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Abstract We use a special duality by perturbation in optimization to find two different bi-duals problems of the non-linear Kirchhoff-Love-von Kármán plate theory. The first gives exactly the intrinsic approach developed by P. G. Ciarlet, the second gives an intrinsic approach implied by a J.J. Telega complementary energy.

Keywords Non-linear Plate theory · Intrinsic Approach · Duality by Perturbation in Optimization

Mathematics Subject Classification (2010) 75K20 · 49J40

1 Introduction

In elasticity theory a major task is the determination of the stress tensor field that arises in a given body in response of applied forces and/or imposed displacements on (a part of) the boundary. In this paper we consider the case of a plate of unit thickness with middle surface ω, a bounded domain in $\mathbb{R}^2$. More precisely we assume that the reference configuration of the plate $\overline{\omega} \times [-\frac{1}{2}, +\frac{1}{2}]$ is a natural state and that the elastic plate is made up with a linear, homogeneous and isotropic material with Lamè coefficients $\mu > 0$ and $\lambda > -\frac{2}{3}\mu$. In the classical approach of the Kirchoff-Love-von Kármán non-linear theory the stress field arising in the plate is achieved through the determination of the displacement field of the middle surface of the plate $\zeta = (\mathbf{u}, w) : \omega \to \mathbb{R}^3$ which is a stationary point of the potential energy functional $J : \zeta \in \mathbb{V} \to J(\zeta) \in \mathbb{R} \cup \{+\infty\}$ given by (27). One then defines the bending moments $\mathbf{M} := -\frac{1}{3}\mathbb{A}\mathbf{H}(w)$ and the stress resultants $\mathbf{N} := \mathbb{A}\mathbf{E}(\zeta)$ (for the notations used see (28), (30) and (29)). The determination of the stress tensor field at every point of the reference configuration of the plate follows as in [4].

Another approach is possible, the so-called *intrinsic approach* where the fields $\mathbf{E}$ and $\mathbf{H}$ are the unknown. In the case of shells and plates it goes back at least to J. L. Synge - W. Z. Chien [22] and has been considerably developed by W. Pietraskiewicz (see e.g. [19]). The mathematical analysis of the intrinsic approach for *linear 3d elasticity* where the unknown is the strain tensor field, has been undertaken by P.G. Ciarlet - P. Ciarlet Jr [5] and then by many other researchers, see e.g. [6], [7], ...

In the case of linearised 3d elasticity, see [8], one can put the three different minimization problems (depending on the choice of the unknown: vector field, stress tensor field, strain tensor field) in the

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perspective of the Legendre-Fenchel duality thanks to the choice of suitable Lagrangians. In Section 2 we extend this perspective to general non-linear problems. For this we use the methods of duality by perturbation developed by I. Ekeland - R. Temam [12] following some ideas of T. Rockafellar [21]; see also J.Toland [24] and G. Auchmuty [2]. More precisely let $(\mathcal{P})$ the problem of evaluating $\inf_{u\in V} J(u)$ where $J : V \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$J(u) = G(\Lambda(u)) -_{V^*} \langle F^*, u\rangle_V \tag{1}$$

When $\Lambda : V \to Y$ is *linear and continuous* the dual problem has been introduced in [12] where the connection with a *first Lagrangian* is also examined; for some further developments see also e.g. [24], [2]. Using this first Lagrangian one can introduce (see Definition 3) a *first bi-dual problem* $(\mathcal{P}^{**})$ which gives a regularization of the initial problem $(\mathcal{P})$. However as has been remarked in [8] one can also introduce a *second Lagrangian* and hence define a *second bi-dual problem* $(\mathcal{P}^{**}_{int})$. This one in the case of linearised elasticity gives the intrinsic formulation. In the present paper the problems $(\mathcal{P}^{**})$ and $(\mathcal{P}^{**}_{int})$ are introduced also when $\Lambda : V \to Y$ is non-linear and satisfies some conditions, see Section 2.2

In the case of the Kirchoff- Love - von Kármán theory of non-linearly elastic plates considered in Section 3 it is possible to make both choices for Λ. The non-linear one, see Section 4, allows to find an intrinsic formulation $(\mathcal{P}^{**}_{intC})$ which is exactly the intrinsic formulation of [9]. A choice of a linear Λ was proposed by J. J. Telega [23] in order to find a complementary energy problem. In Section 5 we remark that for this choice the first bi-dual problem gives a type of regularisation of the original problem and in Section 6 we give also the associated intrinsic formulation $(\mathcal{P}^{**}_{intT})$.

Notations Latin indices vary in the set $\{1, 2\}$ and the summation convention with respect to repeated indices is systematically used in conjunction with this rule. All vector spaces, matrices, etc. considered in this paper are real. Vectors, tensors, vector-valued and tensor valued fields defined over ω are denoted by boldface Roman capitals.

2 Duality by perturbation in optimization

1. We denote $\bar{\mathbb{R}}$ the extended set of real numbers: $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. The dual space of a real Banach space X is denoted X^*, and $_{X^*}\langle\cdot,\cdot\rangle_X$ designates the associated (separating) duality. When there is no ambiguity we only write $\langle\cdot,\cdot\rangle$. The bidual space of X is denoted X^{**}; if X is a reflexive Banach space, X^{**} will be identified with X by means of the usual canonical isometry. $\Gamma(X)$ denotes the set of all functions $g : X \to \bar{\mathbb{R}}$ that are convex, weakly lower semicontinuous (wlsc) and if they take the value $-\infty$ they must be identically $-\infty$. $\Gamma_0(X)$ denotes the subset of proper, convex, wlsc functions of $g \in \Gamma(X)$, i.e. of all functions other than the constants $+\infty$ and $-\infty$. The domain of $g \in \Gamma_0(X)$ is $dom(g) = \{x \in X; g(x) < +\infty\}$. The *indicator function* I_A of a subset A of a set X is the function I_A defined by $I_A(x) := 0$ if $x \in A$ and $I_A(x) := +\infty$ if $x \notin A$.

The *Legendre-Fenchel transform* of $g : X \to \bar{\mathbb{R}}$ is the function $g^* \in \Gamma(X^*)$ defined by:

$$g^* : e \in X^* \to g^*(e) := \sup_{\sigma\in dom(g)} \{\langle e, \sigma\rangle - g(\sigma)\}. \tag{2}$$

One can repeat the process obtaining in this way g^{**}. When X is reflexive the bipolar g^{**} is a function from X to $\bar{\mathbb{R}}$ also called the $\Gamma-$*regularization* of g, it is the largest minorant of g in $\Gamma(X)$. Let us also recall that if one denotes $\bar{g}$ the lsc regularization of g (i.e. the largest l.s.c.minorant of g) then $g^{**} \leq \bar{g} \leq g$. Some basic properties of convex functions and of the Legendre-Fenchel transform when the space X is a reflexive Banach space. are summarized in the following theorem (see, e.g., Ekeland & Temam [12], Dacorogna [11], Brezis [3], ...).

