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Event-triggered control of Korteweg-de Vries equation under averaged measurements

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Abstract

This work addresses distributed event-triggered control law of 1-D nonlinear Korteweg-de Vries (KdV) equation posed on a bounded domain. Such a system, in a continuous framework, is exponentially stabilizable by a linear state feedback as a source term. Here we consider the situation where the feedback is sampled in time and piecewise averaged in space, and an event-triggering mechanism is designed to maintain stability of this infinite dimensional system. Both well-posedness of the closed-loop system and avoiding the Zeno behaviour issues are addressed. Sufficient LMI-based conditions are constructed to guarantee the regional exponential stability. Numerical examples illustrate the efficiency of the method.

Key words: Korteweg-de Vries equation, event-trigger, LMIs.

1 Introduction

In fluid mechanics, the Korteweg-de Vries (KdV) equation is a mathematical model of waves on shallow water surfaces in a rectangular channel, equation in which the effects of dispersion, dissipation and non-linearity are taken into account. When adding a diffusion term, the KdV equation becomes Korteweg-de Vries Burgers (KdVB) equation. The study of KdV/ KdVB systems has been an active research topic because of its potential applications, see e.g. [2–4,6,14,16]. In the field of automatic control, a backstepping approach has been applied in [4,6,16] for the feedback stabilization of KdV equation, and Lyapunov-based arguments have been employed to ensure the stability of the original system under the proposed control law. On the other hand, the survey paper [3] gives a detailed overview of bound-

ary controllability and internal stabilization approaches and results for the KdV equation. One can read in [2] two different approaches (from a Lyapunov functional or from an observability inequality) employed to exponentially stabilize the nonlinear KdV equation via delayed boundary damping terms.

In [14], distributed control of KdVB system has been suggested under point or averaged localised measurements in space but the proof rely strongly on the presence of a diffusion term that is missing in the KdV equation. Such distributed control was introduced for heat equation under point [9] and under averaged [10] measurements. In the latter papers, sampled-data control via time-delay approach and Lyapunov-Krasovskii functionals were studied, and the results of [9,10] were extended to event-triggered control in [22]. However, since the Lyapunov-Krasovskii functionals for sampled-data control depend on the state-derivative (see Chapter 7 of [11]), this method cannot be applied to sampled-data control of KdVB equation. So [14] considered the constant input delay case.

To the best of our knowledge, no event-triggered control of KdV equation has been studied yet. The goal of event-triggering mechanism to a sampled control law is to update the control input only at meaningful instants.

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one obtains a closed-loop system that is globally exponentially stable, as it will be shown later. In this article, we would like to address the question of the robustness of this stability with respect to the presence of both an event triggering in time and a localized averaging in space of the feedback control law. Noticing that we cannot really apply infinite dimensional feedback control law, we will consider here a finite dimensional approximation of (2.2) that still stabilizes the system (see Section 3 below).

More precisely, we will consider that the control law will be implemented in such a way that for all $x \in (0, L)$, for all $t \in [t_k, t_{k+1})$,

$$f(x, t) = -K \sum_{j=1}^N \bar{z}_j(t_k) \mathbb{1}_{\Omega_j}(x), \quad K > \lambda, \quad (2.3)$$

where the sampling times t_k are following an appropriate event trigger law to be given later, while $\{\mathbb{1}_{\Omega_j}\}_j$ are the characteristic functions of the intervals $\{\Omega_j\}_j$ covering $(0, L)$, and $\bar{z}_j(t) = \frac{1}{|\Omega_j|} \int_{\Omega_j} z(x, t) dx$. We will also consider the case that the event-triggered controller does not cover the whole domain $[0, L]$, which is distributed on some parts of subdomains (see (5.1) in Section 5 below).

