Delay-robust stabilization of an n + m hyperbolic PDE-ODE system

Jean Auriol, Federico Bribiesca Argomedo

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Jean Auriol[1], Federico Bribiesca-Argomedo[2]

***Abstract*— In this paper, we study the problem of stabilizing a linear ordinary differential equation through a system of an $n+m$ (hetero-directional) coupled hyperbolic equations in the actuating path. The method relies on the use of a backstepping transform to construct a first feedback to tackle in-domain couplings present in the PDE system and then on a predictive tracking controller used to stabilize the ODE. The proposed control law is robust with respect to small delays in the control signal.**

I. INTRODUCTION

In this paper we present a full-state feedback control design that delay-robustly stabilizes an interconnected system composed of a general system of coupled hetero-directional transport equations coupled through their unactuated boundary to a linear Ordinary Differential Equation (ODE). This delay-robust feedback law is obtained by partially leveraging the backstepping design in [16] and using the explicit mapping between hyperbolic systems and time-delay systems proposed in [5]. Using such a time-delay formulation, we combine a full-state feedback of the Partial Differential Equation (PDE) states (to stabilize the PDE system) with a predictor-based tracking controller (that stabilizes the ODE).

The control of interconnected ODEs and hyperbolic PDEs is an extremely active research topic [8], [10], [16], [27]. This class of systems naturally appears when modeling delays (that can be seen as first-order hyperbolic PDEs) in the actuating and sensing paths of ODEs [8], [9], [17], [29], [30], [33]. For instance, one can consider the problem of the attenuation of mechanical vibrations in drilling applications (see [28] for a review of drilling vibrations models). For such a system, the hyperbolic PDEs model axial and torsional stress propagation (with finite speed of propagation) along the drill string. At the same time, the dynamics of the Bottom hole Assembly (BHA) can be represented by an ODE. The problem of delays acting on ODE systems has been tackled by the Smith predictor [31]. Expressing these delayed terms as PDE states of transport equations [21], numerous related problems have been solved (non-constant and uncertain delays for instance) [7], [9]. To solve the general problem of stabilizing an ODE with a system of first-order linear hyperbolic PDEs in the actuator path, a backstepping transformation has been introduced in [16] to map the original system into a cascade of exponentially stable subsystems. The major drawback of such a transformation is that it cancelled (among other terms) all the reflection terms at the actuated boundary. Although this is mathematically correct, such an approach completely neglects the impact on stability of small delays in the feedback loop (delay-robustness). Regarding the control of linear hyperbolic systems, it has been shown in recent contributions [2] inspired by [14], [23] (in which it has been observed that some feedback systems possess a zero delay margin under linear state feedback) that some boundary reflections should not be compensated by the control action. Thus, a new approach, by leveraging the backstepping design given in [16] and complementing it with a predictor-type feedback has been proposed in [3] to guarantee the delay-robust stabilization of two linear PDEs coupled with an ODE. The control design and robustness analysis techniques developed in [2], [3] rely on a reformulation of the PDEs system as a difference equation. If only two coupled PDEs have been considered in [2], [3], such a mapping between hyperbolic PDEs and difference systems has been explicitly given in [5] for an arbitrary number of PDEs, opening the perspective of adapting on hyperbolic PDE systems the stability analysis methods for time delay systems, such as those developed in [13], [18], [26]. In recent contributions [1], an alternative approach has been considered to solve the problem of delay robust control design of an under-actuated PDE-ODE-PDE for which the PDEs are scalar (the last one being a transport equation). Finally, in [15], the authors consider the stabilization of an ODE-PDE-ODE system (neglecting however the robustness properties of the feedback law).

The main contribution of this paper is the development of a new state-feedback control design (compared to [16]) for a PDE-ODE system, that ensures its delay-robust stabilization. Delay-robustness is guaranteed at the cost of preserving the proximal reflection terms in the target system. The approach of [3] cannot be directly extended due to the non-scalar structure of the problem. In this paper we proceed as follows: (i) Using two successive backstepping transformations (and associated feedback operators) adjusted from [5], [11], we remove most of the in-domain couplings present in the PDEs. Without these in-domain couplings, we can prove that the stability properties of the original system are equivalent to the ones of an ODE coupled with a neutral system. (ii) The stabilization problem of the ODE-neutral system is reduced to the one of a linear ODE system with delayed input by solving a tracking problem. We then design a state-preictor feedback law. (iii) Finally, we study the delay-robustness properties of the system by algebraic means. The paper is organized as follows. In Section II we introduce the model under considerationand the notations. In Section III, we present the stabilization result: using successive backstepping transformations, we rewrite the original system as a distributed delay equation coupled with an ODE.After this reformulation, we design a stabilizing control law, shown to have a non-zero delay margin in Section IV. Finally, some simulation results are presented in Section V.

II. PROBLEM FORMULATION

A. *Definitions and notations*

In this section we detail the notations used through this paper. For any integer $p>0$, $||\cdot||_{\mathbb{R}^p}$ is the classical euclidean norm on $\mathbb{R}^p$. We denote $L^2([0,1],\mathbb{R})$ the space of real-valued square-integrable functions defined on $[0,1]$ with the

[1]Jean Auriol is with University of Calgary, Department of Petroleum Engineering, Calgary, Canada. jean.auriol@ucalgary.ca
[2]Federico Bribiesca-Argomedo is with University of Lyon, INSA Lyon, Laboratoire Ampère CNRS UMR5005, F-69621, Villeurbanne, France

standard L^2 norm, *i.e.*, for any $f \in L^2([0,1],\mathbb{R})$, $||f||^2_{L^2} = \int_0^1 f^2(x)\mathrm{d}x$. The set $L^\infty([0,1],\mathbb{R})$ denotes the space of bounded real-valued functions defined on $[0,1]$ with the standard L^∞ norm, *i.e.*, for any $f \in L^\infty([0,1],\mathbb{R})$, $||f||_{L^\infty} = \sup_{x\in[0,1]} |f(x)|$. In the following, for $(u,v,X) \in (L^2([0,1]))^{n+m} \times \mathbb{R}^p$, we define the norm

$$||(u,v,X)|| = \sum_{i=1}^n ||u_i||_{L^2} + \sum_{i=1}^m ||v_i||_{L^2} + ||X||_{\mathbb{R}^p}. \quad (1)$$

