Effects of Roots of Maximal Multiplicity on the Stability of Some Classes of Delay Differential-Algebraic Systems: The Lossless Propagation Case

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Abstract: It has been observed in several recent works that, for some classes of linear time-delay systems, spectral values of maximal multiplicity are dominant, a property known as multiplicity-induced-dominancy (MID). This paper starts the investigation of whether MID holds for delay differential-algebraic systems by considering a single-delay system composed of two scalar equations. After motivating this problem and recalling some recent results for retarded delay differential equations, we prove that the MID property holds for the delay differential-algebraic system of interest and present some applications.

Keywords: Time-delay equations, stability analysis, spectral methods, root assignment.

1. INTRODUCTION

Time delays are useful modeling tools in a wide range of scientific and technological domains such as biology, chemistry, economics, physics, or engineering. They may represent, for instance, the reaction time of an engineering system, the transfer time of material, energy, or information between parts of a system, the duration of a chemical reaction, or the duration of maturation processes in biology. Due to these applications and the challenging mathematical problems arising in their analysis, systems with time delays have been the subject of much attention by researchers in several fields, in particular since the 1950s and 1960s, such as, for instance, in Bellman and Cooke (1963); Halanay (1966); Pinney (1958). We refer to Diekmann et al. (1995); Gu et al. (2003); Hale and Verduyn Lunel (1993); Insperger and Stépán (2011); Michiels and Niculescu (2007); Stépán (1989) for details on time-delay systems and their applications.

This paper is interested in the analysis of stability and asymptotic behavior of systems with delays of the general form

$$\begin{cases} x'(t) = Ax(t) + \sum_{k=1}^{N} B_k y(t-\tau_k), \\ y(t) = Cx(t) + \sum_{k=1}^{N} D_k y(t-\tau_k), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^{d_x}$, $y(t) \in \mathbb{R}^{d_y}$, N, d_x, and d_y are positive integers, $\tau_1, \dots, \tau_N$ are positive delays, and, for $k \in \{1, \dots, N\}$, A, B_k, C, and D_k are matrices of appropriate dimensions. Systems such as (1) are *delay differential-algebraic systems* since they are written as a system of delay differential equations coupled with a system of algebraic equations with delays. They correspond to a particular class of delay differential-algebraic systems in which delays appear only in the y variable, which is the case in particular in models of lossless propagation phenomena (see Section 2 for an example).

Differential-algebraic systems have been extensively studied in the delay-free setting (see, e.g., Brenan et al. (1989); Coleman (1998); Griepentrog and März (1986); Kumar and Daoutidis (1999); Kunkel and Mehrmann (2006)). This kind of system arises naturally in several situations, such as in some electronic circuit models, in some control problems with constraints, or in the limiting behavior of singularly perturbed systems. Delay differential-algebraic systems have also been considered in the literature, arising in general from systems of transport partial differential equations representing some propagation phenomenon and coupled with static and dynamic boundary conditions (see, e.g., Niculescu (2001); Halanay and Rasvan (1997); Hale and Verduyn Lunel (1993); Brayton (1968)). More specific motivating examples are described in Section 2.

The stability analysis of time-delay systems has attracted much research effort and is an active field (see, e.g., Cooke and van den Driessche (1986); Gu et al. (2003); Michiels and Niculescu (2007); Olgac and Sipahi (2002); Sipahi et al. (2011)). Similarly to the delay-free situation, one may address the asymptotic behavior of a linear time-

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invariant time-delay system through spectral methods by considering the corresponding characteristic function, whose complex roots determine the asymptotic behavior of solutions of the system (see, e.g., Hale and Verduyn Lunel (1993); Michiels and Niculescu (2007); Mori et al. (1982)). The characteristic function of (1) is the function $\Delta : \mathbb{C} \to \mathbb{C}$ defined for $s \in \mathbb{C}$ by

$$\Delta(s) = \det\left(sE - \widehat{A} - \sum_{k=1}^{N} e^{-s\tau_k} \widehat{B}_k\right), \tag{2}$$

where $E, \widehat{A}, \widehat{B}_1, \dots, \widehat{B}_N$ are $(d_x + d_y) \times (d_x + d_y)$ matrices defined by blocks as

$$E = \begin{pmatrix} \mathrm{Id}_{d_x} & 0 \\ 0 & 0 \end{pmatrix}, \quad \widehat{A} = \begin{pmatrix} A & 0 \\ C & -\mathrm{Id}_{d_y} \end{pmatrix}, \quad \widehat{B}_k = \begin{pmatrix} 0 & B_k \\ 0 & D_k \end{pmatrix},$$