Theorem 1 *Let X be a reflexive Banach space, and let $g : X \to \bar{\mathbb{R}}$ be a proper, convex, and wlsc. Then the Legendre-Fenchel transform $g^* : X^* \to \bar{\mathbb{R}}$ of g is also proper, convex, and wlsc. Let*

$$g^{**} : \sigma \in X^{**} \to g^{**}(\sigma) := \sup_{e\in X^*} \{\langle e, \sigma\rangle - g^*(e)\}$$

denote the Legendre-Fenchel transform of g^*. Then (recall that X^{**} is here identified with X) $g^{**} = g$.

The equality $g^{**} = g$ constitutes the *Fenchel-Moreau theorem*; cf. Fenchel [13] and Moreau [17].

2. Let now V, Y be reflexive Banach spaces and $\Lambda : V \to Y$ be a continuous map such that:

(i) $\Lambda(0) = 0$;
(ii) Λ is one-to-one;
(iii) the range $R(\Lambda) = \Lambda(V)$ is weakly sequentially closed;
(iv) the inverse, denoted $\mathcal{F}$, maps weakly convergent sequences into weakly convergent sequences.

Let also be $F^* \in V^*$ and let be $G : Y \to \mathbb{R} \cup \{+\infty\}$ an arbitrary function with the conditions $G(0) = 0$ and $\overline{R(\Lambda)} \subset dom(G)$.

Definition 1 *We denote by* $(\mathcal{P})$ *the problem of evaluating* $\inf_{u\in V} J(u)$ *where* $J : V \to \mathbb{R} \cup \{+\infty\}$ *is defined by* (1) *and we call* $\hat{u} \in V$ *a solution of* $(\mathcal{P})$ *if*

$$J(\hat{u}) \in \mathbb{R} \quad \textit{and } J(\hat{u}) = \inf_{u\in V} J(u) = \inf_{u\in V} \{G(\Lambda(u)) -_{V^*} \langle F^*, u\rangle_V\} \tag{3}$$

Following the presentation by Ekeland-Temam [12] of the idea of perturbed problems (introduced by Rockafellar [21]) we associate to the primal problem $(\mathcal{P})$ the family of perturbations $\Phi : V \times Y \to \mathbb{R} \cup \{+\infty\}$ defined by $\Phi(u, p) := G(\Lambda(u) + p) -_{V^*} \langle F^*, u\rangle_V$; let us explicitly remark that $J(u) = \Phi(u, 0)$. The Legendre-Fenchel transform of $\Phi(u, p)$ in the duality between $V \times Y$ and $V^* \times Y^*$ is given by

$$\Phi^*(u^*, p^*) = \sup_{(u,p)\in V\times Y} \{_{V^*}\langle u^*, u\rangle_V +_{Y^*} \langle p^*, p\rangle_Y - \Phi(u, p)\} \tag{4}$$

As in [12] we introduce the following definition

Definition 2 *The dual problem* $(\mathcal{P}^*)$ *is the problem of evaluating* $\sup_{p^*\in Y^*}\{-\Phi^*(0, p^*)\}$ *and we call* $\hat{p}^* \in Y^*$ *a solution of* $(\mathcal{P}^*)$ *if*

$$-\Phi^*(0, \hat{p}^*)) \in \mathbb{R} \quad \textit{and } -\Phi^*(0, \hat{p}^*)) = \sup_{p^*\in Y^*} \{-\Phi^*(0, p^*)\} \tag{5}$$

From (4) it follows that $\Phi^*(0, p^*) \geq \sup_{u\in V}\{-\Phi(u, 0)\}$ and hence

$$\sup_{p^*\in Y^*} \{-\Phi^*(0, p^*)\} = -\inf_{p^*\in Y^*} \Phi^*(0, p^*) \leq \inf_{u\in V} \Phi(u, 0) = \inf_{u\in V} J(u) \tag{6}$$

Taking into account (3) and (4) it is possible to obtain a more explicit formula for $\Phi^*(0, p^*)$. Indeed:

$$\Phi^*(0, p^*) = \sup_{(u,p)\in V\times Y} \{_{Y^*}\langle p^*, p\rangle_Y +_{V^*} \langle F^*, u\rangle_V - G(\Lambda(u) + p)\}\} \tag{7}$$

Setting, for a *fixed* $u \in V$, $p = q - \Lambda(u)$ one obtains:

$$\Phi^*(0, p^*) = G^*(p^*) + \sup_{u\in V}\{_{V^*}\langle F^*, u\rangle_V -_{Y^*} \langle p^*, \Lambda(u)\rangle_Y\} = G^*(p^*) + H_{F^*}(p^*) \tag{8}$$

where

$$H_{F^*}(p^*) := \sup_{u\in V}\{\ _{V^*}\langle F^*, u\rangle_V -_{Y^*} \langle p^*, \Lambda(u)\rangle_Y\} \tag{9}$$

Hence the dual problem $(\mathcal{P}^*)$ is the problem of evaluating

$$\sup_{p^*\in Y^*} \{-G^*(p^*) - H_{F^*}(p^*)\} \tag{10}$$

and (6) becomes:

$$-\inf_{u\in V} J(u) \leq \inf_{p^*\in Y^*} \{G^*(p^*) + H_{F^*}(p^*)\}. \tag{11}$$

One can remark that when Λ is linear then (9) becomes

$$H_{F^*}(p^*) = \sup_{u\in V}\{ {}_{V^*}\langle F^*, u\rangle_V - {}_{Y^*}\langle \Lambda^T(p^*), u\rangle_Y \} = \begin{cases} 0 & \text{when} \quad \Lambda^T(p^*) = F^* \\ +\infty & \text{elsewhere} \end{cases} \tag{12}$$

To prove the equality in (11) is non-trivial. Let us e.g. recall that when $\Lambda = Identity$ (and hence $V = Y$ and so $V^* = Y^*$) and $G \in \Gamma_0(V)$ then a theorem of Fenchel-Rockafellar (see [3], chap. 1) implies that the inequality is indeed an equality.

3. Following the ideas of [12] we can use the formulation of the problem $(\mathcal{P}^*)$ as a starting point for the introduction of *two* Lagrangians and correspondingly *two* bi-duals problems.

