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# Event-triggered control of Korteweg-de Vries equation under averaged measurements

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## Abstract

This work addresses distributed event-triggered control law of 1-D nonlinear Korteweg-de Vries (KdV) equation posed on a bounded domain. Such a system, in a continuous framework, is exponentially stabilizable by a linear state feedback as a source term. Here we consider the situation where the feedback is sampled in time and piecewise averaged in space, and an event-triggering mechanism is designed to maintain stability of this infinite dimensional system. Both well-posedness of the closed-loop system and avoiding the Zeno behaviour issues are addressed. Sufficient LMI-based conditions are constructed to guarantee the regional exponential stability. A numerical example illustrates the efficiency of the method.

*Key words:* Korteweg-de Vries equation, event-trigger, LMIs.

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## 1 Introduction

In fluid mechanics, the Korteweg-de Vries (KdV) equation is a mathematical model of waves on shallow water surfaces in a rectangular channel, equation in which the effects of dispersion, dissipation and non-linearity are taken into account. When adding a diffusion term, the KdV equation becomes Korteweg-de Vries Burgers (KdVB) equation. The study of KdV/ KdVB systems has been an active research topic because of its potential applications, see e.g. [2–4,6,14,16]. In the field of automatic control, a backstepping approach has been applied in [4,6,16] for the feedback stabilization of KdV equation, and Lyapunov-based arguments have been employed to ensure the stability of the original system under the proposed control law. On the other hand, the survey paper [3] gives a detailed overview of bound-

ary controllability and internal stabilization approaches and results for the KdV equation. One can read in [2] two different approaches (from a Lyapunov functional or from an observability inequality) employed to exponentially stabilize the nonlinear KdV equation via delayed boundary damping terms.

In [14], distributed control of KdVB system has been suggested under point or averaged localised measurements in space but the proof rely strongly on the presence of a diffusion term that is missing in the KdV equation. Such distributed control was introduced for heat equation under point [9] and under averaged [10] measurements. In the latter papers, sampled-data control via time-delay approach and Lyapunov-Krasovskii functionals were studied, and the results of [9,10] were extended to event-triggered control in [23]. However, since the Lyapunov-Krasovskii functionals for sampled-data control depend on the state-derivative (see Chapter 7 of [11]), this method cannot be applied to sampled-data control of KdVB equation. So [14] considered the constant input delay case.

To the best of our knowledge, no event-triggered control of KdV equation has been studied yet. The goal of event-triggering mechanism to a sampled control law is to update the control input only at meaningful instants.

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an event triggering in time and a localized averaging in space of the of the feedback control law. Noticing that we cannot really apply infinite dimensional feedback control law, we will consider here a finite dimensional approximation of (2.2) that still stabilizes the system (see Section 3 below).

More precisely, we will consider that the control law will be implemented in such a way that for all  $x \in (0, L)$ , for all  $t \in [t_k, t_{k+1}[$ ,

$$f(x, t) = -K \sum_{j=1}^N \bar{z}_j(t_k) \mathbb{1}_{\Omega_j}(x), \quad K > \lambda, \quad (2.3)$$

where the sampling times  $t_k$  are following an appropriate event trigger law to be given later, while  $\mathbb{1}_{\Omega_j}$  are the characteristic functions of the intervals  $\Omega_j$  covering  $(0, L)$ , and  $\bar{z}_j(t) = \frac{1}{|\Omega_j|} \int_{\Omega_j} z(x, t) dx$ .

## 2.2 Well-posedness and exponential stabilization result under (2.2)

The proof of existence and regularity of solutions for the KdV equation has been investigated in many references, in particular in the field of controllability studies and even if several results rely on the smallness of the initial and source data (e.g. [3,6]), one can find in [5] the proof of the following general result :

