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Absolute continuity of the spectrum in a twisted Dirichlet-Neumann waveguide

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Quantum waveguide with the shape of planar infinite straight strip and combined Dirichlet and Neumann boundary conditions on the opposite half-lines of the boundary is considered. The absence of the point as well as of the singular continuous spectrum is proved.

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I. INTRODUCTION

Two-dimensional straight waveguides with combined boundary conditions, classical as well as quantum, were considered in a number of papers [1]–[5]. Mostly the existence of isolated eigenvalues was studied. We consider a very special configuration of such quantum waveguide here for which we show the absence of the eigenvalues, including the embedded in the essential spectrum ones, and the absence of singular continuous spectrum.

Let H be the operator that acts as the Laplacian in a straight strip $\Omega := \mathbb{R}\times(0,d)$ with $d>0$ and satisfies Dirichlet boundary conditions on $\partial_D\Omega := [(-\infty,0)\times\{0\}]\cup[(0,\infty)\times\{d\}]$ and Neumann boundary conditions on the other part of the boundary $\partial_N\Omega := [(-\infty,0)\times\{d\}]\cup[(0,\infty)\times\{0\}]$. We understand H as the self-adjoint operator in the Hilbert space $L^2(\Omega)$ generated by the closed form

$$h[\psi] := \int_\Omega |\nabla\psi|^2, \quad \mathsf{D}(h) := \{\psi\in H^1(\Omega)|\ \psi\restriction\partial_D\Omega = 0\}\,. \tag{1.1}$$

One has

$$H\psi = -\Delta\psi, \quad \mathsf{D}(H) = \{\psi\in H^1(\Omega)\,|\,\Delta\psi\in L^2(\Omega)\,,\ \psi\restriction\partial_D\Omega = 0\,,\partial_y\psi\restriction\partial_N\Omega = 0\}\,.$$

Here we denote by (x,y) a generic point in Ω.

The model belongs to the configurations introduced in [6]. Let $E_n := (2n-1)^2\pi^2/(2d)^2$ with $n\in\mathbb{N}^* := \mathbb{N}\setminus\{0\}$ denote the eigenvalues of the Laplacian in $L^2((0,d))$, subject to a Dirichlet boundary condition at 0 and a Neumann boundary condition at d (or vice versa). It is easy to see that

$$\sigma(H) = \sigma_{\rm ess}(H) = [E_1,\infty)\,.$$

In [7] it was shown that the operator H satisfies a Hardy-type inequality $H - E_1 \geq c/(1+x^2)$ with a positive constant c and in [8] the consequences on the behaviour of the heat semigroup e^{-tH} for large times $t>0$ were studied. In particular, it follows that E_1 cannot be an eigenvalue of H. As the last progress, the existence of a scattering stationary wave function was established in [9].

To complete the study of the model, in this paper we study the nature of the essential spectrum and show that the spectrum of H is actually purely absolutely continuous:

Theorem 1. *One has*

$$\sigma_{\rm p}(H) = \varnothing \qquad \textit{and} \qquad \sigma_{\rm sc}(H) = \varnothing\,.$$

The idea of our proof of the absence of point spectrum is based on the (here formal) commutator identity

$$i[H,A] = -2\,\partial_x^2\,, \tag{1.2}$$

where A is the dilation operator in the longitudinal direction acting as

$$A := -\frac{i}{2}\,(x\,\partial_x + \partial_x\,x)\,. \tag{1.3}$$

It follows from (1.2) that if there exists $u\in\mathsf{D}(H)\cap\mathsf{D}(A)$ such that $Hu=\lambda u$ with $\lambda\in\mathbb{R}$, then

$$0 = (u, i[H,A]u) = 2\|\partial_x u\|^2\,,$$

where $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner product and norm in $L^2(\Omega)$, respectively. Consequently, $\partial_x u = 0$ as an element of $L^2(\Omega)$ and therefore necessarily $u=0$. It essentially shows that the point spectrum of H is empty. To prove the other statement of Theorem 1, we employ the positivity of the right-hand side of (1.2), apart from the set of thresholds

$$\mathcal{T} := \{E_k\}_{k\in\mathbb{N}^*}\,, \tag{1.4}$$

with help of the Mourre theory of conjugate operators [10].