for $k \in \{1, \dots, N\}$. Notice that (1) can be rewritten in terms of these matrices as

$$E z'(t) = \widehat{A} z(t) + \sum_{k=1}^{N} \widehat{B}_k z(t - \tau_k),$$

where $z(t) = (x(t)^{\mathrm{T}}, y(t)^{\mathrm{T}})^{\mathrm{T}}$. Similarly to the delay-free case, the exponential behavior of (1) is determined by the *spectral abscissa* γ of Δ, defined by $\gamma = \sup\{\operatorname{Re} s \mid s \in \mathbb{C} \text{ and } \Delta(s) = 0\}$, and all solutions of (1) converge exponentially to 0 if and only if $\gamma < 0$.

The spectral abscissa of Δ is related to the notion of dominant root, defined as follows.

Definition 1. Let $Q : \mathbb{C} \to \mathbb{C}$ and $s_0 \in \mathbb{C}$ be such that $Q(s_0) = 0$. We say that s_0 is a *dominant* (respectively, *strictly dominant*) root of Q if, for every $s \in \mathbb{C} \setminus \{s_0\}$ such that $Q(s) = 0$, one has $\operatorname{Re} s \leq \operatorname{Re} s_0$ (respectively, $\operatorname{Re} s < \operatorname{Re} s_0$).

It follows immediately from the above definition that, if Q admits a dominant root s_0, then $\gamma = \operatorname{Re} s_0$, but dominant roots may not exist in general.

Functions of the form (2) are particular instances of quasipolynomials, whose definition is the following.

Definition 2. A *quasipolynomial* is an entire function Q which can be written under the form

$$Q(s) = \sum_{k=0}^{\ell} P_k(s) e^{\lambda_k s},$$

where ℓ is a positive integer, $\lambda_0, \dots, \lambda_\ell$ are pairwise distinct real numbers, and, for $k \in \{0, \dots, \ell\}$, P_k is a non-zero polynomial of degree d_k. The integer $D = \ell + \sum_{k=0}^{\ell} d_k$ is called the *degree* of Q.

The above definition of the degree of a quasipolynomial is motivated by a classical property, provided in (Pólya and Szegő, 1998, Problem 206.2) and known as the *Pólya–Szegő bound*, which implies that, given a quasipolynomial Q of degree $D \geq 0$, the multiplicity of any root of Q does not exceed D. Recent works such as Boussaada and Niculescu (2016a,b) have provided characterizations of multiple roots of quasipolynomials using approaches based on Birkhoff and Vandermonde matrices.

It has been recently remarked (see, e.g., Boussaada and Niculescu (2016b); Boussaada et al. (2018, 2020); Mazanti et al. (2020a,b)) that, for quasipolynomials coming from some systems with time-delays, real roots of maximal multiplicity are often dominant, a property usually referred to as *multiplicity-induced-dominancy* (MID for short). This property has been shown to hold, in particular, for scalar single-delay differential equations of retarded type (see Mazanti et al. (2020a)), a result we present and explain below in Section 3, and also extended for some systems to the case of complex roots of maximal multiplicity in Mazanti et al. (2020b).

One of the applications of the MID property is in the design of stabilizing feedback controllers for control systems with time delays, as in Boussaada et al. (2020). A major difficulty when addressing this question is that, except in degenerate situations, quasipolynomials have infinitely many roots, and one usually disposes only of finitely many parameters that can be chosen in the controller design. If, however, the MID property holds and these parameters are chosen in such a way as to guarantee the existence of a root of maximal multiplicity, then this root is dominant, and hence determines the asymptotic behavior of the system, allowing for stabilization if one chooses this root with negative real part.

The aim of this paper is to start the investigation of whether the MID property holds for delay differential-algebraic systems of the form (1). For this purpose, we restrict our attention to the first non-trivial situation, corresponding to the case $d_x = d_y = N = 1$ in which both equations in (1) are scalar and the system contains a single delay. This simple-looking, low-dimensional case illustrates many of the subtleties in the analysis of the MID property for delay differential-algebraic systems. Our main result for this system, Theorem 4, shows that the MID property does hold in this setting, providing necessary and sufficient conditions on the system parameters for having a real root of maximal multiplicity and characterizing further the other roots of the characteristic quasipolynomial when these conditions are satisfied.

