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1    **LOCAL TRANSPARENT BOUNDARY CONDITIONS FOR WAVE**  
2    **PROPAGATION IN FRACTAL TREES (I): METHOD AND**  
3    **NUMERICAL IMPLEMENTATION**

4    PATRICK JOLY \* AND MARYNA KACHANOVSKA \*

5    **Abstract.** This work is dedicated to the construction and analysis of high-order transparent  
6 boundary conditions for the weighted wave equation on a fractal tree, which models sound propaga-  
7 tion inside human lungs. This article follows the works [9, 6], aimed at the analysis and numerical  
8 treatment of the model, as well as the construction of low-order and exact discrete boundary con-  
9 ditions. The method suggested in the present work is based on the truncation of the meromorphic  
10 series that approximate the symbol of the Dirichlet-to-Neumann operator, in the spirit of the absorb-  
11 ing boundary conditions of B. Engquist and A. Majda. We analyze its stability and convergence, as  
12 well as present computational aspects of the method. Numerical results confirm theoretical findings.

13    **Key words.** wave equation, fractal, metric graph, Dirichlet-to-Neumann operator

14    **AMS subject classifications.** 35L05, 65M12, 34B45

15    **1. Introduction.** Modelling sound propagation in a human lung is important  
16 for medical diagnostics, see e.g. the Audible Human Project [1], [17] and references  
17 therein. Because the physical phenomenon of wave propagation in a lung is highly  
18 complex and multi-scale, its computational tractability relies on the use of simplified  
19 models. One of such models is based on the geometric representation of a bronchiolar  
20 tree as a self-similar network of tubes, see [15, 5, 16]. An asymptotic analysis of the  
21 3D wave equation posed on such a network, with respect to the thickness of the tubes  
22 tending to zero, leads to a weighed 1D wave equation on a self-similar infinite tree,  
23 see [10, 18]. This model is somewhat non-standard, and its mathematical analysis  
24 was performed in [9], which gave rise to many ideas of the present paper.

25    Because such a tree has infinitely many edges, to perform any kind of numeri-  
26 cal simulations, we need to be able to truncate the computational domain. This is  
27 classically done via introducing 'absorbing' boundary conditions, which, in turn, are  
28 based on an approximation of the Dirichlet-to-Neumann (DtN) operator. The princi-  
29 pal difficulty lies in its time non-locality. One of the methods for approximating the  
30 DtN, based on the convolution quadrature [12, 13], was proposed in [6]. Since the cost  
31 of this method is quadratic in the number of time steps, in this work we suggest an  
32 alternative method to approximate the DtN, based on the classical ideas of Engquist  
33 and Majda [4], namely, approximation of its symbol by rational fractions.

34    Compared to the classical case of free space wave propagation, there are multiple  
35 additional difficulties associated to the model we study. In particular, it describes the  
36 wave propagation in highly heterogeneous media, and exhibits multi-scale phenomena.  
37 As a result, no closed form expression for the Green function is available. Nonetheless,  
38 in [9] it was shown that the symbol of the DtN operator is a meromorphic function,  
39 which satisfies a certain non-linear equation. This symbol can be represented as a  
40 convergent series of rational functions with simple real poles. To approximate the  
41 symbol of the DtN, we truncate this series at several first poles that are closest to the  
42 origin. This approximation leads to high-order approximated transparent boundary  
43 conditions and can be realized via local operators in the time domain. We prove the  
44 stability of such boundary conditions and perform their error analysis; moreover, we

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45 demonstrate how to choose the number of poles to achieve a desired accuracy.

46 This article is organized as follows. Section 2 introduces the notations and the  
 47 problem. Section 3 contains the description of the method (Section 3.1), as well as  
 48 the stability (Section 3.2) and convergence analysis (Section 3.3). Section 4 deals  
 49 with the error control and the complexity of the method. In Section 5 we present  
 50 numerical aspects of the method and numerical results. Finally, Section 6 is dedicated  
 51 to conclusions and open problems.

52 Let us mention that this article is companion of [7], where we develop a quanti-  
 53 tative error analysis for the approximate high-order transparent boundary conditions  
 54 introduced in the this article.

55 **2. Problem setting.** This section is not new, and presents a shortened version  
 56 of the corresponding section in [6]. The definitions can be found as well in [9].

57 **2.1. Notation.** Given  $p \in \mathbb{N}_*$ , let  $\mathcal{T}$  be a  $p$ -adic tree (a collection of vertices and  
 58 edges), which satisfies:

- 59 • it is rooted, i.e. there exists a vertex  $M^*$  designated as a 'root' of the tree;
- 60 • the root vertex  $M^*$  is incident to a single edge ('root edge') of the tree;
- 61 • every edge of the tree has  $p$  children edges ( $p$ -adicity); this in particular  
 62 implies that the tree has infinitely many vertices and edges.

63 For such a tree we can introduce a notion of a generation. A generation  $\mathcal{G}^n$  is a set of  
 64 edges that is defined inductively in the following way:  $\mathcal{G}^0$  contains a root edge only;  
 65  $\mathcal{G}^{n+1}$  contains all children edges of the edges from  $\mathcal{G}^n$ . The  $p^n$  edges belonging to  $\mathcal{G}^n$   
 66 will be denoted by  $\Sigma_{n,k}$ ,  $k = 0, \dots, p^n - 1$ . The edge  $\Sigma_{n,k}$  has  $p$  children

67 (2.1) 
$$\Sigma_{n+1,pk+j}, \quad j = 0, \dots, p-1.$$

69 By  $\mathcal{T}^m$  we denote the subtree of  $\mathcal{T}$  containing the first  $m+1$  generations, i.e.  $\bigcup_{\ell=0}^m \mathcal{G}^\ell$ .

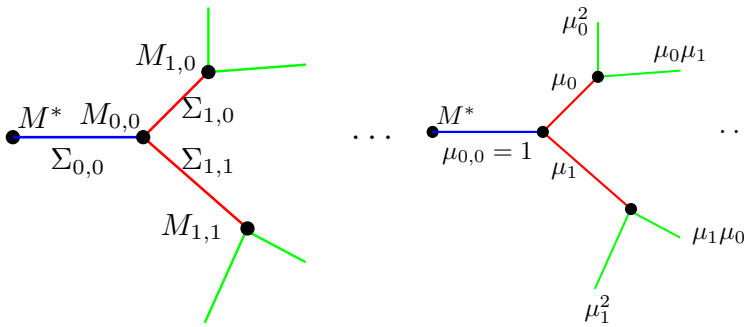


FIGURE 1. Left: A self-similar  $p$ -adic ( $p = 2$ ) infinite tree. In blue we mark the edges that belong to  $\mathcal{G}^0$ , in red the edges of  $\mathcal{G}^1$ , in green the edges of  $\mathcal{G}^2$ . Right: Distribution of weights on the edges of a 2-adic infinite self-similar tree.

70

71 We will study metric trees, i.e. trees in which edges are identified with segments  
 72 on the real line. Thus, to each of the edges  $\Sigma_{n,k}$  one can assign a length  $\ell_{n,k} > 0$ .  
 73 This allows to define a notion of a distance  $d(M, M^*)$  between the given vertex  $M$   
 74 and the root vertex  $M^*$  as a sum of lengths of the edges that connect  $M$  to  $M^*$ .

75 To define a notation for the vertices, similar to  $\Sigma_{n,k}$  for edges, let us consider an  
 76 edge  $\Sigma_{n,k}$ . Provided that it is incident to two vertices  $M_0, M_1$ , let us set  $M_{n,k} :=$   
 77  $\operatorname{argmax}_{V \in \{M_0, M_1\}} d(V, M^*)$ . This notation allows to identify all the vertices in a  
 78 unique manner; a respective illustration is provided in Figure 1, left.

79 Besides its length  $\ell_{n,k}$ , with every edge  $\Sigma_{n,k}$  we associate another quantity, namely  
 80 a weight  $\mu_{n,k} > 0$  (see also Remark 2.3). W.l.o.g., we assume that  $\mu_{0,0} = 1$ .

81 In what follows, we will consider self-similar (fractal) trees, see [9, Definition 2.3].  
 82 To define them, let  $\alpha := (\alpha_0, \dots, \alpha_{p-1}) \in (\mathbb{R}_*^+)^p$  and  $\mu := (\mu_0, \dots, \mu_{p-1}) \in (\mathbb{R}_*^+)^p$ .  
 83 Then the length/weight of the edge  $\Sigma_{n+1,pk+j}$  (see 2.1) is related to the length/weight  
 84 of its parent edge  $\Sigma_{n,k}$  according to the following law:

$$85 \quad \ell_{n+1,pk+j} = \alpha_j \ell_{n,k}, \quad \mu_{n+1,pk+j} = \mu_j \mu_{n,k}, \quad j = 0, \dots, p-1.$$

87 This is illustrated in Figure 1, right.

88 We will call a reference tree the tree, for which the length of its root edge  $\ell_{0,0}$  is  
 89 equal to 1. Unless stated otherwise, we will always assume that  $\mathcal{T}$  is a reference tree.  
 90 Additionally, we will assume that  $|\alpha|_\infty := \max_j \alpha_j < 1$ .

91 **REMARK 2.1.** *This assumption implies that with some  $C > 0$ ,  $d(M_{n,j}, M^*) < C$*   
 92 *for all  $n \in \mathbb{N}$ ,  $j = 0, \dots, p^n - 1$ .*

93 **REMARK 2.2.** *When  $p = 1$ ,  $\mathcal{T}$  can be identified with a bounded interval on  $\mathbb{R}$ .*

94 **REMARK 2.3.** *We refer to [11, 18] for the physical meaning of the weights  $\mu_{n,k}$*   
 95 *in an asymptotic modelling of the human lung. The simplified model of a human lung*  
 96 *that we consider has the following parameters:  $p = 2$  and for all  $i$ ,  $\alpha_i \approx \alpha = 0.84$ , see*  
 97 *[20, p.125] and  $\mu_i = \mu \approx 0.8$ , see [20, p.123] and the asymptotic analysis in [11, 18].*

98 **2.2. The weighted wave equation on a fractal tree.** In order to introduce  
 99 the model under consideration, we define a parameterization of an edge  $\Sigma_{n,j}$  by an  
 100 abscissa  $s \in [0, \ell_{n,j}]$ . The parametrization is introduced so that  $\ell_{n,j}$  is associated to  
 101 the vertex  $M_{n,j}$ . We additionally introduce a measure on  $\Sigma_{n,j}$  as a usual Lebesgue  
 102 measure on the interval  $[0, \ell_{n,j}]$ .

103 Let  $s$  be an abscissa on the tree  $\mathcal{T}$  (for each edge  $\Sigma_{n,j}$  defined as above); formally,  
 104 let us define the weight function  $\mu$  on  $\mathcal{T}$  as follows:  $\mu(s) = \mu_{n,j}$ ,  $s \in \Sigma_{n,j}$  (with an  
 105 obvious abuse of notation). We look for an acoustic pressure  $u : \mathcal{T} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ . On  
 106 each edge,  $u$  satisfies the wave equation with a source term:  $u_{n,j} = u|_{\Sigma_{n,j}}$  solves

$$107 \quad (2.2) \quad \partial_t^2 u_{n,j} - \partial_s^2 u_{n,j} = f_{n,j} \quad \text{on } \Sigma_{n,j}, \quad j = 0, \dots, p^n - 1, \quad n \geq 0,$$

$$108 \quad (2.3) \quad u(M^*, t) = 0, \quad u(., 0) = \partial_t u(., 0) = 0.$$

110 The above introduces an infinite system of PDEs; they are coupled through the fol-  
 111 lowing continuity ( $\mathcal{C}$ ) and Kirchoff ( $\mathcal{K}$ ) conditions at all the vertices, see (2.1) (with  
 112 an obvious abuse of notation)

$$113 \quad (\mathcal{C}) \quad u_{n,j}(M_{n,j}, t) = u_{n+1,pj+k}(M_{n,j}, t), \quad k = 0, \dots, p-1,$$

$$114 \quad (\mathcal{K}) \quad \partial_s u_{n,j}(M_{n,j}, t) = \sum_{k=0}^{p-1} \mu_k \partial_s u_{n+1,pj+k}(M_{n,j}, t), \quad j = 0, \dots, p^n - 1, \quad n \geq 0.$$

116 This problem needs to be equipped with the boundary conditions, obviously, at the  
 117 root vertex  $M^*$  of the tree, as done in (2.3), but also at the 'infinite' boundary of the  
 118 tree. This becomes clearer if one recalls that when  $p = 1$ , the tree  $\mathcal{T}$  can be identified

119 to an interval on the real line, cf. Remark 2.2. If, additionally,  $\mu_0 = 1$ , the problem  
 120 (2.2, 2.3,  $\mathcal{C}$ ,  $\mathcal{K}$ ) reduces to the IVP for the wave equation on an interval, whose well-  
 121 posedness necessitates defining boundary conditions at both ends of the interval. This  
 122 will be done in the weak form, by using a proper Sobolev space framework.

123 **2.3. Dirichlet and Neumann BVPs for (2.2, 2.3,  $\mathcal{C}$ ,  $\mathcal{K}$ ).**

124 **2.3.1. Sobolev Spaces.** For a function  $v : \mathcal{T} \rightarrow \mathbb{R}$ , let

$$125 \quad (2.4) \quad \int_{\mathcal{T}} \mu v := \sum_{n=0}^{\infty} \sum_{k=0}^{p^n-1} \int_{\Sigma_{n,k}} \mu_{n,k} v(s) ds$$

127 Let  $C(\mathcal{T})$  be a space of continuous functions on  $\mathcal{T}$ , and

$$128 \quad C_0(\mathcal{T}) := \{v \in C(\mathcal{T}) : v = 0 \text{ on } \mathcal{T} \setminus \mathcal{T}^m, \text{ for some } m \in \mathbb{N}\}.$$

129 Next, we introduce the following three Hilbert spaces of functions on  $\mathcal{T}$ . First of all,

$$130 \quad L_{\mu}^2(\mathcal{T}) = \{v : v|_{\Sigma_{n,j}} \in L^2(\Sigma_{n,j}), \|v\|_{L_{\mu}^2(\mathcal{T})} < \infty\}, \quad \|v\|^2 = \|v\|_{L_{\mu}^2(\mathcal{T})}^2 = \int_{\mathcal{T}} \mu |v|^2.$$

132 We will denote by  $(\cdot, \cdot)$  the associated scalar product. Next, the weighted Sobolev  
 133 space  $H_{\mu}^1$  is defined in a natural way:

$$134 \quad H_{\mu}^1(\mathcal{T}) := \{v \in C(\mathcal{T}) \cap L_{\mu}^2(\mathcal{T}) : \|\partial_s v\| < \infty\}, \quad \|v\|_{H_{\mu}^1(\mathcal{T})}^2 = \|v\|^2 + \|\partial_s v\|^2.$$

136 Similarly, we define the corresponding spaces  $L_{\mu}^2(\mathcal{T}^m)$  and  $H_{\mu}^1(\mathcal{T}^m)$  on a truncated  
 137 tree  $\mathcal{T}^m$ . The associated  $L_{\mu}^2$ -scalar product will be denoted by  $(\cdot, \cdot)_{\mathcal{T}^m}$ , and the norm  
 138 by  $\|\cdot\|_{\mathcal{T}^m}$ . The remaining space, by analogy with  $H_0^1$ , is defined as the closure:

$$139 \quad H_{\mu,0}^1(\mathcal{T}) := \overline{C_0(\mathcal{T}) \cap H_{\mu}^1(\mathcal{T})}^{\|\cdot\|_{H_{\mu}^1(\mathcal{T})}}.$$

141 **REMARK 2.4.** *Unless stated otherwise, we work with real-valued function spaces,*  
 142 *and indicate explicitly where complex-valued function spaces (and respective Hermitian*  
 143 *scalar products) are used, without changing the notation.*

145 **2.3.2. The BVP problems.** First of all, let us introduce the two additional  
 146 spaces (for brevity denoted in what follows by  $V_n, V_{\mathfrak{d}}$ ):

$$147 \quad V_n(\mathcal{T}) = \{v \in H_{\mu}^1(\mathcal{T}) : v(M^*) = 0\}, \quad V_{\mathfrak{d}}(\mathcal{T}) = \{v \in H_{\mu,0}^1(\mathcal{T}) : v(M^*) = 0\}.$$

149 These spaces differ from  $H_{\mu}^1(\mathcal{T})$  (resp. from  $H_{\mu,0}^1(\mathcal{T})$ ) by the condition at  $M^*$ .

150 As discussed in the end of Section 2.2, let us define the Neumann and Dirichlet  
 151 problems for (2.2, 2.3,  $\mathcal{C}$ ,  $\mathcal{K}$ ), with 'Neumann' ('Dirichlet') referring to the conditions  
 152 at the fractal boundary of the tree.

153 **DEFINITION 2.5** (Time-domain Neumann problem). *Find*

$$154 \quad u_n \in C(\mathbb{R}^+; V_n) \cap C^1(\mathbb{R}^+; L_{\mu}^2(\mathcal{T})), \text{ s.t. } u_n(\cdot, 0) = \partial_t u_n(\cdot, 0) = 0, \text{ and}$$

$$155 \quad (N) \quad (\partial_t^2 u_n, v) + (\partial_s u_n, \partial_s v) = (f, v), \quad \text{for all } v \in V_n.$$

157 DEFINITION 2.6 (Time-domain Dirichlet problem). *Find*

158  $u_{\mathfrak{d}} \in C(\mathbb{R}^+; V_{\mathfrak{d}}) \cap C^1(\mathbb{R}^+; L_{\mu}^2(\mathcal{T})),$  s.t.  $u_{\mathfrak{d}}(\cdot, 0) = \partial_t u_{\mathfrak{d}}(\cdot, 0) = 0,$  and  
 159 (D)  $(\partial_t^2 u_{\mathfrak{d}}, v) + (\partial_s u_{\mathfrak{d}}, \partial_s v) = (f, v),$  for all  $v \in V_{\mathfrak{d}}.$

161 It is easy to see that the above problems imply (2.2, 2.3,  $\mathcal{C}$ ,  $\mathcal{K}$ ); they are well-posed if  
 162  $f \in L_{loc}^1(\mathbb{R}^+; L_{\mu}^2(\mathcal{T})),$  cf. [6, Theorem 1]. It appears that in some cases the solutions  
 163 to (N) and (D) coincide. To explain this result in more detail, let us introduce

164 
$$\langle \mu \alpha \rangle := \sum_{i=0}^{p-1} \mu_i \alpha_i, \quad \left\langle \frac{\mu}{\alpha} \right\rangle \equiv \langle \mu / \alpha \rangle := \sum_{i=0}^{p-1} \frac{\mu_i}{\alpha_i}.$$

166 With the above definitions, we have

167 THEOREM 2.7 ([9]). *If  $\langle \mu \alpha \rangle \geq 1$  or  $\langle \mu / \alpha \rangle \leq 1,$  the spaces  $H_{\mu,0}^1(\mathcal{T})$  and  $H_{\mu}^1(\mathcal{T})$   
 168 coincide, and thus  $u_{\mathfrak{n}} = u_{\mathfrak{d}}.$  Otherwise,  $H_{\mu,0}^1(\mathcal{T}) \subsetneq H_{\mu}^1(\mathcal{T}),$  and  $u_{\mathfrak{n}} \neq u_{\mathfrak{d}}.$*

169 **2.4. Transparent boundary conditions.** Because the tree  $\mathcal{T}$  is structurally  
 170 infinite, in order to perform the numerical simulations, it is necessary to truncate  
 171 the tree to a finite number of generations. However, imposing simple Dirichlet or  
 172 Neumann boundary conditions at the boundary of the truncated tree  $\mathcal{T}^m$  does not  
 173 allow to reach a reasonable accuracy unless  $m$  is sufficiently large, cf. [9]. Because  
 174 the computational costs increase exponentially with  $m$  (since at the  $m$ -th level there  
 175 are  $p^m$  branches of the tree), this leads to a significant computational overhead. The  
 176 main goal of this section is the definition of transparent BCs, which allow to perform  
 177 accurate simulations on trees  $\mathcal{T}^m$  with  $m$  arbitrary small.

