



**HAL**  
open science

# A Survey on Prescription of Multifractal Behavior

Stéphane Seuret

► **To cite this version:**

Stéphane Seuret. A Survey on Prescription of Multifractal Behavior. Uta Freiberg, Ben Hambly, Michael Hinz, Steffen Winter. Fractal Geometry and Stochastics VI, 76, Springer International Publishing, pp.47-70, 2021, Progress in Probability, 978-3-030-59648-4. 10.1007/978-3-030-59649-1\_3 . hal-02458904

**HAL Id: hal-02458904**

**<https://hal.science/hal-02458904v1>**

Submitted on 29 Jan 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A SURVEY ON PRESCRIPTION OF MULTIFRACTAL BEHAVIOR

STÉPHANE SEURET

ABSTRACT. Multifractal behavior has been identified and mathematically established for large classes of functions, stochastic processes and measures. Multifractality has also been observed on many data coming from Geophysics, turbulence, Physics, Biology, to name a few. Developing mathematical models whose scaling and multifractal properties fit those measured on data is thus an important issue. This raises several still unsolved theoretical questions about the prescription of multifractality (i.e. how to build mathematical models with a singularity spectrum known in advance), typical behavior in function spaces, and existence of solutions to PDEs or SPDEs with possible multifractal behavior. In this survey, we gather some of the latest results in this area.

*Dedicated to Alain Arnéodo,  
pioneer in the development of wavelet tools for data analysis.*

## 1. MULTIFRACTALITY BETWEEN PURE AND APPLIED MATHEMATICS

The notion of multifractal functions and measures can be traced back to the interest of physicists in the Hölder singularities structure in fully developed turbulence, which is described in terms of large deviations for the distribution at small scales of Mandelbrot random multiplicative cascades in [35], and in a geometric setting in the version of the so-called multifractal formalism for functions proposed by Frisch and Parisi [23], see Section 4. Another source leading to multifractal ideas is provided by the works of Henschel & Procaccia [26] and Halsey & al. [25]. Since then, multifractal analysis was further developed in dynamical systems theory and geometric measure theory, and has become a standard tool to describe the fine geometric structure of objects possessing nice invariance properties, such as self-similar and self-affine measures and functions, many classes of stochastic processes such as Lévy processes and more general Markov processes, as well as random measures emerging from multiplicative chaos theory.

Let us recall the notion of singularity spectrum of a function, leading to multifractals.

Let  $d \geq 1$  be an integer. Given a real function  $f \in L_{loc}^\infty(\mathbb{R}^d)$  and  $x_0 \in \mathbb{R}^d$ ,  $f$  is said to belong to  $\mathcal{C}^H(x_0)$ , for some  $H \geq 0$ , if there exists a polynomial  $P$  of degree at most  $\lfloor H \rfloor$  and a constant  $C > 0$  such that

$$\text{for } x \text{ close to } x_0, \quad |f(x) - P(x - x_0)| \leq C|x - x_0|^H.$$

**Definition 1.** The *pointwise Hölder exponent* of  $f \in L_{loc}^\infty(\mathbb{R}^d)$  at  $x_0$  is

$$h_f(x_0) = \sup \{H \geq 0 : f \in \mathcal{C}^H(x_0)\},$$

and  $f$  is said to have a Hölder singularity of order  $h_f(x_0)$  at  $x_0$ .

---

*Date:* December 20, 2019.

Research partly supported by the grant ANR MULTIFRACS. .

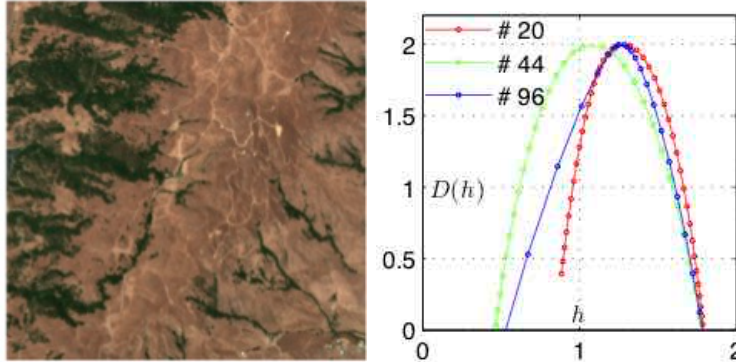


FIGURE 1. Image and estimated multifractal spectrum of different color levels of a satellite image. *Courtesy H. Wendt.*

The *singularity spectrum*  $D_f$  of  $f$  is the map:

$$D_f : H \in [0, \infty] \longmapsto \dim E_f(H), \quad \text{where } E_f(H) := \{x_0 \in \mathbb{R}^d : h_F(x_0) = H\}.$$

The notation  $\dim$  stands for the Hausdorff dimension, and by convention  $\dim \emptyset = -\infty$ .

The multifractal spectrum  $D_f$  encapsulates key information on a given function  $f$ , in particular it carries a description of the distribution of the singularities of  $f$ . But the computation of  $D_f$  often raises deep mathematical questions (for instance, it took almost 130 years to find the multifractal spectrum of the famous Riemann series  $\sum_{n=1}^{+\infty} \frac{\sin(n^2\pi x)}{n^2}$ ), and in most cases the exact value of  $D_f$  happens to be not directly accessible, neither theoretically nor numerically.

Fortunately, the notion of multifractal formalism furnishes a clever way to circumvent this difficulty and to compute the explicit value of the spectrum of large classes of measures and functions. Also, multifractal formalism provides ideas to develop numerical algorithms able to estimate  $D_f$  on real-life data. The main idea is that for very large classes of functions  $f$  (and also for other mathematical objects like measures, stochastic processes - such examples will be given in this paper),  $D_f$  is equal to the Legendre transform of the so-called  $L^q$ -spectrum  $\tau_f$  of  $f$ : this  $L^q$ -spectrum is computed directly using the values of  $f$ , and is numerically accessible. When these two quantities ( $D_f$  and the Legendre transform of  $\tau_f$ ) coincide, it is said that  $f$  satisfies the multifractal formalism. Examples of  $L^q$ -spectra for functions (and measures) based on increments, wavelet coefficients or wavelet leaders, are given in the upcoming sections (see (2), (3), (9) or (10)). The intuition that a multifractal formalism should hold is due to U. Frisch and G. Parisi, we refer the reader to Section 4 for an account on the ideas leading to this formula.

The multifractal formalism, and its validity for many mathematical models, explains the success of the multifractal approach used as classification tool in signal and image processing. Indeed, algorithms have been developed (mainly based on wavelet theory, see [38] for the original WTMM method and more recently [1] for a mathematical study of the wavelet leaders algorithm and the latest developments and algorithms based on wavelet leaders) to estimate numerically  $L^q$ -scaling functions, the stability and efficiency of these algorithms being mathematically grounded. Using these algorithms, it is now established that many data coming from Geophysics, turbulence, Physics, Biology, exhibit non-linear  $L^q$ -scaling functions, which for a given function  $f$  is interpreted thanks to the Frisch-Parisi heuristics

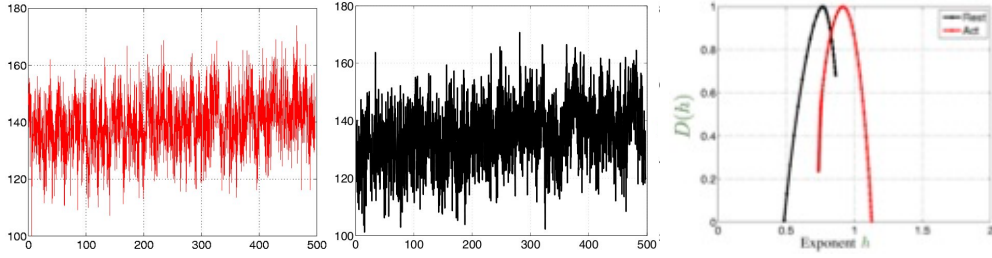


FIGURE 2. Two FMRI signals of a resting (in black) and acting (in red) patient. Comparison between their estimated multifractal spectrum. *Courtesy H. Wendt.*

as a non-trivial singularity spectrum  $D_f$  of  $f$ . Examples of data and estimated singularity spectra are plotted in Figure 1 and 2.

Resuming the above, we have on one side many mathematical objects  $f$  with non-linear  $L^q$ -scaling functions and a non-trivial singularity spectrum  $D_f$ , and on the other side an impressive quantity of signals, images and multivariate, multi-dimensional data whose estimated  $L^q$ -spectra and singularity spectra are non-trivial. It is worth asking which mathematical objects are indeed the most relevant to model the observed data, and how to create models with any reasonable multifractal behavior.

This general problematics can be understood in various ways, and raises several theoretical questions, most of them still being open:

- (i) What are the mappings  $\sigma : \mathbb{R}^+ \rightarrow [0, d] \cup \{-\infty\}$  that are admissible to be a multifractal spectrum, i.e. there exists a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $D_f = \sigma$ ?
- (ii) What are the mappings  $\sigma : \mathbb{R}^+ \rightarrow [0, d] \cup \{-\infty\}$  that are admissible to be a *homogeneous* multifractal spectrum, i.e. there exists a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such for every cube  $I \subset \mathbb{R}^d$  with non-empty interior,  $D_{f_I} = \sigma$  where  $f_I$  stands for the restriction of  $f$  on  $I$ ?
- (iii) Given an admissible (homogeneous or not) singularity spectrum  $\sigma : \mathbb{R}^+ \rightarrow [0, d] \cup \{-\infty\}$ , is there a functional space in which Baire typical functions have  $\sigma$  as singularity spectrum? Do typical functions satisfy a multifractal formalism?
- (iv) Given an admissible (homogeneous or not) singularity spectrum  $\sigma : \mathbb{R}^+ \rightarrow [0, d] \cup \{-\infty\}$ , is there a differential equation, a PDE or a stochastic (P)DE whose solution has  $\sigma$  as singularity spectrum?