The set $\mathcal{C}^p([0,1])$ (with $p \in \mathbb{N} \cup \{\infty\}$) stands for the space of real-valued functions defined on $[0,1]$ that are p times differentiable and whose p-th derivative is continuous. The set $\mathcal{T}$ is defined as

$$\mathcal{T} = \{(x,\xi) \in [0,1]^2 \text{ s.t. } \xi \leq x\}. \quad (2)$$

$L^\infty(\mathcal{T})$ stands for the space of real-valued L^∞ functions on $\mathcal{T}$. For any $(p,q) \in \mathbb{N}$, we denote $\mathcal{M}^{p\times q}(\mathbb{R})$ the set of matrices with p rows and q columns. The symbol I_p (or I if no confusion arises) represents the $p \times p$ identity matrix. We use the notation $\hat{f}(s)$ for the Laplace transform of a function $f(t)$, provided it is well defined. The set $\mathcal{A}$ stands for the convolution Banach algebra of BIBO-stable generalized functions in the sense of [32]. A function $g(\cdot)$ belongs to $\mathcal{A}$ if it can be expressed as $g(t) = g_r(t) + \sum_{i=0}^\infty g_i \delta(t-t_i)$, where $g_r \in L^1(\mathbb{R}^+,\mathbb{R})$, $\sum_{i\geq 0} |g_i| < \infty$, $0 = t_0 < t_1 < \ldots$ and $\delta(\cdot)$ is the Dirac distribution. The associated norm is $\|g\|_{\mathcal{A}} = \|g_r\|_{L^1} + \sum_{i\geq 0} |g_i|$. The set $\hat{\mathcal{A}}$ of Laplace transforms of elements in $\mathcal{A}$ is also a Banach algebra with associated norm $\|\hat{g}\|_{\hat{\mathcal{A}}} = \|g\|_{\mathcal{A}}$.

B. System under consideration

We consider a class of systems consisting of an ODE coupled to a general linear heterodirectional first-order hyperbolic system in the actuation path.

More precisely, we consider systems of the form:

$$\partial_t u(t,x) + \Lambda^+ \partial_x u(t,x) = \Sigma^{++}(x)u + \Sigma^{+-}(x)v, \quad (3)$$
$$\partial_t v(t,x) - \Lambda^- \partial_x v(t,x) = \Sigma^{+-}(x)u + \Sigma^{--}(x)v, \quad (4)$$
$$\dot{X}(t) = AX(t) + Bv(t,0), \quad (5)$$

evolving in $\{(t,x) \text{ s.t. } t>0,\ x \in [0,1]\}$, with the boundary conditions

$$\begin{aligned} u(t,0) &= Qv(t,0) + CX(t), \\ v(t,1) &= Ru(t,1) + V(t), \end{aligned} \quad (6)$$

where $X \in \mathbb{R}^p$ is the ODE state, and $u = (u_1 \ldots u_n)^T, v = (v_1 \ldots v_m)^T$ are the PDE states. The matrices Λ^+ and Λ^- are diagonal ($\Lambda^+ = \mathrm{diag}(\lambda_i)$, $\Lambda^- = \mathrm{diag}(\mu_i)$) and their coefficients satisfy $-\mu_m \leq \cdots \leq -\mu_1 < 0 < \lambda_1 \leq \cdots \leq \lambda_n$. The constant real boundary coupling matrices are defined by $Q = \{q_{ij}\}_{1\leq i\leq n, 1\leq j\leq m}$ and $R = \{\rho_{ij}\}_{1\leq i\leq m, 1\leq j\leq n}$. The spatially-varying inside domain couplings matrices are defined as follows

$$\Sigma^{++}(x) = \{\sigma^{++}_{ij}(x)\}_{1\leq i,j\leq n} \quad \Sigma^{+-}(x) = \{\sigma^{+-}_{ij}(x)\}_{1\leq i\leq n, 1\leq j\leq m}$$
$$\Sigma^{-+}(x) = \{\sigma^{-+}_{ij}(x)\}_{1\leq i\leq m, 1\leq j\leq n} \quad \Sigma^{--}(x) = \{\sigma^{--}_{ij}(x)\}_{1\leq i,j\leq m},$$

their coefficients are assumed to belong in $\mathcal{C}^0([0,1],\mathbb{R})$. We assume that the diagonal terms of the matrices Σ^{++} and Σ^{--} are equal to zero (such coupling terms can be removed using a change of coordinates, see [12] for details). The matrices A, B and C respectively belong to $\mathcal{M}^{p\times p}(\mathbb{R})$, $\mathcal{M}^{p\times m}(\mathbb{R})$ and $\mathcal{M}^{n\times p}(\mathbb{R})$ The initial conditions of the state (u,v) are denoted u_0 and v_0 and are assumed to respectively belong to $(L^2([0,1],\mathbb{R}))^n$ and $(L^2([0,1],\mathbb{R}))^m$. The initial condition of the ODE (5) is denoted X_0. The function V is an input function (control law) that has values in $\mathbb{R}^m$. The resulting system (3)-(6) is well-posed [6, Theorem A.6, page 254]. Remark that this system naturally features several couplings that can be source of instabilities. Finally, we denote $\tau = \frac{1}{\lambda_1} + \frac{1}{\mu_1}$ the sum of the largest transport times in each direction.

C. Control problem

Let us consider the two following assumptions

Assumption 1: The pair (A,B) is stabilizable, i.e. there exists a matrix K such that $A+BK$ is Hurwitz.

Assumption 2: The system defined for all $i \in [1,m]$ by

$$z_i(t) = \sum_{\substack{1\leq l\leq m \\ 1\leq k\leq n}} R_{ik} Q_{kl} z_l(t - \frac{1}{\lambda_k} - \frac{1}{\mu_l}). \quad (7)$$

is exponentially stable

In this paper, we show that if Assumption 1 and Assumption 2 are satisfied, then it is possible to explicitly design a feedback control law $V = \mathcal{K}[(u,v,X)]$ (where $\mathcal{K} : (L^2[0,1])^2 \times \mathbb{R}^p \to \mathbb{R}$ is a linear operator) that **delay-robustly stabilizes** (in the sense of [23]) system (3)-(6), i.e.:

- the state (u,v,X) of the resulting feedback system (3)-(6) exponentially converges to its zero equilibrium (**stabilization problem**), *i.e.* there exist $\kappa_0 \geq 0$ and $\nu > 0$ such that for any initial condition $(u_0, v_0, X_0) \in (L^2[0,1])^2 \times \mathbb{R}^p$

$$||(u,v,X)|| \leq \kappa_0 \mathrm{e}^{-\nu t} ||(u_0,v_0,X_0)||,\ t \geq 0. \quad (8)$$

- the resulting feedback system (3)-(6) is robustly stable with respect to small delays in the loop (**delay-robustness**), *i.e.* there exists $\delta^\star > 0$ such that for any $\delta_i (\in [0,\delta^\star])^m$, the control law whose components are defined by $(V(t-\delta_i))_i$ still stabilizes (3)-(6).