The danger of the formal procedure described above is best illustrated by observing that the same conclusions are obtained for the modified operator H_ε generated by the form (1.1), where $\partial_D\Omega$ is replaced by $\partial_D^\varepsilon\Omega := [(-\infty,-\varepsilon)\times\{0\}]\cup[(\varepsilon,\infty)\times\{d\}]$ with any real ε. But if ε is positive (so that the Neumann boundary conditions overlap) and sufficiently large, then it is known (see [6]) that H_ε admits (discrete) eigenvalues. The reason behind this apparent contradiction is the fact that the function Au does not necessarily belong to $\mathsf{D}(H)$, so the identity (1.2) does not make sense even when applied to $u\in\mathsf{D}(H)$.

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We prove the absence of the point and singular continuous spectrum for a very special configuration of the planar straight quantum waveguide with combined Dirichlet and Neumann boundary conditions. While the specific configuration is essential for the non-existence of discrete eigenvalues, the absence of the singular continuous spectrum is a more robust property. As the used conjugate operator is localised at infinity (acts as zero near the origin $x = 0$), the same proofs can be done for variants of H modified in a bounded subset of Ω. For instance, we could consider an arbitrary finite combination of Dirichlet-Neumann boundary conditions in $(-R, R) \times (0, d)$, or even Robin boundary conditions and perhaps compactly supported potentials. However, the modifications should be such that Proposition 2 below, i.e. the bound of $\|\partial_x \psi\| \leq C\|H\psi\|$ used in the estimate of (3.11), holds. This might be a restriction on the possibility of the waveguide shape local modifications.

We use the Mourre theory in its original form [10]. More advanced exposition can be found in the book [11]. The first application of the Mourre theory in the context of quantum waveguides is due to [12], see also [13], [14] and [15] for further developments.

The organisation of the paper is as follows. In order to justify that the formal argument goes through in our situation $H = H_0$, in Section II we use a cut-off approximation of u both for large and small x and proceed by the method of multipliers in the spirit of [16, 17]. It is interesting that this apparently technical regularisation actually gives an insight into why this procedure for H_ε with positive ε cannot generally work. Finally, in Section III we modify (1.3) to a conjugate operator "localised at infinity" and prove a (non-strict) Mourre estimate.

II. ABSENCE OF THE POINT SPECTRUM

Let us assume that there exists an eigenfunction $u \in \mathsf{D}(H) \subset \mathsf{D}(h)$ and an eigenvalue $\lambda \in \mathbb{R}$ satisfying

$$(H - \lambda)u = 0\,. \tag{2.1}$$

Then for any $v \in \mathsf{D}(h)$

$$h(v, u) - \lambda\,(v, u) = 0\,. \tag{2.2}$$

We would like to construct a special v such that from the last equation would follow $u = 0$ and so there is no eigenvector. More precisely, our choice of v would not lie in $\mathsf{D}(h)$ so we need to construct a sequence of regularised functions $v_n \in \mathsf{D}(h)$ and obtain the result in the limit.

Without loss of generality, we assume that u is real as $\Re u$ and $\Im u$ satisfy (2.1) separately. As a solution of the differential equation $-\Delta u - \lambda u = 0$, $u \in C^\infty(\Omega)$ (cf., e.g., [18, Thm. 2.2 of Chapt. 4] together with the Sobolev embedding theorem). In particular, the derivatives of u and its powers may be calculated as classical.

For the regularisation purposes, let us first define a sequence of functions ($n = 2, 3, 4, \dots$)

$$\varphi_n(x) := \begin{cases} 0 & \text{for } x \leq -2n\,, \\ (x+2n)/n & \text{for } -2n < x < -n\,, \\ 1 & \text{for } -n \leq x \leq -n^{-1}\,, \\ n^2(x+n^{-2})/(1-n) & \text{for } -n^{-1} < x < -n^{-2}\,, \\ 0 & \text{for } -n^{-2} \leq x \leq n^{-2}\,, \\ n^2(x-n^{-2})/(n-1) & \text{for } n^{-2} < x < n^{-1}\,, \\ 1 & \text{for } n^{-1} \leq x \leq n\,, \\ (2n-x)/n & \text{for } n < x < 2n\,, \\ 0 & \text{for } x \geq 2n\,, \end{cases} \tag{2.3}$$

belonging to $H^1(\mathbb{R})$ with the derivatives φ_n defined almost everywhere. Then set

$$v_n(x, y) := \varphi_n(x)(2xu_x(x, y) + u(x, y))\,. \tag{2.4}$$

Now

$$\begin{aligned} v_{nx}(x, y) &= \varphi_n'(x)(2xu_x(x, y) + u(x, y)) + \varphi_n(x)(3u_x(x, y) + 2xu_{xx}(x, y))\,, \\ v_{ny}(x, y) &= \varphi_n(x)(2xu_{xy}(x, y) + u_y(x, y))\,. \end{aligned}$$