178 Our construction is based on the results of [9] (see also [6]). In what follows, we  
 179 fix  $m \geq 1.$  We will assume that

180 ASSUMPTION 2.8. *The source term  $f(\cdot, t), t \geq 0,$  is compactly supported in  $\mathcal{T}^{m-1}.$*

181 **2.4.1. Notations.** We denote by  $\Gamma_m := \{M_{m,j} : j = 0, \dots, p^m - 1\} \equiv \mathbb{R}^{p^m}$  the  
 182 'outer' boundary of  $\mathcal{T}^m.$

183 Let  $\mathbf{V}_{\mu}(\mathcal{T}^m) := \{v \in H_{\mu}^1(\mathcal{T}^m) : v(M^*) = 0\}.$  For  $v \in \mathbf{V}_{\mu}(\mathcal{T}^m),$  we define its trace:

184 (2.5)  $\gamma_m : \mathbf{V}_{\mu}(\mathcal{T}^m) \rightarrow \mathbb{R}^{p^m}, \quad \gamma_m v = (v(M_{m,0}), \dots, v(M_{m,p^m-1})).$

186 Let us additionally introduce, for  $v \in \mathbf{V}_{\mu}(\mathcal{T}^m), f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}, \mathbf{h} \in \mathbb{R}^{p^m},$

187 (2.6) 
$$\int_{\Gamma_m} f_1(\mu) f_2(\alpha) \mathbf{h} \gamma_m v := \sum_{j=0}^{p^m-1} f_1(\mu_{m,j}) f_2(\alpha_{m,j}) h_j v(M_{m,j}).$$

189 **2.4.2. Transparent boundary conditions: definition.** Our goal is to com-  
 190 pute the restriction of the solution  $u_{\mathfrak{a}}$  to the tree  $\mathcal{T}^m.$  This means that we need to  
 191 solve (2.2, 2.3,  $\mathcal{C}$ ,  $\mathcal{K}$ ) for  $n = 0, \dots, m,$  equipped with certain boundary conditions at  
 192  $\Gamma_m.$  These boundary conditions should be chosen so that the exact solution  $u_{\mathfrak{a}}$  satisfy  
 193 them (hence the name 'transparent'). We will express them using DtN operators:

194 (2.7)  $- \mu_{m,j} \partial_s u_{m,j}(M_{m,j}, t) = \mathcal{B}_{m,j}^{\mathfrak{a}}(\partial_t) u_{m,j}(M_{m,j}, t), \quad j = 0, \dots, p^m - 1,$

196 where  $\mathcal{B}_{m,j}^{\mathfrak{a}}(\partial_t), \mathfrak{a} \in \{\mathfrak{d}, \mathfrak{n}\},$  is the exact DtN operator associated to the point  $M_{m,j}$   
 197 (see Remark 2.9 for an explanation of the notation  $\mathcal{B}_{m,j}^{\mathfrak{a}}(\partial_t)$ ). To clarify the meaning  
 198 of  $\mathcal{B}_{m,j}^{\mathfrak{a}}(\partial_t),$  let us introduce the following space:

199  $H_{0,loc}^1(\mathbb{R}^+) := \{v \in H_{loc}^1(\mathbb{R}^+) : v(0) = 0\}.$

201 Then  $\mathcal{B}_{m,j}^{\mathfrak{a}}(\partial_t) \in \mathcal{L}(H_{0,loc}^1(\mathbb{R}^+), L_{loc}^2(\mathbb{R}^+))$ , cf. [8]. To define its action, let us first fix  
 202  $g \in H_{0,loc}^1(\mathbb{R}^+)$ . Let  $\mathcal{T}_k := \mathcal{T}_{m+1,pj+k}$ ,  $k = 0, \dots, p-1$ , be a  $p$ -adic self-similar infinite  
 203 subtree of  $\mathcal{T}$ , whose root vertex is  $M_{m,j}$  and the root edge is  $\Sigma_{m+1,pj+k}$ , cf. (2.1).  
 204 Then the DtN operator is defined as:

$$205 \quad (2.8) \quad \mathcal{B}_{m,j}^{\mathfrak{a}}(\partial_t)g = - \sum_{k=0}^{p-1} \mu_{m,j} \mu_k \partial_s u_{g,k}^{\mathfrak{a}}(M_{m,j}, \cdot),$$

207 where  $u_{g,k}^{\mathfrak{a}} \in C^1(\mathbb{R}^+; L_{\mu}^2(\mathcal{T}_k))$  satisfies:

208 1. if  $\mathfrak{a} = \mathfrak{d}$ ,  $u_{g,k}^{\mathfrak{d}} \in C(\mathbb{R}^+; H_{\mu,0}^1(\mathcal{T}_k))$  solves the Dirichlet problem:

$$210 \quad (2.9) \quad \begin{aligned} & (\partial_t^2 u_{g,k}^{\mathfrak{d}}, v)_{\mathcal{T}_k} + (\partial_s u_{g,k}^{\mathfrak{d}}, \partial_s v)_{\mathcal{T}_k} = 0, \quad \text{for all } v \in V_{\mathfrak{d}}(\mathcal{T}_k), \\ & u_{g,k}^{\mathfrak{d}}(M_{m,j}, t) = g(t), \quad u_{g,k}^{\mathfrak{d}}(\cdot, 0) = \partial_t u_{g,k}^{\mathfrak{d}}(\cdot, 0) = 0. \end{aligned}$$

212 2. if  $\mathfrak{a} = \mathfrak{n}$ ,  $u_{g,k}^{\mathfrak{n}} \in C(\mathbb{R}^+; H_{\mu}^1(\mathcal{T}_k))$  solves (2.9) with  $V_{\mathfrak{d}}$  replaced by  $V_{\mathfrak{n}}$ .

214 Next, we show that all the operators  $\mathcal{B}_{m,j}^{\mathfrak{a}}(\partial_t)$  can be expressed with the help of a  
 215 single operator  $\Lambda_{\mathfrak{a}}(\partial_t)$ , which is the DtN map in the root vertex of the reference tree.

216 **2.4.3. Reference DtN operator.** Provided  $g \in H_{0,loc}^1(\mathbb{R}^+)$ , let us define the  
 217 reference DtN operator on the reference tree  $\mathcal{T}$  (i.e.  $\ell_{0,0} = 1$ ) as follows:

$$218 \quad \Lambda_{\mathfrak{a}}(\partial_t)g(t) = -\partial_s u_g^{\mathfrak{a}}(M^*, t), \quad \text{where}$$

220 1. if  $\mathfrak{a} = \mathfrak{d}$ , the function  $u_g^{\mathfrak{d}} \in C(\mathbb{R}^+; H_{\mu,0}^1(\mathcal{T}))$  solves

$$221 \quad (2.10) \quad \begin{aligned} & (\partial_t^2 u_g^{\mathfrak{d}}, v) + (\partial_s u_g^{\mathfrak{d}}, \partial_s v) = 0, \quad \text{for all } v \in V_{\mathfrak{d}}, \\ & u_g^{\mathfrak{d}}(M^*, t) = g(t), \quad u_g^{\mathfrak{d}}(\cdot, 0) = \partial_t u_g^{\mathfrak{d}}(\cdot, 0) = 0. \end{aligned}$$

223 2. if  $\mathfrak{a} = \mathfrak{n}$ ,  $u_g^{\mathfrak{n}} \in C(\mathbb{R}^+; H_{\mu}^1(\mathcal{T}))$  solves (2.10) where  $V_{\mathfrak{d}}$  is substituted by  $V_{\mathfrak{n}}$ .

224 As the coefficients of (2.10) are independent of time,  $\Lambda_{\mathfrak{a}}(\partial_t)$  is a convolution operator.

226 **REMARK 2.9.** The notations  $\Lambda_{\mathfrak{a}}(\partial_t)$ ,  $\mathcal{B}_{m,j}^{\mathfrak{a}}(\partial_t)$  are adapted from the convolution  
 227 quadrature theory, cf. [14], and indicate that the respective operators are convolution  
 228 operators. A convolution operator  $\mathcal{K}(\partial_t)$  is defined as (where the integral below is  
 229 understood in the sense of a convolution of causal tempered distributions)

$$230 \quad (\mathcal{K}(\partial_t)g)(t) = \int_0^t k(t-\tau)g(\tau)d\tau, \quad g \in H_{0,loc}^1(\mathbb{R}^+).$$

232 The Fourier-Laplace transform of its convolution kernel, namely

$$233 \quad (2.11) \quad \mathcal{K}(\omega) = \mathcal{F}k := \int_0^{\infty} e^{i\omega t} k(t)dt, \quad \omega \in \mathbb{C}^+ = \{z \in \mathbb{C} : \text{Im } z \geq 0\},$$

234 is called the symbol of the operator  $\mathcal{K}(\partial_t)$ . We will use the boldface to distinguish  
 235 between the operators and their symbols. Moreover, given  $\gamma > 0$ , we will use the  
 236 notation  $\mathcal{K}(\gamma\partial_t)$  for a convolution operator with a symbol  $\mathcal{K}(\gamma\omega)$ .

238 **2.4.4. Transparent BCs via the reference DtN.** According to [9], the op-  
 239 erator  $\mathcal{B}_{m,j}^{\mathbf{a}}(\partial_t)$  can be expressed as follows:

$$240 \quad (2.12) \quad \mathcal{B}_{m,j}^{\mathbf{a}}(\partial_t) = \mu_{m,j} \alpha_{m,j}^{-1} \sum_{k=0}^{p-1} \frac{\mu_k}{\alpha_k} \Lambda_{\mathbf{a}}(\alpha_k \alpha_{m,j} \partial_t).$$

242 The above follows from the Kirchoff conditions ( $\mathcal{K}$ ) and a scaling argument (recall  
 243 that  $\alpha_{m,j}$  is the length of the edge  $\Sigma_{m,j}$ ). The expression (2.12) indicates that ap-  
 244 proximating  $\mathcal{B}_{m,j}^{\mathbf{a}}(\partial_t)$  relies on approximating  $\Lambda_{\mathbf{a}}(\partial_t)$ .

245 **2.4.5. The problem posed on the truncated tree  $\mathcal{T}^m$ .** As explained in the  
 246 beginning of Section 2.4, to compute numerically the solution to (D) (or (N)), we  
 247 will use the transparent boundary conditions, constructed in the previous section.  
 248 Consistently with the notation (2.5), let us introduce

$$249 \quad (2.13) \quad \mathcal{B}_m^{\mathbf{a}}(\partial_t) = \text{diag}(\mathcal{B}_{m,0}^{\mathbf{a}}(\partial_t), \dots, \mathcal{B}_{m,p^m-1}^{\mathbf{a}}(\partial_t)).$$

251 After integration by parts applied to (D) (resp. (N)), the variational formulation with  
 252 the transparent BCs reads: with  $\mathbf{a} \in \{\mathfrak{d}, \mathbf{n}\}$ ,

$$253 \quad \text{find } u_m^{\mathbf{a}} \in C(\mathbb{R}^+; \mathbf{V}_{\mu}(\mathcal{T}^m)) \cap C^1(\mathbb{R}^+; \mathbf{L}_{\mu}^2(\mathcal{T}^m)),$$

255 s.t.  $u_m^{\mathbf{a}}(\cdot, 0) = \partial_t u_m^{\mathbf{a}}(\cdot, 0) = 0$  and, for all  $v \in \mathbf{V}_{\mu}(\mathcal{T}^m)$ ,

$$256 \quad (2.14) \quad (\partial_t^2 u_m^{\mathbf{a}}, v)_{\mathcal{T}^m} + (\partial_s u_m^{\mathbf{a}}, \partial_s v)_{\mathcal{T}^m} + \int_{\Gamma_m} \mathcal{B}_m^{\mathbf{a}}(\partial_t) \gamma_m u_m^{\mathbf{a}} \gamma_m v = (f, v)_{\mathcal{T}^m}.$$

257  
 258 In [6] it was proven that this problem is well-posed and that  $u_m^{\mathbf{a}} = u_{\mathbf{a}}|_{\mathcal{T}^m}$ .

259 **THEOREM 2.10** ( Theorem 2.6 in [6]). *For all  $f \in L_{loc}^1((0, \infty); \mathbf{L}_{\mu}^2(\mathcal{T}))$  that satis-  
 260 fies  $\text{supp } f(\cdot, t) \subseteq \mathcal{T}^{m-1}$  for all  $t > 0$ , the problem (2.14) has a unique solution which  
 261 is the restriction of the solution  $u_{\mathbf{a}}$ ,  $\mathbf{a} = \mathfrak{d}$  (resp.  $\mathbf{a} = \mathbf{n}$ ) to (D) (resp. (N)) to  $\mathcal{T}^m$ .*

262 **3. Approximation of transparent boundary conditions.** As shown in Sec-  
 263 tion 2.4.4, with (2.12) the question of approximating  $\mathcal{B}_m^{\mathbf{a}}(\partial_t)$  can be reduced to the  
 264 question of approximating  $\Lambda_{\mathbf{a}}(\partial_t)$ . However, the associated convolution kernel is not  
 265 known neither in the time nor in the frequency domain. This section is dedicated to  
 266 a design of a tractable approximation of the reference DtN. It is organized as follows:  
 267 in Section 3.1 we provide an approximation of  $\Lambda_{\mathbf{a}}(\partial_t)$  and analyze its error. Section  
 268 3.2 deals with the stability and error analysis of the respective coupled formulation.

269 **3.1. Principal idea: truncated reference DtN operator.**

270 **3.1.1. Some preliminaries.** The main idea of our approach relies on a repre-  
 271 sentation of the symbol of the DtN as a meromorphic series. Let us provide more  
 272 details on this. Given a non-negative Hermitian sesquilinear form

$$273 \quad (3.1) \quad a(u, v) := \sum_{n=0}^{\infty} \sum_{k=0}^{p^n-1} \int_{\Sigma_{n,k}} \mu_{n,k} \partial_s u \partial_s \bar{v} ds,$$

275 let us define the following operators (where we work with complex-valued functions):

$$276 \quad (3.2) \quad \mathcal{A}_{\mathbf{a}} : D(\mathcal{A}_{\mathbf{a}}) \rightarrow \mathbf{L}_{\mu}^2(\mathcal{T}), \quad (\mathcal{A}_{\mathbf{a}} u, v) = a(u, v),$$

$$277 \quad (3.3) \quad D(\mathcal{A}_{\mathbf{a}}) = \{v \in V_{\mathbf{a}} : \exists C(v) \geq 0, \text{ s.t. } |a(v, g)| < C(v) \|g\|_{\mathbf{L}_{\mu}^2(\mathcal{T})}, \text{ for all } g \in V_{\mathbf{a}}\}.$$



279 In [9] the following result was shown (the condition  $|\alpha|_\infty < 1$  is necessary for its  
280 validity, see Remark 2.1 and [18] for a counter-example when  $|\alpha|_\infty \geq 1$ ).

281 **THEOREM 3.1** ([9]). *The embedding of  $H_\mu^1(\mathcal{T})$  into  $L_\mu^2(\mathcal{T})$  is compact.*

282 Therefore, the resolvent of  $\mathcal{A}_\mathfrak{a}$  is compact, thus the spectrum of  $\mathcal{A}_\mathfrak{a}$  is a pure point  
283 spectrum. We define the eigenvalues and normalized eigenfunctions as

$$284 \quad (3.4) \quad \mathcal{A}_\mathfrak{a} \varphi_{\mathfrak{a},n} = \omega_{\mathfrak{a},n}^2 \varphi_{\mathfrak{a},n}, \quad \|\varphi_{\mathfrak{a},n}\|_{L_\mu^2(\mathcal{T})} = 1, \quad 0 < \omega_{\mathfrak{a},1}^2 \leq \omega_{\mathfrak{a},2}^2 \leq \dots \rightarrow +\infty,$$

286 The eigenvalues do not vanish, as shown in [9, Remark 1.20]. Using a spectral repre-  
287 sentation of the operator  $\mathcal{A}_\mathfrak{a}$  it is possible to show the following result.

288 **THEOREM 3.2** (Proposition 1.23, discussion after (144) in [9]). *The symbol of*  
289 *the reference DtN operator  $\Lambda_\mathfrak{a}$ ,  $\mathfrak{a} \in \{\mathfrak{n}, \mathfrak{d}\}$ , satisfies*

$$290 \quad (3.5) \quad \Lambda_\mathfrak{a}(\omega) = \Lambda_\mathfrak{a}(0) - \sum_{n=1}^{+\infty} \frac{a_{\mathfrak{a},n} \omega^2}{(\omega_{\mathfrak{a},n})^2 - \omega^2}, \quad a_{\mathfrak{a},n} = \omega_{\mathfrak{a},n}^{-2} (\partial_s \varphi_{\mathfrak{a},n}(M^*))^2.$$

292 *The above series converges uniformly on compact subsets of  $\mathbb{C} \setminus \{\pm \omega_{\mathfrak{a},n}, n \in \mathbb{N}_*\}$ .*

293 Surprisingly, the values of  $\Lambda_\mathfrak{a}(0)$  will play an important role in the error analysis.

294 **PROPOSITION 3.3** (Lemma 5.5, Corollary 5.6 in [9]).  *$\Lambda_\mathfrak{a}(0)$  is given by*

- 295 • *if  $\langle \mu/\alpha \rangle \leq 1$ , then  $\Lambda_\mathfrak{d} \equiv \Lambda_\mathfrak{n}$  and  $\Lambda_\mathfrak{d}(0) = 0$ .*
- 296 • *if  $\langle \mu\alpha \rangle < 1 < \langle \mu/\alpha \rangle$ , then  $\Lambda_\mathfrak{d}(0) = 1 - \langle \mu/\alpha \rangle^{-1}$  and  $\Lambda_\mathfrak{n}(0) = 0$ .*
- 297 • *if  $\langle \mu\alpha \rangle \geq 1$ , then  $\Lambda_\mathfrak{d} \equiv \Lambda_\mathfrak{n}$  and  $\Lambda_\mathfrak{d}(0) = 1 - \langle \mu/\alpha \rangle^{-1}$ .*

302 *Moreover,  $\Lambda_\mathfrak{a}(0) \geq 0$ ,  $\mathfrak{a} \in \{\mathfrak{n}, \mathfrak{d}\}$ .*

303 The representation (3.5) provides an expansion of  $\Lambda_\mathfrak{a}$  into a meromorphic series.  
304 However, because the eigenvalues of  $\mathcal{A}_\mathfrak{a}$  are not necessarily simple, the respective  
305 terms of the series (3.5) may have repeated poles. Moreover, in some cases, the  
306 coefficients  $a_{\mathfrak{a},n}$  may vanish. For our purpose, it is more convenient to rewrite the  
307 series (3.5) in a form where the poles are different, and the residues do not vanish.

