HAL
open science

Lieb–Thirring inequalities for an effective Hamiltonian of bilayer graphene

Philippe H Briet, J.-C Cuenin, L Golinskii, Stanislas Kupin

To cite this version:

Philippe H Briet, J.-C Cuenin, L Golinskii, Stanislas Kupin. Lieb–Thirring inequalities for an effective Hamiltonian of bilayer graphene. Journal of Spectral Theory, 2021, 11 (3), pp.1145-1178. 10.4171/jst/368 . hal-02457720

HAL Id: hal-02457720

https://hal.science/hal-02457720

Submitted on 28 Jan 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

arXiv:1910.04296v2 [math-ph] 30 Oct 2019

LIEB-THIRRING INEQUALITIES FOR AN EFFECTIVE HAMILTONIAN OF BILAYER GRAPHENE

PH. BRIET, J.-C. CUENIN, L. GOLINSKII, AND S. KUPIN

Abstract. Combining the methods of Cuenin [7] and Borichev-Golinskii-Kupin [4], [5], we obtain the so-called Lieb-Thirring inequalities for non-selfadjoint perturbations of an effective Hamiltonian for bilayer graphene.

Introduction and main results

Since the early 2000-s, a certain amount of attention of the mathematical community has been attracted by the spectral properties of complex (non-selfadjoint) perturbations of model operators from mathematical physics. Among relatively recent papers in this direction, we quote articles by Demuth-Hansmann-Katriel [13], Frank [19], [20], Frank-Simon [22], Frank-Sabin [21], Frank-Laptev-Safronov [23], Fanelli-Krejčiřík-Vega [15, 16], Mizutani [40], Fanelli-Krejčiřík [17], Cuenin-Kenig [10] and Lee-Seo [38], dealing with spectral properties of complex Schrödinger operators. Similar problems for Dirac, fractional Schrödinger and other types of operators were treated in Cuenin-Laptev-Tretter [8], Cuenin-Seigl [9], Dubuisson [14], Cuenin [6, 11], Cossetti [12], Ibrogimov-Krejčiřík-Laptev [34] and Hulko [30, 31]. A series of results on spectral analysis of Jacobi matrices can be found in Borichev-Golinskii-Kupin [4, 5] and Golinskii-Kupin [26]-[29].

In the present article, we are interested in the study of perturbations of bilayer graphene Hamiltonian given by

$$D_{bg,m} := \begin{bmatrix} m & 4\partial_z^2 \\ 4\partial_{\bar z}^2 & -m \end{bmatrix}, \tag{0.1}$$

where $m \geq 0$ and

$$\partial_z := \frac{1}{2}\left(\partial_{x_1} + i\partial_{x_2}\right), \quad \partial_{\bar z} := \frac{1}{2}\left(\partial_{x_1} - i\partial_{x_2}\right).$$

As usual, we let

$$L^2(\mathbb{R}^2;\mathbb{C}^2) := \left\{ f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : \|f\|_2^2 = \int_{\mathbb{R}^2} |f(x)|^2\, dx < \infty \right\}$$

to be the standard space of measurable vector-valued functions; here

$$|f(x)| = (|f_1(x)|^2 + |f_2(x)|^2)^{1/2}.$$

Furthermore, let

$$H^2(\mathbb{R}^2;\mathbb{C}^2) := \left\{ f \in L^2(\mathbb{R}^2;\mathbb{C}^2) : \|f\|_{H^2}^2 = \int_{\mathbb{R}^2} (1+|\xi|^2)^2 |\hat f(\xi)|^2\, d\xi < \infty \right\}$$

2010 *Mathematics Subject Classification.* Primary: 35P15; Secondary: 30C35, 47A75, 47B10.

Key words and phrases. effective Hamiltonian of bilayer graphene, complex (non-selfadjoint) perturbation, discrete spectrum, Lieb-Thirring inequalities, Schatten–von Neumann classes.

The research is partially supported by ANR-18-CE40-0035 grant.

be the corresponding second order Sobolev space, where $\hat{f}$ denotes the Fourier transform of a function f, see Section 1.1 for more notation. It is not difficult to see that

$$D_{bg,m} : H^2(\mathbb{R}^2;\mathbb{C}^2) \to L^2(\mathbb{R}^2;\mathbb{C}^2)$$

is a selfadjoint operator. Since

$${D_{bg,m}}^2 = (\Delta^2 + m^2)I_2,$$

the spectral mapping theorem yields $\sigma(D_{bg,m}) := (-\infty, -m] \cup [m, +\infty)$. The resolvent set of $D_{bg,m}$ is denoted by $\rho(D_{bg,m}) := \mathbb{C}\backslash\sigma(D_{bg,m})$.

Detailed discussion of this and other similar operators from the physical point of view can be found in the book of Katznelson [36].

We consider the perturbed operator

$$D_{bg} := D_{bg,m} + V \tag{0.2}$$

with $V \in L^q(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$, $q \geq 1$. Since the perturbation V is not assumed to be selfadjoint, the operator D_{bg} may be non-selfadjoint as well. For the formal definition of $D_{bg,m} + V$ for the class of potentials considered here we allude to the "factorization method" of Kato [35]; see also Gesztesy-Latushkin et al. [24]. A version of Weyl's theorem [24, Theorem 4.5] asserts that

$$\sigma_{ess}(D_{bg}) = \sigma_{ess}(D_{bg,m}) = (-\infty, -m] \cup [m, +\infty), \tag{0.3}$$

where we adopt the convention that $\sigma_{ess}(D_{bg}) := \sigma(D_{bg})\backslash\sigma_d(D_{bg})$ and the discrete spectrum $\sigma_d(D)$ is the set of isolated eigenvalues of D of finite multiplicity.

We shall be interested in distribution properties of the discrete spectrum $\sigma_d(D_{bg})$ of the perturbed operator D_{bg}. Note that $\sigma_d(D_{bg})$ can only accumulate to $\sigma_{ess}(D_{bg})$, and we want to find some quantitative characteristics of the rate of accumulation.

The first step in this direction is to understand better the localization of the discrete spectrum $\sigma_d(D_{bg})$. The well-established Birman-Schwinger operator

$$BS_z := |V|^{1/2}(D_{bg,m} - z)^{-1}V^{1/2}, \quad z \in \rho(D_{bg,m}), \tag{0.4}$$

plays a key role in this problem, see original references by Birman [3], Schwinger [43]. Here, $V(x) = |V(x)|\,U(x)$ is the polar decomposition of the matrix $V(x)$, $|V(x)| := (V(x)^*V(x))^{1/2}$ and $U(x)$ is the corresponding partial isometry. So, $V^{1/2}(x) := |V(x)|^{1/2}U(x)$ for a. e. $x \in \mathbb{R}^2$. The Birman-Schwinger principle [24, Theorem 3.2] says that $z \in \rho(D_{bg,m})$ is an eigenvalue of D_{bg} iff -1 is an eigenvalue of the operator BS_z. In particular, we have the inclusion

$$\sigma_d(D_{bg}) \subset \{z \in \rho(D_{bg,m}) : \|BS_z\| \geq 1\}.$$

Laptev-Ferrulli-Safronov [18, Thm. 1.1] obtain the following interesting result.

Theorem 0.1 ([18]). *Let $D_{bg,m}, D_{bg}$ be as above and $V \in L^q(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$, $1 < q < 4/3$. Then*

(1) For $z \in \rho(D_{bg,m})$,

$$\|BS_z\|^q = \||V|^{1/2}(D_{bg,m} - z)^{-1}V^{1/2}\|^q \leq C_q\|V\|_q^q \frac{(|z-m| + |z+m|)^q}{|z^2 - m^2|^{q-1/2}}. \tag{0.5}$$

(2) In particular,

$$\sigma_d(D_{bg}) \subset \left\{ z : C_q\|V\|_q^q \frac{(|z-m| + |z+m|)^q}{|z^2 - m^2|^{q-1/2}} \geq 1 \right\}.$$

Slightly later, the second author [7, Thm. 1.1, Prop. 2.4] improved the resolvent bound in several respects. First, he showed that the norm of the Birman-Schwinger operator BS_z in the LHS of (0.5) can be taken in an appropriate Schatten-von Neumann class $\mathcal{S}_p$, $p = p(q)$; second, the range of parameter q is extended to

$1 \le q \le 3/2$. It was observed that these results were optimal in a certain sense. We mention also that [7, Prop. A.5] addresses more general situations as compared to [18, Thm. 1.1]; in particular, the former is valid for more general differential operators than the bilayer graphene Hamiltonian.

The key to the Lieb-Thirring type inequalities obtained in this article is a claim similar to [7, Prop. 2.4]. We feel that it is appropriate to give a detailed and a self-contained proof of this result, see Theorem 0.2 below. As compared to [7, Prop. 2.4], we extend the range of parameter q to $1 \le q < \infty$.

Theorem 0.2. *Let $D_{bg,m}, D_{bg}$ be defined in* (0.1), (0.2), *and $m > 0$. For $q \ge 1$ and $\varepsilon > 0$, set*

$$p = p(q,\varepsilon) := \begin{cases} \frac{q}{2-q} + \varepsilon, & 1 \le q < 4/3, \\ \frac{q}{2-q}, & 4/3 \le q \le 3/2, \\ 2q, & q > 3/2. \end{cases} \tag{0.6}$$

(I) Let $1 \le q \le 3/2$. There is a $C_3 > 0$ such that for any $A, B \in L^{2q}(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$, one has

$$\|A(D_{bg,m} - z)^{-1} B\|_{\mathcal{S}_p} \le C_3 \Phi(z) \|A\|_{2q} \|B\|_{2q}, \tag{0.7}$$

where

$$\Phi(z) = \Phi_q(z) := \frac{|z+m| + |z-m|}{|z^2 - m^2|^{q_1}},$$

$z \in \rho(D_{bg,m})$ and $q_1 := 1 - 1/(2q)$.

(II) Let $q > 3/2$. There is a $C_4 > 0$ such that for any $A, B \in L^{2q}(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$, one has

$$\|A(D_{bg,m} - z)^{-1} B\|_{\mathcal{S}_p} \le C_4 \Psi(z) \, \|A\|_{2q} \|B\|_{2q}, \tag{0.8}$$

where

$$\Psi(z) = \Psi_q(z) := \frac{(|z+m| + |z-m|)^{q_2}}{|z^2 - m^2|^{1/q}} \, \frac{1}{d^{1-q_2}(z, \sigma(D_{bg,m}))},$$

$z \in \rho(D_{bg,m})$ and $q_2 := 3/(2q) < 1$. Here, $d(z, \sigma(D_{bg,m}))$ is the distance from z to $\sigma(D_{bg,m})$. The constants C_3, C_4 depend on m, q, ε, but not on $A, B \in L^{2q}(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$.

The above result along with discussion on Birman-Schwinger operators preceding Theorem 0.1 provides the following corollary.

Corollary 0.3.

(1) For $1 \le q \le 3/2$ and $V \in L^q(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$, we have

$$\sigma_d(D_{bg}) \subset \{z : C_3 \Phi(z) ||V||_q \ge 1\}.$$

In particular, the discrete spectrum $\sigma_d(D_{bg})$ is bounded.

(2) For $q > 3/2$ and $V \in L^q(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$, we have

$$\sigma_d(D_{bg}) \subset \{z : C_4 \Psi(z) ||V||_q \ge 1\}.$$

Theorem 0.2 combined with techniques developed in Borichev-Golinskii-Kupin [4], [5] implies the following result.