The *first Lagrangian* of the problem $(\mathcal{P})$ is the function L defined on $V \times Y^*$ by

$$-L(u, p^*) = \sup_{p\in Y}\{{}_{Y^*}\langle p^*, p\rangle_Y - \Phi(u,p)\} = \sup_{p\in Y}\{{}_{Y^*}\langle p^*, p\rangle_Y - G(\Lambda(u)+p) + {}_{V^*}\langle F^*, u\rangle_V\} \tag{13}$$

Setting as before for a *fixed* $u \in V$, $p = q - \Lambda(u)$ one obtains:

$$-L(u, p^*) = G^*(p^*) + {}_{V^*}\langle F^*, u\rangle_V - {}_{Y^*}\langle p^*, \Lambda(u)\rangle_Y \tag{14}$$

and hence

$$-\Phi^*(0, p^*) = \inf_{u\in V} L(u, p^*) \tag{15}$$

and the dual problem $(\mathcal{P}^*)$ can therefore be recast as the problem of evaluating

$$\sup_{p^*\in Y^*}\inf_{u\in V} L(u, p^*)$$

Since

$$\sup_{p^*\in Y^*} L(u,p^*) = \sup_{p^*\in Y^*}\{-G^*(p^*) - {}_{V^*}\langle F^*, u\rangle_V + {}_{Y^*}\langle p^*, \Lambda(u)\rangle_Y\} = G^{**}(\Lambda(u)) - {}_{V^*}\langle F^*, u\rangle_V \tag{16}$$

the problem of evaluating

$$\inf_{u\in V}\sup_{p^*\in Y^*} L(u, p^*)$$

suggest the following definition

Definition 3 *The first bi-dual problem $(\mathcal{P}^{**})$ is the problem of evaluating* $\inf_{u\in V}\{G^{**}(\Lambda(u)) - {}_{V^*}\langle F^*, u\rangle_V\}$ *and we call $\hat{\hat{u}} \in V$ a solution of $(\mathcal{P}^{**})$ if*

$$G^{**}(\Lambda\hat{\hat{u}}) - {}_{V^*}\langle F^*, \hat{\hat{u}}\rangle_V \in \mathbb{R} \quad \textit{and } G^{**}(\Lambda\hat{\hat{u}}) - {}_{V^*}\langle F^*, \hat{\hat{u}}\rangle_V = \inf_{u\in V}\{G^{**}(\Lambda(u)) - {}_{V^*}\langle F^*, u\rangle_V\} \tag{17}$$

Since the function G^{**} is the largest minorant of G in $\Gamma(Y)$ one has:

$$\inf_{u\in V}\{G^{**}(\Lambda(u)) - {}_{V^*}\langle F^*, u\rangle_V\} \le \inf_{u\in V} J(u) \tag{18}$$

Let us also remark that (16) and a property of saddle points imply :

$$\sup_{p^*\in Y^*}(-G^*(p^*)) - H_{F^*}(p^*)) \le \inf_{u\in V}\{G^{**}(\Lambda(u)) - {}_{V^*}\langle F^*, u\rangle_V\} \tag{19}$$

When G is proper, convex and wlsc then the Fenchel-Moreau theorem implies that the $G^{**} = G$ and hence the first bi-dual problem $(\mathcal{P}^{**})$ coincides with the problem $(\mathcal{P})$. In the general situation one can consider that the bi-dual problem $(\mathcal{P}^{**})$ gives a *regularized formulation* of the original problem $(\mathcal{P})$. The very interesting problem of the existence of saddle-points for the Lagrangian (13) and their connections with the problems $(\mathcal{P})$ and $(\mathcal{P}^*)$ is outside this paper and some hints can be found in [12].

4. Thanks to the Fenchel-Moreau theorem one always has $G^* = G^{***}$ and so (8) can be recast as :

$$\begin{aligned}&\Phi^*(0,p^*) = G^{***}(p^*) + H_{F^*}(p^*) = \sup_{D\in Y}\{_{Y^*}\langle p^*, D\rangle_Y - G^{**}(D)\} + H_{F^*}(p^*) = \\ &= -\inf_{D\in Y, v\in V}\{G^{**}(D) +_{Y^*} \langle p^*, \Lambda(v) - D\rangle_Y -_{V^*} \langle F^*, v\rangle_V\}\end{aligned} \tag{20}$$

The *second Lagrangian* of the problem $(\mathcal{P})$ is the function $\hat{\mathcal{L}}$ defined on $Y^* \times Y \times V$ by:

$$-\hat{\mathcal{L}}(p^*, D, v) =_{Y^*} \langle p^*, D - \Lambda(v)\rangle_Y - G^{**}(D) +_{V^*} \langle F^*, v\rangle_V \tag{21}$$

therefore

$$-\Phi^*(0,p^*) = \inf_{D\in Y, v\in V} \hat{\mathcal{L}}(p^*, D, v) \tag{22}$$

and so the dual problem$(\mathcal{P}^*)$ can also be recast as the problem of evaluating

$$\sup_{p^*\in Y^*} \inf_{D\in Y, v\in V} \hat{\mathcal{L}}(p^*, D, v)$$

Let us remark that

$$\sup_{p^*\in Y^*} \hat{\mathcal{L}}(p^*, D, v) = G^{**}(D) + \sup_{p^*\in Y^*} \{_{Y^*}\langle p^*, \Lambda(v) - D\rangle_Y\} -_{V^*} \langle F^*, v\rangle_V. \tag{23}$$

Since

$$\sup_{p^*\in Y^*}\{_{Y^*}\langle p^*, \Lambda(v) - D\rangle_Y\} = \begin{cases} 0 \quad \text{when} \quad \Lambda(v) = D \\ \\ +\infty \quad \text{elsewhere} \end{cases}$$

then

$$\sup_{p^*\in Y^*} \hat{\mathcal{L}}(p^*, D, v) = \begin{cases} \{G^{**}(D) -_{V^*} \langle F^*, \mathcal{F}(D)\rangle_V\} \quad \text{for all}(D,v) \text{ such that} \quad \Lambda(v) = D \\ \\ +\infty \quad \text{elsewhere} \end{cases} \tag{24}$$

We can therefore introduce the following definition of *intrinsic bi-dual problem*

Definition 4 *The problem of evaluating* $\inf_{D\in R(\Lambda)}\{G^{**}(D) -_{V^*} \langle F^*, \mathcal{F}(D)\rangle_V\}$ *is called the intrinsic bi-dual problem* $(\mathcal{P}^{**}_{int})$. *We call* $\hat{D} \in R(\Lambda)$ *a solution of* $(\mathcal{P}^{**}_{int})$ *if* $G^{**}(\hat{D}) -_{V^*} \langle F^*, \mathcal{F}(\hat{D})\rangle_V \in \mathbb{R}$ *and*

$$G^{**}(\hat{D}) -_{V^*} \langle F^*, \mathcal{F}(\hat{D})\rangle_V = \inf_{D\in R(\Lambda)}\{G^{**}(D) -_{V^*} \langle F^*, \mathcal{F}(D)\rangle_V\} \tag{25}$$

Let us stress that the first bi-dual problem is a problem of minimization in the space V and the intrinsic bi-dual problem is a problem of minimization in the weakly sequentially closed subset $R(\Lambda)$ of Y. Since Λ is one-to-one then

$$\inf_{D\in R(\Lambda)}\{G^{**}(D) -_{V^*} \langle F^*, \mathcal{F}(D)\rangle_V\} = \inf_{u\in V}\{G^{**}(\Lambda(u)) -_{V^*} \langle F^*, u\rangle_V\} \tag{26}$$

The analogous of (19) is true for the second bi-dual problem as follows from a property of saddle points or else from (26).

