{\zeta_k} - \sum_{j=1}^d P_{\zeta_k, j}^{\alpha_{k+1}} Y_{\tau_{k+1}}^j + \bar{c}_{\zeta_k}^{\alpha_{k+1}} \right) 1_{\{t < \tau_{k+1} \leq s\}}. \end{aligned} \quad (2.21)$$

Considering the previous equality also with $s = T$, subtracting with the previous one and using the definition of $\mathcal{Y}^\phi, \mathcal{Z}^\phi, \mathcal{K}^\phi, \mathcal{M}^\phi, \mathcal{A}^\phi$ (in particular that, for example, $\mathcal{Y}_u^\phi = Y_u^{\zeta_k}$ on $\{\tau_k \leq u < \tau_{k+1}\}$ for each $k \geq 0$), we obtain, for all $s \in [t, T]$,

$$\begin{aligned} \mathcal{Y}_s^\phi &= \xi^{a_T} + \int_s^T f^{a_u}(u, \mathcal{Y}_u^\phi, \mathcal{Z}_u^\phi) du - \int_s^T \mathcal{Z}_u^\phi dW_u - \int_s^T d\mathcal{M}_u^\phi - \int_s^T d\mathcal{A}_u^\phi \\ &\quad + \left[\left(\mathcal{A}_T^\phi + \mathcal{K}_T^\phi \right) - \left(\mathcal{A}_s^\phi + \mathcal{K}_s^\phi \right) \right]. \end{aligned} \quad (2.22)$$

For any $n \geq 1$, we consider the admissible strategy $\phi^n = (\zeta_0, (\tau_k^n)_{k \geq 0}, (\alpha_k^n)_{k \geq 1})$ defined by $\zeta_0^n = i = \zeta_0, \tau_k^n = \tau_k, \alpha_k^n = \alpha_k$ for $k \leq n$, and $\tau_k^n = T + 1$ for all $k > n$. We set

$\mathcal{Y}^n := \mathcal{Y}^{\phi^n}$, $\mathcal{Z}^n := \mathcal{Z}^{\phi^n}$, and so on.

By (2.22) applied to the strategy ϕ^n , we get, recalling that $\mathcal{A}_t^n = 0$,

$$\begin{aligned} \mathcal{A}_{\tau_n \wedge T}^n &= \mathcal{Y}_t^n - \mathcal{Y}_{\tau_n \wedge T}^n - \int_t^{\tau_n \wedge T} f^{a_s^n}(s, \mathcal{Y}_s^n, \mathcal{Z}_s^n) ds + \int_t^{\tau_n \wedge T} \mathcal{Z}_s^n dW_s \\ &\quad + \int_t^{\tau_n \wedge T} d\mathcal{M}_s^n + \int_t^{\tau_n \wedge T} dA_s^n - \int_t^{\tau_n \wedge T} d\mathcal{K}_s^n. \end{aligned} \quad (2.23)$$

We obtain, for a constant $\Lambda > 0$, recalling that $\mathcal{K}_t^n = 0$,

$$\begin{aligned} \mathbb{E}[|\mathcal{A}_{\tau_n \wedge T}^n|^2] &\leq \Lambda \left(\mathbb{E}[|\mathcal{Y}_t^n|^2 + |\mathcal{Y}_{\tau_n \wedge T}^n|^2 + \int_t^{\tau_n \wedge T} |f^{a_s^n}(s, \mathcal{Y}_s^n, \mathcal{Z}_s^n)|^2 ds \right. \\ &\quad \left. + \int_t^{\tau_n \wedge T} |\mathcal{Z}_s^n|^2 ds + \int_t^{\tau_n \wedge T} d[\mathcal{M}^n]_s + (A_T^\phi - A_t^\phi)^2 + (\mathcal{K}_T^n)^2 \right), \end{aligned} \quad (2.24)$$

with $[\mathcal{M}^n]$ the quadratic variation process of \mathcal{M}^n , using $0 \leq A_T^n - A_t^n \leq A_T^\phi - A_t^\phi$ as the costs are positive. We have

$$\begin{aligned} \mathbb{E}[|\mathcal{Y}_r^n|^2] &\leq \sum_{j=1}^d \mathbb{E}[|Y_r^j|^2] = \mathbb{E}[|Y_r|^2] \leq \mathbb{E}\left[\sup_{t \leq r \leq T} |Y_r|^2\right] = \|Y\|_{\mathbb{S}_d^2(\mathbb{F}^0)}^2, \\ \mathbb{E}\left[\int_t^{\tau_n \wedge T} |\mathcal{Z}_s^n|^2 ds\right] &\leq \|Z\|_{\mathbb{H}_{d \times \kappa}^2(\mathbb{F}^0)}^2, \\ \mathbb{E}\left[\int_t^{\tau_n \wedge T} |f^{a_s^n}(s, \mathcal{Y}_s^n, \mathcal{Z}_s^n)|^2 ds\right] &\leq 4L^2 T \|Y\|_{\mathbb{S}_d^2(\mathbb{F}^0)}^2 + 4L^2 \|Z\|_{\mathbb{H}_d^2(\mathbb{F}^0)}^2 + 2\|f(\cdot, 0, 0)\|_{\mathbb{H}_d^2(\mathbb{F}^0)}^2 \end{aligned}$$

and

$$\mathbb{E}[(\mathcal{K}_T^n)^2] \leq \mathbb{E}[|K_T|^2].$$

Thus, by these estimates and the fact that $A_T^\phi - A_t^\phi \in L^2(\mathcal{F}_T^\infty)$ as ϕ is admissible, there exists a constant $\Lambda_1 > 0$ such that

$$\mathbb{E}[|\mathcal{A}_{\tau_n \wedge T}^n|^2] \leq \Lambda_1 + \Lambda \mathbb{E}\left[\int_t^{\tau_n \wedge T} d[\mathcal{M}^n]_s\right]. \quad (2.25)$$

Using (2.22) applied to ϕ^n , we apply Itô's formula between t and $\tau_n \wedge T$. Since the number of switches is bounded by n , \mathcal{M}^n is a square integrable \mathbb{F}^∞ -martingale orthogonal to W and $[W, \mathcal{M}^n] = 0$, see Remark A.2. As \mathcal{K}^n is non-decreasing and continuous, $[\cdot, \mathcal{K}^n] = 0$. Moreover, we also observe that, for $0 \leq k \leq n-1$, as $\tau_{k+1} \leq \tau_n \wedge T$ is equivalent to $\tau_{k+1} \leq T$ and $\bar{c}_{\zeta_k}^{\alpha_{k+1}} \mathbf{1}_{t < \tau_{k+1} \leq T}$ is $\mathcal{F}_{\tau_{k+1}}^k$ -measurable,

$$\begin{aligned} \mathbb{E}\left[\int_t^{\tau_n \wedge T} d[\mathcal{M}^n, A^n]_s\right] &= \mathbb{E}\left[\sum_{k=0}^{n-1} \left(Y_{\tau_{k+1}}^{\zeta_{k+1}} - \mathbb{E}\left[Y_{\tau_{k+1}}^{\zeta_{k+1}} \mid \mathcal{F}_{\tau_{k+1}}^k\right]\right) \bar{c}_{\zeta_k}^{\alpha_{k+1}} \mathbf{1}_{t < \tau_{k+1} \leq \tau_n \wedge T}\right] \\ &= \sum_{k=0}^{n-1} \mathbb{E}\left[\left(Y_{\tau_{k+1}}^{\zeta_{k+1}} - \mathbb{E}\left[Y_{\tau_{k+1}}^{\zeta_{k+1}} \mid \mathcal{F}_{\tau_{k+1}}^k\right]\right) \bar{c}_{\zeta_k}^{\alpha_{k+1}} \mathbf{1}_{t < \tau_{k+1} \leq T}\right] = 0. \end{aligned}$$

The same argument gives $\mathbb{E}\left[\int_t^{\tau_n \wedge T} d[\mathcal{M}^n, \mathcal{A}^n]_s\right] = 0$ as \mathcal{A}^n is also a sum of $\mathcal{F}_{\tau_{k+1}}^k$ -measurable random variables. Thus, Itô's formula gives, as $\int_t^{\tau_n \wedge T} d[A^n]_s \geq 0$ and $\int_t^{\tau_n \wedge T} d[\mathcal{A}^n]_s \geq 0$,

$$\begin{aligned}
& \mathbb{E}\left[|\mathcal{Y}_t^n|^2 + \int_t^{\tau_n \wedge T} |\mathcal{Z}_s^n|^2 ds + \int_t^{\tau_n \wedge T} d[\mathcal{M}^n]_s\right] \\
& \leq \mathbb{E}\left[|\mathcal{Y}_{\tau_n \wedge T}^n|^2 + 2 \int_t^{\tau_n \wedge T} \mathcal{Y}_s^n f_s^{a_s^n}(s, \mathcal{Y}_s^n, \mathcal{Z}_s^n) ds - 2 \int_t^{\tau_n \wedge T} \mathcal{Y}_s^n dA_s^n\right. \\
& \quad \left.+ 2 \int_t^{\tau_n \wedge T} \mathcal{Y}_s^n d\mathcal{A}_s^n + 2 \int_t^{\tau_n \wedge T} \mathcal{Y}_s^n d\mathcal{K}_s^n\right] \\
& \leq \mathbb{E}[|\mathcal{Y}_{\tau_n \wedge T}^n|^2] + 2\mathbb{E}\left[\int_t^{\tau_n \wedge T} |\mathcal{Y}_s^n f_s^{a_s^n}(s, \mathcal{Y}_s^n, \mathcal{Z}_s^n)| ds\right] + 2\mathbb{E}\left[\int_t^{\tau_n \wedge T} |\mathcal{Y}_s^n| dA_s^n\right] \\
& \quad + 2\mathbb{E}\left[\int_t^{\tau_n \wedge T} |\mathcal{Y}_s^n| d\mathcal{A}_s^n\right] + 2\mathbb{E}\left[\int_t^{\tau_n \wedge T} |\mathcal{Y}_s^n| d\mathcal{K}_s^n\right]. \tag{2.26}
\end{aligned}$$

We have, using Young's inequality, for some $\epsilon > 0$, and (2.25),

$$\begin{aligned}
\mathbb{E}\left[\int_t^{\tau_n \wedge T} |\mathcal{Y}_s^n f_s^{a_s^n}(s, \mathcal{Y}_s^n, \mathcal{Z}_s^n)| ds\right] & \leq \frac{1}{2}\mathbb{E}\left[\int_t^{\tau_n \wedge T} |\mathcal{Y}_s^n|^2 ds\right] + \frac{1}{2}\mathbb{E}\left[\int_t^{\tau_n \wedge T} |f_s^{a_s^n}(s, \mathcal{Y}_s^n, \mathcal{Z}_s^n)|^2 ds\right] \\
& \leq T\left(\frac{1}{2} + 2L^2\right)\|Y\|_{\mathbb{S}_d^2(\mathbb{F}^0)}^2 + 2L^2\|Z\|_{\mathbb{H}_d^2(\mathbb{F}^0)}^2 + \|f(\cdot, 0, 0)\|_{\mathbb{H}_d^2(\mathbb{F}^0)}^2, \\
\mathbb{E}\left[\int_t^{\tau_n \wedge T} |\mathcal{Y}_s^n| dA_s^n\right] & \leq \frac{1}{2}\|Y\|_{\mathbb{S}_d^2(\mathbb{F}^0)}^2 + \frac{1}{2}\mathbb{E}\left[(A_T^\phi - A_t^\phi)^2\right], \\
\mathbb{E}\left[\int_t^{\tau_n \wedge T} |\mathcal{Y}_s^n| d\mathcal{K}_s^n\right] & \leq \frac{1}{2}\|Y\|_{\mathbb{S}_d^2(\mathbb{F}^0)}^2 + \frac{1}{2}\mathbb{E}[|K_T|^2],
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[\int_t^{\tau_n \wedge T} |\mathcal{Y}_s^n| d\mathcal{A}_s^n\right] & \leq \frac{1}{2\epsilon}\|Y\|_{\mathbb{S}_d^2(\mathbb{F}^0)}^2 + \frac{\epsilon}{2}\mathbb{E}[(\mathcal{A}_{\tau_n \wedge T}^n)^2] \\
& \leq \frac{1}{2\epsilon}\|Y\|_{\mathbb{S}_d^2(\mathbb{F}^0)}^2 + \frac{\epsilon}{2}\left(\Lambda_1 + \Lambda\mathbb{E}\left[\int_t^{\tau_n \wedge T} d[\mathcal{M}^n]_s\right]\right).
\end{aligned}$$

Using these estimates together with (2.26) gives, for a constant $C_\epsilon > 0$ independent of n ,

$$(1 - \epsilon\Lambda)\mathbb{E}\left[\int_t^{\tau_n \wedge T} d[\mathcal{M}^n]_s\right] \leq C_\epsilon\left(\|Y\|_{\mathbb{S}_d^2(\mathbb{F}^0)}^2 + \|Z\|_{\mathbb{H}_{d \times \kappa}^2(\mathbb{F}^0)}^2 + \|f(\cdot, 0, 0)\|_{\mathbb{H}_d^2(\mathbb{F}^0)}^2 + \mathbb{E}[|K_T|^2]\right), \tag{2.27}$$

and choosing $\epsilon = \frac{1}{2\Lambda}$ gives that $\mathbb{E}\left[\int_t^{\tau_n \wedge T} d[\mathcal{M}^n]_s\right]$ is upper bounded independently of n .

We also get an upper bound independent of n for $\mathbb{E}[(\mathcal{A}_{\tau_n \wedge T}^n)^2]$ by (2.25).

Since $\int_t^{\tau_n \wedge T} d[\mathcal{M}^n]_s$ (resp. $|\mathcal{A}_{\tau_n \wedge T}^n|^2$) is non-decreasing to $\int_t^T d[\mathcal{M}^\phi]_s$ (resp. to $|\mathcal{A}_T^\phi|^2$), we obtain by monotone convergence the first part of Lemma 2.2, i.e. that \mathcal{M}^ϕ is a square integrable \mathbb{F}^∞ -martingale with $\mathcal{M}_t^\phi = 0$, that \mathcal{A}^ϕ is non-decreasing with $\mathcal{A}_t^\phi = 0$ and $\mathcal{A}_T^\phi \in L^2(\mathcal{F}_T^\infty)$.

We now prove (2.18). Using that $\mathbb{E}[(N_t^\phi)^2 | \mathcal{F}_t^0]$ is almost-surely finite as ϕ is admissible, $\hat{c} > 0$ and $\tau_{k+1} = t = \tau_0$ on $\{\tau_{k+1} \leq t\}$, we compute,

$$\begin{aligned} & \mathbb{E} \left[\left(\sum_{k \geq 0} \left(Y_{\tau_{k+1}}^{\zeta_{k+1}} - \mathbb{E}[Y_{\tau_{k+1}}^{\zeta_{k+1}} | \mathcal{F}_{\tau_{k+1}}^k] \right) 1_{\{\tau_{k+1} \leq t\}} \right)^2 \middle| \mathcal{F}_t^0 \right] \\ & \leq \mathbb{E} \left[\left(\sum_{k \geq 0} \left| Y_t^{\zeta_{k+1}} - \sum_{j=1}^d P_{\zeta_k, j}^{\alpha_{k+1}} Y_t^j \right| 1_{\{\tau_{k+1} \leq t\}} \right)^2 \middle| \mathcal{F}_t^0 \right] \\ & \leq 4|Y_t|^2 \mathbb{E}[(N_t^\phi)^2 | \mathcal{F}_t^0] < +\infty \quad \text{a.s.} \end{aligned} \tag{2.28}$$

□

2.3.2 An optimal strategy

In the following lemma, we show that, since \mathcal{D} has a non-empty interior, the number of switches (hence the cost), following the strategy ϕ^* (recall (2.10)-(2.11)), required to leave any point on the boundary of \mathcal{D} , is square integrable. This result will be used to prove that the cost associated to ϕ^* satisfies $\mathbb{E}[(A_t^{\phi^*})^2 | \mathcal{F}_t^0] < +\infty$ almost-surely.

Lemma 2.3. *Let Assumption 2.1-i)&iii) hold. For $y \in \mathcal{D}$, we define*

$$S(y) = \left\{ 1 \leq i \leq d \mid y_i = \max_{u \in \mathcal{C}} \left\{ \sum_{j=1}^d P_{i,j}^u y_j - \bar{c}_i^u \right\} \right\}, \tag{2.29}$$

and $(u_i)_{i \in S(y)}$ the family of elements of \mathcal{C} given by

$$u_i = \min \arg \max_{u \in \mathcal{C}} \left\{ \sum_{j=1}^d P_{i,j}^u y_j - \bar{c}_i^u \right\}.$$

Consider the homogeneous Markov Chain X on $S(y) \cup \{0\}$ defined by, for $k \geq 0$ and $i, j \in S(y)$,

$$\begin{aligned} \mathbb{P}(X_{k+1} = j \mid X_k = i) &= P_{i,j}^{u_i}, \\ \mathbb{P}(X_{k+1} = 0 \mid X_k = i) &= 1 - \sum_{j \in S(y)} P_{i,j}^{u_i}, \\ \mathbb{P}(X_{k+1} = 0 \mid X_k = 0) &= 1, \\ \mathbb{P}(X_{k+1} = i \mid X_k = 0) &= 0. \end{aligned}$$

Then 0 is accessible from every $i \in S(y)$, meaning that X is an absorbing Markov Chain. Moreover, let $N(y) = \inf \{n \geq 0 \mid X_n = 0\}$. Then $N(y) \in L^2(\mathbb{P}^i)$ for all $i \in S(y)$, where \mathbb{P}^i is the probability satisfying $\mathbb{P}^i(X_0 = i) = 1$.

Proof. Assume that there exists $i \in S(y)$ from which 0 is not accessible. Then every communicating class accessible from i is included in $S(y)$. In particular, there exists a recurrent class $S' \subset S(y)$. For all $i \in S'$, we have $P_{i,j}^{u_i} = 0$ if $j \notin S'$ since S' is recurrent. Moreover, since $S' \subset S(y)$, we obtain, for all $i \in S'$, by definition of $S(y)$,

$$y_i = \sum_{j \in S'} P_{i,j}^{u_i} y_j - \bar{c}_i^{u_i}. \quad (2.30)$$

Since S' is a recurrent class, the matrix $\tilde{P} = (P_{i,j}^{u_i})_{i,j \in S'}$ is stochastic and irreducible. By definition of \mathcal{D} , we have

$$\mathcal{D} \subset \mathbb{R}^{d-|S'|} \times \left\{ z \in \mathbb{R}^{|S'|} \mid z_i \geq \sum_{j \in S'} P_{i,j}^{u_i} z_j - \bar{c}_i^{u_i}, i \in S' \right\} = \mathbb{R}^{d-|S'|} \times \mathcal{D}'.$$

With a slight abuse of notation, we do not renumber coordinates of vectors in \mathcal{D}' . Let $i_0 \in S'$ and let us restrict ourself to the domain \mathcal{D}' . According to Lemma 3.1, \mathcal{D}' is invariant by translation along the vector $(1, \dots, 1)$ of $\mathbb{R}^{|S'|}$. Moreover, Assumption 3.1 is fulfilled since \tilde{P} is irreducible and controls $(u_i)_{i \in S(y)}$ are set. So, Proposition 3.1 yields that $\mathcal{D}' \cap \{z \in \mathbb{R}^{|S'|} \mid z_{i_0} = 0\}$ is a compact convex polytope. Recalling (2.30), we see that $(y_i - y_{i_0})_{i \in S'}$ is a point of $\mathcal{D}' \cap \{z \in \mathbb{R}^{|S'|} \mid z_{i_0} = 0\}$ that saturates all the inequalities. So, $(y_i - y_{i_0})_{i \in S'}$ is an extreme point of $\mathcal{D}' \cap \{z \in \mathbb{R}^{|S'|} \mid z_{i_0} = 0\}$ and all extreme points are given by

$$\mathcal{E} := \left\{ z \in \mathbb{R}^{|S'|} \mid z_i = \sum_{j \in S'} P_{i,j}^{u_i} z_j - \bar{c}_i^{u_i}, i \in S', z_{i_0} = 0 \right\}.$$

Recalling that \mathcal{D}' is compact, \mathcal{E} is a non-empty bounded affine subspace of $\mathbb{R}^{|S'|}$, so it is a singleton. Since $\mathcal{D}' \cap \{z_{i_0} = 0\}$ is a compact convex polytope, it is the convex hull of \mathcal{E} and so it is also a singleton. Hence \mathcal{D}' is a line in $\mathbb{R}^{|S'|}$. Moreover, $|S'| \geq 2$ as $P_{i,i}^u \neq 1$ for all $u \in \mathcal{C}$ and $i \in \{1, \dots, d\}$. Thus $\mathcal{D} \subset \mathbb{R}^{d-|S'|} \times \mathcal{D}'$ gives a contradiction with the fact that \mathcal{D} has a non-empty interior and the first part of the lemma is proved. Finally, we have $N(y) \in L^2(\mathbb{P}^i)$ for all $i \in S(y)$ thanks to Theorem 3.3.5 in [18]. \square

Lemma 2.4. Suppose that Assumption 2.1-i),iii) and Assumption 2.2 are in force. Then the strategy ϕ^* is admissible, recall (2.10)-(2.11).