**Lemma 1.** *For any  $T > 0$ ,  $L > 0$ , if  $z^0 \in L^2(0, L)$  and  $f \in L^1(0, T; L^2(0, L))$ , then the Cauchy problem (2.1) is well posed in the space  $C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$ , meaning that there exists a unique solution  $z$  to the system (2.1) that satisfies, for a constant  $c = c(T, L) > 0$ ,*

$$\begin{aligned} \|z\|_{L^\infty(0, T; L^2(0, L))} + \|z\|_{L^2(0, T; H^1(0, L))} \\ \leq c \|z^0\|_{L^2(0, L)} + c \|f\|_{L^1(0, T; L^2(0, L))}. \end{aligned}$$

The proof of this lemma is detailed in [5] and relies on a fixed point argument for the local well-posedness (as also referenced and described in [3]) of the problem, that allows to handle the non-linearity  $z\partial_x z$ , and on clever a priori estimates of the local solution to extend arbitrarily the time frame and get a global existence and regularity result.

The proof of the well-posedness of the closed-loop system (2.1)-(2.2) stems from the same arguments and is not detailed here. Besides, it is easy to prove its exponential stability, stated here:

**Lemma 2.** *Let  $L > 0$ ,  $T > 0$ ,  $K > \lambda$  and  $z^0 \in L^2(0, L)$ . The closed-loop KdV system (2.1)-(2.2) is exponentially stable, and there exists  $\delta > 0$  such that*

$$\|z(\cdot, t)\|_{L^2(0, L)}^2 \leq e^{-2(K-\lambda)t} \|z^0\|_{L^2(0, L)}^2, \quad \forall t \geq 0.$$

Indeed : define the energy (that will act as a Lyapunov functional) of the solution of a KdV equation by

$$E(t) = \|z(\cdot, t)\|_{L^2(0, L)}^2, \quad \forall t \geq 0. \quad (2.4)$$

Taking the time derivative of  $E(t)$  along (2.1)-(2.2), we have, for any  $t \geq 0$ ,

$$\begin{aligned} \dot{E}(t) &\leq -2(K - \lambda) \int_0^L |z(x, t)|^2 dx - \alpha |\partial_x z(0, t)|^2 \\ &\leq -2(K - \lambda) E(t) \end{aligned}$$

implying  $E(t) \leq e^{-2(K-\lambda)t} E(0)$ ,  $\forall t \geq 0$ .

Furthermore, [3] gathers several internal stabilization results for nonlinear KdV equations, and specifically, the stabilization through a localized distributed internal damping  $f(x, t) = -a(x)z(x, t)$  with  $a \in L^\infty(0, L)$  such that  $a(\cdot) \geq a_0 > \lambda$  in  $\omega \subset (0, L)$ , is actually also true, see e.g. [18,20]. However, in our study, we focus on the case described by (2.3).

As already mentioned before, in this paper we will use a Lyapunov approach to deal with an event-triggered control of the KdV equation under averaged measurements. The next section is devoted to the description of our technical setting.

## 3 Problem formulation and main result

We consider the following closed-loop KdV system:

$$\begin{cases} \partial_t z + z\partial_x z + \partial_x z + \alpha \partial_{xxx} z - \lambda z = -K \sum_{j=1}^N \bar{z}_j(t_k) \mathbb{1}_{\Omega_j}(x), \\ \hspace{15em} \text{in } (0, L) \times [t_k, t_{k+1}[ , k \in \mathbb{N}, \\ z(0, t) = z(L, t) = 0, \partial_x z(L, t) = 0, \hspace{10em} \forall t \geq 0, \\ z(x, 0) = z^0(x), \hspace{15em} \forall x \in (0, L). \end{cases} \quad (3.1)$$

where the chosen control law for (2.1) is (2.3),

$$\bar{z}_j(t_k) = \frac{1}{|\Omega_j|} \int_{\Omega_j} z(x, t_k) dx. \quad (3.2)$$

This closed-loop system is defined under the following assumptions:

- **Space averaging:** As in [1,9,10,15], we assume that the points  $0 = x_0 < x_1 < \dots < x_N = L$  divide the interval  $[0, L]$  into  $N$  intervals  $\Omega_j = [x_{j-1}, x_j[$  covering it all. The width of each sub-interval is supposed to be upper bounded by some constant:  $0 < x_j - x_{j-1} = |\Omega_j| \leq \Delta$  and as expected, the characteristic functions  $\mathbb{1}_{\Omega_j}(x)$  are such that
 
$$\begin{cases} \mathbb{1}_{\Omega_j}(x) = 0, & x \notin \Omega_j, \\ \mathbb{1}_{\Omega_j}(x) = 1, & \text{otherwise,} \end{cases} \quad j = 1, \dots, N. \quad (3.3)$$



ness and regularity of the solution. We proceed by induction.