The paper is organized as follows. Section 2 presents some examples of systems which can be put under the form (1). We then present, in Section 3, a previous result from Mazanti et al. (2020a) on the MID property for retarded delay differential equations, which can be seen as a particular case of (1) in which $N = 1$ and $D_1 = 0$. We briefly recall the strategy of its proof, which serves as inspiration for the analysis of the MID property for a delay differential-algebraic system with scalar unknowns and a single delay in Section 4. An application to one of the examples from Section 2 is provided in Section 5.

Notation. For a given complex number s, we denote by $\overline{s}$, $\operatorname{Re} s$, and $\operatorname{Im} s$ its complex conjugate, real part, and imaginary part, respectively. Given nonnegative integers n, k with $0 \leq k \leq n$, the notation $\binom{n}{k}$ represents the usual binomial coefficient $\frac{n!}{k!(n-k)!}$.

2. MOTIVATING EXAMPLES

In this section, we briefly describe two systems that can be modeled by delay differential-algebraic equations.

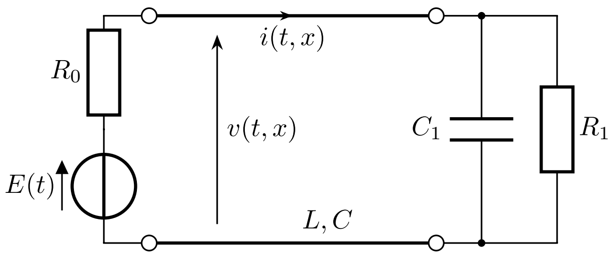

Fig. 1. Electrical circuit with a transmission line

2.1 Electrical circuit with a transmission line

Consider the electrical circuit from Figure 1, in which a voltage source $E(t)$ with internal resistance R_0 is connected, through a lossless transmission line of normalized length 1 and normalized characteristic impedance L and capacitance C, to a parallel association between a capacitor of capacitance C_1 and a resistor of resistance R_1. This electrical circuit can be described by the system

$$\begin{cases} \partial_x v(t,x) + L\partial_t i(t,x) = 0, & t \geq 0,\ x \in (0,1), \\ \partial_x i(t,x) + C\partial_t v(t,x) = 0, & t \geq 0,\ x \in (0,1), \\ v(t,0) = E(t) - R_0 i(t,0), & t \geq 0, \\ i(t,1) = C_1 \partial_t v(t,1) + \dfrac{1}{R_1} v(t,1), & t \geq 0, \end{cases} \tag{3}$$

with suitable initial conditions. Performing the classical change of variables into Riemann invariants

$$u_1(t,x) = \frac{1}{2}\left[v(t,x) + \sqrt{\frac{L}{C}} i(t,x)\right],$$

$$u_2(t,x) = \frac{1}{2}\left[v(t,x) - \sqrt{\frac{L}{C}} i(t,x)\right],$$

and setting $y_1(t) = v(t,1)$ and $y_2(t) = u_2(t,1)$, one verifies that (3) is rewritten as

$$\begin{cases} y_1'(t) = -\dfrac{1}{C_1}\left(\dfrac{1}{R_1} + \sqrt{\dfrac{C}{L}}\right) y_1(t) \\ \qquad - \dfrac{2}{C_1}\sqrt{\dfrac{C}{L}} \rho y_2(t-\tau) + \dfrac{2}{C_1}\sqrt{\dfrac{C}{L}} \dfrac{E(t-\tau/2)}{1 + R_0\sqrt{\frac{L}{C}}}, \\ y_2(t) = y_1(t) + \rho y_2(t-\tau) - \dfrac{E(t-\tau/2)}{1 + R_0\sqrt{\frac{L}{C}}}, \end{cases} \tag{4}$$

where $\rho = \frac{1 - R_0\sqrt{\frac{L}{C}}}{1 + R_0\sqrt{\frac{L}{C}}}$ is the reflection coefficient at the voltage source and $\tau = 2\sqrt{LC}$ represents the round-trip travel time along the transmission line. If one is interested in the stability analysis of (4) under no input, i.e., when $E(t) = 0$ for every t, then (4) reduces to a delay differential-algebraic system of the form (1) with $d_x = d_y = 1$ and $N = 1$. We refer the interested reader to Brayton (1968); Halanay and Rasvan (1997); Niculescu (2001) and references therein for more details on these kind of circuits and their applications, as well as for further examples of engineering systems which can be put under the form of a delay differential-algebraic system (1).