Evidently, $v_n \in \mathsf{D}(h)$ and so satisfies (2.2). Remembering the properties of $\mathsf{D}(H)$ [6], u,u_x,u_y, $u_{xx} + u_{yy} \in L^2(\Omega)$ and $u_{xx}, u_{xy}, u_{yy} \in L^2(\operatorname{supp}\varphi_n \times (0, d))$, we write

$$h(v_n, u) = \int_\Omega \varphi_n'(x)(2xu_x^2 + uu_x)\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \varphi_n(x)(3u_x^2 + 2xu_xu_{xx} + 2xu_yu_{xy} + u_y^2)\,\mathrm{d}x\,\mathrm{d}y\,. \tag{2.5}$$

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Integration by parts with respect to x, and also with respect to y in one case, gives

$$\begin{aligned}\int_\Omega \varphi_n' u u_x \,\mathrm{d}x\,\mathrm{d}y &= -\int_\Omega \varphi_n (u_x^2 + u u_{xx})\,\mathrm{d}x\,\mathrm{d}y \\ &= -\int_\Omega \varphi_n (u_x^2 + u\Delta u - u u_{yy})\,\mathrm{d}x\,\mathrm{d}y \\ &= -\int_\Omega \varphi_n (u_x^2 + u_y^2 + u\Delta u)\,\mathrm{d}x\,\mathrm{d}y\,,\\ \int_\Omega \varphi_n (2x u_x u_{xx} + 2x u_y u_{xy})\,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega \varphi_n x (u_x^2 + u_y^2)_x \,\mathrm{d}x\,\mathrm{d}y \\ &= -\int_\Omega \varphi_n' x (u_x^2 + u_y^2)\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega \varphi_n (u_x^2 + u_y^2)\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

Inserting to (2.5), we get $h(v_n, u) = I_n + J_n$ with

$$I_n := \int_\Omega \varphi_n'(x) x (u_x^2 - u_y^2)\,\mathrm{d}x\,\mathrm{d}y\,, \qquad J_n := \int_\Omega \varphi_n(x)(u_x^2 - u_y^2 - u\Delta u)\,\mathrm{d}x\,\mathrm{d}y\,.$$

By similar calculations,

$$(v_n, u) = \int_\Omega \varphi_n (2xuu_x + u^2)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \varphi_n (x(u^2)_x + u^2)\,\mathrm{d}x\,\mathrm{d}y = -\int_\Omega \varphi_n'(x) x u^2\,\mathrm{d}x\,\mathrm{d}y\,.$$

Looking at the definition (2.3), it is clear that, $|\varphi_n| \le 1$, $\lim_{n\to\infty} \varphi_n(x) = 1$ for every $x \neq 0$ and $|x\varphi_n'(x)| \le 2$ for almost every $x \in \mathbb{R}$. Furthermore, $\varphi_n'(x) \neq 0$ only for

$$x \in (-2n, -n) \cup (-n^{-1}, -n^{-2}) \cup (n^{-2}, n^{-1}) \cup (n, 2n)\,.$$

Consequently,

$$\lim_{n\to\infty} I_n = 0\,, \qquad \lim_{n\to\infty} J_n = \int_\Omega (u_x^2 - u_y^2 - u\Delta u)\,\mathrm{d}x\,\mathrm{d}y = 2\|u_x\|^2\,, \qquad \lim_{n\to\infty} (v_n, u) = 0\,,$$

by the dominated convergence. As

$$0 = h(v_n, u) - \lambda(v_n, u) = I_n + J_n - \lambda(v_n, u) \xrightarrow[n\to\infty]{} 2\,\|u_x\|^2\,,$$

it follows that $u_x = 0$, so u is necessarily x-independent. Now $u = 0$ because $u \in L^2(\Omega)$ and there is no non-zero eigenfunction and no eigenvalue satisfying (2.1). So the relation $\sigma_{\mathrm{p}}(H) = \varnothing$ from Theorem 1 is proved.

