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Open 3-manifolds which are connected sums of closed ones

Laurent Bessières Gérard Besson Sylvain Maillot

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Abstract

We consider open, oriented 3-manifolds which are infinite connected sums of closed 3-manifolds. We introduce some topological invariants for these manifolds and obtain a classification in the case where there are only finitely many summands up to diffeomorphism. This result encompasses both the Kneser-Milnor Prime Decomposition Theorem for closed 3-manifolds and the Kerékjártó-Richards classification theorem for open surfaces.

Keywords. Topology of 3-manifolds

1 Introduction

In [Sco77], P. Scott studied decompositions of open 3-manifolds as (possibly infinite) connected sums of prime manifolds which may or may not be compact. He gave examples of open 3-manifolds that do not have such decompositions. By contrast, in [BBM11] the authors proved that open 3-manifolds which carry certain Riemannian metrics of positive scalar curvature do admit such decompositions, and in addition all summands are compact. In order to recall the result, we introduce some terminology.

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Throughout the paper, we work in the category of smooth, oriented 3-manifolds. We say that two oriented 3-manifolds M_1, M_2 are *isomorphic* if there exists an orientation-preserving diffeomorphism from M_1 to M_2 . For convenience we say that a 3-manifold N has *spherical boundary* if every component of ∂N (if any) is a 2-sphere. Then there is a unique manifold, up to isomorphism, obtained by gluing a 3-ball to every boundary component of N . This operation is called *capping-off*. The capped-off manifold is denoted by \widehat{N} .

Definition 1.1. *Let \mathcal{X} be a class of connected oriented 3-manifolds without boundary. An oriented 3-manifold M is said to be decomposable over \mathcal{X} if M contains a locally finite collection of pairwise disjoint embedded 2-spheres $\{S_i\}$ such that for every connected component C of M split along $\{S_i\}$, the capped-off manifold \widehat{C} is isomorphic to some member of \mathcal{X} or to S^3 .*

The main result of [BBM11] can be formulated as follows: if M is an open 3-manifold which admits a complete Riemannian metric of bounded geometry and uniformly positive scalar curvature, then there exists a finite family \mathcal{F} of spherical manifolds such that M is decomposable over $\mathcal{F} \cup \{S^2 \times S^1\}$. This begs the question of whether such manifolds, and more generally 3-manifolds which are decomposable over some finite collection of closed 3-manifolds, can be classified. In this paper we show that this is the case, using a set of invariants inspired by the work of Kerékjártó [Ker23] and Richards [Ric63] to classify open surfaces:³ the space of ends of M together with a colouring depending on whether a given closed prime 3-manifold appears infinitely many times in any neighbourhood of this end or not.

Recall that a closed 3-manifold K is *prime* if it is connected, not diffeomorphic to S^3 , and whenever K is diffeomorphic to $M_1 \# M_2$, one of the M_i 's is diffeomorphic to S^3 . In this paper we shall use the word ‘prime’ only for *closed* manifolds. The class of oriented prime 3-manifolds will be denoted by \mathcal{P} .

Let K be a (possibly disconnected) compact oriented 3-manifold with spherical boundary. For each $P \in \mathcal{P}$, we denote by $n_P(K)$ the sum over all connected components L of K of the number of summands isomorphic to P in the Kneser-Milnor decomposition of the capped-off manifold \widehat{L} . Thus the Kneser-Milnor theorem implies: every closed oriented 3-manifold is decomposable over some finite collection of prime oriented 3-manifolds, and two closed, connected, oriented 3-manifolds K_1, K_2 are isomorphic if and only if $n_P(K_1) = n_P(K_2)$ for every $P \in \mathcal{P}$.

We now come to the definition of our invariants. Let P be an oriented prime 3-manifold. Let U be an open oriented 3-manifold. Then we define $n_P(U) \in \mathbf{N} \cup \{\infty\}$

³While the Kerékjártó-Richards theorem also applies to nonorientable surfaces, in this paper we restrict attention to orientable manifolds for simplicity.

as the supremum of the numbers $n_P(K)$ where K ranges over all compact submanifolds of U with spherical boundary. Let M be an open connected oriented 3-manifold. An end e of M has *colour* P if for every neighbourhood U of e , with compact boundary, the number $n_P(U)$ is nonzero. (Then it is actually infinite, see below.) The set of ends of M of colour P is denoted by $E_P(M)$. By Lemma 3.2 below, it is a closed subset of $E(M)$. Note that an end may have several colours, i.e. $E_P(M) \cap E_{P'}(M)$ can be nonempty for $P, P' \in \mathcal{P}$. In fact, the closed subsets $E_P(M)$ can be quite arbitrary by our realisation theorem, Theorem 2.5 below.

Let M, M' be open connected oriented 3-manifolds. A map $\phi : E(M) \rightarrow E(M')$ is called *colour-preserving* if for every $P \in \mathcal{P}$, we have $\phi(E_P(M)) = E_P(M')$.

Theorem 1.2 (Main theorem). *Let \mathcal{X} be a finite collection of prime oriented 3-manifolds. Let M, M' be open connected oriented 3-manifolds which are decomposable over \mathcal{X} . Then M is isomorphic to M' if and only if the following conditions hold:*

- (1) *For every $P \in \mathcal{P}$, the numbers $n_P(M)$ and $n_P(M')$ are equal.*
- (2) *There is a colour-preserving homeomorphism $\phi : E(M) \rightarrow E(M')$.*

Remark 1.3. 1. By Theorem 2.3 and Proposition 2.4 below, it suffices in Condition (1) to assume this for all $P \in \mathcal{X} \cup \{S^2 \times S^1\}$, since all the other numbers vanish.

2. Example 1 below shows that the finiteness assumption on \mathcal{X} is necessary.
3. In the case where $n_P(M) < \infty$ for all $P \in \mathcal{P}$, the manifold is determined, up to isomorphism, by $E(M)$ and the numbers $n_P(M)$.

To conclude this introduction, let us mention that the notion of infinite connected sum is also useful in large dimensions, see e.g. [Why01].

The paper is organized as follows: in Section 2, we recall, in a slight more precise form, the notion of connected sum of closed manifolds along a possibly infinite graph, discussed in [BBM11, BBMM], and explain the relationship between this notion and decomposability. We state Theorem 2.3, which allows to turn the graph into a tree, possibly at the expense of adding $S^2 \times S^1$ summands. We also state Proposition 2.4 and Theorem 2.5, and give Example 1 mentioned above. In Section 3, we discuss the properties of special exhaustions of decomposable 3-manifolds called *spherical exhaustions*, which play an important role in all the subsequent proofs. Proposition 2.4 and Theorem 2.5 are proved there. Section 4 is devoted to the proof of Theorem 2.3. Section 5 is dedicated to the proof of Theorem 1.2. Some concluding remarks are gathered in Section 6. We finish

with an appendix reviewing the theory of spaces of ends.