3 Non linearly elastic plates: potential energy formulation

Let us consider a two-dimensional Euclidean space identified by $\mathbb{R}^2$ and such that the two vectors $\mathbf{e}_i$ form an orthonormal basis. Let be ω is a *domain in* $\mathbb{R}^2$, i.e., a bounded, connected, open subset of $\mathbb{R}^2$ whose boundary, denoted γ, is Lipschitz-continuous, the set ω being locally on a single side of γ (see, e.g., Nečas [18]). The outer normal to γ is denoted by $\mathbf{n}$.

Consider an elastic plate of unit thickness with ω as its middle surface, made up with a homogeneous and isotropic elastic material with Lamè coefficients $\mu > 0$ and $\lambda > -\frac{2}{3}\mu$, and whose reference configuration $\overline{\omega} \times [-\frac{1}{2}, +\frac{1}{2}]$ is a natural state. In the classical formulation of the displacement-traction problem of the Kirchoff-Love-von Kármán theory of non-linearly elastic plates the displacement field of the middle surface of the plate $\zeta = (\mathbf{u}, w) : \omega \to \mathbb{R}^3$ is a stationary point of the potential energy functional $J : \zeta \in \mathbb{V} \to J(\zeta) \in \mathbb{R} \cup \{+\infty\}$. The potential energy functional is defined by

$$J(\zeta) := \frac{1}{2} \int_\omega \{\mathbb{A}\mathbf{E}(\zeta)\mathbf{E}(\zeta) + \frac{1}{3}\mathbb{A}\mathbf{H}(w)\mathbf{H}(w)\} d\omega - L(\zeta) \tag{27}$$

where:

$$\mathbb{A} = (a_{ijlm}) \quad \text{with} \quad a_{ijlm} = \frac{4\lambda\mu}{\lambda + 2\mu}\delta_{ij}\delta_{lm} + 2\mu(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}), \tag{28}$$

$$\mathbf{E}(\zeta) = (E_{ij}(\zeta)) \quad \text{with} \quad E_{ij}(\zeta) = \frac{1}{2}(u_{i,j} + u_{j,i} + w_{,i}w_{,j}), \tag{29}$$

$$\mathbf{H}(w) = (H_{ij}(w)) \quad \text{with} \quad H_{ij}(w) = w_{,ij}. \tag{30}$$

The choices of the space $\mathbb{V}$ and of the linear form $L(\zeta)$ depend of the model taken into account. In order to avoid inessential technicalities in the sequel we consider *only the case of a clamped plate* (see e.g. Ciarlet [4],Telega [23]) :

$$\mathbb{V} := \mathbf{H}_0^1(\omega) \times H_0^2(\omega), \quad L(\zeta) := \int_\omega \{\mathbf{F}\mathbf{u} + \phi w\} d\omega =_{\mathbb{V}^*} \langle F^*, \zeta \rangle_{\mathbb{V}} \tag{31}$$

where $\mathbb{V}^* = \mathbf{H}^{-1}(\omega) \times H^{-2}(\omega)$ with the usual notations for the Sobolev spaces and where $\mathbf{F} \in \mathbf{L}^2(\omega)$ and $\phi \in L^2(\omega)$ and $F^* = (\mathbf{F}, \phi)$. Thanks to the Sobolev space embeddings one has $H^1(\omega) \subset L^q(\omega)$ for all $2 \leq q < +\infty$. Hence taking $q = 4$ it follows that $E_{ij}(\zeta) \in L^2(\omega)$ and $J(\zeta) \in \mathbb{R}$.

The *primal problem* $(\mathcal{P})$ is the problem of evaluating $\inf_{\zeta \in \mathbb{V}} J(\zeta)$, i.e. the minimum principle of the total potential energy. A remarkable result (see e.g. Rabier [20]) is:

Theorem 2 *There exists at least one* $\hat{\zeta} \in \mathbb{V}$ *such that* $J(\hat{\zeta}) = \inf_{\zeta \in \mathbb{V}} J(\zeta)$.

Since $J(\zeta)$ is differentiable the stationary points of this functional are those $\zeta \in \mathbb{V}$ such that such that $J'(\zeta) = 0$ where J' denotes the Fréchet derivative of J. Hence they satisfy the equations:

$$-\mathbf{div}(\mathbb{A}\mathbf{E}(\zeta)) = \mathbf{F} \tag{32}$$

$$-div(\mathbb{A}\mathbf{E}(\zeta)\nabla w) + \frac{1}{3} div\,\mathbf{div}(\mathbb{A}\mathbf{H}) = \phi \tag{33}$$

Using the results of section 2 we can introduce *two complementary energy* formulations and *two intrinsic* formulations. The first (section 4) gives the intrinsic formulation developed by P. G. Ciarlet in a series of papers (see e.g. [5], [6],...) while the second one (section 6) is obtained using the Telega complementary energy formulation of section 5.

4 Nonlinear elastic plates: first (Ciarlet) intrinsic formulation

1. The potential energy functional (27) can be recast in the form (1) taking :

$$Y := \mathbb{L}^2_s(\omega) \times \mathbb{L}^2_s(\omega), \tag{34}$$

$$G(\mathbf{E},\mathbf{H}) := \frac{1}{2}\int_\omega \{\mathbb{A}\mathbf{E}\mathbf{E} + \frac{1}{3}\mathbb{A}\mathbf{H}\mathbf{H}\} d\omega, \tag{35}$$

$$\Lambda(\zeta) = \Lambda((\mathbf{u},w)) := (\mathbf{E}(\zeta),\mathbf{H}(w)) = (E_{ij}(\zeta), H_{ij}(w)) \tag{36}$$

where $\mathbf{E}(\zeta)$, resp.$\mathbf{H}(w)$, are given by (29), resp. (30). As previously remarked, thanks to the Sobolev embeddings the map $\zeta \to \Lambda(\zeta)$ is well defined and continuous and the condition $\Lambda(0) = 0$ is satisfied. In order to prove that the map is one-to-one we remark that $\Lambda(\zeta^1) = \Lambda(\zeta^2)$ implies $\mathbf{H}(w^1) = \mathbf{H}(w^2)$ and the Poincaré inequality implies $w^1 = w^2$. Hence $\mathbf{E}(\zeta^1) = \mathbf{E}(\zeta^2)$ implies $\mathbf{e}(\mathbf{u^1}) = \mathbf{e}(\mathbf{u^2})$ where $\mathbf{e}(\mathbf{u}) = (e_{ij}(\mathbf{u}))$ with $e_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$. The Korn inequality implies then $\mathbf{u^1} = \mathbf{u^2}$ and the condition (ii) is proved. The characterization of $R(\Lambda)$ and the proof of properties (iii) and (iv) follow from the non-linear Donati compatibility condition of [9]. For completeness we briefly recall some results of [9]. Given a symmetric matrix vector field $\mathbf{S} \in \mathbb{L}^2_s(\omega)$ a function $w \in H^2(\omega)$ is called an Airy function for $\mathbf{S}$ when