Theorem 0.4. *Let $D_{bg,m}, D_{bg}$ be defined in* (0.1), (0.2), *and $m > 0$. For $q > 1$ and $\varepsilon > 0$, set*

$$\beta = \beta(q,\varepsilon) := \begin{cases} \frac{4q-5}{2(2-q)} + \frac{2q-1}{2q}\varepsilon, & 1 < q < \frac{4}{3}, \\ \frac{4q-5}{2(2-q)}, & \frac{4}{3} \le q \le \frac{3}{2}. \end{cases} \tag{0.9}$$

Assume that $\|V\|_q \le 1$. Then the Lieb–Thirring inequalities for the discrete spectrum $\sigma_d(D_{bg})$ hold:

(I) for $1 \le q \le 3/2$,

$$\sum_{\zeta\in\sigma_d(D_{bg})} d^{1+\varepsilon}(\zeta,\sigma(D_{bg,m}))\,|\zeta^2-m^2|^{\beta} \le C_5\|V\|_q, \tag{0.10}$$

(II) for $q > 3/2$,

$$\sum_{\zeta\in\sigma_d(D_{bg})} \frac{|\zeta|^{2q+1+\varepsilon} d^{2q-2+\varepsilon}(\zeta,\sigma(D_{bg,m}))\,|\zeta^2-m^2|}{(1+|\zeta|)^{2q+1+\varepsilon}} \le C_6\|V\|_q. \tag{0.11}$$

The constants C_5, C_6 *depend on* m, q, ε, *but not on* $V \in L^q(\mathbb{R}^2;\mathrm{Mat}_{2,2}(\mathbb{C}))$.

The counterparts of the above theorems for the case $m = 0$ are given below. Their proofs are similar to Theorems 0.2, 0.4, and therefore they are omitted.

Theorem 0.5. *Let* $D_{bg,0}, D_{bg}$ *be given by* (0.1), (0.2) *and* $z \in \rho(D_{bg,0}) := \mathbb{C}\backslash\mathbb{R}$. *Take an* $\varepsilon > 0$ *and put* $p = p(q,\varepsilon)$ *as in* (0.6).

(I) Let $1 \le q \le 3/2$. *There is a* $C_3' > 0$ *such that for any* $A, B \in L^{2q}(\mathbb{R}^2;\mathrm{Mat}_{2,2}(\mathbb{C}))$, *one has*

$$\|A(D_{bg,0}-z)^{-1}B\|_{\mathcal{S}_p} \le C_3'\,|z|^{-(1-\frac{1}{q})}\|A\|_{2q}\|B\|_{2q}. \tag{0.12}$$

(II) Let $q > 3/2$. *There is a* $C_4' > 0$ *such that for any* $A, B \in L^{2q}(\mathbb{R}^2;\mathrm{Mat}_{2,2}(\mathbb{C}))$, *one has*

$$\|A(D_{bg,0}-z)^{-1}B\|_{\mathcal{S}_p} \le C_4'\,|z|^{-\frac{1}{2q}}|\mathrm{Im}\,z|^{-(1-\frac{3}{2q})}\|A\|_{2q}\|B\|_{2q}, \tag{0.13}$$

Above, $|\mathrm{Im}\,z| = d(z,\mathbb{R})$ *is the distance from* z *to the real line* $\mathbb{R}$. *The constants* C_3', C_4' *depend on* q, ε, *but not on* $A, B \in L^{2q}(\mathbb{R}^2;\mathrm{Mat}_{2,2}(\mathbb{C}))$.

Similarly to Corollary 0.3, we can decribe the regions containg the discrete spectrum $\sigma_d(D_{bg})$ for $m = 0$. In particular, the set is bounded for $1 \le q \le 3/2$ and $V \in L^q(\mathbb{R}^2;\mathrm{Mat}_{2,2}(\mathbb{C}))$.

Theorem 0.6. *Let* $D_{bg,0}, D_{bg}$ *be defined as above. Let* $q > 1$ *and* $\varepsilon > 0$ *be small enough. Assume that* $\|V\|_q \le 1$. *Then the Lieb–Thirring inequalities for the discrete spectrum* $\sigma_d(D_{bg})$ *hold:*

(I) for $1 \le q \le 3/2$,

$$\sum_{\zeta\in\sigma_d(D_{bg})} |\mathrm{Im}\,\zeta|^{1+\varepsilon} \le C_5'\|V\|_q, \tag{0.14}$$

(II) for $q > 3/2$,

$$\sum_{\zeta\in\sigma_d(D_{bg})} \frac{|\mathrm{Im}\,\zeta|^{2-\frac{3}{2q}+\varepsilon}}{(1+|\zeta|)^{1-\frac{3}{2q}+2\varepsilon}} \le C_6'\|V\|_q. \tag{0.15}$$

The constants C_5', C_6' *depend on* q, ε, *but not on* $V \in L^q(\mathbb{R}^2;\mathrm{Mat}_{2,2}(\mathbb{C}))$.

Remark 0.7.

(1) In order to prove the above theorems we need the $\mathcal{S}_p$*-norm of the Birman-Schwinger operator* $\|V_2(D_{bg,m}-z(iy))^{-1}V_1\|_{\mathcal{S}_p}$ *to go to zero when* $y \to +\infty$, *see* (2.4). *For this reason inequality* (0.10) *is obtained for* $1 < q \le 3/2$, *even though the case* $q = 1$ *is treated in Theorem 0.2.*

(2) The assumption $\|V\|_q \le 1$ *does not mean that the perturbation is small. Theorem 0.4 holds uniformly over any bounded in* L^q *set of potentials* V, *i.e.,* 1 *can be replaced with a constant* $C(q,m,\varepsilon)$.

The paper is organized in the following manner. We start Section 1 recalling some basic facts and notation on differential operators. The second part of Section 1 is devoted to the proof of Theorem 0.2. The proof of Theorem 0.4 is in Section 2. Section 3 is an appendix containing results on interpolation between $\mathcal{S}_p$-spaces and the Kato-Selier-Simon lemma.

The space of infinitely differentiable functions on $\mathbb{R}^2$ is denoted by $C^\infty(\mathbb{R}^2)$; $C_0^\infty(\mathbb{R}^2)$ are infinitely differentiable functions with compact support. The notation $L^p(\mathbb{R}^2)$, $1 \le p \le \infty$, stays for the familiar space of p-summable measurable functions. $L_0^\infty(\mathbb{R}^2)$ refers also to functions from $L^\infty(\mathbb{R}^2)$ with compact support. Meaningful constants are written as C_j, C_j', $j = 0, 1, \dots$; technical constants are denoted by c, C, and they change from one relation to another.

1. Resolvent bounds for the bilayer graphene Hamiltonian

1.1. **Fourier transforms.** The purpose of this subsection is to fix some notation and recall some basic properties of the Fourier transformation. For this purpose we temporarily consider the case of arbitrary dimension n. At the end of the subsection we will compute Fourier transforms of some tempered distributions (homogeneous distributions and surface-carried measures) that will play an important role in the next subsection. We refer to Hörmander [32], Sogge [45] for more details on the subject.

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is defined as

$$(\mathcal{F}f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx.$$

Let $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$ denote the Schwartz space, *i.e.,* the space of rapidly decreasing smooth functions on $\mathbb{R}^n$. The Fourier transformation is an isomorphism $\mathcal{F} : \mathscr{S} \to \mathscr{S}$, and its inverse is furnished by the Fourier inversion formula,

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi}\, dx.$$

We use the standard notation $\check{f} := \mathcal{F}^{-1} f$. Hence, $\mathcal{F}$ may be extended to the dual space $\mathscr{S}'$, the space of tempered distributions, by setting $\hat{u}(\phi) = u(\hat{\phi})$ for $u \in \mathscr{S}'$, $\phi \in \mathscr{S}$. Moreover, Plancherel's formula,

$$\|\hat{f}\|_2 = (2\pi)^{n/2} \|f\|_2, \quad f \in \mathscr{S}, \tag{1.1}$$

gives rise to a continuous extension $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

Let $D = \nabla$ be a formal differential operator. The Fourier multiplier $m(D) : \mathscr{S} \to \mathscr{S}'$ associated to a tempered distribution $m \in \mathscr{S}'$ is the operator

$$m(D) f := \mathcal{F}^{-1}(m\hat{f}), \quad f \in \mathscr{S},$$

and (1.1) shows that m is bounded on $L^2(\mathbb{R}^n)$ if and only if $m \in L^\infty(\mathbb{R}^n)$, and $\|m(D)\| = \|m\|_\infty$. We also have

$$(m(D)\varphi)(x) = \check{m} * \varphi = \int_{\mathbb{R}^n} \check{m}(x-y)\varphi(y)\, dy, \quad \varphi \in \mathscr{S}, \tag{1.2}$$

with the understanding that $* : \mathscr{S}' \times \mathscr{S} \to \mathscr{S}'$ is the convolution between a Schwartz function and a tempered distribution. The second identity in (1.2) is in general only formal, but it is rigorous if $\check{m}$ is a regular tempered distribution. To simplify notation, the expression $(m(D))(x)$, refers to the convolution kernel $\check{m}(x)$ of the integral operator in (1.2).

Consider now a smooth real-valued function ρ which we think of as (a normalized power of) a Hamiltonian. Then, for $\lambda \in \mathbb{R}$, we define the level sets of ρ (*i.e.,* the

sets of constant energy) as

$$S_\lambda := \rho^{-1}(\lambda) = \{\xi \in \mathbb{R}^n : \rho(\xi) = \lambda\}. \tag{1.3}$$

These sets play a crucial role in scattering theory, see e.g. Hörmander [33, Ch. XIV]. In the present paper the main feature of S_λ is its nowhere vanishing Gaussian curvature. To ensure that S_λ is in fact a manifold (a curve) we make the assumption that ρ is normalized such that $|\nabla\rho| = 1$ on S_λ. In the following we will only deal with[1] $\rho(\xi) = |\xi|$, in which case S_λ is just the sphere of radius λ. Let $d\sigma_{S_\lambda}$ be the canonical surface measure on S_λ. As usual, $L^2(d\sigma_{S_\lambda})$ is the space of measurable square-summable functions on S_λ. The Fourier restriction operator for S_λ is defined by

$$R(\lambda)\varphi := \hat{\varphi}\big|_{S_\lambda}, \quad \varphi \in \mathscr{S}(\mathbb{R}^n).$$

Its formal adjoint (the Fourier extension operator) is given by

$$R(\lambda)^*\varphi = \widehat{\varphi\, d\sigma_{S_\lambda}}, \quad \varphi \in \mathscr{S}(\mathbb{R}^n).$$

Here, the Fourier transform of the measure $\varphi\, d\sigma_{S_\lambda}$ is defined as

$$\widehat{\varphi\, d\sigma_{S_\lambda}}(x) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(\xi)d\sigma_{S_\lambda}(\xi).$$

The multiplier corresponding to the function $\xi \mapsto |\xi|$ is denoted by $\sqrt{-\Delta}$. Denote by $E_{\sqrt{-\Delta}}(\lambda)$ the (operator-valued) spectral measure associated to this operator, viewed as an unbounded selfadjoint operator on $L^2(\mathbb{R}^n)$. Since its spectrum is absolutely continuous we may write $dE_{\sqrt{-\Delta}}(\lambda) = \frac{dE_{\sqrt{-\Delta}}(\lambda)}{d\lambda}d\lambda$, where the convolution kernel of the density is given by

$$\frac{dE_{\sqrt{-\Delta}}(\lambda)}{d\lambda}(x-y) = (2\pi)^{-n}\int_{|\xi|=\lambda} e^{i(x-y)\cdot\xi}d\sigma_{S_\lambda}(\xi).$$

By a change of variables $\xi = \lambda\xi'$, $|\xi'| = 1$, we see that

$$\frac{dE_{\sqrt{-\Delta}}(\lambda)}{d\lambda} = \frac{\lambda^{n-1}}{(2\pi)^n}R(\lambda)^*R(\lambda), \tag{1.4}$$

where $R(\lambda)$ is the restriction operator discussed above. It is also plain that

$$R(\lambda)f = \lambda^{-n}R(1)(f(\lambda^{-1}\cdot)).$$

Define

$$\chi_+^w(\tau) := \mathbf{1}_{[0,\infty)}(\tau)\tau^w/\Gamma(w+1), \quad w \in \mathbb{C},$$

where Γ is the usual Gamma function.