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2 Connected sums along graphs

In this section we reformulate our notion of decomposable manifolds in terms of performing connected sums of possibly infinitely many closed oriented manifolds along some locally finite graph. This was discussed in previous articles of the authors and F. C. Marques [BBM11, BBMM] (cf. [Sco77].) Here though, we need to be more precise regarding orientations, and fix some notation for future use.

Let I be a subset of \mathbb{N} and $\mathcal{X} = \{X_k\}_{k \in I}$ be a family of closed connected oriented 3-manifolds. By convention we assume henceforth that $0 \in I$ and $X_0 = S^3$. A *coloured graph* is a pair (G, f) where G is a locally finite connected graph, possibly with loops, with vertex set $V(G)$ and f is a map from $V(G)$ to I .

Construction. For each vertex $v \in V(G)$, let Y_v be a copy of $X_{f(v)}$ with $d(v)$ disjoint open 3-balls removed, where $d(v)$ is the degree of v . Thus Y_v is a compact manifold with spherical boundary. Let Y be the disjoint union of all Y_v 's. Then glue the Y_v 's to each other along the edges of G , using an orientation-reversing diffeomorphism. We denote by $M(G, f)$ the resulting oriented 3-manifold; it is well-defined up to isomorphism (see [BBMM], Section 2.1). For each edge a of G there is an embedded 2-sphere S_a in $M(G, f)$ along which the gluing has been done.

Definition 2.1. Let \mathcal{X} , G and f be as above. The oriented manifold $M(G, f)$ is called the manifold obtained by connected sum along *the coloured graph* (G, f) .

Remark 2.2. 1. The manifold $M(G, f)$ is connected and without boundary. It is closed if and only if G is finite.

2. If there is a loop in G at the vertex v , then the natural map from Y_v to $M(G, f)$ is not injective. If there is none, then Y_v can be identified with its image in M .

3. The family $\{S_a\}$ is a locally finite family of embedded, pairwise disjoint 2-spheres in $M(G, f)$ whose dual graph is isomorphic to G . It follows that $M(G, f)$ is decomposable over \mathcal{X} . Conversely, if M is decomposable over \mathcal{X} , then M is isomorphic to the connected sum along the dual graph G of the splitting collection of 2-spheres with appropriate colouring.

In the compact case this definition is slightly nonstandard in the sense that in the usual definition the graph is a *tree*. It is well-known, however, that a connected sum along a finite graph can be made into a connected sum along a tree at the expense of adding vertices with $S^2 \times S^1$ factors. This is also true in the infinite case:

Theorem 2.3. *Let $\mathcal{X} = \{X_k\}$ be a (possibly infinite) family of closed connected oriented manifolds with $X_0 = S^3$. Let M be a connected open oriented 3-manifold. If M is decomposable over \mathcal{X} , then it is isomorphic to a connected sum along a coloured tree of members of $\mathcal{X} \cup \{S^2 \times S^1\}$.*

Our next task is to define combinatorial invariants of coloured trees and explain the relationship between these invariants and the topological invariants of decomposable 3-manifolds introduced earlier. In particular, Proposition 2.4 below shows that the latter can be effectively calculated.

Let $\mathcal{X} = \{P_k\}_{k \in I}$ be a family of pairwise non-isomorphic closed oriented manifolds such that $X_0 = S^3$ and P_k is a prime manifold for all $k > 0$. Let (T, f) be a coloured tree and let $M = M(T, f)$. For every $k \in I \setminus \{0\}$, we denote by $n_k(T, f) \in \mathbf{N} \cup \{\infty\}$ the number of vertices v of T such that $f(v) = k$. For every pre-end $e = U_1 \supset U_2 \supset \dots$ of T we let $n_k(e)$ be the limit as n goes to ∞ of the number of vertices v in U_n such that $f(v) = k$. This passes to a well-defined invariant $n_k(e^*)$ for ends of T . We let $E_k(T, f)$ denote the set of ends e^* of T such that $n_k(e^*) = \infty$.

Proposition 2.4. *Let \mathcal{X} be as above. Let (T, f) be a coloured tree and $M = M(T, f)$. Then the following assertions hold:*

1. *For every $k \in I \setminus \{0\}$ we have $n_{P_k}(M) = n_k(T, f)$.*
2. *There is a homeomorphism $\phi : E(M) \rightarrow E(T)$ such that for every $k \in I \setminus \{0\}$, we have $E_k(T, f) = \phi(E_{P_k}(M))$.*
3. *For every prime oriented manifold P which is not isomorphic to a member of \mathcal{X} , we have $n_P(M) = 0$ and $E_P(M) = \emptyset$.*

Example 1. Let $(P_n)_{n \in \mathbf{N}}$ be a sequence of pairwise non isomorphic oriented 3-manifolds, such that $P_0 = S^3$ and P_n is prime for all $n > 0$. Let Z be the Cayley graph of \mathbf{Z} with standard generators. We define two colourings f_1, f_2 on G by setting $f_1(n) = 0$ if $n < 0$, $f_1(n) = n$ if $n \geq 0$, $f_2(n) = -2n$ if $n < 0$ and $f_2(n) = 2n + 1$ if $n \geq 0$.

Then for every $i \in \mathbf{N} \setminus \{0\}$ we have $n_i(Z, f_1) = n_i(Z, f_2) = 1$, and the subsets $E_i(Z, f_j)$ are empty for all $i > 0, j \in \{1, 2\}$. It follows from Proposition 2.4 that the manifolds $M(Z, f_1)$ and $M(Z, f_2)$ have the same invariants. However, M_1 and M_2 are not isomorphic. Indeed, M_1 has an end that has a neighbourhood diffeomorphic to $S^2 \times \mathbf{R}$, while no end of M_2 has this property.

Finally, we state our realisation theorem.

