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A mathematician's deliberation in reaching the formal world and students' world views of the eigentheory

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In this paper we analyzed a mathematician's journals of 5-day teaching episodes on eigenvalues and eigenvectors in a first-year linear algebra course, as well as his students' responses to a survey. We employed Tall's (2013) three world model, to follow the mathematician's and his students' movements between the three worlds. The study revealed that despite the mathematician's efforts in demonstrating a more holistic view of the concepts, many students found linear algebra very abstract and gravitated more toward symbolic thinking.

Keywords: three worlds of mathematical thinking, eigentheory, concept images, reflections.

Theoretical Background

Building on Tall and Vinner's (1981) notions of *concept images* and *concept definitions*, Vinner (1991, p. 69) added that, "We assume that to acquire a concept means to form a concept image for it. To know by heart a concept definition does not guarantee understanding of the concept. To understand, so we believe, means to have a concept image". Developing these two notions further, Tall's (2010; 2013) three-world model of mathematical thinking (embodied, symbolic, and formal) endeavored to lay out an individual's mathematical journey from childhood to research mathematician. According to Tall (2010), the embodied world is based on "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognize properties and patterns...and other forms of figures and diagrams" (p. 22). In Tall's (2010, p. 22) words, "The world of operational symbolism involves practicing sequences of actions until we can perform them accurately with little conscious effort. It develops beyond the learning of procedures to carry out a given process (such as counting) to the concept created by that process (such as number)". Tall defines thinking in the formal world as that which "builds from lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure" (p. 22). The overall goal of the first author's research program is to: (a) Examine the three-world model of mathematical thinking as a possible lens for understanding how mathematical content is conveyed by a mathematician-teacher with a view toward serving students' learning; (b) Develop and extend the three-world model further. In this study we are focusing on goal (a).

Many researchers have maintained that reflection is an essential part of teaching. For example, according to Dewey (1933), reflection is "active, persistent, and careful consideration of any belief or form of knowledge in the light of the grounds that support it and the further conclusions to which it tends" (p. 9). Using Tall's (2013) model, Stewart, Thompson, and Brady (2017) investigated a

mathematician's (and co-author) movements between the three worlds while teaching algebraic topology. The instructor reported that students experienced the most difficulty in moving from the embodied world into the formal world. Believing the struggle would stimulate mathematical growth in his students, this instructor "refused to give students proofs that were pre-packaged. More specifically, he desired to provide students with intuitions and pictures that would help them understand the conceptual nature of the proof and ultimately lead them to it" (p. 2262). In a similar study, Stewart, Troup, and Plaxco (2018) examined a mathematics educator's (and co-author) movements as well as decision making moments while teaching linear algebra. In a different study, Stewart (2018) created a set of linear algebra tasks designed to help students move between the three worlds. These studies indicate that movements between the worlds is worthy of ongoing investigation.

Since the present study was focused on eigentheory, naturally recent literature on students' reasoning with eigentheory is acknowledged. While several studies have highlighted students' difficulties learning the eigenvalue and eigenvector concept, they have also provided methods whereby these difficulties can be alleviated. For example, Thomas and Stewart (2011) investigated students' conceptual understanding of eigenvalues and eigenvectors. They discovered that students appeared confident with symbolic procedures (i.e., calculating the characteristic polynomial), but not embodied ones (linking diagrams and eigenvector properties). While the students referred to "being stretched," or "unchanging direction," when describing eigenvectors, it appeared very few were able to implement this definition in context. Thomas and Stewart (2011) also noted that despite their relative strength in symbolic manipulation, the students were not able to give justifications for the change of the equation $A\mathbf{x} = \lambda\mathbf{x}$ into the equivalent equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$. They claimed that thinking about eigenvectors as invariant appeared to be a useful embodied concept for the students, even while they operated within the symbolic world. Gol Tabaghi and Sinclair (2013), and Caglayan (2015), reported that usage of dynamic geometric environments (DGEs) appeared to help students learn the eigenvectors and eigenvalues concepts, perhaps by encouraging related embodied thought as Thomas and Stewart (2011) suggested. Salgado and Trigueros (2015) claimed that via a model designed to capture student interest, the students constructed an object conception of eigenvalues and eigenvectors. They presented modeling as a powerful tool that can elucidate the problems students are having with the associated concepts, make learning the concepts easier and more approachable for the students, and supply students with a broader range of strategies with which to approach problems. These studies suggested that learning eigenvalues and eigenvectors requires students to be able to consider multiple aspects of the concept: geometric, algebraic, and structural. The presentation of eigenvectors and eigenvalues geometrically as a stretching direction and a stretching factor seems to be considered particularly useful. Thus, the research questions guiding this study were:

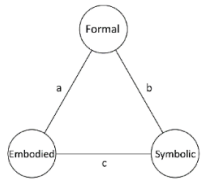
(a) When and why did the teacher decide to move between the three worlds? (b) Were the students willing to move with him and what were their challenges? (c) What were some of the challenges for the mathematician-teacher in moving the class to the formal world?

Methods

This qualitative narrative study is intended to examine a linear algebra instructor's and his students' mathematical thought processes using Tall's (2013) model. The study took place over the course of a semester at a Southwestern research university in the US. The research team consisted of a mathematician specializing in differential geometry (the instructor, postdoctoral fellow, and co-author), two mathematics educators, and an undergraduate research assistant. The analysis of the data focused on the instructor's observations, as recorded through journal entries, over a five-day period, while implementing tasks from the Inquiry-Oriented Linear Algebra (IOLA) curriculum (Wawro et. al, 2013). These 50-minute classes were structured around a sequence of four IOLA tasks using the ideas of "stretch direction" and "stretch factor" of a linear transformation to develop the formal notions of eigenvector and eigenvalue. Several of the requisite concepts, such as bases, coordinates and matrix representations of linear transformations, were covered earlier, so that the IOLA sequence could be used. The instructor was introduced to the three-world model, but taught the class as he originally planned and the research team never made any comments on how or what to teach. Introducing the three worlds was done to help him articulate his thoughts and to provide a language for reflection in his teaching journals. His dual role as a teacher as well as a researcher was indispensable in every

aspect of this research. In addition, the instructor presented Tall's (2013) framework to his class. Throughout the semester, the instructor recorded his thought

1. Recall Tall's division of mathematics roughly into three worlds: embodied, symbolic and formal (see additional sheet for brief descriptions). We can represent the three connections between the worlds schematically with the following graph. Rank the three connections in the graph from most used to least used while you were learning the concepts for this course. Explain your thought process.



2. An acquaintance is planning to take linear algebra soon, and says to you, "I've heard a lot about eigenvalues and eigenvectors, but I have no idea what they are." How would you explain this to them?

processes as well as observations on how his class reacted to his teaching. He also met with the research team regularly throughout the semester and the following summer to discuss these experiences and reflections. This allowed the research team to triangulate data via member checking with the instructor directly and additionally afforded him ample time to share a wide variety of teaching experiences, as well as his reasoning while making these decisions. To collect additional data on the student's perspective, the research team administered a survey (see Figure 1), and provided some excerpts defining the three worlds. The students (16 from a class of 30) were mainly engineering students and were already familiar with the notions of embodied, symbolic and formal. To analyze the instructor's journal, in line with a narrative study, the research team performed a retrospective analysis of the journal (Creswell, 2013) by iteratively coding the data. The team started with a combination of themes developed from the previous study (Stewart, Troup, & Plaxco, 2018) and an open coding (Strauss & Corbin, 1998) scheme to allow for the possibility of discovering new themes unique to this study. The themes were: Teaching, Students, Class Activities, Math (instructor's math, students' math), Reflection, and the Three Worlds. By instructor's math, we mean the math he was doing and talking about, and by students' math, we mean his reflections on students' mathematical abilities, and mathematical conversations in class. Under each theme we considered further fine-grained ideas and assigned codes. For Teaching, we considered IOLA tasks (TtIOLA); other tasks (TtOther); pedagogical decisions (Tpd); responses (Tr); lecture style (Tsl); group work style (Tsgw); and class discussion

Figure 1: The Survey

style (Tscd). The themes that emerged from students' survey were, movements between the three worlds as well as static presence in each world. Other themes were students' elaboration on the worlds' appearance in their own line of thinking, or the course itself, as well as the connections between the concepts.

