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# Stabilizability properties of a linearized water waves system

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## Abstract

We consider the strong stabilization of small amplitude gravity water waves in a two dimensional rectangular domain. The control acts on one lateral boundary, by imposing the horizontal acceleration of the water along that boundary, as a multiple of a scalar input function  $u$ , times a given function  $h$  of the height along the active boundary. The state  $z$  of the system consists of two functions: the water level  $\zeta$  along the top boundary, and its time derivative  $\dot{\zeta}$ . We prove that for suitable functions  $h$ , there exists a bounded feedback functional  $F$  such that the feedback  $u = Fz$  renders the closed-loop system strongly stable. Moreover, for initial states in the domain of the semi-group generator, the norm of the solution decays like  $(1+t)^{-\frac{1}{6}}$ . Our approach uses a detailed analysis of the partial Dirichlet to Neumann and Neumann to Neumann operators associated to certain edges of the rectangular domain, as well as recent abstract non-uniform stabilization results by Chill, Paunonen, Seifert, Stahn and Tomilov (2019).

*Keywords:* Linearized water waves equation, collocated actuators and sensors, Dirichlet to Neumann map, Neumann to Neumann map, operator semigroup, state feedback, strong stabilization, Hilbert's inequality.

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## 1. Notation

Throughout this paper, the notation

$$\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$$

stands for the sets of natural numbers (starting with 1), integers, real numbers and complex numbers, respectively. We denote  $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ .

If  $n, k \in \mathbb{N}$  and  $\mathcal{O} \subset \mathbb{R}^n$  is an open set, then we use the notation  $\mathcal{H}^k(\mathcal{O})$  for the *Sobolev space* formed by the distributions  $f \in \mathcal{D}'(\mathcal{O})$  having the property that  $\partial^\alpha f \in L^2(\mathcal{O})$  for every multi-index  $\alpha \in \mathbb{Z}^n$  with  $\alpha_j \geq 0$  and  $|\alpha| \leq k$ . For  $f \in \mathcal{H}^k(\mathcal{O})$  we set

$$\|f\|_k^2 = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^2}^2. \quad (1.1)$$

Let  $\mathcal{H}^0(\mathcal{O}) := L^2(\mathcal{O})$  and let  $\mathcal{H}^s(\mathcal{O})$ , with  $s > 0$ , denote the fractional order Sobolev spaces obtained by interpolation via fractional powers of a positive operator (see, for instance, Lions and Magenes [1]).

The system considered in this work is described by the linearized equations of water waves in the vertical (in the sense of gravity) rectangular domain

$$\Omega = (0, \pi) \times (-1, 0). \quad (1.2)$$

We set

$$\mathcal{H}_{top}^1(\Omega) = \{f \in \mathcal{H}^1(\Omega) \mid f(x, 0) = 0, x \in (0, \pi)\}, \quad (1.3)$$

where the values at the top boundary are defined in the sense of the Dirichlet trace, as in [1], [2, Sect. 13.6]).

For two functions  $u$  and  $v$  defined on  $[0, \infty)$  and for any  $\tau \geq 0$ , their  $\tau$ -concatenation, denoted by  $u \diamond_\tau v$ , is the function

$$u \diamond_\tau v = \begin{cases} u(t) & \text{for } t \in [0, \tau), \\ v(t - \tau) & \text{for } t \geq \tau. \end{cases}$$

If  $H$  is a Hilbert space,  $\mathcal{D}(A_0)$  is a subspace of  $H$  and  $A_0 : \mathcal{D}(A_0) \rightarrow H$  is a linear operator, then  $A_0$  is called *strictly positive* if  $A_0$  is self-adjoint and there exists  $m_0 > 0$  such that

$$\langle A_0 z, z \rangle_H \geq m_0 \|z\|_H^2 \quad \forall z \in \mathcal{D}(A_0).$$

## 2. The water wave model and its well-posedness

In this work we study the stabilizability of a system describing small-amplitude water waves in a rectangular domain, in the presence of a wave maker. For more details on water waves models we refer to Whitham's book [3, Chapter 13] and to Lannes [4, Chapter 1]. Here we consider the stabilization of linear water waves by an input (the acceleration of the wave maker) acting at one of the lateral edges. We assume that the domain  $\Omega$  is delimited at its top by a free water surface  $\Gamma_s$  and that the bottom  $\Gamma_f$  is flat. The other two components of the boundary of the fluid domain, denoted by  $\Gamma_1$  and  $\Gamma_2$ , are supposed to be vertical, see Figure 1. Moreover, we assume that the fluid fills the rectangular domain  $\Omega$  defined in Section 1, that it is homogeneous, incompressible, inviscid and that it undergoes irrotational flows. There is a wave maker that acts at the left boundary of  $\Omega$ , by injecting (or extracting) fluid in the horizontal direction, at an acceleration determined by the control signal  $u$ .

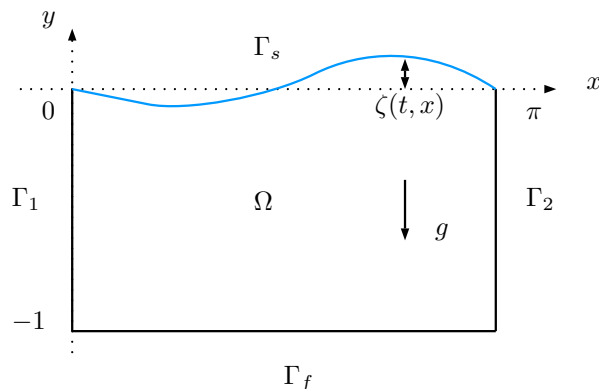


Figure 1: A rectangular domain  $\Omega$  filled with water

The equations of the system, for all  $t \geq 0$ , are:

$$\left\{ \begin{array}{ll} \Delta\phi(t, x, y) = 0 & ((x, y) \in \Omega), \\ \phi(t, x, 0) + \zeta(t, x) = 0 & (x \in (0, \pi)), \\ \frac{\partial\phi}{\partial y}(t, x, 0) = \ddot{\zeta}(t, x) & (x \in (0, \pi)), \\ \frac{\partial\phi}{\partial x}(t, 0, y) = -h(y)u(t) & (y \in (-1, 0)), \\ \frac{\partial\phi}{\partial y}(t, x, -1) = 0 = \frac{\partial\phi}{\partial x}(t, \pi, y) & ((x, y) \in \Omega). \end{array} \right. \quad (2.1)$$

In the above equations  $\phi$  stands for the derivative with respect to time of the velocity potential of the fluid and  $\zeta$  for the elevation of the free surface. The function  $h$  is given and it represents the profile of the acceleration field imposed by the wave maker. Usually we assume that  $\int_{-1}^0 h(y)dy = 0$ , to ensure the conservation of the volume of water. As far as we know, the controllability and stabilizability properties of systems derived from (2.1) have been first studied in Russell and Reid [5] and further in Mottelet [6].

Before stating our well-posedness result for (2.1), we need some background and more notation. First we recall the concept of a well-posed linear control system, following Weiss [7] (where these systems have been called *abstract linear control systems*), see also Tucsnak and Weiss [2].

**Definition 2.1.** *Let  $U$  and  $X$  be Hilbert spaces. A well-posed linear control system with the state space  $X$  and the input space  $U$  is a couple  $(\mathbb{T}, \Phi)$  of families of operators such that*

1.  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  is a strongly continuous operator semigroup (also called a  $C_0$ -semigroup) on  $X$ .
2.  $\Phi = (\Phi_t)_{t \geq 0}$  is a family of bounded linear operators from  $L^2([0, \infty); U)$  to  $X$  (called *input maps*) such that for every  $u, v \in L^2([0, \infty); U)$ ,

$$\Phi_{\tau+t}(u \diamond_{\tau} v) = \mathbb{T}_t \Phi_{\tau} u + \Phi_t v \quad \forall t, \tau \geq 0, \quad (2.2)$$

where we used the concatenation of functions, see Section 1.

For any  $\tau \geq 0$ , let  $\mathbf{P}_{\tau}u$  denote the truncation of  $u : [0, \infty) \rightarrow U$  to  $[0, \tau]$ , setting  $(\mathbf{P}_{\tau}u)(t) = 0$  for  $t > \tau$ . It follows from (2.2) that  $\Phi_{\tau} \mathbf{P}_{\tau} = \Phi_{\tau}$  (causality), and hence  $\Phi_{\tau}$  has a natural extension to  $L^2_{\text{loc}}([0, \infty); U)$ .

Still using the notation from the above definition, if  $z_0 \in X$  and  $u \in L^2_{\text{loc}}([0, \infty); U)$ , then we call the function  $z(t) = \mathbb{T}_t z_0 + \Phi_t u$  the *state trajectory* of the system corresponding to the initial state  $z_0$  and the input  $u$ . Let  $A : \mathcal{D}(A) \rightarrow X$  denote the generator of  $\mathbb{T}$ . For every well-posed linear control system there exists a (usually unbounded) operator  $B$  defined on  $U$  and with values in an extrapolation space that contains  $X$ , with the following property: For any  $z_0 \in X$  and  $u \in L^2_{\text{loc}}([0, \infty); U)$ , the corresponding state trajectory is the unique solution (in the extrapolation space) of the abstract differential equation

$$\dot{z}(t) = Az(t) + Bu(t), \quad (2.3)$$

with initial condition  $z(0) = z_0$ . For details on this see [2, Chapter 4]. The above operator  $B$  is called the *control operator* of the system. This operator is called *bounded* if  $B \in \mathcal{L}(U, X)$  (this is the case of interest in this paper).