The first assumption (stabilizability of the ODE subsystem) is necessary for the stabilizability of the whole system. It can be shown that violating the second assumption results in an open-loop transfer function with an infinite number of poles on the closed right half-plane.Consequently (see [23, Theorem 1.2]), one cannot find any linear state feedback law $V(\cdot)$ that delay-robustly stabilizes (3)-(6). Note that if the delays $\frac{1}{\lambda_k} + \frac{1}{\mu_l}$ are rationally independent a necessary and sufficient condition for this assumption to be satisfied is given by [19, Theorem 6.1] in terms of spectral radius.

III. Design of the control law

In this section we derive a control law that guarantees the stabilization of (3)-(6), following the methodology introduced above. Using two successive backstepping transformations, we map the original system to a simpler target system for which local coupling terms $\Sigma^{\cdot\cdot}$ have been moved to the boundary. The target system is recast as a neutral system for which a predictor-based control law is developed.

A. Backstepping transformations

We derive two backstepping transformations to remove the in-domain coupling terms of (3)-(6). These transforms will introduce some non-local coupling terms yet keep pointwise coupling terms, ensuring a non-zero delay margin for the system.Let us consider the following Volterra transformation adjusted from the one defined in [5], [11], [20]

$$\begin{aligned}\alpha = u- &\int_0^x \left(K^{uu}(x,\xi)u(\xi) + K^{uv}(x,\xi)v(\xi)\right) d\xi \\ &+ \gamma_0(x)X(t), \end{aligned} \tag{9}$$

$$\begin{aligned}\beta = v- &\int_0^x \left(K^{vu}(x,\xi)u(\xi) + K^{vv}(x,\xi)v(\xi)\right) d\xi, \\ &+ \gamma_1(x)X(t), \end{aligned} \tag{10}$$

where the kernels $K^{uu}, K^{uv}, K^{vu}, K^{vv}$ belong to $L^\infty(\mathcal{T})$ (where $\mathcal{T}$ is introduced in (2)), while γ_0 and γ_1 are differentiable matrices defined on $([0,1])$. They satisfy the following set of PDEs

$$\Lambda^+\partial_x K^{uu} + \partial_\xi K^{uu}\Lambda^+ = -K^{uu}\Sigma^{++}(\xi) - K^{uv}\Sigma^{-+}(\xi), \tag{11}$$
$$\Lambda^+\partial_x K^{uv} - \partial_\xi K^{uv}\Lambda^- = -K^{uu}\Sigma^{+-}(\xi) - K^{uv}\Sigma^{--}(\xi), \tag{12}$$
$$\Lambda^-\partial_x K^{vu} - \partial_\xi K^{vu}\Lambda^+ = K^{vu}\Sigma^{++}(\xi) + K^{vv}\Sigma^{-+}(\xi), \tag{13}$$
$$\Lambda^-\partial_x K^{vv} + \partial_\xi K^{vv}\Lambda^- = K^{vu}\Sigma^{+-}(\xi) + K^{vv}\Sigma^{--}(\xi), \tag{14}$$

and ODEs

$$\Lambda^+\gamma_0'(x) = -\gamma_0(x)A + K^{uu}(x,0)\Lambda^+C, \tag{15}$$
$$\Lambda^-\gamma_1'(x) = \gamma_1(x)A - K^{vu}(x,0)\Lambda^+C, \tag{16}$$

with the boundary conditions

$$\Sigma^{++}(x) - \Lambda^+K^{uu}(x,x) + K^{uu}(x,x)\Lambda^+ = 0, \tag{17}$$
$$\Sigma^{+-}(x) - \Lambda^+K^{uv}(x,x) - K^{uv}(x,x)\Lambda^- = 0, \tag{18}$$
$$\Sigma^{-+}(x) + \Lambda^-K^{vu}(x,x) + K^{vu}(x,x)\Lambda^+ = 0, \tag{19}$$
$$\Sigma^{--}(x) + \Lambda^-K^{vv}(x,x) - K^{vv}(x,x)\Lambda^- = 0, \tag{20}$$
$$(K^{vv}(x,0)\Lambda^-)_{ij} = (K^{vu}(x,0)\Lambda^+Q - \gamma_1(x)B)_{ij} \tag{21}$$
$$\gamma_0(0) = 0, \quad \gamma_1(0) = 0, \tag{22}$$

where in equation (21), we have $1 \le j \le i \le m$. To ensure well-posedness of the kernel equations, we add the following artificial boundary conditions for $K^{uu}_{ij}(n \ge i > j \ge 1)$ on $\xi = 0$: $K^{uu}_{ij}(x,0) = k_{ij}(x) \in \mathcal{C}^1([0,1])$. We have the following lemma

Lemma 1: Consider system (11)-(22). There exists a unique solution K^{uu}, K^{uv}, K^{vu} and K^{vv} in $L^\infty(\mathcal{T})$ and differentiable matrices γ_0, γ_1.

Proof: This result follows, with some minor adaptations, from [20] and [16, Theorem 3.2]. The main idea consists in reinterpreting the ODEs as PDEs evolving in the triangular domain $\mathcal{T}$ with horizontal characteristic lines (since there is only an evolution along x) and then solving all the PDEs together. In the case $n = m = 1$, the proof can be found in [3]. Due to page limitation, we do not give the complete proof here. ■

Since the transformation (9)-(10) is a Volterra transformation, it is invertible (see e.g. [22]). The corresponding inverse kernels are denoted $L^{\alpha\alpha}, L^{\alpha\beta}, L^{\beta\alpha}, L^{\beta\beta}, \bar{\gamma}_0$ and $\bar{\gamma}_1$. They have the same regularity.

$$\begin{aligned}u = \alpha- &\int_0^x \left(L^{\alpha\alpha}(x,\xi)\alpha(\xi) + L^{\alpha\beta}(x,\xi)\beta(\xi)\right) d\xi \\ &+ \bar{\gamma}_0(x)X(t), \end{aligned} \tag{23}$$

$$\begin{aligned}v = \beta- &\int_0^x \left(L^{\beta\alpha}(x,\xi)\alpha(\xi) + L^{\beta\beta}(x,\xi)\beta(\xi)\right) d\xi, \\ &+ \bar{\gamma}_1(x)X(t). \end{aligned} \tag{24}$$