Proof. For $n \geq 1$, we consider the admissible strategy $\phi^n = (\zeta_0, (\tau_k^n)_{k \geq 0}, (\alpha_k^n)_{k \geq 1})$ defined by $\zeta_0^n = i = \zeta_0^*$, $\tau_k^n = \tau_k^*$, $\alpha_k^n = \alpha_k^*$ for $k \leq n$, and $\tau_k^n = T + 1$ for all $k > n$. We set $\mathcal{Y}_s^n := \mathcal{Y}_s^{\phi^n}$, $\mathcal{Z}_s^n := \mathcal{Z}_s^{\phi^n}$ and so on, for all $s \in [t, T]$.

By definition of τ^*, α^* , recall (2.10)-(2.11), it is clear that $\mathcal{A}_{s \wedge \tau_n^*}^n = 0$ and that $\int_{\tau_k^* \wedge s}^{\tau_{k+1}^* \wedge s} dK_u^{\zeta_k^*} = 0$ for all $k < n$ and $s \in [t, T]$. The identity (2.22) for the admissible strategy ϕ^n gives

$$\mathcal{Y}_t^n = \mathcal{Y}_{\tau_n^* \wedge T}^n + \int_t^{\tau_n^* \wedge T} f^{a_s^n}(s, \mathcal{Y}_s^n, \mathcal{Z}_s^n) ds - \int_t^{\tau_n^* \wedge T} \mathcal{Z}_s^n dW_s - \int_t^{\tau_n^* \wedge T} d\mathcal{M}_s^n - \int_t^{\tau_n^* \wedge T} dA_s^n.$$

Using similar arguments and estimates as in the proof of Lemma 2.2 (see equation (2.25) and its derivation), we get

$$\mathbb{E}\left[|A_{\tau_n^* \wedge T}^n - A_t^n|^2\right] \leq \Lambda_1 + \Lambda \mathbb{E}\left[\int_t^{\tau_n^* \wedge T} d[\mathcal{M}^n]_s\right], \quad (2.31)$$

and (see equation (2.27) and its derivation), for $\epsilon > 0$,

$$(1 - \epsilon\Lambda) \mathbb{E}\left[\int_t^{\tau_n^* \wedge T} d[\mathcal{M}^n]_s\right] \leq C_\epsilon \left(\|Y\|_{\mathbb{S}_d^2(\mathbb{F}^0)}^2 + \|Z\|_{\mathbb{H}_{d \times \kappa}^2(\mathbb{F}^0)}^2 + \|f(\cdot, 0, 0)\|_{\mathbb{H}_d^2(\mathbb{F}^0)}^2\right). \quad (2.32)$$

Choosing $\epsilon = \frac{1}{2\Lambda}$ gives that $\mathbb{E}\left[\int_t^{\tau_n^* \wedge T} d[\mathcal{M}^n]_s\right]$ and $\mathbb{E}\left[|A_{\tau_n^* \wedge T}^n - A_t^n|^2\right]$ are upper bounded uniformly in n , hence by monotone convergence, we get that $A_T^{\phi^*} - A_t^{\phi^*} \in L^2(\mathcal{F}_T^\infty)$.

It remains to prove that $\mathbb{E}\left[\left(A_t^{\phi^*}\right)^2 \middle| \mathcal{F}_t^0\right] < +\infty$. We have $A_t^{\phi^*} \leq \tilde{c}N_t^{\phi^*}$, and $\mathbb{E}\left[(N_t^{\phi^*})^2 \middle| \mathcal{F}_t^0\right] < +\infty$ a.s. is immediate from Lemma 2.3, since $\mathbb{E}\left[(N_t^{\phi^*})^2 \middle| \mathcal{F}_t^0\right] = \Psi(Y_t)$ with $\Psi(y) = \mathbb{E}^i\left[(N(y))^2\right]$, $y \in \mathcal{D}$, where \mathbb{E}^i is the expectation under the probability \mathbb{P}^i defined in Lemma 2.3. \square

2.3.3 Proof of Theorem 2.1

We now have all the key ingredients to conclude the proof of Theorem 2.1.

1. Let $\phi \in \mathcal{A}_t^i$, and consider the identity (2.22). Since \mathcal{M}^ϕ is a square integrable \mathbb{F}^∞ -martingale, orthogonal to W by Remark A.2, and since $\mathcal{A}_T^\phi + \mathcal{K}_T^\phi \in L^2(\mathcal{F}_T^\infty)$ and the process $\mathcal{A}^\phi + \mathcal{K}^\phi$ is non-negative and non-decreasing, the comparison Theorem A.6 and Remark A.3 give $\mathcal{Y}_t^\phi \geq U_t^\phi$, recall (2.4).

Now, we have

$$\begin{aligned} \mathcal{Y}_t^\phi &= Y_t^i + \sum_{k \geq 0} \left(Y_t^{\zeta_{k+1}} - Y_t^{\zeta_k} \right) 1_{\{\tau_{k+1} \leq t\}} \\ &= Y_t^i + \sum_{k \geq 0} \left(Y_t^{\zeta_{k+1}} - \mathbb{E}\left[Y_t^{\zeta_{k+1}} \middle| \mathcal{F}_{\tau_{k+1}}^k\right] \right) 1_{\{\tau_{k+1} \leq t\}} \\ &\quad - \sum_{k \geq 0} \left(Y_t^{\zeta_k} - \sum_{j=1}^d P_{\zeta_k, j}^{\alpha_{k+1}} Y_{\tau_{k+1}}^j + \bar{c}_{\zeta_k}^{\alpha_{k+1}} \right) 1_{\{\tau_{k+1} \leq t\}} + A_t^\phi. \end{aligned} \quad (2.33)$$

Since $U_t^\phi \leq \mathcal{Y}_t^\phi$ and $\sum_{k \geq 0} \left(Y_t^{\zeta_k} - \sum_{j=1}^d P_{\zeta_k, j}^{\alpha_{k+1}} Y_{\tau_{k+1}}^j + \bar{c}_{\zeta_k}^{\alpha_{k+1}} \right) 1_{\{\tau_{k+1} \leq t\}} \geq 0$, we get

$$\begin{aligned} U_t^\phi - A_t^\phi &\leq Y_t^i + \sum_{k \geq 0} \left(Y_t^{\zeta_{k+1}} - \mathbb{E} \left[Y_t^{\zeta_{k+1}} \middle| \mathcal{F}_{\tau_{k+1}}^k \right] \right) 1_{\{\tau_{k+1} \leq t\}} \\ &\quad - \sum_{k \geq 0} \left(Y_t^{\zeta_k} - \sum_{j=1}^d P_{\zeta_k, j}^{\alpha_{k+1}} Y_{\tau_{k+1}}^j + \bar{c}_{\zeta_k}^{\alpha_{k+1}} \right) 1_{\{\tau_{k+1} \leq t\}} \\ &\leq Y_t^i + \sum_{k \geq 0} \left(Y_t^{\zeta_{k+1}} - \mathbb{E} \left[Y_t^{\zeta_{k+1}} \middle| \mathcal{F}_{\tau_{k+1}}^k \right] \right) 1_{\{\tau_{k+1} \leq t\}}. \end{aligned} \quad (2.34)$$

Using (2.18), we can take conditional expectation on both side with respect to \mathcal{F}_t^0 to obtain the result.

2. Lemma 2.4 shows that the strategy ϕ^\star is admissible. Using (2.22), since $\mathcal{A}^{\phi^\star} = 0$ and $\int_{\tau_k^\star \wedge T}^{\tau_{k+1}^\star \wedge T} dK_u^{\zeta_k^\star} = 0$ for all $k \geq 0$, we obtain

$$\mathcal{Y}_s^{\phi^\star} = \xi^{a_T^\star} + \int_s^T f^{a_u^\star}(u, \mathcal{Y}_u^{\phi^\star}, \mathcal{Z}_u^{\phi^\star}) du - \int_s^T \mathcal{Z}_u^{\phi^\star} dW_u - \int_s^T d\mathcal{M}_u^{\phi^\star} - \int_s^T dA_u^{\phi^\star}. \quad (2.35)$$

By uniqueness from Theorem A.4 and Remark A.3, we get that $\mathcal{Y}_t^{\phi^\star} = U_t^{\phi^\star}$, recall (2.4). We also have

$$\begin{aligned} \mathcal{Y}_t^{\phi^\star} &= Y_t^i + \sum_{k \geq 0} \left(Y_t^{\zeta_{k+1}^\star} - Y_t^{\zeta_k^\star} \right) 1_{\{\tau_{k+1}^\star \leq t\}} \\ &= Y_t^i + \mathcal{M}_t^{\phi^\star} + A_t^{\phi^\star}, \end{aligned}$$

thus $U_t^{\phi^\star} - A_t^{\phi^\star} = \mathcal{Y}_t^{\phi^\star} - A_t^{\phi^\star} = Y_t^i + \mathcal{M}_t^{\phi^\star}$, and taking conditional expectation gives the result. \square

3 Properties of the domain of reflection with signed costs

As observed in the previous section, see in particular Proposition 2.1, uniqueness of solutions to the obliquely reflected BSDE (2.6)-(2.7)-(2.8) in the case of signed costs is deduced from the uniqueness result with positive costs, which follows directly from the control problem representation, see Corollary 2.1. The additional assumption to apply Proposition 2.1 is that the domain \mathcal{D} , where the Y -component of the solution of the reflected BSDE is constrained to take its values, has a non-empty interior. We give in this section some general properties of the domain \mathcal{D} and identify necessary and sufficient conditions linked to the non-emptiness of its interior, which is not trivially obtained in the setting of signed costs.

The first result shows that the domain \mathcal{D} defined in (2.9) is invariant by translation along the vector $\mathbf{1} := (1, \dots, 1) = \sum_{i=1}^d e_i$ and deduces some property for its normal cone. Most of the time, we will thus be able to limit our study to

$$\mathcal{D}_\circ = \mathcal{D} \cap \left\{ y \in \mathbb{R}^d \middle| y_d = 0 \right\}. \quad (3.1)$$

Lemma 3.1. *For all $x \in \mathcal{D}$, we have*

- 1) *for all $h \in \mathbb{R}$, $x + h\mathbf{1} \in \mathcal{D}$,*
- 2) *there is a unique decomposition $x = y^x + z^x$ with $y^x \in \mathcal{D}_\circ$ and $z^x \in \mathbb{R}\mathbf{1} := \{\lambda\mathbf{1} \mid \lambda \in \mathbb{R}\}$,*
- 3) *we have $\mathcal{C}(x) \subset \left\{v \in \mathbb{R}^d \mid \sum_{i=1}^d v_i = 0\right\}$,*
- 4) *$\mathcal{C}(x) = \mathcal{C}(y^x)$, where y^x is given in 2).*

Proof. Fix $x \in \mathcal{D}$.

1. If $i \in \{1, \dots, d\}$, we have, for all $h \in \mathbb{R}$,

$$x_i + h \geq \max_{u \in \mathcal{C}} \left(\sum_{j=1}^d P_{i,j}^u x_j - c_i^u \right) + h = \max_{u \in \mathcal{C}} \left(\sum_{j=1}^d P_{i,j}^u (x_j + h) - c_i^u \right),$$

and thus $x + h\mathbf{1} \in \mathcal{D}$.

2. We set $y^x = x - z^x$ with $z^x = x_d\mathbf{1}$. It is clear that $y_d^x = 0$, and $y^x \in \mathcal{D}$ thanks to the first point. The uniqueness is clear since we have necessarily $z^x = x_d\mathbf{1}$.
3. Point 1. shows that $x \pm \mathbf{1} \in \mathcal{D}$. Let $v \in \mathcal{C}(x)$. Then we have, by definition,

$$0 \geq v^\top (x \pm \mathbf{1} - x) = \pm v^\top \mathbf{1} = \pm \sum_{i=1}^d v_i,$$

and thus, $\sum_{i=1}^d v_i = 0$.

4. Since $x = y^x + x_d\mathbf{1}$, it is enough to show that for all $w \in \mathcal{D}$ and all $a \in \mathbb{R}$, $\mathcal{C}(w) \subset \mathcal{C}(w + a\mathbf{1})$, as taking $(w, a) = (y^x, x_d)$ (resp. $(w, a) = (x, -x_d)$) yields $\mathcal{C}(y^x) \subset \mathcal{C}(y^x + x_d\mathbf{1}) = \mathcal{C}(x)$ (resp. $\mathcal{C}(x) \subset \mathcal{C}(x - x_d\mathbf{1}) = \mathcal{C}(y^x)$).

Let $v \in \mathcal{C}(w)$. We have, for all $z \in \mathcal{D}$, since $\sum_{i=1}^d v_i = 0$ and $v^\top (z - w) \leq 0$,

$$v^\top (z - (w + a\mathbf{1})) = v^\top (z - w) - av^\top \mathbf{1} = v^\top (z - w) \leq 0,$$

and thus $v \in \mathcal{C}(w + a\mathbf{1})$. □

Before studying the domain of reflection, we introduce three examples in dimension 3 of switching problems. On Figure 1, we draw the domain \mathcal{D}_\circ for these three different switching problems to illustrate the impact of the various controlled randomisations on the shape of the reflecting domain.

Example 1: Classical switching problem with a constant cost 1, i.e. $\mathcal{C} = \{1, 2\}$,

$$P^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \bar{c}^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \bar{c}^2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Here, when the agent decides to switch, she directly chooses the new mode which is attained with probability 1, at fixed cost 1.

Example 2: Randomised switching problem with $\mathcal{C} = \{0\}$,

$$P^0 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \text{ and } \bar{c}^0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Here, when the agent decides to switch, the two remaining modes are reached with probability $\frac{1}{2}$.

Example 3: Switching problem with controlled randomisation where $\mathcal{C} = [0, 1]$,

$$P^u = \begin{pmatrix} 0 & u & 1-u \\ 1-u & 0 & u \\ u & 1-u & 0 \end{pmatrix} \text{ and } \bar{c}^0 = \begin{pmatrix} 1-u(1-u) \\ 1-u(1-u) \\ 1-u(1-u) \end{pmatrix} \quad \forall u \in [0, 1]. \quad (3.2)$$

In this example, the transition matrices are given by convex combinations of transition matrices of Example 1. When the agent decides to switch using control $u \in [0, 1]$, one of the other mode is reached with probability u while the last mode is reached with probability $1-u$, with cost $1-u(1-u)$. The agent can increase the probability to reach one given state by taking u close to 0 or 1, but the price to pay will be higher than when u is close to $\frac{1}{2}$, meaning that the two remaining states can be reached with almost equal probability.

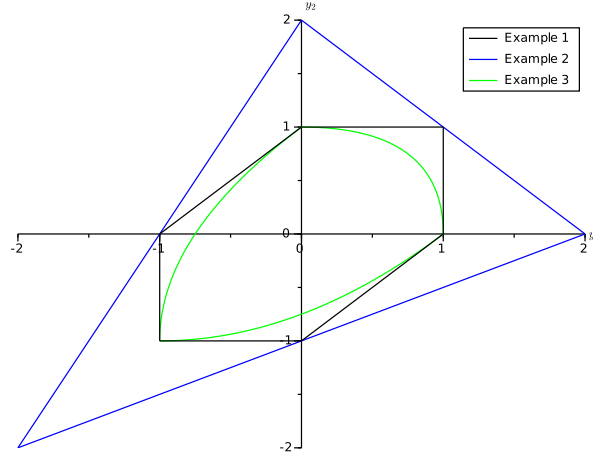


Figure 1: Domain \mathcal{D}_0 for three examples of switching problems with or without controlled randomisation.

Remark 3.1. *For the randomised switching problem, in any dimension, we can replace $(P_{i,j})_{1 \leq j \leq d}$ by $\left(\frac{P_{i,j}}{1-P_{i,i}} \mathbf{1}_{i \neq j}\right)_{1 \leq j \leq d}$ and \bar{c}_i by $\frac{\bar{c}_i}{1-P_{i,i}}$ as soon as $P_{i,i} < 1$, without changing \mathcal{D} . The factor $(1 - P_{i,i})^{-1}$ in the cost has to be seen as the expectation of the geometric law of the number of trials needed to exit state i . So assuming that diagonal terms are zero is equivalent to assume that $P_{i,i} < 1$, for all $1 \leq i \leq d$.*

3.1 The uncontrolled case

In this part, we study the domain \mathcal{D} for a fixed control, which is set to be 0, without loss of generality. The properties of the domain are closely linked in this case to the homogeneous Markov chain, denoted X , associated to the stochastic matrix P . For this part, we shall work with the following assumption.

Assumption 3.1. *The set of control is reduced to $\mathcal{C} = \{0\}$. The Markov chain X with stochastic matrix $P = (P_{i,j})_{1 \leq i,j \leq d} := (P_{i,j}^0)_{1 \leq i,j \leq d}$ is irreducible.*

Our main goal is to find necessary and sufficient conditions to characterize the non-emptiness of the domain \mathcal{D} . To this end, we will introduce some quantities related to the Markov Chain X and the costs vector $\bar{c} := \bar{c}^0$.

For $1 \leq i, j \leq d$, we consider the expected cost along an “excursion” from state i to j :

$$\bar{C}_{i,j} := \mathbb{E} \left[\sum_{n=0}^{\tau_j-1} \bar{c}_{X_n} \middle| X_0 = i \right] = \mathbb{E} \left[\sum_{n=0}^{\tau_j-1} c_{X_n, X_{n+1}} \middle| X_0 = i \right], \quad (3.3)$$

where (as defined in the Introduction) $c = (c_{i,j})_{1 \leq i,j \leq d}$ is a matrix, whose entry $c_{i,j}$ represents the cost applied when j is the drawn state switching from state i , and which is linked to the vector \bar{c} through the relations, for all $1 \leq i \leq d$,

$$\bar{c}_i = \sum_{1 \leq j \leq d} P_{i,j} c_{i,j},$$

and

$$\tau_j := \inf \{n \geq 1 \mid X_n = j\}.$$

We also define

$$C_{j,j} := 0 \text{ and } C_{i,j} = \bar{C}_{i,j} \text{ for } 1 \leq i \neq j \leq d. \quad (3.4)$$

We observe that, introducing $\tilde{\tau}_j := \inf \{n \geq 0 \mid X_n = j\}$, the cost C rewrites as \bar{C} :

$$C_{i,j} := \mathbb{E} \left[\sum_{n=0}^{\tilde{\tau}_j-1} \bar{c}_{X_n} \middle| X_0 = i \right], \text{ for } 1 \leq i, j \leq d.$$

Let us remark that $\mathbb{E}[\tau + \tilde{\tau}] < +\infty$ and so \bar{C} and C are finite since the Markov chain is irreducible recurrent.

Setting $Q = I_d - P$, the domain \mathcal{D} , defined in (2.9), rewrites:

$$\mathcal{D} = \left\{ x \in \mathbb{R}^d \mid Qx + \bar{c} \geq 0 \right\}. \quad (3.5)$$

Since P is irreducible, it is well known (see for example [3, Section 2.5]) that for all $1 \leq i, j \leq d$, the matrix $Q^{(i,j)}$ is invertible, and that we have

$$\tilde{\mu}_i := \det Q^{(i,i)} = (-1)^{i+j} \det Q^{(i,j)} > 0. \quad (3.6)$$

Moreover, $\tilde{\mu}Q = 0$ with $\tilde{\mu} = (\tilde{\mu}_i)_{i=1}^d$, i.e. $\mu := \frac{\tilde{\mu}}{\sum_{i=1}^d \tilde{\mu}_i}$ is the unique invariant probability measure for the Markov chain with transition matrix P .

The main result of this section is the following.

Theorem 3.1. *Let Assumption 3.1 hold. The following conditions are equivalent:*

- i) *The domain \mathcal{D} is non-empty (resp. has a non-empty interior). More precisely, for each $1 \leq i \leq d$, $-C_{\cdot,i} \in \mathcal{D}$ as $[Q(-C_{\cdot,i}) + \bar{c}]_j = \bar{C}_{i,i} \mathbf{1}_{i=j}$.*
- ii) *There exists $1 \leq i \neq j \leq d$ such that $C_{i,j} + C_{j,i} \geq 0$ (resp. $C_{i,j} + C_{j,i} > 0$).*
- iii) *The inequality $\mu \bar{c} \geq 0$ (resp. $\mu \bar{c} > 0$) is satisfied.*
- iv) *For all $1 \leq i \neq j \leq d$, $C_{i,j} + C_{j,i} \geq 0$ (resp. $C_{i,j} + C_{j,i} > 0$).*

The proof of this theorem is postponed to Section 3.1.2, as it requires to prove technical preliminary results, which are given in Section 3.1.1.

3.1.1 Preliminary results

We now state and prove several technical results that are useful to prove Theorem 3.1, see Section 3.1.2.