We would like to formulate the system of equations (2.1) as a well-posed linear control system. This is not obvious, because the equations (2.1) do not even resemble (2.3). We have to define what we mean by the state of our system at some time  $t \geq 0$ : this should be

$$z(t) = \begin{bmatrix} \zeta(t, \cdot) \\ \dot{\zeta}(t, \cdot) \end{bmatrix}. \quad (2.4)$$

To define the state space  $X$ , and also for other arguments, we introduce a scale of Hilbert spaces as follows. We set

$$H = \left\{ \eta \in L^2[0, \pi] \mid \int_0^\pi \eta(x) dx = 0 \right\}, \quad (2.5)$$

which is a Hilbert space when endowed with the inner product inherited from  $L^2[0, \pi]$ . It is known that the family  $(\varphi_k)_{k \in \mathbb{N}}$  defined by

$$\varphi_k(x) = \sqrt{\frac{2}{\pi}} \cos(kx) \quad \forall x \in [0, \pi], \quad (2.6)$$

forms an orthonormal basis in  $H$ . For any  $\eta \in H$ , we denote  $\eta_k = \langle \eta, \varphi_k \rangle$ . The scale of Hilbert spaces  $(H_\alpha)_{\alpha \geq 0}$  are defined by  $H_0 = H$  and

$$H_\alpha = \left\{ \eta \in H \mid \sum_{k \geq 1} k^{2\alpha} |\eta_k|^2 < \infty \right\} \quad (\alpha \geq 0), \quad (2.7)$$

with the inner products  $(\langle \cdot, \cdot \rangle_\alpha)_{\alpha \geq 0}$  defined by  $\langle \eta, \psi \rangle_\alpha = \sum_{k \geq 1} k^{2\alpha} \eta_k \overline{\psi_k}$ , for all  $\eta, \psi \in H_\alpha$ . It is not difficult to check that

$$H_1 = \left\{ \eta \in \mathcal{H}^1(0, \pi) \mid \int_0^\pi \eta(x) dx = 0 \right\}.$$

By interpolation theory (see, for instance, Lions and Magenes [1], Bensoussan et al [8, Part II] and Chandler-Wilde et al [9]) it follows that

$$H_s = \left\{ \eta \in \mathcal{H}^s(0, \pi) \mid \int_0^\pi \eta(x) dx = 0 \right\} \quad \forall s \in (0, 1). \quad (2.8)$$

We define what we mean by a solution of the water wave equations (2.1).

**Definition 2.2.** Given  $u \in L^2_{\text{loc}}[0, \infty)$  and  $h \in L^2[-1, 0]$ , with  $\int_{-1}^0 h(y) dy = 0$ , a couple  $(\phi, \zeta)$  is called a solution of (2.1) if

$$\begin{aligned} \phi &\in L^2_{\text{loc}}([0, \infty); \mathcal{H}^1(\Omega)), \quad \zeta \in C([0, \infty); H_{\frac{1}{2}}) \cap C^1([0, \infty); H), \\ \phi(t, \cdot, 0) + \zeta(t, \cdot) &= 0, \text{ equality in } L^2_{\text{loc}}([0, \infty); L^2[0, \pi]), \end{aligned} \quad (2.9)$$

and for every  $\Psi \in \mathcal{H}^1(\Omega)$  and every  $t > 0$  we have

$$\begin{aligned} \int_0^\pi \dot{\zeta}(t, x) \overline{\Psi(x, 0)} dx - \int_0^\pi \dot{\zeta}(0, x) \overline{\Psi(x, 0)} dx &= \\ \int_0^t \int_\Omega \nabla \phi(\sigma, x, y) \cdot \overline{\nabla \Psi(x, y)} dx dy d\sigma - \int_0^t u(\sigma) \int_{-1}^0 h(y) \overline{\Psi(0, y)} dy d\sigma. \end{aligned} \quad (2.10)$$

**Remark 2.3.** We explain the connection between the water waves equations (2.1) and their variational formulation (2.9)- (2.10). In one direction, assume that  $(\phi, \zeta)$  is a classical solution of (2.1), having the smoothness

$$\phi \in C([0, \infty); \mathcal{H}^2(\Omega)), \quad \zeta \in C([0, \infty); H_{\frac{1}{2}}) \cap C^2([0, \infty); H). \quad (2.11)$$

(If  $u \not\equiv 0$ , this implies that  $h \in \mathcal{H}^{\frac{1}{2}}(-1, 0)$  and  $u$  is continuous.) The equation (2.9) is simply copied from (2.1). We multiply the first equation in (2.1) with  $\overline{\Psi}$  and apply the first Green formula (integration by parts), taking into account the last three lines of (2.1). After this we do simple integration with respect to  $t$ , and we obtain (2.10).

In the opposite direction, let us assume that  $(\phi, \zeta)$  is a solution of (2.9)- (2.10) with the additional regularity (2.11), and  $u$  is continuous. Then (2.10) can be differentiated with respect to the time  $t$ , and after using the first Green formula we obtain

$$\begin{aligned} \int_0^\pi \ddot{\zeta}(t, x) \overline{\Psi(x, 0)} dx &= \int_{\partial\Omega} \left( \frac{\partial}{\partial \nu} \phi \right) \overline{\Psi} d\sigma - \int_\Omega \Delta \phi(t, x, y) \cdot \overline{\Psi(x, y)} dx dy \\ &\quad - u(t) \int_{-1}^0 h(y) \overline{\Psi(0, y)} dy \quad \forall t \geq 0, \end{aligned}$$

where  $\frac{\partial}{\partial \nu}$  denotes the Neumann trace on the entire boundary  $\partial\Omega$ . Considering only functions  $\Psi$  with compact support in  $\Omega$ , we see from the above that we must have  $\Delta \phi = 0$ . After this, we consider test functions  $\Psi$  whose trace is supported on one of the four segments of  $\partial\Omega$ , knowing that these traces are dense in the  $L^2$  space of the relevant segment, see [2, Theorem 13.6.10]. From here we can get that  $\phi$  and  $\zeta$  satisfy also the last three equations in (2.1). (It also follows that  $h \in \mathcal{H}^{\frac{1}{2}}(-1, 0)$ .)

The following result establishes the existence of a well-posed linear control system corresponding to (2.1). For the proof we refer to Section 6.

**Theorem 2.4.** *Let  $h \in L^2[-1, 0]$  be such that  $\int_{-1}^0 h(y) dy = 0$ . Then for every  $u \in L^2_{\text{loc}}[0, \infty)$ ,  $\zeta_0 \in H_{\frac{1}{2}}$  and  $w_0 \in H$ , there exists a unique solution of (2.1) with  $\zeta(0) = \zeta_0$  and  $\dot{\zeta}(0) = w_0$ . Moreover, there exists a well-posed linear control system  $(\mathbb{T}, \Phi)$  with state space  $X = H_{\frac{1}{2}} \times H$  and input space  $U = \mathbb{C}$  such that, setting  $z_0 = \begin{bmatrix} \zeta_0 \\ w_0 \end{bmatrix}$  and using the state from (2.4), we have*

$$z(\tau) = \mathbb{T}_\tau z_0 + \Phi_\tau u \quad \forall \tau \geq 0. \quad (2.12)$$

Finally, the generator  $A$  of  $\mathbb{T}$  is skew-adjoint, with domain  $\mathcal{D}(A) = H_1 \times H_{\frac{1}{2}}$ , and there exists  $B \in \mathcal{L}(\mathbb{C}, X)$  such that for any  $\tau \geq 0$ ,

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-\sigma} B u(\sigma) d\sigma \quad \forall u \in L^2_{\text{loc}}[0, \infty). \quad (2.13)$$

We mention that, according to the above theorem and what we have said around (2.3), the state trajectories of our system are solutions of (2.3), in the sense of [2, Sect. 4.1-4.2], and our control operator  $B$  is bounded.

### 3. Statement of the main result

We recall some commonly used stabilizability concepts, for the particular situation of bounded control and feedback operators.

**Definition 3.1.** *Let  $\Sigma = (\mathbb{T}, \Phi)$  be a well-posed linear control system with state space  $X$  and input space  $U$ . Let  $A$  be the generator of  $\mathbb{T}$  and assume that there exists  $B \in \mathcal{L}(U, X)$  such that (2.13) holds. For some feedback operator  $F \in \mathcal{L}(X, U)$  we denote by  $\mathbb{T}^{\text{cl}}$  the (closed loop) operator semigroup on  $X$  generated by  $A + BF$ . Then the system  $(\mathbb{T}, \Phi)$  is:*

1. Exponentially stabilizable with bounded feedback, if there exists  $F \in \mathcal{L}(X, U)$  such that the semigroup  $\mathbb{T}^{\text{cl}}$  is exponentially stable;
2. Strongly stabilizable with bounded feedback, if there exists  $F \in \mathcal{L}(X, U)$  such that the semigroup  $\mathbb{T}^{\text{cl}}$  is strongly stable;
3. Uniformly stabilizable for smooth data (USSD), if there exists  $F \in \mathcal{L}(X, U)$  and  $f : [0, \infty) \rightarrow [0, \infty)$ , with  $\lim_{t \rightarrow \infty} f(t) = 0$ , such that

$$\|\mathbb{T}_t^{\text{cl}} z_0\|_X \leq f(t) \|z_0\|_{\mathcal{D}(A)} \quad \forall z_0 \in \mathcal{D}(A), t \geq 0. \quad (3.1)$$



If  $f$  in (3.1) can be chosen such that  $\lim_{t \rightarrow \infty} t^m f(t) = 0$  for some  $m \in \mathbb{N}$ , then the USSD property is called polynomial stabilizability.