Theorem 2.5. *Let E be a compact, metrisable, totally disconnected space. Consider a collection E_1, \dots, E_k of closed subsets of E . Let n_1, \dots, n_ℓ be nonnegative integers. Let $\mathcal{X} = \{P_1, \dots, P_k, Q_1, \dots, Q_\ell\}$ be a family of prime manifolds. Then there exists a connected oriented 3-manifold M which is a connected sum along a tree of members of $\mathcal{X} \cup \{S^3\}$ with the following properties:*

- *For every $i \in \{1, \dots, \ell\}$ we have $n_M(Q_i) = n_i$.*
- *There exists a homeomorphism $\phi : E(M) \rightarrow E$ such that $\phi(E_{P_i}(M)) = E_i$ for every $i \in \{1, \dots, k\}$.*

3 Preliminaries

3.1 Spherical exhaustions

Recall that an *exhaustion* of an open manifold M is a sequence (K_n) of compact connected submanifolds of M such that $M = \bigcup_n K_n$, and $\overset{\circ}{K}_{n+1} \supset K_n$ for every n . We may assume that for each n , every component of $M \setminus K_n$ is unbounded, i.e. not contained in any compact subset. A *spherical exhaustion* is an exhaustion (K_n) where each K_n has spherical boundary.

Let $P \in \mathcal{P}$. Let M be an open connected oriented 3-manifold. Let $e = U_1 \supset U_2 \supset \dots$ be a pre-end of M . Then the sequence $(n_P(U_i))$ is non increasing. Let $n_P(e)$ denote the limit of this sequence.

Lemma 3.1. *The number $n_P(e)$ is either 0 or $+\infty$.*

Proof. If $n_P(U_i) = \infty$ for every i , then $n_P(e) = \infty$. Otherwise, there exists i such that $n_P(U_i)$ is finite. Thus there exists a compact submanifold $K \subset U_i$ with spherical boundary such that $n_P(K) = n_P(U_i)$. By definition of a pre-end, there exists $j > i$ such that $K \cap U_j = \emptyset$. Then $n_P(U_j) = 0$, otherwise U_j would contain some compact K' with nonzero $n_P(K')$ and we would have $n_P(K \cup K') = n_P(K) + n_P(K') > n_P(K)$, contradicting the choice of K . \square

Furthermore, if e, e' are equivalent pre-ends, then it is readily checked that $n_P(e) = n_P(e')$. Thus we have a well-defined invariant $n_P(e^*) \in \{0, \infty\}$ for each end e^* of M . We remark that the previously defined set of ends of M of colour P is the set of $e^* \in E(M)$ such that $n_P(e^*) = \infty$.

Lemma 3.2. *The set $E_P(M)$ is a closed subset of $E(M)$.*

Proof. Let $e^* \in E(M) \setminus E_P(M)$ and let $e = U_1 \supset U_2 \supset \dots$ representing e^* . Then $n_P(U_i) = 0$ for all i large enough, which implies that all ends $e'^* \in U_i^*$ satisfy $n_P(e'^*) = 0$. Therefore U_i^* is an open neighbourhood of e^* disjoint from $E_P(M)$, proving that $E(M) \setminus E_P(M)$ is open. \square

3.2 Proof of Proposition 2.4

Let $\mathcal{X} = \{P_k\}_{k \in I}$ be a family of pairwise non-isomorphic closed oriented manifolds such that $X_0 = S^3$ and P_k is prime for all $k > 0$. Let (T, f) be a coloured tree and let $M = M(T, f)$. Recall from Remark 2.2 that for every vertex v in T we have a submanifold Y_v of M and for every edge a of T there is an embedded 2-sphere S_a in M .

We construct a proper continuous map $F : M \rightarrow T$ which naturally induces a homeomorphism $\phi : E(M) \rightarrow E(T)$. We recall that T is endowed with the topology given by the length distance for which the edges are isometric to the interval $[0, 1]$. For each edge a of T , let $Z_a \subset M$ be a tubular neighbourhood of S_a , whose size is chosen so that $Z_a \cap Z_{a'} = \emptyset$ if $a \neq a'$. We also choose a parametrisation $Z_a \simeq S_a \times (-1, 1)$ with $S_a \simeq S_a \times \{0\}$. For each vertex $v \in T$ we define $C_v = Y_v \setminus \bigcup_a Z_a$, which we call the core of Y_v . Notice that C_v is homeomorphic to Y_v . We then define $F : M \rightarrow T$ by

1. $F(C_v) = v$ for each vertex $v \in T$.
2. $F(Z_a) = a$ for each edge $a \in T$. More precisely,

$$F : Z_a \simeq S_a \times (-1, 1) \rightarrow (-1, 1) \simeq (0, 1) \simeq a \setminus \partial a$$

is obtained by composing the projection on the second factor of the parametrisation of Z_a with the map $s \mapsto \frac{1}{2}s + \frac{1}{2}$.

We notice that $F^{-1}(F(Y_v)) = Y_v$, that $F(S_a)$ is the middle point of a and that $F^{-1}(F(S_a)) = S_a$. Let $\phi : E(M) \rightarrow E(T)$ be the continuous map induced by F .

Lemma 3.3. *The map ϕ is a homeomorphism.*

Proof. Let us choose a point $x_v \in C_v$ for each vertex v and, for every edge a joining v to v' , an embedded path $\gamma_a \subset C_v \cup Z_a \cup C_{v'}$ joining x_v to $x_{v'}$ in such a way that $\gamma_a \cap Z_a \simeq \{\theta_a\} \times (-1, 1)$ for some $\theta_a \in S_a$. We then define a continuous proper map $F' : T \rightarrow M$ so that, for every edge $a \in T$, it is a homeomorphism between a and γ_a and $F'(v) = x_v$. The map F' satisfy the following two properties:

- 1'. $F \circ F'(v) = v$ for every vertex $v \in T$.
- 2'. $F \circ F'$ globally preserves each edge $a \in T$.

Let $\phi' : E(T) \rightarrow E(M)$ be the continuous map induced by F' .

By compactness of the spaces of ends it suffices to show that ϕ is bijective. By construction $\phi \circ \phi' = id$, hence ϕ is surjective. Let $e^* \neq e'^* \in E(M)$ be two ends of M respectively represented by the pre-ends $e = U_1 \supset U_2 \supset \dots$ and $e' = U'_1 \supset U'_2 \supset \dots$. We can choose the U_i 's and the U'_i 's so that they are bounded by spheres in the family $\{S_\alpha\}$. Hence they satisfy, for all i , $F^{-1}(F(U_i)) = U_i$ and $F^{-1}(F(U'_i)) = U'_i$. Since $e \neq e'$ there exist two indices i and j such that $U_i \cap U'_j = \emptyset$. Consequently, $F(U_i) \cap F(U'_j) = \emptyset$. It follows that $\phi(e^*) \neq \phi(e'^*)$. Hence ϕ is injective. \square

Proof of Proposition 2.4, assertion 1. Let us assume that $n_k = n_k(T, f)$ is finite. There exists n_k vertices v of T such that $\widehat{Y}_v \simeq P_k$. Let K be the union of these Y_v . Then $n_{P_k}(K) = n_k$ and therefore $n_{P_k}(M) \geq n_k$. Let us assume that $n_{P_k}(M) > n_k$. This means, by definition, that there exists a compact submanifold with spherical boundary $L \subset M$ such that $n_k(L) > n_k$. We may assume that L is connected and a union of Y_v 's. Consequently, \widehat{L} can be written as a connected sum of these \widehat{Y}_v along a subtree T' of T using the restriction f' of f to T' . By Milnor's uniqueness there are exactly $n_{P_k}(L)$ components Y_v such that $\widehat{Y}_v \simeq P_k$. Then $n_k(T', f') \geq n_{P_k}(L)$, and

$$n_k = n_k(T, f) \geq n_k(T', f') \geq n_{P_k}(L) > n_k,$$

a contradiction.