2.2 Delayed output feedback for proper control systems

Consider the linear control system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{5}$$

where $x(t) \in \mathbb{R}^{d_x}$ is the state, $u(t) \in \mathbb{R}^{d_u}$ is the control, $y(t) \in \mathbb{R}^{d_y}$ is the output, d_x, d_u, and d_y are positive integers, and A, B, C, and D are matrices of appropriate dimensions. We consider the problem of stabilizing (5) by a delayed output feedback of the form $u(t) = Ky(t-\tau)$, where $\tau > 0$ is the delay and K is a matrix of appropriate dimension. The closed-loop system is then

$$\begin{cases} \dot{x}(t) = Ax(t) + BKy(t-\tau), \\ y(t) = Cx(t) + DKy(t-\tau), \end{cases} \tag{6}$$

which is under the form (1) with $N = 1$ delay.

3. PRELIMINARY RESULTS ON RETARDED DELAY DIFFERENTIAL EQUATIONS

Consider the delay differential-algebraic system (1) in the particular case $N = 1$ and $D_1 = 0$. Letting $\tau = \tau_1$ and $B = B_1 C$, this system can be written under the form

$$x'(t) = Ax(t) + Bx(t-\tau). \tag{7}$$

System (7) is a system of delay differential equations said to be of *retarded type* since the derivative of highest order only appears in the non-delayed term $x'(t)$.

An important particular case which can be put into the form (7) is that of a scalar retarded delay differential equation of order n,

$$x^{(n)}(t) + \sum_{k=0}^{n-1} a_k x^{(k)}(t) + \sum_{k=0}^{n-1} \alpha_k x^{(k)}(t-\tau) = 0, \tag{8}$$

where n is a positive integer and the coefficients a_k and α_k are real numbers for $k \in \{0, \ldots, n-1\}$. The corresponding characteristic quasipolynomial is the function $\Delta : \mathbb{C} \to \mathbb{C}$ given by

$$\Delta(s) = s^n + \sum_{k=0}^{n-1} a_k s^k + e^{-s\tau} \sum_{k=0}^{n-1} \alpha_k s^k. \tag{9}$$

Notice that, according to Definition 2, Δ is of degree $2n$, and hence any root of Δ has multiplicity at most $2n$.

The MID property has been extensively studied for some retarded differential equations under the form (8), such as in Boussaada et al. (2018, 2020); Boussaada and Niculescu (2016b); Mazanti et al. (2020a,b). In particular, Mazanti et al. (2020a) presents the following result.

Theorem 3. Consider the quasipolynomial Δ from (9) and let $s_0 \in \mathbb{R}$.

(a) The number s_0 is a root of multiplicity $2n$ of Δ if and only if, for every $k \in \{0, \ldots, n-1\}$,

$$\begin{cases} a_k = \dbinom{n}{k}(-s_0)^{n-k} \\ \qquad + (-1)^{n-k} n! \displaystyle\sum_{j=k}^{n-1} \dbinom{j}{k}\dbinom{2n-j-1}{n-1} \dfrac{s_0^{j-k}}{j!\tau^{n-j}}, \\ \alpha_k = (-1)^{n-1} e^{s_0\tau} \displaystyle\sum_{j=k}^{n-1} \dfrac{(-1)^{j-k}(2n-j-1)!}{k!(j-k)!(n-j-1)!} \dfrac{s_0^{j-k}}{\tau^{n-j}}. \end{cases} \tag{10}$$

(b) If (10) is satisfied, then s_0 is a strictly dominant root of Δ.

The proof of this result is detailed in the case $n = 2$ in Mazanti et al. (2020a), the full proof being provided in the extended version of that reference. A first step of the proof is to remark that it suffices to consider the case $s_0 = 0$ and $\tau = 1$, since the general case can be reduced to this setting by performing the translation and scaling of the spectrum represented by the change of variables $z = \tau(s - s_0)$. Part (a) of Theorem 3 can then be obtained by straightforward computations, imposing that $\Delta^{(k)}(0) = 0$ for every $k \in \{0, \dots, 2n-1\}$. The dominance proof for establishing (b) is then carried by providing first a priori bounds on the imaginary part of non-roots with non-negative real part and then using a suitable factorization of Δ to show, using this bound, that such roots cannot exist.