308 **COROLLARY 3.4** (Corollary of Theorem 3.2). *The symbol  $\Lambda_\mathfrak{a}$ ,  $\mathfrak{a} \in \{\mathfrak{n}, \mathfrak{d}\}$  satisfies*

$$309 \quad (3.6) \quad \Lambda_\mathfrak{a}(\omega) = \Lambda_\mathfrak{a}(0) - \sum_{k=1}^{+\infty} \frac{A_{\mathfrak{a},k} \omega^2}{\Omega_{\mathfrak{a},k}^2 - \omega^2},$$

311 *where the poles  $0 < \Omega_{\mathfrak{a},0} < \Omega_{\mathfrak{a},1} < \dots \rightarrow +\infty$ , the coefficients  $A_{\mathfrak{a},k} > 0$  for all*  
312  *$k \geq 1$ , and  $\Lambda_\mathfrak{a}(0)$  is given in Proposition 3.3. The above series converges uniformly*  
313 *on compact subsets of  $\mathbb{C} \setminus \{\pm \Omega_{\mathfrak{a},k}, k \in \mathbb{N}_*\}$ .*

314 *Proof.* The proof of (3.6) is immediate from (3.5). By Theorem 3.2 the proof of  
315 uniform convergence in  $\mathbb{C} \setminus \{\pm \Omega_{\mathfrak{a},k}, k \in \mathbb{N}_*\}$  is a matter of verifying that the series  
316 converges uniformly in  $B_\delta(\pm \omega_{\mathfrak{a},n})$ , where  $\omega_{\mathfrak{a},n} > 0$  is not a pole of  $\Lambda_\mathfrak{a}$ , for  $\delta > 0$   
317 sufficiently small. This follows by a direct computation, thus we omit the details.  $\square$

318 **3.1.2. Reference DtN in the time domain.** The expression (3.6) provides a  
319 convenient way to realize the DtN operator in the time domain. In particular, the  
320 symbol  $\omega \mapsto \frac{i\omega}{\Omega_{\mathfrak{a},\ell}^2 - \omega^2}$  corresponds to the following convolution operator:

$$321 \quad \mathcal{F}\lambda = \frac{i\omega}{\Omega_{\mathfrak{a},\ell}^2 - \omega^2} \mathcal{F}g \iff \frac{d^2}{dt^2} \lambda + \Omega_{\mathfrak{a},\ell}^2 \lambda = \frac{d}{dt} g, \quad \lambda(0) = \frac{d}{dt} \lambda(0) = 0,$$

322

323 where  $g$  is sufficiently smooth with  $g(0) = g'(0) = 0$ . To formalize this result, we need  
 324 the following assumption on  $g$ :

$$325 \quad (3.7) \quad g \in C^2(\mathbb{R}^+), \quad g(0) = \dots = g^{(2)}(0) = 0, \quad \text{and } g^{(3)} \in L^1_{loc}(\mathbb{R}^+).$$

327 LEMMA 3.5. *Let  $g$  satisfy (3.7). Let  $\lambda_{\mathbf{a},\ell}$ ,  $\ell \in \mathbb{N}_*$ , solve the system of ODEs:*

$$329 \quad (3.8) \quad \frac{d^2 \lambda_{\mathbf{a},\ell}}{dt^2} + \Omega_{\mathbf{a},\ell}^2 \lambda_{\mathbf{a},\ell} = \frac{dg}{dt}, \quad \lambda_{\mathbf{a},\ell}(0) = \frac{d\lambda_{\mathbf{a},\ell}}{dt}(0) = 0.$$

331 Then

332 (1) the series  $\sum_{\ell=1}^{\infty} A_{\ell} \lambda'_{\mathbf{a},\ell}(t)$  converges uniformly on compact subsets of  $[0, \infty)$ .

333 (2) if, additionally, for all  $t > 0$ ,  $|g(t)| \leq C(1 + t^n)$ , with some  $C, n \geq 0$ , then

$$334 \quad (3.9) \quad \Lambda(\partial_t)g(t) = \mathbf{\Lambda}(0)g(t) + \sum_{\ell=1}^{\infty} A_{\ell} \lambda'_{\mathbf{a},\ell}(t).$$

336 Before proving the above, let us state the following technical lemma.

337 LEMMA 3.6. *The series  $S_{\mathbf{a}} = \sum_{\ell=1}^{\infty} A_{\mathbf{a},\ell} \Omega_{\mathbf{a},\ell}^{-2}$  converges.*

338 *Proof.* Provided  $r \in (0, \Omega_{\mathbf{a},1})$ , where  $\Omega_{\mathbf{a},1}$  is the smallest positive pole of  $\mathbf{\Lambda}_{\mathbf{a}}(\omega)$ ,

339 the series  $S_{\mathbf{a},r} := \mathbf{\Lambda}_{\mathbf{a}}(r) - \mathbf{\Lambda}_{\mathbf{a}}(0) = \sum_{\ell=1}^{\infty} A_{\mathbf{a},\ell} r^2 (\Omega_{\mathbf{a},\ell}^2 - r^2)^{-1}$  converges, according to

340 Corollary 3.4. We conclude by noticing that  $|S_{\mathbf{a}}| < r^{-2} S_{\mathbf{a},r}$ .  $\square$

341 *Proof of Lemma 3.5. Proof of (1).* As  $g'(0) = 0$ , the solution to (3.8) is given by

$$342 \quad \lambda_{\mathbf{a},\ell}(t) = \Omega_{\mathbf{a},\ell}^{-1} \int_0^t \sin \Omega_{\mathbf{a},\ell}(t - \tau) g'(\tau) d\tau, \quad \text{and} \quad \lambda'_{\mathbf{a},\ell}(t) = \int_0^t \cos \Omega_{\mathbf{a},\ell}(t - \tau) g'(\tau) d\tau,$$

344 Next, we re-express  $\lambda_{\mathbf{a},\ell}$  so that the general term of the series  $\sum_{\ell=1}^{\infty} A_{\mathbf{a},\ell} \lambda'_{\mathbf{a},\ell}$  can be

345 bounded by the general term of the convergent series  $S$ , namely  $A_{\mathbf{a},\ell} \Omega_{\mathbf{a},\ell}^{-2}$ . For this we  
 346 integrate by parts twice, to obtain

$$347 \quad (3.10) \quad \lambda'_{\mathbf{a},\ell}(t) = \Omega_{\mathbf{a},\ell}^{-2} g^{(2)}(t) - \Omega_{\mathbf{a},\ell}^{-2} \int_0^t \cos \Omega_{\mathbf{a},\ell}(t - \tau) g^{(3)}(\tau) d\tau,$$

349 where we used  $g^{(1)}(0) = g^{(2)}(0) = 0$ . Hence,

$$350 \quad \sum_{\ell=1}^{\infty} A_{\mathbf{a},\ell} |\lambda'_{\mathbf{a},\ell}(t)| \leq \sum_{\ell=1}^{\infty} \frac{A_{\mathbf{a},\ell}}{\Omega_{\mathbf{a},\ell}^2} \left( |g^{(2)}(t)| + \int_0^t |g^{(3)}(\tau)| d\tau \right).$$

352 This proves the uniform convergence of  $\sum_{\ell=1}^{\infty} A_{\mathbf{a},\ell} |\lambda'_{\mathbf{a},\ell}(t)|$  on compact subsets of  $\mathbb{R}^+$ .

353 *Proof of (2).* it suffices to prove that the expression in the right-hand side of (3.9) is  
 354  $\mathcal{F}^{-1}(\mathbf{\Lambda}_{\mathbf{a}}(\omega) \mathcal{F}g(\omega))$ . This follows by a direct computation, cf. (3.6) (the polynomial  
 355 bound on  $g$  is used to ensure that  $\omega \mapsto \mathcal{F}g(\omega)$  is analytic in  $\mathbb{C}^+$ ).  $\square$

356 **3.1.3. Approximating the reference DtN in the time domain.** Based on  
 357 the results of the previous section, it is natural to approximate the reference DtN  
 358 operator by truncating the series (3.9) to  $N$  terms:

$$359 \quad (3.11) \quad \Lambda_{\mathbf{a}}^N(\partial_t)g(t) = \Lambda_{\mathbf{a}}(0)g(t) + \sum_{\ell=1}^N A_{\mathbf{a},\ell} \frac{d\lambda_{\mathbf{a},\ell}}{dt}(t),$$

360  
 361 where  $\lambda_{\mathbf{a},\ell}$  are defined in (3.8). The symbol of this operator reads

$$362 \quad (3.12) \quad \Lambda_{\mathbf{a}}^N(\omega) = \Lambda_{\mathbf{a}}(0) - \sum_{\ell=1}^N \frac{A_{\mathbf{a},\ell}\omega^2}{\Omega_{\mathbf{a},\ell}^2 - \omega^2}.$$

363  
 364 To show how the error of truncating (3.11) depends on  $N$ , let us introduce the fol-  
 365 lowing notation for the remainder of the series  $S$  from Lemma 3.6:

$$366 \quad (3.13) \quad r_{\mathbf{a},N} := \sum_{\ell=N+1}^{\infty} \frac{A_{\mathbf{a},\ell}}{\Omega_{\mathbf{a},\ell}^2}.$$

367  
 368 The error then is quantified by the following lemma.

369 **LEMMA 3.7.** *Let  $g$  satisfy (3.7). Then*

$$370 \quad (3.14) \quad |\Lambda_{\mathbf{a}}(\partial_t)g(t) - \Lambda_{\mathbf{a}}^N(\partial_t)g(t)| \leq 2r_{\mathbf{a},N} \int_0^t |g^{(3)}(\tau)| d\tau, \quad \text{for all } t \geq 0.$$

371  
 372 *Proof.* Difference of (3.9) and (3.11) yields

$$373 \quad |\Lambda_{\mathbf{a}}(\partial_t)g(t) - \Lambda_{\mathbf{a}}^N(\partial_t)g(t)| = \left| \sum_{\ell=N+1}^{\infty} A_{\mathbf{a},\ell} \lambda'_{\mathbf{a},\ell}(t) \right|, \text{ from where, by (3.10),}$$

$$374 \quad |\Lambda_{\mathbf{a}}(\partial_t)g(t) - \Lambda_{\mathbf{a}}^N(\partial_t)g(t)| \leq \sum_{\ell=N+1}^{\infty} \frac{A_{\mathbf{a},\ell}}{\Omega_{\mathbf{a},\ell}^2} \left( |g^{(2)}(t)| + \int_0^t |g^{(3)}(\tau)| d\tau \right).$$

375  
 376 We conclude by bounding  $|g^{(2)}(t)|$  by  $\int_0^t |g^{(3)}(\tau)| d\tau$  (recall that  $g^{(2)}(0) = 0$ ).  $\square$

377 The above result shows that the error of the truncation of the DtN operator is con-  
 378 trolled by the remainder  $r_{\mathbf{a},N}$  of the convergent series  $S$ , and thus, as  $N \rightarrow +\infty$ ,  
 379 converges to zero. In the following section we will prove that approximating  $\Lambda_{\mathbf{a}}(\partial_t)$   
 380 by (3.11) in the transparent boundary conditions (2.12) and using the respective ap-  
 381 proximation in the coupled problem (2.14) leads to a stable problem. Moreover, we  
 382 will provide a quantification of the solution error.

### 383 **3.2. An approximate problem on $\mathcal{T}^m$ : formulation and stability.**

384 **3.2.1. Formulation.** Let us consider (2.14) with  $\mathcal{B}_m^{\mathbf{a}}(\partial_t)$  replaced by the trun-  
 385 cated DtN operator  $\mathcal{B}_m^{\mathbf{a},N}(\partial_t)$ , defined as, cf. (2.13),

$$386 \quad \mathcal{B}_m^{\mathbf{a},N}(\partial_t) = \text{diag} \left( \mathcal{B}_{m,0}^{\mathbf{a},N}(\partial_t), \dots, \mathcal{B}_{m,p^m-1}^{\mathbf{a},N}(\partial_t) \right),$$

387

388 where each  $\mathcal{B}_{m,j}^{N,\alpha}(\partial_t)$  is expressed via the truncated reference DtN, like in the definition  
 389 of  $\mathcal{B}_{m,j}^{\alpha}(\partial_t)$  via  $\Lambda_{\alpha}(\partial_t)$ , cf. (2.12):

$$390 \quad (3.15) \quad \mathcal{B}_{m,j}^{N,\alpha}(\partial_t) = \mu_{m,j} \alpha_{m,j}^{-1} \sum_{k=0}^{p-1} \frac{\mu_k}{\alpha_k} \Lambda_{\alpha}^N(\alpha_k \alpha_{m,j} \partial_t).$$

392 These are the operators with the following symbols, cf. also (3.12) for  $\Lambda_{\alpha}^N(\omega)$ :

$$393 \quad (3.16) \quad \mathcal{B}_{m,j}^N(\omega) = \mu_{m,j} \alpha_{m,j}^{-1} \left( \left\langle \frac{\boldsymbol{\mu}}{\boldsymbol{\alpha}} \right\rangle \Lambda_{\alpha}(0) - \sum_{k=0}^{p-1} \frac{\mu_k}{\alpha_k} \sum_{\ell=1}^N \frac{A_{\alpha,\ell} \omega^2}{(\alpha_{m,j}^{-1} \alpha_k^{-1} \Omega_{\alpha,\ell})^2 - \omega^2} \right).$$

395 Replacing  $\mathcal{B}_m^{\alpha}(\partial_t)$  in (2.14) by  $\mathcal{B}_m^{\alpha,N}(\partial_t)$  leads to the following problem: find

$$396 \quad u_m^{\alpha,N} \in C^1(\mathbb{R}^+; L_{\mu}^2(\mathcal{T}^m)) \cap C(\mathbb{R}^+; \mathbf{V}_{\mu}(\mathcal{T}^m)),$$

398 s.t.  $u_m^{\alpha,N}(\cdot, 0) = \partial_t u_m^{\alpha,N}(\cdot, 0) = 0$ , and that satisfies, for all  $v \in \mathbf{V}_{\mu}(\mathcal{T}^m)$ , the following:  
 399

$$400 \quad (3.17a) \quad (\partial_t^2 u_m^{\alpha,N}, v)_{\mathcal{T}^m} + (\partial_s u_m^{\alpha,N}, \partial_s v)_{\mathcal{T}^m} + \int_{\Gamma_m} \mathcal{B}_m^{\alpha,N}(\partial_t) \gamma_m u_m^{\alpha,N} \gamma_m v = (f, v)_{\mathcal{T}^m},$$

402 where  $(\mathcal{B}_m^{\alpha,N}(\partial_t) \gamma_m u_m^{\alpha,N}(t)) \in \mathbb{R}^{p^m}$  is defined as follows:

$$403 \quad (\mathcal{B}_m^{\alpha,N}(\partial_t) \gamma_m u_m^{\alpha,N})(t) = \mathbf{W}_m \mathbf{D}_m \left( \left\langle \frac{\boldsymbol{\mu}}{\boldsymbol{\alpha}} \right\rangle \Lambda_{\alpha}(0) \gamma_m u_m^{\alpha,N}(t) + \sum_{k=0}^{p-1} \frac{\mu_k}{\alpha_k} \sum_{\ell=1}^N A_{\alpha,\ell} \frac{d\boldsymbol{\lambda}_{\ell,k}^{\alpha}}{dt}(t) \right),$$

$$404 \quad (3.17b) \quad \mathbf{D}_m = \text{diag}(\alpha_{m,0}^{-1}, \dots, \alpha_{m,p^m-1}^{-1}), \quad \mathbf{W}_m = \text{diag}(\mu_{m,0}, \dots, \mu_{m,p^m-1}),$$

406 and the vector-valued functions  $\boldsymbol{\lambda}_{\ell,k}^{\alpha} : \mathbb{R}^+ \rightarrow \mathbb{R}^{p^m}$  solve

$$407 \quad (3.17c) \quad \frac{d^2}{dt^2} \boldsymbol{\lambda}_{\ell,k}^{\alpha} + \alpha_k^{-2} \Omega_{\alpha,\ell}^2 \mathbf{D}_m^2 \boldsymbol{\lambda}_{\ell,k}^{\alpha} = \partial_t \gamma_m u_m^{\alpha,N}, \quad \boldsymbol{\lambda}_{\ell,k}^{\alpha}|_{t=0} = \frac{d\boldsymbol{\lambda}_{\ell,k}^{\alpha}}{dt} \Big|_{t=0} = 0.$$

409 **3.2.2. Stability of the formulation (3.17).** The stability of (3.17) is guar-  
 410 anteed by the non-negativity of  $\Lambda_{\alpha}(0)$ , see Proposition 3.3, and of  $A_{\alpha,\ell}$  in (3.16), cf.  
 411 Corollary 3.4. To prove this, we introduce an energy associated with (3.17):

$$412 \quad E_m^{\alpha,N}(t) := E_{m,u}^{\alpha,N}(t) + E_{m,\lambda}^{\alpha,N}(t), \text{ where}$$

$$413 \quad E_{m,u}^{\alpha,N}(t) := \frac{1}{2} \left( \|\partial_t u_m^{\alpha,N}(t)\|_{\mathcal{T}^m}^2 + \|\partial_s u_m^{\alpha,N}(t)\|_{\mathcal{T}^m}^2 + \left\langle \frac{\boldsymbol{\mu}}{\boldsymbol{\alpha}} \right\rangle \Lambda_{\alpha}(0) \int_{\Gamma_m} \frac{\boldsymbol{\mu}}{\boldsymbol{\alpha}} |\gamma_m u_m^{\alpha,N}(t)|^2 \right),$$

$$414 \quad E_{m,\lambda}^{\alpha,N}(t) := \frac{1}{2} \sum_{k=0}^{p-1} \sum_{\ell=1}^N A_{\alpha,\ell} \frac{\mu_k}{\alpha_k} \left( \int_{\Gamma_m} \frac{\boldsymbol{\mu}}{\boldsymbol{\alpha}} |\partial_t \boldsymbol{\lambda}_{\ell,k}^{\alpha}(t)|^2 + \frac{\Omega_{\alpha,\ell}^2}{\alpha_k^2} \int_{\Gamma_m} \frac{\boldsymbol{\mu}}{\boldsymbol{\alpha}^3} |\boldsymbol{\lambda}_{\ell,k}^{\alpha}(t)|^2 \right).$$

420 **THEOREM 3.8 (Stability).** *Let  $f \in L_{loc}^1(\mathbb{R}^+; L_{\mu}^2(\mathcal{T}^m))$ . Then, for all  $T > 0$ ,*

$$421 \quad \sqrt{E_m^{\alpha,N}(t)} \leq C \int_0^T \|f(\tau)\|_{L_{\mu}^2(\mathcal{T}^m)} d\tau, \quad \text{for all } 0 \leq t \leq T,$$

422 where  $C > 0$  does not depend on  $N, m, T, \boldsymbol{\alpha}, \boldsymbol{\mu}$ .

