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Recall and substantiation routines in exam scripts: injective and surjective functions

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In this paper, we focus on first-year university students' engagement with university mathematics discourse in the context of a final examination question on injective and surjective functions. The data consists of twenty-two responses as well as excerpts from an interview with the exam-setting lecturer. Our commognitive analysis focuses on student engagement with recall and substantiation routines: how they recall and provide relevant definitions, and how they substantiate whether a function is injective or surjective. We identify three issues that are exemplified with samples from the data: ambivalent word use, and visual mediation, relating to equivalence; partial justifications regarding the injectivity of $h(n) = 3n$; and, conflation of justifications for a function's surjectivity in \mathbb{Z} with those used for functions in \mathbb{R} . We conclude by discussing the notion of precedent events as evident in students' engagement with routines in a range of mathematical topics.

Keywords: Substantiation, recall, routines, injective, surjective function.

From secondary to university mathematics

Transition from secondary school to university mathematics is a topic of growing interest (Gueudet, 2008). While differences between secondary school and university mathematics vary across countries, there are aspects of this transition which seem to be common in different contexts. In the larger study from which this paper stems (Thoma, 2018), we explore this transition through analyses of Year 1 examination questions, lecturers' perspectives on designing and using these and students' responses. Functions, one of the topics examined in these questions, have been identified as a key mathematical topic in the transition from secondary to university mathematics (Winsløw, Gueudet, Hochmuth, & Nardi, 2018) and research has reported issues around students' learning about functions for a long time (Bansilal, Brijlall & Trigueros, 2017). Analogous attention has been given by researchers to students' deployment of formal definitions and proofs (Selden, 2011), their use of mathematical notation (e.g. Mamolo, 2010) and their difficulties with mathematical objects that have different meanings in various mathematical areas (Kontorovich, 2018). Here, we focus on one aspect of learning about functions that has received relatively little attention: what students think a surjective and an injective function is and how they decide whether a function is injective or surjective.

In this paper¹, we focus on twenty-two students' responses to a Year 1 examination question on injective and surjective functions. We take a discursive approach, the theory of commognition (Sfard, 2008), in analysing the data, and we aim to characterise the routines that students engage in when answering the questions. Building on previously reported work (Thoma & Nardi, 2018a;

¹ An earlier version of this paper was presented as a PME42 Oral Communication (Thoma & Nardi, 2018b).

2018b), we examine the relevant definitions students provide and the procedures they use when substantiating that a given function is (or is not) injective or surjective.

Commognitive routines: recall and substantiation

A discursive perspective which is being increasingly used in university mathematics education research is Sfard's (2008) theory of commognition (Nardi, Ryve, Stadler & Viirman, 2014). In this perspective, mathematics is defined as a discourse (Sfard, 2008) which can be described in terms of the following four characteristics: *word use* (e.g. injective), *visual mediators* (e.g. algebraic and logical symbols), *endorsed narratives* (e.g. definitions) and *routines* (e.g. proving). Sfard (2008) describes three types of routines: *deeds* ("an action resulting in a physical change in objects", p. 236), *rituals* ("creating and sustaining a bond with other people", p. 241) and *explorations* ("producing endorsed narratives", p. 259) with the exploration routines further categorised as *recall* (e.g. recalling the definition of an injective function), *substantiation* (examining whether a given function is injective) and *construction* (e.g. constructing a function that is injective). Of relevance to our analysis here, are exploration routines, which often develop from rituals. This process has been described through the notion of *task-situation* in which an individual feels a need to act and which is taken as "the set of all the characteristics of the precedent events that she considers as requiring replication" (Lavie, Steiner & Sfard, 2018, p. 9). Thus, explorations can emerge as changes in the way learners search for precedents in task situations. In our study, we deploy *task* to mean a part of an examination question which the students are asked to engage with. These task situations are interpreted by the students, who draw on *precedent events*, which include previous examples or exercises each student has engaged with before and which they see as relevant to the present *task*. Sfard (2008) describes the *procedure* (or *course of action*) and the *when* of a *routine*, with the latter further described in terms of its *applicability* and *closing* conditions (pp. 208-209). The applicability conditions are "circumstances in which the routine course of action is likely to be evoked by the person" (p. 209, *ibid*) and closing conditions "[signal] a successful completion of performance" (p. 209, *ibid*).