In (3.1) and also later,  $\|\cdot\|_{\mathcal{D}(A)}$  denotes the graph norm on  $\mathcal{D}(A)$ .

**Remark 3.2.** Note that the property (3.1) does not imply that the semigroup  $\mathbb{T}^{cl}$  is strongly stable. Indeed, consider  $\mathbb{T}_t^{cl}$  to be  $e^{-0.7t}$  times the semigroup from [10, Example 2.3] (based on Zabczyk [11]), with  $\lambda_n = 2^n$ , then it satisfies (3.1) with  $f(t) = Me^{-0.2t}$  (for some  $M > 0$ ) but  $\mathbb{T}^{cl}$  is exponentially growing:  $\|\mathbb{T}_t^{cl}\| = e^{0.3t}$ . However, if  $\mathbb{T}^{cl}$  is a bounded semigroup and (3.1) holds, then it is easy to see that  $\mathbb{T}^{cl}$  is strongly stable.

Here is our main result:

**Theorem 3.3.** *Let  $\Sigma = (\mathbb{T}, \Phi)$  be the well-posed linear control system introduced in Theorem 2.4. Then*

1.  $\Sigma$  is not exponentially stabilizable with bounded feedback;
2.  $\Sigma$  is strongly stabilizable with bounded feedback if and only if  $h$  is a strategic profile, in the sense that

$$\int_{-1}^0 h(y) \cosh [k(y+1)] dy \neq 0 \quad \forall k \in \mathbb{N}; \quad (3.2)$$

In this case, one strongly stabilizing feedback operator is  $F = -B^*$ .

3. If

$$\inf_{k \in \mathbb{N}} \frac{k}{\cosh k} \left| \int_{-1}^0 h(y) \cosh [k(y+1)] dy \right| > 0, \quad (3.3)$$

then the system  $\Sigma$  is USSD. More precisely, the feedback operator  $F = -B^*$  leads to the closed-loop semigroup  $\mathbb{T}^{cl}$  (with generator  $A - BB^*$ ) which is strongly stable and has the following property: there exists  $M > 0$  such that

$$\|\mathbb{T}_t^{cl} z_0\|_X \leq \frac{M}{(1+t)^{\frac{1}{6}}} \|z_0\|_{\mathcal{D}(A)} \quad \forall z_0 \in \mathcal{D}(A), t \geq 0. \quad (3.4)$$

**Remark 3.4.** It is not difficult to check (by integration by parts) that condition (3.3) is satisfied, for instance, if there exists  $\varepsilon \in (0, 1)$  such that

$$\|h'\|_{L^\infty[-1,0]} < \frac{(1-\varepsilon) \tanh 1}{1 - \frac{2}{e}} |h(0)|, \quad (3.5)$$

where  $e = 2.71828\dots$  is the basis of the natural logarithm. Indeed, there are many functions satisfying (3.5) and  $\int_{-1}^0 h(y)dy = 0$ , such as the linear function  $h_1(y) = y + \frac{1}{2}$ , the trigonometric function  $h_2(y) = \cos\left[\frac{1}{2}\pi\left(y + \frac{3}{2}\right)\right]$  and some slightly modified step functions. Compared with other strategic conditions, for instance the constraint condition at rational points in [12], the condition (3.5) is easier to satisfy in practice.

**Remark 3.5.** The first two conclusions in Theorem 3.3 appear partially in [6], with some steps of the proof not given. (For instance the operators  $A_0$  and  $B_0$  that we introduce in Section 5 are used without a detailed construction and proof of their main properties.) As far as we know, the property of the water waves system described in the third point of Theorem 3.3 is new, and gives us more detailed information on the stability of the closed-loop system.

#### 4. Some background on the partial Dirichlet and Neumann maps in a rectangular domain

In this section we consider two boundary value problems for the Laplacian in the rectangular domain  $\Omega = (0, \pi) \times (-1, 0)$  and we define the corresponding solution operators. Note that,  $\Omega$  being a rectangle, we are able to construct these solution operators, as well as the Dirichlet to Neumann and Neumann to Neumann operators (in the next section) in an elementary and explicit way, using the separation of variables and analysis of Fourier or Dirichlet series. Another possible approach to these issues, pursued in [6], is the use of the much more sophisticated theory of elliptic problems in polygonal domains as described, for instance, in Grisvard [13].

We begin by introducing a self-adjoint operator on  $L^2(\Omega)$  which plays an important role in our arguments in this section.

**Proposition 4.1.** *With  $\Omega$  as in (1.2), we consider the operator  $A_1 : \mathcal{D}(A_1) \rightarrow L^2(\Omega)$  defined by*

$$\mathcal{D}(A_1) = \left\{ f \in \mathcal{H}^2(\Omega) \mid \begin{array}{l} f(x, 0) = 0, \quad \frac{\partial f}{\partial y}(x, -1) = 0 \quad x \in (0, \pi) \\ \frac{\partial f}{\partial x}(0, y) = 0, \quad \frac{\partial f}{\partial x}(\pi, y) = 0 \quad y \in (-1, 0) \end{array} \right\},$$

$$A_1 f = -\Delta f \quad \forall f \in \mathcal{D}(A_1).$$

*Then  $A_1$  is a strictly positive operator on  $L^2(\Omega)$ .*

*Proof.* The operator  $A_1$  is obviously symmetric. Moreover, the family

$$\Psi_{kl}(x, y) = \frac{2}{\sqrt{\pi}} \cos(kx) \sin\left[\left(2l - 1\right)\frac{\pi}{2}y\right] \quad \forall k, l \in \mathbb{N}, (x, y) \in \Omega,$$

is an orthonormal basis for  $L^2(\Omega)$  formed of eigenvectors of  $A_1$ , corresponding to the eigenvalues

$$\lambda_{kl} = k^2 + (2l - 1)^2 \frac{\pi^2}{4} \quad \forall k, l \in \mathbb{N}.$$

Let  $g \in L^2(\Omega)$ , so that  $g = \sum_{k,l \in \mathbb{N}} c_{kl} \Psi_{kl}$ , with  $c_{kl} \in l^2(\mathbb{N}^2)$ . This implies that  $f$  defined by

$$f = \sum_{k,l \in \mathbb{N}} \frac{c_{kl}}{k^2 + (2l - 1)^2 \frac{\pi^2}{4}} \Psi_{kl},$$

satisfies  $f \in \mathcal{D}(A_1)$  and  $A_1 f = g$ . Thus the operator  $A_1$  is onto so that (see, for instance, [2, Proposition 3.2.4])  $A_1$  is self-adjoint. Finally, it follows from the first Green formula that

$$\langle A_1 f, f \rangle_{L^2(\Omega)} = \|\nabla f\|_{L^2(\Omega)}^2 \quad \forall f \in \mathcal{D}(A_1).$$

This, together with a version of the Poincaré inequality (see [2, Theorem 13.6.9]), implies that  $A_1$  is strictly positive.  $\square$

**Proposition 4.2.** *For every  $\eta \in L^2[0, \pi]$  there exists a unique function  $D\eta \in L^2(\Omega)$  such that*

$$\int_{\Omega} (D\eta)(x, y) \overline{g(x, y)} dx dy = - \int_0^{\pi} \eta(x) \overline{\frac{\partial(A_1^{-1}g)}{\partial y}(x, 0)} dx \quad \forall g \in L^2(\Omega). \quad (4.1)$$

Moreover, the operator  $\eta \mapsto D\eta$  (called the partial Dirichlet map) is bounded from  $L^2[0, \pi]$  into  $L^2(\Omega)$ .

*Proof.* We first note from Proposition 4.1 that  $A_1^{-1} \in \mathcal{L}(L^2(\Omega), \mathcal{H}^2(\Omega))$ . Thus, by a standard trace theorem the map  $g \mapsto \frac{\partial(A_1^{-1}g)}{\partial y}(\cdot, 0)$  is bounded from  $L^2(\Omega)$  to  $L^2[0, \pi]$ . Consequently, the right-hand side of (4.1) defines an antilinear functional of the argument  $g \in L^2(\Omega)$ , and the result follows by applying the Riesz representation theorem. (See also [2, Sect. 10.6].)  $\square$

**Remark 4.3.** For every  $\eta \in H$ , we have  $D\eta \in C^\infty(\Omega)$  and  $\Delta(D\eta) = 0$ . Indeed, this follows by an argument that is similar to the one used in the proof of [2, Proposition 10.6.2]: We take  $g = \Delta\varphi$  with  $\varphi \in \mathcal{D}(\Omega)$  in (4.1) to see that  $\Delta(D\eta) = 0$  in the sense of distributions. It follows from [2, Remark 13.5.6] that  $D\eta \in \mathcal{H}_{loc}^n(\Omega)$  for every  $n \in \mathbb{N}$ . Then we use the embedding  $\mathcal{H}_{loc}^n(\Omega) \subset C^m(\Omega)$  for  $n > 1 + m$  ( $m \in \mathbb{N}$ ) (see [2, Remark 13.4.5]), so that

indeed  $D\eta \in C^\infty(\Omega)$ , and hence  $\Delta(D\eta) = 0$ . Moreover, if  $D\eta \in C^1(\overline{\Omega})$ , then  $D\eta$  is the unique function in  $C^2(\Omega) \cap C(\overline{\Omega})$  that satisfies, in the classical sense, the following boundary value problem:

$$\begin{cases} \Delta(D\eta)(x, y) = 0 & ((x, y) \in \Omega), \\ (D\eta)(x, 0) = \eta(x), \quad \frac{\partial(D\eta)}{\partial y}(x, -1) = 0 & (x \in (0, \pi)), \\ \frac{\partial(D\eta)}{\partial x}(0, y) = 0, \quad \frac{\partial(D\eta)}{\partial x}(\pi, y) = 0 & (y \in (-1, 0)). \end{cases} \quad (4.2)$$