424 *Proof.* The proof is classical. It suffices to test (3.17a) with  $v = \partial_t u_m^{\mathbf{a},N}$ . Then,  
 425 by (3.17c), with  $\langle \cdot, \cdot \rangle$  denoting the Euclidean scalar product in  $\mathbb{R}^{p^m}$ ,

$$426 \quad \langle \mathbf{W}_m \mathbf{D}_m \partial_t \boldsymbol{\lambda}_{\ell,k}^{\mathbf{a}}, \partial_t \gamma_m u_m^{\mathbf{a},N} \rangle = \frac{1}{2} \frac{d}{dt} \left( \int_{\Gamma_m} \frac{\boldsymbol{\mu}}{\boldsymbol{\alpha}} |\partial_t \boldsymbol{\lambda}_{\ell,k}^{\mathbf{a}}(t)|^2 + \alpha_k^{-2} \Omega_{\mathbf{a},\ell}^2 \int_{\Gamma_m} \frac{\boldsymbol{\mu}}{\boldsymbol{\alpha}^3} |\boldsymbol{\lambda}_{\ell,k}^{\mathbf{a}}(t)|^2 \right).$$

428 This results in the energy identity  $\frac{d}{dt} E_m^{\mathbf{a},N} = (f, \partial_t u_m^{\mathbf{a},N})_{\mathcal{T}^m}$ . The rest follows by a  
 429 straightforward application of a Gronwall's lemma, cf. [8, Appendix E].  $\square$

430 **3.3. Error analysis.** Here we study the error of approximating (2.14) by (3.17)  
 431 as a function of the number of the terms in the truncated series  $N$ , as  $N \rightarrow \infty$ . Let  
 432 us introduce the following energy norm of  $v \in C^0(\mathbb{R}^+; \mathbf{V}_\mu(\mathcal{T}^m)) \cap C^1(\mathbb{R}^+; \mathbf{L}_\mu^2(\mathcal{T}^m))$ :

$$433 \quad \|v\|_{[0,T];\mathcal{T}^m} := \sup_{t \leq T} \|\partial_t v(\cdot, t)\|_{\mathbf{L}_\mu^2(\mathcal{T}^m)} + \sup_{t \leq T} \|\partial_s v(\cdot, t)\|_{\mathbf{L}_\mu^2(\mathcal{T}^m)}.$$

435 The principal result of this section is summarized below.

436 **THEOREM 3.9.** *Let  $m, N \geq 1$ . Let  $f \in W_{loc}^{4,1}(\mathbb{R}^+; \mathbf{L}_\mu^2(\mathcal{T}^m))$  be s.t.  $f^{(j)}(0) = 0$ ,  
 437  $j = 0, \dots, 3$ , and satisfy Assumption 2.8.*

438 *Let  $u_{\mathbf{a}}$ ,  $\mathbf{a} = \mathfrak{d}$  (resp.  $\mathbf{a} = \mathfrak{n}$ ), solve (D) (resp. (N)), and  $u_m^{\mathbf{a},N}$  solve (3.17). Let the  
 439 error be defined as follows:  $\varepsilon_m^{\mathbf{a},N} = u_m^{\mathbf{a},N} - u_{\mathbf{a}}|_{\mathcal{T}^m}$ .*

440 *Then, with  $r_{\mathbf{a},N}$  defined in (3.13), for all  $T > 0$ , it holds:*

$$441 \quad (3.19) \quad \|\varepsilon_m^{\mathbf{a},N}\|_{[0,T];\mathcal{T}^m} \leq C_m r_{\mathbf{a},N} T \|\partial_t^4 \partial_s u\|_{L^1(0,T;\mathbf{L}_\mu^2(\mathcal{T}))},$$

443 where  $C_{\mathbf{a},m} > 0$  is given by

$$444 \quad C_m = \begin{cases} C_{\mathbf{a}} m^2 \eta^m, & \eta = \max(\langle \boldsymbol{\mu} \boldsymbol{\alpha} \rangle, |\boldsymbol{\alpha}|_\infty^2), & \text{if } \langle \boldsymbol{\mu} \boldsymbol{\alpha} \rangle < 1, \\ C_{\mathbf{a}} |\boldsymbol{\alpha}|_\infty^{2m}, & & \text{if } \langle \boldsymbol{\mu} \boldsymbol{\alpha} \rangle \geq 1. \end{cases}$$

446 The constant  $C_{\mathbf{a}} > 0$  does not depend on  $T, m, N$ .

447 Therefore, for fixed  $T, m$ ,  $\|\varepsilon_m^{\mathbf{a},N}(T)\|_{\mathbf{L}_\mu^2(\mathcal{T}^m)} \rightarrow 0$ , as  $N \rightarrow +\infty$ .

448 **REMARK 3.10.** *Theorem 3.9 indicates as well the behaviour of the error  $\varepsilon_m^{\mathbf{a},N}$  as  
 449 a function of  $m$  (the level at which the tree is truncated). The respective bound can  
 450 be translated as a bound for  $\|\varepsilon_m^{\mathbf{a},N}(t)\|_{\mathbf{L}_\mu^2(\mathcal{T}^m)}$ : since  $\varepsilon_m^{\mathbf{a},N}(0) = 0$ , we have*

$$451 \quad (3.20) \quad \|\varepsilon_m^{\mathbf{a},N}(t)\|_{\mathbf{L}_\mu^2(\mathcal{T}^m)} \leq t \sup_{\tau \in (0,t)} \|\partial_\tau \varepsilon_m^{\mathbf{a},N}(\tau)\|.$$

453 Therefore, when  $N, T$  are fixed, as  $m \rightarrow \infty$ ,  $\|\varepsilon_m^{\mathbf{a},N}(t)\|_{\mathbf{L}_\mu^2(\mathcal{T}^m)} \rightarrow 0$ .

454 We consider that this is of less importance, because the complexity of resolution of  
 455 (3.17) increases exponentially with  $m$ . Concurrently, this complexity is linear in  $N$ .

456 The proof of Theorem 3.9 relies on two auxiliary trace lemmas that follow from [9].

457 **LEMMA 3.11.** *Let  $v \in \mathbf{V}_\mu(\mathcal{T}^m)$ . Then, for all  $m \geq 1$ ,*

$$458 \quad (3.21) \quad \int_{\Gamma_m} \boldsymbol{\mu} \boldsymbol{\alpha} |\gamma_m v|^2 \leq C_{\boldsymbol{\alpha},\boldsymbol{\mu}} m^2 \eta^m \|\partial_s v\|_{\mathbf{L}_\mu^2(\mathcal{T}^m)}^2, \quad \eta = \max(\langle \boldsymbol{\mu} \boldsymbol{\alpha} \rangle, |\boldsymbol{\alpha}|_\infty^2),$$

460 where  $C_{\boldsymbol{\alpha},\boldsymbol{\mu}} > 0$  is independent of  $m$ .

461 *Proof.* The proof relies on the following inequality from the proof of [9, Theorem  
462 3.24] (in the notation of [9], see also (117),  $\int_{\Gamma_m} \mu \alpha |\gamma_m v|^2 = \|\Pi v\|_{L_\mu^2(\mathcal{G}_m)}^2$ ):  
463

$$464 \int_{\Gamma_m} \mu \alpha |\gamma_m v|^2 \leq C_m \|\partial_s v\|_{L_\mu^2(\mathcal{T}^m)}^2, \quad C_m = \begin{cases} m^2 \langle \mu \alpha \rangle^m & \text{if } \langle \mu \alpha \rangle = |\alpha|_\infty^2, \\ m |\alpha|_\infty^2 \frac{\langle \mu \alpha \rangle^m - |\alpha|_\infty^{2m}}{\langle \mu \alpha \rangle - |\alpha|_\infty^2}, & \text{if } \langle \mu \alpha \rangle \neq |\alpha|_\infty^2. \end{cases}$$

465  
466 The result follows with  $C_m \leq C m^2 \max(\langle \mu \alpha \rangle^m, |\alpha|_\infty^{2m})$ ,  $C > 0$ .  $\square$   
467

468 LEMMA 3.12. *Let*  $\langle \mu \alpha \rangle \geq 1$ , *and*  $v \in H_\mu^1(\mathcal{T}) = H_{\mu,0}^1(\mathcal{T})$ . *Then, for all*  $m \geq 1$ ,

$$469 (3.22) \quad \int_{\Gamma_m} \mu \alpha^{-1} |\gamma_m v|^2 \leq C_{\alpha, \mu} \|\partial_s v\|_{L_\mu^2(\mathcal{T})}^2,$$

470 where  $C_{\alpha, \mu}$  is independent of  $m$ .

472 *Proof.* See [9, the end of the proof of Theorem 3.18 and the notation (110)].  $\square$

473 *Proof of Theorem 3.9.* First, we remark that the regularity condition on  $f$  ensures  
474 the required regularity of the solution  $u_\alpha$ , see [6]. Let us fix  $N, m$ .

475 *Step 1. Re-expressing*  $\varepsilon_m^{\alpha, N}$ . By taking difference between (3.17) and (2.14), we  
476 see that the error  $\varepsilon_m^{\alpha, N}$  solves the following problem:

477 find  $\varepsilon_m^{\alpha, N} \in C^0(\mathbb{R}^+; V_n(\mathcal{T}^m))$ , s.t.  $\varepsilon_m^{\alpha, N}(0) = \partial_t \varepsilon_m^{\alpha, N}(0) = 0$ , and

$$478 (\partial_t^2 \varepsilon_m^{\alpha, N}, v)_{\mathcal{T}^m} + (\partial_s \varepsilon_m^{\alpha, N}, \partial_s v)_{\mathcal{T}^m} + \int_{\Gamma_m} (\mathcal{B}_m^{\alpha, N}(\partial_t) \gamma_m u_m^{\alpha, N} - \mathcal{B}_m^\alpha(\partial_t) \gamma_m u_\alpha) \gamma_m v = 0,$$

479 for all  $v \in V_n(\mathcal{T}^m)$ . Defining

$$480 (3.23) \quad \zeta_m^{\alpha, N} = (\mathcal{B}_m^\alpha(\partial_t) - \mathcal{B}_m^{\alpha, N}(\partial_t)) \gamma_m u_\alpha,$$

481 we rewrite the above in the form (3.17a):

$$482 (\partial_t^2 \varepsilon_m^{\alpha, N}, v)_{\mathcal{T}^m} + (\partial_s \varepsilon_m^{\alpha, N}, \partial_s v)_{\mathcal{T}^m} + \int_{\Gamma_m} \mathcal{B}_m^{\alpha, N}(\partial_t) \gamma_m \varepsilon_m^{\alpha, N} \gamma_m v = \int_{\Gamma_m} \zeta_m^{\alpha, N} \gamma_m v.$$

483 To derive the error estimates, we will use the energy techniques, like in Theorem 3.8.

484 Let us introduce an energy of the error, cf. the definition (3.18),

$$485 (3.24) \quad \mathcal{E} := \frac{1}{2} \left( \|\partial_t \varepsilon_m^{\alpha, N}\|_{\mathcal{T}^m}^2 + \|\partial_s \varepsilon_m^{\alpha, N}\|_{\mathcal{T}^m}^2 + \left\langle \frac{\mu}{\alpha} \right\rangle \Lambda_\alpha(0) \int_{\Gamma_m} \frac{\mu}{\alpha} |\gamma_m \varepsilon_m^{\alpha, N}|^2 \right) + \mathcal{E}_\lambda,$$

486 with  $\mathcal{E}_\lambda$  defined like in (3.18). Like in the proof of Theorem 3.8, we have

$$487 (3.25) \quad \frac{d}{dt} \mathcal{E}(t) := \int_{\Gamma_m} \zeta_m^{\alpha, N}(t) \partial_t \gamma_m \varepsilon_m^{\alpha, N}(t).$$

488 Integrating the above from 0 to  $T$  results in (since  $\varepsilon_m^{\alpha, N}(0) = \partial_t \varepsilon_m^{\alpha, N}(0) = 0$ ):

$$489 (3.26) \quad \mathcal{E}(T) = \underbrace{\int_{\Gamma_m} \zeta_m^{\alpha, N}(T) \gamma_m \varepsilon_m^{\alpha, N}(T)}_{I_1(T)} - \underbrace{\int_0^T \int_{\Gamma_m} \partial_t \zeta_m^{\alpha, N}(t) \gamma_m \varepsilon_m^{\alpha, N}(t) dt}_{I_2(T)}.$$

490

496 *Step 2. Bounding the right-hand side of (3.26).*

497 *Step 2.1. Bounding  $I_1(T)$ .* From (3.23) and the representation (3.15), it follows

$$498 \quad |I_1(T)| \leq \sum_{k=0}^{p-1} \frac{\mu_k}{\alpha_k} \int_{\Gamma_m} \left( \frac{\mu}{\alpha} |\Lambda_a^N(\alpha_k \alpha \partial_t) \gamma_m u_a(T) - \Lambda_a(\alpha_k \alpha \partial_t) \gamma_m u_a(T)| |\gamma_m \varepsilon_m^{a,N}(T)| \right).$$

500 The goal is to find a bound for the above by bounding (cf. the notation (2.6))

$$501 \quad (3.27) \quad q_{m,j}(t) := \Lambda_a^N(\alpha_k \alpha_{m,j} \partial_t) u_a(M_{m,j}, t) - \Lambda_a(\alpha_k \alpha_{m,j} \partial_t) u_a(M_{m,j}, t).$$

503 Since  $\Lambda_a(\alpha_k \alpha_{m,j} \omega) = \Lambda_a(0) - \sum_{\ell=1}^{\infty} \frac{A_{a,\ell} \omega^2}{(\alpha_k^{-1} \alpha_{m,j}^{-1} \Omega_{a,\ell})^2 - \omega^2}$ , to bound  $q_{m,j}$ , we use the  
504 same argument as in (3.14), see also (3.13):

$$505 \quad |q_{m,j}(t)| \leq \sum_{\ell=N+1}^{\infty} \frac{A_{a,\ell}}{\alpha_k^{-2} \alpha_{m,j}^{-2} \Omega_{a,\ell}^2} \int_0^t |\partial_\tau^3 u_a(M_{m,j}, \tau)| d\tau.$$

507 Therefore,

$$508 \quad |I_1(T)| \leq 2r_{a,N} \sum_{k=0}^{p-1} \frac{\mu_k}{\alpha_k} \int_0^T \int_{\Gamma_m} \left( \frac{\mu}{\alpha} \alpha_k^2 \alpha^2 |\partial_\tau^3 \gamma_m u_a(\tau)| |\gamma_m \varepsilon_m^{a,N}(T)| \right) d\tau$$

$$509 \quad (3.28) \quad = 2r_{a,N} \langle \mu \alpha \rangle \int_0^T \int_{\Gamma_m} \left( \mu \alpha |\partial_\tau^3 \gamma_m u_a(\tau)| |\gamma_m \varepsilon_m^{a,N}(T)| \right) d\tau.$$

511 *Step 2.2. Bounding  $I_2(T)$ .* The same argument as in Step 2.1 yields

$$512 \quad \left| \int_{\Gamma_m} \partial_t \zeta_m^{a,N}(t) \gamma_m \varepsilon_m^{a,N}(t) \right| \leq 2r_{a,N} \langle \mu \alpha \rangle \int_0^t \int_{\Gamma_m} \left( \mu \alpha |\partial_\tau^4 \gamma_m u_a(\tau)| |\gamma_m \varepsilon_m^{a,N}(t)| \right) d\tau, \text{ thus}$$

$$513 \quad (3.29) \quad |I_2(T)| \leq 2r_{a,N} \langle \mu \alpha \rangle \int_0^T \int_0^t \int_{\Gamma_m} \left( \mu \alpha |\partial_\tau^4 \gamma_m u_a(\tau)| |\gamma_m \varepsilon_m^{a,N}(t)| \right) d\tau dt.$$

515 *Step 2.3. Bounding the right-hand side of (3.26).* We use the bounds (3.28) and  
516 (3.29) to bound (3.26) as follows:

$$517 \quad \mathcal{E}(T) \leq 2r_{a,N} \langle \mu \alpha \rangle \int_0^T \int_{\Gamma_m} \left( \mu \alpha |\partial_\tau^3 \gamma_m u_a(\tau)| |\gamma_m \varepsilon_m^{a,N}(T)| \right) d\tau$$

$$518 \quad + 2r_{a,N} \langle \mu \alpha \rangle \int_0^T \int_0^t \int_{\Gamma_m} \left( \mu \alpha |\partial_\tau^4 \gamma_m u_a(\tau)| |\gamma_m \varepsilon_m^{a,N}(t)| \right) d\tau dt$$

$$519 \quad (3.30) \quad \leq 2r_{a,N} \langle \mu \alpha \rangle \int_0^T \int_0^t \int_{\Gamma_m} \left( \mu \alpha |\partial_\tau^4 \gamma_m u_a(\tau)| (|\gamma_m \varepsilon_m^{a,N}(T)| + |\gamma_m \varepsilon_m^{a,N}(t)|) \right) d\tau dt,$$

520

521 where in the last bound we used  $\partial_\tau^3 \gamma_m u_a = \int_0^t \partial_\tau^4 \gamma_m u_a$  (it holds that  $\partial_t^3 \gamma_m u_a|_{t=0} = 0$

522 because of the finite speed of wave propagation and the assumption on the source  $f$ ).

523 *Step 3. Bounding  $\varepsilon_m^{\alpha, N}$  based on (3.30).* Naturally, we would like to apply a  
524 Gronwall inequality to the bound (3.30). The cases  $\langle \mu \alpha \rangle < 1$  and  $\langle \mu \alpha \rangle \geq 1$  will be  
525 treated differently. When  $\langle \mu \alpha \rangle < 1$ , by Theorem 2.7 and Proposition 3.3,  
526

527 (1) if  $\langle \mu / \alpha \rangle > 1$ , then  $\Lambda_\partial \neq \Lambda_n$ ,  $\Lambda_\partial(0) > 0$  and  $\Lambda_n(0) = 0$ .