Lemma 1.1. *Let $z, \zeta \in \mathbb{C}$, $\operatorname{Im} z > 0$. The one-dimensional inverse Fourier transform of the function $\eta_{z,\zeta}(x) := (x-z)^{-\zeta}$, $x \in \mathbb{R}$, is given by*

$$\check{\eta_{z,\zeta}}(\tau) = e^{i(\pi\zeta/2+z\tau)}\chi_+^{\zeta-1}(\tau). \tag{1.5}$$

Proof. After a change of variables, this follows immediately by applying the inverse Fourier transformation to the following identity (see [32], specifically the explanation after Example 7.1.17)

$$\mathcal{F}\left(x \mapsto e^{-\epsilon x}\chi_+^z(x)\right)(\xi) = e^{-i\pi(z+1)/2}(\xi - i\epsilon)^{-z-1}, \quad \epsilon > 0, \quad z \in \mathbb{C}.$$

□

[1]The fact that $\xi \mapsto |\xi|$ is not smooth at $\xi = 0$ is irrelevant for our purposes since (by homogeneity) we will only need smoothness in a neighborhood of the unit sphere $S_1 = \{\xi : |\xi| = 1\}$.

Lemma 1.2. *Let $\beta \in C_0^\infty(\mathbb{R}^n)$ and let S_1 be the unit sphere in $\mathbb{R}^n$. Then the inverse Fourier transform of the surface measure $d\mu := \beta\, d\sigma_{S_1}$ admits the representation*

$$\check{d\mu}(x) = \sum_{\pm} e^{\pm i|x|} a_\pm(|x|) := e^{i|x|} a_+(|x|) - e^{-i|x|} a_-(|x|),$$

where $a_\pm \in C^\infty(\mathbb{R}_+)$ satisfy the symbol bounds

$$|\partial^k a_\pm(s)| \le C_{k\pm}(1+|s|)^{-\frac{n-1}{2}-k}. \tag{1.6}$$

Proof. This is a special case of [45, Theorem 1.2.1]. □

Lemma 1.3. *Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be supported in the annulus $\{1/2 \le |\xi| \le 3/2\}$, and $S = \{\zeta : a \le \operatorname{Re}\zeta \le b\}$ be a vertical strip in $\mathbb{C}$. Then*

$$\left| \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \frac{\chi(\xi)}{(|\xi|-z)^\zeta}\, d\xi \right| \le C e^{\pi^2 |\operatorname{Im}\zeta|^2} (1+|x|)^{-\frac{n+1}{2}+\operatorname{Re}\zeta}, \quad \zeta \in S, \quad |z|=1,$$

where the constant depends on a, b and finitely many derivatives of χ, but is independent of ζ, z.

Proof. It suffices to prove this for $|x| > 1$ since the case $|x| \le 1$ is trivial. Writing the integral in polar coordinates and using Lemma 1.2 we find that

$$\int_{\mathbb{R}^n} e^{-ix\cdot\xi} \frac{\chi(\xi)}{(|\xi|-z)^\zeta}\, d\xi = \sum_{\pm} \int_{-\infty}^{\infty} e^{\pm ir|x|} \frac{r^{n-1} a_\pm(r|x|)}{(r-z)^\zeta}\, dr,$$

where the function $r \mapsto r^{n-1} a_\pm(r|x|)$ is supported in a neighborhood of $r = 1$ and it satisfies

$$|r^{n-1} a_\pm(r|x|)| \le C(1+|x|)^{-\frac{n-1}{2}}$$

for any fixed Schwartz norm $|\cdot|$. Hence, by Lemma 1.2 again, its inverse Fourier transform is bounded by

$$|\mathcal{F}^{-1}\left(r \mapsto r^{n-1} a_\pm(r|x|)\right)(\tau)| \le C_N (1+|\tau|)^{-N} (1+|x|)^{-\frac{n-1}{2}}$$

for any $N > 0$. The convolution theorem and Lemma 1.1 yield

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} e^{\pm ir|x|} \frac{r^{n-1} a_\pm(r|x|)}{(r-z)^\zeta}\, dr \right| \\ &\quad \le C_N e^{\pi|\operatorname{Im}\zeta|} (1+|x|)^{-\frac{n-1}{2}} \int_{-\infty}^{\infty} (1+|\tau-|x||)^{-N} \chi_+^{\operatorname{Re}\zeta-1}(\tau)\, d\tau \\ &\quad \le C e^{\pi|\operatorname{Im}\zeta|} |\Gamma(\zeta)^{-1}| (1+|x|)^{-\frac{n+1}{2}+\operatorname{Re}\zeta}. \end{aligned}$$

The claim now follows from the estimate $|\Gamma(\zeta)^{-1}| \le C e^{\pi^2|\zeta|^2}$; see e.g. formula (11.21) in Muscalu-Schlag [41]. □

1.2. **Resolvent bounds in $\mathcal{S}_p$-norm for bilayer graphene.** We now return to the case $n = 2$ and the bilayer Hamiltonian. The coming bound is a special case of [7, Lemma A.6]. It is crucial for coming resolvent estimates.

In the following, we fix a function $\chi \in C_0^\infty(\mathbb{R}^2)$ supported in the annulus $\{1/2 \le |\xi| \le 3/2\}$ such that, in addition, $\chi(\xi) = 1$ for $3/4 \le |\xi| \le 5/4$.

Proposition 1.4. *Let $1 \le a \le 3/2, t \in \mathbb{R}$, and $z \notin \mathbb{R}_+$. There exists a constant $C_1' > 0$ (depending on χ only) such that*

$$|\chi(D)(\Delta^2 - z)^{-(a+it)}(x)| \le \frac{C_1' e^{\pi^2 t^2}}{(1+|x|)^{3/2-a}}, \quad x \in \mathbb{R}^2, \quad |z| = 1. \tag{1.7}$$

Proof. Set $z^{1/4} = |z|^{1/4} e^{(i\operatorname{Arg} z)/4}$. Clearly the 4-th power complex roots of z are given by $\{i^m z^{1/4}\}$, $m = 0, 1, 2, 3$. Without loss of generality, we suppose that $m = 0$ and $|\operatorname{Arg} z| \le \pi$, or $|\operatorname{Arg} z^{1/4}| \le \pi/4$, the other cases being analogous. Writing

$$(|\xi|^4 - z) = (|\xi| - z^{1/4}) \left(\prod_{k=1}^{3} (|\xi| - i^k z^{1/4}) \right)$$

and absorbing the second factor into χ, we see that it suffices to prove

$$\int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{\tilde{\chi}(\xi; a; t)}{(|\xi| - z^{1/4})^{a+it}} \, d\xi \le \frac{C e^{\pi^2 t^2}}{(1 + |x|)^{3/2-a}},$$

whenever $\tilde{\chi}(\xi; a, t)$ satisfies the bounds

$$\sum_{|\alpha| \le N} \|\partial_\xi^\alpha \tilde{\chi}(\cdot; a, t)\|_\infty \le C_N e^{2\pi|t|}$$

for a fixed, sufficiently large $N > 0$. This follows directly from Lemma 1.3. □

Remark 1.5. *In view of the identity*

$$\frac{1}{|\xi|^2 - z^{1/2}} - \frac{1}{|\xi|^2 + z^{1/2}} = \frac{2z^{1/2}}{|\xi|^4 - z},$$

inequality (1.7) *also follows from a two-dimensional version of estimates (2.23) and (2.25) in Kenig-Ruiz-Sogge* [37]*; see also (44) in Frank-Sabin* [21]*. To keep the article self-contained, we provided the above proof which rests only on the stationary phase method (Lemma 1.2) and formula* (1.5)*.*

Proposition 1.6. *Fix an $\varepsilon > 0$ and set the function χ as above. For $q \ge 1$, let*

$$p = p(q, \varepsilon) := \begin{cases} \frac{q}{2-q} + \varepsilon, & 1 \le q < 4/3, \\ \frac{q}{2-q}, & 4/3 \le q \le 3/2, \\ 2q, & q > 3/2. \end{cases} \tag{1.8}$$

For $A, B \in L^{2q}(\mathbb{R}^2)$, the following bounds hold true:

(I) for $1 \le q \le 3/2$,

$$\|A\chi(D)(\Delta^2 - z)^{-1} B\|_{\mathcal{S}_p} \le C_7 \, \|A\|_{2q} \|B\|_{2q}, \quad |z| = 1; \tag{1.9}$$

(II) for $q > 3/2$

$$\|A\chi(D)(\Delta^2 - z)^{-1} B\|_{\mathcal{S}_p} \le \frac{C_8}{d(z, \mathbb{R}_+)^{1-3/(2q)}} \, \|A\|_{2q} \|B\|_{2q}, \quad |z| = 1. \tag{1.10}$$

Here, $C_j = C_j(q, \varepsilon)$, $j = 7, 8$, are independent of A, B and z.

Proof. The proof relies heavily on interpolation between Schatten-von Neumann classes $\mathcal{S}_p$, $p \ge 1$, presented in Section 3. It is convenient to separate part (I) of the proposition in two cases: *Case I.1* for $1 \le q < 4/3$ and *Case I.2* for $4/3 \le q \le 3/2$. We begin with the proof of *Case I.2*.

Case I.2: $4/3 \le q \le 3/2$**.** Without loss of generality we may assume that $A > 0$ and $B > 0$. At the moment, we suppose also that $A, B \in L^{2q}(\mathbb{R}^2) \cap L_0^\infty(\mathbb{R}^2)$. We wish to apply Corollary 3.4 to the analytic family of operators given by

$$T_\zeta := A^\zeta \chi(D)(\Delta^2 - z)^{-\zeta} B^\zeta$$

on the strip $S = S_{0,a_0} := \{\zeta : 0 \le \operatorname{Re} z \le a_0\}$, with $1 \le a_0 \le 3/2$. Here, $\zeta = a + it$, $0 \le a \le a_0$, and $t \in \mathbb{R}$.

We start by checking assumptions of Corollary 3.4, see also Theorem 3.3. For arbitrary $f, g \in L^2(\mathbb{R}^2)$ we have, by Plancherel's identity,

$$(T_\zeta f, g) = \int_{\mathbb{R}^2} \chi(\xi)(|\xi|^4 - z)^{-\zeta} \widehat{B^\zeta f}(\xi) \overline{\widehat{A^\zeta g}(\xi)} d\xi,$$

which shows that $\zeta \mapsto (T_\zeta f, g)$ is analytic in S. By Cauchy-Schwarz inequality,

$$|(T_\zeta f, g)| \leq \|\chi\|_\infty \|(|\cdot|^4 - z)^{-\zeta}\|_\infty \|B^\zeta f\|_2 \, \|A^\zeta g\|_2.$$

Since $|\arg(|\xi|^4 - z)| \leq 2\pi$, we have that

$$\begin{aligned} |(|\xi|^4 - z)^{-\zeta}| &= \left| \exp(-(a+it)\,(\log \|\xi|^4 - z| + i \arg(|\xi|^4 - z)) \right| \\ &\leq |(|\xi|^4 - z)|^{-a} \exp(2\pi|t|). \end{aligned}$$

Observe that a varies over a compact interval and z is fixed. Putting all this together, we obtain that

$$|(T_\zeta f, g)| \leq C e^{2\pi|t|} ||\chi||_\infty \|A\|_\infty^a \|B\|_\infty^a \|f\|_2 \|g\|_2, \quad \zeta = a + it,$$

showing that (3.2) is satisfied. It also yields that

$$\|T_\zeta\|_{\mathcal{S}_\infty} \leq C e^{2\pi|\operatorname{Im}\zeta|} \tag{1.11}$$

for $\operatorname{Re}\zeta = 0$. Note that T_ζ is compact since we have the Hilbert-Schmidt bound

$$\begin{aligned} \|T_\zeta\|_{\mathcal{S}_2}^2 &= \int_{\mathbb{R}^2_x} \int_{\mathbb{R}^2_y} |A^\zeta(x)|^2 |\mathcal{F}\left(\chi(|\cdot|^4 - z)^{-\zeta}\right)(x-y)|^2 |A^\zeta(x)|^2 dxdy \\ &\leq e^{4\pi|\operatorname{Im}\zeta|} \|\chi(|\cdot|^4 - z)^{-\operatorname{Re}\zeta}\|_1^2 \|A\|_2^{2\operatorname{Re}\zeta} \|B\|_2^{2\operatorname{Re}\zeta}, \end{aligned}$$

and the right hand side is finite by the assumption that $A, B \in L_0^\infty(\mathbb{R}^2)$.