These problems have their counterpart in terms of  $L^q$ -spectra: replacing everywhere  $\sigma : \mathbb{R}^+ \rightarrow [0, d] \cup \{-\infty\}$  by  $\tau : \mathbb{R} \rightarrow \mathbb{R}$ , one may ask for the admissible  $\tau$  that can be the  $L^q$ -spectrum of a function (homogeneous or not), and if such an  $L^q$ -spectrum is typical in some functional space.

The same questions arise when considering probability measures instead of functions. The main difference with the function setting is that there are additional constraints when dealing with measures, see Sections 2 and 4.1.

Although the tools used in the two contexts (functions and measures) are of different nature, a connection between the two situations is provided by the following theorem from [9], based on wavelet analysis.

**Theorem 1.** *Let  $\mu$  be a probability measure on  $\mathbb{R}^d$  such that there exist  $\alpha, C > 0$  satisfying that for every  $x \in \mathbb{R}^d$  and  $0 \leq r \leq 1$ ,  $\mu(B(x, r)) \leq Cr^\alpha$ .*

Consider the function  $F_\mu : \mathbb{R}^d \rightarrow \mathbb{R}$  whose wavelet coefficients are given by  $d_\lambda = \mu(\lambda)$  for every dyadic cube  $\lambda \in \Lambda$  (see Section 4.2 for definitions).

Then the multifractal spectra of  $\mu$  and  $F_\mu$  coincide.

Our purpose here is to provide a survey on recent results and on some open problems related to these various research directions, which combine many ideas coming from (and having applications to) geometric measure theory, functional and harmonic analysis, and real analysis, as well as ergodic theory and dynamical systems.

## 2. PRESCRIPTION OF EXPONENTS AND LOCAL DIMENSIONS

For a given mapping  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  belonging to  $L_{loc}^\infty(\mathbb{R}^d)$ , its associated pointwise Hölder exponent mapping  $h_f : x \mapsto h_f(x)$  may be very erratic, changing violently from one point to the other. Nevertheless  $h_f$  (viewed as a function) is quite well understood, as confirmed by the following theorem by S. Jaffard which provides a full characterization of  $h_f$  [27, 29]. Recall that  $C^{\log}(\mathbb{R}^d)$  is the space of those functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying that there exists  $C > 0$  such that for every  $x, y \in \mathbb{R}^d$  with  $|x - y| \leq 1/2$ ,  $|f(x) - f(y)| \leq C |\log |x - y||^{-1}$ .

**Theorem 2.** *When  $f \in C^{\log}(\mathbb{R}^d)$ , the mapping  $h_f$  is a liminf of a sequence of continuous functions.*

*Conversely, let  $H : \mathbb{R}^d \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  be a liminf of a sequence of continuous functions. There exists a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $f \in C^{\log}(\mathbb{R}^d)$ , such that for every  $x \in \mathbb{R}^d$ ,  $h_f(x) = H(x)$ .*

Let us also mention that in [5] the authors build a continuous nowhere differentiable stochastic process  $(M_x)_{x \geq 0}$  whose pointwise Hölder exponents have the most general form, i.e. the mapping  $x \mapsto h_M(x) \in (0, 1)$  can be any liminf of a sequence of continuous functions.

It is a natural question to investigate the same issues for local dimensions for measures.

**Definition 2.** Let  $\mathcal{M}(K)$  be the set of Borel probability measures on a Borel set  $K \subset \mathbb{R}^d$ . For  $\mu \in \mathcal{M}(\mathbb{R}^d)$ , the support of  $\mu$  is the set

$$\text{Supp}(\mu) = \{x \in \mathbb{R}^d : \mu(B(x, r)) > 0 \text{ for every } r > 0\}.$$

The (lower) local dimension of  $\mu$  at  $x \in \text{Supp}(\mu)$  is

$$(1) \quad h_\mu(x) = \liminf_{r \rightarrow 0^+} \frac{\log \mu(B(x, r))}{\log r}$$

and the singularity spectrum of  $\mu$  is defined for  $H \in \mathbb{R} \cup \{+\infty\}$  by

$$D_\mu(H) = \dim E_\mu(H) \quad \text{where } E_\mu(H) = \{x \in \text{Supp}(\mu) : h_\mu(x) = H\}.$$

It is common (and in many situations, relevant and important) to look at points  $x$  at which (1) turns out to be a limit (and not only a liminf). Nevertheless, in this article only lower local dimensions are considered (we will forget the term "lower" in the following), since we are interested in quantities defined for all  $x \in \text{Supp}(\mu)$ .

**Definition 3.** A function  $f$  (resp. a measure  $\mu$ ) on  $\mathbb{R}^d$  is called homogeneous (in short: HM) if the restriction of  $f$  (resp.  $\mu$ ) on any finite subcube  $I \subset \mathbb{R}^d$  has the same singularity spectrum as  $f$  (resp.  $\mu$ ).

The same definition applies to a function or measure when  $\mathbb{R}^d$  is replaced by  $[0, 1]^d$ .

One could expect that an analog of Theorem 2 should hold for local dimensions of measures. Unfortunately, the situation is not as clear, as proved by the next lemma [17].

**Lemma 3.** *Let  $\mu \in \mathcal{M}(\mathbb{R}^d)$  with a support containing a cube  $U \subset \mathbb{R}^d$ . If the mapping  $x \mapsto h_\mu(x)$  is continuous on  $U$ , then  $h_\mu$  is locally constant and equal to  $d$  on  $U$ .*

Last lemma leads to the two following open problems: What are the admissible mappings  $H : \mathbb{R}^d \rightarrow \mathbb{R}^+$  satisfying  $H = h_\mu$  for some probability measure  $\mu$ ? Given an admissible mapping  $H$ , can one explicitly build a measure  $\mu \in \mathcal{M}(\mathbb{R}^d)$  such that  $h_\mu = H$ ?

Even if all these questions are mathematically relevant and raise delicate questions (in geometric measure theory for instance), in many situations it is even more important to construct functions with prescribed singularity spectrum. This is the case in particular when trying to model real-life data, for which essentially only global quantities (like the  $L^q$ -spectrum) are accessible.

### 3. PRESCRIPTION OF MULTIFRACTAL BEHAVIOR

As expected, the prescription of singularity spectrum for functions or measures is more involved than that of exponents. Indeed, there is no obvious characterization for the admissible singularity spectrum for functions. Yet, using wavelet techniques, S. Jaffard was able to prove the following theorem [28]. Let

$$\mathcal{R} = \left\{ \sigma : \mathbb{R}^+ \rightarrow [0, d] \cup \{-\infty\} : \left\{ \begin{array}{l} \exists \text{ bounded interval } I \subset \mathbb{R}^+ \text{ and } \alpha \in [0, d] \\ \text{such that } \sigma = \alpha \mathbf{1}_I + (-\infty) \mathbf{1}_{\mathbb{R}^+ \setminus I} \end{array} \right. \right\}.$$

**Theorem 4.** *Let  $\sigma : \mathbb{R}^+ \rightarrow [0, d] \cup \{-\infty\}$  be the supremum of a countable sequence of functions  $(\sigma_n)_{n \geq 1} \in \mathcal{R}$ . Then there exists a continuous function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $D_f = \sigma$ .*

Although probably not optimal, this theorem already covers a large class of singularity spectra, certainly sufficient to mimic precisely all the singularity spectra that can be estimated on real data.

In particular, any concave mapping  $\sigma : \mathbb{R}^+ \rightarrow [0, d] \cup \{-\infty\}$  can be written as  $\sup_{n \in \mathbb{N}} \sigma_n$  for some well chosen functions  $\sigma_n \in \mathcal{R}$ , hence it is possible to build a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $D_f = \sigma$ .

The same questions were addressed for measures first in [17] and then in [7]. The admissible singularity spectra for measures are not characterized either, but when compared to spectra of functions, there are additional constraints: if  $d_\mu = \sigma$  for some  $\mu \in \mathcal{M}(\mathbb{R}^d)$ , then  $\sigma(h) \leq \min(h, d)$  (see [13, 39]).

Another surprising constraint obtained in [17] is that the support of the singularity spectrum of a 1-dimensional HM measure contains an interval. We call  $\text{Supp}(\sigma)$  the support of a function  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}$ , and by abuse of notation, if  $\sigma : \mathbb{R} \rightarrow \mathbb{R}^+ \cup \{-\infty\}$ ,  $\text{Supp}(\sigma) = \{H : \sigma(H) \geq 0\}$ .

**Proposition 5.** *For any non-atomic HM probability measure  $\mu \in \mathcal{M}(\mathbb{R})$ ,  $\text{Supp}(D_\mu) \cap [0, 1]$  is necessarily an interval of the form  $[\alpha, 1]$ , where  $0 \leq \alpha \leq 1$ .*

This proposition leads to the following notation: for  $\sigma : \mathbb{R}^+ \rightarrow [0, 1] \cup \{-\infty\}$ , consider the mapping

$$\sigma^\dagger(H) = \max(\sigma(H), 0 \cdot \mathbf{1}_{[\inf(\text{Supp}(\sigma)), \sup(\text{Supp}(\sigma))]}(H)).$$

Essentially,  $\sigma^\dagger$  fills the gaps in the support of  $\sigma$  by replacing the value  $-\infty$  by 0.