528 (2) if  $\langle \mu / \alpha \rangle \leq 1$ , then  $\Lambda_n = \Lambda_\partial$ ,  $\Lambda_\partial(0) = \Lambda_n(0) = 0$ .

529 Let us consider the case (2). As  $\Lambda_n(0) = 0$ , the boundary term in (3.24) does not  
530 control  $\gamma_m \varepsilon_m^{\alpha, N}(T)$ . Hence instead we will use the trace continuity result of Lemma  
531 3.11. Because this strategy will allow to conclude that the error decays exponentially  
532 fast in  $m$ , we will make use of it also in the case (1), when  $\Lambda_n(0)$  does not necessarily  
533 vanish (i.e. when  $\Lambda = \Lambda_\partial$ ).

534 The obtained error bound is valid when  $\langle \mu \alpha \rangle \geq 1$ , but it is non-optimal (it grows  
535 exponentially fast with  $m$ ). That is why we deal with the case  $\langle \mu \alpha \rangle \geq 1$  separately.  
536 Here, by Theorem 2.7 and Proposition 3.3,  $\Lambda_\partial = \Lambda_n$ ,  $\Lambda_n(0) > 0$ . This property allows  
537 to control the boundary term in (3.24) by the energy  $\mathcal{E}$ .

538 *Error bound when  $\langle \mu \alpha \rangle < 1$ .* Application of the Cauchy-Schwarz inequality to both  
539 terms of (3.30) yields  
540

$$541 \quad \mathcal{E}(T) \leq 2r_{\alpha, N} \langle \mu \alpha \rangle \int_0^T \int_0^t \left( \int_{\Gamma_m} \mu \alpha |\partial_\tau^4 \gamma_m u_a(\tau)|^2 \right)^{\frac{1}{2}} \left( \int_{\Gamma_m} \mu \alpha |\gamma_m \varepsilon_m^{\alpha, N}(T)|^2 \right)^{\frac{1}{2}} d\tau dt$$

$$542 \quad + 2r_{\alpha, N} \langle \mu \alpha \rangle \int_0^T \int_0^t \left( \int_{\Gamma_m} \mu \alpha |\partial_\tau^4 \gamma_m u_a(\tau)|^2 \right)^{\frac{1}{2}} \left( \int_{\Gamma_m} \mu \alpha |\gamma_m \varepsilon_m^{\alpha, N}(t)|^2 \right)^{\frac{1}{2}} d\tau dt.$$

543 Next, we apply (3.21) to the terms with  $\gamma_m u_a$  and  $\gamma_m \varepsilon_m^{\alpha, N}$ , which yields, with  $C' > 0$ ,

$$544 \quad \mathcal{E}(T) \leq C' r_{\alpha, N} m^2 \eta^m \int_0^T \int_0^t \|\partial_s \partial_\tau^4 u_a(\tau)\|_{\mathcal{T}^m} \left( \|\partial_s \varepsilon_m^{\alpha, N}(T)\|_{\mathcal{T}^m} + \|\partial_s \varepsilon_m^{\alpha, N}(t)\|_{\mathcal{T}^m} \right) d\tau dt$$

$$545 \quad \leq \sqrt{2} C' r_{\alpha, N} m^2 \eta^m \int_0^T \|\partial_s \partial_t^4 u_a\|_{L^1(0, t; L_\mu^2(\mathcal{T}^m))} \left( \sqrt{\mathcal{E}(T)} + \sqrt{\mathcal{E}(t)} \right) dt.$$

546 A Gronwall's inequality, cf. [8, Appendix E] yields, with  $C > 0$  (independent of  $T$ ):

$$547 \quad (3.31) \quad \sqrt{\mathcal{E}(T)} \leq C r_{\alpha, N} m^2 \eta^m \int_0^T \|\partial_s \partial_t^4 u_a\|_{L^1(0, t; L_\mu^2(\mathcal{T}^m))} dt,$$

548 hence the bound in the statement of the theorem.

549 *Error bound when  $\langle \mu \alpha \rangle \geq 1$ .* Like before, we start by applying the Cauchy-Schwarz



554 inequality to (3.30):

$$555 \quad \mathcal{E}(T) \leq 2r_{\mathbf{a},N} \langle \boldsymbol{\mu} \boldsymbol{\alpha} \rangle \int_0^T \int_0^t \left( \int_{\Gamma_m} \boldsymbol{\mu} \boldsymbol{\alpha}^3 |\partial_\tau^3 \gamma_m u_{\mathbf{a}}(\tau)|^2 \right)^{\frac{1}{2}} \left( \int_{\Gamma_m} \boldsymbol{\mu} \boldsymbol{\alpha}^{-1} |\gamma_m \varepsilon_m^{\mathbf{a},N}(T)|^2 \right)^{\frac{1}{2}} d\tau dt$$

(3.32)

$$556 \quad + 2r_{\mathbf{a},N} \langle \boldsymbol{\mu} \boldsymbol{\alpha} \rangle \int_0^T \int_0^t \left( \int_{\Gamma_m} \boldsymbol{\mu} \boldsymbol{\alpha}^3 |\partial_\tau^4 \gamma_m u_{\mathbf{a}}(\tau)|^2 \right)^{\frac{1}{2}} \left( \int_{\Gamma_m} \boldsymbol{\mu} \boldsymbol{\alpha}^{-1} |\gamma_m \varepsilon_m^{\mathbf{a},N}(t)|^2 \right)^{\frac{1}{2}} d\tau dt.$$

557

558 Remark that, cf. (2.6),

$$559 \quad \int_{\Gamma_m} \boldsymbol{\mu} \boldsymbol{\alpha}^3 |\partial_\tau^4 \gamma_m u_{\mathbf{a}}|^2 \leq \max_j \alpha_{m,j}^4 \int_{\Gamma_m} \boldsymbol{\mu} \boldsymbol{\alpha}^{-1} |\partial_\tau^4 \gamma_m u_{\mathbf{a}}|^2 \leq |\boldsymbol{\alpha}|_\infty^{4m} \int_{\Gamma_m} \boldsymbol{\mu} \boldsymbol{\alpha}^{-1} |\partial_\tau^4 \gamma_m u_{\mathbf{a}}|^2.$$

560

561 Applying Lemma 3.12 to bound the above, we obtain

$$562 \quad (3.33) \quad \int_{\Gamma_m} \boldsymbol{\mu} \boldsymbol{\alpha}^3 |\partial_\tau^4 \gamma_m u_{\mathbf{a}}|^2 \leq C_{\boldsymbol{\alpha},\boldsymbol{\mu}}^2 |\boldsymbol{\alpha}|_\infty^{4m} \|\partial_s \partial_\tau^4 u_{\mathbf{a}}\|^2.$$

563

Moreover,

$$\int_{\Gamma_m} \boldsymbol{\mu} \boldsymbol{\alpha}^{-1} |\gamma_m \varepsilon_m^{\mathbf{a},N}(t)|^2 \leq 2 \langle \boldsymbol{\mu} / \boldsymbol{\alpha} \rangle^{-1} \boldsymbol{\Lambda}_{\mathbf{a}}^{-1}(0) \mathcal{E}(t).$$

564 Thus, with the above and (3.33), the inequality (3.32) can be rewritten as follows:

$$565 \quad \mathcal{E}(T) \leq Cr_{\mathbf{a},N} |\boldsymbol{\alpha}|_\infty^{2m} \int_0^T \int_0^\tau \|\partial_s \partial_\tau^4 u_{\mathbf{a}}(\tau)\|_{\mathcal{T}^m} \left( \mathcal{E}^{\frac{1}{2}}(t) + \mathcal{E}^{\frac{1}{2}}(T) \right) d\tau dt.$$

566

567 To obtain the desired estimate we proceed like in the derivation of (3.31):

$$568 \quad \sqrt{\mathcal{E}(T)} \leq Cr_{\mathbf{a},N} |\boldsymbol{\alpha}|_\infty^{2m} T \|\partial_s \partial_t^4 u_{\mathbf{a}}\|_{L^1(0,T;L_\mu^2(\mathcal{T}^m))}.$$

569

570 *Step 4. Convergence.* Since  $\lim_{N \rightarrow +\infty} r_{\mathbf{a},N} = 0$  as a remainder of the convergent series,

571 cf. Lemma 3.6 and (3.13), by combining (3.19) and (3.20), we see that for fixed  $T, m$ ,  
572 it holds that  $\lim_{N \rightarrow +\infty} \|\varepsilon_m^{\mathbf{a},N}(t)\|_{\mathcal{T}^m} = 0$ .  $\square$

573 **4. Error control and complexity estimates.** All over the section we fix the  
574 simulation time  $T$  and the parameter  $m$  and study the behavior of the error with  $N$ .

**4.1. Error control.** It appears that the error bound provided by Theorem 3.9  
is (at least partially) computable. Let us show how

$$r_{\mathbf{a},N} = \sum_{k=N+1}^{\infty} A_{\mathbf{a},k} \Omega_{\mathbf{a},k}^{-2}, \quad \mathbf{a} \in \{\mathfrak{d}, \mathfrak{n}\}, \text{ see definition (3.13),}$$

575 which controls the error, can be approximated numerically, provided that  $A_{\mathbf{a},k}, \Omega_{\mathbf{a},k}$ ,  
576  $k = 1, \dots, N$ , are known (see Section 5.1). A direct computation using (3.6) yields

$$577 \quad \boldsymbol{\Lambda}_{\mathbf{a}}''(0) = -2 \sum_{k=0}^{\infty} A_{\mathbf{a},k} \Omega_{\mathbf{a},k}^{-2}.$$

578

579 By Lemma 5.5 and Corollary 5.6 in [9], the right-hand side of the above is known  
580 explicitly. Introducing

$$581 \quad \lambda_{\mathcal{N}} = - (1 - \langle \boldsymbol{\mu} \boldsymbol{\alpha} \rangle)^{-1}, \quad \lambda_{\mathcal{D}} = -\frac{1}{3} (\langle \boldsymbol{\mu} \boldsymbol{\alpha} \rangle^2 + \langle \boldsymbol{\mu} \boldsymbol{\alpha} \rangle + 1) (\langle \boldsymbol{\mu} \boldsymbol{\alpha} \rangle^2 - \langle \boldsymbol{\mu} \boldsymbol{\alpha} \rangle)^{-1},$$

583 we have the following:

- 585 • when  $\langle \boldsymbol{\mu} \boldsymbol{\alpha} \rangle \leq 1$ ,  $\Lambda_{\mathfrak{n}}''(0) = \Lambda_{\mathfrak{d}}''(0) = \lambda_{\mathcal{N}}$ ;
- 587 • when  $\langle \boldsymbol{\mu} \boldsymbol{\alpha} \rangle < 1 < \langle \boldsymbol{\mu} \boldsymbol{\alpha} \rangle$ ,  $\Lambda_{\mathfrak{n}}''(0) = \lambda_{\mathcal{N}}$  and  $\Lambda_{\mathfrak{d}}''(0) = \lambda_{\mathcal{D}}$ ;
- 588 • when  $\langle \boldsymbol{\mu} \boldsymbol{\alpha} \rangle \geq 1$ ,  $\Lambda_{\mathfrak{n}}''(0) = \Lambda_{\mathfrak{d}}''(0) = \lambda_{\mathcal{D}}$ .

590 Hence, provided  $A_{\mathfrak{a},k}$  and  $\Omega_{\mathfrak{a},k}$ ,  $k = 1, \dots, N$  (approximated numerically), we compute

$$592 \quad r_{\mathfrak{a},N} = -\frac{1}{2} \left( \Lambda_{\mathfrak{a}}''(0) + 2 \sum_{k=1}^N \frac{A_{\mathfrak{a},k}}{\Omega_{\mathfrak{a},k}^2} \right), \quad \mathfrak{a} \in \{\mathfrak{d}, \mathfrak{n}\},$$

594 with  $\Lambda_{\mathfrak{a}}''(0)$  being given by one of the above expressions.

595 **4.2. Convergence and complexity estimates.** Obviously, the complexity of  
596 the method described in Section 3 depends linearly on the number of poles  $N$  in  
597 (3.17). The estimate of Theorem 3.9 shows that in order to ensure that the error in  
598 the energy norm is  $O(\varepsilon)$ , one should choose  $N = N_{\mathfrak{a},\varepsilon}$ , so that  $r_{\mathfrak{a},N} < \varepsilon$ . Let

$$599 \quad (4.1) \quad N_{\mathfrak{a},\varepsilon} := \min_{N \in \mathbb{N}_*} \{r_{\mathfrak{a},N} < \varepsilon\}.$$

601 In [7] it was shown that  $N_{\mathfrak{a},\varepsilon}$  is related to the following quantity:

$$603 \quad (4.2) \quad P_{\mathfrak{a}}(\lambda) := \#\{k : 0 < \Omega_{\mathfrak{a},k} < \lambda\}.$$

604 More precisely,  $N_{\mathfrak{a},\varepsilon}$  is bounded by

$$605 \quad (4.3) \quad P_{\mathfrak{a}}(C_1 \varepsilon^{-1}) \leq N_{\mathfrak{a},\varepsilon} \leq P_{\mathfrak{a}}(C_2 \varepsilon^{-1}), \quad \text{with some } C_2 > C_1 > 0.$$

607 **4.2.1. Asymptotic estimates on  $N_{\mathfrak{a},\varepsilon}$  and  $r_{\mathfrak{a},N}$ .** As shown by (4.3), to find  
608 an asymptotic upper bound on  $N_{\mathfrak{a},\varepsilon}$ , it is sufficient to obtain a bound on  $P_{\mathfrak{a}}(\lambda)$  as  
609  $\lambda \rightarrow +\infty$ . For this, in [7] we used the fact that the poles in (3.6) are related to the  
610 eigenvalues  $\omega_{\mathfrak{a},k}$  of  $\mathcal{A}_{\mathfrak{a}}$ , cf. Theorem 3.2. More precisely, because the eigenvalues  $\omega_{\mathfrak{a},k}$   
611 (unlike the poles  $\Omega_{\mathfrak{a},k}$ ) are counted with multiplicities, it holds that  $P_{\mathfrak{a}}(\lambda) \leq \#\{k : \omega_{\mathfrak{a},k} < \lambda\}$ . This allows to relate bounds of  $P_{\mathfrak{a}}$  to the asymptotics of the eigenvalue  
612 counting function. It is then not surprising that such a result will depend on the  
613 geometry of the tree  $\mathcal{T}$ . To state it, we define (with  $\sum \equiv \sum_i$ ):

$$615 \quad (4.4) \quad d_s \in (0, \infty) \text{ a unique number s.t. } \sum \alpha_i^{d_s} = 1.$$

616 The existence and uniqueness of  $d_s$  follows by noticing that  $(0, \infty) \ni x \mapsto \sum \alpha_i^x$  is a  
617 strictly monotonically decreasing function with the values on the interval  $(0, p)$ .

618 To state a bound on  $P_{\mathfrak{a}}$ , let us introduce  $\langle \boldsymbol{\alpha} \rangle = \sum \alpha_i$ .

619 **THEOREM 4.1 ([7]).** *There exists  $C_{\mathfrak{a}} > 0$ ,  $\mathfrak{a} \in \{\mathfrak{n}, \mathfrak{d}\}$ , depending on  $\boldsymbol{\alpha}$ ,  $\boldsymbol{\mu}$ , s.t.,*  
620 *for all  $\lambda > 2$ , it holds:*

621

- 622 1. if  $\langle \alpha \rangle < 1$  ( $d_s < 1$ ), then  $P_a(\lambda) \leq C_a \lambda$ .  
623  
624 2. if  $\langle \alpha \rangle = 1$  ( $d_s = 1$ ), then  $P_a(\lambda) \leq C_a \lambda \log \lambda$ .  
625  
626 3. if  $\langle \alpha \rangle > 1$  ( $d_s > 1$ ), then  $P_a(\lambda) \leq C_a \lambda^{d_s}$ .

627 With the above theorem and (4.3), we can obtain an upper bound on the number of  
628 poles in the approximation required to achieve a desired accuracy  $\varepsilon$ .

629 THEOREM 4.2 ([7]). *There exists  $C_a^+ > 0$ , depending only on  $\mu, \alpha$ , such that,*  
630 *for all  $0 < \varepsilon < 1/2$ ,  $N_{a,\varepsilon}$  satisfies:*

- 632 • if  $\langle \alpha \rangle < 1$  ( $d_s < 1$ ),  $N_{a,\varepsilon} \leq C_a^+ \varepsilon^{-1}$ .  
633  
634 • if  $\langle \alpha \rangle = 1$  ( $d_s = 1$ ),  $N_{a,\varepsilon} \leq C_a^+ \varepsilon^{-1} \log \varepsilon^{-1}$ .  
635  
636 • if  $\langle \alpha \rangle > 1$  ( $d_s > 1$ ),  $N_{a,\varepsilon} \leq C_a^+ \varepsilon^{-d_s}$ .

637 Next, we present a bound on  $r_{a,N}$  with respect to  $N$ , which, together with the results  
638 of Theorem 3.9 allows to conclude about the convergence of the method. This bound  
639 is a corollary of Theorem 4.2.

640 THEOREM 4.3 ([7]). *There exists  $c_a^+ > 0$ , depending only on  $\mu, \alpha$ , such that, for*  
641 *all  $N \geq 2$ ,  $r_{a,N}$  satisfies:*

- 643 • if  $\langle \alpha \rangle < 1$  ( $d_s < 1$ ),  $r_{a,N} \leq c_a^+ N^{-1}$ .  
644  
645 • if  $\langle \alpha \rangle = 1$  ( $d_s = 1$ ),  $r_{a,N} \leq c_a^+ N^{-1} \log N$ .  
646  
647 • if  $\langle \alpha \rangle > 1$  ( $d_s > 1$ ),  $r_{a,N} \leq c_a^+ N^{-\frac{1}{d_s}}$ .

648 The following result provides a lower bound on  $P_a, N_{a,\varepsilon}$  and  $r_{a,N}$ .

649 THEOREM 4.4 ([7]). *With some  $c_P, c_n, c_r > 0$ ,  $\varepsilon_0, \lambda_0 > 0$ ,  $N_0 \in \mathbb{N}_*$ , it holds:*  
650 *for all  $\varepsilon < \varepsilon_0$ ,  $\lambda > \lambda_0$ ,  $N > N_0$ ,*

651 
$$P_a(\lambda) > c_P \lambda, \text{ and, by (4.3), } N_{a,\varepsilon} > c_n \varepsilon^{-1}, \quad r_{a,N} > c_r N^{-1}.$$

653 REMARK 4.5. *Since the error bound provided in Theorem 3.9 is only an upper*  
654 *bound, the lower bound on  $r_{a,N}$  given in Proposition 4.4 **does not imply** that the*  
655 *convergence of the method is at best  $O(N^{-1})$ . Nonetheless, in practice, as we will see*  
656 *in Section 5.3, the bound of Theorem 3.9 is close to optimal.*

657 **4.2.2. Numerical experiments.** The goal of this section is to examine numerically  
658 the sharpness of the bounds of Theorem 4.1, as this is equivalent, by (4.3), to  
659 verifying sharpness of Theorems 4.2, 4.3. For this we compute numerically the quantity  
660  $P_a(\lambda)$  defined in (4.2); the poles of  $\Lambda_a(\omega)$  are determined with the help of the  
661 method described in Section 5.1. We consider three cases, as per Theorem 4.1.