Results

In analyzing his 5-day teaching segments, we examined the instructor's (a) movements between Tall's (2013) worlds, (b) pedagogical decisions, and (c) reflections on self and students. In the students' survey we examined (a) their views and preferences on each world, and (b) their thoughts on eigenvalues and eigenvectors and their connections to other concepts. The IOLA unit on eigentheory consists of four one-page worksheets designed to introduce the notions of eigenvalue and eigenvector through the ideas of "stretching factor" and "stretching direction" for a linear transformation.

An analysis of the instructor's journals on teaching eigenvalues and eigenvectors

Day 1: Blending embodiment and symbolism. The first IOLA task started by describing a linear transformation geometrically (see Figure 2), in terms of "stretch directions" and "stretch factors". First, the students were asked to sketch the image of a figure "Z" centered at the origin. Next, they were asked to sketch the image of two vectors and then compute the images numerically. Lastly, they were asked to produce a matrix representation of the linear transformation. This task is primarily situated in the embodied and symbolic worlds. The students must grapple with the action of the linear transformation before

In the direction along the line $y = -3x$, the transformation stretches all points by a factor of two.

In the direction along the line $y = x$, the transformation keeps all points fixed.

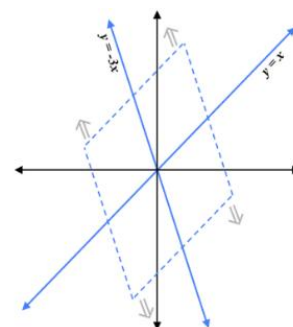


Figure 2: IOLA Task 1

considering a matrix representation.

The instructor noted that "They had a lot of trouble with this. So, after a few minutes, we went through the exercise collectively." Among the concepts that he recalled were "for every linear transformation the zero vectors gets sent to the zero vector," and "points are identified with vectors." In order to show how the image of any vector can be computed, "we converted the two vectors into linear combinations of vectors in the stretching direction. Then used the linearity of the transformation to find their images. I'm not sure if this made sense to them."

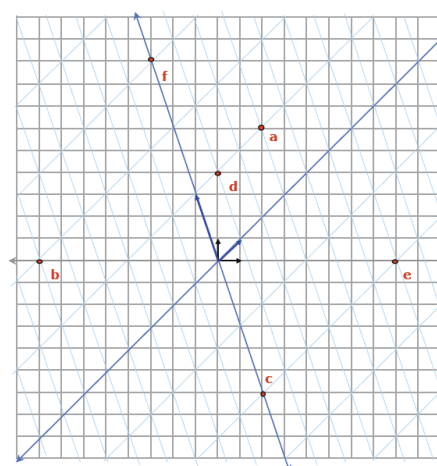


Figure 3: IOLA Task 2

Day 2: Embodied, symbolic and formal thinking. The second IOLA task continued with the same linear transformation introduced in task 1. It presents \mathbb{R}^2 with the standard coordinate grid (referred to as the "black" coordinates) overlaid on the one determined by the stretch directions (referred to as "blue"

coordinates) together with a discrete collection of points (see Figure 3). The students were asked to 1) label each point with its “black” and “blue” coordinates, 2) determine two matrices that will systematically rename points from the blue coordinate system as points in the black coordinate system and vice versa, and 3) compute the images of new points in the black and blue coordinate systems. The instructor quickly noted that “most students don’t have a facility with coordinate vectors. Even if they remembered how to compute the coordinates of a vector, they have not connected that to the picture of a grid determined by a basis.” After expressing some frustration that “Even after the class, another student asked...how was I finding the ‘blue’ coordinates,” he related that “his follow-up question was very good: what does the symbol ‘a’ represent when we write $[a]_{\text{blue}}$. This indicated that he was starting to abstract the notion of a vector and separate it from its various coordinate representations.” Prior to the class, the instructor made the pedagogical decision to present the formal definitions of eigenvalue and eigenvector after this task, and the last sentence of the journal entry for this day is, “Finally, I was able to define eigenvalue and eigenvector.”