2.2 Well-posedness and exponential stabilization result under (2.2)

The proof of existence and regularity of solutions for the KdV equation has been investigated in many references, in particular in the field of controllability studies and even if several results rely on the smallness of the initial and source data (e.g. [3,6]), one can find in [5] the proof of the following general result:

Lemma 1. *For any $T > 0$, $L > 0$, if $z^0 \in L^2(0, L)$ and $f \in L^1(0, T; L^2(0, L))$, then the Cauchy problem (2.1) is well posed in the space $C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$, meaning that there exists a unique solution z to the system (2.1) that satisfies, for a constant $c = c(T, L) > 0$,*

$$\begin{aligned} & \|z\|_{L^\infty(0, T; L^2(0, L))} + \|z\|_{L^2(0, T; H^1(0, L))} \\ & \leq c \|z^0\|_{L^2(0, L)} + c \|f\|_{L^1(0, T; L^2(0, L))}. \end{aligned}$$

The proof of this lemma is detailed in [5] and relies on a fixed point argument for the small time well-posedness (as also referenced and described in [3]) of the problem, that allows to handle the non-linearity $z\partial_x z$, and on clever a priori estimates of the local solution to extend arbitrarily the time frame and get a global existence and regularity result.

The proof of the well-posedness of the closed-loop system (2.1)-(2.2) stems from the same arguments and

is not detailed here. Besides, it is easy to prove its exponential stability, stated here:

Lemma 2. *Let $L > 0$, $T > 0$, $K > \lambda$ and $z^0 \in L^2(0, L)$. The closed-loop KdV system (2.1)-(2.2) is exponentially stable in the sense that*

$$\|z(\cdot, t)\|_{L^2(0, L)}^2 \leq e^{-2(K-\lambda)t} \|z^0\|_{L^2(0, L)}^2, \quad \forall t \geq 0.$$

Indeed : define the energy (that will act as a Lyapunov functional) of the solution of a KdV equation by

$$E(t) = \|z(\cdot, t)\|_{L^2(0, L)}^2, \quad \forall t \geq 0. \quad (2.4)$$

Taking the time derivative of $E(t)$ along (2.1)-(2.2), we have, for any $t \geq 0$,

$$\begin{aligned} \dot{E}(t) & \leq -2(K - \lambda) \int_0^L |z(x, t)|^2 dx - \nu |\partial_x z(0, t)|^2 \\ & \leq -2(K - \lambda) E(t) \end{aligned}$$

implying $E(t) \leq e^{-2(K-\lambda)t} E(0)$, $\forall t \geq 0$.

Furthermore, [3] gathers several internal stabilization results for nonlinear KdV equations, and specifically, the stabilization through a localized distributed internal damping $f(x, t) = -a(x)z(x, t)$ with $a \in L^\infty(0, L)$ such that $a(\cdot) \geq a_0 > \lambda$ in some subdomain ω of $(0, L)$, is actually also true, see e.g. [17,19]. However, in our study, we focus on the case described by (2.3).

As already mentioned before, in this paper we will use a Lyapunov approach to deal with an event-triggered control of the KdV equation under averaged measurements. The next section is devoted to the description of our technical setting.

3 Problem formulation and main result

We consider the following closed-loop KdV system:

$$\begin{cases} \partial_t z + z\partial_x z + \partial_x z + \nu \partial_{xxx} z - \lambda z \\ \quad = -K \sum_{j=1}^N \bar{z}_j(t_k) \mathbb{1}_{\Omega_j}(x) \\ \quad \text{in } (0, L) \times [t_k, t_{k+1}), \quad k \in \mathbb{N}, \\ z(0, t) = z(L, t) = 0, \quad \partial_x z(L, t) = 0, \quad \forall t \geq 0, \\ z(x, 0) = z^0(x), \quad \forall x \in (0, L). \end{cases} \quad (3.1)$$

where the chosen control law for (2.1) is (2.3),

$$\bar{z}_j(t_k) = \frac{1}{|\Omega_j|} \int_{\Omega_j} z(x, t_k) dx. \quad (3.2)$$

On the other hand, f defined by (2.3) can be rewritten as

$$f(x, t) = -K \sum_{j=1}^N \mathbb{1}_{\Omega_j}(x) [z(x, t) - f_j(x, t) - \rho_j(t)],$$

$$\forall x \in [0, L], \forall t \in [t_k, t_{k+1}), \forall k \in \mathbb{N} \quad (4.4)$$

where

$$f_j(x, t) = z(x, t) - \bar{z}_j(t) = z(x, t) - \frac{1}{|\Omega_j|} \int_{\Omega_j} z(x, t) dx,$$

$$\rho_j(t) = \bar{z}_j(t) - \bar{z}_j(t_k) = \frac{1}{|\Omega_j|} \int_{\Omega_j} e_k(x, t) dx.$$