To see this, we take in (4.1)  $g = \Delta f$ , where  $f \in \mathcal{D}(A_1)$ , and use integration by parts, which yields that

$$\begin{aligned} \int_0^\pi \eta(x) \overline{\frac{\partial f}{\partial y}(x, 0)} dx &= \int_0^\pi \left[ (D\eta) \overline{\frac{\partial f}{\partial y}} \right] (x, 0) dx \\ &+ \int_0^\pi \left[ \frac{\partial D\eta}{\partial y} \overline{f} \right] (x, -1) dx + \int_{-1}^0 \left[ \frac{\partial D\eta}{\partial x} \overline{f} \right] (0, y) dy - \int_{-1}^0 \left[ \frac{\partial D\eta}{\partial x} \overline{f} \right] (\pi, y) dy. \end{aligned}$$

If we choose  $f \in \mathcal{D}(A_1)$  such that  $f = 0$  on the lateral boundaries and the bottom of  $\Omega$ , we obtain that  $(D\eta)(x, 0) = \eta(x)$  for almost every  $x \in [0, \pi]$ . By choosing suitable other test functions  $f \in \mathcal{D}(A_1)$  (we omit the details), we can obtain also the remaining three equalities in (4.2).

**Remark 4.4.** The term ‘‘partial Dirichlet map’’ comes from the fact that  $D$  acts on the upper boundary of  $\Omega$  rather than the entire boundary  $\partial\Omega$ .

**Lemma 4.5.** *For every  $\eta \in H$ ,  $D\eta$  is given by*

$$(D\eta)(x, y) = \sum_{k \in \mathbb{N}} \frac{\langle \eta, \varphi_k \rangle}{\cosh k} \varphi_k(x) \cosh [k(y + 1)] \quad \forall x, y \in \Omega, \quad (4.3)$$

where the functions  $\varphi_k$  have been introduced in (2.6). Moreover, for every  $\eta \in H_3$  we have  $D\eta \in C^2(\overline{\Omega})$ .

*Proof.* Using Remark 4.3 it is easily checked that for every  $k \in \mathbb{N}$  we have

$$(D\varphi_k)(x, y) = \sqrt{\frac{2}{\pi}} \frac{\cos(kx) \cosh[k(y + 1)]}{\cosh(k)} \quad \forall x, y \in \Omega. \quad (4.4)$$

On the other hand, we can see that the right-hand side of (4.3) converges in  $L^2(\Omega)$ . This fact, together with (4.4) clearly implies (4.3).

Moreover, for every  $\alpha \in \{0, 1, 2\}$  we have

$$\left| \frac{\partial^{\alpha, 2-\alpha}}{\partial x^\alpha \partial y^{2-\alpha}} \left( \frac{\cos(kx) \cosh(k(y+1))}{\cosh(k)} \right) \right| \leq k^2 \quad \forall k \in \mathbb{N}, x, y \in \Omega.$$

Using the Cauchy-Schwarz inequality and the fact that  $\sum_{k \in \mathbb{N}} 1/k^2 = \pi^2/6$ ,

$$\begin{aligned} \sum_{k \in \mathbb{N}} \left| \frac{\partial^{\alpha, 2-\alpha}}{\partial x^\alpha \partial y^{2-\alpha}} \left( \frac{\langle \eta, \varphi_k \rangle}{\cosh k} \varphi_k(x) \cosh(k(y+1)) \right) \right| \\ \leq \sum_{k \in \mathbb{N}} \frac{1}{k} \cdot k^3 |\langle \eta, \varphi_k \rangle| \leq \frac{\pi}{\sqrt{6}} \|\eta\|_{H_3} \quad \forall \eta \in H_3, x, y \in \Omega. \end{aligned}$$

Combining the last estimate with (4.3), we obtain that indeed  $D\eta \in C^2(\overline{\Omega})$  for every  $\eta \in H_3$ .  $\square$

**Corollary 4.6.** *Let  $\gamma_0 : C(\overline{\Omega}) \rightarrow C[-1, 0]$  be the partial Dirichlet trace operator defined by*

$$(\gamma_0 g)(y) = g(0, y) \quad \forall g \in C(\overline{\Omega}), y \in [-1, 0],$$

and let  $D$  be the map defined in Proposition 4.2. Then  $\tilde{C}_0$  defined by

$$\tilde{C}_0 \eta = \gamma_0 D\eta \quad \forall \eta \in H_3$$

can be uniquely extended to a bounded operator  $C_0 \in \mathcal{L}(H, L^2[-1, 0])$ .

*Proof.* According to Lemma 4.5, we have

$$(\tilde{C}_0 \eta)(y) = \sum_{k \in \mathbb{N}} \sqrt{\frac{2}{\pi}} \frac{\langle \eta, \varphi_k \rangle}{\cosh k} \cosh[k(y+1)] \quad \forall \eta \in H_3, y \in [-1, 0],$$

which implies that there exists a constant  $K > 0$  such that

$$\|\tilde{C}_0 \eta\|_{L^2[-1, 0]}^2 \leq K \sum_{k \in \mathbb{N}} |\langle \eta, \varphi_k \rangle|^2 = K \|\eta\|_H^2 \quad \forall \eta \in H_3,$$

which shows that  $\tilde{C}_0$  can be extended as claimed.  $\square$

**Lemma 4.7.** *The partial Dirichlet map  $D$  defined in Proposition 4.2 is bounded from  $H_{\frac{1}{2}}$  to  $\mathcal{H}^1(\Omega)$ , i.e.  $D \in \mathcal{L}(H_{\frac{1}{2}}, \mathcal{H}^1(\Omega))$ . Moreover,*

$$(D\eta)(x, 0) = \eta(x) \quad \forall \eta \in H_{\frac{1}{2}}, \text{ equality in } L^2[0, \pi], \quad (4.5)$$

$$\int_{\Omega} \nabla(D\eta) \cdot \overline{\nabla \Psi} dx dy = 0 \quad \forall \eta \in H_{\frac{1}{2}}, \Psi \in \mathcal{H}_{top}^1(\Omega). \quad (4.6)$$

*Proof.* According to Lemma 4.5,  $D\eta$  is given by (4.3). Since  $\left\{\sqrt{\frac{2}{\pi}} \sin(kx)\right\}_{k \in \mathbb{N}}$  is an orthonormal basis in  $L^2[0, \pi]$ , we have that for every  $\eta \in H_{\frac{1}{2}}$ ,

$$\begin{aligned} \left\| \frac{\partial(D\eta)}{\partial x} \right\|_{L^2(\Omega)}^2 &= \int_{-1}^0 \int_0^\pi \left| \sum_{k \in \mathbb{N}} \sqrt{\frac{2}{\pi}} \frac{k \langle \eta, \varphi_k \rangle}{\cosh k} \cosh[k(y+1)] \sin(kx) \right|^2 dx dy \\ &\leq \sum_{k \in \mathbb{N}} \frac{|k \langle \eta, \varphi_k \rangle|^2}{\cosh^2 k} \int_{-1}^0 \cosh^2[k(y+1)] dy \\ &= \sum_{k \in \mathbb{N}} \frac{k^2 |\langle \eta, \varphi_k \rangle|^2}{\cosh^2 k} + \sum_{k \in \mathbb{N}} \frac{k |\langle \eta, \varphi_k \rangle|^2}{2 \cosh^2 k} \sinh(2k), \end{aligned}$$

which clearly implies that there exists  $K_1 > 0$  such that

$$\left\| \frac{\partial(D\eta)}{\partial x} \right\|_{L^2(\Omega)} \leq K_1 \|\eta\|_{\frac{1}{2}} \quad \forall \eta \in H_{\frac{1}{2}}.$$

A similar estimate for  $\left\| \frac{\partial(D\eta)}{\partial y} \right\|_{L^2}$  can be obtained in a completely similar manner. Moreover, we know from Proposition 4.2 that  $\|D\eta\|_{L^2}$  is also bounded by a similar estimate. Recalling (1.1) we conclude that  $D \in \mathcal{L}(H_{\frac{1}{2}}, \mathcal{H}^1(\Omega))$ .

Formula (4.5) in the lemma follows from the last part of Lemma 4.5 together with Remark 4.3 and the density of  $H_3$  in  $H_{\frac{1}{2}}$ .