Now, if $n_k = n_k(T, f) = +\infty$, for all $n \in \mathbf{N}$ there exists some $L_n \subset M$ a union of Y_v 's such that $n_{P_k}(L_n) = n$. This shows $n_{P_k}(M) \geq n$ for all n , and hence that $n_{P_k}(M) = +\infty$. \square

Proof of Proposition 2.4, assertion 2.

There remains to prove that $\phi(E_{P_k}(M)) = E_k(T, f)$, which is equivalent to

$$n_k(e^*) = +\infty \iff n_k(\phi(e^*)) = +\infty.$$

We have:

$$n_k(e^*) = +\infty \iff n_k(e) = +\infty \iff n_k(U_i) = +\infty \quad \forall i,$$

where $e = U_1 \supset U_2 \dots$ is a pre-end defining e^* . We may choose the U_i saturated by F , i.e. satisfying $F^{-1}(F(U_i)) = U_i$. Then we set $V_i = F(U_i)$. This is a connected set whose boundary is a finite union of points which are midpoints of edges and images by F of the boundary of U_i , which is a finite union of spheres S_α .

Let (K_n) be an exhaustion of M by compact sets which are saturated by F , i.e. $F^{-1}(F(K_n)) = K_n$. For $n \in \mathbf{N}$, by definition of a pre-end, there exists $i \in \mathbf{N}$ such that $U_i \cap K_n = \emptyset$.

By the choice of saturated sets we have $V_i \cap F(K_n) = \emptyset$. By definition of ϕ the family $V_1 \supset V_2 \supset \dots$ is a pre-end representing $\phi(e^*)$.

For $i \in \mathbf{N}$ let T_i be the subtree of T whose vertices are the v 's such that $Y_v \subset U_i$. The same argument as in the proof of assertion 1 yields

$$n_{P_k}(U_i) = +\infty \iff n_k(T_i) = +\infty.$$

Therefore, by definition of V_i , the vertices of T included in V_i are those of T_i and consequently, for all $i \in \mathbf{N}$,

$$n_{P_k}(U_i) = +\infty \iff n_k(T_i) = +\infty \iff |\{v \in V_i; f(v) = k\}| = +\infty,$$

which shows that $n_k(\phi(e^*)) = +\infty$. □

Proof of Proposition 2.4, assertion 3. If the assertion is not true then there exists a prime manifold $P \notin \mathcal{X}$ and a compact submanifold K of M with spherical boundary such that $n_P(K) \geq 1$. Let $L \subset M$ be a compact connected submanifold with spherical boundary such that $K \subset L$. We may assume that L is a union of Y_v 's. We have that \widehat{L} is homeomorphic to a connected sum of \widehat{K} and $L \setminus \overset{\circ}{K}$. Hence we have $n_P(K) \leq n_P(L) = 0$. The last equality comes from the hypothesis, the fact that L is a union of Y_v 's and thus \widehat{L} is a connected sum of prime manifolds in \mathcal{X} and Milnor's uniqueness theorem. □

It follows immediately that ϕ is colour-preserving and that for every k the numbers $n_k(T, f)$ and $n_{P_k}(M)$ coincide.

3.3 Proof of Theorem 2.5

Proof. It suffices to prove the case when $\mathcal{X} = \{P_1, \dots, P_k\}$, the general case following easily by adding a finite connected sum of the Q_i 's. In order to follow our convention we add S^3 to the family \mathcal{X} by setting $P_0 = S^3$.

Let \mathcal{C} be the Cantor set which we can identify, by abuse of language, to $\{0, 1\}^{\mathbf{N}}$. Let T_0 be the regular dyadic rooted tree with $E(T) = \mathcal{C}$. The vertices of T_0 are parametrised by finite sequences of 0's and 1's.

We identify E with a closed subset of \mathcal{C} and let T be the subtree of T_0 whose vertices are finite prefixes of elements of E . To each vertex v of T we associate a finite set of integers F_v which is the set of $i \in \{1, \dots, k\}$ such that v is a prefix of some element of E_i .

Form a tree T' by attaching to each vertex v of T a finite tree T'_v with as many vertices as elements of F_v . Observe that the obvious retraction from T' to T induces a homeomorphism between $E(T)$ and $E(T')$, so that we can identify these two spaces.

We define a colouring f on T' as follows: for each $v \in T$, set $f(v) = 0$, and send the vertices of T'_v bijectively to F_v . Let M be the connected sum $M(T, f)$. From Proposition 2.4 there exists a homeomorphism $\phi : E(M) \rightarrow E(T)$ such that for every $i \in \{1, \dots, k\}$ we have $\phi(E_{P_k}(M)) = E_k(T', f) = E_k$. \square

4 Proof of Theorem 2.3

Lemma 4.1. *Let M be a connected open oriented 3-manifold which is decomposable over the class of all closed 3-manifolds. Let $A \subset M$ be a connected compact submanifold with spherical boundary such that no component of $M \setminus A$ is bounded.*

Then there is a compact connected submanifold $B \subset M$ with spherical boundary such that:

1. *A is contained in the interior of B .*
2. *Every component U of $M \setminus B$ is unbounded and has connected boundary.*
3. *If C is the closure of a component of $B \setminus A$, then C is a punctured 3-sphere and ∂C has exactly one component contained in ∂B .*

Proof. Let U_1, \dots, U_s be the components of $M \setminus A$.