III. ABSENCE OF THE SINGULAR CONTINUOUS SPECTRUM

Given any $E \in \mathbb{R}$ and $\delta > 0$, P_δ will denote the spectral projection of H onto the interval $(E-\delta, E+\delta)$. We restrict to $E \notin \mathcal{T}$, where the set $\mathcal{T}$ is introduced in (1.4), and choose δ so small that $(E-\delta, E+\delta) \cap \mathcal{T} = \varnothing$. Let H be as above and let A be a self-adjoint operator to be specified in a moment (it will be a regularisation of (1.3)). To apply the abstract theorem of [10] and thus conclude the absence of the singular continuous spectrum of H, it is enough to verify the following properties:

(a) The intersection $\mathsf{D}(A) \cap \mathsf{D}(H)$ is a core of H.

(b) The unitary group e^{itA} leaves the domain of H invariant and

$$\forall \psi \in \mathsf{D}(H)\,, \qquad \sup_{|t|<1} \|H e^{itA}\psi\| < \infty. \tag{3.1}$$

(c) The form

$$\dot{b}[\psi] := i(H\psi, A\psi) - i(A\psi, H\psi)\,, \qquad \mathsf{D}(\dot{b}) := \mathsf{D}(A) \cap \mathsf{D}(H)\,,$$

is bounded from below and closable. Moreover, the self-adjoint operator B associated with the closure b of $\dot{b}$ satisfies

$$\mathsf{D}(B) \supset \mathsf{D}(H)\,.$$

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(d) The operator defined by the form

$$\dot{c}[\psi] := i(B\psi, A\psi) - i(A\psi, B\psi)\,, \qquad \mathsf{D}(\dot{c}) := \mathsf{D}(A) \cap \mathsf{D}(H)\,,$$

extends to an operator

$$C \in \mathscr{B}(\mathsf{D}(H), \mathsf{D}(H)^*)\,,$$

$\mathsf{D}(H)$ being equipped with the graph norm and $\mathsf{D}(H)^*$ being its dual space.

(e) There exists a positive number α and a compact operator K on $L^2(\Omega)$ such that

$$P_\delta B P_\delta \geq \alpha P_\delta + P_\delta K P_\delta\,.$$

Note that B (respectively, C) can be interpreted as a realisation of the commutator $i[H, A]$ (respectively, the double commutator $i[i[H, A], A]$).

A. The Hamiltonian

We begin with establishing some new results about the operator H which will be needed later.

Proposition 1. *For every positive ϵ, the set*

$$\mathcal{C} := \big\{\varphi \in \mathsf{D}(H) \,|\, \exists \phi \in C_0^\infty(\mathbb{R}^2)\,, \quad \varphi \restriction ((-\infty, -\epsilon) \cup (\epsilon, +\infty)) \times (0, d) \,=\, \phi \restriction ((-\infty, -\epsilon) \cup (\epsilon, +\infty)) \times (0, d)\big\}$$

is a core of H.

Proof. Let ψ be an arbitrary function from $\mathsf{D}(H)$. We show that it can be approximated by functions from $\mathcal{C}$. Let ϑ_1, ϑ_2 be functions from $C^\infty(\mathbb{R})$ such that $0 \leq \vartheta_1, \vartheta_2 \leq 1$ and

$$\vartheta_1(x) = 1 \quad \text{for} \quad x < -\epsilon\,, \qquad\qquad \vartheta_1(x) = 0 \quad \text{for} \quad x > -\frac{\epsilon}{2}\,,$$
$$\vartheta_2(x) = 1 \quad \text{for} \quad x > \epsilon\,, \qquad\qquad \vartheta_2(x) = 0 \quad \text{for} \quad x < \frac{\epsilon}{2}\,.$$

Let us define

$$\psi_1 = \vartheta_1 \psi\,, \qquad \psi_2 = \vartheta_2 \psi\,, \qquad \psi_3 = (1 - \vartheta_1 - \vartheta_2)\psi\,,$$

so that

$$\psi = \psi_1 + \psi_2 + \psi_3 \qquad \text{and} \qquad \psi_1, \psi_2, \psi_3 \in \mathsf{D}(H)\,.$$