Lemma 3.2. *Let Assumption 3.1 hold. The mean costs C are given for $1 \leq i \neq j \leq d$ by*

$$C_{i,j} = \left((Q^{(j,j)})^{-1} \bar{c}^{(j)} \right)_{i - \mathbf{1}_{\{i > j\}}} . \quad (3.7)$$

Proof. 1. We first show that for $1 \leq i, j \leq d$:

$$\bar{C}_{i,j} = \bar{c}_i + \sum_{\ell \neq j} \bar{C}_{\ell,j} P_{i,\ell} . \quad (3.8)$$

From (3.3), we have

$$\bar{C}_{i,j} = \mathbb{E} \left[\sum_{n=0}^{+\infty} \bar{c}_{X_n} \mathbf{1}_{\{n < \tau_j\}} \mid X_0 = i \right] = \bar{c}_i + \mathbb{E} \left[\sum_{n=1}^{+\infty} \bar{c}_{X_n} \mathbf{1}_{\{n < \tau_j\}} \mid X_0 = i \right] .$$

Then, since for all $n \geq 1$, $\{X_1 = j\} \cap \{n < \tau_j\} = \emptyset$, we get

$$\bar{C}_{i,j} = \bar{c}_i + \mathbb{E} \left[\sum_{n=1}^{+\infty} \sum_{\ell \neq j} \bar{c}_{X_n} \mathbf{1}_{\{X_1=\ell\}} \mathbf{1}_{\{n < \tau_j\}} \middle| X_0 = i \right].$$

We compute that, for $\ell \neq j$,

$$\mathbb{E} \left[\sum_{n=1}^{+\infty} \bar{c}_{X_n} \mathbf{1}_{\{X_1=\ell\}} \mathbf{1}_{\{n < \tau_j\}} \middle| X_0 = i \right] = \mathbb{E} \left[\sum_{n=1}^{+\infty} \bar{c}_{X_n} \mathbf{1}_{\{n < \tau_j\}} \middle| X_1 = \ell \right] P_{i,\ell}.$$

The proof of (3.8) is then concluded observing that, from the Markov property,

$$\mathbb{E} \left[\sum_{n=1}^{+\infty} \bar{c}_{X_n} \mathbf{1}_{\{n < \tau_j\}} \middle| X_1 = \ell \right] = \bar{C}_{\ell,j}.$$

2. From (3.8), we deduce, recall Definition (3.4), that, for $i \neq j$,

$$C_{i,j} = \bar{c}_i + \sum_{\ell=1}^d C_{\ell,j} P_{i,\ell} = \bar{c}_i + \sum_{\ell \neq j} C_{\ell,j} P_{i,\ell}. \quad (3.9)$$

This equality simply rewrites $Q^{(j,j)} C_{\cdot,j} = \bar{c}^{(j)}$, which concludes the proof. \square

Proposition 3.1. *Let Assumption 3.1 holds and assume furthermore that \mathcal{D} is non-empty. Then,*

1. *the mean cost with respect to the invariant measure is non-negative, namely:*

$$\mu \bar{c} \geq 0. \quad (3.10)$$

2. *For all $1 \leq i, j \leq d$,*

$$\min_{1 \leq i, j \leq d} (C_{i,j} + C_{j,i}) \geq 0. \quad (3.11)$$

3. *The set \mathcal{D}_\circ is compact in $\{y \in \mathbb{R}^d \mid y_d = 0\}$.*

Moreover, if \mathcal{D} has a non-empty interior, then

$$\mu \bar{c} > 0 \quad \text{and} \quad \min_{1 \leq i \neq j \leq d} (C_{i,j} + C_{j,i}) > 0. \quad (3.12)$$

Proof. 1.a We first show the key relation, for all $x \in \mathcal{D}$,

$$-C_{i,j} \leq x_i - x_j \leq C_{j,i}, \quad \text{for } 1 \leq i, j \leq d. \quad (3.13)$$

For $j \in \{1, \dots, d\}$ and $x \in \mathbb{R}^d$, we introduce $\pi^j(x) \in \mathbb{R}^{d-1}$, given by,

$$\pi^j(x)_k = x_{k+1} \mathbf{1}_{\{k \geq j\}} - x_j, \quad k \in \{1, \dots, d-1\}.$$

Let $x \in \mathcal{D}$ and $j \in \{1, \dots, d\}$. For all $i \in \{1, \dots, d\}, i \neq j$, we have, by definition of \mathcal{D} and since $\sum_{k=1}^d P_{i,k} = 1$,

$$x_i - x_j \geq \sum_{k=1}^d P_{i,k} (x_k - x_j) - \bar{c}_i.$$

Thus $\pi_j(x)$ satisfies to

$$Q^{(j,j)} \pi^j(x) \geq -\bar{c}^{(j)}.$$

Since P is irreducible, [9, Proposition 2.20 (iii)] applies to $P^{(j,j)}$ and one obtains $(Q^{(j,j)})^{-1} = \sum_{k \geq 0} (P^{(j,j)})^k \geq 0$. We then obtain, using inequality (3.7)

$$\pi^j(x) \geq -\left(Q^{(j,j)}\right)^{-1} \bar{c}^{(j)} = -C_{\cdot,j}^{(j)}, \quad (3.14)$$

which means $x_i - x_j \geq -C_{i,j}$ for all $i \neq j$.

Let $1 \leq i \neq j \leq d$. The precedent reasoning gives $x_i - x_j \geq -C_{i,j}$ and $x_j - x_i \geq -C_{j,i}$, thus (3.13) is proved.

From (3.13), we straightforwardly obtain (3.11) and the fact that \mathcal{D}_0 is compact in $\{y \in \mathbb{R}^d \mid y_d = 0\}$.

1.b Since \mathcal{D} is non-empty, the following holds for some $x \in \mathbb{R}^d$, recalling (3.5),

$$Qx + \bar{c} \geq 0.$$

Multiplying by μ the previous inequality, we obtain (3.10), since $\mu Q = 0$.

2. Assume now that \mathcal{D} has a non-empty interior and consider $x \in \overset{\circ}{\mathcal{D}}$. Then, for all $1 \leq i \leq d$, we have that $x - \epsilon e_i$ belongs to \mathcal{D} for $\epsilon > 0$ small enough. Thus, we get

$$x_i - \epsilon \geq \sum_{\ell=1}^d P_{i,\ell} x_\ell - \epsilon P_{i,i} - \bar{c}_i$$

and then

$$Qx + \bar{c} \geq \epsilon \min_{1 \leq i \leq d} (1 - P_{i,i}) \sum_{\ell=1}^d e_\ell.$$

Since P is irreducible, $\min_{1 \leq i \leq d} (1 - P_{i,i}) > 0$, and multiplying by μ both sides of the previous inequality we obtain $\mu \bar{c} > 0$.

For any $j \neq i$, since $x - \epsilon e_i \in \mathcal{D}$, we deduce from (3.13), $-C_{i,j} + \epsilon \leq x_i - x_j$. Using again (3.13), we get $-C_{i,j} + \epsilon \leq C_{j,i}$. This proves the right hand side of (3.12). \square

The next lemma, whose proof is postponed to Appendix A.1, links the condition (3.10) to costly round-trip.

Lemma 3.3. *Under Assumption 3.1, the following statements hold, for $1 \leq j \leq d$,*

$$\bar{C}_{jj} = \frac{\mu\bar{c}}{\mu_j}, \quad (3.15)$$

and, for $1 \leq i \neq j \leq d$,

$$C_{i,j} + C_{j,i} = \frac{\mu\bar{c}}{\mu_i} \left(\left[Q^{(j,j)} \right]^{-1} \right)_{i^{(j)}, i^{(j)}}. \quad (3.16)$$

3.1.2 Proof of Theorem 3.1 and useful consequences

We are now in position to prove the main result of this section.

Proof of Theorem 3.1. 1. We first note that in Proposition 3.1 we have proved $i) \implies iv)$. We also remark that $iv) \implies ii)$ trivially, and $ii) \implies iii)$ in a straightforward way from equality (3.16), recalling that $(Q^{(j,j)})^{-1} = \sum_{k \geq 0} (P^{(j,j)})^k \geq 0$. 2. We now study $iii) \implies i)$.

2.a Assume that $\mu\bar{c} \geq 0$. For $1 \leq j \leq d$, we denote $z^j := -C_{\cdot,j}$. Then from (3.9), we straightforwardly observe that, for all $i \neq j$,

$$z_i^j = \sum_{\ell=1}^d z_\ell^j P_{i,\ell} - \bar{c}_i. \quad (3.17)$$

which reads $(Qz^j + \bar{c})_i = 0$.

We now take care of the case $i = j$ by computing, recall $z_j^j = 0$,

$$(Qz^j + \bar{c})_j = z_j^j - \sum_{\ell=1}^d z_\ell^j P_{j,\ell} + \bar{c}_j = \sum_{\ell=1}^d C_{\ell,j} P_{j,\ell} + \bar{c}_j = \bar{C}_{j,j}, \quad (3.18)$$

where we used (3.8) with $i = j$. Then, combining (3.15) and the assumption $\mu\bar{c} \geq 0$ for this step, we obtain $(Qz^j + \bar{c})_j \geq 0$. Equations (3.17) and (3.18) thus imply $z^j \in \mathcal{D}$ and so \mathcal{D} is non-empty.

2.b We assume that $\mu\bar{c} > 0$, which implies that $\bar{C}_{j,j} = \frac{\mu\bar{c}}{\mu_j} > 0$ for all $1 \leq j \leq d$, recalling (3.15). Fix any $j \in \{1, \dots, d\}$ and consider $z^j := -C_{\cdot,j}$ introduced in the previous step. We then set

$$x := z^j + \frac{1}{2(d-1)} \sum_{k \neq j} (z^k - z^j). \quad (3.19)$$

Next, we compute, for $i \neq j$, recalling $(Qz^j + \bar{c})_i = 0$ and $(Qz^j)_i = (Qz^k)_i = -\bar{c}_i$ for $k \neq i, j$ (coming from (3.17)) and $(Qz^i)_i = \bar{C}_{i,i} - \bar{c}_i$ (from (3.18)),

$$\begin{aligned} (Qx + \bar{c})_i &= (Qz^j + \bar{c})_i + \frac{1}{2(d-1)} \sum_{k \neq j} (Qz^k - Qz^j)_i \\ &= 0 + \frac{1}{2(d-1)} (Qz^i - Qz^j)_i = \frac{1}{2(d-1)} ((Qz^i)_i + \bar{c}_i) = \frac{1}{2(d-1)} \bar{C}_{i,i} > 0. \end{aligned}$$

For $i = j$, we compute, using the same identities,

$$\begin{aligned} (Qx + \bar{c})_j &= (Qz^j + \bar{c})_j + \frac{1}{2(d-1)} \sum_{k \neq j} (Qz^k - Qz^j)_j \\ &= \bar{C}_{j,j} + \frac{1}{2(d-1)} \sum_{k \neq j} (-\bar{c}_j + \bar{c}_j - \bar{C}_{j,j}) = \frac{\bar{C}_{j,j}}{2} > 0. \end{aligned}$$

Combining the two previous inequalities, we obtain that

$$Qx + \bar{c} \geq \frac{\delta}{2} \mathbf{1} \quad \text{with} \quad \delta = \min_{1 \leq i \leq d} \bar{C}_{i,i}.$$

From this, we easily deduce that $x + B(0, \frac{\delta}{4 \sup_i \|Q_{i,\cdot}\|_2}) \subset \mathcal{D}$, which proves that \mathcal{D} has a non-empty interior. \square

To conclude this part, we give two useful consequences of Theorem 3.1.

First, we provide extra conditions that are linked to the non-emptiness of the domain \mathcal{D} .

We then completely characterize the domain \mathcal{D}_0 as the convex hull of d points whose coordinates are explicit, see Proposition 3.3. This result will only be used in the proof of Theorem 4.2, which provides existence to solutions to (2.6)-(2.7)-(2.8) in a Markovian framework in the setting of Assumption 3.1 with signed costs.

Proposition 3.2. *Let Assumption 3.1 hold. The following assertions are equivalent:*

- i) \mathcal{D} is non-empty,
- ii) For all $1 \leq i, j, k \leq d$, the following holds

$$C_{j,k} \leq C_{j,i} + C_{i,k}, \tag{3.20}$$

- iii) For any round trip of length less than d , i.e. $1 \leq n \leq d$, $1 \leq i_1 \neq \dots \neq i_n \leq d$, we have

$$\sum_{k=1}^{n-1} C_{i_k, i_{k+1}} + C_{i_n, i_1} \geq 0. \tag{3.21}$$

Proof. 1. $i) \implies ii)$ is a direct consequence of Theorem 3.1 and (3.13). Indeed, let us set $1 \leq i, j, k \leq d$. If $j = k$, this comes directly from the fourth item of Theorem 3.1 as $C_{j,j} = 0$. Otherwise, the second inequality of (3.13) used together with $x = -C_{\cdot,k}$, which is in \mathcal{D} by the first item of Theorem 3.1, writes

$$-C_{i,k} + C_{j,k} \leq C_{j,i}, \text{ for all } 1 \leq i, j \leq d,$$

which concludes the proof for this step.

2. $ii) \implies iii)$ is direct since $C_{i,i} = 0$ for all $1 \leq i \leq d$. Finally $iii) \implies i)$ is already proved in Theorem 3.1 for a 2-state round trip. \square

Proposition 3.3. *Let Assumption 3.1 hold and assume that \mathcal{D} has a non-empty interior. Define $\theta_{\cdot,j} = C_{\cdot,j} - C_{d,j}\mathbf{1}$, for all $1 \leq j \leq d$. Then $(-\theta_{\cdot,j})_{1 \leq j \leq d}$ are affinely independent and \mathcal{D}_\circ is the convex hull of these points.*

Proof. We know from Theorem 3.1 that $-C_{\cdot,j} \in \mathcal{D}$ for all $1 \leq j \leq d$. The invariance by translation along $\mathbf{1}$ of the domain proves that $-\theta_{\cdot,j}$ is in \mathcal{D}_\circ . More precisely, we obtain from (3.9) that,

$$\theta_{i,j} - \sum_{\ell=1}^d \theta_{\ell,j} P_{i,\ell} = \bar{c}_i, \quad \text{for } 1 \leq i \neq j \leq d. \quad (3.22)$$

1. We now prove that $(\theta_{\cdot,j})_{1 \leq j \leq d}$ are affinely independent. We consider thus $\alpha \in \mathbb{R}^d$ such that

$$\sum_{j=1}^d \alpha_j = 0 \quad \text{and} \quad z := \sum_{j=1}^d \alpha_j \theta_{\cdot,j} = 0, \quad (3.23)$$

and we aim to prove that $\alpha_j = 0$, for $j \in \{1, \dots, d\}$. To this end, we compute, for $i \in \{1, \dots, d\}$, using (3.22) and $\theta_{i,i} = C_{i,i} - C_{d,i} = -C_{d,i}$ by definition,

$$\begin{aligned} 0 = z_i &= \sum_{j=1}^d \alpha_j \theta_{i,j} = \sum_{j \neq i} \alpha_j \theta_{i,j} + \alpha_i \theta_{i,i} = \bar{c}_i \sum_{j \neq i} \alpha_j + \sum_{j \neq i} \alpha_j \sum_{\ell=1}^d \theta_{\ell,j} P_{i,\ell} - \alpha_i C_{d,i} \\ &= \bar{c}_i \sum_{j \neq i} \alpha_j + \sum_{\ell=1}^d P_{i,\ell} \sum_{j \neq i} \alpha_j \theta_{\ell,j} - \alpha_i C_{d,i} \\ &= \bar{c}_i \sum_{j \neq i} \alpha_j + \sum_{\ell=1}^d P_{i,\ell} \sum_{j=1}^d \alpha_j \theta_{\ell,j} - \sum_{\ell=1}^d P_{i,\ell} \alpha_i \theta_{\ell,i} - \alpha_i C_{d,i}. \end{aligned}$$

Using (3.23), $\sum_{\ell=1}^d P_{i,\ell} = 1$ and (3.9), we obtain

$$\begin{aligned} 0 &= -\bar{c}_i \alpha_i + \sum_{\ell=1}^d P_{i,\ell} z_\ell - \alpha_i \sum_{\ell=1}^d P_{i,\ell} \theta_{\ell,i} - \alpha_i C_{d,i} = -\alpha_i \left(\bar{c}_i + \sum_{\ell=1}^d P_{i,\ell} \theta_{\ell,i} + C_{d,i} \right) \\ &= -\alpha_i \left(\bar{c}_i + \sum_{\ell=1}^d (C_{\ell,i} - C_{d,i}) P_{i,\ell} + C_{d,i} \right) = -\alpha_i \left(\bar{c}_i + \sum_{\ell=1}^d C_{\ell,i} P_{i,\ell} \right) \\ &= -\alpha_i \bar{C}_{i,i}. \end{aligned}$$

We thus deduce that $\alpha_i = 0$ since $\bar{C}_{i,i} = \frac{\mu \bar{c}}{\mu_i} > 0$ from Theorem 3.1, which concludes the proof for this step.

2. We now show that \mathcal{D}_\circ is the convex hull of points $(-\theta_{\cdot,j})_{1 \leq j \leq d}$, which are affinely independent from the previous step. For $y \in \mathbb{R}^d \cap \{y_d = 0\}$, there exists thus a unique

$(\lambda_1, \dots, \lambda_{d-1}) \in \mathbb{R}^{d-1}$ such that $y = \sum_{j=1}^d -\lambda_j \theta_{\cdot,j}$, with $\lambda_d = 1 - \sum_{j=1}^{d-1} \lambda_j$. Assuming that $y \in \mathcal{D}$, we have that

$$v := Qy + \bar{c} = \sum_{j=1}^d -\lambda_j Q\theta_{\cdot,j} + \bar{c} = \sum_{j=1}^d \lambda_j [Q(-\theta_{\cdot,j}) + \bar{c}] \geq 0.$$

Since, from Theorem 3.1, $[Q(-\theta_{\cdot,j}) + \bar{c}]_i = [Q(-C_{\cdot,j}) + \bar{c}]_i + C_{d,j}(Q\mathbf{1})_i = 0$ for all $i \neq j$, we get, for all $1 \leq i \leq d$,

$$v_i = \lambda_i ([Q(-\theta_{\cdot,i})]_i + \bar{c}_i) \geq 0.$$

Recalling that $[Q(-\theta_{\cdot,i})]_i + \bar{c}_i \geq 0$ as $-\theta_{\cdot,i} \in \mathcal{D}$, we obtain $\lambda_i \geq 0$ which concludes the proof. \square

3.2 The setting of controlled randomisation

In this part we adapt Assumption 3.1 in the following natural way.

Assumption 3.2. *For all $u \in \mathcal{C}$, the Markov chain with stochastic matrix $P^u := (P_{i,j}^u)_{1 \leq i,j \leq d}$ is irreducible.*

For each $u \in \mathcal{C}$, let μ^u the unique invariant probability distribution of P^u .

We then consider the matrix \hat{C} defined, for all $(i,j) \in \{1, \dots, d\}$, by

$$\hat{C}_{i,j} := \min_{u \in \mathcal{C}} C_{i,j}^u, \quad (3.24)$$

recall the Definition of $C_{i,j}^u$ for a fixed control in (3.4). Let us note that $\hat{C}_{i,j}$ is well defined in \mathbb{R} under Assumption 3.2 since \mathcal{C} is compact and $u \mapsto C_{i,j}^u$ is continuous for all $1 \leq i, j \leq d$.

The following result is similar to Proposition 3.1 but in the context of switching with controlled randomisation.

Proposition 3.4. *Let Assumption 3.2 hold and assume that \mathcal{D} is non-empty (resp. has a non-empty interior). Then,*

$$\min_u \mu^u \bar{c}^u \geq 0 \text{ (resp. } > 0) \quad \text{and} \quad \min_{1 \leq i \neq j \leq d} (\hat{C}_{i,j} + \hat{C}_{j,i}) \geq 0 \text{ (resp. } > 0). \quad (3.25)$$

Moreover, the set \mathcal{D}_\circ is compact in $\{y \in \mathbb{R}^d \mid y_d = 0\}$.

Proof. 1. Let $x \in \mathcal{D}$. From (3.13), we have for each $u \in \mathcal{C}$, $-C_{i,j}^u \leq x_i - x_j \leq C_{j,i}^u$. Minimising (resp. maximising) the upper (resp. lower) bound with respect to $u \in \mathcal{C}$, we then obtain

$$-\hat{C}_{i,j} \leq x_i - x_j \leq \hat{C}_{j,i}. \quad (3.26)$$

From this, we deduce that \mathcal{D}_\circ is compact in $\{y \in \mathbb{R}^d \mid y_d = 0\}$ and we get the right hand side of (3.25).

We also have that, for all $u \in \mathcal{C}$,

$$Q^u x + \bar{c}^u \geq 0,$$

then multiplying by μ^u we obtain $\mu^u \bar{c}^u \geq 0$. This leads to $\min_u \mu^u \bar{c}^u \geq 0$.