The result concerning the prescription of singularity spectrum of measures obtained in [17] is the following.

**Theorem 6.** *Let  $\sigma : \mathbb{R}^+ \rightarrow [0, 1] \cup \{-\infty\}$  be the supremum of a countable sequence of functions  $(\sigma_n)_{n \geq 1} \in \mathcal{R}$  satisfying in addition that for every  $n \geq 1$ , calling  $I_n$  the interval on which  $\sigma_n$  is not  $-\infty$ ,*

- $I_n \subset [0, 1]$ ,
- $I_n$  is closed,
- $\sigma_n(x) \leq x$  for  $x \in I_n$ .

*Then:*

- (i) *There exists  $\mu \in \mathcal{M}(\mathbb{R})$  such that  $D_\mu = \sigma$ .*
- (ii) *There exists a HM measure  $\mu \in \mathcal{M}(\mathbb{R})$  with support equal to  $[0, 1]$  such that  $D_\mu = \sigma^\dagger$ , and  $D_\mu(1) = 1$ .*

Observe that although the class of singularity spectra obtained here is quite large, only local dimensions less than 1 are dealt with, and only the one-dimensional case is covered.

Theorem 6 is completed by the result by Barral [7].

**Theorem 7.** *Let  $\sigma : \mathbb{R}^+ \rightarrow [0, d] \cup \{-\infty\}$  be an upper semi-continuous function with support included in  $[\alpha, \beta]$  for some  $0 < \alpha < \beta < +\infty$ , satisfying  $\sigma(h) \leq h$  for every  $h \in [\alpha, \beta]$ , and such that  $\sigma(h) = h$  for some  $h$ . Then there exists  $\mu \in \mathcal{M}(\mathbb{R}^d)$  such that  $D_\mu = \sigma$ .*

In the last theorem, Barral was also able to build measures that were "homogeneous" in the sense that the restriction of  $\mu$  to any bounded cube  $I \subset \mathbb{R}^d$  such that  $\mu(I) \neq 0$  has the same singularity spectrum as  $\mu$  itself. A comparison between Theorems 6 and 7 yields that (at least) in dimension 1, the measures constructed by Barral are necessarily not supported by a full interval (their support is a Cantor-like set), otherwise  $\sigma$  should be replaced by  $\sigma^\dagger$ .

Theorems 4, 6 and 7 are not entirely satisfying. Indeed,

- the construction used in Theorem 4 does not guarantee that the corresponding spectrum is homogeneous. Homogeneous spectra are yet very common (for instance, trajectories of stationary processes usually exhibit homogeneous spectra).
- in the three previous theorems, even if the prescribed spectrum is concave, the corresponding function or measure *a priori* does not satisfy a multifractal formalism.
- the functions and measures built along the proofs of Theorems 4 and 6 are not "typical" in any sense, and may essentially appear, from the modeling standpoint, as mathematical extreme toy examples.

These issues will be addressed in the next sections.

#### 4. PRESCRIPTION OF MULTIFRACTAL FORMALISMS

Let us very quickly recall the intuition by Frisch & Parisi [23], who studied the velocity  $v$  of a turbulent fluid in a bounded domain  $\Omega \subset \mathbb{R}^3$ . More precisely, inspired by the seminal works by Kolmogorov on turbulent fluids and the study of the local fluctuations of their velocity, Frisch and Parisi were interested in the moments of the increments of  $v$  defined by

$$(2) \quad \text{for every } q \in \mathbb{R}, \quad S_v(q, l) = \int_{\Omega} |v(x+l) - v(x)|^q dx.$$

For real data,  $q$  being fixed, it has been observed that when  $|l|$  becomes small,  $S_v(q, l)$  obeys a scaling law:

$$S_v(q, l) \sim |l|^{\zeta_v(q)} \quad \text{for some exponent } \zeta_v(q) \in \mathbb{R}.$$

The mapping  $q \mapsto \zeta_v(q)$  is called the scaling function of the velocity of the fluid. It can be seen that if  $v$  is modeled at small scales by a fractional Brownian motion of index  $H_0$  (as did Kolmogorov for instance), then  $\zeta_v(q)$  is linear with slope  $H_0$ . However, in the

1980's, numerical experiments for the velocity show that  $\zeta_v(q)$  is non-linear and concave. The seminal idea by Frisch and Parisi consists in interpreting this non-linearity in terms of multifractality of  $v$ , via the following heuristic argument.

Replacing Hausdorff by box dimension, and making all kind of rough approximations (i.e. assuming that limits exist, etc), for all points  $x \in \mathbb{R}^3$  at which  $h_v(x) = H$ , one has  $|v(x+l) - v(x)| \sim |l|^H$  for small  $l$ . Since  $\dim E_v(H) = D_v(H)$ , there should exist approximately  $|l|^{-D_v(H)}$  cubes of size  $l$  in the domain  $\Omega$  containing points  $x$  which are singularities of order  $H$  for the velocity  $v$ . All these intuitions lead to the estimates

$$S(q, l) = \int_{\Omega} |v(x+l) - v(x)|^q dx \sim \sum_H |l|^{qH} |l|^{-D_v(H)} |l|^3 \sim \sum_H |l|^{qH - D_v(H) + 3}.$$

When  $|l| \rightarrow 0$ , the greatest contribution is obtained for the smallest exponent:

$$\zeta_v(q) = \inf_H (qH - D_v(H) + 3).$$

The corresponding mapping  $q \mapsto \zeta_v(q)$  is called the  $L^q$ -spectrum or the scaling function of  $v$  - soon we will see more relevant formulas for  $\zeta_v(q)$  and how to define it for measures.

By inverse Legendre transform, one deduces that

$$D_v(H) = \inf_{q \in \mathbb{R}} (qH - \zeta_v(q) + 3)$$

which justifies that  $D_v$  has a concave shape.

It is striking that despite the series of crude approximations, this intuition has proved to hold true in many (if not most of) situations, after some renormalization and suitable choices for the scaling functions.

**Definition 4.** We call multifractal formalism any formula relating the singularity spectrum of a function (or a measure) to a scaling function via a Legendre transform.

For almost 30 years now, many efforts have been made to prove the validity of multifractal formalism(s) in various functional spaces, for many mathematical objects (self-similar or self-affine functions and measures) including random processes (Mandelbrot cascades, Gaussian multiplicative chaos, Lévy processes). This line of research was constantly followed and fostered by applications which gave mathematicians lots of signals and physical phenomena to study and work on, see Figures 1 and 2. In particular, stable algorithms to estimate  $L^q$ -spectra of data have been developed, furnishing to the scientific community many robustly analyzed sets of data [1].

A remaining question though lies in the existence of a functional setting in which a given multifractal behavior would be "generic". This is known after [30] as the *Frisch-Parisi conjecture*, which can be formulated as follows:

**Conjecture 1.** Given any admissible concave mapping  $\sigma : \mathbb{R}^+ \rightarrow [0, d] \cup \{-\infty\}$ , is there a functional space in which typical functions have  $\sigma$  as singularity spectrum and satisfy a multifractal formalism?

Notice that ideas leading to a multifractal formalism can also be found in thermodynamics (see [25, 26] and the large literature around thermodynamical formalism). This outlines the universality of the approach consisting in describing local fluctuations via the (Legendre transform of) global statistical quantities computed directly on the object (function, measure, random process) under consideration.

From now on, and without loss of generality, we restrict our statements to measures and functions supported in the cube  $[0, 1]^d$ .



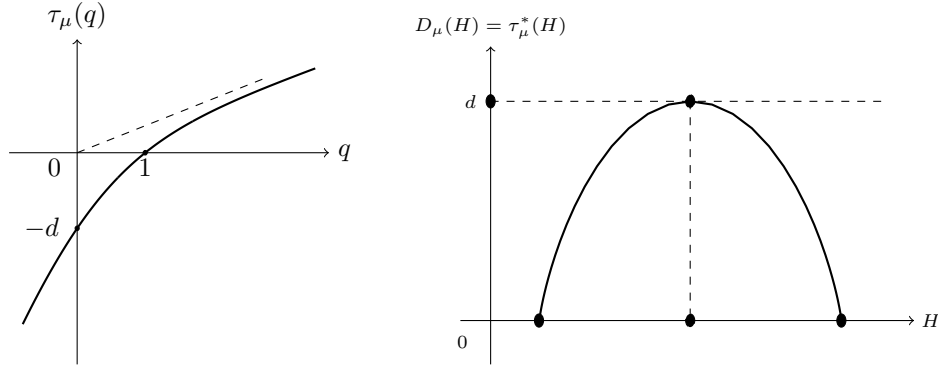


FIGURE 3. **Left:**  $L^q$ -spectrum of a measure  $\mu$  on  $[0, 1]$ . **Right:** The corresponding singularity spectrum of  $\mu$  when it satisfies a multifractal formalism.