Proof of Theorem 1

Writing $V_\mu(t) = \mu \int_0^L x |z(x, t)|^2 dx$ with $\mu > 0$, we define the following functional (see [2]):

$$V(t) \triangleq E(t) + V_\mu(t) = \int_0^L |z(x, t)|^2 dx + \mu \int_0^L x |z(x, t)|^2 dx. \quad (4.5)$$

First, this Lyapunov functional candidate $V(t)$ is equivalent to the energy of the system $E(t)$ in the sense that

$$E(t) \leq V(t) \leq (1 + \mu L)E(t). \quad (4.6)$$

Then, let us estimate its time derivative. For $t \in [t_k, t_{k+1})$, substituting (4.4) into (3.1) and differentiating $V(t)$ along (3.1), one gets

$$\begin{aligned} \dot{V}(t) &= \dot{E}(t) + \dot{V}_\mu(t) \\ &= 2 \int_0^L z(x, t) \partial_t z(x, t) dx + 2\mu \int_0^L x z(x, t) \partial_t z(x, t) dx \\ &= 2 \int_0^L (1 + \mu x) z(x, t) [-\nu \partial_{xxx} z(x, t) - z(x, t) \partial_x z(x, t) \\ &\quad - \partial_x z(x, t) + \lambda z(x, t) - K z(x, t)] dx \\ &\quad + 2K \sum_{j=1}^N \int_{\Omega_j} (1 + \mu x) z(x, t) [f_j(x, t) + \rho_j(t)] dx. \end{aligned}$$

Hence,

$$\begin{aligned} \dot{V}(t) &= -\nu |\partial_x z(0, t)|^2 - 3\mu\nu \int_0^L |\partial_x z(x, t)|^2 dx \\ &\quad + \mu \int_0^L |z(x, t)|^2 dx + \frac{2}{3}\mu \int_0^L z^3(x, t) dx \\ &\quad - 2(K - \lambda) \int_0^L (1 + \mu x) |z(x, t)|^2 dx \\ &\quad + 2K \sum_{j=1}^N \int_{\Omega_j} (1 + \mu x) z(x, t) [f_j(x, t) + \rho_j(t)] dx. \end{aligned}$$

Using (4.3), for any $\lambda_0 > 0$ we can deduce that

$$\begin{aligned} \dot{V}(t) &\leq \dot{V}(t) + \lambda_0 \left[\gamma E(t) + \gamma_0 E(0) e^{-2\theta t} - \|e_k(\cdot, t)\|_{L^2(0, L)}^2 \right] \\ &\leq -3\mu\nu \int_0^L |\partial_x z(x, t)|^2 dx - (2K - \mu - \lambda_0\gamma - 2\lambda) \int_0^L |z(x, t)|^2 dx \\ &\quad + \frac{2}{3}\mu \int_0^L z^3(x, t) dx - (2K - 2\lambda)\mu \int_0^L x |z(x, t)|^2 dx \\ &\quad + 2K \sum_{j=1}^N \int_{\Omega_j} (1 + \mu x) z(x, t) [f_j(x, t) + \rho_j(t)] dx \\ &\quad + \lambda_0\gamma_0 E(0) e^{-2\theta t} - \lambda_0 \|e_k(\cdot, t)\|_{L^2(0, L)}^2. \end{aligned} \quad (4.7)$$

Several estimates can now be obtained to deal with each of these terms and bring this into a quadratic form. First, Cauchy-Schwarz inequality and Sobolev's inequality (see Lemma A.3) leads to

$$\begin{aligned} \int_0^L z^3(x, t) dx &\leq \|z(\cdot, t)\|_{L^\infty(0, L)}^2 \int_0^L |z(x, t)| dx \\ &\leq L\sqrt{L} \|\partial_x z(\cdot, t)\|_{L^2(0, L)} \|z(\cdot, t)\|_{L^2(0, L)} \end{aligned} \quad (4.8)$$

Then from Lemma A.4, Wirtinger's inequality yields

$$\lambda_1 \left[\|\partial_x z(\cdot, t)\|_{L^2(0, L)}^2 - \frac{\pi^2}{L^2} \|z(\cdot, t)\|_{L^2(0, L)}^2 \right] \geq 0. \quad (4.9)$$

for any $\lambda_1 > 0$.