(i) *Initialization.* On the first time interval, (3.1) reads

$$\begin{cases} \partial_t z + z \partial_x z + \partial_x z + \alpha \partial_{xxx} z - \lambda z = -K \sum_{j=1}^N \bar{z}_j^0 \mathbb{1}_{\Omega_j}(x), \\ \quad \quad \quad \forall x \in (0, L), t \in [0, t_1], \\ z(0, t) = z(L, t) = 0, \partial_x z(L, t) = 0, \quad \quad \forall t \geq 0, \\ z(x, 0) = z^0(x), \quad \quad \quad \forall x \in (0, L), \end{cases}$$

where  $\bar{z}_j^0 = \bar{z}_j(0) = \frac{1}{|\Omega_j|} \int_{\Omega_j} z^0(x) dx$ , and  $K > \lambda$ . This is a nonlinear KdV equation with initial data  $z^0 \in L^2(0, L)$

and source term  $f = -K \sum_{j=1}^N \bar{z}_j^0 \mathbb{1}_{\Omega_j} \in L^1(0, t_1; L^2(0, L))$ .

Lemma 1 allows to conclude that there exists a unique solution  $z \in C([0, t_1]; L^2(0, L)) \cap L^2(0, t_1; H_0^1(0, L))$  to the latter system.

(ii) *Heredity.* Let us only highlight that the previously obtained solution satisfies  $z(t_1) \in L^2(0, L)$  so that system (3.1) considered on the next time interval  $[t_1, t_2[$  has an initial condition  $z(t_1) \in L^2(0, L)$  and a source term  $-K \sum_{j=1}^N \bar{z}_j^1 \mathbb{1}_{\Omega_j} \in L^1(t_1, t_2; L^2(0, L))$  where  $\bar{z}_j^1 = \bar{z}_j(t_1)$ .

Therefore, the same argument using Lemma 1 holds again and the heredity is proved similarly at any step  $k \in \mathbb{N}$ .

(iii) *Conclusion.* By induction, for any  $k \in \mathbb{N}$ ,  $z \in C([t_k, t_{k+1}]; L^2(0, L)) \cap L^2(t_k, t_{k+1}; H^1(0, L))$ . Therefore, from the extension by continuity at the instants  $t_k$ , one can conclude that (3.1) has a unique solution  $z \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$ .

We aim at showing that the event-triggering mechanism (3.4) rules out the Zeno behaviour, where an infinite number of updates may occur in a finite amount of time. It is actually sufficient to show that for a given  $T > 0$ , there exists  $\tau^* > 0$  such that all the sampling instants  $t_k \leq T$  complying to (3.4) satisfy  $t_{k+1} - t_k \geq \tau^*$ .

Let us denote by  $e_k$  the deviation from the continuous time position: for any  $x \in [0, L]$  and  $t \in (0, T)$ , there exists  $k \in \mathbb{N}$  such that  $t \in [t_k, t_{k+1})$ , and we set

$$e_k(x, t) \triangleq z(x, t) - z(x, t_k). \quad (4.1)$$

Since the solution of closed-loop system (3.1) satisfies  $z \in C([0, T]; L^2(0, L))$  and  $[0, T]$  is a compact set, this error function  $e_k$  is uniformly continuous in time with values in  $L^2(0, L)$ . This means that for any  $\epsilon > 0$  there exists  $\tau^* > 0$  such that for all  $t, s \in [0, T]$ , if  $|t - s| < \tau^*$  then we have  $\|e_k(\cdot, t) - e_k(\cdot, s)\|_{L^2(0, L)} < \epsilon$ .