2. Then, results concerning the non-empty interior framework can be obtained as in the proof of Proposition 3.1. \square

3.2.1 The case of controlled costs only

To conclude this section, we consider the case where there exists a transition matrix P such that $P^u \equiv P$ for all $u \in \mathcal{C}$, i.e. the controller only controls the costs. In this setting, it is intuitively clear that, optimally, one chooses the control $u \in \mathcal{C}$ for which the cost to pay is the smallest. Let us start by introducing the minimal controlled mean cost:

$$\hat{c}_i := \min_{u \in \mathcal{C}} \bar{c}_i^u, \quad \text{for } 1 \leq i \leq d.$$

We thus have

$$\begin{aligned} \mathcal{D} &:= \left\{ x \in \mathbb{R}^d \mid (Qx)_i + \bar{c}_i^u \geq 0, \text{ for all } u \in \mathcal{C}, 1 \leq i \leq d \right\} \\ &= \left\{ x \in \mathbb{R}^d \mid (Qx)_i + \hat{c}_i \geq 0, \text{ for all } 1 \leq i \leq d \right\}. \end{aligned}$$

Using the result of Proposition 3.1 with the new costs \hat{c} , we know that a necessary and sufficient condition for \mathcal{D} to be non-empty is $\mu \hat{c} \geq 0$. Moreover, the matrix C is defined here by

$$C_{i,j} = \left((Q^{(j,j)})^{-1} \hat{c}^{(j)} \right)_{i-1_{\{i>j\}}}, \quad 1 \leq i \neq j \leq d, \quad (3.27)$$

and $C_{i,i} = 0$, for $1 \leq i \leq d$. Comparing the above expression with the definition of \hat{C} in (3.24), we observe that $C_{i,j} \leq \hat{C}_{i,j}$, $1 \leq i, j \leq d$. The following example confirms that

$$\min_{1 \leq i \neq j \leq d} (\hat{C}_{i,j} + \hat{C}_{j,i}) \geq 0,$$

recall Proposition 3.4, is not a sufficient condition in this context for non-emptiness of the domain.

Example 3.1. Set $\mathcal{C} = \{0, 1\}$,

$$P = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}, \quad \bar{c}^0 = \begin{pmatrix} -0.5 \\ 1.2 \\ 0.7 \end{pmatrix} \quad \text{and} \quad \bar{c}^1 = \begin{pmatrix} 1.5 \\ 0.2 \\ 0.2 \end{pmatrix}.$$

Observe that $\mu = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\hat{c} = (-0.5, 0.2, 0.2)^\top$. Then, one computes that

$$\min_{1 \leq i \neq j \leq d} (\hat{C}_{i,j} + \hat{C}_{j,i}) > 0 \quad \text{but} \quad \mu \hat{c} < 0.$$

4 Existence results for obliquely reflected BSDEs

In this section, we study the obliquely reflected BSDE (2.6)-(2.7)-(2.8) associated to the *switching problem with controlled randomisation*. We first prove existence results for the BSDE in the Markovian framework, see Section 4.1, and then in the non-Markovian framework, see Section 4.2, relying on the approach in [6]. Existence results in [6] are obtained for general obliquely reflected BSDEs where the oblique reflection is specified through an operator H that transforms, on the boundary of the domain, the normal cone into the oblique direction of reflection. Thus, the main difficulty is to construct this operator H with some specific properties needed to apply the existence theorems of [6]. This task is carried out successfully for the *randomised switching problem* in the Markovian framework. We also consider an example of *switching problem with controlled randomisation* in this framework. In the non-Markovian framework, which is more challenging as more properties are required on H , we prove the well-posedness of the BSDE for some examples of *randomised switching problem*.

4.1 The Markovian framework

We now introduce a Markovian framework, and prove that a solution to (2.6)-(2.7)-(2.8) exists for the *randomised switching problem* under Assumption 3.1 and a technical copositivity hypothesis, see Assumption 4.2 below. We also investigate the example of *switching problem with controlled randomisation* given by (3.2).

To this effect, we rely on the existence theorem obtained in [6], which we recall next. For all $(t, x) \in [0, T] \times \mathbb{R}^q$, let $X^{t,x}$ be the solution to the following SDE:

$$dX_s = b(s, X_s)ds + \sigma(s, X_s)dW_s, \quad s \in [t, T], \quad (4.1)$$

$$X_t = x. \quad (4.2)$$

We are interested in the solutions $(Y^{t,x}, Z^{t,x}, K^{t,x}) \in \mathbb{S}_d^2(\mathbb{F}^0) \times \mathbb{H}_{d \times \kappa}^2(\mathbb{F}^0) \times \mathbb{A}_d^2(\mathbb{F}^0)$ of (2.6)-(2.7)-(2.8), where the terminal condition satisfies $\xi = g(X_T^{t,x})$, and the driver satisfies $f(\omega, s, y, z) = \psi(s, X_s^{t,x}(\omega), y, z)$ for some deterministic measurable functions g, ψ . We next give the precise set of assumptions we need to obtain our results.

For sake of completeness, we recall here the existence result proved in [6], see also [8].

Assumption 4.1. *There exist $p \geq 0$ and $L \geq 0$ such that*

i) The function ψ satisfies

$$|\psi(t, x, y, z)| \leq L(1 + |x|^p + |y| + |z|).$$

Moreover, $\psi(t, x, \cdot, \cdot)$ is continuous on $\mathbb{R}^d \times \mathbb{R}^{d \times \kappa}$ for all $(t, x) \in [0, T] \times \mathbb{R}^q$.

ii) $(b, \sigma) : [0, T] \times \mathbb{R}^q \rightarrow \mathbb{R}^q \times \mathbb{R}^{q \times \kappa}$ is a measurable function satisfying, for all $(t, x, y) \in [0, T] \times \mathbb{R}^q \times \mathbb{R}^q$,

$$\begin{aligned} |b(t, x)| + |\sigma(t, x)| &\leq L(1 + |x|), \\ |b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| &\leq L|x - y|. \end{aligned}$$

iii) $g : \mathbb{R}^q \rightarrow \mathcal{D}$ is measurable and for all $(t, x) \in [0, T] \times \mathbb{R}^q$, we have

$$|g(x)| \leq L(1 + |x|^p).$$

iv) Let $\mathcal{X} = \{\mu(t, x; s, dy) \mid x \in \mathbb{R}^q \text{ and } 0 \leq t \leq s \leq T\}$ be the family of laws of $X^{t,x}$ on \mathbb{R}^q , i.e., the measures such that $\forall A \in \mathcal{B}(\mathbb{R}^q)$, $\mu(t, x; s, A) = \mathbb{P}(X_s^{t,x} \in A)$. There exists $a \in \mathbb{R}^q$ such that, for any $t \in [0, T)$, any $\delta \in [0, T - t)$ and for $\mu(0, a; t, dy)$ -almost every $x \in \mathbb{R}^q$, there exists an application $\phi_{t,x} : [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ such that:

$$(a) \quad \forall k \geq 1, \phi_{t,x} \in L^2([t + \delta, T] \times [-k, k]^q; \mu(0, a; s, dy)ds),$$

$$(b) \quad \mu(t, x; s, dy)ds = \phi_{t,x}(s, y)\mu(0, a; s, dy)ds \text{ on } [t + \delta, T] \times \mathbb{R}^q.$$

v) $H : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ is a measurable function, and there exists $\eta > 0$ such that, for all $(y, y') \in \mathbb{R}^d \times \mathbb{R}^d$ and $v \in \mathfrak{n}(\mathfrak{P}_{\mathcal{D}}(y))$, we have

$$v^\top H(y)v \geq \eta,$$

$$|H(y')| \leq L.$$

Moreover, H is continuous on \mathcal{D} .

Remark 4.1. Due to Aronson estimates on the density function of $X^{t,x}$, Assumption 4.1-iv) is true for all $a \in \mathbb{R}^q$ as soon as σ is uniformly elliptic: see [15, Section 28] for a proof.

The existence result in the Markovian setting reads as follows.

Theorem 4.1 ([6], Theorem 4.1). Under Assumption 4.1, there exists a solution $(Y^{t,x}, Z^{t,x}, \Psi^{t,x}) \in \mathbb{S}_d^2(\mathbb{F}^0) \times \mathbb{H}_{d \times \kappa}^2(\mathbb{F}^0) \times \mathbb{H}_d^2(\mathbb{F}^0)$ of the following system

$$Y_s = g(X_T^{t,x}) + \int_s^T \psi(u, X_u^{t,x}, Y_u, Z_u)du - \int_s^T Z_u dW_u - \int_s^T H(Y_u)\Psi_u du, s \in [t, T], \quad (4.3)$$

$$Y_s \in \mathcal{D}, \quad \Psi_s \in \mathcal{C}(Y_s), \quad t \leq s \leq T, \quad (4.4)$$

$$\int_t^T 1_{\{Y_s \notin \partial \mathcal{D}\}} |\Psi_s| ds = 0. \quad (4.5)$$

The main point to invoke Theorem 4.1 is then to construct a function $H : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ which satisfies Assumption 4.1-v) and such that

$$H(y)v \in \mathcal{C}_o(y), \quad (4.6)$$

for all $y \in \mathcal{D}$ and $v \in \mathcal{C}(y)$, where $\mathcal{C}_o(y)$ is the cone of directions of reflection, given here by

$$\mathcal{C}_o(y) := - \sum_{i=1}^d \mathbb{R}_+ e_i 1_{\{y_i = \max_{u \in \mathcal{C}} \{\sum_{j=1}^d P_{i,j}^u y_j - \bar{c}_i^u\}\}}. \quad (4.7)$$

If Assumption 4.1 i), ii), iii), iv) is also satisfied, we obtain the existence of a solution to (4.3)-(4.4)-(4.5). Setting $K_s^{t,x} := -\int_t^s H(Y_u^{t,x}) \Psi_u^{t,x} du$ for $t \leq s \leq T$ shows that $(Y^{t,x}, Z^{t,x}, K^{t,x})$ is a solution to (2.6)-(2.7)-(2.8). Indeed, (2.6)-(2.7) are clearly satisfied by definition of $K^{t,x}$ and by (4.3)-(4.4). Let us check that (2.8) is also satisfied. We have, for each $1 \leq i \leq d$,

$$\begin{aligned} & - \int_t^T \left(Y_s^{t,x,i} - \sup_{u \in \mathcal{C}} \left\{ \sum_{j=1}^d P_{i,j}^u Y_s^{t,x,j} - \bar{c}_i^u \right\} \right) dK_s^{t,x,i} \\ &= \int_t^T \mathbf{1}_{\{Y_s^{t,x} \notin \partial \mathcal{D}\}} \left(Y_s^{t,x,i} - \sup_{u \in \mathcal{C}} \left\{ \sum_{j=1}^d P_{i,j}^u Y_s^{t,x,j} - \bar{c}_i^u \right\} \right) (H(Y_s^{t,x}) \Psi_s^{t,x})^i ds \\ &+ \int_t^T \mathbf{1}_{\{Y_s^{t,x} \in \partial \mathcal{D}\}} \left(Y_s^{t,x,i} - \sup_{u \in \mathcal{C}} \left\{ \sum_{j=1}^d P_{i,j}^u Y_s^{t,x,j} - \bar{c}_i^u \right\} \right) (H(Y_s^{t,x}) \Psi_s^{t,x})^i ds. \end{aligned}$$

On $\{Y_s^{t,x} \notin \partial \mathcal{D}\}$, we have $\Psi_s^{t,x} = 0$ a.s. by (4.5), which shows that the first integral is zero.

On $\{Y_s^{t,x} \in \partial \mathcal{D}\}$, we have $H(Y_s^{t,x}) \Psi_s^{t,x} \in \mathcal{C}_o(Y_s^{t,x})$ by (4.4) and (4.6), and if moreover $Y_s^{t,x,i} > \sup_{u \in \mathcal{C}} \left\{ \sum_{j=1}^d P_{i,j}^u Y_s^{t,x,j} - \bar{c}_i^u \right\}$, then the i th coordinate of $H(Y_s^{t,x}) \Psi_s^{t,x}$ is zero by definition of $\mathcal{C}_o(Y_s^{t,x})$. This proves that the second integral is also zero, and (2.8) is proved.

4.1.1 Well-posedness result in the uncontrolled case

We need to introduce the following technical assumption in order to construct H satisfying Assumption 4.1-v) and (4.6).

Assumption 4.2. For all $1 \leq i \leq d$, the matrix $Q^{(i,i)}$ is strictly copositive, meaning that for all $0 \leq x \in \mathbb{R}^{d-1}, x \neq 0$, we have

$$x^\top Q^{(i,i)} x > 0. \quad (4.8)$$

Our main result for this section is the following theorem.

Theorem 4.2. Suppose that Assumption 3.1 and Assumption 4.2 are satisfied and that \mathcal{D} has a non-empty interior.

Then, there exists $H : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ satisfying Assumption 4.1-v).

Consequently, if Assumption 4.1 i), ii), iii), iv) also holds, for all $(t, x) \in [0, T] \times \mathbb{R}^q$, there exists a solution to (2.6)-(2.7)-(2.8) with $\xi = g(X_T^{t,x})$ and $f(\omega, s, y, z) = \psi(s, X_s^{t,x}(\omega), y, z)$. Moreover, this solution is unique if we suppose also Assumption 2.1-ii).

Proof. We first observe that uniqueness follows from Proposition 2.1 as, under Assumption 3.1, irreducibility implies Assumption 2.1-iii) namely $P_{i,i} \neq 1$ for all $1 \leq i \leq d$. We now focus on proving existence of a solution, which amounts to exhibit a convenient H function. The construction of such a function is done in three steps.

By Proposition 3.3, we know that \mathcal{D}_\circ is the convex hull of the family of points $(y^i = -\theta_{\cdot,i})_{1 \leq i \leq d}$ defined by

$$y^i := (C_{d,i} - C_{j,i})_{1 \leq j \leq d}, \quad 1 \leq i \leq d. \quad (4.9)$$

To define $H(y^i)$ for each $1 \leq i \leq d$ so that (4.6) is met at these points, we compute the cones $\mathcal{C}(y^i)$, which is the goal of the first step below.

In the second step, we show that, for each $1 \leq i \leq d$, there exists a unique matrix $H(y^i)$ satisfying Assumption 4.1-v) and (4.6). Using Proposition 3.3, we then extend H to \mathcal{D}_\circ by linear combinations and we check that Assumption 4.1-v) and (4.6) are still satisfied.

In the third step, we extend H to \mathcal{D} and then to \mathbb{R}^d . To construct H on \mathcal{D} , we set $H(x) = H(y^x)$ for each $x \in \mathcal{D}$, where x writes uniquely $x = y^x + z^x$ with $y^x \in \mathcal{D}_\circ$ and $z^x \in \mathbb{R}\mathbf{1}$, as proved by the second item in Lemma 3.1. We eventually define H on \mathbb{R}^d by projection onto \mathcal{D} .

1. We start by computing the outward normal cone $\mathcal{C}(y)$ for all $y \in \mathcal{D}_\circ$. Let us fix $y \in \mathcal{D}_\circ$. Thanks to Proposition 3.3, there exists a unique $(\lambda_i)_{1 \leq i \leq d} \in [0, 1]^d$ such that

$$y = \sum_{i=1}^d \lambda_i y^i, \quad \sum_{i=1}^d \lambda_i = 1.$$

Let us denote $\mathcal{E}_y = \{1 \leq i \leq d \mid \lambda_i > 0\}$. We will show that

$$\mathcal{C}(y) = \sum_{j \notin \mathcal{E}_y} \mathbb{R}_+ n_j, \quad (4.10)$$

where $n_i := (-Q_{i,j})_{1 \leq j \leq d}$, and with the convention $\mathcal{C}(y) = \{0\}$ when $\mathcal{E}_y = \{1, \dots, d\}$. Let us remark that the result is obvious when $\mathcal{C}(y) = \{0\}$, since, in this case, y is in the interior of \mathcal{D} . So we will assume in the following that $\mathcal{C}(y) \neq \{0\}$.

1.a. First, let us show that for any $1 \leq i \leq d$, $(n_j)_{j \neq i}$ is a basis of $\left\{v \in \mathbb{R}^d \mid \sum_{k=1}^d v_k = 0\right\}$.

Let $1 \leq i \neq j \leq d$. Since $\sum_{k=1}^d n_{j,k} = -\sum_{k=1}^d Q_{j,k} = \sum_{k=1}^d P_{j,k} - 1 = 0$, we have $n_j \in \left\{v \in \mathbb{R}^d \mid \sum_{k=1}^d v_k = 0\right\}$. Since it is a hyperplane of \mathbb{R}^d and that the family $(n_j)_{j \neq i}$ has $d-1$ elements, it is enough to show that the vectors are linearly independent. We observe that the matrix whose lines are the $n_j^{(i)}$, $j \neq i$, is $-Q^{(i,i)}$. Since P is irreducible, $Q^{(i,i)}$ is invertible. The vectors $n_j^{(i)}$, $j \neq i$ form a basis of \mathbb{R}^{d-1} , hence the vectors $(n_j)_{j \neq i}$ form a basis of $\left\{v \in \mathbb{R}^d \mid \sum_{k=1}^d v_k = 0\right\}$, as $(n_j)_{j \neq i}$ is the image of $(n_j^{(i)})_{j \neq i}$ under the linear isomorphism $\mathbb{R}^{d-1} \rightarrow \left\{v \in \mathbb{R}^d \mid \sum_{k=1}^d v_k = 0\right\}$, $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d) \mapsto (x_1, \dots, x_{i-1}, -\sum_{j \neq i} x_j, x_{i+1}, \dots, x_d)$.

1.b. We now fix $j \notin \mathcal{E}_y$ and we will show that $n_j \in \mathcal{C}(y)$. For any $z \in \mathcal{D}$, by definition of \mathcal{D} , we have

$$\bar{c}_j \geq \sum_{k=1}^d P_{j,k} z_k - z_j = n_j^\top z,$$

and for all $i \in \mathcal{E}_y$, by definition of y^i , we have

$$\bar{c}_j = \sum_{k=1}^d P_{j,k} y_k^i - y_j^i = n_j^\top y^i,$$

recalling (3.22) and the fact that $y^i = -\theta_{\cdot,i}$. This gives $n_j^\top(z-y) = n_j^\top z - \sum_{i \in \mathcal{E}_y} \lambda_i n_j^\top y^i \leq 0$, hence $n_j \in \mathcal{C}(y)$.

1.c. We now set $i = \min \mathcal{E}_y$. Since $(n_j)_{j \neq i}$ is a basis of $\left\{v \in \mathbb{R}^d \mid \sum_{i=1}^d v_i = 0\right\} \supset \mathcal{C}(y)$, see Lemma 3.1, for $v \in \mathcal{C}(y)$ there exists a unique $\alpha = (\alpha_j)_{j \neq i} \in \mathbb{R}^{d-1}$ such that $v = \sum_{j \neq i} \alpha_j n_j = (n_j)_{j \neq i} \alpha$, where we identify $(n_j)_{j \neq i}$ with the matrix whose columns are given by the family of vectors. We will show here that $\alpha_\ell = 0$ for all $\ell \in \mathcal{E}_y \setminus \{i\}$ and $\alpha_\ell \geq 0$ for all $\ell \notin \mathcal{E}_y$.

For all $z \in \mathcal{D}$, previous calculations yield:

$$0 \geq \alpha^\top [(n_j)_{j \neq i}]^\top (z - y) = -\alpha^\top Q^{(i,\cdot)}(z - \sum_{\ell \in \mathcal{E}_y} \lambda_\ell y^\ell) = -\alpha^\top \left[Q^{(i,\cdot)} z - \sum_{\ell \in \mathcal{E}_y} \lambda_\ell Q^{(i,\cdot)} y^\ell \right].$$

Let us recall that for any $j \neq i$, we have $y^j = -\theta_{\cdot,j}$. So computations made in the proof of Proposition 3.3 yield $Q^{(i,\cdot)} y^j + \bar{c}^{(i)} = \frac{\mu \bar{c}}{\mu_j} e_j$, and $Q^{(i,\cdot)} y^i + \bar{c}^{(i)} = 0$. Thus, the previous inequality becomes

$$\begin{aligned} 0 &\leq \alpha^\top \left[Q^{(i,\cdot)} z - \sum_{\ell \in \mathcal{E}_y \setminus \{i\}} \lambda_\ell \left(\frac{\mu \bar{c}}{\mu_\ell} e_\ell - \bar{c}^{(i)} \right) + \lambda_i \bar{c}^{(i)} \right] \\ &= \alpha^\top \left[Q^{(i,\cdot)} z + \bar{c}^{(i)} - \sum_{\ell \in \mathcal{E}_y \setminus \{i\}} \lambda_\ell \frac{\mu \bar{c}}{\mu_\ell} e_\ell \right]. \end{aligned} \quad (4.11)$$

By taking $z = y^j$ in (4.11), with $j \in \mathcal{E}_y \setminus \{i\}$, we get

$$0 \leq \alpha^\top \left[\frac{\mu \bar{c}}{\mu_j} e_j - \sum_{\ell \in \mathcal{E}_y \setminus \{i\}} \lambda_\ell \frac{\mu \bar{c}}{\mu_\ell} e_\ell \right] \quad (4.12)$$

and so, we can sum, over j , previous inequality with positive weights λ_j , to obtain

$$0 \leq \left(1 - \sum_{j \in \mathcal{E}_y \setminus \{i\}} \lambda_j \right) \alpha^\top \left[\sum_{\ell \in \mathcal{E}_y \setminus \{i\}} \lambda_\ell \frac{\mu \bar{c}}{\mu_\ell} e_\ell \right].$$

Then $0 \leq \alpha^\top \left[\sum_{\ell \in \mathcal{E}_y \setminus \{i\}} \lambda_\ell \frac{\mu \bar{c}}{\mu_\ell} e_\ell \right]$ since $\lambda_i > 0$. Moreover, we have also $0 \geq \alpha^\top \left[\sum_{\ell \in \mathcal{E}_y \setminus \{i\}} \lambda_\ell \frac{\mu \bar{c}}{\mu_\ell} e_\ell \right]$ by taking $z = y^i$ in (4.11), which yields

$$\alpha^\top \left[\sum_{\ell \in \mathcal{E}_y \setminus \{i\}} \lambda_\ell \frac{\mu \bar{c}}{\mu_\ell} e_\ell \right] = 0. \quad (4.13)$$

We recall that $\mu\bar{c} > 0$ since \mathcal{D} has a non-empty interior (see Theorem 3.1). Inserting (4.13) into (4.12) yields that $\alpha_j \geq 0$ for all $j \in \mathcal{E}_y \setminus \{i\}$, which, combined with (4.13) allows to conclude that $\alpha_j = 0$ for all $j \in \mathcal{E}_y \setminus \{i\}$.