Moreover, since $\int_{\Omega_j} f_j(x, t) dx = 0$, from Lemma A.5, Poincaré's inequality rewrites

$$\|f_j(\cdot, t)\|_{L^2(\Omega_j)}^2 \leq \frac{\Delta^2}{\pi^2} \|\partial_x z(\cdot, t)\|_{L^2(\Omega_j)}^2,$$

bringing for any $\lambda_2 > 0$

$$\lambda_2 \sum_{j=1}^N \left[\|\partial_x z(\cdot, t)\|_{L^2(\Omega_j)}^2 - \frac{\pi^2}{\Delta^2} \|f_j(\cdot, t)\|_{L^2(\Omega_j)}^2 \right] \geq 0. \quad (4.10)$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \sum_{j=1}^N \int_{\Omega_j} \rho_j^2(t) &= \sum_{j=1}^N \rho_j^2(t) |\Omega_j| \leq \sum_{j=1}^N \frac{1}{|\Omega_j|} \left(\int_{\Omega_j} e_k(x, t) dx \right)^2 \\ &\leq \sum_{j=1}^N \int_{\Omega_j} e_k^2(x, t) dx = \int_0^L e_k^2(x, t) dx. \end{aligned} \quad (4.11)$$

Hence,

$$\lambda_0 \left[\|e_k(\cdot, t)\|_{L^2(0, L)}^2 - \sum_{j=1}^N \int_{\Omega_j} \rho_j^2(t) \right] \geq 0. \quad (4.12)$$

Set $\eta(x, t) = \text{col}\{z(x, t), f_j(x, t), \rho_j(t)\}$. Substituting (4.8) and (4.11) into (4.7), and adding (4.9), (4.10) and

(4.12) to $\dot{V}(t)$, we obtain

$$\begin{aligned} \dot{V}(t) + 2\delta V(t) &\leq \sum_{j=1}^N \int_{\Omega_j} \eta(x, t)^\top \Phi(x) \eta(x, t) \\ &- \left(3\mu\nu - \lambda_1 - \lambda_2 - \frac{2\mu}{3} L\sqrt{L} \|z(\cdot, t)\|_{L^2(0, L)} \right) \|\partial_x z(\cdot, t)\|_{L^2(0, L)}^2 \\ &- 2(K - \lambda - \delta)\mu \int_0^L x |z(x, t)|^2 dx + \lambda_0 \gamma_0 E(0) e^{-2\theta t}, \end{aligned} \quad (4.13)$$

where

$$\Phi(x) = \begin{bmatrix} \phi_{11} & K(1 + \mu x) & K(1 + \mu x) \\ * & -\lambda_2 \frac{\pi^2}{\Delta^2} & 0 \\ * & * & -\lambda_0 \end{bmatrix}$$

and $\phi_{11} = -2K + 2\lambda + \mu + \lambda_0 \gamma - \lambda_1 \frac{\pi^2}{L^2} + 2\delta$ as in (3.8). Applying Schur complement theorem [11], one gets that $\Phi(x) \prec 0$ is equivalent to

$$\phi_{11} + K^2(1 + \mu x)^2 \left(\frac{\Delta^2}{\lambda_2 \pi^2} + \lambda_0^{-1} \right) < 0,$$

that also writes

$$\begin{aligned} -2K + 2\lambda + \mu + \lambda_0 \gamma - \lambda_1 \frac{\pi^2}{L^2} \\ + 2\delta + K^2(1 + \mu x)^2 \left(\frac{\Delta^2}{\lambda_2 \pi^2} + \lambda_0^{-1} \right) < 0 \end{aligned} \quad (4.14)$$