662 **Case 1 of Theorem 4.1:**  $d_s < 1$ . We take  $\alpha = (0.5, 0.2)$ ,  $\mu = (1, 2)$ , cf. Figure 2,  
663 left. The numerical results confirm the bounds of Theorem 4.1:  $P(\lambda) = O(\lambda)$ .

664 **Case 2 of Theorem 4.1:**  $d_s = 1$ . The numerical experiment for the case  $\alpha =$   
665  $(0.4, 0.6)$ ,  $\mu = (0.5, 0.3)$ , cf. Figure 2, right, indicates that the upper bound is sharp  
666 in this case.

667 **Case 3 of Theorem 4.1:**  $d_s > 1$ . We study two cases:

- 668 •  $\alpha = (0.45, 0.73)$ ,  $\mu = (0.5, 0.5)$ ,  $d_s \approx 1.34$ . See Figure 3, left.  
669 •  $\alpha = (0.8, 0.8)$ ,  $\mu = (0.4, 0.6)$ ,  $d_s \approx 3.11$ . See Figure 3, right.

670 Two phenomena can be observed:

- 671 • when the tree is not symmetric, i.e. when  $\alpha_0 \neq \alpha_1$ , the upper bound of  
672 Theorem 4.1 seems to be sharp, i.e.  $P_a(\lambda) = O(\lambda^{d_s})$ ;

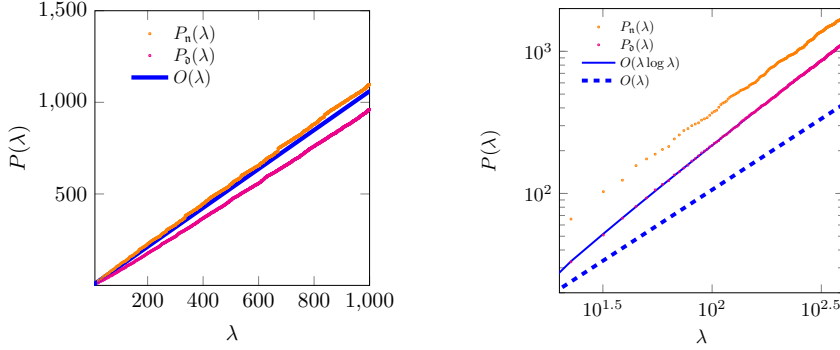


FIGURE 2.  $P_a(\lambda)$  vs its theoretical bounds given by Theorem 4.1 for the case when  $\sum_{i=0}^{n-1} \alpha_i \leq 1$ . Left:  $\alpha = (0.5, 0.2)$ ,  $\mu = (1, 2)$ . Right:  $\alpha = (0.6, 0.4)$  and  $\mu = (1, 0.5)$ . Remark that in the right figure the scale is logarithmic.

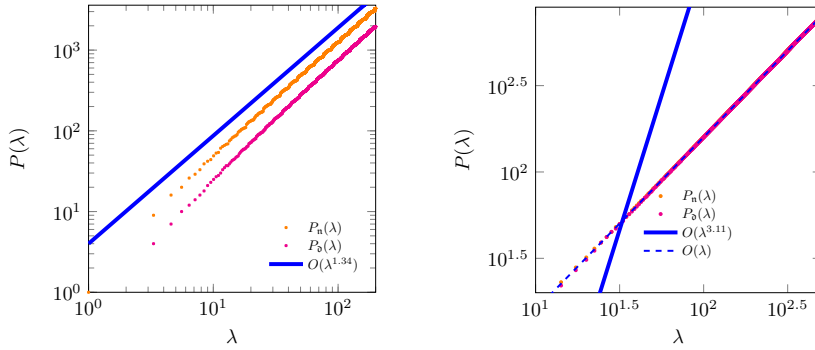


FIGURE 3. Comparison of  $P_a(\lambda)$ ,  $a \in \{0, n\}$ , and the bounds of Theorem 4.1 for different values of  $\alpha$ . Left:  $\alpha = (0.45, 0.73)$ ,  $\mu = (0.5, 0.5)$ . Right:  $\alpha = (0.8, 0.8)$ ,  $\mu = (0.4, 0.6)$ . In this latter case on the plot scale the difference between  $P_0$  and  $P_n$  is almost invisible.

673

- in the case of the symmetric tree (cf. Figure 3, right), we observe that the upper bound of Theorem 4.1 is not sharp. Numerically, the behaviour of  $P_a$  is closer to the lower bound  $P_a(\lambda) = O(\lambda)$ . This is justified in [7].

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Let us finally remark that in the experiment with  $\alpha = (0.45, 0.73)$ , on the interval  $(0, 200)$  we computed more than 3200 poles for the Neumann problem.

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## Conclusions.

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1. Numerical experiments suggest that when the tree  $\mathcal{T}$  is not symmetric (i.e.  $\alpha_i \neq \alpha_j$ , for  $i \neq j$ ), the upper bound of Theorem 4.1 is sharp. This fails in the presence of symmetries.
2. When the upper bound of Theorem 4.1 is sharp, and  $d_s > 1$ , the number of poles on intervals of a fixed length  $(a, a + \ell)$  increases as  $a \rightarrow +\infty$ . To see this, suppose that  $P_a(\lambda) = C_a \lambda^{d_s} + o(\lambda^{d_s})$ , with  $C_a > 0$ , and assume that the number of poles on  $(a, a + \ell)$  is bounded by a constant  $M > 0$  as  $a \rightarrow +\infty$ .

687 This would imply that on the interval  $(0, n\ell)$ , as  $n \rightarrow +\infty$ , there are at  
 688 most  $nM$  poles, and thus  $P_{\mathbf{a}}(n\ell) \leq nM$ . However, by assumption,  $P_{\mathbf{a}}(n\ell) =$   
 689  $C_{\mathbf{a}}(n\ell)^{d_s} + o(n^{d_s})$ , with  $d_s > 1$ , and thus we arrive at the contradiction.

690 **5. Numerical resolution of (3.17). Numerical experiments.** In this sec-  
 691 tion we address the numerical aspects of the resolution of (3.17):

- 692 • in Section 5.1 we outline one strategy to compute  $\Omega_{\mathbf{a},k}$  and  $A_{\mathbf{a},k}$ ;
- 693 • in Section 5.2, we present a stable discretization of (3.17);
- 694 • Section 5.3 is dedicated to the numerical experiments.

695 **5.1. Computing poles and zeros in the approximation  $\Lambda_{\mathbf{a}}^N$ .** In order to  
 696 use the approximation (3.15), (3.12), it is necessary to be able to evaluate  $\Omega_{\mathbf{a},k}$ , the  
 697 poles of  $\Lambda_{\mathbf{a}}$ , and the respective (scaled) residues  $A_{\mathbf{a},k}$ . Because  $\Lambda_{\mathbf{a}}(\omega)$  can be efficiently  
 698 evaluated for each  $\omega \in \mathbb{C}^+$  using the method described in [6] (and for  $\omega \in \mathbb{C}^-$ , we have  
 699  $\Lambda_{\mathbf{a}}(\omega) = (\Lambda_{\mathbf{a}}(\omega^*))^*$ ), it would be rather natural to use classical contour integration  
 700 techniques for computing the poles and the residues of  $\Lambda_{\mathbf{a}}$ . However,

- 701 • the location of poles is not known. This is aggravated by the fact that the  
 702 poles and the zeros of  $\Lambda_{\mathbf{a}}$  interlace (this can be proven rigorously), and thus  
 703 a straightforward use of the argument principle does not seem to be possible.
- 704 • poles of  $\Lambda_{\mathbf{a}}$  can be located very close to each other (which poses difficulties  
 705 in choosing an integration contour for computing residues), see Section 4.2.2.
- 706 • evaluating  $\Lambda_{\mathbf{a}}$  close to the real axis may be inaccurate, because of the proxim-  
 707 ity to the poles, cf. the error estimates in [6]. Let us remark that in practice  
 708 we found this much less of a problem than the two previous issues.

709 To overcome (at least some of) these difficulties, we suggest to use an alternative  
 710 strategy described in the sections that follow.

711 **5.1.1. An auxiliary function.** Let us introduce an auxiliary function (which,  
 712 since  $\Lambda_{\mathbf{a}}(\omega)$  is the symbol of the DtN, can be viewed as a Robin-to-Robin operator):

$$713 \quad (5.1) \quad \mathbf{g}_{\mathbf{a}}(\omega) := (-\omega^{-1}\Lambda_{\mathbf{a}}(\omega) - i) (-\omega^{-1}\Lambda_{\mathbf{a}}(\omega) + i)^{-1}.$$

715 The following proposition relates the location of poles of  $\Lambda_{\mathbf{a}}$  to the points  $\omega$  where  
 716  $\operatorname{Re} \mathbf{g}_{\mathbf{a}}(\omega) = 0$  and the values of coefficients  $A_{\mathbf{a},k}$  to the derivatives of  $\mathbf{g}_{\mathbf{a}}$  in these points.

717 **PROPOSITION 5.1.** *The function  $\mathbf{g}_{\mathbf{a}}$  is meromorphic in  $\mathbb{C}$ . Moreover,*

- 718 (1) *if  $\omega_0$  is a pole of  $\mathbf{g}_{\mathbf{a}}$ , then  $\operatorname{Im} \omega_0 < 0$ ;*
- 719 (2)  *$|\mathbf{g}_{\mathbf{a}}(\omega)| = 1$  when  $\omega \in \mathbb{R}$ ;*
- 720 (3) *let  $\omega_0 \in \mathbb{R}$ . Then  $\mathbf{g}_{\mathbf{a}}(\omega_0) = 1$  if and only if  $\omega_0$  is a pole of  $\Lambda_{\mathbf{a}}(\omega)$ .*
- 721 (4) *the coefficient  $A_{\mathbf{a},\ell}$ , cf. (3.6), is given by  $A_{\mathbf{a},\ell} = 4i(\mathbf{g}'_{\mathbf{a}}(\Omega_{\mathbf{a},\ell}))^{-1}$ .*

722 *Proof.* The function  $\mathbf{g}_{\mathbf{a}}$  is meromorphic because  $\Lambda_{\mathbf{a}}$  is such, cf. Theorem 3.2.

723 *Proof of (2).* (2) follows from the property  $\omega^{-1}\Lambda_{\mathbf{a}}(\omega) \in \mathbb{R}$  for  $\omega \in \mathbb{R}$ , cf. (3.6).

724 *Proof of (3).* We prove  $\implies$ , while the other implication is immediate. Let  
 725  $\mathbf{g}_{\mathbf{a}}(\omega_0) = 1$ . Assume that  $\omega_0$  is not a pole of  $\omega^{-1}\Lambda_{\mathbf{a}}(\omega)$ . We have  $\omega_0^{-1}\Lambda_{\mathbf{a}}(\omega_0) \in \mathbb{R}$ ;  
 726 set  $z = -\omega_0^{-1}\Lambda_{\mathbf{a}}(\omega_0) + i$ . Then  $\operatorname{Im} z = 1$ , but  $\mathbf{g}_{\mathbf{a}}(\omega_0) = z^*z^{-1} = 1$ , which implies that  
 727  $\operatorname{Im} z = 0$ , thus a contradiction.

728 *Proof of (1).* By (2),  $\mathbf{g}_{\mathbf{a}}(\omega)$  has no poles on  $\mathbb{R}$ , it thus remains to show that it  
 729 has no poles in  $\mathbb{C}^+$ . Because, by Corollary 3.4,  $\omega \mapsto \omega^{-1}\Lambda_{\mathbf{a}}(\omega)$  is analytic in  $\mathbb{C}^+$ ,  $\mathbf{g}_{\mathbf{a}}$   
 730 may have a poles in  $\omega_0 \in \mathbb{C}^+$  if and only if  $\omega_0^{-1}\Lambda_{\mathbf{a}}(\omega_0) = i$ . This is impossible by  
 731 Theorem 5.9 in [9]: for all  $\omega \in \mathbb{C}^+$ , we have  $\operatorname{Im}(\omega^{-1}\Lambda_{\mathbf{a}}(\omega)) < 0$ .

732 *Proof of (4).* In the vicinity of the pole  $\Omega_{\mathbf{a},\ell}$  it holds that

$$733 \quad \omega^{-1}\mathbf{\Lambda}_{\mathbf{a}}(\omega) = -\frac{A_{\mathbf{a},\ell}\omega}{\Omega_{\mathbf{a},\ell}^2 - \omega^2} + O(1), \text{ and}$$

$$734 \quad (\omega^{-1}\mathbf{\Lambda}_{\mathbf{a}}(\omega))' = -\frac{2A_{\mathbf{a},\ell}\Omega_{\mathbf{a},\ell}^2}{(\omega^2 - \Omega_{\mathbf{a},\ell}^2)^2} + O((\omega^2 - \Omega_{\mathbf{a},\ell}^2)^{-1}).$$

735

Inserting the above into

$$\mathbf{g}'_{\mathbf{a}}(\omega) = -\frac{2i(\omega^{-1}\mathbf{\Lambda}_{\mathbf{a}}(\omega))'}{(-\omega^{-1}\mathbf{\Lambda}_{\mathbf{a}}(\omega) + i)^2},$$

736 we deduce that  $\lim_{\omega \rightarrow \Omega_{\mathbf{a},\ell}} \mathbf{g}'_{\mathbf{a}}(\omega) = 4iA_{\mathbf{a},\ell}^{-1}$ .  $\square$

737 One draws two important conclusions from Proposition 5.1:

- 738 • to find the poles of  $\mathbf{\Lambda}_{\mathbf{a}}$  it suffices to find  $\omega \in \mathbb{R}_*$  s.t.  $\mathbf{g}_{\mathbf{a}}(\omega) = 1$ .
- 739 • to compute  $A_{\mathbf{a},\ell}$  it suffices to compute  $\mathbf{g}'_{\mathbf{a}}(\Omega_{\mathbf{a},\ell})$ .

740 All of the above requires a method for evaluation of  $\mathbf{g}_{\mathbf{a}}$  on the real axis. For computing  
741  $\mathbf{g}_{\mathbf{a}}$  we will use the same ideas as in [6] for evaluating  $\mathbf{\Lambda}_{\mathbf{a}}$  in  $\mathbb{C}^+$ . We start by writing  
742 a non-linear equation satisfied by  $\mathbf{g}_{\mathbf{a}}$ .

743 **5.1.2.  $\mathbf{g}_{\mathbf{a}}$  as a solution of a non-linear equation.** An equation for  $\mathbf{g}_{\mathbf{a}}$  can be  
744 obtained from the equation for  $\mathbf{\Lambda}_{\mathbf{a}}$  in [9].

745 LEMMA 5.2 (Lemmas 5.3, 5.5 in [9]). *The symbol of the reference DtN operator*  
746  $\mathbf{\Lambda} = \mathbf{\Lambda}_{\mathbf{a}}$ ,  $\mathbf{a} \in \{\mathbf{n}, \mathbf{d}\}$ , *is a unique even solution of the problem: find  $\mathbf{\Lambda} : \mathbb{C} \rightarrow \mathbb{C}$ , s.t.*

$$747 \quad (5.2) \quad \mathbf{\Lambda}(\omega) = -\omega \frac{\tan \omega - \mathbf{F}_{\alpha,\mu}(\omega)}{\tan \omega \mathbf{F}_{\alpha,\mu}(\omega) + \omega}, \quad \mathbf{F}_{\alpha,\mu}(\omega) = \sum_{i=0}^{p-1} \frac{\mu_i}{\alpha_i} \mathbf{\Lambda}(\alpha_i \omega),$$

748

749 *that is analytic in the origin, and whose value  $\mathbf{\Lambda}(0)$  is specified by Proposition 3.3.*

750 The above equation allows to obtain an equation similar to (5.2) satisfied by  $\mathbf{g}_{\mathbf{a}}$ . For  
751 this we 1) re-express  $\mathbf{\Lambda}_{\mathbf{a}}$  via  $\mathbf{g}_{\mathbf{a}}$  from (5.1); 2) replace  $\mathbf{\Lambda}_{\mathbf{a}}(\alpha_i \omega)$  in the right-hand side  
752 of (5.2) by the obtained expression; 3) substitute the obtained expression for  $\mathbf{\Lambda}_{\mathbf{a}}(\omega)$   
753 into the right hand side of (5.1). This procedure yields

$$754 \quad (5.3) \quad \mathbf{g}_{\mathbf{a}}(\omega) = -e^{2i\omega} \frac{1 - \mathbf{g}_{\alpha,\mu}(\omega)}{1 + \mathbf{g}_{\alpha,\mu}(\omega)}, \quad \mathbf{g}_{\alpha,\mu}(\omega) = \sum_{j=0}^{p-1} \mu_j \frac{1 + \mathbf{g}_{\mathbf{a}}(\alpha_j \omega)}{1 - \mathbf{g}_{\mathbf{a}}(\alpha_j \omega)}.$$

755

756 Using the connection between (5.2) and (5.3) it is easy to obtain the following result.

757 LEMMA 5.3. *The function  $\mathbf{g}_{\mathbf{a}}$ ,  $\mathbf{a} \in \{\mathbf{d}, \mathbf{n}\}$  is a unique even meromorphic solution*  
758 *of the equation (5.3) that is analytic in the origin, satisfies  $\mathbf{g}_{\mathbf{a}}(\omega)\mathbf{g}_{\mathbf{a}}(-\omega) = 1$  for all*  
759  *$\omega \in \mathbb{C}$ , as well as the following condition in the origin:*

- 760 • when  $\langle \mu/\alpha \rangle \leq 1$ ,  $\mathbf{g}_{\mathbf{d}}(0) = \mathbf{g}_{\mathbf{n}}(0) = -1$ .
- 761 • when  $\langle \mu\alpha \rangle < 1 < \langle \mu/\alpha \rangle$ ,  $\mathbf{g}_{\mathbf{n}}(0) = -1$  and  $\mathbf{g}_{\mathbf{d}}(0) = 1$ ;
- 762 • when  $\langle \mu\alpha \rangle \geq 1$ ,  $\mathbf{g}_{\mathbf{d}}(0) = \mathbf{g}_{\mathbf{n}}(0) = 1$ .