Defining the L^∞ matrix $G_1(x) = \gamma_0(x)B - K^{uu}(x,0)\Lambda^+ Q + K^{uv}(x,0)\Lambda^-$, and the **upper-triangular** matrix $G_2(x) = -K^{vu}(x,0)\Lambda^+Q + K^{vv}(x,0)\Lambda^- + \gamma_1(x)B$, the invertible transformation (9)-(10) maps the original system (3)-(6) to the target system

$$\partial_t\alpha(t,x) + \Lambda^+\partial_x\alpha(t,x) = G_1(x)\beta(t,0), \tag{25}$$
$$\partial_t\beta(t,x) - \Lambda^-\partial_x\beta(t,x) = G_2(x)\beta(t,0), \tag{26}$$
$$\dot{X}(t) = AX(t) + B\beta(t,0), \tag{27}$$

along with the boundary conditions

$$\alpha(t,0) = Q\beta(t,0) + CX(t), \tag{28}$$
$$\begin{aligned}\beta(t,1) = &R\alpha(t,1) + V(t) + (R\bar{\gamma}_0(1) - \bar{\gamma}_1(1))X(t) \\ &+ \int_0^1 \left(N^\alpha(\xi)\alpha(t,\xi) + N^\beta(\xi)\beta(t,\xi)\right) d\xi, \end{aligned} \tag{29}$$

where $N^\alpha(\xi) = L^{\beta\alpha}(1,\xi) - RL^{\alpha\alpha}(1,\xi)$ and $N^\beta(\xi) = L^{\beta\beta}(1,\xi) - RL^{\alpha\beta}(1,\xi)$. The associated initial condition, denoted (α_0, β_0, X_0), is related to the initial condition (u_0, v_0, X_0) by the transformation (9)-(10). Let us now consider the following transformation which is invertible due to its cascade structure

$$\alpha(t,x) = w(t,x), \tag{30}$$
$$\beta(t,x) = z(t,x) - \int_0^1 F(x,\xi)z(t,\xi)d\xi, \tag{31}$$

where the kernel F is a strict upper triangular matrix (i.e $F_{ij}(x,\xi) = 0$ if $i \ge j$) defined on $\mathcal{T}_1 = \{(x,\xi) \in [0,1]^2\}$ by the following set of PDEs

$$\Lambda^-F_x(x,\xi) + F_\xi(x,\xi)\Lambda^- = 0, \tag{32}$$

along with the boundary conditions

$$F(x,0) = G_2(x)(\Lambda^-)^{-1}, \quad F(0,\xi) = 0. \tag{33}$$

It is straightforward to show that this set of PDEs admits a unique solution $F \in L^\infty([0,1] \times [0,1])^{m\times m}$ whose components can explicitly be obtained by the method of characteristics [11]. Differentiating (30)-(31) with respect to time and space, we obtain that the system (25)-(29) (and consequently the system (3)-(6)) is equivalent to

$$\partial_t w(t,x) + \Lambda^+\partial_x w(t,x) = G_1(x)z(t,0), \tag{34}$$
$$\partial_t z(t,x) - \Lambda^-\partial_x z(t,x) = G_3(x)z(t,1), \tag{35}$$
$$\dot{X}(t) = AX(t) + Bz(t,0), \tag{36}$$

with the boundary conditions

$$w(t,0) = Qz(t,0) + CX(t), \tag{37}$$
$$\begin{aligned}z(t,1) = &Rw(t,1) + V(t) + (R\bar{\gamma}_0(1) - \bar{\gamma}_1(1))X(t) \\ &+ \int_0^1 N^\alpha(\xi)w(t,\xi) + N^z(\xi)z(t,\xi)d\xi, \end{aligned} \tag{38}$$

where G_3 is the unique L^∞solution of the equation $G_3(x) = F(x,1)\Lambda^- + \int_0^1 F(x,\xi)G_3(\xi)d\xi$ (which can be directly solved due to the triangular structure of F). and where $N^z(\xi) = F(1,\xi)+N^\beta(\xi) - \int_0^1 N^\beta(\nu)F(\nu,\xi)d\nu$. Note that G_3 is **upper-triangular**. For the control design of the target system (34)-(36) with the boundary conditions (37)-(38), we decompose the control input $V(t)$ as $V(t) = V^1(t) + V^0(t)$, where $V^1(\cdot)$ will be designed in the next sectionsand where $V^0(t)$ is defined by,

$$\begin{aligned} V^0(t) = &-(R\bar{\gamma}_0(1) - \bar{\gamma}_1(1))X(t) \\ &- \int_0^1 (N^\alpha(\xi)w(t,\xi) + N^z(\xi)z(t,\xi))\mathrm{d}\xi, \end{aligned} \quad (39)$$

so that (38) rewrites $z(t,1) = Rw(t,1)+V^0(t)$. Remark that, due to the invertibility of the backstepping transformations (9)-(10) and (30)-(31), $V^0(t)$ can be expressed in terms of u, v and X.

B. A neutral system satisfied by $z(t,1)$

Using the specific structure of the target system (34)-(38), it is possible to express the state $z(t,1)$ as the solution of a neutral system. We have the following theorem.

Theorem 1: There exists $G_4(\cdot)$ an $L^\infty([0,\tau])$ function such that for all $1 \le j \le m$ and all $1 \le i \le n$ and all $t \ge \tau$

$$\begin{aligned} z_j(t,x) &= z_j(t - \frac{1-x}{\mu_j}, 1) + \sum_{k=1}^m \int_0^{\frac{1-x}{\mu_j}} (G_3)_{jk}(x+\mu_j s) \\ &\cdot z_k(t-s,1)ds, \end{aligned} \quad (40)$$

$$\begin{aligned} w_i(t,x) &= \sum_{k=1}^m (Q)_{ik} z_k(t - \frac{x}{\lambda_i} - \frac{1}{\mu_k}, 1) + \sum_{k=1}^p C_{ik}X_k(t- \\ &\frac{x}{\lambda_i}) + \sum_{l=1}^m \int_0^\tau (G_4)_{il}(x,\eta) z_l(t-\eta,1)d\eta. \end{aligned} \quad (41)$$