**4.1. Prescription of multifractal formalism for measures.** In case of measures  $\mu \in \mathcal{M}([0, 1]^d)$ , the formula for the  $L^q$ -spectrum is quite standard and given by

$$(3) \quad \tau_\mu(q) = \liminf_{j \rightarrow +\infty} \frac{1}{-j} \log_2 \sum_{\lambda \in \mathcal{D}_j: \mu(\lambda) \neq 0} \mu(\lambda)^q,$$

where  $\mathcal{D}_j$  stands for the set of dyadic cubes  $\lambda_{j,k} = 2^{-j}k + [0, 2^{-j}]^d$ ,  $k \in \mathbb{Z}^d$ , of generation  $j \in \mathbb{Z}$  (i.e. dyadic cubes with side-length equal to  $2^{-j}$ ). It is easily seen that  $\tau_\mu$  is always concave, non-decreasing, and that  $-d \leq \tau_\mu(0^+) \leq \tau_\mu(1) = 0$ . In addition, the support of  $\tau_\mu$  is equal to  $\mathbb{R}$  when  $\limsup_{r \rightarrow 0^+} \frac{\log(\inf\{\mu(B(x,r)): x \in \text{Supp}(\mu)\})}{\log r} < +\infty$ , and it is  $[0, +\infty)$  when the same quantity is infinite [7].

Recall that the Legendre transform of a mapping  $\tau : \mathbb{R} \rightarrow \mathbb{R}$  (used in the previous section) is defined for  $H \geq 0$  as

$$\tau^*(H) := \inf_{q \in \mathbb{R}} (qH - \tau(q)).$$

Barral solved in [7] the following inverse problem.

**Theorem 8.** *Let  $\tau : \mathbb{R} \rightarrow \mathbb{R}$  be concave, non-decreasing, with  $-d \leq \tau(0^+) \leq \tau(1) = 0$ . There exists a probability measure  $\mu \in \mathcal{M}([0, 1]^d)$  compactly supported, such that  $\tau_\mu = \tau$  and  $\mu$  satisfies the multifractal formalism, i.e.  $D_\mu = \tau^*$ .*

See Figure 3 for an illustration.

The drawback of this first important step is that the measure constructed by Barral in [7] has again a Cantor-like set as support (so it is not fully supported on  $[0, 1]^d$ ), hence is not suitable to model any real-life signal supported by, say, an interval. The result is reinforced in the upcoming paper [10], in which we build fully supported measures satisfying a prescribed multifractal formalism, which in addition are almost-doubling in the following sense.

A Borel set function is a mapping  $\mu$  associating with every Borel set  $B \subset [0, 1]^d$  a positive real number  $\mu(B) \in [0, +\infty]$ . A Borel set function  $\mu$  is *almost-doubling* when there exists a non increasing function  $\theta : (0, 1] \rightarrow \mathbb{R}^+ \setminus \{0\}$  such that :

- $\theta(1) = 0$  and  $\lim_{r \rightarrow 0^+} \frac{\theta(r)}{\log(r)} = 0$
- there is a constant  $C \geq 1$  such that for all  $x \in [0, 1]^d$  and  $r \in (0, 1]$  one has

$$(4) \quad C^{-1}e^{-\theta(r)}\mu(B(x, r)) \leq \mu(B(x, 2r)) \leq Ce^{\theta(r)}\mu(B(x, r)).$$

When  $\theta \equiv 0$ , then  $\mu$  is said to be *doubling*.

Doubling and almost-doubling measures occupy a special place in geometric measure theory since they are easier to deal with in many situations - such properties guarantee a certain stability of the values of  $\mu$  in the sense that  $\mu(B)$  and  $\mu(B')$  have comparable values as soon as  $B$  and  $B'$  are two balls of comparable radii that are close to each other. It is thus important to investigate the possible combination of these properties with the multifractal ones, as done in the following theorem proved in [10].

**Theorem 9.** *Let  $\tau : \mathbb{R} \rightarrow \mathbb{R}$  be concave, non-decreasing, with  $-d = \tau(0^+) \leq \tau(1) = 0$ .*

*Then there exists an HM almost doubling measure  $\mu \in \mathcal{M}([0, 1]^d)$  with full support in  $[0, 1]^d$  such that  $\tau_\mu = \tau$  and  $\mu$  satisfies the multifractal formalism, i.e.  $D_\mu = \tau^*$ .*

Although Gibbs measures associated with Hölder regular potentials and smooth maps provide examples of doubling measures with non-trivial multifractal behavior, it may seem surprising that the almost doubling property (which, as said above, limits the local variations of a measure) does not constitute a constraint from the multifractal formalism standpoint: every (admissible) concave mapping can be obtained as the singularity spectrum of a compactly supported probability measure satisfying the multifractal formalism.

Theorem 9 leaves open interesting questions in ergodic theory and dynamical systems, and geometric measure theory, which to the best of our knowledge are not completely addressed yet:

- (i) Can the almost doubling property be simplified in a "simple" doubling property in Theorem 9?
- (ii) Given an almost doubling measure  $\mu$ , is there a doubling measure  $\tilde{\mu}$  with same multifractal behavior as  $\mu$ ?
- (iii) Is it possible to find a Hölder potential on a suitable dynamical system such that the associated invariant measure satisfies the multifractal formalism with a  $L^q$ -spectrum given in advance?

**Remark 1.** In Theorem 9, it is possible to impose additional conditions on the measures  $\mu$  so that the same result ( $D_\mu = \tau^*$ ) holds. One useful condition, which will be used later, is the following.

**Definition 5.** Let  $\Theta$  be the set of non decreasing functions  $\theta : \mathbb{N} \rightarrow \mathbb{R}_+^*$  such that:

- (i)  $\theta(j) = o(j)$  as  $j \rightarrow \infty$
- (ii)  $\theta(0) = 0$
- (iii) for all  $\varepsilon > 0$ , there exists  $j_\varepsilon \in \mathbb{N}$  such that for all  $j' \geq j \geq j_\varepsilon$ ,  $\theta(j') - \theta(j) \leq \varepsilon(j' - j)$ .

A measure  $\mu \in \mathcal{M}([0, 1]^d)$  (or  $\mu \in \mathcal{M}(\mathbb{R}^d)$ ) satisfies property (P) if there exist  $C, s_1, s_2 > 0$  such that:

(P1) for all  $j \in \mathbb{N}$  and  $\lambda \in \mathcal{D}_j$ , if  $\mu(\lambda) \neq 0$ , then

$$(5) \quad C^{-1}2^{-js_2} \leq \mu(\lambda) \leq C2^{-js_1}.$$

(P2) There exists  $\theta \in \Theta$  such that for all  $j, j' \in \mathbb{N}$  with  $j' \geq j$ , and all  $\lambda, \tilde{\lambda} \in \mathcal{D}_j$  such that  $\mu(\lambda) \neq 0$ ,  $\mu(\tilde{\lambda}) \neq 0$ ,  $\partial\lambda \cap \partial\tilde{\lambda} \neq \emptyset$ , and  $\lambda' \in \mathcal{D}_{j'}$  such that  $\lambda' \subset \lambda$ :

$$(6) \quad C^{-1}2^{-\theta(j)}2^{(j'-j)s_1}\mu(\lambda') \leq \mu(\tilde{\lambda}) \leq C2^{\theta(j)}2^{(j'-j)s_2}\mu(\lambda').$$

Heuristically, this last condition yields for every dyadic cube  $\lambda \in \mathcal{D}_j$  a control of the  $\mu$ -mass of the cubes  $\tilde{\lambda} \in \mathcal{D}_{\tilde{j}}$  with  $\tilde{j} \geq j$  and  $\tilde{\lambda} \subset 3\lambda$ . It is easily checked on self-similar measures satisfying an open-set condition for instance.

In [10], it is proved that there exist measures satisfying (P) for which the conclusion of Theorem 9 holds.

**4.2. Prescription of multifractal formalism for functions.** While the definition of the  $L^q$ -spectrum for measures is quite standard and intuitive, finding a suitable formula for the  $L^q$ -spectrum of functions is not straightforward. Indeed, one easily sees that equation (2) does not allow one to catch and describe the local regularity characteristics of smooth functions (with pointwise exponents greater than 1). Many alternative formulas have been proposed, and most of them are based on wavelets. It is thus useful at this point to set the notation concerning wavelets coefficients and wavelet leaders.

Let  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$  be a scaling function and consider an associated family of smooth wavelets  $\Psi = \{\psi^{(i)}\}_{i=1, \dots, 2^d-1}$  belonging to  $C^r(\mathbb{R}^d)$ , with  $r \in \mathbb{N}^*$  (for a general construction, see [37, Ch. 3]). For simplicity, we assume that  $\Phi$  and the wavelets  $\Psi$  are compactly supported [19]. For every  $j \in \mathbb{Z}$ , recall that  $\mathcal{D}_j$  is the set of dyadic cubes of generation  $j$ , i.e. if  $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$  and

$$\lambda_{j,k} := \prod_{i=1, \dots, d} [k_i 2^{-j}, (k_i + 1) 2^{-j}) \subset \mathbb{R}^d$$

then  $\mathcal{D}_j = \{\lambda_{j,k} : k \in \mathbb{Z}^d\}$  (see the beginning of Section 4.1). Further we consider the set

$$\Lambda_j = \{\lambda = (i, j, k) : k \in \mathbb{Z}^d, i \in \{1, \dots, 2^d - 1\}\},$$

and  $\Lambda = \bigcup_{j \in \mathbb{Z}} \Lambda_j$ . By abuse of notation,  $\lambda \in \Lambda_j$  will still be called a dyadic cube of generation  $j$  and identified with  $\lambda = \lambda_{j,k} \in \mathcal{D}_j$ .