4. MID FOR A DELAY DIFFERENTIAL-ALGEBRAIC EQUATION

We consider in this section the delay differential-algebraic equation

$$\begin{cases} x'(t) = ax(t) + by(t-\tau), \\ y(t) = cx(t) + dy(t-\tau), \end{cases} \tag{11}$$

where $x(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, and a, b, c, d are real coefficients. System (11) corresponds to (1) with $d_x = d_y = N = 1$ and the explicit computation of its characteristic quasipolynomial Δ from (2) yields

$$\Delta(s) = s - a - e^{-s\tau}\left(sd - ad + bc\right). \tag{12}$$

Note that Δ is a quasipolynomial of degree 3. The main result we prove in this paper is the following counterpart of Theorem 3.

Theorem 4. Consider the quasipolynomial Δ from (12) and let $s_0 \in \mathbb{R}$.

(a) The number s_0 is a root of multiplicity 3 of Δ if and only if the coefficients a, b, c, d, the root s_0, and the delay τ satisfy the relations

$$a = s_0 + \frac{2}{\tau}, \quad d = -e^{s_0\tau}, \quad bc = -\frac{4}{\tau}e^{s_0\tau}. \tag{13}$$

(b) If (13) is satisfied, then s_0 is a dominant root of Δ. Moreover, for every other complex root s of Δ, one has $\operatorname{Re} s = s_0$.

(c) Let $\Xi = \{\xi \in \mathbb{R} \mid \tan\xi = \xi\}$. If (13) is satisfied, then the set of roots of Δ is $\{s_0 + i\frac{2}{\tau}\xi \mid \xi \in \Xi\}$.

Remark 5. With respect to Theorem 3, Theorem 4 provides, in its part (c), additional information on the location of the other roots of Δ. On the other hand, s_0 is not strictly dominant in this case.

Remark 6. In the particular case $s_0 = 0$ and $\tau = 1$, (13) yield $a = 2$, $d = -1$, and $bc = -4$. The corresponding quasipolynomial (12) is then given by

$$\widehat{\Delta}(z) = z - 2 + e^{-z}(z+2). \tag{14}$$

For general $s_0 \in \mathbb{R}$ and $\tau > 0$, one may reduce to the above setting by performing the translation and scaling of the spectrum represented by the change of variables $z = \tau(s - s_0)$.

The proof of Theorem 4 follows the same general line of that of Theorem 3, but extra properties should be proved in order to obtain the additional conclusions of Theorem 4. The main properties we need for the proof are provided in Appendix A.

Proof of Theorem 4. Let $\widetilde{\Delta}$ be the quasipolynomial obtained from Δ by setting

$$\widetilde{\Delta}(z) = \tau\Delta\left(\frac{z}{\tau} + s_0\right) \tag{15}$$

for $z \in \mathbb{C}$. Then

$$\widetilde{\Delta}(z) = z + b_0 + e^{-z}(\beta_1 z + \beta_0)$$

with

$$\begin{aligned} b_0 = \tau(s_0 - a), \quad \beta_1 = -de^{-s_0\tau}, \\ \beta_0 = \tau e^{-s_0\tau}(ad - bc - ds_0). \end{aligned} \tag{16}$$

It follows immediately from relation (15) that s_0 is a root of multiplicity 3 of Δ if and only if 0 is a root of multiplicity 3 of $\widetilde{\Delta}$. Since $\widetilde{\Delta}$ is a quasipolynomial of degree 3, 0 is a root of multiplicity 3 of $\widetilde{\Delta}$ if and only if $\widetilde{\Delta}(0) = \widetilde{\Delta}'(0) = \widetilde{\Delta}''(0) = 0$. We compute

$$\begin{aligned} \widetilde{\Delta}'(z) &= 1 + e^{-z}(-\beta_1 z - \beta_0 + \beta_1), \\ \widetilde{\Delta}''(z) &= e^{-z}(\beta_1 z + \beta_0 - 2\beta_1), \end{aligned}$$

and thus 0 is a root of multiplicity 3 of $\widetilde{\Delta}$ if and only if

$$b_0 + \beta_0 = 0, \quad 1 - \beta_0 + \beta_1 = 0, \quad \beta_0 - 2\beta_1 = 0.$$

One immediately verifies that the above linear system of equations on (b_0, β_1, β_0) admits a unique solution, given by $(b_0, \beta_1, \beta_0) = (-2, 1, 2)$. Using (16), one concludes that s_0 is a root of multiplicity 3 of Δ if and only if (13) holds, concluding the proof of (a). Notice moreover that, under (13), one has $\widetilde{\Delta} = \widehat{\Delta}$, where $\widehat{\Delta}$ is the quasipolynomial defined in (14).