It is sufficient to approximate ψ_1, ψ_2 by functions from $\mathcal{C}$. It is known that $\psi_1, \psi_2 \in H^2(\Omega)$, see [6]. Let us extend them to $H^2(\mathbb{R} \times (-d, 2d))$ first. To keep the boundary conditions, let us choose extensions symmetric with respect to the Neumann parts of the boundary and antisymmetric with respect to the Dirichlet parts. Notice, that in half-planes where the functions are zero it means the same. So we define

$$\psi_1(x, y) = -\psi_1(x, -y) \quad \text{for} \quad -d < y < 0\,, \qquad \psi_1(x, y) = \psi_1(x, 2d - y) \quad \text{for} \quad d < y < 2d\,,$$
$$\psi_2(x, y) = \psi_2(x, -y) \quad \text{for} \quad -d < y < 0\,, \qquad \psi_2(x, y) = -\psi_2(x, 2d - y) \quad \text{for} \quad d < y < 2d\,.$$

The extended functions are in $H^2(\mathbb{R} \times (-d, 0))$ and $H^2(\mathbb{R} \times (d, 2d))$. As the traces of functions and the normal derivatives on the boundaries of Ω from both sides coincide, the extended functions are in $H^2(\mathbb{R} \times (-d, 2d))$. In fact, we used a special case of [19, Thm 4.26] and its proof. Then extend them to $H^2(\mathbb{R}^2)$ which is possible over the straight boundary.

Further, we need to approximate ψ_1 and ψ_2 by C^∞ functions. We use the standard mollifications, see, e.g. [19, Lem. 3.15],

$$J_\eta \psi_k(x) = \int_{\mathbb{R}^2} j_\eta(x - y)\psi_k(y)\,\mathrm{d}y \qquad (k = 1, 2),$$

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where

$$j_\eta(x) = \eta^{-2} j(x/\eta)\,, \qquad j \in C_0^\infty(B(1))\,, \qquad j \geq 0\,, \qquad \int_{\mathbb{R}^2} j(x) d^2x = 1\,.$$

Let us consider only $0 < \eta < \min(d, \epsilon/2)$ for $\operatorname{supp} j_\eta \subset B(\eta)$. Then $J_\eta \psi_{1,2} \in H^2(\mathbb{R} \times (-d, 2d))$ and approach $\psi_{1,2}$ there as $\eta \to 0^+$. These function are in $\mathsf{D}(H)$ if they satisfy the corresponding boundary conditions at $\partial\Omega$ which are easily verified for the usual symmetric choice of $j_\eta(x, y) = j_\eta(x, -y)$.

Let us show it here for the case of Neumann boundary condition on $(0, +\infty) \times \{0\}$. The trace exists as $J_\eta \psi_2 \in H^2(\mathbb{R} \times (-d, 2d))$ and we can simply calculate

$$\partial_2 J_\eta \psi_2(x, 0) = \int_{\mathbb{R}^2} \partial_2 j_\eta(x - x', -y') \psi_2(x', y')\, \mathrm{d}x'\, \mathrm{d}y' = \int_{\mathbb{R}^2} \partial_2 j_\eta(x - x', -y') \psi_2(x', -y')\, \mathrm{d}x'\, \mathrm{d}y'$$
$$= \int_{\mathbb{R}^2} \partial_2 j_\eta(x - x', y') \psi_2(x', y')\, \mathrm{d}x'\, \mathrm{d}y' = -\int_{\mathbb{R}^2} \partial_2 j_\eta(x - x', -y') \psi_2(x', y')\, \mathrm{d}x'\, \mathrm{d}y' = -\partial_2 J_\eta \psi_2(x, 0)$$

and the required boundary condition $\partial_2 J_\eta \psi_2(x, 0)$ at $x > 0$ follows. The other boundary conditions are verified similarly.