To prove (4.6) first we assume that  $\eta \in H_3$  so that, according to Remark 4.3,  $D\eta$  is the unique classical solution of (4.2). Multiplying the first equation in (4.2) by  $\bar{\Psi} \in \mathcal{H}_{top}^1(\Omega)$  and integrating by parts, it follows that (4.6) holds for  $\eta \in H_3$ . Using the density of  $H_3$  in  $H_{\frac{1}{2}}$  and the fact that  $D \in \mathcal{L}(H_{\frac{1}{2}}, \mathcal{H}^1(\Omega))$ , it follows that indeed (4.6) holds for all  $\eta \in H_{\frac{1}{2}}$ .  $\square$

The second important map constructed in this section is a partial Neumann map. To this aim, recall the space  $\mathcal{H}_{top}^1(\Omega)$  introduced in (1.3) and notice that, due the version of the Poincaré inequality in [2, Theorem 13.6.9], the sesquilinear form on  $\mathcal{H}_{top}^1(\Omega)$  given by

$$a[f, g] = \int_{\Omega} \nabla f \cdot \bar{\nabla} g \, dx \, dy \quad \forall f, g \in \mathcal{H}_{top}^1(\Omega) \quad (4.7)$$

defines an inner product on  $\mathcal{H}_{top}^1(\Omega)$  which is equivalent to the one inherited from  $\mathcal{H}^1(\Omega)$ . These facts, combined with the continuity of the Dirichlet trace (as an operator from  $\mathcal{H}^1(\Omega)$  to  $L^2(\partial\Omega)$ ), imply the following:

**Proposition 4.8.** For every  $v \in L^2[-1, 0]$  there exists a unique function  $Nv \in \mathcal{H}_{top}^1(\Omega)$  such that

$$\int_{\Omega} \nabla(Nv) \cdot \overline{\nabla g} \, dx \, dy = \int_{-1}^0 v(y) \overline{g(0, y)} \, dy \quad \forall g \in \mathcal{H}_{top}^1(\Omega). \quad (4.8)$$

Moreover, the operator  $N$ , called a partial Neumann map, is linear and bounded from  $L^2[-1, 0]$  to  $\mathcal{H}_{top}^1(\Omega)$ .

*Proof.* The results follow from the Lax-Milgram theorem by using the sesquilinear form  $a[\cdot, \cdot]$  introduced in (4.7) (see also [14, Proposition 7.1]).  $\square$

**Remark 4.9.** The above proposition can be formulated also as follows: for every  $v \in L^2[-1, 0]$  the boundary value problem

$$\begin{cases} \Delta f(x, y) = 0 & ((x, y) \in \Omega), \\ f(x, 0) = 0, \quad \frac{\partial f}{\partial y}(x, -1) = 0 & (x \in (0, \pi)), \\ \frac{\partial f}{\partial x}(0, y) = -v, \quad \frac{\partial f}{\partial x}(\pi, y) = 0 & (y \in (-1, 0)), \end{cases} \quad (4.9)$$

admits a unique weak solution  $f = Nv \in \mathcal{H}_{top}^1(\Omega)$ . If  $f \in C^2(\overline{\Omega})$  and  $v \in C[-1, 0]$ , then  $f = Nv$  is the unique classical solution of (4.9).

We note that the sequence  $(\psi_k)_{k \in \mathbb{N}}$  defined by

$$\psi_k(y) = \sqrt{2} \cos \left[ (2k-1) \frac{\pi}{2} (y+1) \right] \quad \forall k \in \mathbb{N}, \, y \in [-1, 0], \quad (4.10)$$

is an orthonormal basis in  $L^2[-1, 0]$  (see [2, [Sect. 2.6]]). We can use this basis to construct the scale of Hilbert spaces  $(\mathcal{U}_\beta)_{\beta \geq 0}$  defined by  $\mathcal{U}_0 = L^2[-1, 0]$  and (for  $\beta > 0$ )

$$\mathcal{U}_\beta = \left\{ v \in \mathcal{U}_0 \mid \sum_{k \in \mathbb{N}} (2k-1)^{2\beta} \left| \int_{-1}^0 v(y) \psi_k(y) \, dy \right|^2 < \infty \right\}, \quad (4.11)$$

with the inner products  $(\langle \cdot, \cdot \rangle_\beta)_{\beta \geq 0}$  given, for every  $v, \chi \in \mathcal{U}_\beta$ , by

$$\langle v, \chi \rangle_{\mathcal{U}_\beta} = \sum_{k \in \mathbb{N}} (2k-1)^{2\beta} \left( \int_{-1}^0 v(y) \psi_k(y) \, dy \right) \overline{\left( \int_{-1}^0 \chi(y) \psi_k(y) \, dy \right)}.$$

**Lemma 4.10.** *Let  $N$  be the operator defined in Proposition 4.8. Then for every  $v \in L^2[-1, 0]$  and every  $(x, y) \in \Omega$  we have*

$$(Nv)(x, y) = \sum_{k \in \mathbb{N}} a_k \cosh \left[ (2k-1) \frac{\pi}{2} (x - \pi) \right] \cos \left[ (2k-1) \frac{\pi}{2} (y + 1) \right], \quad (4.12)$$

with convergence in  $\mathcal{H}_{top}^1(\Omega)$ , where

$$a_k = \frac{2\sqrt{2} \langle v, \psi_k \rangle}{(2k-1)\pi \sinh \left[ (2k-1) \frac{\pi^2}{2} \right]} \quad \forall k \in \mathbb{N}.$$

Moreover, for every  $v \in \mathcal{U}_2$  we have  $Nv \in C^2(\bar{\Omega})$ .

*Proof.* By using Remark 4.9 and separation of variables, we see that

$$(N\psi_k)(x, y) = \frac{2\psi_k(y) \cosh \left[ (2k-1) \frac{\pi}{2} (x - \pi) \right]}{(2k-1)\pi \sinh \left[ (2k-1) \frac{\pi^2}{2} \right]} \quad \forall k \in \mathbb{N}, \quad (4.13)$$

for all  $(x, y) \in \Omega$ . Since  $Nv = \sum_{k \in \mathbb{N}} \langle v, \psi_k \rangle N\psi_k$ , this clearly implies (4.12), with convergence in  $\mathcal{H}_{top}^1(\Omega)$  due to Proposition 4.8. For every  $j \in \{0, 1, 2\}$ ,

$$\left| \frac{\partial^{j, 2-j}}{\partial x^j \partial y^{2-j}} (N\psi_k)(x, y) \right| \leq \frac{\sqrt{2}}{2} \pi (2k-1) \quad \forall k \in \mathbb{N}, (x, y) \in \Omega,$$

so that for every  $v \in \mathcal{U}$ , the series  $Nv = \sum_{k \in \mathbb{N}} \langle v, \psi_k \rangle N\psi_k$  converges in  $C^2(\bar{\Omega})$  if the sequence  $k \langle v, \psi_k \rangle$  is in  $l^1$ . For this (by an argument similar to the one in the proof of Lemma 4.5) it is sufficient if the sequence  $k^2 \langle v, \psi_k \rangle$  is in  $l^2$ , which is precisely the condition  $v \in \mathcal{U}_2$ .  $\square$

## 5. Partial Dirichlet to Neumann and Neumann to Neumann maps

In this section we give an explicit construction of the operators allowing us to recast (2.1) as a well-posed linear control system. Recall the orthonormal basis  $(\varphi_k)_{k \in \mathbb{N}}$  in  $H$  introduced in (2.6) and the corresponding spaces  $H_\alpha$ . First we note a direct consequence of Proposition 4.2 and of Lemma 4.5.

**Corollary 5.1.** *Let  $\gamma_1 : C^1(\bar{\Omega}) \rightarrow C[0, \pi]$  be the partial Neumann trace operator defined by*

$$(\gamma_1 f)(x) = \frac{\partial f}{\partial y}(x, 0) \quad \forall f \in C^1(\bar{\Omega}), x \in [0, \pi].$$



Then  $\tilde{A}_0$  defined by

$$\tilde{A}_0\eta = \gamma_1 D\eta \quad \forall \eta \in H_3,$$

where  $D$  is the Dirichlet map defined in Proposition 4.2, is a linear bounded map from  $H_3$  to  $C[0, \pi]$ . Moreover, we have

$$\tilde{A}_0\varphi_k = k \tanh(k)\varphi_k \quad \forall k \in \mathbb{N}.$$

We are now in a position to define a partial Dirichlet to Neumann map.

**Proposition 5.2.** *The operator  $\tilde{A}_0$  introduced in Corollary 5.1 has a unique continuous extension to an operator  $A_0 : H_1 \rightarrow H$ . This extension is strictly positive and  $\mathcal{D}(A_0^{\frac{1}{2}}) = H_{\frac{1}{2}}$ . For each  $k \in \mathbb{N}$ , we have  $A_0\varphi_k = \lambda_k\varphi_k$ , where*

$$\lambda_k = k \tanh(k) \quad \forall k \in \mathbb{N} \quad (5.1)$$

and

$$A_0\eta = \sum_{k \in \mathbb{N}} \lambda_k \langle \eta, \varphi_k \rangle \varphi_k \quad \forall \eta \in H_1. \quad (5.2)$$

*Proof.* It is clear from the previous proposition that  $A_0$  fits into the class of diagonalizable operators discussed in [2] in Proposition 3.2.9 and the remarks after it, and in Proposition 3.4.8 of the same book.  $\square$

**Proposition 5.3.** *Let  $A_0$  and  $D$  be the operators introduced in Propositions 5.2 and 4.2, respectively. Let  $\gamma \in H_{\frac{1}{2}}$  and  $\Psi \in \mathcal{H}^1(\Omega)$  be such that*

$$\Psi(x, 0) = \gamma(x), \quad \text{equality in } L^2[0, \pi].$$

Then for every  $\eta \in H_{\frac{1}{2}}$  we have  $D\eta \in \mathcal{H}^1(\Omega)$  and

$$\langle A_0^{\frac{1}{2}}\eta, A_0^{\frac{1}{2}}\gamma \rangle = \langle \nabla(D\eta), \nabla\Psi \rangle_{L^2(\Omega)}. \quad (5.3)$$

*Proof.* First we assume that  $\eta \in H_3$ , so that according to Lemma 4.5 we have  $D\eta \in C^2(\bar{\Omega})$ . Then (5.3) follows by a simple integration by parts and Proposition 5.2. The fact that  $D\eta \in \mathcal{H}^1(\Omega)$  for every  $\eta \in H_{\frac{1}{2}}$  has already been proved in Lemma 4.7. Finally, to prove that (5.3) still holds for  $\eta \in H_{\frac{1}{2}}$  it suffices to use the density of  $H_3$  in  $H_{\frac{1}{2}}$ , combined with Lemma 4.7.  $\square$

**Corollary 5.4.** *With  $\gamma_1$  as in Corollary 5.1, define the operator  $\tilde{B}_1$  by*

$$\tilde{B}_1 v = \gamma_1 N v \quad \forall v \in \mathcal{U}_2,$$

where  $N$  is the Neumann map introduced in Proposition 4.8. Then  $\tilde{B}_1$  is a bounded linear operator from  $\mathcal{U}_2$  to  $C[0, \pi]$ . Moreover, we have

$$\left( \tilde{B}_1 \psi_k \right) (x) = \frac{(-1)^k \sqrt{2}}{\sinh \left[ (2k-1) \frac{\pi^2}{2} \right]} \cosh \left[ (2k-1) \frac{\pi}{2} (x - \pi) \right], \quad (5.4)$$

for all  $k \in \mathbb{N}$ ,  $x \in [0, \pi]$ , where the functions  $\psi_k$  have been defined in (4.10).