To prove (b), it suffices to show that every root of $\widehat{\Delta}$ lies on the imaginary axis. Note first that

$$\widehat{\Delta}(z) = z^3 \int_0^1 t(1-t)e^{-zt}\,\mathrm{d}t, \tag{17}$$

as one immediately verifies by integrating by parts. Assume, to obtain a contradiction, that there exists a root $z_0 \in \mathbb{C}$ of $\widehat{\Delta}$ such that $\operatorname{Re} z_0 \neq 0$. Writing $z_0 = \sigma + i\omega$ for $\sigma, \omega \in \mathbb{R}$ with $\sigma \neq 0$, one may assume, with no loss of generality thanks to Corollary 10 in Appendix A, that $\sigma > 0$ and $\omega > 0$. By Lemma 11 in Appendix A, one has $0 < \omega < 2$.

Using the fact that z_0 is a non-zero root of $\widehat{\Delta}$, one obtains from (17) and taking the imaginary part that

$$\int_0^1 t(1-t)e^{-\sigma t}\sin(\omega t)\,\mathrm{d}t = 0.$$

Since $0 < \omega < 2$, the function $t \mapsto t(1-t)e^{-\sigma t}\sin(\omega t)$ is strictly positive in $(0, 1)$, which contradicts the above equality. Hence (b) is proved.

Finally, (c) follows immediately from the relation between $\widehat{\Delta}$ and Δ under (13), the fact that all roots of $\widehat{\Delta}$ lie on the imaginary axis, and Lemma 12 in Appendix A.

Remark 7. With respect to other results on multiplicity-induced-dominancy for delay differential equations such as Theorem 3, Theorem 4 provides, in its part (c), the additional information of the location of *all* roots of Δ. The set Ξ of the real roots of the equation $\tan\xi = \xi$ is infinite, discrete, and can be written as $\Xi = \{\xi_k \mid k \in \mathbb{Z}\}$, where $(\xi_k)_{k\in\mathbb{Z}}$ is the increasing sequence of the roots of $\tan\xi = \xi$ with the convention that $\xi_0 = 0$. In particular, for every $k \in \mathbb{Z}$, one has $\xi_k \in \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$ and

$\xi_{-k} = -\xi_k$. We also recall that $\xi_k = k\pi + \frac{\pi}{2} + o(1)$ as $k \to +\infty$.

5. APPLICATION TO THE DELAYED OUTPUT FEEDBACK FOR PROPER CONTROL SYSTEMS

Consider system (6) with $d_x = 1$, $d_y = 2$, $d_u = 1$, which we write as

$$\begin{cases} \dot{x}(t) = ax(t) + bk_1y_1(t-\tau) + bk_2y_2(t-\tau), \\ y_1(t) = c_1x(t) + d_1k_1y_1(t-\tau) + d_1k_2y_2(t-\tau), \\ y_2(t) = c_2x(t) + d_2k_1y_1(t-\tau) + d_2k_2y_2(t-\tau). \end{cases} \tag{18}$$

From (2), we compute its characteristic quasipolynomial

$$\begin{aligned} \Delta(s) = s - a - e^{-s\tau}\big((d_1k_1 + d_2k_2)s \\ - k_1(ad_1 - bc_1) - k_2(ad_2 - bc_2)\big). \end{aligned} \tag{19}$$

Even though (18) is not under the form (11), its characteristic quasipolynomial (19) is of the same form of that of (12), and thus Theorem 4 can be applied to (19). We wish to design the parameters k_1 and k_2 and obtain conditions on the delay τ in order to achieve maximal multiplicity of some root $s_0 < 0$. Then Theorem 4 will ensure the dominance of this root, implying the exponential stability of the system.

Conditions (13) can be rewritten in the context of (19) as

$$\begin{aligned} & a = s_0 + \frac{2}{\tau}, \\ & d_1k_1 + d_2k_2 = -e^{s_0\tau}, \\ & (ad_1 - bc_1)k_1 + (ad_2 - bc_2)k_2 = \left(\frac{2}{\tau} - s_0\right) e^{s_0\tau}. \end{aligned} \tag{20}$$

Note that the first equation in (20) can be rewritten as $s_0 = a - \frac{2}{\tau}$, hence one may stabilize the system by a root of maximal multiplicity only if $a \leq 0$ or $\tau < \frac{2}{a}$.