Now we apply (4.11) with $z = y^j$ for $j \notin \mathcal{E}_y$: hence $0 \leq \alpha_j \frac{\mu\bar{c}}{\mu_j}$ for all $j \notin \mathcal{E}_y$, which concludes the proof of (4.10).

2. Then, we construct H on \mathcal{D}_\circ . Let us start by $H(y^i)$ for any $1 \leq i \leq d$. Fix $1 \leq i \leq d$, and let $B^i \in \mathbb{R}^{(d-1) \times (d-1)}$ be the base change matrix from $(-n_j^{(i)})_{j \neq i}$ to the canonical basis of \mathbb{R}^{d-1} . We set $H(y^i) := I^i B^i \Pi^i$, with $I^i : \mathbb{R}^{d-1} \rightarrow \mathbb{R}^d$ and $\Pi^i : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$ the linear maps defined by

$$I^i(x_1, \dots, x_{d-1}) = (x_1, \dots, x_{i-1}, 0, x_i, \dots, x_{d-1}), \quad (4.14)$$

$$\Pi^i(x_1, \dots, x_d) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d). \quad (4.15)$$

Now we set $H(y) := \sum_{i \in \mathcal{E}_y} \lambda_i H(y^i)$. Let us take $v \in \mathcal{C}(y)$. Thanks to (4.10), we know that $v = \sum_{j=1}^d \alpha_j n_j$ for some $(\alpha_j)_{1 \leq j \leq d} \in (\mathbb{R}_+)^d$ and such that $\alpha_j = 0$ when $j \in \mathcal{E}_y$. Since $n_k = -Q_k^\top$, for all $1 \leq k \leq d$, we have $v = -Q^\top \alpha$. By construction, we get that

$$H(y)v = - \sum_{j \notin \mathcal{E}_y} \alpha_j e_j = -\alpha \in \mathcal{C}_o(y).$$

It remains to check that Assumption 4.1-v) is fulfilled. If $v \neq 0$, which is equivalent to $\alpha \neq 0$, we have, for $i \in \mathcal{E}_y$,

$$v^\top H(y)v = \alpha^\top Q \alpha = (\alpha^{(i)})^\top Q^{(i,i)} \alpha^{(i)} > 0,$$

due to Assumption 4.2 and the fact that $\alpha_i = 0$.

3. We have constructed H on \mathcal{D}_\circ with the required properties. Finally, we set $H(x) = H(x - x_d \mathbf{1})$ for all $x \in \mathcal{D}$ and $H(x) = H(\mathfrak{P}_\mathcal{D}(x))$ for $x \in \mathbb{R}^d$ and the proof is finished. \square

Remark 4.2. i) Assumption 4.2 is satisfied as soon as P is symmetric and irreducible. Indeed, $Q^{(i,i)}$ is then non-singular, symmetric and diagonally dominant, hence positive definite, for all $i \in \{1, \dots, d\}$.

ii) In dimension 3, if the Markov chain is irreducible, then Assumption 4.2 is automatically satisfied. Indeed, under irreducibility, all diagonal coefficients are different from 1. Then, as already noticed in Remark 3.1, the domain \mathcal{D} can be defined with an irreducible matrix P whose diagonal entries are all equal to 0. We then set

$$P = \begin{pmatrix} 0 & p & 1-p \\ q & 0 & 1-q \\ r & 1-r & 0 \end{pmatrix} \quad (4.16)$$

for some $p, q, r \in [0, 1]$ satisfying $0 \leq p + q, 1 + r - p, 2 - (q + r) < 2$ by irreducibility. Thus, for $i = 1$ for example,

$$Q^{(1,1)} + \left(Q^{(1,1)}\right)^\top = \begin{pmatrix} 2 & -(p+q) \\ -(p+q) & 2 \end{pmatrix} \quad (4.17)$$

is non-singular, symmetric and diagonally dominant, hence positive definite. Thus $x^\top Q^{(1,1)}x = \frac{1}{2}x^\top \left(Q^{(1,1)} + (Q^{(1,1)})^\top \right) x > 0$ for all $x \neq 0$.

iii) However, in dimension greater than 3, it is not always possible to construct a function H satisfying Assumption 4.1. For example in dimension 4, consider the following matrix:

$$P = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 1 - \frac{\sqrt{3}}{2} \\ 1 - \frac{\sqrt{3}}{2} & 0 & \sqrt{3} - 1 & 1 - \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}, \quad (4.18)$$

together with positive costs c to ensure that the domain has a non-empty interior. It is an irreducible stochastic matrix, and let us consider the extremal point y^4 such that

$$y_4^4 = 0, \quad (4.19)$$

$$y_1^4 = \frac{\sqrt{3}}{2}y_2^4 - c_1, \quad (4.20)$$

$$y_2^4 = \left(1 - \frac{\sqrt{3}}{2}\right)y_1^4 + (\sqrt{3} - 1)y_3^4 - c_2, \quad (4.21)$$

$$y_3^4 = y_2^4 - c_3. \quad (4.22)$$

We have $\mathcal{C}(y^4) = \mathbb{R}_+(-1, \frac{\sqrt{3}}{2}, 0, 1 - \frac{\sqrt{3}}{2})^\top + \mathbb{R}_+(1 - \frac{\sqrt{3}}{2}, -1, \sqrt{3} - 1, 1 - \frac{\sqrt{3}}{2})^\top + \mathbb{R}_+(0, 1, -1, 0)^\top =: \sum_{i=1}^3 \mathbb{R}_+ n_i$.

If $H(y^4)$ satisfies $H(y^4)n_1 = (-1, 0, 0, 0)$, $H(y^4)n_2 = (0, -1, 0, 0)$ and $H(y^4)n_3 = (0, 0, -1, 0)$, consider $v = \frac{1}{2}n_1 + n_2 + \frac{\sqrt{3}}{2}n_3 \in \mathcal{C}(y^4)$. Then it is easy to compute $v^\top H v = 0$, hence it is not possible to construct $H(y^4)$ at this point satisfying Assumption 4.1.

4.1.2 An example of switching problem with controlled randomization

We assume here that $\mathcal{C} = [0, 1]$ and we consider the example of *switching problem with controlled randomisation* given by (3.2). Since the cost functions are positive, \mathcal{D} has a non-empty interior.

Theorem 4.3. *There exists a function $H : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$ that satisfies Assumption 4.1-v) and such that*

$$H(y)v \in \mathcal{C}_o(y), \quad \forall y \in \mathcal{D}, v \in \mathcal{C}(y).$$

Consequently, if we assume that Assumption 4.1 i), ii), iii), iv) is fulfilled, for all $(t, x) \in [0, T] \times \mathbb{R}^q$, there exists a solution to (2.6)-(2.7)-(2.8) with $\xi = g(X_T^{t,x})$ and $f(\omega, s, y, z) = \psi(s, X_s^{t,x}(\omega), y, z)$. Moreover, this solution is unique if Assumption 2.1-ii) holds also.

Proof. We first observe that uniqueness follows once again from Proposition 2.1.

1. We start by constructing H on the boundary of \mathcal{D} . Recalling Lemma 3.1, it is enough to construct it on its intersection with \mathcal{D}_o which is made up of 3 vertices

$$y^1 = (1, 0, 0), \quad y^2 = (0, 1, 0), \quad y^3 = (-1, -1, 0),$$

and three edges that are smooth curves. We denote \mathcal{E}_1 (respectively \mathcal{E}_2 and \mathcal{E}_3) the curve between y^1 and y^2 (respectively between y^2 and y^3 and between y^3 and y^1). Let us construct $H(y^1)$ and $H(y^2)$: we must have

$$H(y^1) \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -b \\ -a & 0 \end{pmatrix}, \quad H(y^2) \begin{pmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -c & 0 \\ 0 & 0 \\ 0 & -d \end{pmatrix},$$

with $a, b, c, d > 0$. Let us set $a = b = c = d = 1$. Then we can take

$$H(y^1) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad H(y^2) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

We define now H on \mathcal{E}_1 . We denote $(x_s)_{s \in [0,1]}$ a continuous parametrization of \mathcal{E}_1 such that $x_0 = y^1$ and $x_1 = y^2$. For all $s \in [0, 1]$, we also denote R_s the matrix that send the standard basis to a local orthonormal basis at point x_s defined as: the two first vectors are in the plane $\{z = 0\}$, the first one is orthogonal to \mathcal{E}_1 at the point x_s , pointing towards the exterior of \mathcal{D}_0 , the third one is e_3 , while the second one is tangent to \mathcal{E}_1 at the point x_s and is directed such that this basis has a direct orientation. We have in particular $Q_0 = \text{Id}$. Then we just have to set

$$H(x_s) = R_s[sH(y^1) + (1-s)R_1^{-1}H(y^2)R_1]R_s^{-1}.$$

We can check that, by construction, Assumption 4.1-v) and (4.6) are fulfilled for points on \mathcal{E}_1 . Moreover, we are able to construct by the same method H on y^3 , and then on \mathcal{E}_2 and \mathcal{E}_3 , satisfying Assumption 4.1-v) and (4.6).

2. By using Lemma 3.1 we can extend H on all the boundary of \mathcal{D} . Finally, we can extend H by continuity on the whole space \mathbb{R}^3 by following [6, Remark 2.1]. \square

4.2 The non-Markovian framework

We now switch to the non-Markovian case, which is more challenging. We prove the well-posedness of the obliquely reflected BSDE in the uncontrolled setting for two cases: problems in dimension 3 and the example of a symmetric transition matrix P , in any dimension.

We first recall Proposition 3.1 in [6] that gives an existence result for non-Markovian obliquely reflected BSDEs and the corresponding assumptions, see Assumption 4.3 below. Let us remark that the non-Markovian case is more challenging for our approach as it requires more structural conditions on H , which must be symmetric and smooth in this case.

Assumption 4.3. *There exists $L > 0$ such that*

- i) $\xi := g((X_t)_{t \in [0,T]})$ with $g : C([0,T], \mathbb{R}^q) \rightarrow \mathcal{D}$ a bounded uniformly continuous function and X solution of the SDE (4.1) where $(b, \sigma) : [0, T] \times \mathbb{R}^q \rightarrow \mathbb{R}^q \times \mathbb{R}^{q \times \kappa}$ is

a measurable function satisfying, for all $(t, x, y) \in [0, T] \times \mathbb{R}^q \times \mathbb{R}^q$,

$$\begin{aligned} |\sigma(t, x)| &\leq L, \\ |b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| &\leq L|x - y|. \end{aligned}$$

ii) $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times \kappa} \rightarrow \mathbb{R}^d$ is a $\mathcal{P}(\mathbb{F}^0) \otimes \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d \times \kappa})$ -measurable function such that, for all $(t, y, y', z, z') \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d \times \kappa} \times \mathbb{R}^{d \times \kappa}$,

$$|f(t, y, z) - f(t, y', z')| \leq L(|y - y'| + |z - z'|).$$

Moreover we have

$$\operatorname{ess\,sup}_{\omega \in \Omega, t \in [0, T]} \mathbb{E} \left[\int_t^T |f(s, 0, 0)|^2 ds \middle| \mathcal{F}_t^0 \right] \leq L.$$

iii) $H : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ is valued in the set of symmetric matrices Q satisfying

$$|Q| \leq L, \quad L|v|^2 \geq v^\top Q v \geq \frac{1}{L}|v|^2, \quad \forall v \in \mathbb{R}^d. \quad (4.23)$$

H is a \mathcal{C}^1 -function and H^{-1} is a \mathcal{C}^2 -function satisfying

$$|\partial_y H| + |H^{-1}| + |\partial_y H^{-1}| + |\partial_{yy}^2 H^{-1}| \leq L.$$

From this assumption is deduced the following general existence result in the non-Markovian setting.

Theorem 4.4 ([6], Proposition 3.1). *We assume that \mathcal{D} has a non-empty interior. Under Assumption 4.3, there exists a solution $(Y, Z, \Psi) \in \mathbb{S}_d^2(\mathbb{F}^0) \times \mathbb{H}_{d \times \kappa}^2(\mathbb{F}^0) \times \mathbb{H}_d^2(\mathbb{F}^0)$ of the following system*

$$Y_s = \xi + \int_s^T f(u, Y_u, Z_u) du - \int_s^T Z_u dW_u - \int_s^T H(Y_u) \Psi_u du, \quad s \in [0, T], \quad (4.24)$$

$$Y_s \in \mathcal{D}, \quad \Psi_s \in \mathcal{C}(Y_s), \quad 0 \leq s \leq T, \quad (4.25)$$

$$\int_0^T 1_{\{Y_s \notin \partial \mathcal{D}\}} |\Psi_s| ds = 0. \quad (4.26)$$

Remark 4.3. i) *The assumption on the terminal condition is slightly less general than the one needed in [6] (see Assumption **SB**(i) and Corollary 2.2 in [6]). One could get a more general result by assuming that $\mathbb{E}[\xi^0 | \mathcal{F}]$ is a BMO martingale such that its bracket has sufficiently large exponential moment.*

ii) *We cannot use Theorem 3.1 in [6] since the domain \mathcal{D} is not smooth enough (see Assumption **SB**(iv) in [6]). Consequently, we have to assume the extra assumption that ξ is bounded.*

iii) *The uniqueness result for this part is also obtained by invoking Proposition 2.1.*

iv) *The discussion following Theorem 4.1 shows, once again in this non-Markovian setting, that a solution to (4.24)-(4.25)-(4.26), with H satisfying to $H(y)v \in \mathcal{C}_o(y)$ for all $y \in \mathcal{D}$ and $v \in \mathcal{C}(y)$, induces a solution to (2.6)-(2.7)-(2.8).*

4.2.1 Existence of solutions in dimension 3

We focus in this part on the uncontrolled case $\mathcal{C} = \{0\}$, in dimension $d = 3$ and in the irreducible case. Thus, there is a unique transition matrix given by

$$P := P^0 = \begin{pmatrix} 0 & p & 1-p \\ q & 0 & 1-q \\ r & 1-r & 0 \end{pmatrix}, \quad (4.27)$$

for some $p, q, r \in [0, 1]$. Indeed, as already mentioned in Remark 4.2 for the Markovian case, irreducibility implies that diagonal entries are different from 1, and Remark 3.1 explains how to obtain a matrix with diagonal entries set to 0.

Theorem 4.5. *Let us assume that $0 < p, q, r < 1$ and that \mathcal{D} has a non-empty interior. Then there exists a function $H : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$ that satisfies Assumption 4.3-iii) and such that*

$$H(y)v \in \mathcal{C}_o(y), \quad \forall y \in \mathcal{D}, v \in \mathcal{C}(y). \quad (4.28)$$

Consequently, if we assume that Assumption 4.3-i) & (ii) are fulfilled, then there exists a solution to the obliquely reflected BSDE (2.6)-(2.7)-(2.8). Moreover this solution is unique if we assume also Assumption 2.1-ii).

Proof. We first observe that uniqueness follows once again from Proposition 2.1. Concerning existence, once again we exhibit a convenient H . Thanks to Lemma 3.1, it is enough to construct H only on $\mathbb{R}^3 \cap \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$. We start by \mathcal{D}_o , which is a triangle with three vertices $v^i = (v_1^i, v_2^i, v_3^i)$, $i = 1, 2, 3$, defined as

$$v_1^1 = pv_2^1 + (1-p)v_3^1 - c_1, v_1^2 = pv_2^2 + (1-p)v_3^2 - c_1, v_2^3 = qv_1^3 + (1-q)v_3^3 - c_2, \quad (4.29)$$

$$v_2^1 = qv_1^1 + (1-q)v_3^1 - c_2, v_3^2 = rv_1^2 + (1-r)v_2^2 - c_3, v_3^3 = rv_1^3 + (1-r)v_2^3 - c_3, \quad (4.30)$$

$$v_3^1 = 0, v_3^2 = 0, v_3^3 = 0. \quad (4.31)$$

Let us now construct H on each vertex. We consider first the point v^1 . It is easy to compute its outward normal cone, which is given by

$$\mathcal{C}(v^1) = \mathbb{R}_+(-1, p, 1-p)^\top + \mathbb{R}_+(q, -1, 1-q)^\top. \quad (4.32)$$

The matrix $H(v^1)$ must satisfy

$$H(v^1) \begin{pmatrix} -1 & q \\ p & -1 \\ 1-p & 1-q \end{pmatrix} = \begin{pmatrix} -a & 0 \\ 0 & -b \\ 0 & 0 \end{pmatrix} \quad (4.33)$$

for some $a, b > 0$. Taking $a = \frac{1}{q}, b = \frac{1}{p}$, we consider, for any $\alpha > 0$,

$$H(v^1) = - \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -q & pq \\ pq & -p \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha \end{pmatrix} \quad (4.34)$$

$$= \frac{1}{pq(1-pq)} \begin{pmatrix} \alpha + p & \alpha + pq & \alpha \\ \alpha + pq & \alpha + q & \alpha \\ \alpha & \alpha & \alpha \end{pmatrix}. \quad (4.35)$$

It is easy to check that this matrix $H(v^1)$ is symmetric and positive definite for any $\alpha > 0$, so we can set $\alpha = 1$ in the following. Similarly, we construct H on vertices v^2, v^3 ,

$$H(v^2) = \frac{1}{r(1-p)(1-r(1-p))} \begin{pmatrix} 1 + (1-p) & 1 & 1 + r(1-p) \\ 1 & 1 & 1 \\ 1 + r(1-p) & 1 & 1 + r \end{pmatrix}, \quad (4.36)$$

$$H(v^3) = \frac{1}{(1-q)(1-r)(1-(1-q)(1-r))} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 + (1-q) & 1 + (1-q)(1-r) \\ 1 & 1 + (1-q)(1-r) & 1 + (1-r) \end{pmatrix}.$$

We can extend H on all \mathcal{D}_o by convex combination, i.e. linear interpolation. Thus H stays valued in the set of positive definite symmetric matrices and is smooth enough. Thanks to Lemma 3.1, the extension to \mathcal{D} is straightforward.

We could also define H outside $\mathcal{D} \cap \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$ by linear interpolation but we will lose the boundedness and the positivity of H . Nevertheless we can find a bounded and convex, \mathcal{C}^2 open neighborhood \mathcal{V} of \mathcal{D} , small enough, such that H (still defined by linear interpolation) stays valued in the set of positive definite symmetric matrices on $\bar{\mathcal{V}}$. Then we define $H(y)$ for $y \notin \bar{\mathcal{V}}$ by $H(\mathcal{P}_{\bar{\mathcal{V}}}(y))$. In this way, H is a bounded function with values in the set of positive definite symmetric matrices, that satisfies (4.23), (4.28) and that is $\mathcal{C}^0(\mathbb{R}^2) \cap \mathcal{C}^2(\mathbb{R}^2 \setminus \partial\mathcal{V})$ smooth, with $\partial\mathcal{V}$ the boundary of \mathcal{V} . Finally, we just have to mollify H in a neighborhood of $\partial\mathcal{V}$, small enough to stay outside $\mathcal{D} \cap \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$. \square

Remark 4.4. When $pqr(1-p)(1-q)(1-r) = 0$, one can show that it is not possible to construct a function H that satisfies Assumption 4.3-iii) and (4.28).