For every $1 \leq i \leq s$ we construct a submanifold C_i as follows. If U_i has connected boundary, then we let C_i be a closed collar neighbourhood of ∂U_i in U_i . Otherwise, let $\Sigma_1, \dots, \Sigma_t$ be the components of ∂U_i . For each $1 \leq j \leq t - 1$ pick a smooth arc γ_j properly embedded in \bar{U}_i and connecting Σ_j to Σ_{j+1} . Choose a small closed tubular neighbourhood C_i in \bar{U} of $\Sigma_1 \cup \dots \cup \Sigma_t \cup \gamma_1 \cup \dots \cup \gamma_{t-1}$. Then C_i is a punctured 3-sphere and its boundary is the union of all Σ_i 's and some 2-sphere $S \subset U$.

Doing this for every i , we set $B = A \cup \bigcup_{1 \leq i \leq s} C_i$. \square

We can now prove Theorem 2.3.

Proof. The decomposability hypothesis implies that there exists an exhaustion (A_n) of M by compact connected 3-submanifolds with spherical boundary, such that each capped-off manifold \widehat{A}_n is a finite connected sum of members of \mathcal{X} . Without loss of generality, we assume that for every n , each component of $M \setminus A_n$ is unbounded. Applying Lemma 4.1 to each A_n and possibly extracting a subsequence and reindexing, we get an exhaustion $\{B_n\}$ by submanifolds with spherical boundary with the following properties:

1. For each n , the submanifold A_n is contained in the interior of B_n .
2. For each n , every component of $M \setminus B_n$ is unbounded and has connected boundary.
3. If C is the closure of a component of $B_n \setminus A_n$ for some n , then C is a punctured 3-sphere and ∂C has exactly one component contained in ∂B_n .

It follows that for every n , the capped-off manifold \widehat{B}_n is a connected sum of \widehat{A}_n with a finite number of $S^2 \times S^1$ factors.

Let \mathcal{S} be the collection of all components of the boundaries of all B_n 's. Then \mathcal{S} is a locally finite collection of pairwise disjoint, separating, embedded 2-spheres in M . The dual graph T of \mathcal{S} is therefore a tree. It is now easy to attach a finite tree to each vertex of T and get a tree T' such that M is isomorphic to $M(T, f)$ for a suitable colouring f , arguing as in the end of the proof of Theorem 2.5. \square

5 Proof of the Main Theorem

5.1 Setting up the proof

Let $\{P_i\}_{0 \leq i \leq r}$ be a finite family of closed oriented 3-manifolds with $P_0 = S^3$, $P_1 = S^2 \times S^1$, and all other P_k 's are prime, not isomorphic to P_1 and pairwise nonisomorphic. For simplicity we denote the invariants $n_{P_k}(M)$ by $n_k(M)$ and $E_{P_k}(M)$ by $E_k(M)$, when M is a 3-manifold.

Let M, M' be two open connected oriented manifolds which are decomposable over this family. We suppose that $n_k(M) = n_k(M')$ for all $0 < k \leq r$, and that there is a colour-preserving homeomorphism $\phi : E(M) \rightarrow E(M')$. We fix once and for all such a homeomorphism. We want to construct an orientation-preserving diffeomorphism (henceforth called an *isomorphism*) from M to M' .

Definition 5.1. *Let A be a compact, connected submanifold of M with spherical boundary. We say that a component U of $M \setminus A$ is good if the following properties are satisfied:*

1. *The set U is unbounded and has connected boundary.*
2. *For every $0 < k \leq r$, the number $n_k(U)$ is either 0 or $+\infty$.*

The set A is said to be good if every component of $M \setminus A$ is good and a spherical exhaustion $\{A_n\}$ is good if each A_n is good.

Notice that if A is good, then each component of ∂A is separating. Also, for every k , if $n_k(M)$ is finite, then $n_k(M \setminus A) = 0$. In other words, every prime factor which appears only finitely many times in the prime decomposition of M is contained in A .

The idea is inspired by [Ric63] (cf also [Ker23]) and consists in constructing two good exhaustions (A_n) of M and (A'_n) of M' together with isomorphisms $\Phi_n : A_n \rightarrow A'_n$ such that $\Phi_{n+1}|_{A_n} = \Phi_n$ and satisfying the following additional property: for every n , every component U of $M \setminus A_n$ and every component U' of $M' \setminus A'_n$, if $\Phi_n(\partial U) = \partial U'$ then $\phi(U^*) = U'^*$. It is then clear that $\Phi = \lim_{n \rightarrow \infty} \Phi_n$ is the desired map.

5.2 Existence of a good exhaustion

Lemma 5.2. *The manifold M admits a good exhaustion.*

Proof. This follows by iteration from the following claim:

Claim 1. For every bounded subset $A \subset M$, there exists a good submanifold B such that A is contained in the interior of B .

To prove the claim, let $\{K_n\}$ be an ancillary spherical exhaustion of M . Without loss of generality we assume that for every n , each component of $M \setminus K_n$ is unbounded, and denote these components by $U_{n,1}, \dots, U_{n,s(n)}$. The sets $U_{n,i}^*$ form a basis for the topology of $E(M)$. Any two of these sets are either disjoint or contained in one another. Every end $e^* \in E(M)$ is represented by a pre-end $e = V_1 \supset V_2 \supset \dots$ where the V_j are taken among the $U_{n,i}$'s (see Appendix A).

For any $e^* \in E(M)$ with $e = V_1 \supset V_2 \supset \dots$ there exists $N \in \mathbf{N}$ such that for any $0 < k \leq r$, we have either

- $n_k(V_i) = 0$, $\forall i \geq N$, (if $e^* \notin E_P(M)$), or
- $n_k(V_i) = +\infty$ for all i , (if $e^* \in E_P(M)$).

For every end e^* we choose such a number N large enough so that $V_N \cap \bar{A} = \emptyset$, and set $V(e^*) := V_N$. The collection $(V(e^*))_{e^* \in E(M)}$ is an open covering of the compact space $E(M)$. Therefore there exists a finite collection of ends e_1^*, \dots, e_p^* such that $V(e_1^*)^* \cup \dots \cup V(e_p^*)^* = E(M)$. We set

$$B = M \setminus (V(e_1^*) \cup \dots \cup V(e_p^*)).$$

By construction, B is a compact, connected submanifold of M with spherical boundary and contains A in its interior. Moreover, for every component U of $M \setminus B$, the open set U is unbounded and we have $n_k(U) \in \{0, \infty\}$ for all $0 < k \leq r$.

If for some U the boundary of U is disconnected, argue as in the proof of Theorem 2.3 and add something to B to solve the problem. Then B is good. \square

5.3 Proof of the Main Theorem

By Lemma 5.2, we have good exhaustions $\{B_n\}$ of M and $\{B'_n\}$ of M' .