Here, we explore how students recall the definitions of an injective and surjective function and how they engage in substantiation routines, in order to examine whether functions defined on the integers are (or are not) injective or surjective. Previous research has documented "how lack of flexibility in working across different representations influences students' encounter with the concept of function" (Nardi, 2008, p. 161) and has posited that students' prior experience with functions and sets may have contributed to this lack of flexibility (Bansilal et al., 2017). However, there is little insight from prior research into students' engagement with injective and surjective functions in various numerical domains (e.g. reals and integers). In the UK educational context of the study we report in this paper, engagement with various numerical domains is different in school and university mathematics. In school mathematics, students are initially introduced to the domain of integers. Later, they are introduced to real numbers and discursive activity with integers is subsumed within that of the reals. The tasks they engage with here are in the context of real numbers. At university, students are expected to engage with various numerical domains, often within the same task. Being aware of this variety and of the various restrictions within each domain is therefore crucial.

Our study aims to provide some insight into this through exploring the following research question:

What engagement in mathematical routines can we observe in students' scripts when they examine whether given functions are injective or surjective?

To answer this research question, we examine students' use of mathematical terminology and notation which may be incompatible with the mathematical discourses they are expected to engage with when responding to the task at hand. We also examine the closing and applicability conditions of the routines, in which the students are required to engage in when solving the question within the various mathematics discourses present in the question (e.g. discourse of functions, integers, reals). We examine the written scripts of the students which capture the output of students' engagement with these mathematical discourses.

The exam question, the participating students and the exam-setter's intentions

The data on which we focus are examination scripts from 22 students taking an exam on a *Sets, Numbers and Probability* Year 1 module. In this module, the focus is on *Sets, Numbers and Theory* in the first semester and on *Probability* in the second semester. The final examination consists of six questions, three coming from each part of the module, one compulsory and two optional from each of the two parts of the module. Here, we focus on one sub question of one of the optional questions from *Sets, Numbers and Theory*, on surjective and injective functions. During the year, the students were asked to engage with the properties of injectivity and surjectivity in exercise sheets and coursework with functions defined in different numerical domains. In this paper, we discuss students' responses to this examination sub question:

Suppose A and B are sets and $f: A \rightarrow B$ is a function. Define what is meant by f being *surjective* and what is meant by f being *injective*. For each of the following functions decide whether it is injective, surjective (or both, or neither). Give brief reasons for your answers.

(a) $g: \mathbb{R} \rightarrow \mathbb{R}$ where $g(x) = 1/(1 + \sin^2(x))$ for $x \in \mathbb{R}$.

(b) $h: \mathbb{Z} \rightarrow \mathbb{Z}$ where $h(n) = 3n$ for $n \in \mathbb{Z}$.

Fifty-four students took part in the final examination and the marks of their responses to the whole question (the first sub question is given above and the second dealt with modular arithmetic) ranged from 0 to 20 marks with an average of 14.31. The scripts of 22 students were selected by the first author to represent a variety of marks (for more information on the selection process, see Thoma, and Nardi, 2017, Fig. 3 on p. 2269). We analyse students' scripts according to the definitions they recall for injective and surjective functions, and their substantiations that the given functions are (or are not) surjective or injective. In our discussion of the students' scripts, we bear in mind also the following selected quote from an interview, conducted soon after the exam, with the lecturer who taught the module and set the examination task. This quote illustrates the lecturer's concern with a perceived discrepancy between "knowing how to write down" a definition for injectivity and knowing what injectivity "means". We return to this quote towards the end of paper.

"somehow they know what injective means, they just don't know how to write down the definition (...) it is a very strange experience to see that a student knows what injective means but can't write down what it means, it's something about maybe not even about mathematics, it's

about language and about logic (...) They need to see this transition between (...) the symbols and the meaning and the logic of things and it's one of the most important things and it is one of the hardest things to teach"

Recall and substantiation routines in students' scripts

In the sub question under consideration, students are first asked to provide the definition of injective and surjective function (recall); then, to determine whether two functions are injective or surjective and provide brief explanations for their choices (substantiation). Here, we highlight three issues that emerged from the analysis of the scripts: ambivalent word use, and visual mediation, relating to the object of equivalence (\Leftrightarrow) instead of implication (\Rightarrow) in the definition of injective function; partial justification of the injectivity of $h(n) = 3n$; and, conflation of justifications for a function's surjectivity in \mathbb{Z} with those used for functions in \mathbb{R} . We discuss each case providing also examples from students' scripts.