Consequently, combining this equation with (38), there exists $m \times m$ $L^\infty([0,\tau])$-functions denoted $G_j^l(\cdot)$ $(1 \le j,l \le m)$ such that for all $t \ge \tau$, for all $1 \le i \le m$ we have

$$\begin{aligned} z_i(t,1) &= V_i^1(t) + \sum_{k=1}^n \sum_{l=1}^m R_{ik}Q_{kl} z_l(t - \frac{1}{\lambda_k} - \frac{1}{\mu_l}, 1) + \\ &\sum_{l=1}^m \int_0^\tau G_i^l(s) z_l(t-s)ds + \sum_{k,l=1}^{n,p} R_{ik}C_{kl}X_l(t - \frac{1}{\lambda_k}) \end{aligned} \quad (42)$$

Proof: The proof follows the same steps as the ones of [5, Theorem 4]. It relies on the method of characteristics. It is not given here, due to the page limitation. ■

In the following we decompose the control law $V^1(t)$ as $V^1(t) = V_{BS}(t) + V_{ODE}(t)$ where V_{BS} is defined for all $1 \le i \le m$ by

$$\begin{aligned} (V_{BS})_i(t) = &-\sum_{k=1}^n \sum_{l=1}^p R_{ik}C_{kl}X_l(t - \frac{1}{\lambda_k}) \\ &- \int_0^\tau G_i^l(s) z_l(t-s)ds, \end{aligned} \quad (43)$$

while $V_{ODE}(\cdot)$ has to be designed for the stabilization of the ODE dynamics. Remark that V_{BS} can be expressed as delayed values of u, v and X. With this control law, equation (42) rewrites

$$z_i(t,1) = \sum_{k=1}^n \sum_{l=1}^m R_{ik}Q_{kl} z_l(t - \frac{1}{\lambda_k} - \frac{1}{\mu_l}, 1) + V_{ODE}(t). \quad (44)$$

As the principal part of system (44) generates an exponentially stable semigroup (due to Assumption 2), the control law V_{BS} would guarantee the stabilization of $z(t,1)$ in the absence of the ODE. The objective now is to design V_{ODE} such that the ODE state also converges to zero. If such a stabilizing control law only depends on $X(\cdot)$, then once the ODE has been stabilized, we are brought to the previous situation and have the convergence of $z(t,1)$ to zero, and consequently of the system (w,z), due to the cascade structure of system (34)-(38) (and using the invertibility of the backstepping transformations, the stability of the system (u,v)).

Remark 1: Contrary to [16] ,we do not cancel the principal part $\sum_{k=1}^n \sum_{l=1}^m R_{ik}Q_{kl}z_l(t - \frac{1}{\lambda_k} - \frac{1}{\mu_l}, 1)$ in the control law to guarantee the existence of some delay-margins (see [2] for details). However, as it is done in [5, Theorem 5], we could cancel a small part of these reflection terms to improve the convergence rate.

C. A tracking problem solved by a predictor

To stabilize the ODE (36), we would like to directly control $z(t,0)$, and ideally we would like $z(t,0) = KX(t)$ (where K is defined in Assumption 1 and is such that $A + BK$ is Hurwitz). This corresponds to a tracking problem. Using equation (35) (and the fact that G_3 is upper triangular), we have for all $1 \le i \le m$

$$\begin{aligned} z_i(t,0) &= z_i(t - \frac{1}{\mu_i}, 1) \\ &+ \sum_{k=i+1}^m \int_0^{\frac{1}{\mu_i}} (G_3)_{ik}(\mu_i s) z_k(t-s,1)ds. \end{aligned}$$

This yields

$$\begin{aligned} z_i(t,1) &= z_i(t + \frac{1}{\mu_i}, 0) \\ &- \sum_{k=i+1}^m \int_0^{\frac{1}{\mu_i}} (G_3)_{ik}(\mu_i s) z_k(t + \frac{1}{\mu_i} - s, 1)ds, \end{aligned} \quad (45)$$

and using the cascade structure of the integral part, we can express $z(t,1)$ as a function of **future values** of $z(\cdot,0)$ which may lead to a non causal problem. Let us denote $\frac{1}{\bar{\mu}} = \sum_{k=1}^m \frac{1}{\mu_k}$ and $\bar{z}(t,0) = z(t + \frac{1}{\bar{\mu}}, 0)$. If instead of tracking $z(\cdot,0)$, we choose to track $\bar{z}(\cdot,0)$, the problem becomes causal as equation (45) rewrites $z_i(t,1) = \bar{z}_i(t - \frac{1}{\bar{\mu}} + \frac{1}{\mu_i}, 0) - \sum_{k=i+1}^m \int_0^{\frac{1}{\mu_i}} (G_3)_{ik}(\mu_i s)\bar{z}_k(t + \frac{1}{\mu_i} - \frac{1}{\bar{\mu}} - s, 1)ds$. Thus, if we want $\bar{z}(\cdot,0)$ to be equal to an arbitrary function $\zeta(\cdot)$, it immediately implies

$$z_m(t,1) = \zeta_m(t - \frac{1}{\bar{\mu}} + \frac{1}{\mu_m}), \quad (46)$$

$$\vdots$$

$$\begin{aligned} z_1(t,1) &= \zeta_1(t - \frac{1}{\bar{\mu}} + \frac{1}{\mu_1}) \\ &- \sum_{k=2}^m \int_0^{\frac{1}{\mu_1}} (G_3)_{1k}(\mu_1 s) z_k(t + \frac{1}{\mu_1} - s, 1)ds. \end{aligned} \quad (47)$$

In what follows, the function ζ will be chosen to stabilize the ODE system. Using equations (46)-(47), it becomes possible to express the state $z(t,1)$ as a function of $\zeta(\cdot)$. More precisely, we have the following lemma

Lemma 2: There exist $L^\infty([0,\frac{1}{\bar\mu}])$-functions denoted $H^l_j(\cdot)$ $(1\le j,l\le m)$ that only depend on G_3 and on ζ, such that, if for all $1\le i\le m$ we have

$$z_i(t,1)=\chi(t)=\zeta_i(t-\frac{1}{\bar\mu}+\frac{1}{\mu_i}) + \sum_{k=i+1}^{m}\int_{\frac{1}{\bar\mu}-\sum_{l=k}^m\frac{1}{\mu_l}}^{\frac{1}{\bar\mu}} H_{ik}(s)\zeta_k(t-s)ds, \quad (48)$$

then $\bar z_i(t,0)=\zeta_i(t)$ and consequently $z_i(t,0)=\zeta_i(t-\frac{1}{\bar\mu})$.