On the vertical line $\{\zeta : \operatorname{Re}\zeta = a_0\}$, Proposition 1.4 and Hardy-Littlewood-Sobolev inequality (see Lieb-Loss [39, Sect. 4.3]) yield that

$$\begin{aligned} \|T_{a_0+it}\|_{\mathcal{S}_2}^2 &\leq \int_{\mathbb{R}^2_x} \int_{\mathbb{R}^2_y} |\chi(D)(\Delta^2 - z)^{-(a_0+it)}(x-y)|^2 |A(x)|^{2a_0} |B(y)|^{2a_0} \, dxdy \\ &\leq C e^{2\pi^2 t^2} \int_{\mathbb{R}^2_x} \int_{\mathbb{R}^2_y} \frac{1}{|x-y|^{3-2a_0}} |A(x)|^{2a_0} |B(y)|^{2a_0} \, dxdy \\ &\leq C e^{2\pi^2 t^2} \||A|^{2a_0}\|_s \||B|^{2a_0}\|_s, \end{aligned}$$

where $2/s + (3 - 2a_0)/2 = 2$, or $s = 4/(1 + 2a_0)$. In particular,

$$\||A|^{2a_0}\|_s = \|A\|_{8a_0/(1+2a_0)}^{2a_0},$$

the same equality holding for $\||B|^{2a_0}\|_s$. Hence, gathering the above computations, we arrive at the bound

$$\|T_\zeta\|_{\mathcal{S}_2} \leq C e^{\pi^2 |\operatorname{Im}\zeta|^2} \|A\|_{8a_0/(1+2a_0)}^{a_0} \|B\|_{8a_0/(1+2a_0)}^{a_0} \quad \text{for} \quad \operatorname{Re}\zeta = a_0. \tag{1.12}$$

We recall now Corollary 3.4 (see also Theorem 3.3) with parameters chosen as

$$\zeta := 1, \quad 1 = \gamma \cdot a_0 + (1-\gamma) \cdot 0, \quad \frac{1}{s_\gamma} = \frac{\gamma}{2} + \frac{(1-\gamma)}{\infty} = \frac{\gamma}{2},$$

to interpolate between (1.11) and (1.12). Solving first for γ and then for s_γ yields $\gamma = 1/a_0$ and $s_\gamma = 2a_0$. Corollary 3.4 then implies that

$$\|A\chi(D)(\Delta^2 - z)^{-1} B\|_{\mathcal{S}_{2a_0}} \leq C_7 \, \|A\|_{8a_0/(1+2a_0)} \|B\|_{8a_0/(1+2a_0)},$$

which is exactly (1.9) with $4/3 \leq q \leq 3/2$ if one puts $2q = 8a_0/(1 + 2a_0)$.

To sum up, we proved (1.9) for $4/3 \leq q \leq 3/2$ and $A, B \in L^{2q}(\mathbb{R}^2) \cap L_0^\infty(\mathbb{R}^2)$. It remains to get rid of the assumption that $A, B \in L_0^\infty(\mathbb{R}^2)$. The proof relies essentially on the fact that the constant C_7 from (1.9) *does not depend* on A, B. We proceed by a limiting argument. Let $A, B \in L^{2q}(\mathbb{R}^2)$. For $n \in \mathbb{N}$, define

$$E_n = \{x \in \mathbb{R}^2 : |x| + |A(x)| + |B(x)| \leq n\}$$

and set the "truncations" of A, B to be

$$A_n = A\mathbf{1}_{E_n}, \quad B_n = B\mathbf{1}_{E_n}.$$

Let $P_n : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ be the corresponding orthogonal projection

$$P_n f = \mathbf{1}_{E_n} f, \quad f \in L^2(\mathbb{R}^2).$$

The elementary properties of L^{2q}-integrable functions yield that for any $f \in L^2(\mathbb{R}^2)$, we have

$$\lim_{n\to+\infty} \|P_n f - f\|_2 = 0.$$

Recalling [25, Thm. 5.2] and inequality (1.9) for functions from $L^{2q}(\mathbb{R}^2) \cap L_0^\infty(\mathbb{R}^2)$, we obtain

$$\begin{aligned}\|A\chi(D)(\Delta^2 - z)^{-1}B\|_{\mathcal{S}_p} &= \sup_n \|P_n\big(A\chi(D)(\Delta^2 - z)^{-1}B\big)P_n\|_{\mathcal{S}_p} \\ &= \sup_n \|A_n\chi(D)(\Delta^2 - z)^{-1}B_n\|_{\mathcal{S}_p} \le C_7\|A_n\|_{2q}\|B_n\|_{2q} \le C_7\|A\|_{2q}\|B\|_{2q}.\end{aligned}$$

Case I.2 follows.

Case II: $q > 3/2$**.** As before, we may assume without loss of generality that $A, B \in L^{2q}(\mathbb{R}^2) \cap L_0^\infty(\mathbb{R}^2)$, and that $A, B > 0$.

Let $S := S_{0,a_0} := \{a + it : 0 \le a \le a_0 = 2q/3,\ t \in \mathbb{R}\}$. Notice that $q > 3/2$ implies that $a_0 = 2q/3 > 1$. Consider the analytic family of operators

$$T_\zeta = A^\zeta \chi(D)(\Delta^2 - z)^{-1}B^\zeta,$$

defined on S. For $\operatorname{Re}\zeta = a_0$, inequality (1.9) applied with $p_0 = 3, q_0 = 3/2$ instead of p, q yields

$$\|T_\zeta\|_{\mathcal{S}_3} \le C_3\|A^{2q/3}\|_3\|B^{2q/3}\|_3 = C_3\|A\|_{2q}^{2q/3}\|B\|_{2q}^{2q/3} \tag{1.13}$$

for $\operatorname{Re}\zeta = a_0$. On the other hand, since for $\operatorname{Re}\zeta = 0$ we have $|A^\zeta| = |B^\zeta| = 1$ a.e. on $\mathbb{R}^2$, we also see that

$$\|T_\zeta\|_{\mathcal{S}_\infty} \le \frac{\|\chi\|_\infty}{d(z, \mathbb{R}_+)}. \tag{1.14}$$

by the spectral theorem for Δ^2. Compactness of T_ζ follows by the same argument as in *Case I.1*. Interpolating in between (1.13) and (1.14), with

$$\zeta := 1, \quad 1 = \frac{2q}{3} \cdot \gamma + 0 \cdot (1 - \gamma) = \frac{2q}{3}\gamma,$$

we get $\gamma = 3/(2q) \in (0, 1)$ and consequently

$$\frac{1}{p_{0\gamma}} = \frac{\gamma}{3} + \frac{(1-\gamma)}{\infty} = \frac{\gamma}{3},$$

which means that $p_{0\gamma} = 2q$. That is,

$$\|A\chi(D)(\Delta^2 - z)^{-1}B\|_{\mathcal{S}_{2q}} \le \frac{C_8}{d(z, \mathbb{R}_+)^{1-\gamma}}\|A\|_{2q}\|B\|_{2q},$$

By the same limiting argument as before, we get relation (1.10).

Case I.1: $1 \le q \le 4/3$**.** Let $\tilde\chi$ be a cutoff function with the same support properties as χ and such that $\tilde\chi = 1$ on the support of χ; in particular, $\tilde\chi\chi = \chi$.

Let $A, B \in L^2(\mathbb{R}^2)$. We start by proving that

$$\|A\chi(D)\frac{dE_{\sqrt{-\Delta}}(\lambda)}{d\lambda}\tilde\chi(D)B\|_{\mathcal{S}_1} \le C\,\|A\|_2\|B\|_2. \tag{1.15}$$

Indeed, using (1.4), we re-write the operator on the left hand side of (1.15) as

$$A\chi(D)\frac{dE_{\sqrt{-\Delta}}(\lambda)}{d\lambda}\tilde\chi(D)B = \frac{\lambda^{n-1}}{(2\pi)^n}\big(R(\lambda)\chi(D)A\big)^*\big(R(\lambda)\tilde\chi(D)B\big). \tag{1.16}$$

The kernel of the operator $R(\lambda)\chi(D)A : L^2(\mathbb{R}^2) \to L^2(S_\lambda)$ is given by

$$(R(\lambda)\chi(D)A)(\xi, x) = \chi(\xi)e^{ix\xi}A(x), \quad x \in \mathbb{R}^2, \xi \in S_\lambda,$$

and thus

$$\|R(\lambda)\chi(D)A\|_{\mathcal{S}_2}^2 = \int_{\mathbb{R}^2_x}\int_{S_{\lambda,\xi}} |\chi(\xi)A(x)|^2\, dxd\sigma_{S_\lambda}(\xi) = \|\chi\|^2_{L^2(S_\lambda)}\|A\|_2^2 \le C\|A\|_2^2.$$

Since the same bound holds for $R(\lambda)\tilde{\chi}(D)B$, Hölder's inequality for $\mathcal{S}_p$-classes yields (1.15).

Set $0 < a_0 < 1$. Using the formula

$$(\Delta^2 - z)^{-(a_0+it)} = \int_{\mathbb{R}} (\lambda^4 - z)^{-(a_0+it)}\, dE_{\sqrt{-\Delta}}(\lambda).$$

inequality (1.15) and the fact that the functions $\|\chi_j\|_{S_\lambda}$ are supported on the set where $1/2 \le \lambda \le 3/2$, we get the bound

$$\|A\chi(D)(\Delta^2 - z)^{-(a_0+it)}\chi(D)B\|_{\mathcal{S}_1} \le C\,\frac{e^{2\pi|t|}}{(1-a_0)}\|A\|_2\|B\|_2. \tag{1.17}$$

On the other hand, from (1.7), we see that

$$|\chi(D)(\Delta^2 - z)^{-3/2+it}(x)| \le C_1' e^{\pi^2 t^2},$$

that is, the kernel of $\chi(D)(\Delta^2 - z)^{-3/2+it}(x)$ is uniformly bounded with respect to the "space variable" $x \in \mathbb{R}^2$. The Hilbert-Schmidt bound for integral operators implies immediately

$$\|A\chi(D)(\Delta^2 - z)^{-3/2+it}B\|_{\mathcal{S}_2} \le C\,\|A\|_2\|B\|_2. \tag{1.18}$$

Let $0 < \varepsilon < 1/2$ be fixed. Suppose, as in *Cases I.2 and II*, that $A, B \in L^2(\mathbb{R}^d) \cap L_0^\infty(\mathbb{R}^2)$. Furthermore, set

$$T_\zeta := A\chi(D)^2(\Delta^2 - z)^{-\zeta}B$$

and $S = S_{a_0,b_0} := \{\zeta : a_0 \le \operatorname{Re}\zeta \le b_0\}$ to be the vertical strip with

$$a_0 = \frac{(1-2\varepsilon)}{(1-\varepsilon)} < 1, \quad b_0 = 3/2 > 1.$$

As previously, the family (T_ζ) on S_{a_0,b_0} satisfies the assumptions of Theorem 3.3 and we can interpolate between (1.17) and (1.18). More precisely, for the parameters of the corollary we take $\zeta := 1$ and

$$1 = \frac{1-2\varepsilon}{1-\varepsilon}\gamma + \frac{3}{2}(1-\gamma),$$

i.e., $\gamma = (1-\varepsilon)/(1+\varepsilon)$. Hence the relation

$$\frac{1}{s_\gamma} = \frac{\gamma}{1} + \frac{(1-\gamma)}{2}$$

gives $s_\gamma = 1 + \varepsilon$. To sum up, we arrive at

$$\|A\chi(D)(\Delta^2 - z)^{-1}B\|_{\mathcal{S}_{1+\varepsilon}} \le C\varepsilon^{-(1-\varepsilon)/(1+\varepsilon)}\|A\|_2\|B\|_2. \tag{1.19}$$

We interpolate once again in between (1.19) and (1.9) for $q = 4/3$ to obtain (1.9) for $1 \le q < 4/3$. Passing from $A, B \in L^{2q}(\mathbb{R}^2) \cap L_0^\infty(\mathbb{R}^2)$ to general $A, B \in L^{2q}(\mathbb{R}^2)$ is carried out as in the previous cases. □

We introduce some notation before going to the proof of Theorem 0.2. Let

$$k(u)^4 := (u^2 - m^2),$$

where we use the principal branch of 4-th complex root, so that $k(u) = (u^2 - m^2)^{1/4} \in \mathbb{R}_+$ for $u = x \in \mathbb{R}, x > m$. Furthermore,

$$\zeta(u) := \frac{u+m}{k(u)^2} = \left(\frac{u+m}{u-m}\right)^{1/2}, \; u \ne \pm m$$

with the standard choice of the branch of the square complex root.