Thus, the following reasoning by contraposition holds:  $\forall \epsilon > 0, \exists \tau^* > 0, \forall t, s \in [0, T]$ ,

$$\|e_k(\cdot, t) - e_k(\cdot, s)\|_{L^2(0, L)} \geq \epsilon \implies |t - s| \geq \tau^*. \quad (4.2)$$

Since  $e_k(t_k) = 0$ , we have

$$\|e_k(\cdot, t_{k+1}) - e_k(\cdot, t_k)\|_{L^2(0, L)} = \|e_k(\cdot, t_{k+1})\|_{L^2(0, L)}.$$

Next the substitution  $t \rightarrow t_{k+1}$  and  $s \rightarrow t_k$  into (4.2), together with the definition of  $t_{k+1}$  in (3.4), leads to

$$\begin{aligned} \|e_k(\cdot, t_{k+1})\|_{L^2(0, L)} &\geq \gamma E(t_{k+1}) + \gamma_0 E(0) e^{-2\theta t_{k+1}} \\ &\geq \gamma_0 \|z^0\|_{L^2(0, L)}^2 e^{-2\theta T} \end{aligned}$$

implying that  $|t_{k+1} - t_k| \geq \tau^*$ . Indeed, given  $z^0 \neq 0$ , we choose  $\epsilon = \gamma_0 \|z^0\|_{L^2(0, L)}^2 e^{-2\theta T}$  so that there exists  $\tau^* > 0$ , depending on  $z^0, \theta, \gamma_0, \gamma$  and  $T$  for which for any  $k$  such that  $t_k, t_{k+1} \in [0, T]$ , one has  $t_{k+1} - t_k \geq \tau^*$ , so that the Zeno behaviour is avoided.  $\square$

## 4.2 Regional stability analysis

Now we focus on the regional stability analysis of the closed-loop system and prove Theorem 1. Let us mention two things.

On the one hand, the event-triggering mechanism (3.4) yields that the event-triggering error function is bounded on each time sub-interval as follows:  $\forall t \in [t_k, t_{k+1})$

$$\|e_k(\cdot, t)\|_{L^2(0, L)}^2 \leq \gamma E(t) + \gamma_0 E(0) e^{-2\theta t}. \quad (4.3)$$

On the other hand,  $f(x, t)$  defined by (2.3) can be rewritten as

$$\begin{aligned} f(x, t) &= -K \sum_{j=1}^N \mathbb{1}_{\Omega_j}(x) [z(x, t) - f_j(x, t) - \rho_j(t)], \\ \forall x \in [0, L], \forall t \in [t_k, t_{k+1}), \forall k \in \mathbb{N} \end{aligned} \quad (4.4)$$

where

$$\begin{aligned} f_j(x, t) &= z(x, t) - \bar{z}_j(t) = z(x, t) - \frac{1}{|\Omega_j|} \int_{\Omega_j} z(x, t) dx, \\ \rho_j(t) &= \bar{z}_j(t) - \bar{z}_j(t_k) = \frac{1}{|\Omega_j|} \int_{\Omega_j} e_k(x, t) dx. \end{aligned}$$

### Proof of Theorem 1

Writing  $V_\mu(t) = \mu \int_0^L x |z(x, t)|^2 dx$  for  $\mu > 0$ , we define the following functional (using simplified notations [2]):

$$V(t) \triangleq E(t) + V_\mu(t) = \int_0^L |z(x, t)|^2 dx + \mu \int_0^L x |z(x, t)|^2 dx. \quad (4.5)$$

First, this Lyapunov functional candidate  $V(t)$  is equivalent to the energy of the system  $E(t)$  in the sense that

$$E(t) \leq V(t) \leq (1 + \mu L)E(t). \quad (4.6)$$

Then, let us estimate its time derivative. For  $t \in [t_k, t_{k+1})$ , substituting (4.4) into (3.1) and differentiating  $V(t)$  along (3.1), one gets

$$\begin{aligned} \dot{V}(t) &= \dot{E}(t) + \dot{V}_\mu(t) \\ &= 2 \int_0^L z(x, t) \partial_t z(x, t) dx + 2\mu \int_0^L x z(x, t) \partial_t z(x, t) dx \\ &= 2 \int_0^L (1 + \mu x) z(x, t) [-\alpha \partial_{xxx} z(x, t) - z(x, t) \partial_x z(x, t) \\ &\quad - \partial_x z(x, t) + \lambda z(x, t) - K z(x, t)] dx \\ &\quad + 2K \sum_{j=1}^N \int_{\Omega_j} (1 + \mu x) z(x, t) [f_j(x, t) + \rho_j(t)] dx. \end{aligned}$$