Let us denote $\delta_i = ad_i - bc_i$ for $i \in \{1, 2\}$. The second and third equations of (20) can be rewritten as

$$\begin{pmatrix} d_1 & d_2 \\ \delta_1 & \delta_2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{4}{\tau} - a \end{pmatrix} e^{a\tau - 2}. \tag{21}$$

This system admits at least one solution if and only if $d_1\delta_2 \neq d_2\delta_1$ or $\delta_i = \left(a - \frac{4}{\tau}\right) d_i$ for $i \in \{1, 2\}$, with exactly one solution in the first case and infinitely many solutions in the second case. This discussion can be concentrated in the following result.

Proposition 8. Consider system (18) for given real parameters a, b, c_1, c_2, d_1, and d_2 and a positive delay τ, and assume that either $a \leq 0$ or $\tau < \frac{2}{a}$. Let $\delta_i = ad_i - bc_i$ for $i \in \{1, 2\}$ and assume moreover that either $d_1\delta_2 \neq d_2\delta_1$ or $\delta_i = \left(a - \frac{4}{\tau}\right) d_i$ for $i \in \{1, 2\}$. Then there exist real parameters k_1, k_2 such that (18) is exponentially stable, with exponential decay rate $a - \frac{2}{\tau}$. Moreover, k_1 and k_2 are solutions of the linear system (21).

Consider, as an example, the case $a = 1$, $b = 1$, $c_1 = 2$, $c_2 = 1$, $d_1 = 1$, $d_2 = 2$, and $\tau = \frac{3}{2}$. Since $a = 1$, the corresponding open-loop system is unstable. Note that the inequality $\tau < \frac{2}{a}$ is indeed satisfied and the exponential decay rate one may obtain with Proposition 8 is $a - \frac{2}{\tau} = -\frac{1}{3}$. One computes, using Proposition 8, the feedback parameters $k_1 \approx -0.87610$ and $k_2 \approx 0.13478$. Figure 2 presents a numerical simulation of the solutions of (20) with these parameters and with initial conditions $x(0) = 1$ and $y_1(t) = y_2(t) = 0$ for $t < 0$. One observes, as expected, that solutions converge to 0 exponentially.

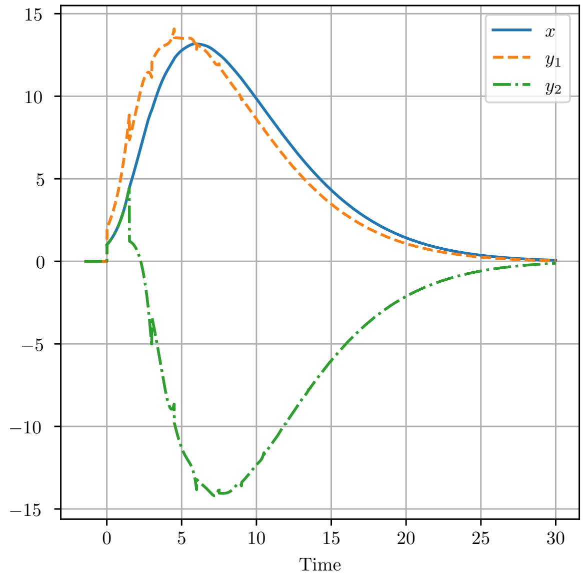

Fig. 2. Trajectories of (18) with the designed delayed output feedback.

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Appendix A. SOME TECHNICAL RESULTS

We present in this appendix technical results used in the proof of Theorem 4. We start by providing some properties of the quasipolynomial $\widehat{\Delta}$ from (14). The first one is the following identity, whose proof is straightforward.

Lemma 9. Let $\widehat{\Delta}$ be given by (14). Then, for every $z \in \mathbb{C}$, one has

$$\widehat{\Delta}(-z) = -e^{z}\widehat{\Delta}(z).$$

As a consequence of the previous identity, one immediately obtains the following symmetry property of the roots of $\widehat{\Delta}$.

Corollary 10. Let $\widehat{\Delta}$ be given by (14) and assume that $z_0 \in \mathbb{C}$ is such that $\widehat{\Delta}(z_0) = 0$. Then $\widehat{\Delta}(z_0) = \widehat{\Delta}(\overline{z}_0) = \widehat{\Delta}(-z_0) = \widehat{\Delta}(-\overline{z}_0) = 0$.

We also need the following a priori bound on the imaginary part of the roots of $\widehat{\Delta}$ outside of the imaginary axis.