$$S_{22} = w_{,11}, \quad S_{12} = -w_{,12} \quad S_{11} = w_{,22} \tag{37}$$

One can prove (see e.g. [14]) that such an Airy function w exists if and only if $\mathbf{S}$ verifies the compatibility condition

$$\mathbf{divS} = 0 \quad \text{in} \quad \mathbf{H}^{-1}(\omega); \tag{38}$$

moreover w is uniquely determined by the conditions

$$\int_\omega w d\omega = \int_\omega w_i d\omega = 0. \tag{39}$$

The linear map that associates to $\mathbf{S} \in \mathbb{L}^2_s(\omega)$ satisfying (38) the uniquely determined Airy function $w \in H^2(\omega)$ verifying (39) well be denoted Φ. The non-linear Donati compatibility condition is the object of the following theorem.

Theorem 3 *Let there be given two matrix fields* $(\mathbf{E},\mathbf{H}) \in \mathbb{L}^2_s(\omega) \times \mathbb{L}^2_s(\omega)$*. Then there exists a uniquely determined vector field* $\zeta = (\mathbf{u}, w) \in \mathbf{H}^1_0(\omega) \times H^2_0(\omega)$ *such that*

$$\begin{cases} \frac{1}{2}(u_{i,j} + u_{j,i} + w_{,i}w_{,j}) = E_{ij} \quad & in \quad L^2(\omega) \\ w_{,ij} = H_{ij} \quad in \quad L^2(\omega) \end{cases} \tag{40}$$

if and only if the following non-linear Donati compatibility conditions are satisfied:

$$\begin{cases} \int_\omega H_{ij}T_{ij} d\omega = 0 \quad \textit{for all} \quad \mathbf{T} = (T_{ij}) \in \mathbb{L}^2_s(\omega) \quad \textit{such that} \quad div\mathbf{div}\mathbf{T} = 0 \ \textit{in} \quad \mathbf{H}^{-2}(\omega), \\ \int_\omega \{E_{ij}S_{ij} + (det\mathbf{H})\Phi(\mathbf{S}\} d\omega = 0 \quad \textit{for all} \quad \mathbf{S} = (S_{ij}) \in \mathbb{L}^2_s(\omega) \ \textit{such that}\ \mathbf{divS} = 0 \ \ \textit{in}\ \mathbf{H}^{-1}(\omega) \end{cases} \tag{41}$$

This result allows the characterization of $\Lambda(\mathbb{V})$:

$$\Lambda(\mathbb{V}) = \{(\mathbf{E},\mathbf{H}) \in \mathbb{L}^2_s(\omega) \times \mathbb{L}^2_s(\omega) \quad \text{satysfiyng (41) } \} \tag{42}$$

One can also prove that $\Lambda(\mathbb{V})$ is sequentially weakly closed in $\mathbb{L}^2_s(\omega) \times \mathbb{L}^2_s(\omega)$ and that the map $\mathcal{F} : \Lambda(\mathbb{V}) \subset \mathbb{L}^2_s(\omega) \times \mathbb{L}^2_s(\omega) \longrightarrow \zeta = (\mathbf{u}, w) \in \mathbf{H}^1_0(\omega) \times H^2_0(\omega)$ defined in this way transforms weakly convergent sequences in $\mathbb{L}^2_s(\omega) \times \mathbb{L}^2_s(\omega)$ into weakly convergent sequences in $\mathbf{H}^1_0(\omega) \times H^2_0(\omega)$.

2. To state the complementary energy formulation (i.e. the dual problem) $(\mathcal{P}^*)$ we remark at first that $Y = Y^*$ with the duality pairing associated to the scalar product and that

$$G^*(\mathbf{E}^*,\mathbf{H}^*) := \frac{1}{2}\int_\omega \{\mathbb{A}^*\mathbf{E}^*\mathbf{E}^* + 3\mathbb{A}^*\mathbf{H}^*\mathbf{H}^*\} d\omega \tag{43}$$

where

$$\mathbb{A}^* = (a^*_{ijlm}) \quad \text{with} \quad a^*_{ijlm} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\delta_{lm} + \frac{1}{2\mu}(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}), \tag{44}$$

As far as it concerns

$$H_{F^*}(\mathbf{E}^*, \mathbf{H}^*) := \sup_{\zeta\in\mathbb{V}}\{\int_\omega \{\mathbf{F}\mathbf{u} + \phi w\}d\omega - \int_\omega \mathbf{E}^*\mathbf{E}(\zeta)d\omega - \int_\omega \mathbf{H}^*\mathbf{H}(w)d\omega\}$$

we remark that

$$H_{F^*}(\mathbf{E}^*, \mathbf{H}^*) = \sup\{1_{\mathcal{A}\mathbf{F}}, \quad \sup_{w\in H^2_0(\omega)}\{\int_\omega \phi w d\omega - \int_\omega \mathbf{E}^*\frac{1}{2}\nabla w\otimes\nabla w d\omega - \int_\omega \mathbf{H}^*\mathbf{H}(w)d\omega\}\} \tag{45}$$

where:

$$\mathcal{A}\mathbf{F} = \{\mathbf{E}^* \in \mathbf{H}^1_0(\omega); div\mathbf{E}^* = -\mathbf{F}\} \tag{46}$$

The study of the complementary energy formulation is outside the aim of this paper.

3. The definition (35) and the properties of the tensor $\mathbb{A}$ imply that $G(\mathbf{E}, \mathbf{H})$ is proper, convex, and wlsc in Y. Hence the Fenchel-Moreau theorem implies that $G^{**} = G$ and that *the bi-dual problem* $(\mathcal{P}^{**})$ *coincides with the primal* $(\mathcal{P})$.

4. Recalling the definition 4 the *Ciarlet's intrinsic bi-dual problem* $(\mathcal{P}^{**}_{intC})$ is the problem of evaluating

$$\inf_{(\mathbf{E},\mathbf{H})\in\Lambda(\mathbb{V})}\{G(\mathbf{E}, \mathbf{H}) - L(\mathcal{F}((\mathbf{E}, \mathbf{H}))\} \tag{47}$$

and it coincides with the intrinsic formulation studied in [9].