Finally, let $\Phi_R \in C_0^\infty(\mathbb{R}^2)$, $\Phi_R(x, y) = \Phi_{1R}(x)\Phi_2(y)$, where Φ_{1R} is a suitable function with the support in $(-R - 1, R + 1)$ and the value 1 in $(-R, R)$ while Φ_2 is a function with the support in $(-d/2, 3d/2)$ and the value 1 in $(-d/4, 5d/4)$. Then $\phi = \Phi_R(J_\eta \psi_1 + \psi_3 + J_\eta \psi_2) \in \mathcal{C}$ is an arbitrarily good approximation of ψ in $\mathsf{D}(H)$ with the graph norm choosing η sufficiently small and R large enough. So $\mathcal{C}$ is a core of H. □

Proposition 2. *There exists a positive constant C such that, for every $\psi \in \mathsf{D}(H)$,*

$$\|\partial_x \psi\| \leq C \|H\psi\|\,, \qquad \|\partial_y \psi\| \leq C \|H\psi\|\,. \tag{3.2}$$

Moreover, for every positive ϵ, there exists a positive constant C_ϵ such that, for every $\psi \in \mathsf{D}(H)$,

$$\|\chi_\varepsilon \partial_x^2 \psi\| \leq C_\epsilon \|H\psi\|\,, \qquad \|\chi_\varepsilon \partial_x \partial_y \psi\| \leq C_\epsilon \|H\psi\|\,. \tag{3.3}$$

where χ_ϵ denotes the characteristic function of the set $\Omega \setminus [(-\epsilon, \epsilon) \times (0, d)]$.

Proof. Given any $g \in L^2(\Omega)$, let $\psi \in \mathsf{D}(H)$ be the unique solution of the resolvent equation $H\psi = g$ (the problem is well defined because $0 \notin \sigma(H)$). The weak formulation reads

$$\forall v \in \mathsf{D}(h)\,, \qquad (\partial_x v, \partial_x \psi) + (\partial_y v, \partial_y \psi) = (v, g)\,. \tag{3.4}$$

Choosing $v := \psi$ in (3.4), we get

$$E_1 \|\psi\|^2 \leq \|\partial_x \psi\|^2 + \|\partial_y \psi\|^2 = (\psi, g) \leq \|\psi\| \|g\|\,.$$

Consequently, $\|\psi\| \leq E_1^{-1} \|g\|$, $\|\partial_x \psi\|^2 \leq E_1^{-1} \|g\|^2$ and $\|\partial_y \psi\|^2 \leq E_1^{-1} \|g\|^2$. This proves (3.2).

To establish (3.3), we follow the ideas of standard elliptic regularity (see, e.g., [20, Sec. 6.3]). Let $\xi \in C_0^\infty(\mathbb{R})$ be such that $0 \leq \xi \leq 1$, $\xi(x) = 0$ if $|x| \leq \epsilon/2$ and $\xi(x) = 1$ if $|x| \geq \epsilon$. Now we choose $v := -\partial_x^{-h}(\xi^2 \partial_x^h \psi)$ in (3.4), where

$$\partial_x^h \varphi(x, y) := \frac{\varphi(x + h, y) - \varphi(x, y)}{h}\,,$$

is the difference quotient of $\varphi \in L^2(\Omega)$ in the direction x. With an abuse of notation (followed also at other places in the paper), we denote here by the same symbol ξ the function on $\mathbb{R}$ as well as $\xi \otimes 1$ on Ω. Choosing $|h| \leq \epsilon/2$, we have $v \in \mathsf{D}(h)$ (it is only important to ensure the Dirichlet boundary conditions). Using the integration-by-parts formula for the difference quotients, (3.4) yields

$$|\, \|\xi \partial_x^h \partial_x \psi\|^2 + 2\, (\xi' \partial_x^h \psi, \xi \partial_x^h \partial_x \psi) + \|\xi \partial_x^h \partial_y \psi\|^2 | = |(v, g)| \leq \|v\| \|g\|\,. \tag{3.5}$$

To deal with the right-hand side, we write

$$\|v\|^2 = \|\partial_x^{-h}(\xi^2 \partial_x^h \psi)\|^2 \leq \|\partial_x(\xi^2 \partial_x^h \psi)\|^2 \leq 2\|\xi^2 \partial_x^h \partial_x \psi\|^2 + 2k_\epsilon^2 \|\partial_x^h \psi\|^2 \leq 2\|\xi \partial_x^h \partial_x \psi\|^2 + 2k_\epsilon^2 \|\partial_x \psi\|^2\,,$$

where $\|(\xi^2)'\|_\infty \leq 2\|\xi'\|_\infty =: k_\epsilon$. On the left-hand side, we use

$$2\, |(\xi' \partial_x^h \psi, \xi \partial_x^h \partial_x \psi)| \leq 2\|\xi' \partial_x^h \psi\| \|\xi \partial_x^h \partial_x \psi\| \leq k_\epsilon \|\partial_x \psi\| \|\xi \partial_x^h \partial_x \psi\|\,.$$