Since we need that property for all $x \in [0, L]$, and since we have $1 \leq (1 + \mu x)^2 \leq (1 + \mu L)^2$, then it proves

$$\Phi(L) \prec 0 \implies \Phi(x) \prec 0, \forall x \in [0, L].$$

Hence, denoting $\Phi = \Phi(L)$ so that (3.6) holds, we have proved that

$$\sum_{j=1}^N \int_{\Omega_j} \eta(x, t)^\top \Phi(x) \eta(x, t) \leq 0. \quad (4.15)$$

A final step as to be performed to handle the non-quadratic estimate (4.8). Let us first assume that

$$\|z(\cdot, t)\|_{L^2(0, L)} < R, \forall t \geq 0. \quad (4.16)$$

Under assumptions (3.5)-(3.6) and (4.16), from (4.13) and (4.15) and choosing $K > \lambda + \delta$, we obtain

$$\dot{V}(t) + 2\delta V(t) \leq \lambda_0 \gamma_0 E(0) e^{-2\theta t} \leq \lambda_0 \gamma_0 V(0) e^{-2\theta t}, \forall t \geq 0.$$

Now let $\theta > \delta$. Then, for all $t \geq 0$ we can write

$$\begin{aligned} V(t) &\leq e^{-2\delta t} V(0) + \lambda_0 \gamma_0 e^{-2\delta t} V(0) \int_0^t e^{-2(\theta - \delta)s} ds \\ &\leq e^{-2\delta t} V(0) + \frac{\lambda_0 \gamma_0 V(0)}{2(\theta - \delta)} [e^{-2\delta t} - e^{-2\theta t}] \\ &\leq \left(1 + \frac{\lambda_0 \gamma_0}{2(\theta - \delta)} \right) e^{-2\delta t} V(0) - \frac{\lambda_0 \gamma_0}{2(\theta - \delta)} V(0) e^{-2\theta t} \end{aligned}$$

From (4.6) it follows that

$$\begin{aligned} E(t) &\leq \left(1 + \frac{\lambda_0 \gamma_0}{2(\theta - \delta)} \right) (1 + \mu L) e^{-2\delta t} E(0) \\ &\quad - \frac{\lambda_0 \gamma_0}{2(\theta - \delta)} E(0) e^{-2\theta t}, \end{aligned} \quad (4.17)$$

which implies (3.9).

In order to end the proof of Theorem 1, we need to prove that (4.16) holds. On the one hand, for $t = 0$, inequality (4.16) holds by hypothesis in Theorem 1, so that $E(0) < \frac{1}{\Gamma}$. On the other hand, let (4.16) be false for some $t > 0$ and let t^* be the smallest instant such that $E(t^*) \geq R^2$. Since E is continuous in time, we have $E(t^*) = R^2$ and $E(t) < R^2$ for $t \in [0, t^*)$. Therefore, the feasibility of inequality (3.5) and LMI (3.6) guarantee that (4.17) is true for all $t \in [0, t^*)$. Hence, by continuity,

$$\begin{aligned} E(t) &\leq \left(1 + \frac{\lambda_0 \gamma_0}{2\theta - 2\delta} \right) (1 + \mu L) e^{-2\delta t} E(0) \\ &\leq \left(1 + \frac{\lambda_0 \gamma_0}{2\theta - 2\delta} \right) (1 + \mu L) \frac{1}{\Gamma}, \quad \forall t \in [0, t^*]. \end{aligned}$$

The above inequality, together with the assumption (3.7), implies

$$E(t) \leq \left(1 + \frac{\lambda_0 \gamma_0}{2\theta - 2\delta} \right) (1 + \mu L) \frac{1}{\Gamma} < R^2$$

for all $t \in [0, t^*)$, which contradicts the definition of t^* . Therefore, (4.16) holds.

Note that the feasibility of the strict LMI (3.6) with $\delta = 0$ implies its feasibility with a slightly larger $\delta_0 > 0$. Therefore, if the strict LMI (3.6) holds for $\delta = 0$, then the closed-loop system is exponentially stable with a small decay rate.