Lemma 11. Let $\widehat{\Delta}$ be given by (14) and assume that $z_0 \in \mathbb{C}$ is such that $\operatorname{Re} z_0 \neq 0$ and $\widehat{\Delta}(z_0) = 0$. Then $\lvert\operatorname{Im} z_0\rvert < 2$.

Proof. Let $z_0 \in \mathbb{C}$ be as in the statement. We write $z_0 = \sigma + i\omega$ with $\sigma, \omega \in \mathbb{R}$ and $\sigma \neq 0$. Thanks to Corollary 10, we assume, with no loss of generality, that $\sigma > 0$. Since z_0 is a root of $\widehat{\Delta}$, one has $e^{-z_0}(z_0+2) = 2-z_0$ and thus, in particular, $\lvert z_0+2\rvert^2 = e^{2\sigma}\lvert 2-z_0\rvert^2$, which yields $(\sigma+2)^2+\omega^2 = e^{2\sigma}\left((2-\sigma)^2+\omega^2\right)$.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = e^{2x}\left((2-x)^2+\omega^2\right) - (x+2)^2 - \omega^2$. Note that $f(0) = f(\sigma) = 0$. Since f is differentiable and $\sigma > 0$, the mean value theorem yields the existence of $x_* \in (0, \sigma)$ such that $f'(x_*) = 0$. We compute $f'(x) = 2e^{2x}\left((2-x)(1-x)+\omega^2\right) - 2(x+2)$, and thus

$$e^{2x_*}\left((2-x_*)(1-x_*)+\omega^2\right) = x_* + 2.$$

Since one has further that $e^{2x_*} > 1$, one deduces that

$$(2-x_*)(1-x_*)+\omega^2 < x_* + 2,$$

which is equivalent to

$$x_*^2 - 4x_* + \omega^2 < 0.$$

Letting $g : \mathbb{R} \to \mathbb{R}$ be the polynomial $g(x) = x^2 - 4x + \omega^2$, since $\lim_{x\to\pm\infty} g(x) = +\infty$, the above inequality implies that g must admit two distinct real roots, and thus its discriminant is positive, i.e., $16 - 4\omega^2 > 0$, which is equivalent to $\omega^2 < 4$. Thus $|\omega| < 2$, as required.

As a final technical result, we provide the following characterization of the roots of $\widehat{\Delta}$ on the imaginary axis.

Lemma 12. Let $\widehat{\Delta}$ be given by (14), Ξ be as in the statement of Theorem 4(c), and $\zeta \in \mathbb{R}$. Then $i\zeta$ is a root of $\widehat{\Delta}$ if and only if $\frac{\zeta}{2} \in \Xi$.

Proof. Note that $i\zeta$ is a root of $\widehat{\Delta}$ if and only if

$$i\zeta - 2 + e^{-i\zeta}(i\zeta + 2) = 0,$$

which is the case if and only if

$$\begin{cases} -2 + 2\cos\zeta + \zeta\sin\zeta = 0, \\ \quad \zeta + \zeta\cos\zeta - 2\sin\zeta = 0. \end{cases}$$

The above system is equivalent to

$$R_{-\zeta}\begin{pmatrix} 2 \\ \zeta \end{pmatrix} = \begin{pmatrix} 2 \\ -\zeta \end{pmatrix}, \tag{A.1}$$

where, for $\theta \in \mathbb{R}$, R_θ is the rotation matrix in $\mathbb{R}^2$, defined by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Recalling that $R_\theta^{-1} = R_{-\theta}$ and $R_{\theta_1}R_{\theta_2} = R_{\theta_1+\theta_2}$ for every $(\theta, \theta_1, \theta_2) \in \mathbb{R}^3$, one deduces that (A.1) is equivalent to

$$R_{-\frac{\zeta}{2}}\begin{pmatrix} 1 \\ \frac{\zeta}{2} \end{pmatrix} = R_{\frac{\zeta}{2}}\begin{pmatrix} 1 \\ -\frac{\zeta}{2} \end{pmatrix}.$$

One then immediately verifies that the above system is equivalent to

$$-\sin\left(\frac{\zeta}{2}\right) + \frac{\zeta}{2}\cos\left(\frac{\zeta}{2}\right) = \sin\left(\frac{\zeta}{2}\right) - \frac{\zeta}{2}\cos\left(\frac{\zeta}{2}\right),$$

which holds if and only if

$$\tan\left(\frac{\zeta}{2}\right) = \frac{\zeta}{2},$$

i.e., if and only if $\frac{\zeta}{2} \in \Xi$, as required.