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Consequently, (3.5) yields

$$\begin{aligned}(1-\delta_1-2\delta_2)\|\xi\partial_x^h\partial_x\psi\|^2+\|\xi\partial_x^h\partial_y\psi\|^2 &\le k_\epsilon^2\left(\frac{1}{\delta_1}+2\delta_2\right)\|\partial_x\psi\|^2+\frac{1}{\delta_2}\|g\|^2\\ &\le\left[k_\epsilon^2\left(\frac{1}{\delta_1}+2\delta_2\right)E_1^{-1}+\frac{1}{\delta_2}\right]\|g\|^2\end{aligned}$$

with any positive numbers δ_1 and δ_2, where the second inequality employs (3.2) with the explicitly given constant. Choosing δ_1 and δ_2 sufficiently small, the left-hand side is a sum of two non-negative terms and the desired claims follows by further estimating $\|\xi\partial_x^h\partial_x\psi\|^2\ge\|\chi_\epsilon\partial_x^h\partial_x\psi\|$ (and similarly for the other norm) and by sending h to 0. □

B. The conjugate operator

Let $f_1^\pm\in C^\infty(\mathbb{R})$ be such that $0\le f_1^\pm\le 1$, $f_1^\pm(x)=0$ if $\pm x\le 1$ and $f_1^\pm(x)=1$ if $\pm x\ge 2$. For every $n\ge 1$, we define $f_n^\pm(x):=f_1^\pm(x/n)$ and $F_n^\pm(x):=\int_0^x f_n^\pm(\xi)\,\mathrm{d}\xi$. Finally, we set $f_n:=f_n^-+f_n^+$ and $F_n:=F_n^-+F_n^+$. Notice that $F_n^\pm(x)\sim x$ as $x\to\pm\infty$ and that $\|(f_n^\pm)^{(m)}\|_\infty=n^{-m}\|(f_1^\pm)^{(m)}\|_\infty$.

With these preliminaries, we define

$$\dot{A}_\parallel:=-\frac{i}{2}\left(F_n(x)\,\partial_x+\partial_x\,F_n(x)\right),\qquad \mathsf{D}(\dot{A}_\parallel):=C_0^\infty(\mathbb{R})\,, \tag{3.6}$$

where F_n is understood as an operator of multiplication. The following considerations are full analogy of [14, Props. 6.1–2]. However, as there is a difference in the cut-off at zero instead of the cut-off at infinity, we give the proofs here.

The operator $\dot{A}_\parallel$ is essentially self-adjoint in $L^2(\mathbb{R})$. This is a consequence of [11, Prop. 7.6.3, part (a)] and its proof. In our special case, it can be also seen directly that the deficiency indices of $\dot{A}_\parallel$ are zero due the properties of function F_n.

Let $A_\parallel$ denote the (self-adjoint) closure of $\dot{A}_\parallel$. Using the Hilbert-space identification $L^2(\Omega)\cong L^2(\mathbb{R})\times L^2((0,d))$, we set

$$A:=A_\parallel\otimes 1\,, \tag{3.7}$$

which is a self-adjoint operator in $L^2(\Omega)$.

For any fixed $x\in\mathbb{R}$, consider the initial-value problem

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}u(t,x)=F_n(u(t,x))\,,\\ \quad u(0,x)=x\,.\end{cases} \tag{3.8}$$

By classical results (see, e.g., [21, Thm. 4.1 of Chapt. V]), (3.8) admits a unique global solution in $C^\infty(\mathbb{R}^2)$. One has

$$\partial_x u(t,x)=e^{\int_0^t f_n(u(s,x))\,\mathrm{d}s}>0 \tag{3.9}$$

for every $t\in\mathbb{R}$ and $x\in\mathbb{R}$. Define

$$(W(t)\varphi)(x,y):=|\partial_x u(t,x)|^{1/2}\,\varphi(u(t,x),y)\,. \tag{3.10}$$

Proposition 3. *W is a strongly continuous unitary group on $L^2(\Omega)$ with the generator* (3.7).

Proof. It is clear from (3.8) that $u(t,0)=0$ for $t\in\mathbb{R}$, and $u(t,x)\gtrless 0$ for $x\gtrless 0$. Using the properties of f_n, the relation (3.9) is now improved to

$$\partial_x u(t,x)\ge e^{-|t|}$$

for every $t,x\in\mathbb{R}$ and

$$\lim_{x\to\pm\infty}u(t,x)=\pm\infty.$$

The unitarity of $W(t)$ then follows from its construction (3.10).