Ambivalent word-use and visual mediation relating to the object of equivalence

In the scripts of four students, there is ambivalent word use and visual mediation relating to equivalence (either using the logical symbol \Leftrightarrow or phrases that signal equivalence) in the definition of injective function, illustrating confusion between the definition of injective function and the definition of a function. In the definition for an injective function, student [02] writes "one to one relationship" and comments on the relationship between the elements of the domain and the

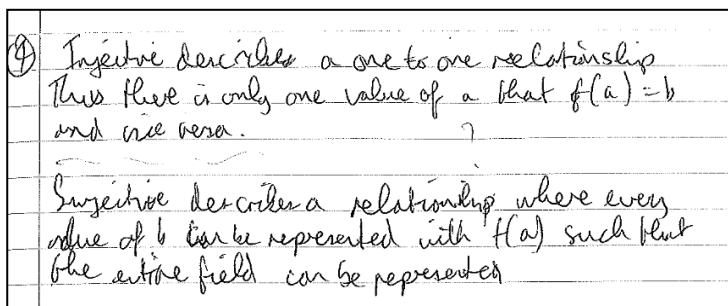


Figure 1: Student [02]'s definitions

codomain of the function (Figure 1). However, later, the student, in trying to clarify what this "one to one relationship" is, writes "f(a)=b and vice versa". This phrase is not further explained and is also used when the student examines whether the given function h is injective, signaling a confusion between the definition of a

function (which must be single valued) and the definition of an injective function. Additionally, we note that the logical quantifiers "for every b " and "there exists a " in the definitions of injectivity and surjectivity are missing from student [02]'s response. This absence of logical quantifiers and the ambivalent use of logical phrases ("vice versa") highlights a difficulty with the logical connections between the various mathematical objects that are part of the definitions. Furthermore, we observe the difficulty to state the definition formally. The phrases used by the student (e.g. "only one value") are ambiguous and are not clearly showcasing whether the student discusses injectivity, which would be "at most one value" or bijectivity ("there exists one and only one value"). The ambiguity between the single valued function and the injective function is also illustrated in definitions given by student [06] (Figure 2) and two

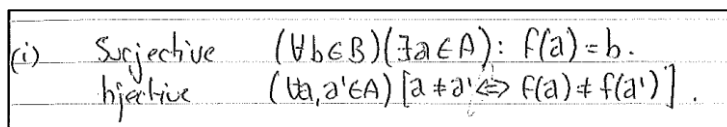


Figure 2: Student [06]'s definitions

more students.

This use of equivalence (either in the form of the symbol (Figure 2) or using the phrase “vice versa” (Figure 1)) in the definition of injective function highlights difficulties in students’ engagement when recalling and writing the definitions. This may lead to difficulties in the substantiation part of the question, illustrating the close connection between definitions of the properties of the functions and the substantiation routines. This ambivalence between definitions and substantiation routines could be explained by considering students’ precedent events. The focus in secondary school is on the properties of a mathematical object. However, the students are rarely required to provide a set of conditions to serve as a definition of a mathematical object. In that occasion providing more conditions and properties is not considered incorrect. This is not the case at university level. The expectation at university mathematics is to provide a minimal set of conditions which suffices for the function to be injective without reiterating the definition of function. The examples from the students’ scripts discussed above, highlight the need to alert the students to the characteristics of the routine in the context of university mathematics.

We now turn to students’ engagement with substantiation routines in examining whether the function which is defined in integers is injective or surjective.