4.2.2 Existence of solutions for a symmetric multidimensional example

We focus in this part on the uncontrolled case $\mathcal{C} = \{0\}$, in dimension $d \geq 3$ with a unique transition matrix P given by

$$P_{i,j} = \frac{1}{d-1} \mathbf{1}_{i \neq j}.$$

Theorem 4.6. Assume that \mathcal{D} has a non-empty interior. There exists a function $H : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ that satisfies Assumption 4.3-iii) and such that

$$H(y)v \in \mathcal{C}_o(y), \quad \forall y \in \mathcal{D}, v \in \mathcal{C}(y).$$

Consequently, if we assume that Assumption 4.3-i) & ii) are fulfilled, then there exists a solution to the obliquely reflected BSDE (2.6)-(2.7)-(2.8). Moreover this solution is unique if we assume also Assumption 2.1-ii).

Proof. The proof follows exactly the same lines as the proof of Theorem 4.5. \mathcal{D}_0 is a convex polytope with d vertices $(y^i)_{1 \leq i \leq d}$ satisfying: for all $1 \leq i \leq d$,

$$y_\ell^i = \sum_{j \neq i} \frac{1}{d-1} y_\ell^j - \bar{c}_i, \quad \forall i \neq \ell, \quad \text{and} \quad y_d^i = 0.$$

Let us construct H on vertex y^d . It is easy to compute its outward normal cone, which is positively generated by vectors f^1, \dots, f^{d-1} where

$$f_i^k = -\mathbf{1}_{i=k} + \frac{1}{d-1} \mathbf{1}_{i \neq k}.$$

For any $1 \leq k \leq d-1$, we impose $H(y^d)f^k = -\alpha_k e_k$ with $\alpha_k > 0$. We can check that it is true with $\alpha_k = 1$ for all $1 \leq k \leq d-1$, if we set, for any $a > 0$,

$$H(y^d) = \begin{pmatrix} a & a - \frac{d-1}{d} & a - 2\frac{d-1}{d} & \vdots \\ & \ddots & & \vdots \\ a - \frac{d-1}{d} & a & & \vdots \\ a - 2\frac{d-1}{d} & \dots & \dots & a - 2\frac{d-1}{d} \end{pmatrix}.$$

Since $\frac{d-1}{d}$ is an eigenvalue of $H(y^d)$ with multiplicity $d-2$, $\det H(y^d) = (a - 2\frac{d-1}{d})(d-1)(\frac{d-1}{d})^{d-2}$ and $\text{Tr}(H(y^d)) = da - 2\frac{d-1}{d}$, $H(y^d)$ is a positive definite symmetric matrix as soon as $a > 2\frac{d-1}{d}$. Thus we can set $a = 2$. By simple permutations of rows and columns in $H(y^d)$ we can construct easily $H(y^k)$ for any $1 \leq k \leq d$. We then follow the proof of Theorem 4.5 to extend H from vertices of \mathcal{D}_0 to the whole space. \square

A Appendix

A.1 Proof of Lemma 3.3

For all $\mathcal{I}, \mathcal{J} \subset \{1, \dots, d\}^2$, let $\text{ad}[Q^{(\mathcal{I}, \mathcal{J})}]$ be the adjunct matrix of $Q^{(\mathcal{I}, \mathcal{J})}$.

For $1 \leq j \leq d$, we denote, for ease of presentation, $\mathfrak{Q}^j := \text{ad}[Q^{(j, j)}]$, and we have

$$\mathfrak{Q}_{i(j), \ell(j)}^j = (-1)^{i(j) + \ell(j)} \det Q^{(\{j, \ell\}, \{j, i\})}. \quad (\text{A.1})$$

for all $(\ell, i) \in \{1, \dots, d\} \setminus \{j\}$. For all $1 \leq i \neq j \leq d$, we define

$$C_{i,j} := \left((Q^{(j,j)})^{-1} \bar{c}^{(j)} \right)_{i-1_{\{i > j\}}} \quad \text{and} \quad C_{j,j} = 0.$$

Using \mathfrak{Q}^j the adjunct matrix of $Q^{(j,j)}$, we observe then, for latter use,

$$C_{i,j} = \frac{1}{\mu_j} \sum_{\ell \neq j} \mathfrak{Q}_{i(j), \ell(j)}^j \bar{c}_\ell, \quad \text{for } i \neq j. \quad (\text{A.2})$$

Proof. 1. We first show that (3.3) holds true. From (3.3), we observe that

$$\begin{aligned}\bar{C}_{j,j} &= \mathbb{E} \left[\sum_{n=0}^{\tau_j-1} \bar{c}_{X_n} \mid X_0 = j \right] \\ &= \mathbb{E} \left[\sum_{n=0}^{\tau_j-1} \sum_{\ell=1}^d \bar{c}_\ell \mathbf{1}_{\{X_n=\ell\}} \mid X_0 = j \right].\end{aligned}$$

Thus,

$$\bar{C}_{j,j} = \sum_{\ell=1}^d \bar{c}_\ell \gamma_\ell^j \quad \text{with} \quad \gamma_\ell^j = \mathbb{E} \left[\sum_{n=0}^{\tau_j-1} \mathbf{1}_{\{X_n=\ell\}} \mid X_0 = j \right].$$

From [20, Theorem 1.7.5], we know that $\gamma_\ell^j = \frac{\mu_\ell}{\mu_j}$.

2. We prove (3.16) assuming the following for the moment: for all distinct $1 \leq i, j, k \leq d$,

$$\mu_i \mathfrak{Q}_{i(j),k(j)}^j + \mu_j \mathfrak{Q}_{j(i),k(i)}^i = \mathfrak{Q}_{j(i),j(i)}^i \mu_k = \mathfrak{Q}_{i(j),i(j)}^j \mu_k. \quad (\text{A.3})$$

Let $1 \leq i \neq j \leq d$. We have, using (A.2),

$$\begin{aligned}C^{i,j} + C^{j,i} &= \frac{1}{\mu_j} \left(\mathfrak{Q}^j \bar{c}^{(j)} \right)_{i(j)} + \frac{1}{\mu_i} \left(\mathfrak{Q}^i \bar{c}^{(i)} \right)_{j(i)} \\ &= \frac{1}{\mu_j} \sum_{k \neq j} \mathfrak{Q}_{i(j),k(j)}^j \bar{c}_k + \frac{1}{\mu_i} \sum_{k \neq i} \mathfrak{Q}_{j(i),k(i)}^i \bar{c}_k \\ &= \frac{1}{\mu_j} \mathfrak{Q}_{i(j),i(j)}^j \bar{c}_i + \frac{1}{\mu_i} \mathfrak{Q}_{j(i),j(i)}^i \bar{c}_j + \sum_{k \neq i,j} \frac{\mu_i \mathfrak{Q}_{i(j),k(j)}^j + \mu_j \mathfrak{Q}_{j(i),k(i)}^i}{\mu_i \mu_j} \bar{c}_k.\end{aligned}$$

Using the previous point and the fact that $\mathfrak{Q}_{i(j),i(j)}^j = \mathfrak{Q}_{j(i),j(i)}^i$, we get

$$C^{i,j} + C^{j,i} = \mathfrak{Q}_{i(j),i(j)}^j \left(\frac{\mu_i \bar{c}_i + \mu_j \bar{c}_j}{\mu_i \mu_j} + \sum_{k \neq i,j} \frac{\mu_k \bar{c}_k}{\mu_i \mu_j} \right) = \frac{\mathfrak{Q}_{i(j),i(j)}^j}{\mu_i \mu_j} \mu \bar{c}$$

which is the result we wanted to prove.

3. We now prove (A.3).

Let $i, j \in \{1, \dots, d\}$ and $i \neq j$. We observe first, using (A.1), that

$$\mathfrak{Q}_{i(j),i(j)}^j = \det Q^{(\{j,i\},\{j,i\})} = \mathfrak{Q}_{j(i),j(i)}^i.$$

For $k \in \{1, \dots, d\} \setminus \{i, j\}$, we denote by $k_{ij} \in \{1, \dots, d-2\}$ (resp. i_{jk}, j_{ik}) the index such that:

$$Q_{k,\cdot} = Q_{k_{ij},\cdot}^{(\{j,i\},\{j,i\})} \quad (\text{resp.} \quad Q_{i,\cdot} = Q_{i_{jk},\cdot}^{(\{j,k\},\{j,i\})}, Q_{j,\cdot} = Q_{j_{ik},\cdot}^{(\{k,i\},\{j,i\})}), \quad (\text{A.4})$$

namely

$$k_{ij} = k - \mathbf{1}_{\{k>i\}} - \mathbf{1}_{\{k>j\}}, \quad i_{jk} = i - \mathbf{1}_{\{i>k\}} - \mathbf{1}_{\{i>j\}} \quad \text{and} \quad j_{ik} = j - \mathbf{1}_{\{j>k\}} - \mathbf{1}_{\{j>i\}}.$$

Let σ_k be the permutation of $\{1, \dots, d-2\}$ given by

$$\begin{pmatrix} 2 & \dots & k_{ij} & 1 & k_{ij} + 1 & \dots & d-2 \\ 1 & \dots & k_{ij} - 1 & k_{ij} & k_{ij} + 1 & \dots & d-2 \end{pmatrix}$$

which is the composition of $k_{ij}-1$ transpositions. Applying σ_k^{-1} to the row of $Q^{(\{j,i\},\{j,i\})}$, we obtain a matrix denoted simply $Q_{\sigma_k(\cdot),\cdot}^{(\{j,i\},\{j,i\})}$ whose first row is $Q_{k,\cdot}^{(\{j,i\},\{j,i\})}$, and we have

$$\det Q_{\sigma_k(\cdot),\cdot}^{(\{j,i\},\{j,i\})} = (-1)^{k_{ij}-1} \det Q_{\sigma_k(\cdot),\cdot}^{(\{j,i\},\{j,i\})}.$$

Since $\mu Q = 0$, we have $Q_{k,\cdot} = -\sum_{\ell \neq k} \frac{\mu_\ell}{\mu_k} Q_{\ell,\cdot}$ and then,

$$\det Q_{\sigma_k(\cdot),\cdot}^{(\{j,i\},\{j,i\})} = -\sum_{\ell \neq k} \frac{\mu_\ell}{\mu_k} \begin{vmatrix} Q_{\ell,\cdot} \\ Q_{\sigma_k(2),\cdot}^{(\{j,i\},\{j,i\})} \\ \vdots \\ Q_{\sigma_k(d-2),\cdot}^{(\{j,i\},\{j,i\})} \end{vmatrix} = -\frac{\mu_i}{\mu_k} \begin{vmatrix} Q_{i,\cdot} \\ Q_{\sigma_k(2),\cdot}^{(\{j,i\},\{j,i\})} \\ \vdots \\ Q_{\sigma_k(d-2),\cdot}^{(\{j,i\},\{j,i\})} \end{vmatrix} - \frac{\mu_j}{\mu_k} \begin{vmatrix} Q_{j,\cdot} \\ Q_{\sigma_k(2),\cdot}^{(\{j,i\},\{j,i\})} \\ \vdots \\ Q_{\sigma_k(d-2),\cdot}^{(\{j,i\},\{j,i\})} \end{vmatrix}.$$

Let σ_i (resp. σ_j) be constructed as σ_k but with i_{jk} (resp. j_{ik}) instead of k_{ji} then one observes

$$\begin{aligned} \det Q_{\sigma_k(\cdot),\cdot}^{(\{j,i\},\{j,i\})} &= -\frac{\mu_i}{\mu_k} \det Q_{\sigma_i(\cdot),\cdot}^{(\{j,k\},\{j,i\})} - \frac{\mu_j}{\mu_k} \det Q_{\sigma_j(\cdot),\cdot}^{(\{i,k\},\{j,i\})} \\ &= -\frac{\mu_i}{\mu_k} (-1)^{i_{jk}-1} \det Q_{\{j,k\},\{j,i\}} - \frac{\mu_j}{\mu_k} (-1)^{j_{ik}-1} \det Q_{\{i,k\},\{j,i\}}. \end{aligned}$$

We compute that

$$(-1)^{i_{jk}-1+k_{ij}-1+i^{(j)}+k^{(j)}} = -1 \quad \text{and} \quad (-1)^{j_{ik}-1+k_{ij}-1+j^{(i)}+k^{(i)}} = -1,$$

leading to

$$\mu_k \mathfrak{Q}_{i^{(j)},i^{(j)}}^j = \mu_i \mathfrak{Q}_{i^{(j)},k^{(j)}}^j + \mu_j \mathfrak{Q}_{j^{(i)},k^{(i)}}^i.$$

□

A.2 Enlargement of a filtration along a sequence of increasing stopping times

We fix an admissible strategy $\phi \in \mathcal{A}$ and we study the associated filtrations $\mathbb{F}^n, n \in \mathbb{N} \cup \{+\infty\}$ which are constructed in Section 2.1. In all this section we assume that $N_T^\phi < +\infty$ a.s. Let us remark that this is satisfied as soon as Assumption 2.1-i) is in

force, which is the case whenever the results of this section are applied.

For each $0 \leq n < +\infty$, we define a new filtration $\mathbb{G}^n = (\mathcal{G}_t^n)_{t \geq 0}$ by the relations $\mathcal{G}_t^0 = \mathcal{F}_t^0, t \geq 0$ and for $1 \leq n < +\infty$, $\mathcal{G}_t^n = \mathcal{F}_t^0 \vee \sigma(\mathfrak{U}_i, i \leq n) = \mathcal{G}_t^{n-1} \vee \sigma(\mathfrak{U}_n), t \geq 0$. The difference between filtrations \mathbb{F}^n and \mathbb{G}^n is that the random variables $\mathfrak{U}_i, 1 \leq i \leq n$ are already known at time 0 in \mathbb{G}^n , while in the filtration \mathbb{F}^n , each \mathfrak{U}_i is only known at the switching time τ_i , for each $1 \leq i \leq n$.

A.2.1 Representation Theorems

The goal of this section is to derive Integral Representation Theorems for filtrations $\mathbb{F}^n, n \in \mathbb{N} \cup \{+\infty\}$.

We first recall, see [1, Theorem 1.11]:

Theorem A.1 (Lévy). *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space with \mathbb{F} not necessarily right-continuous. Let $\xi \in L^1(\mathcal{F})$ and X a \mathbb{F} -supermartingale.*

1. *We have $\mathbb{E}[\xi | \mathcal{F}_t] \rightarrow \mathbb{E}[\xi | \mathcal{F}_\infty]$ a.s. and in L^1 , as $t \rightarrow \infty$.*
2. *If t_m decreases to t , we have $X_{t_m} \rightarrow X_{t+}$ a.s. and in L^1 as $m \rightarrow \infty$.*

In particular, if $X_t = \mathbb{E}[\xi | \mathcal{F}_t]$, we get that $\mathbb{E}[\xi | \mathcal{F}_{t_m}] \rightarrow \mathbb{E}[\xi | \mathcal{F}_{t+}]$ a.s. and in L^1 as $m \rightarrow \infty$, for t_m decreasing to t .

We now recall an important notion of coincidence of filtrations between two stopping times, introduced in [1, Definition 1.28]. This will be useful for our purpose in the sequel. Let $\mathfrak{H}^1 = (\mathcal{H}_t^1)_{t \geq 0}$ and $\mathfrak{H}^2 = (\mathcal{H}_t^2)_{t \geq 0}$ be two filtrations, and let S, T two \mathfrak{H}^1 -stopping times, which are also \mathfrak{H}^2 -stopping times. We set

$$[S, T[:= \{(\omega, s) \in \Omega \times \mathbb{R}_+ \mid S(\omega) \leq s < T(\omega)\},$$

and we say that \mathfrak{H}^1 and \mathfrak{H}^2 *coincide* on $[S, T[$ if

1. for each $t \geq 0$ and each \mathcal{H}_t^1 -measurable variable ξ , there exists a \mathcal{H}_t^2 -measurable variable χ such that $\xi \mathbf{1}_{\{S \leq t < T\}} = \chi \mathbf{1}_{\{S \leq t < T\}}$,
2. for each $t \geq 0$ and each \mathcal{H}_t^2 -measurable variable χ , there exists a \mathcal{H}_t^1 -measurable variable ξ such that $\chi \mathbf{1}_{\{S \leq t < T\}} = \xi \mathbf{1}_{\{S \leq t < T\}}$.

We now study the right-continuity of the filtration \mathbb{G}^n for some $n \geq 0$. Using its specific structure, it is easy to compute conditional expectations. Lévy's theorem then allows to obtain the right-continuity.

Lemma A.1. *Let $0 \leq n < +\infty$.*

1. *If $\xi \in L^1(\mathcal{F}_\infty^0)$ and $\xi' \in L^1(\sigma(\mathfrak{U}_i, 1 \leq i \leq n))$, then for $t \geq 0$, we have $\mathbb{E}[\xi \xi' | \mathcal{G}_t^n] = \mathbb{E}[\xi | \mathcal{F}_t^0] \xi'$.*
2. *\mathbb{G}^n is right-continuous.*

Proof.

1. If $F \in \mathcal{F}_t^0$ and $F' \in \sigma(\mathcal{U}_i, 1 \leq i \leq n)$, since $\sigma(\mathcal{U}_1, \dots, \mathcal{U}_n) \perp \mathcal{F}_\infty^0$ (recall that \mathcal{F}_∞^0 is generated by the Brownian motion path), we have, using at the last equality that $\mathcal{F}_t^0 \subset \mathcal{F}_\infty^0$ is also independent from $\sigma(\mathcal{U}_1, \dots, \mathcal{U}_n)$,

$$\begin{aligned} \mathbb{E}[\xi \xi' \mathbf{1}_{F \cap F'}] &= \mathbb{E}[\xi \mathbf{1}_F] \mathbb{E}[\xi' \mathbf{1}_{F'}] \\ &= \mathbb{E}[\mathbb{E}[\xi | \mathcal{F}_t^0] \mathbf{1}_F] \mathbb{E}[\xi' \mathbf{1}_{F'}] \\ &= \mathbb{E}[\xi' \mathbb{E}[\xi | \mathcal{F}_t^0] \mathbf{1}_{F \cap F'}]. \end{aligned}$$

Since $\{F \cap F' | F \in \mathcal{F}_t^0, F' \in \sigma(\mathcal{U}_i, 1 \leq i \leq n)\}$ is a π -system generating \mathcal{G}_t^n , the result follows by a monotone class argument.

2. Let $t \geq 0$ and t_m decreasing to t . We have, using Lévy's Theorem, the previous point and the right-continuity of \mathbb{F}^0 ,

$$\begin{aligned} \mathbb{E}[\xi \xi' | \mathcal{G}_{t+}^n] &= \lim_m \mathbb{E}[\xi \xi' | \mathcal{G}_{t_m}^n] = \lim_m \xi' \mathbb{E}[\xi | \mathcal{F}_{t_m}^0] \\ &= \xi' \mathbb{E}[\xi | \mathcal{F}_t^0] = \mathbb{E}[\xi \xi' | \mathcal{G}_t^n]. \end{aligned}$$

By a monotone class argument, we have $\mathbb{E}[\xi | \mathcal{G}_{t+}^n] = \mathbb{E}[\xi | \mathcal{G}_t^n]$ for all bounded \mathcal{G}_∞^n -measurable ξ , hence the right-continuity of \mathbb{G}^n follows. \square

Using the previous lemma, we show how to compute conditional expectations in \mathbb{F}^n for all $n \in \mathbb{N} \cup \{+\infty\}$, and that these filtrations are right-continuous.

Proposition A.1. *1. For all $0 \leq n \leq m < +\infty$, \mathbb{F}^n , \mathbb{F}^m and \mathbb{F}^∞ coincide on $\llbracket 0, \tau_{n+1} \rrbracket$.*

For all $0 \leq n < +\infty$, \mathbb{F}^n and \mathbb{G}^n coincide on $\llbracket \tau_n, +\infty \rrbracket$.

2. *For all $0 \leq n < +\infty$ and $t \geq 0$, we have, for $\xi \in L^1(\mathcal{F}_\infty^{n+1})$:*

$$\mathbb{E}[\xi | \mathcal{F}_t^{n+1}] = \mathbb{E}[\xi | \mathcal{F}_t^n] \mathbf{1}_{\{t < \tau_{n+1}\}} + \mathbb{E}[\xi | \mathcal{G}_t^{n+1}] \mathbf{1}_{\{\tau_{n+1} \leq t\}}. \quad (\text{A.5})$$

Let $t \geq 0$ such that $\sum_{n=0}^{+\infty} \mathbb{P}(\tau_n \leq t < \tau_{n+1}) = 1$. Then, for $\xi \in L^1(\mathcal{F}_\infty^\infty)$,

$$\mathbb{E}[\xi | \mathcal{F}_t^\infty] = \sum_{n=0}^{+\infty} \mathbb{E}[\xi | \mathcal{F}_t^n] \mathbf{1}_{\{\tau_n \leq t < \tau_{n+1}\}}.$$

3. *For all $0 \leq n < +\infty$, \mathbb{F}^n is right-continuous.*

4. *The filtration \mathbb{F}^∞ is right-continuous on $[0, T]$.*

Proof.