Day 3: Reinforcing Day 2. The instructor made the pedagogical decision to use day 3 to recap and consolidate the various embodied, symbolic and formal aspects of eigentheory that the students have thus far encountered. His journal entries for this day contain almost no mention of the students, but instead focused on the mathematical connections that he aimed to convey to the students. First, he showed how the two matrix representations of the linear transformations are related by conjugation by the change of coordinates matrix. Then, to demonstrate how “The ‘stretch factors’ and ‘stretch directions’ correspond to eigenvalues and eigenvectors, respectively,” he used GeoGebra to show “how you can spot the stretch directions by moving around a vector, or by looking at what happens to the unit circle. Through several examples, he “presented the definition of eigenvalue/vector as a way to find the stretch factors and directions.” He even pushed these ideas into the realm of infinite dimensional vector spaces by considering differentiation operators on function spaces.

Day 4: Symbolic and formal thinking. He returned to the IOLA sequence with task 3 (a standard textbook exercise). For three distinct two-by-two matrices, the students were asked to 1) find the stretch factors given the stretch directions 2) find the stretch directions given the stretch factors, and 3) find both the stretch factors and directions. The instructor “expected most students launch themselves into finding the eigenvalues and eigenvectors.” But, “Instead, I was surprised to see how many were unsure where to start.” He made the pedagogical decision to guide the class and used the opportunity to present several connections.

Day 5: Symbolic and formal thinking. The fourth IOLA task considers a single linear transformation of \mathbb{R}^3 , presented as a matrix. It is found that a certain stretch factor has two stretch directions, i.e. the corresponding eigenspace is two-dimensional. The third and final part posed a rather provocative question: given that 2 and 3 are stretch factors and the former has two distinct stretch directions, could there be additional stretch factors? He observed that “every student's work that I saw was the same. To decide if there was another eigenvalue or stretch direction they all computed the characteristic polynomial to see if there was another root.” Although this was a valid approach he was eager to present a more sophisticated approach that connected to earlier concepts.

“I then presented a solution that crucially uses the fact that all three eigenvectors form a basis for \mathbb{R}^3 . I did not get very much feedback from the class on whether they were internalizing this.” In summary, the instructor used four IOLA tasks, supplementing with lecture where necessary. His goal was to build a concept image of eigentheory by presenting the fundamental notions via all three worlds of mathematical thinking. Moreover, he emphasized the formal definition of eigenvalues. For activities situated in the embodied and symbolic worlds the students were encouraged to explore independently, while the active guidance provided by the instructor was primarily to connect those worlds back to the formal. Although, many of his pedagogical decisions were dictated by time pressure, he made sure each day to bring the class closer and closer to the formal world.

An analysis of the student survey

In response to describing the meaning of eigenvalues and eigenvectors most students gave a symbolic view of the concepts (9/16). Most students also included a symbolic representation, namely the equation $A\mathbf{x} = \lambda\mathbf{x}$, with the exception of S14 and S16, who only used words. We noticed that their reasoning within this world (symbolic) were mostly reasonable.

S11: $\det(\lambda I - A) = 0$. An eigenvalue is the variable λ in the equations $\det(\lambda I - A) = 0$ or in the equation $L\mathbf{x} = \lambda\mathbf{x}$. Eigenvector is the variable \mathbf{x} in $L\mathbf{x} = \lambda\mathbf{x}$. Eigenvector of L associated to the eigenvalue λ .

S16: An eigenvector is a vector associated with a matrix. An eigenvector multiplied by this matrix equals a scalar multiple of the eigenvector. The values that eigenvector is scaled by are called eigenvalues.