Hence,

$$\begin{aligned} \dot{V}(t) &= -\alpha |\partial_x z(0, t)|^2 - 3\mu\alpha \int_0^L |\partial_x z(x, t)|^2 dx \\ &\quad + \mu \int_0^L |z(x, t)|^2 dx + \frac{2}{3}\mu \int_0^L z^3(x, t) dx \\ &\quad - 2(K - \lambda) \int_0^L (1 + \mu x) |z(x, t)|^2 dx \\ &\quad + 2K \sum_{j=1}^N \int_{\Omega_j} (1 + \mu x) z(x, t) [f_j(x, t) + \rho_j(t)] dx. \end{aligned}$$

Using (4.3) we can deduce that

$$\begin{aligned} \dot{V}(t) &\leq \dot{V}(t) + [\gamma E(t) + \gamma_0 E(0) e^{-2\theta t} - \|e_k(\cdot, t)\|_{L^2(0, L)}^2] \\ &\leq -3\mu\alpha \int_0^L |\partial_x z(x, t)|^2 dx - (2K - \mu - \gamma - 2\lambda) \int_0^L |z(x, t)|^2 dx \\ &\quad + \frac{2}{3}\mu \int_0^L z^3(x, t) dx - (2K - 2\lambda)\mu \int_0^L x |z(x, t)|^2 dx \\ &\quad + 2K \sum_{j=1}^N \int_{\Omega_j} (1 + \mu x) z(x, t) [f_j(x, t) + \rho_j(t)] dx \\ &\quad + \gamma_0 E(0) e^{-2\theta t} - \|e_k(\cdot, t)\|_{L^2(0, L)}^2. \end{aligned} \quad (4.7)$$

Several estimates can now be obtained to deal with each of these terms and bring this into a quadratic form. First, Cauchy-Schwarz inequality and Sobolev's inequality (see Lemma A.3) leads to

$$\begin{aligned} \int_0^L z^3(x, t) dx &\leq \|z(\cdot, t)\|_{L^\infty(0, L)}^2 \int_0^L |z(x, t)| dx \\ &\leq L\sqrt{L} \|\partial_x z(\cdot, t)\|_{L^2(0, L)} \|z(\cdot, t)\|_{L^2(0, L)} \end{aligned} \quad (4.8)$$

Then from Lemma A.4, Wirtinger's inequality yields

$$\|z(\cdot, t)\|_{L^2(0, L)}^2 \leq \frac{L^2}{\pi^2} \|\partial_x z(\cdot, t)\|_{L^2(0, L)}^2.$$

Thus, for any  $\lambda_1 > 0$ ,

$$\lambda_1 \left[ \|\partial_x z(\cdot, t)\|_{L^2(0, L)}^2 - \frac{\pi^2}{L^2} \|z(\cdot, t)\|_{L^2(0, L)}^2 \right] \geq 0. \quad (4.9)$$

Moreover, since  $\int_{\Omega_j} f_j(x, t) dx = 0$ , from Lemma A.5, Poincaré's inequality rewrites

$$\|f_j(\cdot, t)\|_{L^2(\Omega_j)}^2 \leq \frac{\Delta^2}{\pi^2} \|\partial_x z(\cdot, t)\|_{L^2(\Omega_j)}^2,$$

bringing for any  $\lambda_2 > 0$

$$\lambda_2 \sum_{j=1}^N \left[ \|\partial_x z(\cdot, t)\|_{L^2(\Omega_j)}^2 - \frac{\pi^2}{\Delta^2} \|f_j(\cdot, t)\|_{L^2(\Omega_j)}^2 \right] \geq 0. \quad (4.10)$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \sum_{j=1}^N \int_{\Omega_j} \rho_j^2(t) &= \sum_{j=1}^N \rho_j^2(t) |\Omega_j| \leq \sum_{j=1}^N \frac{1}{|\Omega_j|} \left( \int_{\Omega_j} e_k(x, t) dx \right)^2 \\ &\leq \sum_{j=1}^N \int_{\Omega_j} e_k^2(x, t) dx = \int_0^L e_k^2(x, t) dx. \end{aligned} \quad (4.11)$$