Remark 4. *It must be stressed that the present Lyapunov function cannot work for the case of a simple sampled-data control under averaged measurement and that the event-triggered law is critical in the proof of stability.*

Remark 5. *Given $K > \lambda + \delta$, the LMI conditions of Theorem 1 are always feasible for small enough γ , γ_0 , Δ and large enough λ_0 such that $\lambda_0 \gamma$ is small. By Schur complement, $\Phi \prec 0$ is equivalent to (4.14) with $x = L$. The latter holds for $\mu = \lambda_1 = \gamma = \Delta = 0$, and large*

Then for any initial function $z^0 \in L^2(0, L)$ satisfying $\|z^0\|_{L^2(0, L)} < \frac{1}{\sqrt{\Gamma}}$, the closed-loop system (5.5) under the event-triggering mechanism (3.4) is exponentially stable in the sense that (3.9) holds. Moreover, if the above LMIs hold with $\delta = 0$, then the closed-loop system is exponentially stable with a small enough decay rate.

Proof. Consider $V(t)$ given by (4.5). Differentiating $V(t)$ along (5.5), for any $\lambda_0 > 0$ one gets

$$\begin{aligned} \dot{V}(t) &\leq \dot{V}(t) + \lambda_0 \left[\gamma E(t) + \gamma_0 E(0) e^{-2\theta t} - \|e_k(\cdot, t)\|_{L^2(0, L)}^2 \right] \\ &\leq -\nu |\partial_x z(0, t)|^2 - 3\mu\nu \int_0^L |\partial_x z(x, t)|^2 dx \\ &\quad + (\mu + \lambda_0 \gamma) \int_0^L |z(x, t)|^2 dx + \frac{2}{3} \mu \int_0^L z^3(x, t) dx \\ &\quad + 2\lambda \int_0^L (1 + \mu L) |z(x, t)|^2 dx - 2K \sum_{j=1}^N z^2(\bar{x}_t^j, t) |\tilde{\Omega}_j| \\ &\quad - 2K\mu \sum_{j=1}^N \int_{\Omega_j} x \mathbb{1}_{\tilde{\Omega}_j}(x) z(x, t) z(\bar{x}_t^j, t) dx \\ &\quad + 2K \sum_{j=1}^N \int_{\Omega_j} \mathbb{1}_{\tilde{\Omega}_j}(x) (1 + \mu x) z(x, t) \tilde{\rho}_j(t) dx \\ &\quad + \lambda_0 \gamma_0 E(0) e^{-2\theta t} - \lambda_0 \|e_k(\cdot, t)\|_{L^2(0, L)}^2. \end{aligned} \quad (5.12)$$

Cauchy-Schwarz's inequality yields

$$\begin{aligned} \int_{\Omega_j} [\mathbb{1}_{\tilde{\Omega}_j}(x) \tilde{\rho}_j(t)]^2 dx &= |\tilde{\Omega}_j| \tilde{\rho}_j^2(t) = \frac{1}{|\tilde{\Omega}_j|} \left[\int_{\tilde{\Omega}_j} e_k(x, t) dx \right]^2 \\ &\leq \int_{\tilde{\Omega}_j} |e_k(x, t)|^2 dx \leq \int_{\Omega_j} |e_k(x, t)|^2 dx \end{aligned}$$

so that

$$\lambda_0 \sum_{j=1}^N \int_{\Omega_j} \left[|e_k(x, t)|^2 - [\mathbb{1}_{\tilde{\Omega}_j}(x) \tilde{\rho}_j(t)]^2 \right] dx \geq 0.$$

From $\tilde{\Omega}_j \subset \Omega_j$, one has $\beta_1 > 0$ such that

$$\beta_1 \sum_{j=1}^N \int_{\Omega_j} \left[|z(x, t)|^2 - [\mathbb{1}_{\tilde{\Omega}_j}(x) z(x, t)]^2 \right] dx \geq 0$$

Wirtinger's inequality leads to (4.9) and

$$\begin{aligned} &\int_{\Omega_j} [z(x, t) - z(\bar{x}_t^j, t)]^2 dx \\ &= \int_{x_{j-1}}^{\bar{x}_t^j} [z(x, t) - z(\bar{x}_t^j, t)]^2 dx + \int_{\bar{x}_t^j}^{x_j} [z(x, t) - z(\bar{x}_t^j, t)]^2 dx \\ &\leq \frac{4(\bar{x}_t^j - x_{j-1})^2}{\pi^2} \int_{x_{j-1}}^{\bar{x}_t^j} |\partial_x z(x, t)|^2 dx + \frac{4(x_j - \bar{x}_t^j)^2}{\pi^2} \int_{\bar{x}_t^j}^{x_j} |\partial_x z(x, t)|^2 dx. \end{aligned} \quad (5.13)$$