For every cube  $\lambda = (i, j, k) \in \Lambda$ , we denote by  $\psi_\lambda$  the function  $x \mapsto \psi^{(i)}(2^j x - k)$ . The set of functions  $2^{dj/2} \psi_\lambda$ ,  $j \in \mathbb{Z}$ ,  $\lambda \in \Lambda_j$ , forms a Hilbert basis of  $L^2(\mathbb{R}^d)$ , so that every  $f \in L^2(\mathbb{R}^d)$  can be expanded as

$$f = \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} d_\lambda \psi_\lambda, \quad \text{with } d_\lambda = \int_{\mathbb{R}^d} 2^{dj} \psi_\lambda(x) f(x) dx,$$

where equality holds in  $L^2$  (we will work with smooth functions, so equality will also hold pointwise). Observe that we choose an  $L^\infty$  normalization for the so-called *wavelet coefficients*  $(d_\lambda)_{\lambda \in \Lambda}$  of  $f \in L^2(\mathbb{R}^d)$  (more generally, of  $f \in L^p(\mathbb{R}^d)$  for some  $p \in [1, \infty]$ ). For  $f \in L^2(\mathbb{R}^d)$ , define also for  $k \in \mathbb{Z}^d$

$$(7) \quad \beta(k) = \int_{\mathbb{R}^d} f(x) \Phi(x - k) dx.$$

Finally, for a function  $f \in L^p(\mathbb{R}^d)$  with  $p \in [1, \infty]$  whose wavelet coefficients are denoted by  $(d_\lambda)_{\lambda \in \Lambda}$ , the wavelet leader associated with  $\lambda \in \mathcal{D}_j$  is

$$d_\lambda^L = \sup_{\lambda' \in \Lambda, \lambda' \subset 3\lambda} |d_{\lambda'}|,$$

where for  $\lambda \in \mathcal{D}_j$ ,  $3\lambda$  stands for the cube with same center as  $\lambda$  and radius  $\frac{3}{2}2^{-j}$  (it is the cube that contains  $\lambda$  as well as its  $2^d - 1$  neighbors in  $\mathcal{D}_j$ ). While wavelet coefficients are usually sparse (only a few coefficients carry the important information about  $f$ ), wavelet leaders possess a strong hierarchical structure since  $0 \leq d_{\lambda'}^L \leq d_\lambda^L$  when  $\lambda' \subset \lambda$ .

**Remark 2.** Although the notations for wavelet coefficients and wavelet leaders do not mention the function  $f$ , they highly depend on  $f$  and we should never forget about it!

Wavelet coefficients and wavelet leaders characterize the pointwise Hölder exponents: indeed, if  $f \in C^\epsilon(\mathbb{R}^d)$  for some  $\epsilon > 0$ , then for every  $x_0 \in [0, 1]^d$  one has

$$(8) \quad h_f(x_0) = \liminf_{j \rightarrow \infty} \frac{\log d_{\lambda_j(x_0)}^L}{\log(2^{-j})},$$

where  $\lambda_j(x_0)$  is the unique cube  $\lambda \in \mathcal{D}_j$  that contains  $x_0$  (see [31]).

It was quite clear from the beginning that a formula based on increments like (2) was not stable neither mathematically nor numerically. To circumvent this difficulty, the idea of introducing wavelets (whose computation requires local means, bringing simultaneously a numerical stability crucial for applications and a natural connection with characterizations of standard functional spaces, see Section 5) was introduced by Alain Arnéodo and his collaborators. Two formulations are nowadays recognized to be the most relevant:

- Formula based on wavelets:

$$(9) \quad T_f(q, j) = \sum_{\lambda \in \Lambda_j: d_\lambda \neq 0} |d_\lambda|^q \longrightarrow \eta_f(q) = \liminf_{j \rightarrow +\infty} \frac{\log_2 T_f(q, j)}{-j}.$$

- Formula based on wavelet leaders:

$$(10) \quad L_f(q, j) = \sum_{\lambda \in \mathcal{D}_j: d_\lambda^L \neq 0} |d_\lambda^L|^q \longrightarrow L_f(q) = \liminf_{j \rightarrow +\infty} \frac{\log_2 L_f(q, j)}{-j}.$$

Even if wavelets brought some stability in the computations, wavelet leaders are now recognized as the most efficient, relevant and numerically exploitable measurements of local and global regularity. In particular, the hierarchical structure of wavelet leaders (i.e.  $0 \leq d_\lambda \leq d_{\lambda'}$  as soon as  $\lambda \subset \lambda'$ ) makes all computations easier and more stable [1].

**Definition 6.** The wavelet multifractal formalism WMF (*resp.* wavelet leader multifractal formalism WLMF) is satisfied for a function  $f$  on an interval  $J \subset \mathbb{R}^+$  when  $D_f(H) = (\eta_f)^*(H)$  (*resp.*  $D_f(H) = (L_f)^*(H)$ ) for every  $H \in J$ .

We also say that a function  $f$  satisfies the weak wavelet leader multifractal formalism (weak-WLMF) on an interval  $J \subset \mathbb{R}^+$  when there exists an increasing sequence  $(j_n)_{n \geq 1}$  of integers such that if  $\tilde{L}_f(q) = \liminf_{n \rightarrow +\infty} \frac{\log_2 L_f(q, j_n)}{-j_n}$ , then  $D_f(H) = (\tilde{L}_f)^*(H)$  for every  $H \in J$ .

**Remark 3.** The above definition of formalisms depends *a priori* on the chosen wavelets  $\Psi$ . Actually it does not depend on  $\Psi$  in the increasing part of the multifractal spectrum [31], but it does in the decreasing part. For simplicity, we do not mention this dependence in the notations.

Let  $\mathcal{S}_d$  be the set of admissible singularity spectra for functions satisfying a multifractal formalism, i.e.

$$(11) \quad \mathcal{S}_d = \left\{ \sigma : \mathbb{R}^+ \rightarrow [0, d] \cup \{-\infty\} : \left\{ \begin{array}{l} \sigma \text{ is compactly supported in } (0, +\infty), \text{ concave,} \\ \text{with maximum equal to } d. \end{array} \right\} \right\}.$$

We are now able to state the result on multifractal formalism prescription for functions.

**Theorem 10.** *For every mapping  $\sigma \in \mathcal{S}_d$ , there exists a function  $f \in L^2(\mathbb{R}^d)$  satisfying the WLMF and whose singularity spectrum is equal to  $\sigma$ .*

*Proof.* Observe that if a function  $f$  has its wavelet coefficients  $d_\lambda$  given by  $\mu(\lambda)$  for some probability measure  $\mu \in \mathcal{M}([0, 1]^d)$ , then for every choice of  $\alpha, \beta > 0$ , the function  $f_{\alpha, \beta}$  whose wavelet coefficients are  $\tilde{d}_\lambda := d_\lambda^\alpha 2^{-j\beta}$  satisfies

$$\text{for every } H \geq 0, \quad D_{f_{\alpha, \beta}}(H) = D_f \left( \frac{H - \beta}{\alpha} \right).$$

This simply follows from (8) and the fact that  $h_{f_{\alpha, \beta}}(x_0) = \alpha h_f(x_0) + \beta$  for all  $x_0$ .

Let  $\sigma : \mathbb{R}^+ \rightarrow [0, d] \cup \{-\infty\} \in \mathcal{S}_d$  be a mapping satisfying the conditions to be a singularity spectrum of a function satisfying a multifractal formalism.

Let  $\alpha, \beta$  be two strictly positive real numbers such that the mapping  $\sigma_{\alpha, \beta}(H) = \sigma(\alpha H + \beta)$  satisfies  $\sigma_{\alpha, \beta}(H) \leq H$  and there exists  $H_0 > 0$  such that  $\sigma_{\alpha, \beta}(H_0) = H_0$ . The existence of  $(\alpha, \beta)$  is an exercise (notice that  $(\alpha, \beta)$  need not be unique).

Theorem 9 provides us with a measure  $\mu$  satisfying the multifractal formalism for measures and  $D_\mu = \sigma_{\alpha, \beta}$ .

Then, Theorem 1 yields that the function  $F_\mu$  whose wavelet coefficients are given by  $d_\lambda = \mu(\lambda)$  has the same singularity spectrum as  $\mu$ . In addition, comparing (3) with (10), and using the hierarchical structure of the measure (i.e.  $\mu(\lambda') \leq \mu(\lambda)$  whenever  $\lambda' \subset \lambda$ ), one sees that  $\tau_\mu(q) = L_{F_\mu}(q)$  for every  $q \in \mathbb{R}$ , hence  $F_\mu$  satisfies the WLMF.

Finally, using the first remark of this proof, the function  $F$  whose wavelet coefficients equal  $\mu(\lambda)^\alpha 2^{-j\beta}$  has its singularity spectrum equal to  $\sigma$  and satisfies the WLMF.  $\square$

We thus have a complete answer for the prescription of multifractal formalism for functions. But at this point, one may have the feeling that the functions we built are mathematical toy examples. The purpose of the last sections is to explain that for any choice of concave admissible mapping  $\sigma$ , there are natural functional spaces in which typical functions have exactly  $\sigma$  as singularity spectrum. This confirms and strengthens the overall presence of multifractals in most of science fields, and reinforces the position of multifractal machinery as legitimate tool in signal processing and data analysis.