Partial justifications regarding injectivity of $h(n) = 3n$

In the scripts of ten students, we have partial justifications regarding the injectivity of the function h defined in \mathbb{Z} . One of these scripts is in Figure 3. Student [22] claims that $h(n)$ is not surjective by providing a counterexample and then tries to prove that $h(n)$ is injective. We note here that [22] writes “surjective” instead of “injective” in the second part of the script. When discussing why the

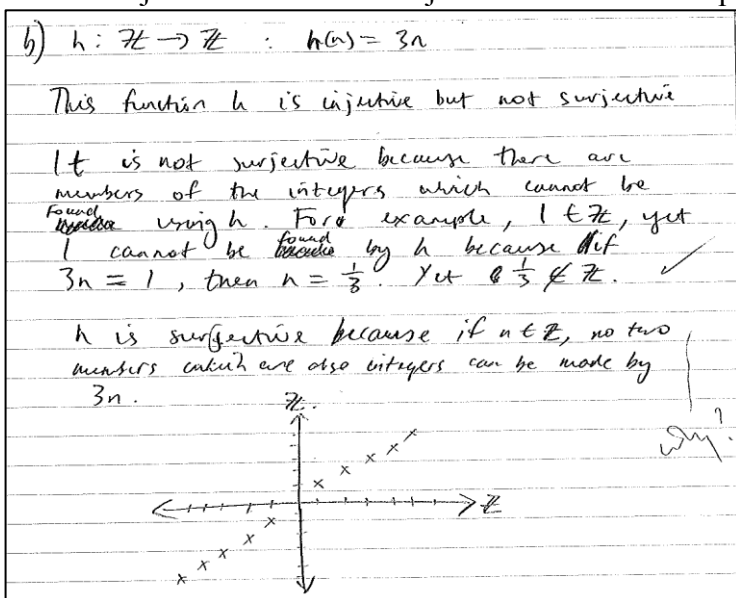


Figure 3: Student [22]’s response to (b)

function is injective s/he says that “no two numbers which are also integers can be made by $3n$ ” and provides a graph to support this claim. In the graph produced by the student, and from their answer regarding surjectivity, we can see that [22] sees $h(n)$ as a discrete function. However, the argument accompanying the graph (“ h is surjective because ... can be made by $3n$ ”) is not sufficient. The relationship between two elements of the codomain and the corresponding elements of the domain is not clear. We note, that the difficulty here is in the closing conditions of the routine.

Conflation of justifications for a functions' surjectivity in \mathbb{Z} and in \mathbb{R}

In seven student scripts, the procedure of substantiation that an integer function is or is not surjective is conflated with procedures that could be used when a function is defined in \mathbb{R} , not in \mathbb{Z} . One such example is shown in Figure 4. When substantiating, the student does not use the definition of the injective function but relies on other procedures. S/he talks about “turning points on this continuous function” and “continuously increasing”. These properties of the function are describing functions which are continuous and defined in \mathbb{R} . This conflation of functions in \mathbb{R} and \mathbb{Z} is also illustrated in the graph produced by student [11]. We observe that the arguments are based on the graph of the function, even though there is a symbol n at the x -axis, the values in the x -axis, y -axis and the line showing the function is a straight line without gaps. This engagement is

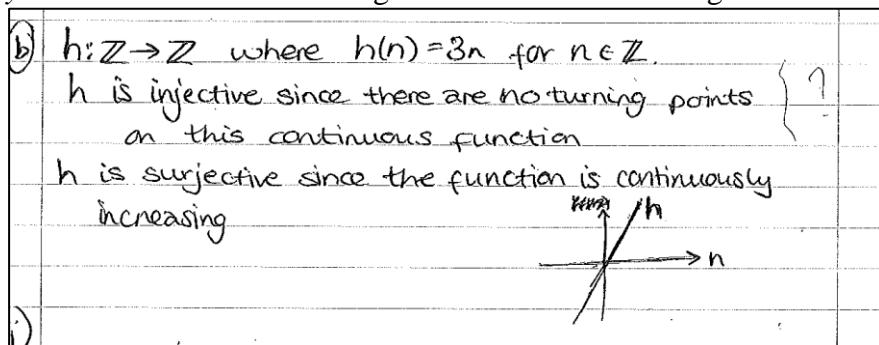


Figure 4: Student [11]’s script to (b)

ritualistic as the students seem to be drawing on familiar precedents: in the past, the majority of the tasks the students engaged with were dealing with functions in \mathbb{R} . The students are examining the function but do not take into account the domain as a significant precedent identifier regarding

the routine that they should follow. This suggests that the student does not consider the integers as the domain of this function but thinks of the function as a function in the reals, highlighting that the applicability conditions of the substantiation routine are not being examined and thus a routine being used typically for a continuous function is used here for a discrete one. We should note that there is a relationship between properties of functions in \mathbb{R} which are restricted in \mathbb{Z} . However, if this relationship is used, then this relationship between the real function and its restriction to integers should be examined as part of the justification provided by the student.