From (5.7) and (5.13), it follows that

$$\int_{\Omega_j} [z(x, t) - z(\bar{x}_t^j, t)]^2 dx \leq \frac{4l^2}{\pi^2} \int_{\Omega_j} |\partial_x z(x, t)|^2 dx,$$

which implies

$$\beta_2 \sum_{j=1}^N \int_{\Omega_j} \left[|\partial_x z(x, t)|^2 - \frac{\pi^2}{4l^2} [z(x, t) - z(\bar{x}_t^j, t)]^2 \right] dx \geq 0$$

for some constant $\beta_2 > 0$.

Set $\tilde{\eta}(x, t) = \{z(x, t), z(\bar{x}_t^j, t), \mathbb{1}_{\tilde{\Omega}_j}(x) z(x, t), \mathbb{1}_{\tilde{\Omega}_j}(x) \rho_j(t)\}$. Using (4.8), (4.9), (5.7), (5.12) and applying S-procedure, we have

$$\begin{aligned} &\dot{V}(t) + 2\delta V(t) \\ &\leq \dot{V}(t) + 2\delta V(t) + \lambda_0 \left[\gamma E(t) + \gamma_0 E(0) e^{-2\theta t} - \|e_k(\cdot, t)\|_{L^2(0, L)}^2 \right] \\ &\quad + \lambda_1 \left[\|\partial_x z(\cdot, t)\|_{L^2(0, L)}^2 - \frac{\pi^2}{L^2} \|z(\cdot, t)\|_{L^2(0, L)}^2 \right] \\ &\quad + \lambda_0 \sum_{j=1}^N \int_{\Omega_j} \left[|e_k(x, t)|^2 - [\mathbb{1}_{\tilde{\Omega}_j}(x) \rho_j(t)]^2 \right] dx \\ &\quad + \beta_1 \sum_{j=1}^N \int_{\Omega_j} \left[|z(x, t)|^2 - [\mathbb{1}_{\tilde{\Omega}_j}(x) z(x, t)]^2 \right] dx \\ &\quad + \beta_2 \sum_{j=1}^N \int_{\Omega_j} \left[|\partial_x z(x, t)|^2 - \frac{\pi^2}{4l^2} [z(x, t) - z(\bar{x}_t^j, t)]^2 \right] dx \\ &\leq \sum_{j=1}^N \int_{\Omega_j} \tilde{\eta}(x, t)^\top \tilde{\Phi}(x) \tilde{\eta}(x, t) + \lambda_0 \gamma_0 E(0) e^{-2\theta t} \\ &\quad - \left(3\mu\nu - \lambda_1 - \beta_2 - \frac{2\mu}{3} L\sqrt{L} \|z(\cdot, t)\|_{L^2(0, L)} \right) \|\partial_x z(\cdot, t)\|_{L^2(0, L)}^2, \end{aligned}$$

where

$$\tilde{\Phi}(x) = \begin{bmatrix} \phi_{11} & \beta_2 \frac{\pi^2}{4l^2} & 0 & K(1 + \mu x) \\ * & -2K\bar{\Delta} - \beta_2 \frac{\pi^2}{4l^2} & -K\mu x & 0 \\ * & * & -\beta_1 & 0 \\ * & * & * & -\lambda_0 \end{bmatrix},$$

with ϕ_{11} as in (5.11). Thus, by Schur complement, the LMIs (5.8), (5.9) yield (3.9). \square

Fig. 5 demonstrates the time evolution of $\ln(E(t))$ for the open-loop system, the closed-loop system under continuous-time controller, and the closed-loop system under the event-triggered controller. The simulations show that the event-triggered controller improves the performance.