1.3. **Proof of Theorem 0.2.** In order to distinguish the variable refered to in operators $\partial_z, \partial_{\bar z}$ and the spectral parameter of the operator $D_{bg,m}$, the latter will be denoted by $u \in \rho(D_{bg,m})$ in this subsection.

We consider first *Case I* of the theorem, *i.e.,* $1 \le q \le 3/2$. Let $A, B \in L^{2q}(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$, that is

$$A(x) = [A_{jl}(x)]_{j,l=1,2}, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

and $A_{jl}(x) \in L^{2q}(\mathbb{R}^2)$. Recalling the identities

$$4\partial_z\partial_{\bar z} = 4\partial_{\bar z}\partial_z = \left(\partial_{x_1}^2 + \partial_{x_2}^2\right)^2 = \Delta^2,$$

we readily see

$$\begin{aligned} {D_{bg,m}}^2 - u^2 &= \begin{bmatrix} m & 4\partial_{\bar z}^2 \\ 4\partial_z^2 & -m \end{bmatrix}^2 - u^2 = \begin{bmatrix} \Delta^2 + (m^2 - u^2) & 0 \\ 0 & \Delta^2 + (m^2 - u^2) \end{bmatrix} \\ &= (\Delta^2 - k(u)^4) I_2. \end{aligned}$$

For $k(u)^4 \in \mathbb{C}\backslash\mathbb{R}_+$, we have

$$(D_{bg,m} - u)^{-1} = (\Delta^2 - k(u)^4)^{-1}(D_{bg,m} + u).$$

We are interested in Schatten-von Neumann properties of Birman-Schwinger operator of the bilayer Hamiltionian, *i.e.,*

$$BS_u := [BS_{u,jl}]_{j,l=1,2} = A(D_{bg,m} - u)^{-1}B = A(\Delta^2 - k(u)^4)^{-1}(D_{bg,m} + u)B.$$

Of course, a bound of the form

$$\|BS_u\|_{\mathcal{S}_p} \le C(u)\|A\|_{2q}\|B\|_{2q},$$

see (0.7), (0.8), will follow if we prove it "entry-by-entry", that is

$$\|BS_{u,jl}\|_{\mathcal{S}_p} \le C(u)\|A\|_{2q}\|B\|_{2q}, \quad j, l = 1, 2.$$

We shall do the computation for the entry $BS_{u,11}$; the bounds for other entries of the operator BS_u are obtained in a similar way. We have

$$\begin{aligned} BS_{u,11} &= (m+u)A_{11}(\Delta^2 - k(u)^4)^{-1}B_{11} + 4A_{11}(\Delta^2 - k(u)^4)^{-1}\partial_{\bar z}^2 B_{21} \\ &\quad + 4A_{12}(\Delta^2 - k(u)^4)^{-1}\partial_z^2 B_{11} + (m-u)A_{12}(\Delta^2 - k(u)^4)^{-1}B_{21}. \end{aligned} \tag{1.20}$$

To simplify the following computations, we use a homogeneity argument; in detail: let $f \in L^s(\mathbb{R}^2), s > 0,\ f = f(x), x \in \mathbb{R}^2$. Set $x = ay, a > 0, y \in \mathbb{R}^2$. We write $g(y) = f(ay)$; to make the writing of differential operators more precise, we write x- or y-subindex to indicate the variable the differential operator is computed with. For instance Δ_x and Δ_y are the Laplacians computed with respect to x and y, respectively.

It is plain that for $j = 1, 2$

$$\begin{aligned} \partial_{y_j} g(y) &= a\partial_{x_j} f(ay) = a\partial_{x_j} f(x), \\ \partial_{y_j^2}^2 g(y) &= a^2\partial_{x_j^2}^2 f(ay) = a^2\partial_{x_j^2}^2 f(x). \end{aligned}$$

In particular, $\partial_{z,y} g = a\partial_{z,x} f,\ \partial_{z,y}^2 g = a^2\partial_{z,x}^2 f,\ \Delta_y^2 g = a^4\Delta_x^2 f$, etc.

Furthermore, one has

$$\|g\|_s^s = \int_{\mathbb{R}_y^2} |g(y)|^s\, dy = \int_{\mathbb{R}_y^2} |f(ay)|^s\, dy = a^{-2}\int_{\mathbb{R}_x^2} |f(x)|^s\, dx = a^{-2}\|f\|_s^s, \tag{1.21}$$

or $\|g\|_s = a^{-2/s}\|f\|_s$.

Suppose that $k(u) \neq 0$ and write $k(u)^4$ as $k(u)^4 = |k(u)|^4 e^{i\varphi}$. We assume also that $e^{i\varphi} \neq 1$; the case $e^{i\varphi} = 1$ can be obtained by a standard argument passing to the limit in relations (1.9), (1.10). So, putting $a = 1/|k(u)|$,

$$(\Delta_x^2 - k(u)^4)f(x) = |k(u)|^4(|k(u)|^{-4}\Delta_x^2 - e^{i\varphi})f(x) = |k(u)|^4(\Delta_y^2 - e^{i\varphi})g(y),$$

where $g(y) = f(ay),\ x = ay$. In the same way,

$$\partial_{z,x}^2 f(x) = |k(u)|^2 \partial_{z,y}^2 g(y), \qquad \partial_{\bar z,x}^2 f(x) = |k(u)|^2 \partial_{\bar z,y}^2 g(y).$$

Set $\tilde A_{jl}(y) = A_{jl}(ay)$ and $\tilde B_{jl}(y) = B_{jl}(ay)$ for $j, l = 1, 2$. Turning back to (1.20), we rewrite it as

(1.22)

$$BS_{u,11} = \frac{1}{|k(u)|^2}\Big(\frac{(m+u)}{|k(u)|^2}\tilde A_{11}(y)(\Delta_y^2 - e^{i\varphi})^{-1}\tilde B_{11}(y) + 4\tilde A_{11}(y)(\Delta_y^2 - e^{i\varphi})^{-1}\partial_{\bar z,y}^2 \tilde B_{21}(y)$$
$$+ 4\tilde A_{12}(y)(\Delta_y^2 - e^{i\varphi})^{-1}\partial_{z,y}^2 \tilde B_{11}(y) + \frac{(m-u)}{|k(u)|^2}\tilde A_{12}(y)(\Delta_y^2 - e^{i\varphi})^{-1}\tilde B_{21}(y)\Big).$$

Suppose momentarily that we could prove the following estimates,

$$\begin{aligned} \|\tilde A_{11}(\Delta_y^2 - e^{i\varphi})^{-1}\tilde B_{11}\|_{\mathcal S_p} &\le C\|\tilde A_{11}\|_{2q}\|\tilde B_{11}\|_{2q}, \\ \|\tilde A_{11}(\Delta_y^2 - e^{i\varphi})^{-1}\partial_{\bar z,y}^2\tilde B_{21}\|_{\mathcal S_p} &\le C\|\tilde A_{11}\|_{2q}\|\tilde B_{21}\|_{2q}, \\ \|\tilde A_{12}(\Delta_y^2 - e^{i\varphi})^{-1}\partial_{z,y}^2\tilde B_{11}\|_{\mathcal S_p} &\le C\|\tilde A_{12}\|_{2q}\|\tilde B_{11}\|_{2q}, \\ \|\tilde A_{12}(\Delta_y^2 - e^{i\varphi})^{-1}\tilde B_{21}\|_{\mathcal S_p} &\le C\|\tilde A_{12}\|_{2q}\|\tilde B_{21}\|_{2q}, \end{aligned} \tag{1.23}$$

Recall that $|(m+u)/|k(u)|^2| = |\zeta(u)|$ and $|(m-u)/|k(u)|^2| = |\zeta(u)|^{-1}$, while

$$1 \le C(|\zeta(u)| + |\zeta(u)|^{-1}), \quad u \in \mathbb{C}.$$

Plugging these bounds in (1.22) implies

$$\begin{aligned} \|BS_{u,11}\|_{\mathcal S_p} &\le \frac{C}{|k(u)|^2}(1 + |\zeta(u)| + |\zeta(u)|^{-1})\|\tilde A\|_{2q}\|\tilde B\|_{2q} \\ &\le \frac{C}{|k(u)|^2}(|\zeta(u)| + |\zeta(u)|^{-1})\|\tilde A\|_{2q}\|\tilde B\|_{2q} \\ &= C(|\zeta(u)| + |\zeta(u)|^{-1})|k(u)|^{2/q-2}\|A\|_{2q}\|B\|_{2q}, \end{aligned} \tag{1.24}$$

where we used the rescaling (1.21) in the last line. We notice that

$$(|\zeta(u)| + |\zeta(u)|^{-1})|k(u)|^{2/q-2} \le C\Phi_q(u), \quad u \in \rho(D_{bg,m}).$$

Hence (1.24) is exactly the formula claimed in (0.7).

Consequently, it remains to prove (1.23). Set

$$m_1(\xi) := \frac{1}{(|\xi|^4 - e^{i\varphi})}, \quad m_2(\xi) := \frac{(\xi_1 \pm i\xi_2)^2}{(|\xi|^4 - e^{i\varphi})}.$$

Furthermore, take $\chi_1 \in C_0^\infty(\mathbb{R}^2)$ with the properties: $0 \le \chi_1(x) \le 1$ for all $x \in \mathbb{R}^2$, χ_1 is supported in $\{x \in \mathbb{R}^2 : 1/2 \le |x| \le 3/2\}$ and $\chi_1(x) = 1$ for $x \in \{x \in \mathbb{R}^2 : 3/4 \le |x| \le 5/4\}$. Let $\chi_2 := 1 - \chi_1$; by definition $\chi_1 + \chi_2 = 1$ is a smooth partition of unity. Rewriting (1.23) in terms of symbols of differential operators, we shall show that

$$\|\tilde A\chi_l(D)m_j(D)\tilde B\|_{\mathcal S_p} \le C\|\tilde A\|_{2q}\|\tilde B\|_{2q}, \quad l, j = 1, 2.$$

For $1 \le q \le 3/2$, the bound for $l = 1$ is exactly *Case I* of Proposition 1.6.

Consider the case $l = 2$ now. Notice that for the range of q's we are interested in, one can always choose $\varepsilon > 0$ small enough so that $p = p(q, \varepsilon) \ge q$. Thus we shall prove the bound

$$\|\tilde A\chi_2(D)m_j(D)\tilde B\|_{\mathcal S_q} \le C\|\tilde A\|_{2q}\|\tilde B\|_{2q}, \quad j = 1, 2,$$

which is stronger than (1.23). Notice that

$$\begin{aligned} |\chi_2(\xi)m_1(\chi)| &= \Big|\frac{\chi_2(\xi)}{|\xi|^4 - e^{i\varphi}}\Big| \le \frac{C}{(1+|\xi|^2)}, \\ |\chi_2(\xi)m_2(\chi)| &= \Big|\frac{\chi_2(\xi)(\xi_1 \pm i\xi_2)^2}{|\xi|^4 - e^{i\varphi}}\Big| \le \frac{C}{(1+|\xi|^2)}. \end{aligned}$$

Lemma 3.1 applied to the operator $\tilde{A}\chi_2(D)m_j(D)\tilde{B}$ gives

$$\|\tilde{A}\chi_2(D)m_j(D)\tilde{B}\|_{\mathcal{S}_q} \le \|(1+|\xi|^2)^{-1}\|_q \, \|\tilde{A}\|_{2q}\|B\|_{2q}, \quad j = 1, 2,$$

as needed.

Let us turn to *Case II*, $q > 3/2$. The proof closely follows the proof of Proposition 1.6, *Case II*. It consists in interpolation in between bounds for parameters $q = 3/2$ (*i.e., Case I*), and $q = \infty$.