5 Non linear elastic plates: Telega complementary energy and bidual formulation

1. Following the procedure of Telega [23], the potential energy functional (27) can be recast in the form (1) taking :

$$Y := \mathbb{L}^2_s(\omega) \times \mathbf{L}^4(\omega) \times \mathbb{L}^2_s(\omega). \tag{48}$$

$$\Lambda(\zeta) = \Lambda((\mathbf{u}, w)) := (\mathbf{e}(\mathbf{u}), \nabla w, \mathbf{H}(w)) \tag{49}$$

where $\mathbf{H}(w)$ is given by (30), $\mathbf{e}(\mathbf{u}) = (e_{ij}(\mathbf{u}))$ with $e_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$ and $\nabla w = (w_{,1}, w_{,2})$ so that $\mathbf{E} = \mathbf{e} + \frac{1}{2}\nabla w\otimes\nabla w$. Hence the potential energy functional (27) can be written $J(\zeta) = G(\Lambda(\zeta)) - L(\zeta)$ where:

$$G(\Lambda(\zeta) := \frac{1}{2}\int_\omega \{\mathbb{A}(\mathbf{e} + \frac{1}{2}\nabla w\otimes\nabla w)(\mathbf{e} + \frac{1}{2}\nabla w\otimes\nabla w) + \frac{1}{3}\mathbb{A}\mathbf{H}(w)\mathbf{H}(w)\}d\omega \tag{50}$$

As in Sect. 4 thanks to the Sobolev embeddings the map $\zeta \to \Lambda(\zeta)$ is well defined and continuous and the condition $\Lambda(0) = 0$ is satisfied. The map is one-to-one and has closed range $R(\Lambda)$ since the Korn inequality implies $\|\mathbf{u}\|_{\mathbf{H}^1_0(\omega)} \le c\|\mathbf{e}(\mathbf{u})\|_{\mathbb{L}^2_s(\omega)}$ and the Poincaré inequality implies $\|w\|_{H^2_0(\omega)} \le c\|\mathbf{H}(w)\|_{\mathbb{L}^2_s(\omega)}$.

In order to find the dual problem $(\mathcal{P}^*)$ one remarks at first that Y is a reflexive space with

$$Y^* = \mathbb{L}^2_s(\omega) \times \mathbf{L}^{4/3}(\omega) \times \mathbb{L}^2_s(\omega).$$

and that their canonical duality is :

$$_{Y^*}\langle(\mathbf{f}, \mathbf{p}, \mathbf{M}), \ (\mathbf{g}, \mathbf{q}, \mathbf{N})\rangle_Y = \int_\omega \{\mathbf{f}\mathbf{g} + \mathbf{p}\mathbf{q} + \mathbf{M}\mathbf{N}\}d\omega \tag{51}$$

Hence

$$G^*(\mathbf{f}^*, \mathbf{p}^*, \mathbf{M}^*) := \sup_{(\mathbf{g},\mathbf{q},\mathbf{N})\in Y}\{\int_\omega \{\mathbf{f}^*\mathbf{g} + \mathbf{p}^*\mathbf{q} + \mathbf{M}^*\mathbf{N}\}d\omega \ - \ G(\mathbf{g}, \mathbf{q}, \mathbf{N})\} \tag{52}$$

where

$$G(\mathbf{g}, \mathbf{q}, \mathbf{N}) := \int_\omega \Phi(\mathbf{g}(x), \mathbf{q}(x), \mathbf{N}(x))d\omega \tag{53}$$

with

$$\Phi(\mathbf{g},\mathbf{q},\mathbf{N}) = \Phi_1(\mathbf{f},\mathbf{p}) + \Phi_2(\mathbf{M}) \tag{54}$$

and

$$\Phi_1(\mathbf{g},\mathbf{q}) := \frac{1}{2}\mathbb{A}(\mathbf{g}+\frac{1}{2}\mathbf{q}\otimes\mathbf{q})(\mathbf{g}+\frac{1}{2}\mathbf{q}\otimes\mathbf{q}) \qquad \Phi_2(\mathbf{N}) := \frac{1}{6}\mathbb{A}\mathbf{N}\mathbf{N} \tag{55}$$

For further use we remark that $\Phi_2(\mathbf{N})$ is convex and that the Hessian matrix of $\Phi_1(\mathbf{f},\mathbf{p})$ is positive semi-definite on the set :

$$\mathcal{K} := \{(\mathbf{g},\mathbf{q}) \in \mathbb{M}^2_s \times \mathbb{R}^2; \mathbb{A}(\mathbf{g}+\frac{1}{2}\mathbf{q}\otimes\mathbf{q}) \in \mathbb{M}^2_s \qquad \text{is positive semi-definite}\} \tag{56}$$

Since the plate is made of an homogeneous material $\mathbb{A}$ is constant on ω and since $\Phi(\mathbf{g},\mathbf{q},\mathbf{N})$ is continuous with respect to $(\mathbf{g},\mathbf{q},\mathbf{N})$, the mapping $\Phi : \omega \times \mathbb{R}^8(\equiv \omega \times \mathbb{M}^2_s \times \mathbb{R}^2 \times \mathbb{M}^2_s) \to \bar{\mathbb{R}}$ defined by (54) is a Carathéodory function and hence a normal integrand (see e.g. [12], Proposition 1.1, chap. VIII). Moreover the assumption on the elasticity coefficients imply $\Phi(\mathbf{g},\mathbf{q},\mathbf{N}) \geq 0$ and so $G(\mathbf{g},\mathbf{q},\mathbf{N}) : Y \to \bar{\mathbb{R}}$ is positive and lower semi-continuous. Since $G(\mathbf{g}_0,\mathbf{q}_0,\mathbf{N}_0) < +\infty$ when $\mathbf{g}_0,\mathbf{q}_0,\mathbf{N}_0$ are constant in ω one can apply the Proposition 2.1, chap. IX of [12] to the computation of $G^*(\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*)$:

$$G^*(\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*) = \int_\omega \Phi^*(\mathbf{f}^*(x),\mathbf{p}^*(x),\mathbf{M}^*(x))d\omega \tag{57}$$

where $\Phi^*(\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*)$ is the polar of the function $\Phi(\mathbf{g},\mathbf{q},\mathbf{N})$ and is given for $(\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*) \in \mathbb{M}^2_s \times \mathbb{R}^2 \times \mathbb{M}^2_s$ by

$$\Phi^*(\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*) = \sup_{(\mathbf{g},\mathbf{q},\mathbf{N})\in\mathbb{M}^2_s\times\mathbb{R}^2\times\mathbb{M}^2_s} \{\mathbf{f}^*\mathbf{g} + \mathbf{p}^*\mathbf{q} + \mathbf{M}^*\mathbf{N} - \Phi(\mathbf{g},\mathbf{q},\mathbf{N})\} \tag{58}$$

Thanks to Proposition 1.2, chap.VIII of [12] one has that Φ^* is also a normal integrand on $\omega\times\mathbb{M}^2_s\times\mathbb{R}^2\times\mathbb{M}^2_s$.