*Proof.* This follows from Lemma 4.10 and the formula (4.13) for  $N\psi_k$ .  $\square$

We are now ready to define a Neumann to Neumann map.

**Theorem 5.5.** *The operator  $\tilde{B}_1$  introduced in Corollary 5.4 can be extended in a unique manner to a linear bounded operator  $B_1 : L^2[-1, 0] \rightarrow L^2[0, \pi]$ . Moreover, for every  $v \in L^2[-1, 0]$  with  $\int_{-1}^0 v(y) dy = 0$  we have that  $B_1 v \in H$ , where  $H$  is defined in (2.5). Finally,*

$$\begin{aligned} \int_0^\pi (B_1 v)(x) \overline{\Psi(x, 0)} dx &= \int_\Omega \nabla(Nv)(x, y) \cdot \overline{\nabla \Psi(x, y)} dx dy \\ &\quad - \int_{-1}^0 v(y) \overline{\Psi(0, y)} dy \quad \forall \Psi \in \mathcal{H}^1(\Omega). \end{aligned} \quad (5.5)$$

*Proof.* For any  $v \in L^2[-1, 0]$  we set

$$b_k = \frac{(-1)^k \sqrt{2}}{\sinh \left[ (2k-1) \frac{\pi^2}{2} \right]}, \quad v_k = \langle v, \psi_k \rangle,$$

and notice that these sequences are in  $l^2$  and  $\|(v_k)\|_{l^2} = \|v\|_{L^2[-1, 0]}$ . From (5.4) it follows that if  $v \in \mathcal{U}_2$  then for every  $x \in [0, \pi]$  we have

$$(\tilde{B}_1 v)(x) = \sum_{k \in \mathbb{N}} b_k v_k \cosh \left[ (2k-1) \frac{\pi}{2} (x - \pi) \right] = \frac{f(x) + g(x)}{2}, \quad (5.6)$$

where

$$f(x) = \sum_{k \in \mathbb{N}} b_k v_k \exp \left[ (2k-1) \frac{\pi}{2} (x - \pi) \right],$$

$$g(x) = \sum_{k \in \mathbb{N}} b_k v_k \exp \left[ (2k-1) \frac{\pi}{2} (\pi - x) \right]. \quad (5.7)$$

On one hand, from  $0 \leq \exp \left[ (2k-1) \frac{\pi}{2} (x - \pi) \right] \leq 1$  for all  $x \in [0, \pi]$ , by using Cauchy-Schwarz we obtain that there exists  $C_1 > 0$  such that

$$\int_0^\pi |f(x)|^2 dx \leq C_1 \|v\|_{L^2[-1,0]}^2 \quad \forall v \in \mathcal{U}_2. \quad (5.8)$$

On the other hand, from (5.7) it follows that

$$\int_0^\infty |g(x)|^2 dx = \frac{1}{\pi} \sum_{k,l \in \mathbb{N}} \frac{c_k \bar{c}_l v_k \bar{v}_l}{k+l-1},$$

where  $c_k = b_k \exp \left[ (2k-1) \frac{\pi^2}{2} \right]$  for all  $k \in \mathbb{N}$ . Using that  $|c_k| \leq |c_1| < \sqrt{10}$  for all  $k \in \mathbb{N}$ , together with Hilbert's inequality, see for instance [15, Chapter IX] or the nice survey [16], we obtain that

$$\int_0^\infty |g(x)|^2 dx \leq 10 \sum_{k \in \mathbb{N}} |v_k|^2 \quad \forall v \in \mathcal{U}_2. \quad (5.9)$$

Putting together (5.6), (5.8) and (5.9), it follows that there exists  $C > 0$  such that

$$\|\tilde{B}_1 v\|_{L^2[0,\pi]}^2 \leq C \|v\|_{L^2[-1,0]}^2 \quad \forall v \in \mathcal{U}_2.$$

The above estimate, combined with the density of  $\mathcal{U}_2$  in  $L^2[-1,0]$ , implies that indeed  $\tilde{B}_1$  admits a unique extension  $B_1 \in \mathcal{L}(L^2[-1,0], L^2[0,\pi])$ .

Assume again that  $v \in \mathcal{U}_2$ . Then, according to Remark 4.9 and to Lemma 4.10 we have that  $f = Nv$  is a classical solution of (4.9), so that for every  $v \in \mathcal{U}_2$  we have

$$\begin{aligned} 0 &= \int_{\Omega} \Delta(Nv)(x,y) \overline{\Psi(x,y)} dx dy \\ &= \int_{-1}^0 v(y) \overline{\Psi(0,y)} dy + \int_0^\pi (B_1 v)(x) \overline{\Psi(x,0)} dx - \int_{\Omega} \nabla(Nv) \cdot \overline{\nabla \Psi} dx dy. \end{aligned}$$

Thus (5.5) holds for  $v \in \mathcal{U}_2$  and by density for  $v \in L^2[-1,0]$ . Using that  $\int_{-1}^0 v(y) dy = 0$  and taking  $\Psi = 1$  in (5.5) we obtain that  $B_1 v$  indeed satisfies the condition  $\int_0^\pi (B_1 v)(x) dx = 0$ , which implies that  $B_1 v \in H$ .  $\square$

**Remark 5.6.** An alternative proof of (5.9) can be given using the Carleson measure criterion for admissibility, see for instance [2, Sect. 5.3]. To this aim, consider the Hilbert space  $\tilde{X} = l^2$ , the strictly negative operator  $\tilde{A} = \text{diag} \left( -\frac{(2k-1)\pi}{2} \right)$  and the observation functional  $\tilde{C} = [c_1 \ c_2 \ c_3 \ \dots]$ . Then according to the aforementioned criterion,  $\tilde{C}$  is an admissible observation operator for the operator semigroup generated by  $\tilde{A}$ , and (5.9) follows.

The above theorem clearly implies the following result:

**Corollary 5.7.** *Let  $h \in L^2[-1, 0]$ , with  $\int_{-1}^0 h(y) dy = 0$  and let  $B_0$  be the operator defined by*

$$B_0 u = u B_1 h \quad \forall u \in \mathbb{C}.$$

*Then  $B_0 \in \mathcal{L}(\mathbb{C}, H)$ . Moreover, we have*

$$\int_0^\pi (B_0 u)(x) \overline{\Psi(x, 0)} dx = u \int_\Omega \nabla(Nh) \cdot \overline{\nabla \Psi} dx dy - u \int_{-1}^0 h(y) \overline{\Psi(0, y)} dy, \quad (5.10)$$

*for all  $u \in \mathbb{C}$  and  $\Psi \in \mathcal{H}^1(\Omega)$ . In particular,*

$$B_0^* \eta = - \int_{-1}^0 h(y) \overline{(C_0 \eta)(y)} dy \quad \forall \eta \in H, \quad (5.11)$$

*where  $C_0 = \gamma_0 D$  is the operator introduced in Corollary 4.6.*

*Proof.* The fact that  $B_0 \in \mathcal{L}(\mathbb{C}, H)$  and (5.10) follows from Theorem 5.5 (in particular from (5.5)) with  $v = uh$ . Moreover, taking  $\Psi = D\eta$  with  $\eta \in H_{\frac{1}{2}}$  (see Lemma 4.7) in (5.10), we see that for every  $u \in \mathbb{C}$ ,

$$\langle B_0 u, \eta \rangle = u \langle B_1 h, \eta \rangle = -u \int_{-1}^0 h(y) \overline{(C_0 \eta)(y)} dy + u \int_\Omega \nabla(Nh) \cdot \overline{\nabla(D\eta)} dx dy.$$

Using (4.6) it follows that the last term in the right-hand side of the above equation is zero, so that we obtain (5.11).  $\square$

## 6. Proof of the main results

Throughout this section we denote by  $X$  the Hilbert space  $H_{\frac{1}{2}} \times H$ , where  $H$  and  $(H_\alpha)_{\alpha>0}$  have been defined in (2.5) and (2.7), respectively. We also introduce the linear operator  $A : \mathcal{D}(A) \rightarrow X$  with  $\mathcal{D}(A) = H_1 \times H_{\frac{1}{2}}$  and

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad \text{i.e.,} \quad A \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} \psi \\ -A_0 \varphi \end{bmatrix} \quad \forall \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A), \quad (6.1)$$

where  $A_0 = \gamma_1 D$  is the strictly positive operator on  $H$ , with domain  $H_1$ , which has been introduced in Proposition 5.2. We redefine the inner product on  $H_{\frac{1}{2}}$  as

$$\langle x, z \rangle_{\frac{1}{2}} = \left\langle A_0^{\frac{1}{2}} x, A_0^{\frac{1}{2}} z \right\rangle,$$

which is equivalent to the original inner product on  $H_{\frac{1}{2}}$ . Then  $A$  is skew-adjoint on  $X$  (see, for instance, [2, Proposition 3.7.6]), so that, according to Stone's theorem (see, for instance [2, Section 3.7]),  $A$  generates a group  $\mathbb{T} = (\mathbb{T}_t)_{t \in \mathbb{R}}$  of unitary operators on  $X$ . Moreover, we recall that  $h \in L^2[-1, 0]$ , with  $\int_{-1}^0 h(y) dy = 0$  and that we have introduced the input space  $U = \mathbb{C}$ . Let  $B \in \mathcal{L}(U, X)$  be given by

$$B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, \quad (6.2)$$

where  $B_0 \in \mathcal{L}(U, H)$  is as in Corollary 5.7. Clearly  $B \in \mathcal{L}(U, X)$ .