Assume that $A > 0$ and $B > 0$. Fix $q > 3/2$ and let $p = p(q) := 2q$. This choice implies in particular that $2q/3 > 1$. Set $a_0 = 0, b_0 = 2q/3$ and consider the strip

$$S := \{\zeta = a + it : a_0 \le a \le b_0, \ t \in \mathbb{R}\}.$$

The family of operators

$$T_\zeta = A^\zeta (D_{bg,m} - u)^{-1} B^\zeta,$$

is analytic on S. Apply (0.7) with $q_0 = 3/2$ in place of q to the family T_ζ on $\operatorname{Re}\zeta = b_0 = 2q/3$; that is

$$\|A^{2q/3+it}(D_{bg,m} - u)^{-1}B^{2q/3+it}\|_{\mathcal{S}_3} \le C\Phi(u)\|A\|_{2q}^{2q/3}\|B\|_{2q}^{2q/3}, \tag{1.25}$$

where we used that $\|A^{2q/3+it}\|_3 = \|A\|_{2q}^{2q/3}$, and the same relation holds for B. Notice that $q_{01} = 1 - 1/(2q_0) = 2/3$. For $\operatorname{Re}\zeta = a_0 = 0$, we have the trivial bound

$$\|A^{it}(D_{bg,m} - u)^{-1}B^{it}\|_{\mathcal{S}_\infty} \le \frac{1}{d(u, \sigma(D_{bg,m}))}. \tag{1.26}$$

As in Proposition 1.6, we interpolate between (1.25) and (1.26) using Theorem 3.3 with parameters $\zeta := 1$ and

$$1 = \frac{2q}{3}\gamma + (1-\gamma)\,0, \quad \frac{1}{p_\gamma} = \frac{3}{\gamma} + \frac{(1-\gamma)}{\infty} = \frac{1}{2q}.$$

Hence, $\gamma = 3/(2q)$ and $p_\gamma = 2q$. Claim (0.8) follows, and this finishes the proof of the theorem. □

2. Lieb–Thirring inequalities for bilayer graphene

In what follows we always assume that $m > 0$. We begin with the standard Zhukovsky transform

$$z = z(w) = \frac{m}{2}\Big(w + \frac{1}{w}\Big), \tag{2.1}$$

which maps the upper half-plane $\mathbb{C}_+$ onto the domain $\rho(D_{bg,m})$. Since

$$|z(w) \pm m| = \frac{m}{2|w|}\,|w \pm 1|^2,$$

we have

$$\begin{aligned} |z+m| + |z-m| &= \frac{m}{2|w|}\Big(|w+1|^2 + |w-1|^2\Big) = \frac{m}{|w|}\big(1 + |w|^2\big), \\ |z^2 - m^2|^{\frac{1}{2}} &= \frac{m}{2|w|}\,|w^2 - 1|. \end{aligned} \tag{2.2}$$

The distortion [42, Cor. 1.4] for the Zhukovsky transform reads as

$$\frac{d(z,\sigma(D_{bg,m}))}{\operatorname{Im} w} \asymp |z'(w)| = \frac{m|w^2-1|}{2|w|^2} = \frac{|z^2-m^2|^{1/2}}{|w|}\,, \qquad w\in\mathbb{C}_+. \tag{2.3}$$

2.1. **Proof of Theorem 0.4, Case I:** $1<q\le 3/2$**.** We have, by (2.2),

$$\Phi(z(w)) = C(1+|w|^2)\,\frac{|w|^{p_1}}{|w^2-1|^{2q_1}}\,, \qquad p_1 := 2q_1-1 = 1-\frac{1}{q}>0.$$

The bound (0.7) in the variable w reads

$$\|V_2(D_{bg,m}-z(w))^{-1}V_1\|_{\mathcal{S}_p} \le C_9(1+|w|^2)\,\frac{|w|^{p_1}}{|w^2-1|^{2q_1}}\,\|V\|_q, \quad w\in\mathbb{C}_+, \tag{2.4}$$

where $V_2 = A := |V|^{1/2}$ and $V_1 = B := V^{1/2}$, see the discussion preceding (0.4). For $w=iy$, $y>0$,

$$\|V_2(D_{bg,m}-z(iy))^{-1}V_1\|_{\mathcal{S}_p} \le C_9\left(\frac{y}{1+y^2}\right)^{p_1}\|V\|_q < \frac{C_9}{y^{p_1}}\,\|V\|_q. \tag{2.5}$$

We proceed with the *regularized perturbation determinant*

$$H(w) := \det_p\big(I+V_2(D_{bg,m}-z(w))^{-1}V_1\big), \qquad w\in\mathbb{C}_+,$$

which admits the bounds, see [44, Thm. 9.2]

$$\log|H(w)| \le \Gamma_p\,\|V_2(D_{bg,m}-z(w))^{-1}V_1\|^p_{\mathcal{S}_p} \tag{2.6}$$

and

$$|H(w)-1| \le \varphi\big(\|V_2(D_{bg,m}-z(w))^{-1}V_1\|_{\mathcal{S}_p}\big), \tag{2.7}$$

where

$$\varphi(x) := x\exp\{\Gamma_p(x+1)^p\}, \quad x\ge 0.$$

Denote

$$h(w) = h_y(w) := \frac{H(yw)}{H(iy)}\,, \qquad h(i)=1, \tag{2.8}$$

$y\ge 1$ is chosen later on.

Proposition 2.1. *Assume that*

$$\|V\|_q \le 1. \tag{2.9}$$

Then there is a constant $C_{10}=C_{10}(m,q,\varepsilon)$ so that for $y=C_{10}$ the following holds

$$\log|h(w)| \le C_{11}\,\frac{(1+|w|)^{4pq_1}}{|w^2-y^{-2}|^{2pq_1}}\,\|V\|_q, \qquad w\in\mathbb{C}_+.$$

Proof. Without loss of generality we assume that $C_9>1$. If $y^{p_1}\ge C_9\ge C_9\|V\|_q$, we have, by (2.5),

$$\|V_2(D_{bg,m}-z(iy))^{-1}V_1\|_{\mathcal{S}_p} \le \frac{C_9}{y^{p_1}}\,\|V\|_q \le \|V\|_q \le 1. \tag{2.10}$$

An obvious bound $\varphi(x)\le \exp\{2^p\Gamma_p\}\,x$, $0\le x\le 1$, implies, in view of (2.10),

$$\varphi\big(\|V_2(D_{bg,m}-z(iy))^{-1}V_1\|_{\mathcal{S}_p}\big) \le e^{2^p\Gamma_p}\,\|V_2(D_{bg,m}-z(iy))^{-1}V_1\|_{\mathcal{S}_p},$$

and so, by (2.7),

$$1-|H(iy)| \le |H(iy)-1| \le \frac{C_9e^{2^p\Gamma_p}}{y^{p_1}}\,\|V\|_q \le \frac{1}{2},$$

as soon as $y^{p_1} \geq 2C_9 \exp\{2^p\Gamma_p\} =: C_{12}$. The case $|H(iy)| > 1$ being trivial, we continue with the case $\frac{1}{2} \leq |H(iy)| \leq 1$. Hence,

$$|H(iy)| \geq \frac{1}{2}\,, \quad \log|H(iy)| \geq -2\big(1-|H(iy)|\big) \geq -C_{12}\,\frac{\|V\|_q}{y^{p_1}}\,. \tag{2.11}$$

A combination of (2.6), (2.4), and (2.11) leads to the bound

$$\begin{aligned}
\log|h(w)| &= \log|H(yw)| - \log|H(iy)| \\
&\leq C(1+y|w|)^{2p}\frac{(y|w|)^{pp_1}}{|y^2w^2-1|^{2pq_1}}\,\|V\|_q^p + C_{12}\,\frac{\|V\|_q}{y^{p_1}} \\
&\leq C_{13}\left[\frac{(1+|w|)^{2p}|w|^{pp_1}}{|w^2-y^{-2}|^{2pq_1}}\,\frac{\|V\|_q^p}{y^{pp_1}} + \frac{\|V\|_q}{y^{p_1}}\right] \\
&\leq C_{13}\frac{\|V\|_q}{y^{p_1}}\left[\frac{(1+|w|)^{2p}|w|^{pp_1}}{|w^2-y^{-2}|^{2pq_1}} + 1\right].
\end{aligned}$$

As $2p + pp_1 - 4pq_1 = -pp_1 < 0$, we have for $y \geq 1$

$$(1+|w|)^{2p}|w|^{pp_1} + |w^2-y^{-2}|^{2pq_1} \leq (1+|w|)^{2p+pp_1} + (1+|w|)^{4pq_1} < 2(1+|w|)^{4pq_1}.$$

The result follows with $y = C_{10} = C_{12}^{1/p_1}$, $C_{11} = 2C_{13}$. □

It is well known that the Lieb–Thirring inequalities agree with the Blaschke type conditions for the zeros of the corresponding perturbation determinants. So, the next step is an application of [5, Thm. 4.4] to the above function h. The input parameters are

$$a = 0, \quad b = 2pq_1, \quad c_j = 0; \quad x_1' = y^{-1},\ x_2' = -y^{-1}, \quad K = C\|V\|_q,$$

$$d_1 = d_2 = d = 2pq_1 = \begin{cases} \frac{2q-1}{2-q} + (2-\frac{1}{q})\varepsilon, & 1 < q < \frac{4}{3}; \\ \frac{2q-1}{2-q}, & \frac{4}{3} \leq q \leq \frac{3}{2}. \end{cases}$$

The output parameters in [5, Thm. 4.4] are

$$l = \{l\}_{a,\varepsilon} = 0, \quad (d-1+\varepsilon)_+ = \frac{3q-3}{2-q} + \omega_q\varepsilon, \quad l_1 = \frac{4q-2}{2-q} + \tau_q\varepsilon,$$

with

$$\omega_q = \begin{cases} \frac{3q-1}{q}, & 1 < q < \frac{4}{3}; \\ 1, & \frac{4}{3} \leq q \leq \frac{3}{2}. \end{cases} \qquad \tau_q = \begin{cases} \frac{6q-1}{q}, & 1 < q < \frac{4}{3}; \\ 1, & \frac{4}{3} \leq q \leq \frac{3}{2}. \end{cases}$$

So, the Blaschke type condition of [5, Thm. 4.4] takes the form

$$\sum_{\xi\in Z(h)} \frac{(\operatorname{Im}\xi)^{1+\varepsilon}}{(1+|\xi|)^{l_1}}\,|\xi^2-y^{-2}|^{(d-1+\varepsilon)_+} \leq C_{14}\|V\|_q, \tag{2.12}$$

and, since the "test point" y in Proposition 2.1 does not depend on V, the constant $C_{14}(m,q,\varepsilon)$ does not depend on V either.