## 5. TYPICAL MULTIFRACTAL BEHAVIOR IN CLASSICAL FUNCTIONAL SPACES

As emphasized above, it is possible to find mathematical models that mimic large classes of multifractal behavior, in particular including all concave singularity spectra. This last part of the results is key, since for real-life data (multi-dimensional and/or multivariate signals, images, ...) only estimates for the  $L^q$ -spectrum are numerically accessible (based on log-log plots on a well-chosen range of scales). Indeed, the standard paradigm is to assume that the discrete data  $f$  (say, a signal) is obtained from discrete samples of a mathematical model obeying a multifractal formalism, and to consider that the Legendre transform of the estimated  $L^q$ -spectrum contains relevant information regarding the distribution of the singularities of  $f$  (somehow extrapolating on Frisch-Parisi heuristics). This Legendre transform is thus viewed as an "approximation" of the singularity spectrum of the data, although the meaning of the singularity spectrum of a discretized signal is not made precise. The obtained estimated singularity spectrum of the data  $f$  possesses various characteristics (values of the largest and the smallest exponents, locations of the maximum, curvature of the concave spectrum at its maximum,...) which are then used as classification tools between numerous samples of a physical, medical,... phenomenon. This has proven to be relevant in various fields going from medicine (heart-beat rate and X-ray analysis) and turbulence [32] to, recently, more surprising areas (paintings analysis [2], text analysis [33]).

Inspired by these applications, it is thus key to investigate whether the mathematical objects we regularly meet satisfy a multifractal formalism (so that all these heuristics described above lie on solid mathematical grounds). In this survey, we focus on "typical" objects in the sense of Baire: in a Baire space  $E$ , a property  $\mathcal{P}$  of elements  $x \in E$  is *typical* or *generic* when the set  $\{x \in E : x \text{ satisfies } \mathcal{P}\}$  is a residual set, i.e. its complement is included in a first Baire category set (a union of countably many nowhere dense sets in  $E$ ).

Regularity properties of typical functions have been explored since the pioneer works of Banach [6] or Mazurkiewicz [36] for instance. The seminal result concerning multifractal properties of typical functions is due to Buczolich and Nagy, who proved the following [14].

**Theorem 11.** *Let  $\text{Mon}([0,1])$  be the set of continuous monotone functions  $f : [0,1] \rightarrow \mathbb{R}$  equipped with the supremum norm of functions. Typical functions in  $\text{Mon}([0,1])$  are multifractal with singularity spectrum equal to  $D_f(H) = H \cdot \mathbf{1}_{[0,1]}(H) + (-\infty) \cdot \mathbf{1}_{(1,+\infty]}(H)$ .*

Theorem 11 was the starting point of an abundant literature on the subject, examples of which are given in the following. The method consists first in finding an upper bound for the singularity spectrum of all functions in  $\text{Mon}([0,1])$  (here, the diagonal  $\sigma(H) = H$ ), then an explicit function  $F_{typ}$  whose local behavior is the one suspected to be typical, and finally to construct a countable sequence  $(A_n)_{n \geq 1}$  of sets of functions, dense in  $\text{Mon}([0,1])$ , which are for a given  $n$ , really close to  $F_{typ}$  at a given scale (depending on  $n$ ). If the parameters are correctly settled, the intersection of the  $(A_n)_{n \geq 1}$  will be the set of typical functions with multifractal behavior similar to that of  $F_{typ}$ .

The proof is based on a careful analysis on local oscillations of functions, and simultaneous constructions of Cantor-like sets  $E_f(H)$  carrying the sets of points with pointwise Hölder exponent equal to  $H$ , for every  $f \in \bigcap_{n \geq 1} A_n$ .

After Theorem 11, the first direction consisted in exploring the typical behavior in other standard functional spaces. The first, spectacular, results were obtained by Jaffard [30], who implemented the same strategy as [14] but added wavelet tools to deal with the important examples of Hölder and Besov spaces.

**Theorem 12.** *1) Let  $\alpha > 0$  and consider the space  $C^\alpha([0,1]^d)$  of  $\alpha$ -Hölder functions on  $[0,1]^d$ . Typical functions in  $C^\alpha([0,1]^d)$  are monofractal and satisfy*

$$D_f(H) = d \cdot \mathbf{1}_{\{\alpha\}}(H) + (-\infty) \cdot \mathbf{1}_{[0,+\infty] \setminus \{\alpha\}}(H).$$

*2) Let  $p \geq 1$  and  $s > d/p$ , and consider the Besov space  $B_q^{s,p}([0,1]^d)$ . Typical functions in  $B_q^{s,p}([0,1]^d)$  are multifractal and satisfy*

$$D_f(H) = p(H - (s - d/p)) \cdot \mathbf{1}_{[s-d/p,s]}(H) + (-\infty) \cdot \mathbf{1}_{[0,+\infty] \setminus [s-d/p,s]}(H).$$

*In addition, typical functions satisfy the WLMF.*

See figure 4 for an illustration.

Theorems 11 and 12 are striking since they underline the preeminence of multifractal properties for "everyday" functions. Jaffard also described the multifractal behavior of typical functions belonging to countable intersections of Besov spaces, leading to a first answer to the Frisch-Parisi Conjecture. Although these results were a giant step in the domain, only increasing singularity spectra with restricted shapes can be obtained and the typical functions do not obey a satisfactory multifractal formalism. Let us also mention that Besov spaces with indices  $s < d/p$  were also considered in [30].

Other directions have been investigated. The most natural one concerns probability measures: typical multifractal properties were explored in [15] for measures supported on

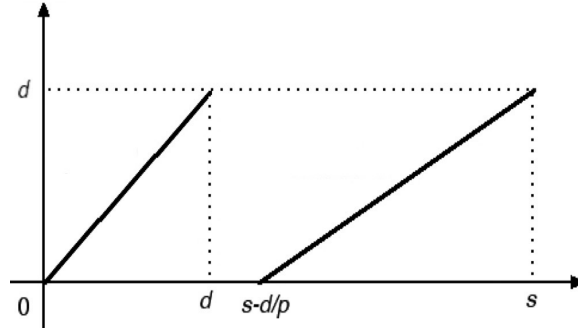


FIGURE 4. Typical singularity spectra of measures supported on  $[0, 1]^d$  (**Left**) and of functions in  $B_q^{s,p}(\mathbb{R}^d)$  (**Right**).

$[0, 1]^d$  and these results were extended by Bayart [11] for measures supported on general compact sets.

**Theorem 13.** *Let  $K \subset \mathbb{R}^d$  be a compact set, and let  $\mathcal{M}(K)$  be the set of probability measures on  $K$ .*

*A typical measure  $\mu \in \mathcal{M}(K)$  satisfies for any  $H \in [0, \dim(K))$ ,  $D_\mu(H) = H$ .*

*In addition, when the  $\dim(K)$ -Hausdorff measure of  $K$  is strictly positive, then typical measures satisfy  $D_\mu(\dim(K)) = \dim(K)$  and obey the multifractal formalism.*

Another extension of typical monotone functions is provided by the set of monotone increasing in several variables: A function  $f : [0, 1]^d \rightarrow \mathbb{R}$  is continuous monotone increasing in several variables (in short: MISV) if for all  $i \in \{1, \dots, d\}$ , the coordinate functions

$$f^{(i)}(t) = f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_d)$$

are continuous monotone increasing. The set of MISV functions is denoted by  $\text{MISV}^d$ .

With Z. Buczolich, we also investigated the set  $\mathcal{CC}^d$  of continuous convex functions  $f : [0, 1]^d \rightarrow \mathbb{R}$ .

Equipped with the supremum norm  $\|\cdot\|$ ,  $\mathcal{CC}^d$  and  $\text{MISV}^d$  are separable complete metric spaces. In [16] and [18] we obtained the following results.

**Theorem 14.** 1) *Typical functions in  $\text{MISV}^d$  satisfy*

$$D_f(H) = (d - 1 + H) \cdot \mathbf{1}_{[0,1]}(H) + (-\infty) \cdot \mathbf{1}_{[0,+\infty] \setminus [0,1]}(H).$$

2) *Typical functions  $f \in \mathcal{CC}^d$  satisfy*

$$D_f(H) = (d - 1) \cdot \mathbf{1}_{\{0\}}(H) + (d - 2 + H) \cdot \mathbf{1}_{[1,2]}(H) + (-\infty) \cdot \mathbf{1}_{[0,+\infty] \setminus [1,2] \cup \{0\}}(H).$$

See Figure 5 for a comparison between typical multifractal behavior in various functional spaces. This shall also be compared to Figure 3.

It appears clearly that in all the previous situations, the singularity spectra of typical functions have the same shape: it is an affine, increasing, mapping, with no decreasing part.

Other functional spaces, called  $S_\nu$  spaces were built in [3], in which typical functions all exhibit a singularity spectrum which is *visibly increasing* in the sense of [34], enlarging the class of possible typical multifractal behavior in functional spaces. In addition, these typical functions do not satisfy a multifractal formalism in the sense of Definition 4.

In order to break this limitation (no decreasing part in the singularity spectrum), new (and natural) functional spaces have been introduced in [10].

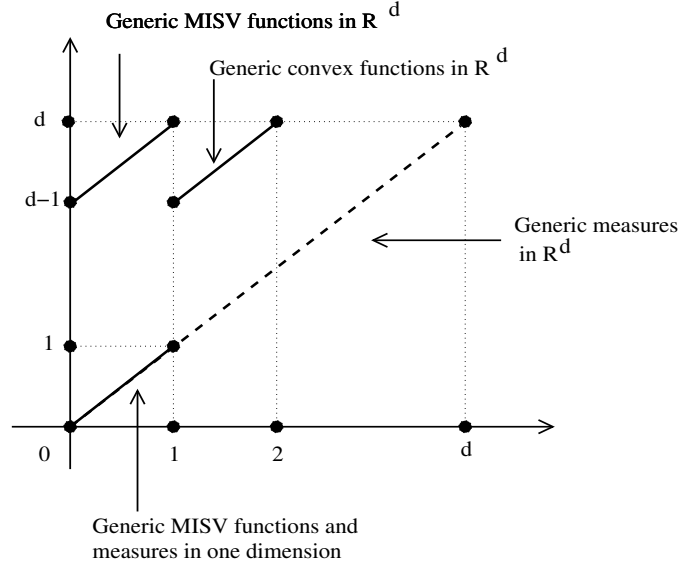


FIGURE 5. Typical singularity spectra for measures, MISV and convex functions

6. BESOV SPACES IN MULTIFRACTAL ENVIRONMENT

Since standard functional spaces do not fulfill our requirements (i.e. typical functions in such spaces do not exhibit concave singularity spectra), it is natural to ask whether there are other functional spaces in which typical functions have any singularity spectrum given in advance, and satisfy a multifractal formalism. This solves the Frisch-Paris conjecture as stated in Conjecture 1.