Our goal is to construct by induction two good exhaustions $\{A_n\}$ of M and $\{A'_n\}$ of M' together with a sequence of isomorphisms $\Phi_n : A_n \rightarrow A'_n$ with the following properties:

1. For every n , the isomorphism Φ_{n+1} is an extension of Φ_n .
2. For every $p \leq n$, every component U of $M \setminus A_p$ and every component U' of $M' \setminus A'_p$, if $\Phi_p(\partial U) = \partial U'$ then $\phi(U^*) = (U')^*$.

We denote by (H_n) the n -th induction step.

The first step of the induction is to show that there exist good subsets $A_0 \subset M$ and $A'_0 \subset M'$ which are isomorphic and satisfying assertion 2 above. Since the argument is similar to part of the induction step, we postpone it until the end of the proof.

We do the construction for n even and for n odd we exchange the roles played by M and M' . Therefore, for n even, we assume that the construction of A_n , A'_n and Φ_n has been done and we proceed to the next step. We set $A'_{n+1} = B'_m$, for m large enough so that $A'_{n+1} \supset A'_n$. Our goal is now to define A_{n+1} together with $\Phi_{n+1} : A_{n+1} \rightarrow A'_{n+1}$ satisfying properties 1 and 2. The first step is to prove that for m large enough B_m contains a submanifold isomorphic to A'_{n+1} .

For a connected component S of ∂A_n , we denote by U_S the component of $M \setminus A_n$ bounded by S , and by U'_S the component of $M' \setminus A'_n$ bounded by $S' := \Phi_n(S)$. The map $U_S \mapsto U'_S$ is a bijection between the set of components of $M \setminus A_n$ and the set of components of $M' \setminus A'_n$. Similarly, for every m large enough so that $B_m \supset A_n$, we denote by $C_{m,S}$ the component of $B_m \setminus A_n$ such that $C_{m,S} \cap A_n = S$, that is $C_{m,S} = \overline{U_S} \cap B_m$. Similarly, C'_S is the component of $A'_{n+1} \setminus A'_n$ such that $C'_S \cap A'_n = S'$, that is $C'_S = \overline{U'_S} \cap A'_{n+1}$. The map $C_{m,S} \mapsto C'_S$ is again a bijection between the set of components of $B_m \setminus A_n$ and the set of components of $A'_{n+1} \setminus A'_n$.

Lemma 5.3. *There exists $m_0 \in \mathbf{N}$ such that for any integer $m \geq m_0$, for any $0 < k \leq r$, for any connected component S of ∂A_n , we have*

$$n_k(C_{m,S}) \geq n_k(C'_S).$$

Proof. Let k and S be such that $n_k(C'_S) > 0$. Using the inclusion $C'_S \subset \overline{U'_S}$ and the fact that A'_n is good, it follows that $n_{U'_S}(P) = +\infty$, which in turn implies that $(U'_S)^* \cap E_k(M') \neq \emptyset$.

By the induction assumption (H_n) , the corresponding component U_S of $M \setminus A_n$ satisfies $\phi(U_S^*) = (U'_S)^*$ and, since we also have $\phi(E_k(M)) = E_k(M')$, we deduce that $U_S^* \cap E_k(M) \neq \emptyset$, which implies $n_k(U_S) = +\infty$. Since $C_{m,S} = \overline{U_S} \cap B_m$, we have $n_k(U_S) = \lim_{m \rightarrow \infty} n_k(C_{m,S})$. Hence, for all m large enough depending on S and k , we have $n_k(C_{m,S}) \geq n_k(C'_S)$.

Since there are only finitely many S and k , we can choose m large enough so that this inequality holds for all of them. \square

Next, for m large enough, we will define A_{n+1} by adding to A_n a piece of $C_{m,S}$ for each S . Let $(U_{i,m})$ be the collection of connected components of $M \setminus B_m$ and (V'_j) the collection of connected components of $M' \setminus A'_{n+1}$.

Lemma 5.4. *There exists $m_1 \in \mathbb{N}$ such that for every integer $m \geq m_1$, for each component $U_{i,m}$ of $M \setminus B_m$, there exists a component V'_j of $M' \setminus A'_{n+1}$ such that $\phi(U_{i,m}^*) \subset V'^*_j$.*

Proof. The collection of open sets $(U_{i,m}^*)$, for $m \geq m_0$, is a basis for the topology of $E(M)$. The sets V'^*_j form an open cover of $E(M')$, so the sets $\phi^{-1}(V'^*_j)$ form an open cover of $E(M)$. Let $e^* \in E(M)$. There exists j such that $e^* \in \phi^{-1}(V'^*_j)$. Since $(U_{i,m}^*)$ is a basis for the topology, there exist i and $n \geq m_0$ such that

$$e^* \in U_{i,n}^* \subset \phi^{-1}(V'^*_j).$$

We choose such a $U_{i,n}^*$ and call it $U(e^*)$. Since $E(M)$ is compact, from the collection $(U(e^*))_{e^* \in E(M)}$ we extract a finite covering of $E(M)$,

$$E(M) \subset U_{i_1, n_1}^* \cup \dots \cup U_{i_k, n_k}^*.$$

Now, we set $m_1 = \max_j \{n_j\}$. For every $m \geq m_1$, each $U_{i,m}$ is included in one U_{i_s, n_s} hence, for each $m \geq m_1$ and for each component $U_{i,m}$ of $M \setminus B_m$, there exists a component V'_j of $M' \setminus A'_{n+1}$ such that $\phi(U_{i,m}^*) \subset V'^*_j$. \square

Lemma 5.5. *For each integer $m \geq m_1$, for every 2-sphere S , there exists a submanifold $C_S \subset C_{m,S}$ with spherical boundary such that the following properties hold:*

- (1) *Each boundary sphere of ∂C_S separates M .*
- (2) *There exists an isomorphism $\Phi_{n+1,S} : C_S \rightarrow C'_S$ whose restriction to S coincides with Φ_n .*
- (3) *Any component U of $M \setminus C_S$ not bounded by S is good. Furthermore, if V' is the component of $M' \setminus C'_S$ bounded by $\Phi_{n+1,S}(\partial U)$, then $\phi(U^*) = V'^*$.*

Once this lemma is proved, for $m = \max\{m_0, m_1\}$, we define $A_{n+1} = A_n \cup_S C_S$ and we extend Φ_n into $\Phi_{n+1} : A_{n+1} \rightarrow A'_{n+1}$ by setting $\Phi_{n+1} = \Phi_{n+1,S}$ on C_S .