The symbolic world seemed the most comfortable world for students in which to express their thoughts, and some even felt that moving from the formal world to the symbolic world “solves most problems” (S10). In general students used more positive language in describing this world. For example one student wrote, I feel as though I thrive in the symbolic world. For linear algebra, the matrices and vectors make the most sense to me in symbolic form (S14). The students’ views on the usefulness of the embodied world were mixed. For example, one student wrote, “I am able to understand easier if I can visualize something” (S15). “The embodied world really does not help me understand what is going on in problems” (S14). At one end of the spectrum, a student wrote, “my actions in the symbolic world are determined by understanding in embodied world. I refer to the embodied world to initially understand the formal world, but usually don’t go back.” (S10). Another student claimed that “the embodied world is useful for getting the big idea” (S12) and understanding the question before moving to the symbolic. In agreement another student wrote: “we took complex problems and simplified them to embody and represent a bigger picture” (S9). At the other end of the spectrum, one student went so far as to isolate the embodied world completely, writing that the “embodied and symbolic worlds” (S16), seldom interacted in her/his viewpoint. Students did not elaborate as much on the formal world. One student believed that the formal world is useful, or at least prevalent in class, but difficult to work with (S12). Others believed that “most of the class was abstract proofs” (S8), “learning concepts required a lot of formal proofs” (S13), and “linear algebra relies on a formal understanding” (S16). As for moving between the worlds, one

student wrote: “I start in symbolic and move to formal for proofs” (S14). Although, most comments were directed toward the class or the course in general, one student conveyed her/his thoughts as, “Formal is the worst that I have a problem with” (S15). In connecting between the concepts, students referred to determinants, linear operators, linear transformations, basis, special solutions, system response, system stability, and even music.

Discussion and concluding remarks

The mathematician-teacher in this study has negotiated the mathematical journey himself, and knows the path well. He viewed the formal world as the destination and wished to bring his students there. However, this was not straightforward and required time and perseverance. Since the instructor’s objective was a mathematical treatment of eigentheory, he used IOLA tasks to present a web of connections surrounding the formal definitions. Although, he valued the embodied and symbolic worlds as part of the concept image, the instructor’s goal of reaching the formal world became apparent in many of his journal writings. For example, his decision to present the definitions of eigenvalue and eigenvector at precisely the midpoint of the unit reflected their significance, representing a single idea uniting the various notions from all three worlds. A mathematical understanding of eigentheory (to him) involved primarily the definitions and how those definitions manifested themselves in the embodied, symbolic, and formal worlds. For example, he was able to think of a definition of eigenvector in symbols as described by the equation $A\mathbf{x} = \lambda\mathbf{x}$, in the embodied world as a picture of an image vector collinear with its preimage, and additionally various properties related to eigenvalues and eigenvectors in the formal world. The instructor believed the more connections between eigentheory and other linear algebraic concepts that he can convey to the students, the more robust their concept image. In Tall’s (2013) view “formal mathematics is more powerful than the mathematics of embodiment and symbolism, which are constrained by the context in which the mathematics is used” (p. 18). It was interesting to notice that even a mathematician that values all three worlds of mathematical thinking, still gravitates toward the formal world as the most important part of a mathematical concept. He seemed to consider them complementary to the formal world. Also, what appears “rote” and part of his “everyday” mode of thinking is completely foreign to the typical undergraduate linear algebra student. Hence the connections between the formal definitions and surrounding concepts that appeared so strong to the instructor were quite tenuous with the students. Although the IOLA curriculum was utilized to highlight the embodied world and draw out its connections to the other two worlds of mathematical thinking, nonetheless, the students gravitated toward symbolic thinking. We speculate on two reasons for this. First, living comfortably in all three worlds and moving between them is a big hurdle for most novice students of mathematics. Second, a significant part of students’ motivation is derived from “answering the question”. As such, students may value the world that seems most helpful in doing so. From the mathematician-instructor’s point of view the world most amenable to evaluation is the symbolic world. Hence the class setting establishes an incentive for students to remain safely in the symbolic world, and any desire to branch out must be internally motivated.

Our study suggests exploring ways of motivating students to achieve a more holistic understanding of linear algebra concepts across the three worlds. The three-world model gave the mathematician-

instructor a language that he felt accurately reflected his own thought processes. It also empowered the students by providing them a language to express their mathematical thought processes.

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