1. Let $t \geq 0$ be fixed. If ξ is \mathcal{F}_t^n -measurable, since $\mathcal{F}_t^n \subset \mathcal{F}_t^m \subset \mathcal{F}_t^\infty$ for $m \geq n$, taking $\chi = \xi$ gives a \mathcal{F}_t^m -measurable (resp. \mathcal{F}_t^∞ -measurable) random variable such that $\xi \mathbf{1}_{\{t < \tau_{n+1}\}} = \chi \mathbf{1}_{\{t < \tau_{n+1}\}}$. Conversely, if χ is a \mathcal{F}_t^m -measurable random variable, then

$$\chi = f(\tilde{\chi}, \mathfrak{U}_1 \mathbf{1}_{\{\tau_1 \leq t\}}, \dots, \mathfrak{U}_m \mathbf{1}_{\{\tau_m \leq t\}}),$$

for a measurable f and a \mathcal{F}_t^0 -measurable variable $\tilde{\chi}$. Since $\mathfrak{U}_k \mathbf{1}_{\{\tau_k \leq t\}} = 0$ on $\{t < \tau_{n+1}\}$ when $k \geq n$, one gets:

$$\begin{aligned} \chi \mathbf{1}_{\{t < \tau_{n+1}\}} &= f(\tilde{\chi}, \mathfrak{U}_1 \mathbf{1}_{\{\tau_1 \leq t\}}, \dots, \mathfrak{U}_n \mathbf{1}_{\{\tau_n \leq t\}}, 0, \dots, 0) \mathbf{1}_{\{t < \tau_{n+1}\}} \\ &=: \xi \mathbf{1}_{\{t < \tau_{n+1}\}}, \end{aligned}$$

where ξ is \mathcal{F}_t^n -measurable.

Last, let χ be a \mathcal{F}_t^∞ -measurable variable. Then $\chi = f(\tilde{\chi}, \mathfrak{U}_{i_1} \mathbf{1}_{\{\tau_{i_1} \leq t\}}, \dots, \mathfrak{U}_{i_N} \mathbf{1}_{\{\tau_{i_N} \leq t\}})$ for some $N \geq 0$ and $1 \leq i_1 \leq \dots \leq i_N$, and the same arguments applies.

The proof of the second claim is straightforward as one remarks that for $t \geq 0$ and $1 \leq n < \infty$, the equality $f(\xi, \mathfrak{U}_1, \dots, \mathfrak{U}_n) \mathbf{1}_{\{\tau_n \leq t\}} = f(\xi, \mathfrak{U}_1 \mathbf{1}_{\{\tau_1 \leq t\}}, \dots, \mathfrak{U}_n \mathbf{1}_{\{\tau_n \leq t\}}) \mathbf{1}_{\{\tau_n \leq t\}}$ holds, since the random times $\tau_i, i \geq 0$ are non-decreasing.

This concludes the proof of this first item by definition of coincidence of filtrations.

2. Let $0 \leq n < +\infty$ and $\xi \in L^1(\mathcal{F}_\infty^{n+1})$. We have $\mathbb{E}[\xi | \mathcal{F}_t^{n+1}] = \mathbb{E}[\xi | \mathcal{F}_t^{n+1}] \mathbf{1}_{\{t < \tau_{n+1}\}} + \mathbb{E}[\xi | \mathcal{F}_t^{n+1}] \mathbf{1}_{\{\tau_{n+1} \leq t\}}$, and we compute both terms separately.

Since \mathbb{F}^n and \mathbb{F}^{n+1} coincide on $\llbracket 0, \tau_{n+1} \rrbracket$, we have $\mathbb{E}[\xi | \mathcal{F}_t^{n+1}] \mathbf{1}_{\{t < \tau_{n+1}\}} = \tilde{\xi} \mathbf{1}_{\{t < \tau_{n+1}\}}$ for a \mathcal{F}_t^n -measurable variable $\tilde{\xi}$. In particular, the left hand side is also \mathcal{F}_t^n -measurable. Hence $\mathbb{E}[\xi | \mathcal{F}_t^{n+1}] \mathbf{1}_{\{t < \tau_{n+1}\}} = \mathbb{E}[\mathbb{E}[\xi | \mathcal{F}_t^{n+1}] \mathbf{1}_{\{t < \tau_{n+1}\}} | \mathcal{F}_t^n] = \mathbb{E}[\xi | \mathcal{F}_t^n] \mathbf{1}_{\{t < \tau_{n+1}\}}$. Similarly, since \mathbb{F}^{n+1} and \mathbb{G}^{n+1} coincide on $\llbracket \tau_{n+1}, +\infty \rrbracket$, we have $\mathbb{E}[\xi | \mathcal{G}_t^{n+1}] \mathbf{1}_{\{\tau_{n+1} \leq t\}} = \hat{\xi} \mathbf{1}_{\{\tau_{n+1} \leq t\}}$ for a \mathcal{F}_t^{n+1} -measurable variable $\hat{\xi}$. In particular, the left hand side is \mathcal{F}_t^{n+1} -measurable. Hence $\mathbb{E}[\xi | \mathcal{G}_t^{n+1}] \mathbf{1}_{\{\tau_{n+1} \leq t\}} = \mathbb{E}[\mathbb{E}[\xi | \mathcal{G}_t^{n+1}] \mathbf{1}_{\{\tau_{n+1} \leq t\}} | \mathcal{F}_t^{n+1}] = \mathbb{E}[\xi | \mathcal{F}_t^{n+1}] \mathbf{1}_{\{\tau_{n+1} \leq t\}}$.

Let $t \geq 0$ such that $\sum_n \mathbb{P}(\tau_n \leq t < \tau_{n+1}) = 1$. We have, since \mathbb{F}^∞ and \mathbb{F}^n coincide on $\llbracket 0, \tau_{n+1} \rrbracket$, using the same arguments as before,

$$\mathbb{E}[\xi | \mathcal{F}_t^\infty] = \sum_n \mathbb{E}[\xi | \mathcal{F}_t^\infty] \mathbf{1}_{\{\tau_n \leq t < \tau_{n+1}\}} = \sum_n \mathbb{E}[\xi | \mathcal{F}_t^n] \mathbf{1}_{\{\tau_n \leq t < \tau_{n+1}\}}.$$

3. We prove by induction that \mathbb{F}^n is right-continuous. Since \mathbb{F}^0 is the augmented Brownian filtration, the result is true for $n = 0$.

Assume now that $\mathbb{F}^{n-1}, n \geq 1$, is right-continuous. Let $t \geq 0, \xi \in L^1(\mathcal{F}_\infty^n)$ and $(t_m)_{m \geq 0}$ such that $t_m \geq t_{m+1}$ and $\lim_m t_m = t$. We have, using the previous point

and the right-continuity of \mathbb{F}^{n-1} and \mathbb{G}^n :

$$\begin{aligned}
\mathbb{E}[\xi | \mathcal{F}_{t^+}^n] &= \lim_m \mathbb{E}[\xi | \mathcal{F}_{t_m}^n] \\
&= \lim_m \mathbb{E}[\xi | \mathcal{F}_{t_m}^n] \mathbf{1}_{\{t_m < \tau_n\}} + \mathbb{E}[\xi | \mathcal{F}_{t_m}^n] \mathbf{1}_{\{\tau_n \leq t_m\}} \\
&= \lim_m \mathbb{E}[\xi | \mathcal{F}_{t_m}^{n-1}] \mathbf{1}_{\{t_m < \tau_n\}} + \mathbb{E}[\xi | \mathcal{G}_{t_m}^n] \mathbf{1}_{\{\tau_n \leq t_m\}} \\
&= \mathbb{E}[\xi | \mathcal{F}_t^{n-1}] \mathbf{1}_{\{t < \tau_n\}} + \mathbb{E}[\xi | \mathcal{G}_t^n] \mathbf{1}_{\{\tau_n \leq t\}} \\
&= \mathbb{E}[\xi | \mathcal{F}_t^n].
\end{aligned}$$

4. Let $t < T, \xi \in L^1(\mathcal{F}_\infty^\infty), (t_m)_{m \geq 0}$ such that $T > t_m > t_{m+1}$ and $\lim_m t_m = t$. We have, by Lévy's Theorem and the first point,

$$\begin{aligned}
\mathbb{E}[\xi | \mathcal{F}_{t^+}^\infty] &= \lim_m \mathbb{E}[\xi | \mathcal{F}_{t_m}^\infty] = \lim_m \sum_{n=0}^{+\infty} \mathbb{E}[\xi | \mathcal{F}_{t_m}^\infty] \mathbf{1}_{\{\tau_n \leq t_m < \tau_{n+1}\}} \\
&= \lim_m \sum_{n=0}^{+\infty} \mathbb{E}[\xi | \mathcal{F}_{t_m}^n] \mathbf{1}_{\{\tau_n \leq t_m < \tau_{n+1}\}}.
\end{aligned}$$

Fix $\omega \in \Omega$. We have that $t_m < T < \tau_{N(\omega)+1}(\omega)$ with $N(\omega) := N_T^\phi(\omega)$ assumed to be almost-surely finite, hence

$$\begin{aligned}
\mathbb{E}[\xi | \mathcal{F}_{t^+}^\infty](\omega) &= \lim_m \sum_{n=0}^{+\infty} \mathbb{E}[\xi | \mathcal{F}_{t_m}^n](\omega) \mathbf{1}_{\{\tau_n(\omega) \leq t_m < \tau_{n+1}(\omega)\}} \\
&= \lim_m \sum_{n=0}^{N(\omega)+1} \mathbb{E}[\xi | \mathcal{F}_{t_m}^n](\omega) \mathbf{1}_{\{\tau_n(\omega) \leq t_m < \tau_{n+1}(\omega)\}} \\
&= \sum_{n=0}^{N(\omega)+1} \lim_m \mathbb{E}[\xi | \mathcal{F}_{t_m}^n](\omega) \mathbf{1}_{\{\tau_n(\omega) \leq t_m < \tau_{n+1}(\omega)\}} \\
&= \sum_{n=0}^{+\infty} \lim_m \mathbb{E}[\xi | \mathcal{F}_{t_m}^n](\omega) \mathbf{1}_{\{\tau_n(\omega) \leq t_m < \tau_{n+1}(\omega)\}}.
\end{aligned}$$

Finally using the right-continuity of each \mathbb{F}^n , we get

$$\mathbb{E}[\xi | \mathcal{F}_{t^+}^\infty] = \sum_{n=0}^{+\infty} \lim_m \mathbb{E}[\xi | \mathcal{F}_{t_m}^n] \mathbf{1}_{\{\tau_n \leq t_m < \tau_{n+1}\}} = \sum_{n=0}^{+\infty} \mathbb{E}[\xi | \mathcal{F}_t^n] \mathbf{1}_{\{\tau_n \leq t < \tau_{n+1}\}} = \mathbb{E}[\xi | \mathcal{F}_t^\infty],$$

which proves that \mathbb{F}^∞ is right-continuous on $[0, T]$. \square

Lemma A.2. Let $0 \leq n < +\infty$ and $\xi \in L^1(\mathcal{F}_\infty^n)$. Let σ be a \mathbb{F}^n -stopping time. We have:

$$\mathbb{E}[\xi | \mathcal{F}_\sigma^{n+1}] = \mathbb{E}[\xi | \mathcal{F}_\sigma^n]. \tag{A.6}$$

Proof. Assume first that $\sigma = s$ is deterministic.

Let $\tilde{\xi} = \psi(\chi, \mathfrak{U}_{n+1} \mathbf{1}_{\{\tau_{n+1} \leq s\}})$ be a \mathcal{F}_s^{n+1} -measurable bounded variable, where χ is \mathcal{F}_s^n -measurable and ψ is deterministic and bounded measurable. We need to show

$$\mathbb{E}[\xi \tilde{\xi}] = \mathbb{E}[\mathbb{E}[\xi | \mathcal{F}_s^n] \tilde{\xi}].$$

We have, with $\hat{\psi}(y) := \int \psi(y, x) \mathbb{P}_{\mathfrak{U}_{n+1}}(dx) = \int_0^1 \psi(y, x) dx$ (as $\mathbb{P}_{\mathfrak{U}_{n+1}}$, the law of \mathfrak{U}_{n+1} , is the uniform distribution on $[0, 1]$ by definition),

$$\begin{aligned} \mathbb{E}[\mathbb{E}[\xi | \mathcal{F}_s^n] \psi(\chi, \mathfrak{U}_{n+1} \mathbf{1}_{\{\tau_{n+1} \leq s\}})] &= \mathbb{E}[\mathbb{E}[\xi | \mathcal{F}_s^n] \psi(\chi, 0) \mathbf{1}_{\{s < \tau_{n+1}\}}] + \mathbb{E}[\mathbb{E}[\xi | \mathcal{F}_s^n] \psi(\chi, \mathfrak{U}_{n+1}) \mathbf{1}_{\{\tau_{n+1} \leq s\}}] \\ &= \mathbb{E}[\xi \psi(\chi, 0) \mathbf{1}_{\{s < \tau_{n+1}\}}] + \mathbb{E}[\xi \hat{\psi}(\chi) \mathbf{1}_{\{\tau_{n+1} \leq s\}}], \end{aligned}$$

and the same computation with ξ instead of $\mathbb{E}[\xi | \mathcal{F}_s^n]$ gives the same result.

Let σ be a \mathbb{F}^n -stopping time, and let $\xi_s = \mathbb{E}[\xi | \mathcal{F}_s^n] = \mathbb{E}[\xi | \mathcal{F}_s^{n+1}]$. Since \mathbb{F}^n (or \mathbb{F}^{n+1}) is right-continuous, there exists a right-continuous modification of $(\xi_s)_{s \geq 0}$. Applying Doob's Theorem twice gives $\xi_\sigma = \mathbb{E}[\xi | \mathcal{F}_\sigma^n]$ and $\xi_\sigma = \mathbb{E}[\xi | \mathcal{F}_\sigma^{n+1}]$, hence we get the result. \square

We are now in position to prove an Integral Representation Theorem in the filtration \mathbb{F}^n , for all $0 \leq n < +\infty$.

Proposition A.2. *Let $0 \leq n < +\infty$ and $\xi \in L^2(\mathcal{F}_T^n)$. Then there exists a process $\psi \in \mathbb{H}^2(\mathbb{F}^n)$ such that*

$$\xi = \mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_n}^n] + \int_{T \wedge \tau_n}^T \psi_s dW_s.$$

Proof. We prove the theorem by induction on $n \geq 0$, following ideas from [2]. The case $n = 0$ is the usual Martingale Representation Theorem in the augmented Brownian filtration \mathbb{F}^0 .

Assume now that the statement is true for all $\xi \in L^2(\mathcal{F}_T^{n-1})$ ($n \geq 1$). Let $\xi \in L^2(\mathcal{F}_T^n)$. Since $\mathcal{F}_T^n = \mathcal{F}_T^{n-1} \vee \sigma(\mathfrak{U}_n \mathbf{1}_{\{\tau_n \leq T\}})$, we get that $\xi = \lim_{m \rightarrow \infty} \xi_m$ in $L^2(\mathcal{F}_T^n)$, with $\xi_m = \sum_{i=1}^{l_m} \chi_m^i \zeta_m^i$ and $(\chi_m^i, \zeta_m^i) \in L^\infty(\mathcal{F}_T^{n-1}) \times L^\infty(\sigma(\mathfrak{U}_n \mathbf{1}_{\{\tau_n \leq T\}}))$ for all $m \geq 0$ and $1 \leq i \leq l_m$. By induction, there exist \mathbb{F}^{n-1} -predictable processes $\psi^{i,m}$ such that

$$\chi_m^i = \mathbb{E}[\chi_m^i | \mathcal{F}_{T \wedge \tau_{n-1}}^{n-1}] + \int_{T \wedge \tau_{n-1}}^T \psi_s^{i,m} dW_s.$$

Since τ_n is a \mathbb{F}^{n-1} -stopping time with $\tau_n \geq \tau_{n-1}$ and $\int_0^T \psi_s^{i,m} dW_s$ is a square integrable \mathbb{F}^{n-1} -martingale, we get

$$\begin{aligned} \mathbb{E}[\chi_m^i | \mathcal{F}_{T \wedge \tau_n}^{n-1}] &= \mathbb{E}\left[\mathbb{E}[\chi_m^i | \mathcal{F}_{T \wedge \tau_{n-1}}^{n-1}] + \int_{T \wedge \tau_{n-1}}^{T \wedge \tau_n} \psi_s^{i,m} dW_s + \int_{T \wedge \tau_n}^T \psi_s^{i,m} dW_s \middle| \mathcal{F}_{T \wedge \tau_n}^{n-1}\right] \\ &= \mathbb{E}[\chi_m^i | \mathcal{F}_{T \wedge \tau_{n-1}}^{n-1}] + \int_{T \wedge \tau_{n-1}}^{T \wedge \tau_n} \psi_s^{i,m} dW_s, \end{aligned}$$

and the two previous relation give

$$\chi_m^i = \mathbb{E}[\chi_m^i | \mathcal{F}_{T \wedge \tau_n}^{n-1}] + \int_{T \wedge \tau_n}^T \psi_s^{i,m} dW_s.$$

Since $\zeta_m^i \in L^\infty(\sigma(\mathcal{U}_n \mathbf{1}_{\{\tau_n \leq T\}})) \subset L^2(\mathcal{F}_{T \wedge \tau_n}^n)$, we get

$$\zeta_m^i \int_{T \wedge \tau_n}^T \psi_s^{i,m} dW_s = \int_{T \wedge \tau_n}^T \zeta_m^i \psi_s^{i,m} dW_s.$$

In addition, since $\chi_m^i \in L^\infty(\mathcal{F}_T^{n-1}) \subset L^1(\mathcal{F}_\infty^{n-1})$ and $T \wedge \tau_n$ is a \mathbb{F}^{n-1} -stopping time, Lemma A.2 gives $\mathbb{E}[\chi_m^i | \mathcal{F}_{T \wedge \tau_n}^n] = \mathbb{E}[\chi_m^i | \mathcal{F}_{T \wedge \tau_n}^{n-1}]$. Since $\zeta_m^i \in L^2(\mathcal{F}_{T \wedge \tau_n}^n)$, we get

$$\zeta_m^i \mathbb{E}[\chi_m^i | \mathcal{F}_{T \wedge \tau_n}^{n-1}] = \zeta_m^i \mathbb{E}[\chi_m^i | \mathcal{F}_{T \wedge \tau_n}^n] = \mathbb{E}[\chi_m^i \zeta_m^i | \mathcal{F}_{T \wedge \tau_n}^n].$$

Combining the three last equalities and the definition of ξ_m , we obtain

$$\begin{aligned} \xi_m &= \sum_{i=1}^{l_m} \chi_m^i \zeta_m^i \\ &= \sum_{i=1}^{l_m} \mathbb{E}[\chi_m^i \zeta_m^i | \mathcal{F}_{T \wedge \tau_n}^n] + \sum_{i=1}^{l_m} \int_{T \wedge \tau_n}^T \zeta_m^i \psi_s^{i,m} dW_s \\ &= \mathbb{E}[\xi_m | \mathcal{F}_{T \wedge \tau_n}^n] + \int_{T \wedge \tau_n}^T \psi_s^m dW_s, \end{aligned}$$

where $\psi^m := \sum_{i=1}^{l_m} \zeta_m^i \psi^{i,m} \in \mathbb{H}^2(\mathbb{F}^n)$.

Finally, since $\xi_m \rightarrow \xi$ in $L^2(\mathcal{F}_T^n)$, we get that $\mathbb{E}[\xi_m | \mathcal{F}_{T \wedge \tau_n}^n] \rightarrow \mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_n}^n]$ in $L^2(\mathcal{F}_T^n)$, hence $\int_{T \wedge \tau_n}^T \psi_s^m dW_s$ converges to a limit $\int_{T \wedge \tau_n}^T \psi_s dW_s$ for a process $\psi \in \mathbb{H}^2(\mathbb{F}^n)$. \square

Theorem A.2. *Let $0 \leq n < +\infty$ and $\xi \in L^2(\mathcal{F}_T^n)$. There exist some processes $\psi^k \in \mathbb{H}^2(\mathbb{F}^k)$, for all $0 \leq k \leq n$, such that:*

$$\begin{aligned} \xi &= \mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_0}^0] + \sum_{k=0}^{n-1} \int_{T \wedge \tau_k}^{T \wedge \tau_{k+1}} \psi_s^k dW_s + \int_{T \wedge \tau_n}^T \psi_s^n dW_s \\ &\quad + \sum_{k=0}^{n-1} \left(\mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_{k+1}}^{k+1}] - \mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_{k+1}}^k] \right) \\ &= \mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_0}^0] + \int_{T \wedge \tau_0}^T \Psi_s^n dW_s + \sum_{k=0}^{n-1} \left(\mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_{k+1}}^{k+1}] - \mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_{k+1}}^k] \right), \end{aligned}$$

with $\Psi_t^n := \sum_{k=0}^{n-1} \psi_t^k \mathbf{1}_{\{T \wedge \tau_k < t \leq T \wedge \tau_{k+1}\}} + \psi_t^n \mathbf{1}_{\{T \wedge \tau_n < t \leq T\}}$.