Let  $\mathcal{B}(\mathbb{R}^d)$  be the Borel sets included in  $\mathbb{R}^d$ , and let us introduce the set of Hölder set functions

$$(12) \quad \mathcal{C}(\mathbb{R}^d) := \left\{ \mu : \mathcal{B}(\mathbb{R}^d) \rightarrow \mathbb{R}^+ \text{ such that } \left\{ \begin{array}{l} \exists s_1, s_2 \geq 0, \forall I \subset \mathbb{R}^d \text{ with } |I| \leq 1, \\ |I|^{s_2} \leq \mu(I) \leq |I|^{s_1} \end{array} \right. \right\}.$$

For  $\mu \in \mathcal{C}(\mathbb{R}^d)$  and  $s \in \mathbb{R}$ , we write

$$\begin{aligned} \mu^s(I) &= \mu(I)^s, \\ \mu^{(s)}(I) &= \mu(I)|I|^s. \end{aligned}$$

We will use the following notation: for  $x, y \in \mathbb{R}^d$ ,  $B[x, y]$  is the smallest Euclidean ball that contains  $x$  and  $y$ .

**Definition 7.** Let  $h \in \mathbb{R}^d$ ,  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ , and consider the finite difference operator  $\Delta_h f : x \mapsto f(x + h) - f(x)$ . Define for  $n \geq 2$  by iteration  $\Delta_h^n f := \Delta_h(\Delta_h^{n-1} f)$ .

For every set function  $\mu \in \mathcal{C}(\mathbb{R}^d)$ , let us introduce for  $n \geq 2$

$$(13) \quad \Delta_h^{\mu, n} f(x) = \frac{\Delta_h^n f(x)}{\mu(B[x, x + nh])}.$$

The  $\mu$ -adapted  $n$ -th order modulus of continuity of  $f$  on  $\mathbb{R}^d$  is defined for  $t > 0$  by

$$(14) \quad \omega_n^\mu(f, t)_p = \sup_{t/2 \leq |h| \leq t} \|\Delta_h^{\mu, n} f\|_{L^p(\mathbb{R}^d)}.$$



It is trivial to check that that when  $\mu(I) = 1$  for every set  $I$ , then  $\omega_n^\mu(f, t)_p$  coincides with the so-called homogeneous  $n$ -th order modulus of continuity of  $f$

$$\omega_n(f, t)_p = \sup_{t/2 \leq |h| \leq t} \|\Delta_h^n f\|_{L^p(\mathbb{R}^d)}.$$

**Definition 8.** Let  $\mu \in \mathcal{C}(\mathbb{R}^d)$  associated with exponents  $0 < s_1 \leq s_2$  in (12).

Let  $n \geq s_2$ . For  $1 \leq p, q \leq +\infty$ , the Besov space in  $\mu$ -environment  $B_q^{\mu, p}(\mathbb{R}^d)$  is the space of those functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $\|f\|_{L^p(\mathbb{R}^d)} < +\infty$  and

$$(15) \quad |f|_{B_q^{\mu, p}} = \|2^{jd/p} (\omega_n^\mu(f, 2^{-j})_p)_{j \geq 1}\|_{\ell^q(\mathbb{N})} < +\infty.$$

Finally, let us introduce the spaces

$$(16) \quad \tilde{B}_q^{\mu, p}(\mathbb{R}^d) = \bigcap_{0 < \varepsilon < s_1/2} B_q^{\mu^{(-\varepsilon)}, p}(\mathbb{R}^d).$$

The reader can check that  $B_q^{\mu, p}(\mathbb{R}^d)$ , when endowed with the topology induced by the norm  $\|f\|_{B_q^{\mu, p}} = \|(\beta(k))_{k \in \mathbb{Z}^d}\|_p + |f|_{B_q^{\mu, p}}$ , forms a Banach space (recall (7) for the definition of  $\beta(k)$ ).

The intuition behind Definition 8 consists in introducing some space-dependent constraints that will create heterogeneity at all scales. Indeed, when a function  $f$  belongs to  $B_q^{\mu, p}(\mathbb{R}^d)$ , its oscillations  $\Delta_h^n f(x)$  must be very small in certain regions (around points  $x$  where  $\mu(B(x, r)) \sim r^\alpha$  with  $\alpha$  large), while in other regions (where  $\mu(B(x, r)) \sim r^\alpha$  with  $\alpha$  small) the control of the oscillations can be relaxed.

In [10], a wavelet characterization of  $B_q^{\mu, p}(\mathbb{R}^d)$  and  $\tilde{B}_q^{\mu, p}(\mathbb{R}^d)$  is proved when  $\mu$  is an almost-doubling set function satisfying condition (P) (recall equation (4) and Remark 1). Observe indeed that Definition 5 of the condition (P) for measures can easily be extended for set functions  $\mu \in \mathcal{C}(\mathbb{R}^d)$ .

For this, let us introduce a second semi-norm for  $f \in L^p(\mathbb{R}^d)$  : we set

$$|f|_{p, q, \mu} = \|(A_j)_{j \geq 1}\|_{\ell^q(\mathbb{N})}, \text{ where } A_j = \left( \sum_{\lambda \in \Lambda_j} \left| \frac{d_\lambda}{\mu(\lambda)} \right|^p \right)^{1/p}.$$

The following inequalities are proved in [10].

**Theorem 15.** *Let  $\mu \in \mathcal{C}(\mathbb{R}^d)$  be an almost doubling set function satisfying condition (P), and let  $\Phi$  be a scaling function associated with wavelets  $\Psi$  (see Section 4.2).*

*Let  $p \geq 1$ , and  $q \in [1, +\infty]$ .*

*Assume that the wavelets  $\Psi$  are compactly supported, belong to the standard Besov space  $B_q^{s, p}(\mathbb{R}^d)$  for some  $s > d/p + s_2$ , and possess at least  $[s_2] + 1$  vanishing moments ( $s_1$  and  $s_2$  are the exponents associated with  $\mu$  in (12)).*

*For every  $0 < \varepsilon < s_1$ , there exists a constant  $C > 1$  (not depending on  $f$ ) such that*

$$(17) \quad \|f\|_{L^p} + |f|_{B_q^{\mu, p}} \leq C(\|f\|_{L^p} + |f|_{\mu^{(+\varepsilon)}, p, q})$$

$$(18) \quad \|f\|_{L^p} + |f|_{\mu, p, q} \leq C(\|f\|_{L^p} + |f|_{B_q^{\mu^{(+\varepsilon)}, p}}).$$

*Moreover, when  $\mu$  is doubling, (17) and (18) hold for  $\varepsilon = 0$ , and the norms  $\|f\|_{L^p} + |f|_{p, q, \mu}$  and  $\|f\|_{L^p} + |f|_{B_q^{\mu, p}}$  are equivalent.*

Last theorem supports the idea that  $\tilde{B}_q^{\mu, p}(\mathbb{R}^d)$  is the right space to work with, since it is characterized by wavelet coefficients, while the spaces  $B_q^{\mu, p}(\mathbb{R}^d)$  are not (unless  $\mu$  is doubling).

The main theorem in [10] is the following.

**Theorem 16.** *Let  $\sigma \in \mathcal{S}_d$  be an admissible singularity spectrum (recall (11)). Call  $H_s$  the smallest value at which  $\sigma(H) = d$ .*

*There exists an almost doubling set function  $\mu \in \mathcal{C}(\mathbb{R}^d)$  satisfying condition (P) and  $p \in [1, +\infty]$  such that for every  $q \in [1, +\infty]$ , typical functions  $f \in \tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  possess the following properties:*

- $D_f = \sigma$
- $f$  satisfies the WLMF for every  $H \leq H_s$ .
- $f$  satisfies the weak-WLMF for every  $H > H_s$ .

*In addition:*

- when  $p = +\infty$ , typical functions in  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$  satisfy  $D_f = D_\mu$ .
- when  $\mu$  is doubling, the same holds for  $B_q^{\mu,p}(\mathbb{R}^d)$  instead of  $\tilde{B}_q^{\mu,p}(\mathbb{R}^d)$ .

Also, given  $\sigma \in \mathcal{S}_d$ , from the proof in [10] it can be checked that the couple  $(\mu, p)$  in Theorem 16 is not unique.

Theorem 16 brings a solution to the Frisch-Parisi conjecture (Conjecture 1). The fact that the (strong) multifractal formalism holds only for the increasing part of the singularity spectrum (when  $H \leq H_s$ ) seems to be unavoidable. A heuristic explanation of the weak validity of the multifractal formalism in the decreasing part of the spectrum (and not the full validity) is that functions have usually sparse wavelet representations, generating very large values for negative values of  $q$  for  $L_f(q, j)$  on some values of  $j$ .