Proof: The proof is straightforward by recursion, the initialization $(i=m+1)$ corresponding to (46). ■

Let us assume that we have $z(t,0)=\zeta(t-\frac{1}{\bar\mu})$. The ODE system (36) rewrites for $t\ge\tau+\frac{1}{\bar\mu}$

$$\dot X(t)=AX(t)+B\zeta(t-\frac{1}{\bar\mu}), \quad (49)$$

which is a finite-dimensional system with a delayed input $\zeta(\cdot)$. Different methods [34] can be used to design a control law that stabilizes equation (49). A classical result from [24] states that any control law that stabilizes such an equation is equivalent to a predictor. More precisely, let us define the function P by

$$P(t)=K\left(\mathrm{e}^{\frac{A}{\bar\mu}}X(t)+\int_{t-\frac{1}{\bar\mu}}^{t}\mathrm{e}^{A(t-\nu)}BP(\nu)d\nu\right), \quad (50)$$

It can be verified that if $\zeta(t)=P(t)$, then the system $\dot X = AX+B\zeta(t-\frac{1}{\bar\mu})$ is exponentially stable (as in this case we have $\zeta(t)=P(t)=KX(t+\frac{1}{\bar\mu})$). Consequently, we want $z(t,0)$ to converge to $\zeta(t-\frac{1}{\bar\mu})=P(t-\frac{1}{\bar\mu})$. This gives us the corresponding value of $z(t,1)$, using Lemma 2. Let us define for all $1\le i\le m$ the control law $(V_{ODE})_i(\cdot)$ by

$$(V_{ODE})_i(t)=\chi_i(t) - \sum_{k=1}^{n}\sum_{l=1}^{m}R_{ik}Q_{kl}\chi_l(t-\frac{1}{\lambda_k}-\frac{1}{\mu_l}), \quad (51)$$

where $\chi_i(t)=P_i(t-\frac{1}{\bar\mu}+\frac{1}{\mu_i})+\sum_{k=i+1}^m\int_{\frac{1}{\bar\mu}-\sum_{l=i}^m\frac{1}{\mu_l}}^{\frac{1}{\bar\mu}}H_{ik}(s)P_k(t-s)ds$, where the function H_{ik} are defined in Lemma 2. We can now conclude this section with the following theorem.

Theorem 2: The feedback law

$$V(t)=V_{ODE}(t)+V_{BS}(t)+V^0(t), \quad (52)$$

where V_{ODE} V_{BS} and V^0 are respectively defined by (51), (43) and (39), exponentially stabilizes (in the sense of equation (8)) the system (3)-(6) to its zero-equilibrium.

Proof: Using the computations done above (and in particular equation (44)), and denoting $e(t)=z(t,1)-\chi(t)$, we have for all $1\le i\le m$

$$e_i(t)=\sum_{k=1}^n\sum_{l=1}^m R_{ik}Q_{kl}(e_l(t-\frac{1}{\lambda_k}-\frac{1}{\mu_l},1)).$$

Thus, due to Assumption 2, this implies the convergence of the function e to zero and consequently, the convergence of $z(t,1)$ to $\chi(t)$. Using Lemma 2, we obtain the convergence of $z(t,0)$ to $P(t-\frac{1}{\bar\mu})$. This implies the stabilization of $X(t)$. Furthermore, the state-predictor $P(t)$ converges to $KX(t+\bar\mu)$, which implies that $P(t)$ (and consequently $\chi(t)$ and $V_{ODE}(t)$) exponentially converges to zero. Consequently, due to (44), $z(t,1)$ exponentially vanishes. Using (35), this implies that $z(t,\cdot)$ converges L^2-exponentially to zero. This implies from (34) and the boundary condition (37), that $w(t,)$ converges also L^2-exponentially to zero. This yields the existence of $0<\kappa_0$ such that $||(w,z,X)||\le\kappa_0\mathrm{e}^{-\nu t}||(w_0,z_0,X_0)||$. Thus, the control law $V(t)$ ensures the exponential stabilization of (34)-(38). Due to the invertibility of the backstepping transformations (9)-(10) and (30)-(31), it is straightforward to prove the stabilization of (3)-(6). ■

IV. Delay-robust stabilization

In this section we prove the delay-robustness of the control law designed in the previous section. To do so, we show that the associated characteristic equation has all its zeros on the left-half plane. Let us consider a vector of positive delays $\delta=(\delta_1\quad\delta_2\quad\cdots\quad\delta_m)^T$ acting on the actuation input $V(\cdot)$ defined in (52) (each δ_i acting on the corresponding component of V). Using equations (40) and (41) and doing some change of variables (see [5] for details) we have $(\int_0^1 N^\alpha w(t,\xi)+N^z z(t,\xi)d\xi)_i=\sum_{k=1}^m\int_0^\tau(G_5)_{ik}(s)z_k(t-s,1)ds+\sum_{l=1}^p\int_0^\tau(G_X)_{il}(s)X_l(t-s)ds$, where for all $1\le i\le m$, $1\le k\le m$ and $1\le l\le p$, the functions $(G_5)_{ik}$ and $(G_X)_{il}$ are L^∞ functions defined in $[0,\tau]$. Thus, in presence of the delayed control law $V(\cdot)$, we now get for all $1\le i\le m$ and all $t\ge\tau+\max\{\delta_i\}$, denoting $y(t)=z(t,1)$, we have

$$\begin{aligned}
y_i(t)=&\sum_{k,l=1}^{n,m}R_{ik}Q_{kl}y_l(t-\frac{1}{\lambda_k}-\frac{1}{\mu_l})+\sum_{l=1}^m\int_0^\tau G_i^l y_l(t-s)ds\\
&+\sum_{k=1}^n\sum_{l=1}^p R_{ik}C_{kl}X_l(t-\frac{1}{\lambda_k})+\sum_{l=1}^p(R\bar\gamma_0(1)-\bar\gamma_1(1))_{il}X_l(t)+\\
&\sum_{k=1}^m\int_0^\tau(G_5)_{ik}y_k(t-s)ds+\sum_{l=1}^p\int_0^\tau(G_X)_{il}(s)X_l(t-s,1)ds\\
&-\sum_{l=1}^m\int_0^\tau G_i^l y_l(t-s-\delta_i)ds-\sum_{k,l=1}^{n,p}R_{ik}C_{kl}X_l(t-\frac{1}{\lambda_k}-\delta_i)\\
&-\sum_{l=1}^p(R\bar\gamma_0(1)-\bar\gamma_1(1))_{il}X_l(t-\delta_i)-\sum_{k=1}^m\int_0^\tau(G_5)_{ik}(s)\cdot\\
&y_k(t-s-\delta_i)ds-\sum_{l=1}^p\int_0^\tau(G_X)_{il}(s)X_l(t-s-\delta_i,1)ds\\
&+\chi_i(t-\delta_i)-\sum_{k=1}^n\sum_{l=1}^m R_{ik}Q_{kl}\chi_l(t-\frac{1}{\lambda_k}-\frac{1}{\mu_l}-\delta_i)
\end{aligned} \quad (53)$$