Proof. This is an immediate consequence of the previous theorem. \square

Last, we extend this theorem to obtain an Integral Representation Theorem in \mathbb{F}^∞ . We now fix $\xi \in L^2(\mathcal{F}_T^\infty)$ and consider the filtration $(\mathcal{F}_T^n)_{n \geq 0}$. By Lévy's Theorem, since $\mathcal{F}_T^\infty = \bigvee_{n \geq 0} \mathcal{F}_T^n$, we get

$$\mathbb{E}[\xi | \mathcal{F}_T^n] \rightarrow_{n \rightarrow \infty} \mathbb{E}[\xi | \mathcal{F}_T^\infty] = \xi, \text{ a.s.} \quad (\text{A.7})$$

For all $n \geq 0$, using Theorem A.2, we can write:

$$\begin{aligned} \mathbb{E}[\xi | \mathcal{F}_T^n] &= \mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_0}^0] + \sum_{k=0}^{n-1} \int_{T \wedge \tau_k}^{T \wedge \tau_{k+1}} \psi_s^{n,k} dW_s + \int_{T \wedge \tau_n}^T \psi_s^{n,n} dW_s \\ &\quad + \sum_{k=0}^{n-1} \left(\mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_{k+1}}^{k+1}] - \mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_{k+1}}^k] \right). \end{aligned}$$

Lemma A.3. *We have $\psi^{n,k} = \psi^{k,k}$ on $[T \wedge \tau_k, T \wedge \tau_{k+1})$, for all $n \geq k$.*

Proof. It follows easily by induction, comparing $\mathbb{E}[\xi | \mathcal{F}_T^k]$ and $\mathbb{E}[\mathbb{E}[\xi | \mathcal{F}_T^n] | \mathcal{F}_T^k]$ and using Itô's isometry. □

For all $n \geq 0$, we define $\psi^n := \psi^{n,n}$. Thus we have, for all $n \geq 0$,

$$\begin{aligned} \mathbb{E}[\xi | \mathcal{F}_T^n] &= \mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_0}^0] + \sum_{k=0}^{n-1} \int_{T \wedge \tau_k}^{T \wedge \tau_{k+1}} \psi_s^k dW_s + \int_{T \wedge \tau_n}^T \psi_s^n dW_s \\ &\quad + \sum_{k=0}^{n-1} \left(\mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_{k+1}}^{k+1}] - \mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_{k+1}}^k] \right). \end{aligned}$$

We set, for $0 \leq s \leq T$,

$$\begin{aligned} \Psi_s &= \sum_{k=0}^{+\infty} \psi_s^k \mathbf{1}_{\{T \wedge \tau_k \leq s < T \wedge \tau_{k+1}\}}, \\ \Psi_s^n &= \Psi_s \mathbf{1}_{\{s \leq T \wedge \tau_{n+1}\}} + \psi_s^n \mathbf{1}_{\{T \wedge \tau_{n+1} < s\}}, \text{ and} \\ \Delta_s^k &:= \mathbb{E}[\xi | \mathcal{F}_{s \wedge \tau_{k+1}}^{k+1}] - \mathbb{E}[\xi | \mathcal{F}_{s \wedge \tau_{k+1}}^k], \end{aligned}$$

so that

$$\mathbb{E}[\xi | \mathcal{F}_T^n] = \mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_0}^0] + \int_{T \wedge \tau_0}^T \Psi_s^n dW_s + \sum_{k=0}^{n-1} \Delta_T^k.$$

Theorem A.3 (Integral Representation Theorem for \mathbb{F}^∞). *Suppose that Assumption 2.1-i) holds. For $\xi \in L^2(\mathcal{F}_T^\infty)$, we have*

$$\xi = \mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_0}^0] + \int_{T \wedge \tau_0}^T \Psi_s dW_s + \sum_{k=0}^{+\infty} \Delta_T^k.$$

Proof. By definition (2.2) of $N := N_T^\phi$, we have $T < \tau_{n+1}$ on $\{n \geq N\}$. Thus,

$$\mathbf{1}_{\{N \leq n\}} \int_{T \wedge \tau_0}^T \Psi_s^n dW_s = \left(\int_{T \wedge \tau_0}^{T \wedge \tau_{n+1}} \Psi_s dW_s + \int_{T \wedge \tau_{n+1}}^T \psi_s^n dW_s \right) \mathbf{1}_{\{N \leq n\}} = \mathbf{1}_{\{N \leq n\}} \int_{T \wedge \tau_0}^T \Psi_s dW_s.$$

Moreover, if $k \geq n$, we have, since $T \wedge \tau_{k+1} = T$,

$$\begin{aligned} \Delta_T^k \mathbf{1}_{\{N \leq n\}} &= \left(\mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_{k+1}}^{k+1}] - \mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_{k+1}}^k] \right) \mathbf{1}_{\{N \leq n\}} \\ &= \left(\mathbb{E}[\xi | \mathcal{F}_T^{k+1}] - \mathbb{E}[\xi | \mathcal{F}_T^k] \right) \mathbf{1}_{\{N \leq n\}}. \end{aligned}$$

Applying (A.5) to $\chi = \mathbb{E}[\xi | \mathcal{F}_T^{k+1}]$, we get

$$\chi = \mathbb{E}[\chi | \mathcal{F}_T^{k+1}] = \mathbb{E}[\chi | \mathcal{F}_T^k] \mathbf{1}_{\{T < \tau_{k+1}\}} + \mathbb{E}[\chi | \mathcal{G}_T^{k+1}] \mathbf{1}_{\{\tau_{k+1} \leq T\}}.$$

Since $T < \tau_{n+1} \leq \tau_{k+1}$ on $\{N \leq n\}$, we finally obtain

$$\mathbb{E}[\xi | \mathcal{F}_T^{k+1}] \mathbf{1}_{\{N \leq n\}} = \chi \mathbf{1}_{\{N \leq n\}} = \mathbb{E}[\chi | \mathcal{F}_T^k] \mathbf{1}_{\{N \leq n\}} = \mathbb{E}[\xi | \mathcal{F}_T^k] \mathbf{1}_{\{N \leq n\}},$$

which gives $\Delta_T^k \mathbf{1}_{\{N \leq n\}} = 0$. Thus:

$$\begin{aligned} \mathbb{E}[\xi | \mathcal{F}_T^n] \mathbf{1}_{\{N \leq n\}} &= \left(\mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_0}^0] + \int_{T \wedge \tau_0}^T \Psi_s^n dW_s + \sum_{k=0}^{n-1} \Delta_T^k \right) \mathbf{1}_{\{N \leq n\}} \\ &= \left(\mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_0}^0] + \int_{T \wedge \tau_0}^T \Psi_s dW_s + \sum_{k=0}^{+\infty} \Delta_T^k \right) \mathbf{1}_{\{N \leq n\}}. \end{aligned}$$

Since N is assumed to be finite almost surely, we have $\mathbf{1}_{\{N \leq n\}} \rightarrow_{n \rightarrow \infty} 1$ almost surely. Sending n to infinity in the previous equation, using (A.7), we finally obtain

$$\xi = \mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_0}^0] + \int_{T \wedge \tau_0}^T \Psi_s dW_s + \sum_{k=0}^{+\infty} \Delta_T^k.$$

□

Remark A.1. Using the classical Martingale Representation Theorem in the Brownian filtration, there exists a \mathbb{F}^0 -predictable process ψ^{-1} such that

$$\mathbb{E}[\xi | \mathcal{F}_{T \wedge \tau_0}^0] = \mathbb{E}[\xi] + \int_0^{T \wedge \tau_0} \psi_s^{-1} dW_s.$$

Hence one easily obtains the representation, for $\xi \in L^2(\mathcal{F}_T^n)$ with $n \in \mathbb{N} \cup \{+\infty\}$

$$\xi = \mathbb{E}[\xi] + \int_0^T \tilde{\Psi}_s dW_s + \sum_{k=0}^{n-1} \Delta_T^k, \quad (\text{A.8})$$

where $\tilde{\Psi} := \psi^{-1} \mathbf{1}_{\{0 \leq \cdot \leq T \wedge \tau_0\}} + \Psi$.

Remark A.2. The process $\sum_{k=0}^{+\infty} \Delta^k$ is a purely discontinuous martingale and $\int_0^T \tilde{\Psi}_s dW_s$ is a continuous one, so computing (co)variations gives,

$$\left[\int_0^\cdot \tilde{\Psi}_s dW_s, \sum_{k=0}^{+\infty} \Delta^k \right]_t = 0, \quad (\text{A.9})$$

$$\left[\int_0^\cdot \tilde{\Psi}_s dW_s \right]_t = \int_0^t \Psi_s^2 ds, \quad (\text{A.10})$$

$$\left[\sum_{k=0}^{+\infty} \Delta^k \right]_t = \sum_{\tau_{k+1} \leq t} |\Delta_t^k|^2. \quad (\text{A.11})$$

In particular, the martingales $\int_0^\cdot \tilde{\Psi}_s dW_s$ and $\sum_k \Delta^k$ are orthogonal.

A.2.2 Backward Stochastic Differential Equations

Using the results from the previous section, in particular that the filtrations $\mathbb{F}^n, n \in \mathbb{N} \cup \{+\infty\}$, are right-continuous and that a Martingale Representation Theorem is available in these filtrations, we recall that (switched) BSDEs with Lipschitz driver have a unique solution and that a comparison theorem is available.

Let $n \in \mathbb{N} \cup \{+\infty\}$. Let ξ be a \mathcal{F}_T^n -measurable random variable and $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times \kappa} \rightarrow \mathbb{R}^d$ a measurable function. We assume here that ξ and f are standard parameters [12, Section 5]:

- $\xi \in L^2(\mathcal{F}_T^n)$,
- $f(\cdot, 0, 0) \in \mathbb{H}_d^2(\mathbb{F}^n)$,
- There exists $C > 0$ such that

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq C (|y_1 - y_2| + |z_1 - z_2|).$$

Under these hypothesis, since \mathbb{F}^n is right-continuous (see Proposition A.1), one can prove (see [12, Theorem 5.1]):

Theorem A.4. *There exists a unique solution $(Y, Z, M) \in \mathbb{S}_d^2(\mathbb{F}^n) \times \mathbb{H}_{d \times \kappa}^2(\mathbb{F}^n) \times \mathbb{H}_d^2(\mathbb{F}^n)$ such that M is a martingale with $M_0 = 0$, orthogonal to the Brownian motion, and satisfying*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s - \int_t^T dM_s.$$

Using Theorem A.2 and Theorem A.3, M is of the form

$$M_t = \mathbb{E}[\xi] + \sum_{k=0}^{n-1} \left(\mathbb{E} \left[\xi \middle| \mathcal{F}_{t \wedge \tau_{k+1}}^{k+1} \right] - \mathbb{E} \left[\xi \middle| \mathcal{F}_{t \wedge \tau_{k+1}}^k \right] \right),$$

where n corresponds to the underlying filtration \mathbb{F}^n .

When $d = 1$, one can easily deal with linear BSDEs in \mathbb{F}^n , and the specific form of its solutions allows to prove a Comparison Theorem. The proofs follow closely [12, Theorem 2.2].

Theorem A.5. *Let (b, c) be a bounded $(\mathbb{R} \times \mathbb{R}^\kappa)$ -valued predictable process and let $a \in \mathbb{H}^2(\mathbb{F}^n)$. Let $\xi \in L^2(\mathcal{F}_T^n)$ and let $(Y, Z, M) \in \mathbb{S}^2(\mathbb{F}^n) \times \mathbb{H}_\kappa^2(\mathbb{F}^n) \times \mathbb{H}^2(\mathbb{F}^n)$ be the unique solution to*

$$Y_t = \xi + \int_t^T (a_s Y_s + b_s Z_s + c_s) ds - \int_t^T Z_s dW_s - \int_t^T dM_s.$$

Let $\Gamma \in \mathbb{H}^2(\mathbb{F}^n)$ the solution to

$$\Gamma_t = 1 + \int_0^t \Gamma_s a_s ds + \int_0^t \Gamma_s b_s dW_s.$$

Then, for all $t \in [0, T]$, one has almost surely,

$$Y_t = \Gamma_t^{-1} \mathbb{E} \left[\Gamma_T \xi + \int_t^T \Gamma_s c_s ds \middle| \mathcal{F}_t^n \right].$$

Proof. We fix $t \in [0, T]$ and we apply Itô's formula to the process $Y_t \Gamma_t$:

$$d(Y_t \Gamma_t) = Y_{t-} d\Gamma_t + \Gamma_{t-} dY_t + d[Y, \Gamma]_t.$$

Since Γ is continuous, we get $[Y, \Gamma]_t = \langle Y^c, \Gamma^c \rangle_t + \sum_{s \leq t} (\Delta Y_s)(\Delta \Gamma_s) = \langle Y^c, \Gamma \rangle_t$, thus,

$$d(Y_t \Gamma_t) = \Gamma_t (b_t Y_t + Z_t) dW_t + \Gamma_t dM_t - \Gamma_t c_t dt.$$

We define a martingale by $N_t = \int_0^t \Gamma_s (b_s Y_s + Z_s) dW_s + \int_0^t \Gamma_s dM_s$, and the previous equality gives

$$Y_T \Gamma_T = Y_t \Gamma_t - \int_t^T \Gamma_s c_s ds + N_T - N_t.$$

Taking conditional expectation with respect to \mathcal{F}_t^n on both sides gives the result. \square

Theorem A.6. *Let (ξ, f) and (ξ', f') two standard parameters. Let $(Y, Z, M) \in \mathbb{S}^2(\mathbb{F}^n) \times \mathbb{H}_\kappa^2(\mathbb{F}^n) \times \mathbb{H}^2(\mathbb{F}^n)$ (resp. (Y', Z', M')) the solution associated with (ξ, f) (resp. (ξ', f')). Assume that*

- $\xi \geq \xi'$ a.s.,
- $f(\cdot, Y', Z') \geq f'(\cdot, Y', Z') d\mathbb{P} \times dt$ a.e.

Then $Y_t \geq Y'_t$ almost surely for all $t \in [0, T]$.

Proof. Since f is Lipschitz, we consider the bounded processes a, b and c defined, for $t \in [0, T]$, by:

$$\begin{aligned} a_t &= \frac{f(t, Y_t, Z_t) - f(t, Y'_t, Z_t)}{Y_t - Y'_t} \mathbf{1}_{\{Y_t \neq Y'_t\}}, \\ b_t &= \frac{(f(t, Y'_t, Z_t) - f(t, Y'_t, Z'_t))(Z_t - Z'_t)^\top}{|Z_t - Z'_t|^2} \mathbf{1}_{\{Z_t \neq Z'_t\}}, \\ c_t &= f(t, Y'_t, Z'_t) - f'(t, Y'_t, Z'_t), \end{aligned}$$

Setting $\delta Y_t = Y_t - Y'_t$, $\delta Z_t = Z_t - Z'_t$ and $\delta M_t = M_t - M'_t$, we observe that $(\delta Y, \delta Z, \delta M)$ is the solution to the following linear BSDE:

$$\delta Y_t = \delta Y_T + \int_t^T (a_s \delta Y_s + b_s \delta Z_s + c_s) ds - \int_t^T \delta Z_s dW_s - \int_t^T d\delta M_s. \quad (\text{A.12})$$

Using the previous Theorem, we get $Y_t = \Gamma_t^{-1} \mathbb{E} \left[\delta Y_T \Gamma_T + \int_t^T \Gamma_s c_s ds \middle| \mathcal{F}_t^n \right]$. By definition, Γ is a positive process, and δY_t and c are non negative by hypothesis, hence $Y_t \geq 0$. \square

Remark A.3. In Section 2, see (2.4) and (2.35), we apply Theorems A.4 and A.6 to more general BSDEs in \mathbb{F}^n , $n \in \mathbb{N} \cup \{+\infty\}$, of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s - \int_t^T dM_s - \int_t^T dA_s, \quad (\text{A.13})$$

where ξ and f are standard parameters and where A is a given non-decreasing process with $A_T - A_0 \in L^2(\mathcal{F}_T^n)$. Let $(\tilde{\xi} := \xi + A_T, \tilde{f}(\omega, t, y, z) := f(\omega, t, y - A_t(\omega), z))$, which is easily seen to be standard parameters. Then Theorem A.4 implies that there exists a unique solution $(\tilde{Y}, \tilde{Z}, \tilde{M}) \in \mathbb{S}^2(\mathbb{F}^n) \times \mathbb{H}_\kappa^2(\mathbb{F}^n) \times \mathbb{H}^2(\mathbb{F}^n)$ to

$$\tilde{Y}_t = \tilde{\xi} + \int_t^T \tilde{f}(s, \tilde{Y}_s, \tilde{Z}_s) ds - \int_t^T \tilde{Z}_s dW_s - \int_t^T d\tilde{M}_s, \quad (\text{A.14})$$

and setting $(Y, Z, M) := (\tilde{Y} - A, \tilde{Z}, \tilde{M})$, we easily see that it is a solution to (A.13). In addition, if (Y^i, Z^i, M^i) ($i = 1, 2$) are two solutions to (A.13), then $(\tilde{Y}^i, \tilde{Z}^i, \tilde{M}^i) := (Y^i + A, Z^i, M^i)$ are two solutions to (A.14) and the uniqueness result from Theorem A.4 gives $Z^1 = Z^2$, $M^1 = M^2$ and $Y^1 + A = Y^2 + A$, i.e. $Y^1 = Y^2$, so (A.13) admits a unique solution.

These considerations also allow to extend the comparison theorem to this setting, which is used in the proof of Theorem 2.1, see Section 2.3.3.

References

- [1] Anna Aksamit and Monique Jeanblanc. *Enlargement of filtration with finance in view*. Springer, 2017.

- [2] Jürgen Amendinger. Martingale representation theorems for initially enlarged filtrations. *Stochastic Processes and their Applications*, 89(1):101–116, 2000.
- [3] Philippe Biane. Polynomials associated with finite markov chains. In *In Memoriam Marc Yor-Séminaire de Probabilités XLVII*, pages 249–262. Springer, 2015.
- [4] René Carmona and Michael Ludkovski. Valuation of energy storage: An optimal switching approach. *Quantitative finance*, 10(4):359–374, 2010.
- [5] Jean-François Chassagneux, Romuald Elie, and Idris Kharroubi. A note on existence and uniqueness for solutions of multidimensional reflected BSDEs. *Electronic Communications in Probability*, 16:120–128, 2011.
- [6] Jean-François Chassagneux and Adrien Richou. Obliquely reflected backward stochastic differential equations. In *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, volume 56, pages 2868–2896. Institut Henri Poincaré, 2020.
- [7] Jakša Cvitanic and Ioannis Karatzas. Backward stochastic differential equations with reflection and Dynkin games. *The Annals of Probability*, 24(4):2024–2056, 1996.
- [8] Tiziano De Angelis, Giorgio Ferrari, and Saïd Hamadène. A note on a new existence result for reflected BSDEs with interconnected obstacles. *arXiv preprint arXiv:1710.02389*, 2017.
- [9] Claude Dellacherie, Servet Martinez, and Jaime San Martín. *Inverse M-matrices and ultrametric matrices*, volume 2118. Springer, 2014.
- [10] Boualem Djehiche, Saïd Hamadène, and Alexandre Popier. A finite horizon optimal multiple switching problem. *SIAM Journal on Control and Optimization*, 48(4):2751–2770, 2009.
- [11] Nicole El Karoui, Christophe Kapoudjian, Étienne Pardoux, Shige Peng, and Marie-Claire Quenez. Reflected solutions of backward SDEs, and related obstacle problems for PDEs. *the Annals of Probability*, pages 702–737, 1997.
- [12] Nicole El Karoui, Shige Peng, and Marie-Claire Quenez. Backward stochastic differential equations in finance. *Mathematical Finance*, 7(1):1–71, 1997.
- [13] Anne Gegout Petit and Étienne Pardoux. Equations différentielles stochastiques rétrogrades réfléchies dans un convexe. *Stochastics: An International Journal of Probability and Stochastic Processes*, 57(1-2):111–128, 1996.
- [14] Saïd Hamadène and Monique Jeanblanc. On the starting and stopping problem: application in reversible investments. *Mathematics of Operations Research*, 32(1):182–192, 2007.

- [15] Saïd Hamadène, Jean-Pierre Lepeltier, and Shige Peng. BSDEs with continuous coefficients and stochastic differential games. *Pitman Research Notes in Mathematics Series*, pages 115–128, 1997.
- [16] Saïd Hamadène and Jianfeng Zhang. Switching problem and related system of reflected backward SDEs. *Stochastic Processes and their applications*, 120(4):403–426, 2010.
- [17] Ying Hu and Shanjian Tang. Multi-dimensional BSDE with oblique reflection and optimal switching. *Probab. Theory Related Fields*, 147(1-2):89–121, 2010.
- [18] John G Kemeny and James Laurie Snell. *Finite Markov Chains: With a New Appendix" Generalization of a Fundamental Matrix"*. Springer, 1981.
- [19] Randall Martyr. Finite-horizon optimal multiple switching with signed switching costs. *Math. Oper. Res.*, 41(4):1432–1447, 2016.
- [20] James Robert Norris. *Markov chains*. Number 2. Cambridge university press, 1998.