We are now in a position to prove our main well-posedness result:

*Proof of Theorem 2.4.* With the above notation for  $X$ ,  $A$ ,  $U$  and  $B$  we consider, for each  $\tau \geq 0$ , the map  $\Phi_\tau$  defined by (2.13), which is clearly linear and bounded from  $L^2([0, \infty); U)$  into  $X$ . Let  $z_0 = \begin{bmatrix} \zeta_0 \\ w_0 \end{bmatrix} \in X$ , let  $u \in L^2_{loc}([0, \infty); U)$  and define  $z(t) = \begin{bmatrix} \zeta \\ w \end{bmatrix} \in C([0, \infty); X)$  by (2.12). Then, according to a classical result (see, for instance, [2, Remark 4.1.2]), for every  $t \geq 0$  and  $\psi \in \mathcal{D}(A)$  we have

$$\langle z(t) - z_0, \psi \rangle_X = \int_0^t [-\langle z(\sigma), A\psi \rangle_X + \langle Bu(\sigma), \psi \rangle_X] d\sigma.$$

Setting  $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$ , with  $\psi_1 \in H_1$  and  $\psi_2 \in H_{\frac{1}{2}}$  and using the specific structure (6.1), (6.2) of  $A$  and  $B$ , the last formula implies that

$$\begin{aligned} \langle A_0^{\frac{1}{2}}(\zeta(t) - \zeta_0), A_0^{\frac{1}{2}}\psi_1 \rangle + \langle w(t) - w_0, \psi_2 \rangle &= - \left\langle \int_0^t A_0^{\frac{1}{2}}\zeta(\sigma) d\sigma, A_0^{\frac{1}{2}}\psi_2 \right\rangle \\ &+ \left\langle \int_0^t w(\sigma) d\sigma, A_0\psi_1 \right\rangle + \int_0^t \langle B_0 u(\sigma), \psi_2 \rangle d\sigma, \end{aligned} \quad (6.3)$$

for every  $t \geq 0$ ,  $\psi_1 \in H_1$ ,  $\psi_2 \in H_{\frac{1}{2}}$ . The above formula holds, in particular, for  $\psi_2 = 0$  and arbitrary  $\psi_1 \in H_1$ , which yields that

$$\zeta(t) - \zeta_0 = \int_0^t w(\sigma) d\sigma \quad \forall t \geq 0,$$

so that  $w(t) = \dot{\zeta}(t)$ , for all  $t \geq 0$ . Inserting the last two formulas in (6.3), we obtain that

$$\langle \dot{\zeta}(t) - w_0, \psi_2 \rangle = - \int_0^t \langle A_0^{\frac{1}{2}} \zeta(\sigma), A_0^{\frac{1}{2}} \psi_2 \rangle d\sigma + \int_0^t \langle B_0 u(\sigma), \psi_2 \rangle d\sigma, \quad (6.4)$$

where  $t \geq 0$ ,  $w_0 = \dot{\zeta}(0)$  and  $\psi_2 \in H_{\frac{1}{2}}$ . (This formula (6.4) is the weak form of the equation  $\ddot{\zeta} = -A_0 \zeta + B_0 u$ .) Let  $\Psi \in \mathcal{H}^1(\Omega)$  be such that  $\int_0^\pi \Psi(x, 0) dx = 0$  and then  $\psi_2(x) = \Psi(x, 0)$  is a function in  $H_{\frac{1}{2}}$ . By combining (6.4) and (5.10) it follows that

$$\begin{aligned} \langle \dot{\zeta}(t) - w_0, \psi_2 \rangle &= - \int_0^t \langle A_0^{\frac{1}{2}} \zeta(\sigma), A_0^{\frac{1}{2}} \psi_2 \rangle d\sigma \\ &+ \int_0^t u(\sigma) \int_\Omega \nabla(Nh) \cdot \overline{\nabla \Psi} dx dy d\sigma - \int_0^t u(\sigma) \int_{-1}^0 h(y) \overline{\Psi(0, y)} dy d\sigma. \end{aligned} \quad (6.5)$$

On the other hand, from Proposition 5.3 it follows that  $D\zeta(\sigma) \in \mathcal{H}^1(\Omega)$  and

$$\langle A_0^{\frac{1}{2}} \zeta(\sigma), A_0^{\frac{1}{2}} \psi_2 \rangle = \langle \nabla(D\zeta(\sigma)), \nabla \Psi \rangle \quad \forall \Psi \in \mathcal{H}^1(\Omega), \psi_2(x) = \Psi(x, 0).$$

The above formula, when combined with (6.5), and setting

$$\phi(t, \cdot, \cdot) = - [D\zeta(t)](\cdot, \cdot) + u(t)(Nh)(\cdot, \cdot) \quad \forall t \geq 0, \quad (6.6)$$

implies that  $\phi(t, \cdot, \cdot) \in \mathcal{H}^1(\Omega)$  and  $(\phi, \zeta)$  satisfies (2.10) for every  $\Psi \in \mathcal{H}^1(\Omega)$  with  $\int_0^\pi \Psi(x, 0) dx = 0$ . On the other hand,  $(\phi, \zeta)$  obviously satisfies (2.10) if  $\Psi$  is a constant function, thus  $(\phi, \zeta)$  satisfies (2.10) for every  $\Psi \in \mathcal{H}^1(\Omega)$ . Moreover, according to Lemma 4.7, Proposition 4.8 and the above definition of  $\phi$ , we have that  $\phi \in L_{loc}^2([0, \infty), \mathcal{H}^1(\Omega))$  and (2.9) holds, so that  $(\phi, \zeta)$  is a solution of (2.1) in the sense of Definition 2.2.

Conversely, assume that  $(\phi, \zeta)$  is a solution of (2.1) in the sense of Definition 2.2, with  $\zeta(0) = \zeta_0 \in H_{\frac{1}{2}}$  and  $\dot{\zeta}(0) = w_0 \in H$ . Using the fact that (2.10) holds, in particular, for  $\Psi \in \mathcal{H}_{top}^1(\Omega)$  it follows that for every  $t \geq 0$  and every  $\Psi \in \mathcal{H}_{top}^1(\Omega)$  we have

$$\int_\Omega \nabla \phi(t, x, y) \cdot \overline{\nabla \Psi(x, y)} dx dy - u(t) \int_{-1}^0 h(y) \overline{\Psi(0, y)} dy = 0.$$

Using the notation

$$\tilde{\phi}(t, \cdot, \cdot) = \phi(t, \cdot, \cdot) - u(t)(Nh)(\cdot, \cdot), \quad (6.7)$$

where  $N$  is the Neumann map defined in Proposition 4.8, it follows that

$$\int_{\Omega} \nabla \tilde{\phi}(t, x, y) \cdot \overline{\nabla \Psi(x, y)} dx dy = 0 \quad \forall \Psi \in \mathcal{H}_{top}^1(\Omega).$$

The last formula holds, in particular, for  $\Psi \in \mathcal{D}(A_1)$ , where  $\mathcal{D}(A_1)$  has been defined in Proposition 4.1, so that an integration by parts yields that

$$\int_{\Omega} \tilde{\phi}(t, x, y) \overline{\Delta \Psi(x, y)} dx dy = \int_0^{\pi} \tilde{\phi}(t, x, 0) \overline{\frac{\partial \Psi}{\partial y}(x, 0)} dx \quad \forall \Psi \in \mathcal{D}(A_1). \quad (6.8)$$

Moreover, according to Definition 2.2 and Proposition 4.8, we have (2.9) and  $(Nh)(x, 0) = 0$  for  $x \in [0, \pi]$ , so that from (6.8) it follows that

$$\int_{\Omega} \tilde{\phi}(t, x, y) \overline{\Delta \Psi(x, y)} dx dy = - \int_0^{\pi} \zeta(t, x) \overline{\frac{\partial \Psi}{\partial y}(x, 0)} dx \quad \forall \Psi \in \mathcal{D}(A_1).$$

Comparing the above formula with the definition (4.1) of the Dirichlet map, with  $g = \overline{\Delta \Psi} = -\overline{A_1 \Psi}$ , and recalling that  $A_1$  is onto, it follows that

$$\tilde{\phi}(t, \cdot, \cdot) = - [D\zeta(t)](\cdot, \cdot) \quad \forall t \geq 0.$$

The last formula and (6.7) yield that again (6.6) holds.