Taking the Laplace transfom of (53), we obtain

$$F(s,\delta)(\hat y(s)-\hat\chi(s))=(I-\Delta)(H_1(s)\hat X(s)+H_2(s)\hat\chi(s)),$$

where for all $1\le i,j\le m$, for all $1\le l\le p$

$$(F(s,\delta))_{ij}=\eta_i^j-\sum_{k=1}^n R_{ik}Q_{kj}\mathrm{e}^{-(\frac{1}{\lambda_k}+\frac{1}{\mu_j})s}-(1-\mathrm{e}^{-\delta_i s})\left(\int_0^\tau(G_5)_{ij}(s)\mathrm{e}^{-\nu s}d\nu+\int_0^\tau G_i^j(s)\mathrm{e}^{-\nu s}d\nu\right), \quad (54)$$

$$(H_1(s))_{il} = (\sum_{k=1}^{n} R_{ik}C_{kl}\mathrm{e}^{-\frac{s}{\lambda_k}} + (R\bar{\gamma}_0(1) - \bar{\gamma}_1(1))_{il}$$

$$+ \int_0^{\tau} (G_X)_{il}(s)\mathrm{e}^{-s\nu}d\nu), \tag{55}$$

$$(H_2(s))_{ij} = -(\eta_i^j - \sum_{k=1}^{n} R_{ik}Q_{kj}\mathrm{e}^{-(\frac{1}{\lambda_k}+\frac{1}{\mu_j})s})$$

$$+ \int_0^{\tau} G_i^j(s)\mathrm{e}^{-\nu s}d\nu + \int_0^{\tau} (G_5)_{ij}(s)\mathrm{e}^{-\nu s}d\nu, \tag{56}$$

with $\Delta_{i,j} = \eta_i^j \mathrm{e}^{-\delta_i s}$, where η_i^j stands for the Kronecker symbol which is equal to one if $i = j$, and is equal to zero otherwise. To obtain the characteristic equation from (54), we need first to prove F is invertible on the Right Half Plane (RHP). From [32, Theorem 1], we know that $F(\cdot, \delta) \in \mathcal{A}$ has a unique inverse in $\mathcal{A}$ if and only if $\inf_{\mathrm{Re}(s)\geq 0} |\det(F(s,\delta))| > 0$. We have the following lemma on invertibility of $F_1(s,\delta)$ in $\hat{\mathcal{A}}$ (where the Banach algebra $\hat{\mathcal{A}}$ is defined in section II-A).

Lemma 3: There exists $\delta^\star \in (0, \tau]$ such that

$$\inf_{\delta\in([0,\delta^\star])^m} \inf_{\mathrm{Re}(s)\geq 0} |\det(F(s,\delta))| > 0. \tag{57}$$

Proof: The proof is analogous to the one of [Theorem 9][5] and is a consequence of Assumption 2 and of the fact that the δ_i is arbitrarily small. ■

We now obtain the characteristic equation associated to (54). We get from (35) that$\hat{z}(t,0) = M(s)\hat{z}(t,1)$, where M is the upper-triangular matrix defined for all $1 \leq i \leq j \leq m$ by $(M(s))_{ij} = \mathrm{e}^{-\frac{1}{\mu_i}s}\eta_i^j + \int_0^{\frac{1}{\mu_i}} (G_3)_{ij}(\mu_i s)\mathrm{e}^{-s\nu}d\nu$. By construction (see equation (46)-(47)), we also have $\hat{\chi}(s) = M(s)\mathrm{e}^{-\frac{s}{\mu}}\hat{P}(s)$ where $\hat{P}$ is the Laplace transform of the predictor state feedback given in (50), namely $\hat{P}(s) = K_0(s)\hat{X}(s)$ where

$$K_0(s) = \left[I - K(sI-A)^{-1}(I - \mathrm{e}^{-(sI-A)\frac{1}{\mu}})B\right]^{-1} K\mathrm{e}^{\frac{A}{\mu}}.$$

Taking the Laplace transform of (36), we obtain for $s \in \mathbb{C}$ such that $\mathrm{Re}(s) \geq 0$

$$\begin{aligned}
&(sI - A - BK_0(s)\mathrm{e}^{-\frac{s}{\mu}})\hat{X}(s) = B\hat{z}(s,0) \\
&- BK_0(s)\mathrm{e}^{-\frac{s}{\mu}}\hat{X}(s) = BM(s)(\hat{z}(s,1) - \hat{\chi}(s)) \\
&= BM(s)(F(s,\delta))^{-1}(I-\Delta)(H_1(s) + H_2(s)M(s) \\
&\quad K_0(s)\mathrm{e}^{-\frac{s}{\mu}})\hat{X}(s),
\end{aligned} \tag{58}$$

as F is invertible. We are now finally able to prove that the control law $V(t)$ as defined in (52) delay-robustly stabilizes the system (3)-(6) by proving that the characteristic equation (58) does not have any root on the RHP.

Theorem 3: The control law $V(t)$ defined in Theorem 2 delay-robustly stabilizes the system (3)-(6). That is, there exists $\delta^\star > 0$ such that, for all $\delta_i \in ([0,\delta^\star])^m$, the control law defined for all $1 \leq i \leq m$ by $V(t) = (V_{ODE}(t-\delta_i) + V_{BS}(t-\delta_i) + V_0(t-\delta_i))_i$ exponentially stabilizes the system (3)-(6).