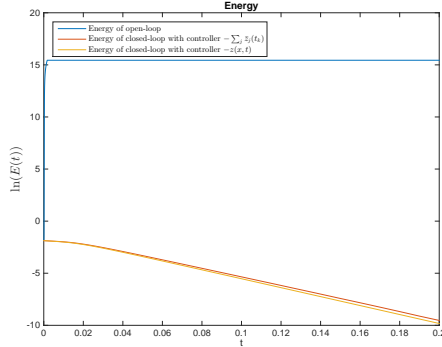


Fig. 5. $\ln(E(t))$ of the open-loop system, closed-loop system under continuous-time/event-triggered controller

Example 2: For the event-triggered control law (5.1) under averaged and localized measurements, we verify LMI conditions of Proposition 2 with $K = L = 1$, $\lambda = 0.5$, $\lambda_0 = 1$, $\nu = 0.3$, $\delta = 0.4$, $R = 0.5$, $l = 0.2$, $\bar{\Delta} = 1/3$. We find that the closed-loop system under event-triggering mechanism (3.4) with $\theta = 2$, $\gamma = 0.0013$ and $\gamma_0 = 0.02$ is exponentially stable for $\mu = 0.0235$ and for any initial values satisfying $\|z^0\|_{L^2(0,1)} < \frac{1}{\sqrt{4.5603}} \approx 0.46$. We proceed further with numerical simulations of the closed-loop KdV system under the event-triggering mechanism

$$t_{k+1} = \inf \left\{ t \geq t_k \mid \|e_k\|_{L^2(0,L)}^2 \geq 0.0013E(t) + 0.02E(0)e^{-4t} \right\}.$$

Let $x_0 = 0$, $x_1 = 0.3$, $x_2 = 0.6$, $x_3 = 0.9$ and $x_4 = 1$. Set $\tilde{x}_1 = 0.1$, $\tilde{x}_2 = 0.2$, $\tilde{x}_3 = 0.4$, $\tilde{x}_4 = 0.5$, $\tilde{x}_5 = 0.7$, $\tilde{x}_6 = 0.8$, $\tilde{x}_7 = 0.9$, $\tilde{x}_8 = 1$. The simulations show that the state of closed-loop KdV system converges to zero (see Fig. 6).

7 Conclusion

The present work discusses event-triggered control of the nonlinear KdV equation. An event-triggering mechanism has been proposed to reduce the number of control update. By constructing an appropriate Lyapunov functional, sufficient LMI-based conditions have been investigated while ensuring that the closed-loop system is regionally exponentially stable. The avoidance of Zeno behaviour is guaranteed. The presented method gives efficient tools for various event-triggered controller and observer design problems for nonlinear PDEs.

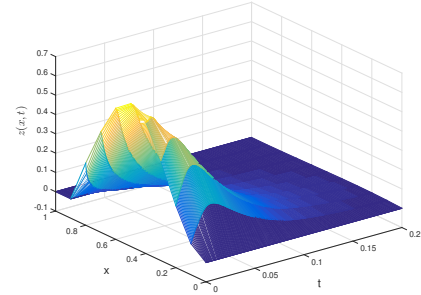


Fig. 6. State of the closed-loop system with the event-triggered control law (5.1) distributed over subdomains

Appendix A

Lemma A.3 (Sobolev embedding and inequality). *The embedding $H^1(0, L) \subset C([0, L])$ is compact and for any $g \in H_0^1(0, L)$, it holds $\|g\|_{L^\infty(0,L)} \leq \sqrt{L}\|g'\|_{L^2(0,L)}$.*

Lemma A.4 (Wirtinger inequality [12]). *Assume that $g \in H^1(0, L)$ with $g(0) = 0$ or $g(L) = 0$. Then $\|g\|_{L^2(0,L)}^2 \leq \frac{4L^2}{\pi^2}\|g'\|_{L^2(0,L)}^2$. Moreover, if $g \in H_0^1(0, L)$, then $\|g\|_{L^2(0,L)}^2 \leq \frac{L^2}{\pi^2}\|g'\|_{L^2(0,L)}^2$.*

Lemma A.5 (Poincaré inequality [10,12]). *Assume that $g \in H^1(0, L)$ with $\int_0^L g(x)dx = 0$. Then $\|g\|_{L^2(0,L)}^2 \leq \frac{L^2}{\pi^2}\|g'\|_{L^2(0,L)}^2$.*

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