In terms of the zeros of H we have

$$\xi \in Z(h) \ \Leftrightarrow\ y\,\xi = \lambda \in Z(H), \qquad \xi = \frac{\lambda}{y}\,,$$

and as $y = C_{10}$ is a constant, condition (2.12) does not alter

$$\sum_{\lambda\in Z(H)} \frac{(\operatorname{Im}\lambda)^{1+\varepsilon}}{(1+|\lambda|)^{l_1}}\,|\lambda^2-1|^{(d-1+\varepsilon)_+} \leq C_{15}\|V\|_q. \tag{2.13}$$

It remains to get back to the spectral variable $z \in \rho(D_{bg,m})$, keeping in mind that for the discrete spectrum of D_{bg} the equivalence holds

$$\zeta \in \sigma_d(D_{bg}) \ \Leftrightarrow\ \lambda \in Z(H).$$

To make the final result transparent, we invoke the main result [7, Theorem 1.1], which claims, in particular, that the discrete spectrum $\sigma_d(D_{bg})$ is bounded, that is, $|\zeta| \le C_{16}$, $\forall \zeta \in \sigma_d(D_{bg})$. In the Zhukovsky variable the latter means

$$0 < c \le |\lambda| \le C < \infty, \qquad \forall \lambda \in Z(H). \tag{2.14}$$

So we can neglect the term $1 + |\lambda|$ in (2.12). Next, as in (2.2),

$$|\zeta^2 - m^2| = \frac{m^2}{4}\,\frac{|\lambda^2 - 1|^2}{|\lambda|^2} \;\Rightarrow\; c|\lambda^2 - 1| \le |\zeta^2 - m^2|^{1/2} \le C|\lambda^2 - 1|.$$

Finally, the distortions (2.3) and (2.14) imply

$$c \operatorname{Im} \lambda \le \frac{d(\zeta, \sigma(D_{bg,m}))}{|\zeta^2 - m^2|^{1/2}} \le C \operatorname{Im} \lambda.$$

Case I of Theorem 0.4 is proved. □

2.2. **Proof of Theorem 0.4, Case II:** $q > 3/2$**.** We use the distortion (2.3) to obtain the bound similar to (2.4)

$$\|V_2(D_{bg,m} - z(w))^{-1} V_1\|_{\mathcal{S}_p} \le C_9 \frac{(1+|w|)^{2q_2}}{(\operatorname{Im} w)^{p_2}}\,\frac{|w|^{p_3}}{|w^2-1|^{p_4}}\,\|V\|_q, \quad w \in \mathbb{C}_+, \tag{2.15}$$

where

$$p = 2q, \quad p_2 := 1 - q_2 = 1 - \frac{3}{2q} > 0, \quad p_3 := 2 - \frac{5}{2q}\,, \quad p_4 := 1 + \frac{1}{2q}\,.$$

Note that $p_3 - p_2 = p_1$. For $w = iy$, $y > 0$, the bound is exactly the same as (2.5)

$$\|V_2(D_{bg,m} - z(iy))^{-1} V_1\|_{\mathcal{S}_p} < \frac{C_9}{y^{p_1}}\,\|V\|_q. \tag{2.16}$$

We argue as in the proof of Proposition 2.1 to obtain the bound for h (2.8)

$$\log|h(w)| \le C_{11}\,\frac{|w|^{pp_2}(1+|w|)^{2pp_4}}{(\operatorname{Im} w)^{pp_2}\,|w^2 - y^{-2}|^{pp_4}}\,\|V\|_q. \tag{2.17}$$

Indeed,

$$\begin{aligned}
\log|h(w)| &= \log|H(yw)| - \log|H(iy)| \\
&\le C\frac{(1+y|w|)^{2pq_2}(y|w|)^{pp_3}}{(\operatorname{Im} yw)^{pp_2}|y^2w^2-1|^{pp_4}}\,\|V\|_q^p + C_{12}\,\frac{\|V\|_q}{y^{p_1}} \\
&\le C_{13}\left[\frac{(1+|w|)^{2pq_2}|w|^{pp_3}}{(\operatorname{Im} w)^{pp_2}|w^2-y^{-2}|^{pp_4}}\,\frac{\|V\|_q^p}{y^{pp_1}} + \frac{\|V\|_q}{y^{p_1}}\right] \\
&\le C_{13}\frac{\|V\|_q}{y^{p_1}}\left[\frac{(1+|w|)^{2pq_2}|w|^{pp_3}}{(\operatorname{Im} w)^{pp_2}|w^2-y^{-2}|^{pp_4}} + 1\right].
\end{aligned}$$

Next,

$$\begin{aligned}
&(1+|w|)^{2pq_2}|w|^{pp_3} + (\operatorname{Im} w)^{pp_2}|w^2-y^{-2}|^{pp_4} \le (1+|w|)^{2pq_2}|w|^{pp_3} + |w|^{pp_2}(1+|w|^2)^{pp_4} \\
&\le |w|^{pp_2}\Big((1+|w|)^{2pq_2}|w|^{pp_1} + (1+|w|)^{2pp_4}\Big) \le 2|w|^{pp_2}(1+|w|)^{2pp_4},
\end{aligned}$$

and (2.17) follows.

The computation with [5, Thm. 4.4] is a bit more complicated now. The input parameters are

$$\begin{gathered}
a = pp_2 = 2q - 3 > 0, \quad b = pp_4 = 2q + 1, \quad x_1' = y^{-1}, \quad x_2' = -y^{-1}, \quad x_1 = 0, \\
c_1 = pp_2 = a, \quad c_j = 0,\ j \ge 2, \quad d_1 = d_2 = d = pp_4 = b, \quad K = C\|V\|_q.
\end{gathered}$$

The output parameters in [5, Thm. 4.4] are

$$l = a, \quad \{l\}_{a,\varepsilon} = -a, \quad (d - 1 + \varepsilon)_+ = 2q + \varepsilon, \quad l_1 = 2 + 4q + 4\varepsilon,$$

so the Blaschke type condition takes the form

$$\sum_{\xi\in Z(h)} \frac{(\operatorname{Im}\xi)^{a+1+\varepsilon}}{(1+|\xi|)^{2+4q+4\varepsilon}} \frac{|\xi^2-y^{-2}|^{2q+\varepsilon}}{|\xi|^a} \le C_{14}\|V\|_q.$$

After the change of variable $\lambda = y\,\xi = C_{10}\xi$, we come to

$$\sum_{\lambda\in Z(H)} \frac{(\operatorname{Im}\lambda)^{a+1+\varepsilon}}{(1+|\lambda|)^{2+4q+4\varepsilon}} \frac{|\lambda^2-1|^{2q+\varepsilon}}{|\lambda|^a} \le C_{15}\|V\|_q. \tag{2.18}$$

As before, the final step relies on the distortion relations for the Zhukovsky transform. Indeed, separate the upper-half plane $\mathbb{C}_+$ in three regions $\Omega_1 := \{\lambda \in \mathbb{C}_+ : c \le |\lambda| \le C\}$, $\Omega_2 := \{\lambda \in \mathbb{C}_+ : |\lambda| \ge C\}$ and $\Omega_3 := \{\lambda \in \mathbb{C}_+ : |\lambda| \le c\}$ with constants c, C chosen as $0 < c < 1 < C < +\infty$. It is clear that

$$\sum_{\lambda\in Z(H)\cap\Omega_1} (\operatorname{Im}\lambda)^{a+1+\varepsilon}\,|\lambda^2-1|^{2q+\varepsilon} \le C \sum_{\lambda\in Z(H)\cap\Omega_1} \frac{(\operatorname{Im}\lambda)^{a+1+\varepsilon}}{(1+|\lambda|)^{2+4q+4\varepsilon}} \frac{|\lambda^2-1|^{2q+\varepsilon}}{|\lambda|^a}.$$

On the other hand, one has $|\zeta(\lambda)| \asymp |\lambda|$ for $\lambda \in \Omega_2$, and $|\zeta(\lambda)| \asymp |\lambda|^{-1}$ for $\lambda \in \Omega_3$. Using these relations along with inequalities given next to (2.14), we cut the sum (2.18) in parts corresponding to domains Ω_i, $i = 1, 2, 3$, and rewrite these partial sums in terms of ζ-variable.

Case II of Theorem 0.4 is proved as well. □

3. Some technical tools: interpolation theorems and Kato-Selier-Simon lemma

3.1. **Kato-Selier-Simon lemma.** Recall the notation introduced in Section 1.1. We have the following proposition usually called Kato-Selier-Simon lemma.

Proposition 3.1 ([44, Thm. 4.1])**.**

(1) Let $f, g \in L^q(\mathbb{R}^d)$, $d \ge 1$. Then, for $2 \le q < \infty$, $f(x)g(D) \in \mathcal{S}_q$, and

$$\|f(x)g(D)\|_{\mathcal{S}_q} \le (2\pi)^{-d}\|f\|_q\|g\|_q.$$

(2) Let $f \in L^q(\mathbb{R}^d)$, $d \ge 1$, and $A, B \in L^{2q}(\mathbb{R}^d)$. For $2 \le q < \infty$,

$$\|A(x)f(D)B(y)\|_{\mathcal{S}_q} \le (2\pi)^{-d}\|f\|_q\,\|A\|_{2q}\|B\|_{2q}.$$

The first claim of the above proposition is in Simon [44, Thm. 4.1]; the second claim is a "symmetrized" version of the first one and it is proved similarly.

3.2. **Interpolation theorem for bounded analytic families.** In this subsection, we follow mainly the presentation of Zhu [47, Ch. 2].

Let X_0, X_1 be two Banach spaces. We say that the pair X_0, X_1 is compatible, if there is a topological Hausdorff space X containing both X_0 and X_1. We have the following theorem.

Theorem 3.2 ([47, Thm. 2.4])**.** *Let X_0, X_1 be a pair of compatible Banach spaces, idem for Y_0, Y_1. For a γ, $0 < \gamma < 1$, there are Banach spaces X_γ, Y_γ,*

$$X_\gamma = [X_0, X_1]_\gamma, \qquad Y_\gamma = [Y_0, Y_1]_\gamma,$$

interpolating in between X_0 and X_1 and Y_0 and Y_1, respectively, in the following sense.

Let $T : X_0 + X_1 \to Y_0 + Y_1$ be a bounded *linear map such that*

$$\begin{aligned} \|Tx\|_{Y_0} &\le C_0\|x\|_{X_0}, \quad x \in X_0,\\ \|Tx\|_{Y_1} &\le C_1\|x\|_{X_1}, \quad x \in X_1. \end{aligned}$$

Then T induces a linear map $T_\gamma : X_\gamma \to Y_\gamma$ with the property

$$\|T_\gamma\| \le C_0^\gamma C_1^{1-\gamma}.$$

Saying "interpolation" we mean "complex interpolation" throughout the article. For instance, we have

$$[L^{p_0}(\mathbb{R}^d), L^{p_1}(\mathbb{R}^d)]_\gamma = L^{p_\gamma}(\mathbb{R}^d), \tag{3.1}$$

where $1 \le p_0, p_1 \le \infty$, $1/p_\gamma = \gamma/p_0 + (1-\gamma)/p_1$, and $d \ge 1$, see [47, Thm. 2.5].

It is important that a similar construction holds for "non-commutative" L^p-spaces as well. That is, denoting by $\mathcal{S}_p$ the Schatten-von Neumann classes of compact operators, we have

$$[\mathcal{S}_{p_0}, \mathcal{S}_{p_1}]_\gamma = \mathcal{S}_{p_\gamma},$$

where $1 \le p_0, p_1 \le \infty$ and $1/p_\gamma = \gamma/p_0 + (1-\gamma)/p_1$. A proof of this result is in [47, Thm. 2.6]. Much more information and further references on the interpolation theory of Banach spaces are in monographs Bennett-Sharpley [1] and Bergh-Löfström [2].

For $1 \le p_{01}, p_{02} \le +\infty$, it is plain to see that

$$L^{p_{01}}(\mathbb{R}^d_x) \times L^{p_{02}}(\mathbb{R}^d_y) \simeq L^{p_{01}}(\mathbb{R}^d_x) \dotplus L^{p_{02}}(\mathbb{R}^d_y), \quad x, y \in \mathbb{R}^d,$$

and so interpolation (3.1) holds for these spaces as well. This observation is often applied to an operator $\mathcal{A}$ of the form

$$\mathcal{A} : L^{p_{01}}(\mathbb{R}^d) \times L^{p_{02}}(\mathbb{R}^d) \to \mathcal{S}_{q_{01}}, \quad 1 \le q_{01} \le +\infty,$$

see Section 1.

3.3. **Interpolation theorem for general analytic families.** Following Gohberg-Krein [25, Ch. III.13], we present a generalized version of interpolation in between $\mathcal{S}_p$-spaces.

Let $a, b \in \mathbb{R},\ a < b$ and

$$S = \{\zeta : a \le \operatorname{Re}\zeta \le b\}$$

be a vertical strip in the complex plane. For a Hilbert space H, we say that a family of bounded operators $(T_\zeta)_{\zeta \in S}, T_\zeta : H \to H$ is is *analytic* on S, if $(T_\zeta f, g)$ is analytic on an open neighborhood of S for any fixed $f, g \in H$.