Set

$$\eta(x, t) = \text{col}\{z(x, t), f_j(x, t), \rho_j(t)\}.$$

Substituting (4.8) and (4.11) into (4.7), and adding (4.9) and (4.10) to  $\dot{V}(t)$ , we obtain

$$\begin{aligned} \dot{V}(t) + 2\delta V(t) &\leq \sum_{j=1}^N \int_{\Omega_j} \eta(t)^\top \Phi(x) \eta(t) + \gamma_0 E(0) e^{-2\theta t} \\ &\quad - \left( 3\mu\alpha - \lambda_1 - \lambda_2 - \frac{2\mu}{3} L\sqrt{L} \|z(\cdot, t)\|_{L^2(0, L)} \right) \|\partial_x z(\cdot, t)\|_{L^2(0, L)}^2 \\ &\quad - 2(K - \lambda - \delta)\mu \int_0^L x |z(x, t)|^2 dx, \end{aligned} \quad (4.12)$$

where

$$\Phi(x) = \begin{bmatrix} \phi_{11} & K(1 + \mu x) & K(1 + \mu x) \\ * & -\lambda_2 \frac{\pi^2}{\Delta^2} & 0 \\ * & * & -1 \end{bmatrix}$$

and  $\phi_{11} = -2K + 2\lambda + \mu + \gamma - \lambda_1 \frac{\pi^2}{L^2} + 2\delta$  as in (3.7).





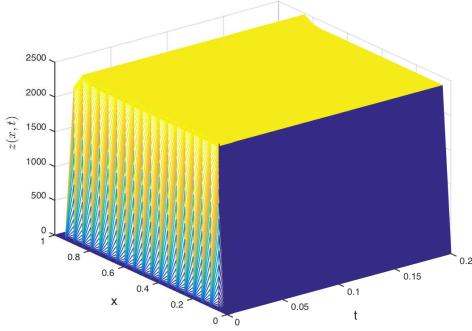


Fig. 1. State of the open-loop system with  $\lambda = 0.5$

For the event-triggered control law (2.3) under averaged measurements, we verify LMI conditions of Theorem 1 with  $K = L = 1$ ,  $\lambda = 0.5$ ,  $\alpha = 0.3$ ,  $\delta = 0.4$ ,  $\Delta = 0.1$ ,  $R = 0.5$ . We find that the closed-loop system under event-triggering mechanism (3.4) with  $\theta = 2$ ,  $\gamma = 0.00029$  and  $\gamma_0 = 0.02$  is exponentially stable for  $\mu = 0.5401$  and for any initial values satisfying  $\|z^0\|_{L^2(0,1)} < \frac{1}{\sqrt{6.1997}} \approx 0.4$ .

A finite difference method is used to illustrate the effect of the proposed event-triggered control law. The steps of space and time are chosen as 0.05 and 0.0001, respectively. Fig. 1 illustrates the evolution of the state of the open-loop KdV system. It is seen that the open-loop system is unstable. Fig. 2 illustrates the evolution of the state of the closed-loop KdV system under the event-triggering mechanism

$$t_{k+1} = \inf \left\{ t \geq t_k \mid \|e_k(\cdot, t)\|_{L^2(0,L)}^2 \geq 0.00029 \cdot E(t) + 0.02E(0)e^{-4t} \right\}.$$

with the control law (2.3) where

$$\bar{z}_j(t_k) = 10 \int_{\Omega_j} z(\zeta, t_k) d\zeta, \quad t \in [t_k, t_{k+1})$$

subject to  $x_j - x_{j-1} = |\Omega_j| = \Delta = 0.1$ . It shows that the state of closed-loop KdV system under the event-triggered controller converges exponentially to zero. Fig. 3 shows that the release time and release interval by event-triggering for  $t \in [0, 0.2]$ . Fig. 4 demonstrates the time evolution of  $\ln(E(t))$  for the open-loop system, the closed-loop system under continuous-time controller, and the closed-loop system under the event-triggered controller. The simulations show that the event-triggered controller improves the performance.