Proof of Lemma 5.5. Let us consider a component S of ∂A_n and recall that C'_S is the component of $A'_{n+1} \setminus \overset{\circ}{A}'_n$ whose intersection with A'_n is the sphere $S' = \Phi_n(S)$. Thus $\partial C'_S$ consists of finitely many 2-spheres S', S'_1, \dots, S'_ℓ where $S'_j \subset \partial A'_{n+1}$, each separating M' . We relabel by V'_j the component of $M' \setminus A'_{n+1}$ bounded by S'_j . Notice that $\{V'_1, \dots, V'_\ell\}$ is a partition of $(U'_S)^*$.

Recall that $C_{m,S}$ is the component of $B_m \setminus \overset{\circ}{A}_n$ such that $C_{m,S} \cap A_n = S$, and that $\phi(U^*_S) = (U^*_S)^*$. We partition $\partial C_{m,S}$ into $S \cup \Gamma_1 \cup \dots \cup \Gamma_\ell$ where each Γ_j is a union of 2-spheres of ∂B_m , and label the Γ_j 's in such a way that if $\partial \overline{U_{i,m}} \subset \Gamma_j$ for some i and j , then $\phi(U^*_{i,m}) \subset V^{j*}$. It follows that if V_j denotes the union of the $U_{i,m}$'s such that $\partial \overline{U_{i,m}} \subset \Gamma_j$, then $\{V^*_1, \dots, V^*_\ell\}$ is a partition of U^*_S and $\phi(V^{j*}) = V^{j*}$.

We now consider the closed oriented 3-manifolds \widehat{C} and \widehat{C}' obtained from $C_{m,S}$ and C'_S by gluing 3-balls to their boundary components. They both have a Kneser-Milnor decomposition in elements of \mathcal{X} :

$$\widehat{C} \simeq \overbrace{P_1 \# \dots \# P_1}^{n_1} \# \overbrace{P_2 \# \dots \# P_2}^{n_2} \dots \# \overbrace{P_r \dots \# P_r}^{n_r} \# P_0$$

with $n_i = n_i(\widehat{C})$, and

$$\widehat{C}' \simeq \overbrace{P_1 \# \dots \# P_1}^{n'_1} \# \overbrace{P_2 \# \dots \# P_2}^{n'_2} \dots \# \overbrace{P_r \dots \# P_r}^{n'_r} \# P_0.$$

with $n'_i = n_i(\widehat{C}')$.

From the previous arguments, $n_k \geq n'_k \geq 0$ for each $0 < k \leq r$. By construction C'_S is a submanifold of \widehat{C}' and $\widehat{C}' \setminus \overset{\circ}{C}'_S$ is a collection of 3-balls D', D'_1, \dots, D'_ℓ bounded by S', S'_1, \dots, S'_ℓ , each sphere separating \widehat{C}' . Consider now the following connected sum

$$\widehat{C}' \# N'_1 \# \dots \# N'_\ell.$$

where each N'_j is a closed manifold chosen below and is attached to D'_j . We claim that we can choose N'_j so that

- (a) $\widehat{C}' \# N'_1 \# \dots \# N'_\ell \approx \widehat{C}$,
- (b) For all $0 < k \leq r$, if $n_k(N'_j) > 0$ then $n_k(V'_j) = \infty$.

The manifolds N'_j are defined as connected sums of members of \mathcal{X} . The fact that we can find manifolds N'_j satisfying (a) follows from the above observation that each prime

component of the Kneser-Milnor decomposition of \widehat{C}' appears at least as many times as in the prime decomposition of \widehat{C} . It then suffices to divide the supernumerary prime components of \widehat{C} (if any) into the N'_j 's. For (b) we first remark that, for $0 < k \leq r$,

$$n_k(\widehat{C}) > 0 \implies n_k(U_S) > 0 \implies n_k(U_S) = +\infty \implies n_k(U'_S) = +\infty,$$

the last implication following from (H_n) , the previous one from goodness.

Finally,

$$n_k(U'_{S'}) = +\infty \implies \exists j, \quad n_k(V'_j) = +\infty.$$

Summarising, for any prime component P_k of \widehat{C} there exists $j \in \{1, \dots, \ell\}$ such that $n_k(V'_j) = +\infty$. For each supernumerary P_k , we select $j = j(P_k)$ such that $n_k(V'_j) = +\infty$. We define N'_j to be the connected sum of S^3 with the supernumerary manifolds P_k for which $j(P_k) = j$. It follows that the N'_j 's satisfy (b).

Let us denote by $f : \widehat{C} \rightarrow \widehat{C}' \# N'_1 \# \dots \# N'_\ell$ an isomorphism. Let $D, D_1, \dots, D_s \subset \widehat{C}$ be the compact 3-balls such that $\widehat{C} = C_{m,S} \cup D \cup D_1 \cup \dots \cup D_s$ and $\partial C_{m,S}$ consists of the boundary of these disks, where $S = \partial D$.

The images $f(D_j)$ are disjoint compact 3-balls in $\widehat{C}' \# N'_1 \# \dots \# N'_\ell$. These balls can be isotoped anywhere in $\widehat{C}' \# N'_1 \# \dots \# N'_\ell$. Hence, up to composing f with finitely many diffeomorphisms we may assume that,

$$f(D) = D', \quad (\text{thus } f(S) = S')$$

and that

$$\partial D_i \subset \Gamma_j \implies f(D_i) \subset N'_j.$$

Then $f(C_{m,S}) \supset C'_S$. We then define,

$$C_S := f^{-1}(C'_S) \subset C_{m,S}, \quad S_j := f^{-1}(S'_j) \subset \partial C_S \subset C_{m,S}.$$

Recall that S'_j separates M' , hence also separates $\widehat{C}' \# N'_1 \# \dots \# N'_\ell$. It follows that S_j separates \widehat{C} , hence also $C_{m,S} = \widehat{C} \setminus \text{Int}(D \cup D_1 \cup \dots \cup D_s)$. As each boundary sphere of $C_{m,S}$ separates M , it follows that each S_j separates M , which implies property (1) of Proposition 5.5. To verify property (2) let us notice that we may deform $f|_{C_S}$ near S into a diffeomorphism $\Phi_{n+1,S} : C_S \rightarrow C'_{S'}$ that satisfies

$$\Phi_{n+1,S|_S} = \Phi_{n|_S} : S \longrightarrow S' = \Phi_n(S).$$

Indeed, the space of orientation-preserving diffeomorphisms of S^2 is path-connected and this follows, for example, from [Sma59]. To check property (3) observe that the connected

component U_j of $M \setminus C_S$ bounded by S_j contains the union of connected components V_j of $M \setminus B_m$ bounded by Γ_j and that $U_j \setminus V_j \subset B_m$ is bounded. Therefore $U_j^* = V_j^*$, from which we infer that $\phi(U_j^*) = \phi(V_j^*) = V_j'^*$ as required. This concludes the proof of Lemma 5.5. As explained before, extending Φ_n by $\Phi_{n+1,S}$ in order to define Φ_{n+1} concludes the proof of the induction.