Returning to the lecturer’s quote, it seems that the phrase “they know what it means” indicates the students’ ability to decide whether a function is injective or surjective. This routine of deciding whether an object has certain properties has precedents in secondary school mathematics. However, in university mathematics, the substantiation routine is at least as important as deciding on the properties of a mathematical object, and the lecturer would like to see the substantiation of students’ decisions relying on an accurately recalled definition, which often was not the case. This new routine of substantiation is challenging for students for many reasons; in the precedent space acquired in secondary school, decision tasks do not typically require the recall or reconstruction of definitions, nor do they require substantiation based on a definition. At university level, the students must become familiar with recalling and providing definitions, focusing on a minimal set of conditions rather than providing all the properties that they can recall relating to the mathematical object in question. In the definition of injective function, they must identify the domain and the

codomain of the function, engage with different elements in the domain and with logical quantifiers that connect statements with elements from the domain and the codomain, and use all these in their substantiation.

The potency of a commognitive lens: Routines and precedent events

Our analysis suggests that, in these exam scripts, students face difficulties when they recall the definition of an injective function, particularly in relation to word use and visual mediation relating to the object of equivalence pertaining to the definition of injectivity and to conflating injective functions and the fact that a function is single valued. Furthermore, the closing conditions of the substantiation routines are not met. This may relate to differences between secondary and university mathematics in the UK context of our study. While in the context of secondary mathematics, students have engaged with examining whether a mathematical object has a property or not; they have not necessarily done so with the explicit requirements for rigour and logical connectedness expected in a Year 1 examination at university. Finally, students' ways of exploring whether function $h(n)$ is surjective indicate that they base their responses on their prior experiences which are usually with functions defined in the reals. However, these procedures are not necessarily applicable in the context of the integers and this results in conflating the discourses on the two. We see the students' responses as underlain by a commognitive conflict: integers are seen as a subset of reals and not as a domain in which there is no closure for the operation of division.

The analysed students' scripts and the comment from the lecturer illustrate the importance of the precedent events. In aiming to deritualise students' engagement with these routines, a more nuanced search for the precedent events is needed – and one which focuses clearly on the identifiers of the precedent events. This is illustrated in the comments of the lecturer as well as those student scripts which showcase ambivalent word use and visual mediation. Both of these cases highlight the difference with the precedent events in secondary school regarding the definition of a mathematical object. Then, the scripts illustrating partial justifications are also showing that students' engagement with precedent events at secondary school, where the justification whether an object satisfies a given identity does not necessarily require the depth of justification needed at university level. Finally, in the last category, the importance of the identifiers regarding the precedent events is further elaborated, as it showcases that the focus should not only be on the object (the function) but also on the domain of the function which is a substantial precedent identifier in terms of the routine required.

Our results offer further insight into students' engagement with routines of proving and recalling definitions, and the relationship of the two in the context of a question on injective and surjective functions. While previous research regards students' prior experience with functions and sets (Bansilal et al., 2017), our results highlight the importance of examining the applicability conditions of substantiation routines in different mathematical contexts (e.g. integers and reals). These results illuminate instances where the lecturers could highlight more explicitly the differences between university and secondary school discourses; and, discourses of reals and integers. Similar results to the ones reported in the paper with students' engagement with recall and substantiation routines are visible to students' responses to other examination questions (Thoma, 2018) relating to: the

definitions of reflexive, symmetric, and transitive relations and the substantiation of these properties; and, recalling Fermat's Little Theorem and applying it to find the remainder of a power (e.g., 27^{313}) divided by a prime number. We credit the commognitive lens for the insights into the student scripts that our analysis allows, and particularly the recent efforts (Lavie et al, 2018) to associate task situations, precedent events and the ways students engage with mathematical routines.

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