763 *Proof.* The proof is left to the reader. Remark that the condition  $\mathbf{g}_{\mathbf{a}}(\omega)\mathbf{g}_{\mathbf{a}}(-\omega) =$   
764  $1$  corresponds to the evenness of  $\mathbf{\Lambda}_{\mathbf{a}}$ .  $\square$

765 **REMARK 5.4.** *Because we are interested in calculating  $\mathbf{g}_a$  on the real axis based*  
766 *on (5.3), it is important to check whether (5.3) is well-defined for all  $\omega \in \mathbb{R}$ .*  
767 *First, let us remark that if  $\lim_{\omega \rightarrow \omega_0} \mathbf{g}_{\alpha, \mu}(\omega) = \infty$ ,  $\mathbf{g}_a(\omega_0)$  is well-defined:  $\mathbf{g}_a(\omega_0) = e^{2i\omega_0}$ .*  
768 *It is possible to prove that  $\lim_{\omega \rightarrow \omega_0} \mathbf{g}_{\alpha, \mu}(\omega) = \infty$  if and only if, for some  $\ell$ ,  $\mathbf{g}_a(\alpha_\ell \omega_0) = 1$ .*  
769 *Second, we remark that the denominator of (5.3) cannot vanish: otherwise this would*  
770 *have implied that in such points  $\omega$ ,  $\mathbf{g}_a(\omega) = \infty$  which contradicts Proposition 5.1 (1)*  
771 *and the uniqueness Lemma 5.3. We thus rewrite (5.3) for  $\omega \in \mathbb{R}$  as follows:*

$$772 \quad (5.4) \quad \mathbf{g}_a(\omega) = e^{2i\omega} \begin{cases} \frac{\mathbf{g}_{\alpha, \mu}(\omega) - 1}{\mathbf{g}_{\alpha, \mu}(\omega) + 1}, & \text{if } \mathbf{g}_a(\alpha_j \omega) \neq 1, \quad \forall j, \\ 1, & \text{otherwise.} \end{cases}$$

774 **5.1.3. A method for calculating  $\mathbf{g}_a$  in a point  $\omega \in \mathbb{R}$ .** Let us discuss how  
775 to compute  $\mathbf{g}_a(\omega)$  in a point  $\omega \in \mathbb{R}$ . We consider two cases:  $|\omega| < r$  and  $|\omega| \geq r$ , for  
776 a fixed small enough  $r$ .

777 **Description of the method.**

778 *Case  $|\omega| < r$ .* By Proposition 5.1,  $\mathbf{g}_a$  is analytic in the vicinity of the origin. Thus  
779 it can be approximated using the truncated Taylor expansion:

$$780 \quad (5.5) \quad \mathbf{g}_a(\omega) \approx \mathbf{g}_a^N(\omega) := \sum_{n=0}^N \omega^n g_n^a, \quad \text{for a fixed } N > 0,$$

782 where  $g_n^a$ ,  $n \in \mathbb{N}$  are the Taylor coefficients of  $\mathbf{g}_a$  in  $\omega = 0$ . They can be found  
783 by power matching from (5.1) and the known recursive expressions for the Taylor  
784 coefficients of  $\Lambda_a$  in the origin given in [9].

785 *Case  $|\omega| \geq r$ .* The expression (5.4) shows that, provided  $\omega \in \mathbb{R}$ , knowing  $\mathbf{g}_a(z)$   
786 for  $|z| < |\alpha|_\infty |\omega|$  (recall that  $|\alpha|_\infty < 1$ ) allows to compute  $\mathbf{g}_a(\omega)$ . In this sense, the  
787 equation (5.4) resembles (5.2). Hence for computing  $\mathbf{g}_a(\omega)$ , we can employ the same  
788 method as for computing  $\Lambda_a(\omega)$  in [6]. We will not present it here, as it is lengthy  
789 and its application to evaluating  $\mathbf{g}_a$  is straightforward. It is based on the two ideas:

- 791 1. To compute  $\mathbf{g}_a(\omega)$ , by (5.4), it suffices to compute  $\mathbf{g}_a(\alpha_j \omega)$ , for  $j = 0, \dots, p-1$ ,  
792 and next use (5.4). Remark that  $|\alpha_j \omega| \leq |\alpha|_\infty |\omega| < |\omega|$ . The same reasoning  
793 can be applied to each of  $\mathbf{g}_a(\alpha_j \omega)$ ,  $j = 0, \dots, p-1$ .  
794 Further application of this idea allows to reduce the question of evaluation of  
795  $\mathbf{g}_a(\omega)$  to the question of computing

$$796 \quad \mathbf{g}_a(\alpha_{i_1} \cdots \alpha_{i_L} \omega), \quad i_1, \dots, i_L \in \{0, \dots, p-1\},$$

798 where  $L$  is such that  $|\alpha|_\infty^L |\omega| < r$ . With such  $L$ ,  $|\alpha_{i_1} \cdots \alpha_{i_L} \omega| \leq |\alpha|_\infty^L |\omega| < r$ .

- 799 2. the values  $\mathbf{g}_a(\alpha_{i_1} \cdots \alpha_{i_L} \omega)$  are then evaluated using (5.5):

$$800 \quad (5.6) \quad \mathbf{g}_a(\alpha_{i_1} \cdots \alpha_{i_L} \omega) \approx \mathbf{g}_a^N(\alpha_{i_1} \cdots \alpha_{i_L} \omega).$$

802 **Preservation of the property  $|\mathbf{g}_a(\omega)| = 1$ .** As seen from the above, the method  
803 of [6] is based on a repeated application of (5.4); let us prove that it preserves the  
804 property  $|\mathbf{g}_a(\omega)| = 1$ .

805 **PROPOSITION 5.5.** *Let  $\omega \in \mathbb{R}_*$  be fixed. Let  $(g_j)_{j=0}^{p-1} \in \mathbb{C}^p$  be s.t.  $|g_j| = 1$  for all*  
806  *$0 \leq j \leq p-1$ . Then  $g \in \mathbb{C}$  given by*

$$807 \quad (5.7) \quad g = e^{2i\omega} \begin{cases} \frac{\mathbf{g}_{\alpha, \mu}(\omega) - 1}{\mathbf{g}_{\alpha, \mu}(\omega) + 1}, & \text{if } \forall j, g_j \neq 1, \\ 1, & \text{otherwise,} \end{cases} \quad g_{\alpha, \mu} = \sum_{j=0}^{p-1} \mu_j \frac{1 + g_j}{1 - g_j},$$

809 *satisfies  $|g| = 1$ .*

810 *Proof.* The result being obvious if for some  $j$   $g_j = 1$ , let us prove it in the opposite  
811 case. A simple computation yields

$$812 \quad |g|^2 = \frac{(1 - \operatorname{Re} g_{\alpha,\mu})^2 + (\operatorname{Im} g_{\alpha,\mu})^2}{(1 + \operatorname{Re} g_{\alpha,\mu})^2 + (\operatorname{Im} g_{\alpha,\mu})^2}.$$

814 It remains to show that  $\operatorname{Re} g_{\alpha,\mu} = 0$ . This follows by a direct computation:

$$815 \quad \operatorname{Re} g_{\alpha,\mu} = \sum_{j=0}^{p-1} \mu^j \frac{1 - |g_j|^2}{|1 - g_j|^2} = 0, \text{ since } |g_j| = 1 \text{ for all } j. \quad \square$$

817 Assume that for all  $i_1, \dots, i_L$ , it holds that  $|\mathbf{g}_{\mathbf{a}}^N(\alpha_{i_1} \cdots \alpha_{i_L} \omega)| = 1$  (cf. (5.6)). Since  
818 the approximation of  $\mathbf{g}_{\mathbf{a}}$ , namely  $\tilde{\mathbf{g}}_{\mathbf{a}}$ , is computed by a repeated application of (5.4),  
819 according to the above lemma, it holds that  $|\tilde{\mathbf{g}}_{\mathbf{a}}(\omega)| = 1$ .

820 **REMARK 5.6.** *For the moment we have only numerical evidence of convergence*  
821 *of the method, as well as of its stability when  $|\mathbf{g}_{\mathbf{a}}^N(\alpha_{i_1} \cdots \alpha_{i_L} \omega)| = 1 \pm \varepsilon$ , for  $\varepsilon$  small.*

822 **Complexity.** From the results of [6], it follows that for a fixed  $r > 0$ ,  $N \in \mathbb{N}$ , the  
823 asymptotic complexity (as  $|\omega| \rightarrow +\infty$ ) of the method scales as  $O(\log^{p+1} |\omega|)$ .

824 **5.1.4. Computing poles and residues of  $\Lambda_{\mathbf{a}}$ .** Computation of the poles of  
825  $\Lambda_{\mathbf{a}}$  and the coefficients  $A_{\mathbf{a},\ell}$  is based on the results of Proposition 5.1. Let us show  
826 how to compute the poles of  $\Lambda_{\mathbf{a}}$  on the interval  $(0, L)$ . First we subdivide  $(0, L)$   
827 into small intervals and interpolate  $\mathbf{g}_{\mathbf{a}}$  on each of these intervals using the Chebyshev  
828 interpolation. The resulting piecewise-Chebyshev interpolant is denoted by  $\mathbf{g}_{\mathbf{a},c}$ .

829 Next, we proceed according to Proposition 5.1. The Chebyshev interpolants do  
830 not preserve the property  $|\mathbf{g}_{\mathbf{a}}(\omega)| = 1$ , and therefore, instead of finding the points  
831 where  $|\mathbf{g}_{\mathbf{a},c}(\omega)| = 1$ , we

- 832 1) compute zeros  $z_k$  of the polynomial interpolant  $\operatorname{Im} \mathbf{g}_{\mathbf{a},c}(\omega)$  (by Proposition  
833 5.1 (2), (3), if  $\operatorname{Im} \mathbf{g}_{\mathbf{a}}(\omega) = 0$ , then  $\operatorname{Re} \mathbf{g}_{\mathbf{a}}(\omega) = \pm 1$ );
- 834 2) check whether  $\operatorname{Re} \mathbf{g}_{\mathbf{a},c}(z_k) > c > 0$ . If this is the case, we consider that  $z_k$  is  
835 an approximation to the pole of  $\Lambda_{\mathbf{a}}$ .

836 Then the evaluation of the coefficients  $A_{\mathbf{a},\ell}$  is done by computing the derivatives of  
837  $\mathbf{g}_{\mathbf{a},c}$ , cf. Proposition 5.1(4).

838 **REMARK 5.7.** *The reason why we subdivide the original interval  $(0, L)$  into mul-*  
839 *tiple sub-intervals and interpolate  $\mathbf{g}_{\mathbf{a}}$  on the sub-intervals is the following: despite the*  
840 *fact that  $\mathbf{g}_{\mathbf{a}}$  is smooth, it may oscillate rapidly (depending on the values  $\alpha, \mu$ ), and*  
841 *hence require a high degree polynomial interpolant. This is illustrated in Figure 4.*

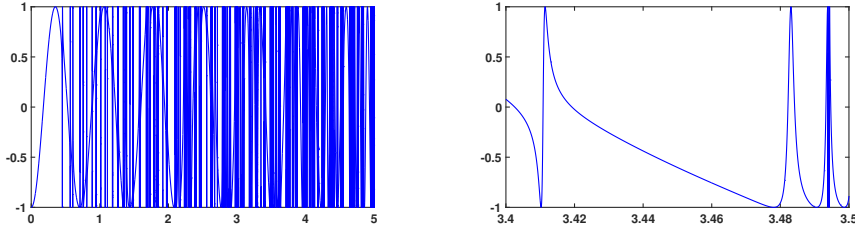


FIGURE 4. *Left: dependence of  $\operatorname{Re} \mathbf{g}_{\mathbf{n}}(\omega)$  on  $\omega$ ;  $\alpha = (0.8, 0.75)$  and  $\mu = (0.5, 0.5)$ . Right: the close-up for the interval  $(3.4, 3.5)$ . In this case  $\Lambda_{\mathbf{n}}$  has about 1100 poles on  $(0, 5)$ .*



842 The implementation of this algorithm was done using the Chebfun [3, 19], which  
 843 allows to construct a highly accurate approximation of  $\mathbf{g}_{\mathbf{a},c}$  and contains an automated  
 844 procedure for choosing the degree of the interpolant, see [2] and the monograph [19].  
 845 If the Chebfun fails to construct an accurate interpolant, we further subdivide the  
 846 interpolation interval.

847 **REMARK 5.8.** *As remarked above, the Chebyshev interpolants do not preserve the*  
 848 *property  $|\mathbf{g}_{\mathbf{a}}(\omega)| = 1$ , but this does not seem to pose significant problems in practice.*

849 **5.2. Discretization.** In this section we discuss the discretization of (3.17), start-  
 850 ing with the semi-discretization in space, and then show a discretization in time. Next,  
 851 we discuss its stability and convergence. All over this section we fix  $m$  and  $N$ . In the  
 852 definition of the discretized quantities, we will omit the indices  $N, m, \mathbf{a}$ .

853 **5.2.1. Semi-discretization in space.** Let  $U_h \subset V_n(\mathcal{T}^m)$  be an extension of  
 854 the Lagrange  $\mathbb{P}_1$  space to the case of fractal trees, defined like in [6]. By  $\mathbf{u}(t) \in$   
 855  $\mathbb{R}^{N_s}$  we denote the respective vector of the degrees of freedom (nodal values) that  
 856 approximates  $u_m^{N,\mathbf{a}}(t)$ , and by  $\boldsymbol{\lambda}_{i,k}^h(t) \in \mathbb{R}^{p^m}$  an approximation of  $\boldsymbol{\lambda}_{n,k}^{\mathbf{a}}(t) \in \mathbb{R}^{p^m}$ .  
 857 The mass and stiffness matrices are denoted by  $\mathbf{M}$  and  $\mathbf{K}$  (remark that they are  
 858 constructed with respect to the weighted  $L_{\mu}^2(\mathcal{T})$  product). Let also the matrix  $\mathbf{P}$  be  
 859 defined as  $\mathbf{P}_{j\ell} = \varphi_{\ell}(M_{m,j})$ .

860 *Formulation.* The discretization in space of (3.17) in the algebraic form reads:  
 861 find  $\mathbf{u} \in C^1([0, \infty); \mathbb{R}^{N_s})$ , s.t.  $\mathbf{u}(0) = \partial_t \mathbf{u}(0) = 0$  and

$$\begin{aligned} & \mathbf{M} \partial_t^2 \mathbf{u} + \mathbf{K} \mathbf{u} + \mathbf{P}^T \mathbf{W}_m \mathbf{D}_m \langle \boldsymbol{\mu} / \boldsymbol{\alpha} \rangle \boldsymbol{\Lambda}_{\mathbf{a}}(0) \mathbf{P} \mathbf{u} \\ & + \mathbf{P}^T \mathbf{W}_m \mathbf{D}_m \sum_{k=0}^{p-1} \sum_{i=0}^{N-1} A_{\mathbf{a},i} \frac{\mu_k}{\alpha_k} \partial_t (\boldsymbol{\lambda}_{i,k}^h) = \mathbf{M} \mathbf{f}^n, \\ & \partial_t^2 \boldsymbol{\lambda}_{i,k}^h + \alpha_k^{-1} \Omega_{\mathbf{a},i}^2 \mathbf{D}_m^2 \boldsymbol{\lambda}_{i,k}^h = \mathbf{P} \mathbf{u}, \quad \boldsymbol{\lambda}_{i,k}^h(0) = \partial_t \boldsymbol{\lambda}_{i,k}^h(0) = 0. \end{aligned} \tag{5.8}$$

862 **5.2.2. Discretization in Time.** Let us describe how we discretize in time the  
 863 approximate problem (5.8). To obtain a stable discretization, the main idea is to use  
 864 the explicit **leapfrog** discretization for the volumic terms, and the implicit **trape-**  
 865 **zoid rule** discretization of the boundary terms. First, however, we introduce some  
 866 notation. Provided a time step  $\Delta t$ , let  $v^n$  be an approximation to  $v(\cdot, n\Delta t)$ . Let  
 867 **zoid rule** discretization of the boundary terms. First, however, we introduce some  
 868 notation. Provided a time step  $\Delta t$ , let  $v^n$  be an approximation to  $v(\cdot, n\Delta t)$ . Let

$$\begin{aligned} & D_{\Delta t} v^n = \frac{v^{n+1} - v^{n-1}}{2\Delta t}, \quad D_{\Delta t}^2 v^n = \frac{v^{n+1} - 2v^n + v^{n-1}}{(\Delta t)^2}, \quad D_{\Delta t} v^{n+\frac{1}{2}} = \frac{v^{n+1} - v^n}{\Delta t}, \\ & \{v^n\}_{1/4} = \frac{v^{n+1} + 2v^n + v^{n-1}}{4}, \quad v^{n+1/2} = \frac{v^n + v^{n+1}}{2}. \end{aligned} \tag{5.9}$$

872 *Formulation.* For simplicity we will assume that the source term  $f$  in (3.17)  
 873 satisfies  $f \in C^1([0, \infty); L_{\mu}^2(\mathcal{T}^m))$ ,  $f(0) = f'(0) = 0$ . The discretization of (3.17a)  
 874 reads: given  $\mathbf{u}^0 = 0$ ,  $\mathbf{u}^1 = 0 \in \mathbb{R}^{N_s}$ , find  $(\mathbf{u}^n)_{n \in \mathbb{N}} \subset \mathbb{R}^{N_s}$ , s.t.

$$\begin{aligned} & \mathbf{M} D_{\Delta t}^2 \mathbf{u}^n + \mathbf{K} \mathbf{u}_N^n + \mathbf{P}^T \langle \boldsymbol{\mu} / \boldsymbol{\alpha} \rangle \boldsymbol{\Lambda}_{\mathbf{a}}(0) \mathbf{W}_m \mathbf{D}_m \mathbf{P} \{ \mathbf{u}^n \}_{1/4} \\ & + \mathbf{P}^T \mathbf{W}_m \mathbf{D}_m \sum_{k=0}^{p-1} \sum_{i=1}^N A_{\mathbf{a},i} \frac{\mu_k}{\alpha_k} D_{\Delta t} (\boldsymbol{\lambda}_{i,k}^h)^n = \mathbf{M} \mathbf{f}^n, \end{aligned} \tag{5.9a}$$

$$D_{\Delta t}^2 (\boldsymbol{\lambda}_{i,k}^h)^n + \alpha_k^{-1} \Omega_{\mathbf{a},i}^2 \mathbf{D}_m^2 \left\{ (\boldsymbol{\lambda}_{i,k}^h)^n \right\}_{1/4} = D_{\Delta t} \mathbf{P} \mathbf{u}^n, \tag{5.9b}$$

$$(\boldsymbol{\lambda}_{i,k}^h)^0 = (\boldsymbol{\lambda}_{i,k}^h)^1 = 0. \tag{5.9c}$$

880 *Stability.* The formulation (5.9a-5.9c) is stable under the CFL condition

$$881 \quad (5.10) \quad C_{CFL} = \left(\frac{\Delta t}{2}\right)^2 \rho(\mathbf{M}^{-1/2} \mathbf{K} \mathbf{M}^{-1/2})^{-1} < 1,$$