We only need now to explain why assertion (H_0) is true. Let us set $A'_0 := B'_0$. As (B_n) is an exhaustion of M and $n_k(M) = n_k(M')$, for all $0 < k \leq r$, one can find $m_0 \in \mathbf{M}$ such that, for all $0 < k \leq r$ and for all $m \geq m_0$,

$$n_k(B_m) \geq n_k(A'_0).$$

We denote by S'_1, \dots, S'_l the components of $\partial A'_0$ and by V_j the components of $M' \setminus A'_0$ bounded by S'_j , for $j = 1, \dots, l$. As in Lemma 5.5, there exists m_1 such that, for all $m \geq m_1$, if $(U_{i,m})$ denotes the components of $M \setminus B_m$, then, for all i and j ,

$$\phi(U_{i,m})^* \subset (V'_j)^*.$$

Let us fix such $m \geq \sup\{m_0, m_1\}$. We can now argue as in Lemma 5.5 with A'_0 instead of C'_S to define a submanifold $A_0 \subset B_m$ and an isomorphism $\Phi_0 : A_0 \rightarrow A'_0$ such that $S_j := \Phi_0^{-1}(S'_j)$ separates M and such that the component U_j of $M \setminus A_0$ bounded by S_j is good and satisfies

$$\Phi(U_j^*) = (V'_j)^*.$$

□

6 Final remark

The above results exhibit a subtle interaction between the combinatorial structure of trees and the topological structure of open decomposable 3-manifolds. It is an interesting question to make this relation more precise and in particular to try to find the more efficient tree representing, up to homeomorphism, a given decomposable 3-manifold. If the manifold is closed, then a linear, finite tree is a good option but if it is open we do not have, at the moment, any hint regarding an optimal description.

A Space of ends

Let X be a connected, locally compact, metrisable, separable topological space. We say that a subset of X is *bounded* if it is contained in some compact subset. Otherwise it is

unbounded. A *pre-end* of X is a decreasing sequence

$$e = U_1 \supset U_2 \supset \dots,$$

of open subsets of X such that:

1. Every U_i is connected, unbounded, and has compact boundary, and
2. For every bounded $A \subset X$, the set $A \cap U_i$ is eventually empty.

Let $e = U_1 \supset U_2 \supset \dots$ and $e' = U'_1 \supset U'_2 \supset \dots$ be pre-ends of X . We say that e is *contained in* e' , and write $e \subset e'$ if for every $n' \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $U_n \subset U'_{n'}$. We define an equivalence relation by letting $e \sim e'$ if $e \subset e'$ and $e' \subset e$. The equivalence class of e is denoted by e^* . We call *end* of X such an equivalence class, and denote by $E(X)$ the set of ends of X .

For every open subset $U \in X$ with compact boundary we set

$$U^* := \{e^* \in E(X); e = U_1 \supset U_2 \supset \dots, \text{ and } U_i \subset U \quad \forall i \text{ large enough}\}.$$

Those sets form a basis for a topology on $E(X)$. Hereafter the set $E(X)$ is endowed with this topology and called the *space of ends* of X . Freudenthal [Fre31] proved that this space is compact, metrisable and totally disconnected. Hence it embeds into the Cantor set (see e.g. [Kec95, 7.B and 7.D]).

The following technical result is well-known. Having being unable to find a reference, we provide a proof.

Lemma A.1. *Let M be a connected open manifold and (K_n) be an exhaustion of M . Let $\{U_{n,i}\}$ be the collection of all unbounded components with compact boundary of all the sets $M \setminus K_n$. Then the topology on $E(M)$ is generated by the $(U_{i,n}^*)$'s.*

Proof. First we show that any end of M can be represented by $U_1 \supset U_2 \supset \dots$ where $U_j \in \{U_{n,i}\}$.

Let $e = U'_1 \supset U'_2 \supset \dots$ be a pre-end of M . By compactness of $\partial U'_j$, one has $\partial U'_j \subset \overset{\circ}{K}_n$ for n large enough. Fix an increasing sequence $n(j)$ such that $\partial U'_j \subset \overset{\circ}{K}_{n(j)}$ for all j . As U'_j is unbounded, it meets $M \setminus K_{n(j)}$, hence some component $U_{n(j),i}$. Notice that any component $U_{n(j),i}$ of $M \setminus K_{n(j)}$ which meets U'_j is contained in U'_j : if not, there exists a path $\gamma \subset U_{n(j),i}$ going from U'_j to its complement, hence crossing $\partial U'_j \subset \overset{\circ}{K}_{n(j)} \subset M \setminus U_{n(j),i}$, a contradiction. Now, for each j choose $m(j)$ large enough so that $K_{n(j)} \cap U'_{m(j)} = \emptyset$. Then select the component $U_{n(j),i(j)}$ of $M \setminus K_{n(j)}$ which meets $U'_{m(j)}$. By connectedness we have

$$U'_j \supset U_{n(j),i(j)} \supset U'_{m(j)}.$$

Setting $U_j := U_{n(j),i(j)}$, the pre-end $U_1 \supset U_2 \supset \dots$ is equivalent to e . The family $\{U_{i,n}^*\}$ is an open cover of $E(X)$. Any two U, V in $\{U_{n,i}\}$ satisfy $U \cap V = \emptyset$ or $U \subset V$ or $U \supset V$. The same property holds for U^*, V^* in $\{U_{n,i}^*\}$ and implies that $\{U_{n,i}^*\}$ is a basis for some topology.

Let U^* be an element of a basis for the topology of $E(X)$, and let $e^* \in U^*$. We can represent e^* by $e = U_1 \supset U_2 \dots$ where $U_j \in \{U_{n,i}\}$ and by definition there exists j such that $U \supset U_j$. Then $U^* \supset U_j^* \ni e^*$. This shows that $\{U_{n,i}^*\}$ is a basis for the topology of $E(X)$. \square

Let e be a pre-end of X and U be an open subset of X . We say that U is a *neighbourhood* of e^* if $e^* \in U^*$. Thus a pre-end is a non-increasing sequence of neighbourhoods of the associated end.

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