In order to complete the formulation of the dual problem one has to compute explicitly

$$H_L(\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*) := \sup_{(\mathbf{u},w)\in\mathbb{V}} \{{}_{Y^*}\langle(\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*),\ (\mathbf{e}(\mathbf{u},\nabla w,\mathbf{H}(w))\rangle_Y - \int_\omega \{\mathbf{F}\mathbf{u} + \phi w\} d\omega\} \tag{59}$$

Using (48) and taking in (59) at first $\mathbf{u} \in \mathcal{D}(\omega;\mathbb{R}^2)$ and $w = 0$ and then taking $\mathbf{u} = \mathbf{0}$ and $w \in \mathcal{D}(\omega)$ one finds

$$H_L(\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*) = \begin{cases} 0 \ \text{ if } \ \mathbf{div}\mathbf{f}^* + \mathbf{F} = \mathbf{0} \text{ and } div(\mathbf{div}\mathbf{M}^* - \mathbf{p}^*) + \phi = 0 \ \text{ in } \omega \\ +\infty \qquad \text{elsewhere} \end{cases} \tag{60}$$

where the equations are to be intended in the distribution sense (the first in $H^{-1}(\omega)$ and the second in$H^{-2}(\omega)$) . For simplicity we define

$$\Sigma_{ad} := \{(\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*) \in Y^*; div\mathbf{f}^* + \mathbf{F} = \mathbf{0} \qquad \text{and} \qquad div(\mathbf{div}\mathbf{M}^* - \mathbf{p}^*) + \phi = 0\} \tag{61}$$

and so (60) an be restated as follows:

$$H_L(\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*) = \begin{cases} 0 \ \text{ if } \ (\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*) \in \Sigma_{ad} \\ +\infty \qquad \text{elsewhere} \end{cases} \tag{62}$$

For the computation of $\Phi^*(\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*)$ let us remark at first that

$$\Phi^*(\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*) = \Phi^*_1(\mathbf{f}^*,\mathbf{p}^*) + \Phi^*_2(\mathbf{M}^*) \tag{63}$$

and hence as usual

$$\Phi^*_2(\mathbf{M}^*) = \frac{3}{2}\mathbb{A}^{-1}\mathbf{M}^*\mathbf{M}^*. \tag{64}$$

The critical points of $\mathcal{F}(\mathbf{g},\mathbf{q}) := \mathbf{f}^*\mathbf{g} + \mathbf{p}^*\mathbf{q} - \Phi_1(\mathbf{g},\mathbf{q})$ satisfy $\mathbf{f}^* = \dfrac{\partial \Phi_1}{\partial \mathbf{g}} = \mathbb{A}(\mathbf{g} + \frac{1}{2}\mathbf{q}\otimes\mathbf{q})$ and $\mathbf{p}^* = \dfrac{\partial \Phi_1}{\partial \mathbf{q}} = \mathbb{A}(\mathbf{g} + \frac{1}{2}\mathbf{q}\otimes\mathbf{q})\mathbf{q} = \mathbf{f}^*\mathbf{q}$. They are maxima only when $\mathbf{f}^*$ is positive semi-definite and hence one gets as in Telega [23]:

$$\Phi_1^*(\mathbf{f}^*,\mathbf{p}^*) = \sup_{(\mathbf{g},\mathbf{q})\in\mathbb{M}_s^2\times\mathbb{R}^2} \mathcal{F}(\mathbf{g},\mathbf{q}) =$$

$$= \begin{cases} \frac{1}{2}\{\mathbb{A}^{-1}\mathbf{f}^*\mathbf{f}^* + (\mathbf{f}^*)^{-1}\mathbf{p}^*\mathbf{p}^*\} & \text{if } \mathbf{f}^* \text{ positive definite} \\ \frac{1}{2}\{\mathbb{A}^{-1}\mathbf{f}^*\mathbf{f}^* + \dfrac{\|\mathbf{p}^*\|^2}{Tr(\mathbf{f}^*)}\} & \text{if } \mathbf{f}^* \text{ positive semi-definite, } dim\ ker(\mathbf{f}^*) = 1 \text{ and } \mathbf{f}^*\mathbf{p}^* = Tr(\mathbf{f}^*)\mathbf{p}^* \\ 0 & \text{if } \mathbf{f}^* = \mathbf{p}^* = 0 \\ +\infty & \text{elsewhere} \end{cases} \tag{65}$$

Collecting (62), (63), (65), and (64) we get that the dual problem $(\mathcal{P}^*)$ is the problem of evaluating

$$\sup_{(\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*)\in\Sigma_{ad}} \left(- \int_\omega \Phi^*(\mathbf{f}^*(x),\mathbf{p}^*(x),\mathbf{M}^*(x))d\omega \right) \tag{66}$$

Let us point out that $\Phi^*(\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*) \geq 0$ and hence $G^*(\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*)$ is positive, lower semi-continuous.

2. In order to find the *bi-dual problem* let us remark that if we take $\mathbf{f}^* = \mathbb{M}^* = \mathbf{1}$ and $\mathbf{p}^* = \mathbf{0}$ then $\int_\omega \Phi^*(\mathbf{1},\mathbf{0},\mathbf{1})d\omega < +\infty$. Therefore, applying the Proposition 2.2 chap.IX of [12], the Γ-regularization G^{**} of G is given by:

$$G^{**}(\mathbf{g},\mathbf{q},\mathbf{N}) := \int_\omega \Phi^{**}(\mathbf{g}(x),\mathbf{q}(x),\mathbf{N}(x))d\omega\} \tag{67}$$

where

$$\Phi^{**}(\mathbf{g},\mathbf{q},\mathbf{N}) = \sup_{(\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*)\in\mathbb{M}_s^2\times\mathbb{R}^2\times\mathbb{M}_s^2} \{\mathbf{f}^*\mathbf{g} + \mathbf{p}^*\mathbf{q} + \mathbf{M}^*\mathbf{N} - \Phi^*(\mathbf{f}^*,\mathbf{p}^*,\mathbf{M}^*)\} \tag{68}$$

Taking into account (63) we get

$$\Phi^{**}(\mathbf{g},\mathbf{q},\mathbf{N}) = \Phi_1^{**}(\mathbf{g},\mathbf{q}) + \Phi_2(\mathbf{N}) \tag{69}$$

since $\Phi_2(\mathbf{N})$ is convex (and hence $\Phi_2^{**}(\mathbf{N}) = \Phi_2(\mathbf{N})$). The function

$$\Phi_1^{**}(\mathbf{g},\mathbf{q}) = \sup_{(\mathbf{f}^*,\mathbf{p}^*)\in\mathbb{M}_s^2\times\mathbb{R}^2} \{\mathbf{f}^*\mathbf{g} + \mathbf{p}^*\mathbf{q} - \Phi_1^*(\mathbf{f}^*,\mathbf{p}^*)\} \tag{70}$$

is the the $\Gamma-$*regularization* of $\Phi_1(\mathbf{g},\mathbf{q})$ given by

$$\Phi_1^{**}(\mathbf{g},\mathbf{q}) = \begin{cases} \frac{1}{2}\mathbb{A}(\mathbf{g} + \frac{1}{2}\mathbf{q}\otimes\mathbf{q})(\mathbf{g} + \frac{1}{2}\mathbf{q}\otimes\mathbf{q}) & \text{if } \mathbb{A}(\mathbf{g} + \frac{1}{2}\mathbf{q}\otimes\mathbf{q}) \text{ positive semi-definite} \\ 0 & \text{elsewhere} \end{cases} \tag{71}$$