Proof: The closed-loop characteristic equation associated to (58) can be written

$$p(s) = \det(F_0(s) - E_0(s)(I-\Delta)H_0(s)) = 0, \tag{59}$$

where

$$\begin{aligned}
F_0(s) &= sI - A - BK_0(s)\mathrm{e}^{-\frac{s}{\mu}}, \\
E_0(s) &= BM(s)(F(s,\delta))^{-1}, \\
H_0(s) &= (H_1(s) + H_2(s)M(s)K_0(s)\mathrm{e}^{-\frac{s}{\mu}})
\end{aligned}$$

The holomorphic function $F_0(s)$ has all its roots in the left-half complex plane as the system $\dot{X} = AX + BP(t - \frac{1}{\mu})$ is exponentially stable. As the holomorphic functions M, H_1, H_2 and F^{-1} are bounded in the right-half complex plane (see Lemma 3 for the boundedness of F^{-1}), so are $E_0(s)$ and $H_0(s)$ and the function $\det(E_0(s)(I - \Delta)H_0(s))$. As the holomorphic function $F_0(s)$ goes to infinity for $|s|$ large enough, there exists $M_2 > 0$ such that $\forall s \in \Omega_1 = \{s \in \mathbb{C},\ \Re(s) \geq 0 \text{ and } |s| \geq M_2\}$

$$|\det(F_0(s) - E_0(s)(I-\Delta)H_0(s))| > 0$$

Let us now consider $s \in \mathbb{C} \in \Omega_2 = \{s \in \mathbb{C},\ \Re(s) \geq 0 \text{ and } |s| \leq M_2\}$. By contradiction, assume that there exists $s \in \mathbb{C}$, $s \neq 0$ and $\mathrm{Re}(s) \geq 0$, such that $p(s) = 0$. There exists $\eta \neq 0$ such that $F_0(s)\eta = E_0(s)(1-\Delta)H_0(s)\eta$. This yields

$$\eta^* F_0^*(s)F_0(s)\eta = \eta^* H_0^*(s)(I-\Delta)^* E_0^*(s)E_0(I-\Delta)H_0(s)\eta,$$

where * denotes the conjugate transpose. Since $F(s)$ is non singular in $\mathbb{C}^+$, there exists $M_3 > 0$ such that $M_3 < \eta^* F^*(s)F(s)\eta$. Similarly, $H_0(s)$ and $E_0(s)$ are bounded in $\mathbb{C}^+$, so that there exists $M_4 > 0$ such that

$$\begin{aligned}
M_3 &\leq \eta^* H_0^*(I-\Delta)^* E_0^* E_0(I-\Delta)H_0\eta \\
&\leq \max_i m|1 - \mathrm{e}^{-\delta_i s}|^2 M_4.
\end{aligned}$$

Construct $\delta_r(s) = \frac{\bar{\delta}}{|s|}$, for some $\bar{\delta} > 0$ such that $\mathrm{e}^{\bar{\delta}} < 1 + \sqrt{\frac{M_3}{mM_4}}$. It follows that for any $\delta_i \leq \delta_r(s)$,

$$|1 - \mathrm{e}^{-\delta s}| \leq \mathrm{e}^{\bar{\delta}} - 1 < \sqrt{\frac{M_3}{mM_4}}. \tag{60}$$

The function $p(s)$ can only have a finite number of zeros in Ω_2 (isolated zeros theorem). As the quantity $\delta^\star = \min_s \delta_r(s)$ is strictly positive, equation (60) holds for any $\delta_i \leq \delta^\star$. This is a contradiction with the previous inequality. Consequently, there does not exist any $s \in \mathbb{C}^+$ such that $p(s) = 0$. This implies delay-robust stability as the asymptotic vertical chain of zeros of $p(s)$ can not be the imaginary axis (the principal term of $p(s)$ being stable). ■

V. Simulation results

In this section we illustrate our results with simulations Let us consider the unstable system (3)-(6) for which the coefficients are defined by

$$\begin{aligned}
&\Lambda^+ = 1,\ \Lambda^- = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},\ \Sigma^{++} = 0,\ \Sigma^{+-} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \\
&\Sigma^{-+} = \begin{pmatrix} 0.8 \\ 0.8 \end{pmatrix},\ \Sigma^{--} = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}\ Q = \begin{pmatrix} 0.3 & 0.5 \end{pmatrix}, \\
&R = \begin{pmatrix} 0.5 & 0.6 \end{pmatrix}^T,\ A = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, B = \begin{pmatrix} 0.1 & 0 \\ 0.1 & 0.1 \end{pmatrix}, \\
&C = \begin{pmatrix} 0.1 & 0.1 \end{pmatrix}.
\end{aligned} \tag{61}$$

The parameters values are chosen such that

- the ODE and the PDE open-loop system are unstable,
- the reflection terms Q and R satisfy Assumption 2.

We consider the norm $||\cdot||$ defined by (1). The initial condition is chosen as a C^1 function. The algorithm we use is adapted from the one proposed in [4]. Using the method of characteristics, we write the integral equations associated to the PDE-system (11)-(22). Finally, the original system (3)-(6) is simulated using a Godunov's discretization scheme. The predictor is adjusted from the one presented in [25]. Figure 1 pictures the $||\cdot||$-norm of the state (u, v, X) in open-loop and using the control law (52) in presence of a 0.05-s delay.

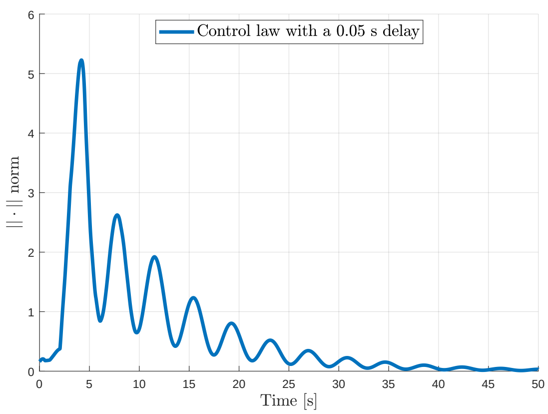

Fig. 1. Evolution of the $||\cdot||$ norm of the system (3)-(6) for the parameters (61) in presence of a 0.05s delay.

VI. Concluding remarks

In this paper, we have developed a delay-robust stabilizing feedback control law for a system composed of a PDE coupled with an ODE through its boundary. Our approach consists of a first backstepping-based feedback and a second prediction based feedback. This second feedback control is obtained after solving a tracking problem after reformulation of the PDE-ODE system as a neutral system. This mixed strategy is proved to robust to small delays as we preserve the proximal reflection terms in the target PDE systems used for the backstepping design. We will consider in future works the delay-robustness properties of the output-feedback controller.

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