Now we take  $\psi_2 \in H_{\frac{1}{2}}$  and we recall from Lemma 4.7 that  $D\psi_2 \in \mathcal{H}^1(\Omega)$  and that  $(D\psi_2)(x, 0) = \psi_2(x)$  for  $x \in [0, \pi]$ . We can thus choose  $\Psi = D\psi_2$  in (2.10) and using Proposition 5.3 and Corollary 5.7, it follows that  $\zeta$  satisfies (6.4). This easily implies that  $z = \begin{bmatrix} \zeta \\ \zeta \end{bmatrix}$  satisfies (2.12).  $\square$

In order to prove Theorem 3.3, we need the following preliminary result on the eigenvalues and the eigenvectors of the operator  $A$  introduced at the beginning of this section.

**Lemma 6.1.** *Let  $(\lambda_k)_{k \in \mathbb{N}}$  and  $(\varphi_k)_{k \in \mathbb{N}}$  be the sequences defined in (5.1) and (2.6), respectively. We extend the sequences  $\mu_k = (\sqrt{\lambda_k})_{k \in \mathbb{N}}$  and  $(\varphi_k)_{k \in \mathbb{N}}$  to  $\mathbb{Z}^*$  by setting*

$$\mu_{-k} = -\mu_k, \quad \varphi_{-k} = -\varphi_k \quad \forall k \in \mathbb{N}.$$

*Then the family  $\{\phi_k\}_{k \in \mathbb{Z}^*}$  defined by*

$$\phi_k = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{i\mu_k} \varphi_k \\ \varphi_k \end{bmatrix} \quad \forall k \in \mathbb{Z}^* \quad (6.9)$$

is an orthonormal basis in  $X$  formed of eigenvectors of the operator  $A$  defined in (6.1). Moreover, for each  $k \in \mathbb{Z}^*$ ,  $A\phi_k = i\mu_k\phi_k$ . Finally, there exists  $\varepsilon > 0$  such that for every  $\omega \in \mathbb{R}$  with  $|\omega| \geq 1$ , the interval  $\left[\omega - \frac{\varepsilon}{|\omega|}, \omega + \frac{\varepsilon}{|\omega|}\right]$  contains at most one element of the sequence  $(\mu_k)_{k \in \mathbb{Z}^*}$ .

*Proof.* According to Proposition 5.2, the family  $(\varphi_k)_{k \in \mathbb{N}}$  defined in (2.6) is an orthonormal basis in  $H$  formed of eigenvectors of  $A_0$  and for each  $k \in \mathbb{N}$ ,  $A_0\varphi_k = \lambda_k\varphi_k$ , where  $(\lambda_k)_{k \in \mathbb{N}}$  have been defined in (5.1). Using the structure (6.1) of  $A$  and a classical result (see, for instance, [2, Section 3.7]), it follows that  $A$  is diagonalizable, with the eigenvalues  $(i\mu_k)_{k \in \mathbb{Z}^*}$  corresponding to the orthonormal basis of eigenvectors  $(\phi_k)_{k \in \mathbb{Z}^*}$ .

Next we prove the last assertion in the lemma by a contradiction argument. Note that for  $k \in \mathbb{N}$ ,  $\mu_k \approx \sqrt{k}$ , with exponentially vanishing approximation error. If we assume that the assertion in the lemma is false, we obtain the existence of a positive sequence  $(\omega_n)$  with  $\omega_n \rightarrow \infty$  and of a sequence  $(k_n)$  in  $\mathbb{N}$  with  $k_n \rightarrow \infty$  such that

$$\{\mu_{k_n}, \mu_{k_n+1}\} \subset \left[\omega_n - \frac{1}{n\omega_n}, \omega_n + \frac{1}{n\omega_n}\right]. \quad (6.10)$$

This implies that  $\lim_{n \rightarrow \infty} \omega_n(\mu_{k_n+1} - \mu_{k_n}) = 0$ . Combining this fact with (6.10), it follows that

$$\lim_{n \rightarrow \infty} \mu_{k_n}(\mu_{k_n+1} - \mu_{k_n}) = 0. \quad (6.11)$$

On the other hand, it is not difficult to check, using (5.1), that

$$\lim_{k \rightarrow \infty} \mu_k(\mu_{k+1} - \mu_k) = \frac{1}{2},$$

which clearly contradicts (6.11) and thus ends the proof.  $\square$

We are now in a position to prove our main stabilizability result.

*Proof of Theorem 3.3.* 1. The first assertion follows directly from Curtain and Zwart [17, Theorem 5.2.6], since  $A$  has infinitely many unstable eigenvalues. Alternatively, we can apply the main result of Gibson [18] or Guo, Guo and Zhang [19, Theorem 3].

2. To prove the second assertion, notice that, since the adjoint of the operator  $B$  defined in (6.2) is  $B^* = \begin{bmatrix} 0 & B_0^* \end{bmatrix}$ , we see from (5.11) and (6.9) that

$$B^*\phi_k = \frac{-1}{\sqrt{2}} \int_{-1}^0 h(y) \overline{(C_0\varphi_k)(y)} dy \quad \forall k \in \mathbb{Z}^*,$$



where  $C_0 = \gamma_0 D$ . Using (4.4), we get from the above that

$$B^* \phi_k = -\frac{1}{\sqrt{\pi}} \int_{-1}^0 h(y) \frac{\cosh[k(y+1)]}{\cosh k} dy \quad \forall k \in \mathbb{Z}^*. \quad (6.12)$$

Assume now that  $h$  is a strategic profile, i.e., (3.2) holds. Then clearly

$$B^* \phi_k \neq 0 \quad \forall k \in \mathbb{Z}^*.$$

According to [2, Proposition 6.9.1] the pair  $(A^*, B^*)$  is approximately observable in infinite time (we have used that the eigenvalues of  $A$  are distinct). Now it follows from the main result of Benchimol [20] that the semigroup generated by  $A - BB^*$  is strongly stable.

Conversely, let us assume that  $h$  is not a strategic profile, i.e., that  $h$  does not satisfy assumption (3.2). Then from (6.12) there exists a  $k \in \mathbb{N}$  such that  $B^* \phi_k = 0$ . Since  $A^* = -A$ , it follows that for every  $F \in \mathcal{L}(X, U)$ ,

$$(A^* + F^* B^*) \phi_k = -i\mu_k \phi_k.$$

Let  $\mathbb{T}^{cl}$  denote the semigroup generated by  $A + BF$ , we have

$$(\mathbb{T}_t^{cl})^* \phi_k = e^{-i\mu_k t} \phi_k \quad \forall t \geq 0,$$

which implies that  $|\langle \mathbb{T}_t^{cl} \phi_k, \phi_k \rangle| = 1$  for all  $t \geq 0$ , so that  $\mathbb{T}^{cl}$  is not strongly stable. We have thus shown that if  $(A, B)$  is strongly stabilizable, then  $h$  satisfies (3.2), which ends the proof of the second assertion.

3. To prove the third assertion, first we notice that by combining (3.3) and (6.12) it follows that there exists  $M_0 > 0$  such that

$$|B^* \phi_k| \geq \frac{M_0}{|k|} \quad \forall k \in \mathbb{Z}^*. \quad (6.13)$$

We introduce, for every  $s \in \mathbb{R}$  and  $\delta > 0$ , the vector space  $WP_{s,\delta}(A)$ , called *wave package of frequency  $s$  and width  $\delta$  associated to the operator  $A$* , which is defined by

$$WP_{s,\delta}(A) = \begin{cases} \{0\} & \text{if } |\mu_k - s| \geq \delta \text{ for all } k \in \mathbb{Z}^*, \\ \text{span } \{\phi_k \mid k \in \mathbb{Z}^* \text{ and } |\mu_k - s| < \delta\} & \text{else.} \end{cases}$$

According to Lemma 6.1 there exists  $\varepsilon > 0$  such that, setting

$$\delta(s) = \frac{\varepsilon}{|s| + 1} \quad \forall s \in \mathbb{R}, \quad (6.14)$$

we either have that  $WP_{s,\delta(s)}(A) = \{0\}$  or

$$WP_{s,\delta(s)}(A) = \text{span} \{\phi_{k(s)}\},$$

where  $k(s)$  is the unique element of  $\mathbb{Z}^*$  such that

$$s - \delta(s) < \mu_{k(s)} < s + \delta(s).$$

Using the fact that  $\mu_k = \sqrt{k \tanh(k)}$  and  $\mu_{-k} = -\mu_k$  for  $k \in \mathbb{N}$ , together with (6.13), it follows that there exists  $M_1 > 0$  such that

$$|B^*\phi| \geq \frac{M_1}{(|s|+1)^2} \|\phi\|_X \quad \forall \phi \in WP_{s,\delta(s)}(A), \quad s \in \mathbb{R}.$$

We have thus obtained that the pair  $(A, B)$  satisfies the assumptions of Theorem 1.1 in [21] with  $\delta$  given by (6.14) and

$$\gamma(s) = \frac{M_1}{(|s|+1)^2} \quad \forall s \in \mathbb{R}.$$

We can apply Theorem 1.1 in [21] to conclude that the semigroup  $\mathbb{T}^{cl}$  generated by  $A - BB^*$  satisfies (3.4).  $\square$

**Remark 6.2.** For the proof of the second assertion we could use (instead of Benchimol [20]) the stronger result of Batty and Vu [22], where  $A$  generates a contraction semigroup and  $B$  is still bounded. An even more general result is in the recently published [23], where  $A$  generates a contraction semigroup and  $B$  may be very unbounded (not even admissible).

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