## 6 Conclusion

The present work discusses event-triggered control of the nonlinear KdV equation. An event-triggering mech-

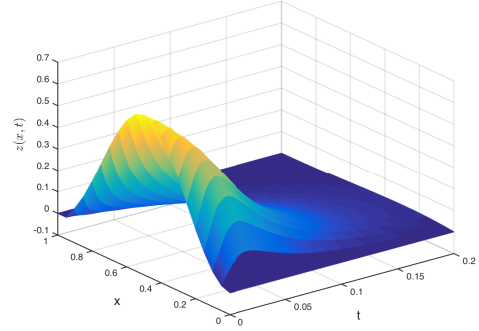


Fig. 2. State of the closed-loop system with the event-triggered control law under averaged measurements

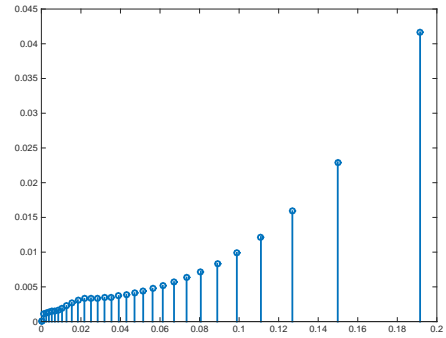


Fig. 3. Release instants and release interval by event-triggering

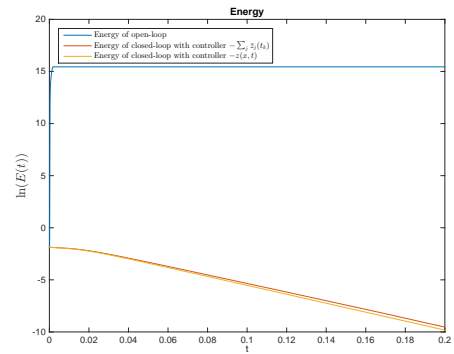


Fig. 4.  $\ln(E(t))$  of the open-loop system, closed-loop system under continuous-time/event-triggered controller

anism has been proposed to reduce the number of control update. By constructing an appropriate Lyapunov functional, sufficient LMI-based conditions have been investigated while ensuring that the closed-loop system is regionally exponentially stable. The avoidance of Zeno behaviour is guaranteed. The presented method gives efficient tools for various event-triggered controller and observer design problems for nonlinear PDEs.

## Appendix A

**Lemma A.3** (Sobolev embedding and inequality). *The embedding  $H^1(0, L) \subset C([0, L])$  is compact and for any  $g \in H_0^1(0, L)$ , it holds  $\|g\|_{L^\infty(0, L)} \leq L\|g'\|_{L^2(0, L)}$ .*

**Lemma A.4** (Wirtinger inequality [12]). *Assume that  $g \in H^1(0, L)$  with  $g(a) = 0$  or  $g(b) = 0$ . Then*

$$\|g\|_{L^2(0, L)}^2 \leq \frac{4L^2}{\pi^2} \|g'\|_{L^2(0, L)}^2.$$

Moreover, if  $g \in H_0^1(0, L)$ , then

$$\|g\|_{L^2(0, L)}^2 \leq \frac{L^2}{\pi^2} \|g'\|_{L^2(0, L)}^2.$$

**Lemma A.5** (Poincaré inequality [17]). *Assume that  $g \in H^1(0, L)$  with  $\int_0^L g(x)dx = 0$ . Then*

$$\|g\|_{L^2(0, L)}^2 \leq \frac{L^2}{\pi^2} \|g'\|_{L^2(0, L)}^2.$$

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