The *bi-dual problem* $(\mathcal{P}^{**})$ is therefore the problem of evaluating

$$\inf_{(\mathbf{u},w)\in\mathbb{V}} \{\int_\omega \Phi^{**}(\mathbf{e}(\mathbf{u}),\nabla w,\mathbf{H}(w))d\omega - L(\mathbf{u},w)\} \tag{72}$$

In order to compare the first bi-dual problem and the minimum principle of the total potential energy it is useful to express $\Phi^{**}(\mathbf{e}(\mathbf{u}),\nabla w,\mathbf{H}(w))$ using the notations of (27) . One has from (29), (69), (71):

$$\Phi^{**}(\mathbf{e}(\mathbf{u}),\nabla w,\mathbf{H}(w)) = \begin{cases} \frac{1}{2}\mathbb{A}\mathbf{E}(\zeta)\mathbf{E}(\zeta) + \frac{1}{3}\mathbb{A}\mathbf{H}(w)\mathbf{H}(w) & \text{if } \mathbb{A}\mathbf{E}(\zeta) \text{ positive semi-definite} \\ \frac{1}{3}\mathbb{A}\mathbf{H}(w)\mathbf{H}(w) & \text{elsewhere} \end{cases} \tag{73}$$

It is interesting to remark that $\mathbb{A}\mathbf{E}(\zeta)$ is the second order tensor of membrane stresses. In order that this bi-dual problem has a solution one has to add some suitable conditions on $L(\mathbf{u},w)$. This is outside the present research.

6 Non linearly elastic plates: second (Telega) intrinsic formulation

For this formulation one needs the characterization of $R(\Lambda) \subset \mathbb{L}^2_s(\omega) \times \mathbf{L}^4(\omega) \times \mathbb{L}^2_s(\omega)$ which is given by the following theorems.

Theorem 4 *Let be given a tensor field* $\mathbf{g} = (g_{ij}) \in \mathbb{L}^2_s(\omega)$. *Then there exists a uniquely determined vector field* $\mathbf{u} = (u_i) \in \mathbf{H}^1_0(\omega)$ *such that*

$$\frac{1}{2}(u_{i,j} + u_{j,i}) = g_{ij} \tag{74}$$

if and only if the following linear Donati compatibility condition is satisfied:

$$\int_\omega g_{ij} S_{ij} d\omega = 0 \quad \textit{for all} \quad \mathbf{S} = (S_{ij}) \in \mathbb{L}^2_s(\omega) \textit{ such that } \mathbf{divS} = 0 \quad \textit{in } \mathbf{H}^{-1}(\omega) \tag{75}$$

The proof of this theorem is known; see e.g. Geymonat and Suquet [16], Geymonat and Krasucki [15] or Amrouche, Ciarlet, Gratie and Kesavan [1].

Theorem 5 *Let be given* $\mathbf{q} = (q_i) \in \mathbf{L}^4(\omega)$ *and* $\mathbf{N} = (N_{ij}) \in \mathbb{L}^2_s(\omega)$. *Then there exists a uniquely determined* $w \in H^2_0(\omega)$ *such that*

$$q_i = w_{,i} \qquad \textit{and} \qquad N_{ij} = w_{,ij} \tag{76}$$

if and only if the following linear compatibility condition is satisfied:

$$\int_\omega \{\mathbf{qp}^* + \mathbf{NM}^*\} d\omega = 0 \;\; \textit{for all } (\mathbf{p}^*, \mathbf{M}^*) \in \mathbf{L}^{4/3}(\omega) \times \mathbb{L}^2_s(\omega) \;\; \textit{such that } \; div(\mathbf{divM}^* - \mathbf{p}^*) = 0 \tag{77}$$

the last equation being intended in $H^{-2}(\omega)$.

Proof. The operator $\Lambda_2 : H^2_0(\omega) \to \mathbf{L}^4(\omega) \times \mathbb{L}^2_s(\omega)$ with $\Lambda_2(w) := (\nabla w, \mathbf{H}(w)$ is linear, continuous and with closed range (thanks to the Poincaré inequality) . Hence the Banach closed graph theorem implies $R(\Lambda_2) = (Ker\Lambda_2^*)^\perp$. The characterization of Λ_2^* result from the following equation

$$\int_\omega \{\nabla w \mathbf{p}^* + \mathbf{H}(w)\mathbf{M}^*\} d\omega =_{H^2_0} \langle w, -div\mathbf{p}^* + div\mathbf{divM}^*\rangle_{H^{-2}} \qquad \text{for all } w \in H^2_0(\omega), \mathbf{p}^* \in \mathbf{L}^{4/3}(\omega), \mathbf{M}^* \in \mathbb{L}^2_s(\omega)$$

and hence (77). Q.E.D.

Thanks to these theorems one can define a linear and continuous map $\mathcal{F} : (\mathbf{g}, \mathbf{q}, \mathbf{N}) \in R(\Lambda) \longrightarrow \zeta = (\mathbf{u}, w) \in \mathbf{H}^1_0(\omega) \times H^2_0(\omega)$ and the *second (Telega) intrinsic problem* $(\mathcal{P}^{**}_{intT})$ becomes the problem of evaluating

$$\inf_{(\mathbf{g},\mathbf{q},\mathbf{N}) \in R(\Lambda)} \{G^{**}(\mathbf{g}, \mathbf{q}, \mathbf{N}) - L(\mathcal{F}(\mathbf{g}, \mathbf{q}, \mathbf{N}))\} \tag{78}$$

where $G^{**}(\mathbf{g}, \mathbf{q}, \mathbf{N})$ is given by (67), (69), (71).

7 Concluding remarks

1. The conditions (iii) and (iv) concerning the map $\Lambda : V \to Y$ are very natural in order that the intrinsic problem $(\mathcal{P}^{**}_{int})$ has a solution.

2. For linearly elastic plates the two intrinsic problems coincide. Indeed $\mathbf{E}(\zeta) = \mathbf{e}(\mathbf{u}) = (u_{ij}) = \frac{1}{2}(u_{i,j} + u_{j,i})$ is linear and so in both formulations one has $Y = \mathbb{L}^2_s(\omega) \times \mathbb{L}^2_s(\omega)$ and G is quadratic, convex, coercive.

3. One could consider other boundary conditions, e. g. mixed one or pure traction. However the verification of the conditions (i)-(iv) can be non trivial, also in the case of linearly elastic plates. For instance for pure traction in the linear case P.G. Ciarlet- S. Mardare [6] introduce a suitable quotient space in order to take into account of $ker(\Lambda)$. The situation is still more complicated when these AA consider non-linearly elastic plates.

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8 References

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