883 where  $\rho(A)$  is the spectral radius of a matrix  $A$ . To see this, let us first introduce the  
884 notation:  $\langle \mathbf{v}, \mathbf{q} \rangle_A = \langle A\mathbf{v}, \mathbf{q} \rangle$ ,  $\|\mathbf{v}\|_A^2 = \langle \mathbf{v}, \mathbf{v} \rangle_A$ . Let us define

$$885 \quad \mathbb{E}^{n+1/2} = \frac{1}{2} \left( \left\| D_{\Delta t} \mathbf{u}^{n+1/2} \right\|_{\mathbf{M}}^2 - \left(\frac{\Delta t}{2}\right)^2 \left\| D_{\Delta t} \mathbf{u}^{n+1/2} \right\|_{\mathbf{K}}^2 \right) + \frac{1}{2} \left\| \mathbf{u}^{n+1/2} \right\|_{\mathbf{K}}^2$$

$$886 \quad + \frac{1}{2} \langle \boldsymbol{\mu} / \boldsymbol{\alpha} \rangle \Lambda_{\mathbf{a}}(0) \left\| \mathbf{P} \mathbf{u}^{n+1/2} \right\|_{\mathbf{W}_m \mathbf{D}_m}^2 + \sum_{i=1}^N A_{\mathbf{a},i} \sum_{k=0}^{p-1} \frac{\mu_k}{\alpha_k} \mathbb{E}_{i,k}^{n+1/2},$$

$$887 \quad \mathbb{E}_{i,k}^{n+1/2} = \frac{1}{2} \left\| D_{\Delta t} (\boldsymbol{\lambda}_{i,k}^h)^{n+1/2} \right\|_{\mathbf{W}_m \mathbf{D}_m}^2 + \frac{\alpha_k^{-1} \Omega_{\mathbf{a},i}^2}{2} \left\| \mathbf{D}_m (\boldsymbol{\lambda}_{i,k}^h)^{n+1/2} \right\|_{\mathbf{W}_m \mathbf{D}_m}^2.$$

889 The condition (5.10) ensures that  $\mathbb{E}^{n+1/2} \geq 0$ .

890 **THEOREM 5.9** (Stability of (5.9)). *Let  $(\mathbf{u}^n)_{n \in \mathbb{N}}$  solve (5.9), and let (5.10) hold*  
891 *true. Then, with  $C > 0$  depending on  $\boldsymbol{\alpha}, \boldsymbol{\mu}$  and the problem (Dirichlet or Neumann),*

$$892 \quad \sqrt{\mathbb{E}^{n+1/2}} \leq C \Delta t \sum_{k=0}^n \|\mathbf{f}^k\|_{\mathbf{M}}, \quad n \in \mathbb{N}.$$

894 *Proof.* The result is obtained by testing the equation (5.9a) with  $D_{\Delta t} \mathbf{u}^n$ . The  
895 only 'non-classical' terms are related to  $\boldsymbol{\lambda}$ , and can be handled using (5.9b):

$$896 \quad \left( \mathbf{W}_m \mathbf{D}_m D_{\Delta t} (\boldsymbol{\lambda}_{i,k}^h)^n, D_{\Delta t} \mathbf{P} \mathbf{u}^n \right) = \left( \mathbf{W}_m \mathbf{D}_m D_{\Delta t} (\boldsymbol{\lambda}_{i,k}^h)^n, D_{\Delta t}^2 (\boldsymbol{\lambda}_{i,k}^h)^n \right)$$

$$897 \quad + \left( \mathbf{W}_m \mathbf{D}_m D_{\Delta t} (\boldsymbol{\lambda}_{i,k}^h)^n, \alpha_k^{-1} \Omega_{\mathbf{a},i}^2 \mathbf{D}_m^2 \left\{ (\boldsymbol{\lambda}_{i,k}^h)^n \right\}_{1/4} \right).$$

899 The above yields :

$$900 \quad \mathbb{E}^{n+1/2} - \mathbb{E}^{n-1/2} = \Delta t \left\langle \mathbf{f}^n, \frac{\mathbf{u}^{n+1} - \mathbf{u}^{n-1}}{\Delta t} \right\rangle_{\mathbf{M}},$$

902 which can be bounded using a discrete Gronwall inequality, see [8, Appendix E].

903 **REMARK 5.10.** *As discussed in [6], the CFL condition (5.10) coincides with the*  
904 *CFL condition for a  $\mathbb{P}_1$ -discretization of a non-weighted wave equation (since the*  
905 *weights are piecewise-constant, and  $u_m^{\mathbf{a},N}$  satisfies (2.2) on each branch). Moreover,*  
906 *in our case the CFL condition is not affected by the DtN approximation, because the*  
907 *related terms are discretized with the implicit trapezoid rule.*

908 **5.2.3. Remarks on convergence.** Like for the CQ discretization in [6], it is  
909 not difficult to demonstrate that (5.9) is of second order in time and first order in  
910 space, when measuring the error in the energy norm, with the constants depending  
911 on the computational time  $T$  polynomially and on some  $W^{\ell,1}(0, T; L_{\mu}^2(\mathcal{T}^m))$ -norm of  
912  $f$ . The convergence estimates can be shown to be independent of  $N$ . As the proof is  
913 classical, we will not state the respective result here.

914 **5.3. Numerical Results.** All the experiments of this section are performed on  
915 the reference tree. Moreover, we use the scheme (5.9a-5.9c) with the mass-lumped  
916 finite elements (and all the norms are computed using mass-lumped matrices).

917 **REMARK 5.11.** *All over this section, we will omit the indices  $m$  and  $\mathbf{a}$ ,*  
918 *in order to make the notations lighter.*

919 **5.3.1. Validity of the method.** To validate the correctness of the approach,  
 920 we compare it to a highly accurate convolution quadrature approximation of the  
 921 transparent boundary conditions, cf. [6]. In particular, we truncate the tree to 3  
 922 generations, and compute the solution on the tree  $\mathcal{T}^m$ ,  $m = 2$ , with the help of the  
 923  $N$ -term transparent boundary conditions. The reference solution  $u_{ref}$  is computed  
 924 on the truncated tree  $\mathcal{T}^{m+1}$  (i.e. the tree with 4 generations), with the help of the  
 925 convolution quadrature method, with the same discretization parameters. In what  
 926 follows we will denote by  $u_N^n$  (resp.  $u_{ref}^n$ ) the solution to (5.9) at the time step  $n$ .

927 We solve the Dirichlet problem for  $\alpha = (0.3, 0.6)$ ,  $\mu = (0.5, 1)$ . As a source we  
 928 take the function supported on the root edge of the tree

930 (5.11) 
$$f(s, t) = 10^5 \exp(-\sigma(s - 0.5)^2 - \sigma(t - 0.5)^2)(s - 0.5), \quad \sigma = 10^3.$$

In all the cases we choose the discretization with  $h = 10^{-4}$  and  $\Delta t \approx 9.9 \cdot 10^{-5}$ . The above function is approximately band-limited in time with the maximal frequency in its Fourier transform being  $\omega_{max} \approx 107$  (we cut-off at  $10^{-5}$ -accuracy). This implies that the maximal frequency present in the Fourier transform of (2.12) is roughly  $\omega_{max} |\alpha|_\infty^m \approx 0.6^3 \cdot 107 \approx 23$ . Thus,  $N$  should be chosen large enough to ensure that all the poles inside the interval  $(0, 23)$  are included into the approximation (3.15), i.e.  $N \geq 27$ . A more precise error control is achieved by computing the value  $r_N \equiv r_{\partial, N}$  as described in Section 4.1. In particular,

$$r_{100} \approx 9 \cdot 10^{-3}, \quad r_{250} \approx 4.1 \cdot 10^{-3}, \quad r_{500} \approx 2.2 \cdot 10^{-3}, \quad r_{950} \approx 1.2 \cdot 10^{-3}.$$

931 We choose the above values of  $N$ , and plot the dependence of the error

932 (5.12) 
$$e_N^n = \frac{\|u_N^n - u_{ref}^n\|_{L_\mu^2(\mathcal{T}^{m-1})}}{\max_\ell \|u_{ref}^\ell\|_{L_\mu^2(\mathcal{T}^{m-1})}},$$

933

on time  $n\Delta t$  in Figure 5. The dependence of the solution  $u_N$  on time evaluated in

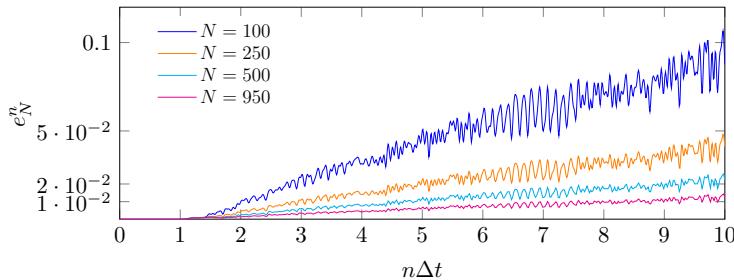


FIGURE 5. Dependence of the error  $e_N^n$  defined in (5.12) on time  $n\Delta t$  for different values of the truncation parameter  $N$ .

934

935 one point of the tree is shown in Figure 6.

936 **5.3.2. Convergence rates.** In this section we study the convergence rates of  
 937 the method, according to the results of Theorems 4.3, 3.9.

938 To verify the result of Theorem 4.3, we conduct four numerical experiments, which  
 939 cover all three cases of Theorem 4.3. We compute the solution  $u_N$  to the Neumann  
 940 problem on a truncated tree  $\mathcal{T}^m$ , with  $m = 2$ , with the help of the approximated

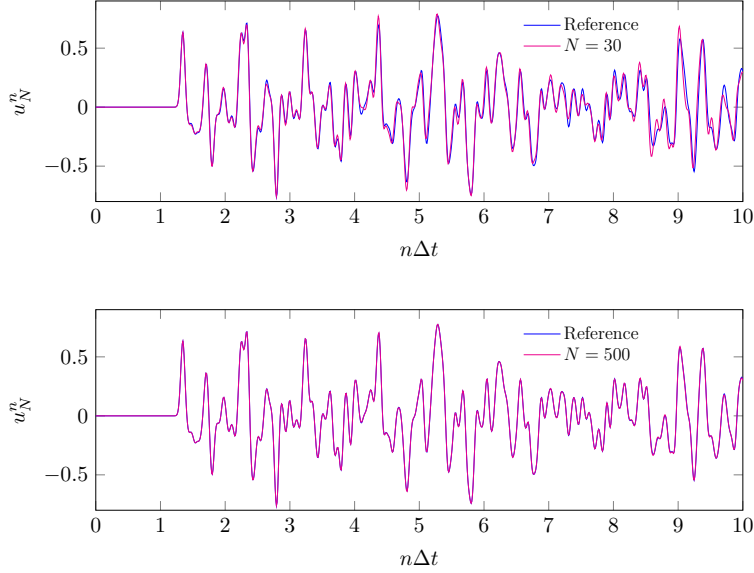


FIGURE 6. *Top: dependence of the solution  $u_N^n$  measured in the middle of the edge  $\Sigma_{2,0}$  of the tree on time  $n\Delta t$ . Top:  $N = 30$ , bottom:  $N = 500$*

941 transparent boundary conditions (3.15), for different values of  $N$ , and compare it to  
 942 the reference solution  $u_{ref}$  computed with the help of the convolution quadrature  
 943 method [6] on the truncated tree  $\mathcal{T}^m$ . In all the experiments we use the discretization  
 944 with the spatial step  $h = 10^{-4}$  and the time step  $\Delta t = 9.9 \cdot 10^{-5}$ . As a source  
 945 term we take (5.11) with  $\sigma = 10^2$  supported on the root branch of the tree. All the  
 946 computations are done on the time interval  $(0, T)$ , with  $T = 10$ , divided into  $N_t$  time  
 steps.

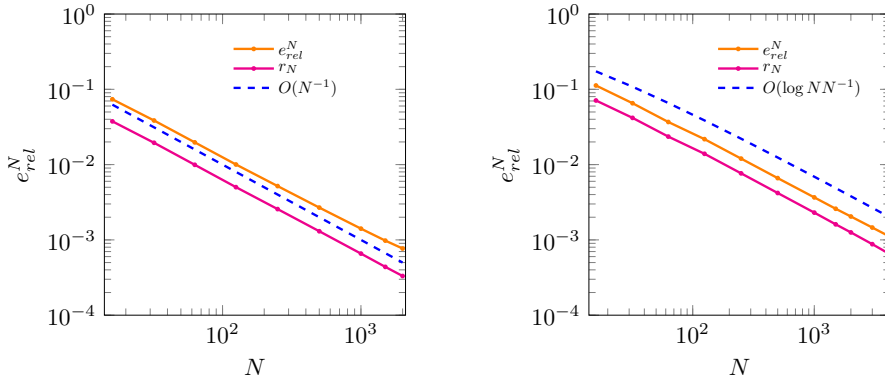


FIGURE 7. *Relative error (5.13) depending on  $N$ . Left:  $\alpha = (0.2, 0.5)$ ,  $\mu = (0.6, 0.1)$  ( $d_s < 1$ , with  $r_N \leq CN^{-1}$ ). Right:  $\alpha = (0.7, 0.3)$ ,  $\mu = (0.3, 0.6)$  ( $d_s = 1$ , with  $r_N \leq CN^{-1} \log N$ ).*

947

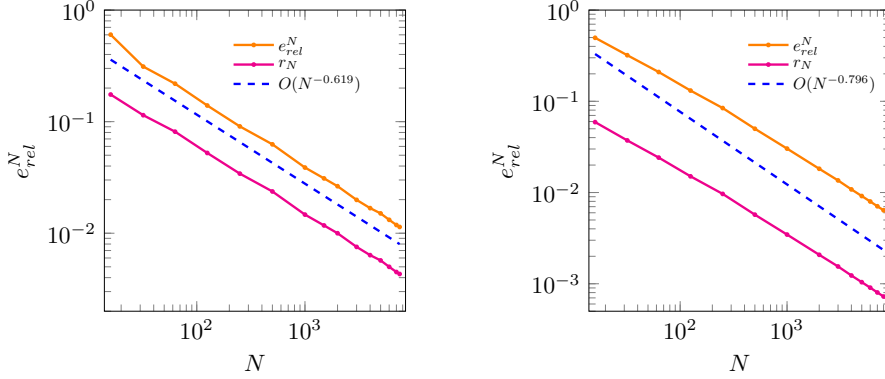


FIGURE 8. Relative error (5.13) depending on  $N$ . Left:  $\alpha = (0.7, 0.6)$ ,  $\mu = (0.3, 0.6)$  ( $d_s \approx 1.615$ ,  $r_N \leq CN^{-\frac{1}{d_s}}$ ). Right:  $\alpha = (0.5, 0.65)$ ,  $\mu = (2, 1)$  ( $d_s \approx 1.256$ ,  $r_N \leq CN^{-\frac{1}{d_s}}$ ). In this latter case the Dirichlet and Neumann problems coincide.

948 We measure the dependence of the following relative error on the order  $N$  of the  
949 transparent boundary conditions:

$$950 \quad (5.13) \quad e_{rel}^N := \frac{e_{abs}^N}{\max_{\ell=0, \dots, N_t} \|u_{ref}^\ell\|_{L_\mu^2(\mathcal{T}^{m-1})}}, \quad e_{abs}^N = \max_{\ell=0, \dots, N_t} \|u_N^\ell - u_{ref}^\ell\|_{L_\mu^2(\mathcal{T}^{m-1})}.$$

951

952 We compare the quantity  $e_{rel}^N$  to the quantity  $r_N \equiv r_{n,N}$ , computed numerically as  
953 described in Section 4.1, as well as a theoretical upper bound given in Theorem 4.3.  
954 The results are given in Figures 7, 8. In these figures we observe in particular that  
955 the numerically computed value  $r_N$  provides an excellent estimate for the convergence  
rates, as expected, and can be potentially used as an error estimator. As a complement

Value of $N$	Numerical convergence rate $d$			
$N_k$	$\alpha = (0.2, 0.5)$	$\alpha = (0.7, 0.3)$	$\alpha = (0.7, 0.6)$	$\alpha = (0.5, 0.65)$
16	-	-	-	-
32	0.94	1.15	0.95	0.63
63	0.99	1.15	0.52	0.63
125	0.99	0.99	0.66	0.68
250	0.95	1.1	0.62	0.64
500	0.95	1.0	0.53	0.75
1000	0.93	1.0	0.69	0.72
2000	0.88	0.98	0.56	0.73
4000	-	0.96	0.65	0.75
7000	-	-	0.62	0.76
<b>Theoretical <math>d</math></b>	1	1	0.62	0.796

TABLE 1

Numerically measured convergence rates in different experiments. In the particular case of  $\alpha = (0.7, 0.3)$  (where the convergence is  $O(N^{-1} \log N)$ ), the quantity provided in the above table is defined

$$as \ d = \frac{\log\left(\frac{e_{rel}^{N_{k+1}}}{e_{rel}^{N_k}}\right)}{\log\left(\frac{N_{k+1}^{-1} \log N_{k+1}}{N_k^{-1} \log N_k}\right)}.$$

956  
 957 to Figures 7, 8, we present the numerically estimated order of convergence associated  
 958 to different experiments in Table 1. We observe a rather good agreement with the  
 959 theoretical convergence estimates, especially in the cases  $\alpha = (0.7, 0.3)$  and  $\alpha =$   
 960  $(0.7, 0.6)$ . In the case  $\alpha = (0.5, 0.65)$ , where the numerically established convergence  
 961 rates are somewhat different from the theoretical one, it appears that the numerical  
 962 convergence rates are quite close to the ones measured from the values  $r_N$ . Most  
 963 likely the discrepancy between the theoretical and the numerical rate is related to the  
 964 fact that the asymptotic regime has not been reached for the range of  $N$  considered.  
 965 Finally, in the case  $\alpha = (0.2, 0.5)$ , we remark that the convergence order deteriorates  
 966 slightly. Because there exists a discrepancy between the convergence rates measured  
 967 from the numerical error and the ones measured for the numerically computed value  
 968  $r_N$  (where it is very close to 1), we think that either it is related to the accuracy  
 969 of computation of the poles and residues in the method, or the discretization error  
 970 becomes significant in this case.

971 **6. Conclusions and Open Questions.** In this work, we have constructed  
 972 transparent boundary conditions for the weighted wave equation on a self-similar  
 973 one-dimensional fractal tree. The approach presented here is alternative to the con-  
 974 volution quadrature [6] and is based on the truncation of the meromorphic series  
 975 representing the symbol of the DtN operator. The complexity of the method depends  
 976 on the number of poles in the truncated series; we have presented estimates on the  
 977 number of poles, required to achieve a desired accuracy  $\varepsilon$ . While the convergence  
 978 in term of the number of poles is rather slow, one of the advantages of this method  
 979 is that its cost does not increase with time (unlike the convolution quadrature ap-  
 980 proach). Our future efforts are directed towards improving the convergence of the  
 981 technique, based on approximation of the remainder of the meromorphic series.

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984

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