Let us conclude this section by mentioning that a deeper study of the  $B_q^{\mu,p}$  and  $\tilde{B}_q^{\mu,p}$  spaces is performed in [10], leading to results that have their own interest. More precisely, a uniform upper bound for the singularity spectrum of all functions in  $B_q^{\mu,p}$  and  $\tilde{B}_q^{\mu,p}$  is found, as well as the singularity spectrum of typical functions in these spaces for large classes of almost-doubling measures  $\mu$ . Without giving details on the results, it appears that the singularity spectra  $D_f$  of typical functions  $f$  may have very different shapes depending on the initial measure  $\mu$ , and the proofs involve many arguments coming from geometric measure theory, ergodic theory and harmonic analysis.

## 7. PERSPECTIVES

First of all, we are far from being exclusive on generic dimensional results in analysis (see for instance [22, 24]), and many other regularity properties shall definitely be studied from the Baire genericity standpoint.

In this survey we focused on the notion of Baire genericity - the same issues can (and must) be addressed in the *prevalence* sense. Many results regarding prevalent multifractal properties have been obtained, see [4, 21, 20, 40, 41] amongst many references, and asking whether prevalent properties coincide with generic ones can sometimes bring some surprises (when they do not coincide).

Finally, one challenging research direction consists in establishing multifractal properties for (classes of) solutions to ordinary or partial differential equations, as well as for the stochastic counterparts. Indeed, multifractal ideas originate from the study of turbulence and other physical phenomena that are ruled by ODEs, SDEs or (S)PDEs, and it would be a fair return to demonstrate the multifractality of (some of) those functions that are solutions to such equations. A few examples already exist (i.e., Burgers equation with a Brownian motion as initial condition [12] and large classes of stochastic jump diffusions [8, 42]), but they are only a first step toward a systematic multifractal analysis of solutions to (partial)

differential equations, which will certainly require the development of new techniques and approaches.

#### ACKNOWLEDGMENTS

The author wish to thank the organizers of the fantastic conference "Fractal Geometry and Stochastics 6", and the referee for all her/his relevant and useful comments on this text.

#### REFERENCES

- [1] P. Abry, S. Jaffard, and H. Wendt. Irregularities and scaling in signal and image processing: Multifractal analysis. In M. Frame Ed., editor, *Benoit Mandelbrot: A Life in Many Dimensions*, pages 31–116. World Scientific, 2015.
- [2] P. Abry, H. Wendt, and S. Jaffard. When Van Gogh meets Mandelbrot: Multifractal classification of painting's texture. *Signal Processing*, 93(3):554–572, mars 2013.
- [3] J.-M. Aubry, S. Bastin, F. Dispa, and S. Jaffard. The spaces  $s_\nu$ : new spaces defined with wavelet coefficients and related to multifractal analysis. *Int. J. Appl. Math. Statist.*, 7:82–95, Feb. 2007.
- [4] J.-M. Aubry, D. Maman, and S. Seuret. Local behavior of traces of besov functions: Prevalent results. *J. Func. Anal.*, 264(3):631–660, 2013.
- [5] A. Ayache, S. Jaffard, and Murad S. Taqqu. Wavelet construction of generalized multifractional processes. *Rev. Mat. Iberoamericana*, 23(1):327–370, 2007.
- [6] S. Banach. Über die Baire'sche kategorie gewisser funktionenmengen. *Studia Math.*, 3:174–179, 1931.
- [7] J. Barral. Inverse problems in multifractal analysis of measures. *Ann. Ec. Norm. Sup*, 48(6):1457–1510, 2015.
- [8] J. Barral, N. Fournier, S. Jaffard, and S. Seuret. A pure jump Markov process with a random singularity spectrum. *Ann. Probab.*, 38(5):1924–1946, 2010.
- [9] J. Barral and S. Seuret. From multifractal measures to multifractal wavelet series. *J. Fourier Anal. Appl.*, 11(5):589–614, 2005.
- [10] J. Barral and S. Seuret. Besov spaces in multifractal environment, and the Frisch-Parisi conjecture. *preprint*, 2019.
- [11] F. Bayart. Multifractal spectra of typical and prevalent measures. *Nonlinearity*, 26:353–367, 2013.
- [12] J. Bertoin and S. Jaffard. Solutions multifractales de l'équation de burgers. *Matapli*, 52:19–28, 1997.
- [13] G. Brown, G. Michon, and J. Peyrière. On the multifractal analysis of measures. *J. Stat. Phys.*, 66:775–790, 1992.
- [14] Z. Buczolich and J. Nagy. Hölder spectrum of typical monotone continuous functions. *Real Anal. Exchange*, pages 133–156, 1999.
- [15] Z. Buczolich and S. Seuret. Typical borel measures on  $[0, 1]^d$  satisfy a multifractal formalism. *Nonlinearity*, 23(11):7–13, 2010.
- [16] Z. Buczolich and S. Seuret. Hölder spectrum of functions monotone in several variables. *J. Math. Anal. Appl.*, 1:110–126, 2011.
- [17] Z. Buczolich and S. Seuret. Measures and functions with prescribed singularity spectrum. *J. Fractal Geometry*, 1(3):295–333, 2014.
- [18] Z. Buczolich and S. Seuret. Multifractal properties of typical convex functions. *Monatshefte für Mathematik*, 2019.
- [19] I. Daubechies. *Ten Lectures on Wavelets*. SIAM, 1992.
- [20] A. Fraysse. Regularity criteria of almost every function in a Sobolev space. *J. Func. Anal.*, pages 1806–1821, 2010.
- [21] A. Fraysse and S. Jaffard. How smooth is almost every function in a Sobolev space? *Rev. Mat. Iberoamericana*, 22:663–682, 2006, 22, pp.663-682.
- [22] A. Fraysse, S. Jaffard, and J.-P. Kahane. Quelques propriétés génériques en analyse. *Notes aux CRAS, Série I, Math.*, 340:645–651, 2005.
- [23] U. Frisch and D. Parisi. Fully developed turbulence and intermittency in turbulence, and predictability in geophysical fluid dynamics and climate dynamics. *International school of Physics "Enrico Fermi", course 88, edited by M. Ghil, North Holland*, pages 84–88, 1985.
- [24] V. Gruslys, V. Jonusas, V. Mijović, O. Ng, L. Olsen, and I. Petrykiewicz. Dimensions of prevalent continuous functions. *Monatshefte für Mathematik*, 166(2):153–180, 2012.
- [25] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia, and B.I. Shraiman. Fractal measures and their singularities: the characterisation of strange sets. *Phys. Rev. A*, 33:1141–1151, 1986.

- [26] H.G. Hentschel and I. Procaccia. The infinite number of generalized dimensions of fractals and strange attractors. *Physica D*, pages 435–444, 1983.
- [27] S. Jaffard. Exposants de Hölder en des points donnés et coefficients d’ondelettes. *C. R. Acad. Sci. Paris Série I*, 308:79–81, 1989.
- [28] S. Jaffard. Construction de fonctions multifractales ayant un spectre de singularités prescrit. *C.R.A.S.*, 315(1):19–24, 1992.
- [29] S. Jaffard. Functions with prescribed Hölder exponent. *Appl. Comput. Harmon. Anal.*, 2:400–401, 1995.
- [30] S. Jaffard. On the Frisch-Parisi conjecture. *J. Math. Pures Appl.*, 79(6):525–552, 2000.
- [31] S. Jaffard. Wavelet techniques in multifractal analysis. In *Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot*, Proc. Symposia in Pure Mathematic. AMS Providence, RI, 2004.
- [32] B. Lashermes, S. Roux, S. Jaffard, and P. Abry. Comprehensive multifractal analysis of turbulent velocity using wavelet leaders. *Eur. Phys. J. B.*, 61(2):201–215, 2008.
- [33] R. Leonarduzzi, P. Abry, S. Jaffard, H. Wendt, L. Gournay, T. Kyriacopoulou, C. Martineau, and C. Martinez.  $p$ -leader multifractal analysis for text type identification. In *IEEE Int. Conf. Acoust., Speech, and Signal Proces. (ICASSP)*, New Orleans, USA, march 2017.
- [34] D. Maman and S. Seuret. Fixed points for the multifractal spectrum map. *Constructive approximation*, 43(3):337–356, 2016.
- [35] B. B. Mandelbrot. Multiplications aléatoires itérées et distributions invariantes par moyennes pondérées. *C. R. Acad. Sci. Paris*, 278:289–292 et 355–358, 1974.
- [36] S. Mazurkiewicz. Sur les fonctions non dérivables. *Studia Math.*, 3:92–94, 1931.
- [37] Y. Meyer. *Ondelettes et opérateurs I*. Hermann, 1990.
- [38] J.-F. Muzy, E. Bacry, and A. Arnéodo. Multifractal formalism for fractal signals: The structure-function approach versus the wavelet-transform modulus-maxima method. *Phys. Rev. E*, 47:875–884, 1993.
- [39] L. Olsen. A multifractal formalism. *Adv. Math.*, 116:92–195, 1995.
- [40] L. Olsen. Fractal and multifractal dimensions of prevalent measures. *Indiana Univ. Math. J.*, 59 (2):661–690, 2010.
- [41] L. Olsen. Prevalent  $L^q$ -dimensions of measures. *Math. Proc. Cambridge Philos. Soc.*, 149(3):553–571, 2010.
- [42] X. Yang. Multifractality of jump diffusion processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 54(4):2042–2074, 2018.

STÉPHANE SEURET, UNIVERSITÉ PARIS-EST, LAMA (UMR 8050), UPEMLV, UPEC, CNRS, F-94010, CRÉTEIL, FRANCE

*E-mail address:* [seuret@u-pec.fr](mailto:seuret@u-pec.fr)