Theorem 3.3 ([25, Thm. 13.1])**.** *Let $(T_\zeta)_{\zeta \in S}$ be an analytic family of operators. Assume that for any $f, g \in H$*

$$\log|(T_\zeta f, g)| \le C_{1;f,g} e^{C_{2;f,g}|\operatorname{Im}\zeta|}, \quad \zeta \in S, \tag{3.2}$$

where the constants $C_{j;f,g},\ j = 1, 2$ depend on f, g, but not on $\zeta \in S$, and

$$0 \le C_{2;f,g} < \frac{\pi}{(b-a)}.$$

Furthermore, suppose that

(1) for $\operatorname{Re}\zeta = a$, $T_\zeta \in \mathcal{S}_{p_0}$, with $1 \le p_0 \le \infty$ and

$$\|T_\zeta\|_{\mathcal{S}_{p_0}} \le C_0.$$

(2) for $\operatorname{Re}\zeta = b$, $T_\zeta \in \mathcal{S}_{p_1}$, with $1 \le p_1 < p_0$ and

$$\|T_\zeta\|_{\mathcal{S}_{p_1}} \le C_1.$$

Take an $x \in (a, b)$ and write it as $x = \gamma\, a + (1-\gamma)\, b,\ \gamma \in (0, 1)$. For $\zeta \in S, \operatorname{Re}\zeta = x$ we have that $T_\zeta \in \mathcal{S}_{p_\gamma}$, and moreover

$$\|T_\zeta\|_{\mathcal{S}_{p_\gamma}} \le C_0^\gamma\, C_1^{1-\gamma},$$

where $1/p_\gamma = \gamma/p_0 + (1-\gamma)/p_1$.

We often use the following corollary of the above theorem.

Corollary 3.4. *Let $(T_\zeta)_{\zeta\in S}$ be an analytic family of operators satisfying the assumption of Theorem 3.3 with conditions (1), (2) replaced by the following assumptions:*

(1') for $\operatorname{Re}\zeta = a$, $T_\zeta \in \mathcal{S}_{p_0}$, with $1 \le p_0 \le \infty$ and

$$\|T_\zeta\|_{\mathcal{S}_{p_0}} \le C_0 e^{A_0|\operatorname{Im}\zeta|^2}.$$

(2') for $\operatorname{Re}\zeta = b$, $T_\zeta \in \mathcal{S}_{p_1}$, with $1 \le p_1 < p_0$ and

$$\|T_\zeta\|_{\mathcal{S}_{p_1}} \le C_1 e^{A_1|\operatorname{Im}\zeta|^2},$$

for some constants $A_0, A_1 \ge 0$.

As above, for an $x = \gamma\, a + (1-\gamma)\, b \in (a,b)$, $\gamma \in (0,1)$ and $\zeta \in S, \operatorname{Re}\zeta = x$ we have that $T_\zeta \in \mathcal{S}_{p_\gamma}$, and moreover

$$\|T_x\|_{\mathcal{S}_{p_\gamma}} \le C''\, C_0^{\gamma}\, C_1^{1-\gamma},$$

where $1/p_\gamma = \gamma/p_0 + (1-\gamma)/p_1$. The constant C'' depends on a, b, C_0, C_1, A_0 and A_1.

The corollary follows immediately by applying Theorem 3.3 to the analytic family of operators $\tilde{T}_\zeta = e^{\max(A_0,A_1)\zeta^2}\, T_\zeta,\ \zeta \in S$.

Acknowledgments. The third and the fourth authors were partially supported by the grant ANR-18-CE40-0035. A part of this research was done during a visit of the third author to the Institute of Mathematics of Bordeaux (IMB UMR5251) of University of Bordeaux. He is grateful to the institution for the hospitality.

References

[1] Bennett, C.; Sharpley, R. Interpolation of operators. Pure and Applied Mathematics, Vol. 129. Academic Press, Inc., Boston, MA, 1988.

[2] Bergh, J.; Löfström, J. Interpolation spaces. An introduction. Grundlehren der Mathematischen Wissenschaften, no. 223. Springer-Verlag, Berlin-New York, 1976.

[3] Birman, M. On the spectrum of singular boundary-value problems. Mat. Sb. (N.S.) 55 (97) 1961 125-174 (Russian); translated in: Amer. Math. Soc. Trans. 53 (1966), 23-80.

[4] Borichev, A.; Golinskii, L.; Kupin, S. A Blaschke-type condition and its application to complex Jacobi matrices. Bull. Lond. Math. Soc. 41 (2009), no. 1, 117-123.

[5] Borichev, A.; Golinskii, L.; Kupin, S. On zeros of analytic functions satisfying non-radial growth conditions. Rev. Mat. Iberoam. 34 (2018), no. 3, 1153-1176.

[6] Cuenin, J.-C. Eigenvalue bounds for Dirac and fractional Schrödinger operators with complex potentials. J. Funct. Anal. 272 (2017), no. 7, 2987-3018.

[7] Cuenin, J.-C. Eigenvalue estimates for bilayer graphene. Ann. Henri Poincaré 20 (2019), no. 5, 1501-1516.

[8] Cuenin, J.-C.; Laptev, A.; Tretter, C. Eigenvalue estimates for non-selfadjoint Dirac operators on the real line. Ann. Henri Poincaré 15 (2014), no. 4, 707-736.

[9] Cuenin, J.-C.; Siegl, P. Eigenvalues of one-dimensional non-self-adjoint Dirac operators and applications. Lett. Math. Phys. 108 (2018), no. 7, 1757-1778.

[10] Cuenin, J.-C.; Kenig, C. E. L^p resolvent estimates for magnetic Schrödinger operators with unbounded background fields. Comm. Partial Differential Equations 42 (2017), no. 2, 235-260.

[11] Cuenin, J.-C. Estimates on complex eigenvalues for Dirac operators on the half-line. Integral Equations Operator Theory 79 (2014), no. 3, 377-388.

[12] Cossetti, L. Uniform resolvent estimates and absence of eigenvalues for Lamé operators with complex potentials. J. Math. Anal. Appl. 455 (2017), no. 1, 336-360.

[13] Demuth, M.; Hansmann, M.; Katriel, G. On the discrete spectrum of non-selfadjoint operators. J. Funct. Anal. 257 (2009), no. 9, 2742-2759.

[14] Dubuisson, C. On quantitative bounds on eigenvalues of a complex perturbation of a Dirac operator. Integral Equations Operator Theory 78 (2014), no. 2, 249-269.

[15] Fanelli, L.; Krejčiřík, D.; Vega, L. Spectral stability of Schrödinger operators with subordinated complex potentials. J. Spectr. Theory 8 (2018), no. 2, 575-604.

[16] Fanelli, L.; Krejčiřík, D.; Vega, L. Absence of eigenvalues of two-dimensional magnetic Schrödinger operators. J. Funct. Anal. 275 (2018), no. 9, 2453-2472.

[17] Fanelli, L.; Krejčiřík, D. Location of eigenvalues of three-dimensional non-self-adjoint Dirac operators. Lett. Math. Phys. 109 (2019), no. 7, 1473-1485.
[18] Ferrulli, F.; Laptev, A.; Safronov, O. Eigenvalues of the bilayer graphene operator with a complex valued potential, submitted, arxiv: *https://arxiv.org/abs/1612.05304*.
[19] Frank, R. Eigenvalue bounds for Schrödinger operators with complex potentials. Bull. Lond. Math. Soc. 43 (2011), no. 4, 745-750.
[20] Frank, R. Eigenvalue bounds for Schrödinger operators with complex potentials. III. Trans. Amer. Math. Soc., 370 (2018) no. 1, 219-240.
[21] Frank, R.; Sabin, J. Restriction theorems for orthonormal functions, Strichartz inequalities, and uniform Sobolev estimates. Amer. J. Math. 139 (2017), no. 6, 1649-1691.
[22] Frank, R.; Simon, B. Eigenvalue bounds for Schrödinger operators with complex potentials. II. J. Spectr. Theory 7 (2017), no. 3, 633-658.
[23] Frank, R. L.; Laptev, A.; Safronov, O. On the number of eigenvalues of Schrödinger operators with complex potentials. J. Lond. Math. Soc. (2) 94 (2016), no. 2, 377-390.
[24] Gesztesy, F.; Latushkin, Y.; Mitrea, M.; Zinchenko, M. Nonselfadjoint operators, infinite determinants, and some applications. Russ. J. Math. Phys. 12 (2005), no. 4, 443-471.
[25] Gohberg, I.; Krein, M. Introduction to the theory of linear nonselfadjoint operators. Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18. American Mathematical Society, Providence, R.I., 1969.
[26] Golinskii, L.; Kupin, S. On discrete spectrum of complex perturbations of finite band Schrödinger operators. Recent trends in analysis, 113-121, Theta Ser. Adv. Math., 16, Theta, Bucharest, 2013.
[27] Golinskii, L.; Kupin, S. On complex perturbations of infinite band Schrödinger operators. Methods Funct. Anal. Topology 21 (2015), no. 3, 237-245.
[28] Golinskii, L.; Kupin, S. A remark on analytic Fredholm alternative. Lett. Math. Phys. 107 (2017), no. 3, 467-474.
[29] Golinskii, L.; Kupin, S. On non-selfadjoint perturbations of infinite band Schrödinger operators and Kato method. Harmonic analysis, function theory, operator theory, and their applications, 171-182, Theta Ser. Adv. Math., 19, Theta, Bucharest, 2017.
[30] Hulko, A. On the number of eigenvalues of the biharmonic operator on $\mathbb{R}^3$ perturbed by a complex potential. Rep. Math. Phys. 81 (2018), no. 3, 373-383.
[31] Hulko, A. On the number of eigenvalues of the discrete one-dimensional Dirac operator with a complex potential. Anal. Math. Phys. 9 (2019), no. 1, 639-654.
[32] Hörmander, L. The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis. Classics in Mathematics. Springer-Verlag, Berlin, 2003.
[33] Hörmander, L. The analysis of linear partial differential operators. II. Differential operators with constant coefficients. Classics in Mathematics. Springer-Verlag, Berlin, 2005.
[34] Ibrogimov, O. O.; Krejčiřík, D.; Laptev, A. Sharp bounds for eigenvalues of biharmonic operators with complex potentials in low dimensions. arXiv: https://arxiv.org/abs/1903.01810.
[35] Kato, T. Wave operators and similarity for some non-selfadjoint operators. Math. Ann. 162 (1965/66), 258-279.
[36] Katsnelson, M. Graphene: carbon in two dimensions. Cambridge Univ. Press, 2012.
[37] Kenig, C. E.; Ruiz, A.; Sogge, C. D. Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators. Duke Math. J. 55 (1987), no. 2, 329-347.
[38] Lee, Y.; Seo, I. A note on eigenvalue bounds for Schrödinger operators. J. Math. Anal. Appl. 470 (2019), no. 1, 340-347.
[39] Lieb, E.; Loss, M. Analysis. Second edition. Graduate Studies in Mathematics, Vol. 14. AMS, Providence, RI, 2001.
[40] Mizutani, H. Eigenvalue bounds for non-self-adjoint Schrödinger operators with the inverse-square potential. J. Spectr. Theory 9 (2019), no. 2, 677-709.
[41] Muscalu, C.; Schlag, W. Classical and multilinear harmonic analysis, I. Cambridge University Press, Vol. 137, Cambridge, 2013.
[42] Pommerenke, Ch. Boundary behaviour of conformal maps. Grundlehren der Mathematischen Wissenschaften, Vol. 299. Springer-Verlag, Berlin, 1992.
[43] Schwinger, J. On the bound states for a given potential, Proc. Nat. Acad. Sci. U.S.A. 47 (1961), 122-129.
[44] Simon, B. Trace ideals and their applications. Mathematical Surveys and Monographs, Vol. 120. AMS, Providence, RI.
[45] Sogge, C. Fourier integrals in classical analysis. Cambridge University Press, Cambridge, 1993.
[46] Stein, E. Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton Mathematical Series, Vol. 43. Princeton University Press, Princeton, NJ, 1993.

[47] Zhu, K. Operator theory in function spaces. Second edition. Mathematical Surveys and Monographs, Vol. 138. AMS, Providence, RI, 2007.

Center of Theoretical Physics, CNRS, Aix-Marseille University and University of Toulon, Luminy Campus, 163 ave. de Luminy, 13288 Marseille Cedex 9, France
E-mail address: briet@cpt.univ-mrs.fr

Institute of Mathematics, University of Munich, Theresien str. 39, D-80333 Munich, Germany
E-mail address: cuenin@math.lmu.edu

Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire, LE11 3TU United Kingdom
E-mail address: J.Cuenin@lboro.ac.uk

B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Science ave., 61103 Kharkiv, Ukraine
E-mail address: golinskii@ilt.kharkov.ua

IMB, CNRS, Université de Bordeaux, 351 ave. de la Libération, 33405 Talence Cedex, France
E-mail address: skupin@math.u-bordeaux1.fr