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The equations (3.8) together with the unicity of their solution implies the relation

$$u(t, u(s, x)) = u(t + s, x)\,,$$

from which the group property

$$W(t)W(s) = W(t + s)$$

follows.

It is sufficient to verify the strong continuity of $W(t)$ at $t = 0$. The continuity of $W(t)\varphi$ is easily seen for $\varphi \in C_0^\infty(\Omega)$ and then extends to $\varphi \in L^2(\Omega)$ by the density argument as $\|W(t)\| = 1$.

Direct calculations show

$$\frac{d}{dt}W(t)\varphi_{|t=0} = i(\dot{A}_\| \otimes 1)\varphi$$

for $\varphi \in C_0^\infty(\Omega)$. As the generator of the group W is self-adjoint, it equals A necessarily. □

The following proposition establishes property (b).

Proposition 4. $\mathsf{D}(H)$ *is stable under the action of* e^{itA} *and* (3.1) *holds.*

Proof. Let $\varphi \in \mathsf{D}(H)$. We need to check that then $e^{itA}\varphi = W(t)\varphi \in \mathsf{D}(H)$, for every $t \in \mathbb{R}$. We have seen in the previous proof that the map $\mathbb{R} \ni x \mapsto u(t, x) \in \mathbb{R}$ leaves $\mathbb{R}^+$ and $\mathbb{R}^-$ invariant. So $e^{itA}\varphi$ satisfies the required boundary conditions at $\partial_D\Omega$ and $\partial_N\Omega$.

Equation (3.9) implies that the derivatives $\partial_x u$, $\partial_x^2 u$, $\partial_x^3 u$ are bounded in x for a fixed t. Then $e^{itA}\varphi \in H^1(\Omega)$. Let us calculate

$$\begin{aligned}\Delta e^{itA}\varphi = W(t)\Delta\varphi + (\partial_x u)^{\frac{1}{2}}((\partial_x u)^2 - 1)\partial_1^2\varphi(u, y) + 2(\partial_x u)^{\frac{1}{2}}(\partial_x^2 u)\partial_1\varphi(u, y)\\ +(\partial_x u)^{\frac{1}{2}}\left(\frac{1}{2}(\partial_x u)^{-1}\partial_x^3 u - \frac{1}{4}(\partial_x u)^{-2}(\partial_x^2 u)^2\right)\varphi(u, y)\,.\end{aligned}$$

Every terms on the right-hand side are clearly square integrable, possibly except of the second one. However, $\partial_x u(t, x) = 1$ for $|x| < e^{-|t|}n$ according to (3.9) and the properties of f_n. So the second term is also square integrable as $\partial_1^2\varphi \in L^2(\Omega \setminus ((u(t, -e^{-|t|}n), u(t, e^{-|t|}n)) \times (0, d)))$, see [6]. Now the relation $e^{itA}\varphi \in D(H)$ is proved. Further, the continuity of the used bounds with respect to t implies (3.1). □

The following proposition establishes property (a).

Proposition 5. $\mathsf{D}(A) \cap \mathsf{D}(H)$ *is dense in* $\mathsf{D}(H)$ *for the graph norm associated with* H.

Proof. The claim follows from Proposition 1 and the fact that $\mathcal{C} \subset \mathsf{D}(A)$. □

C. The first commutator

Let $\psi \in \mathsf{D}(A) \cap \mathsf{D}(H)$. Using the formula (3.7) with (3.6) and integrating by parts, we compute

$$\begin{aligned}\dot{b}[\psi] &= 2\Re(-\partial_x^2\psi - \partial_y^2\psi, F_n\partial_x\psi + \tfrac{1}{2}F_n'\psi)\\ &= -\int_\Omega F_n\partial_x|\partial_x\psi|^2 - \Re\int_\Omega F_n'\overline{\partial_x^2\psi}\psi - 2\Re\int_\Omega F_n\overline{\partial_y^2\psi}\partial_x\psi - \Re\int_\Omega F_n'\overline{\partial_y^2\psi}\psi\\ &= \int_\Omega F_n'|\partial_x\psi|^2 + \int_\Omega F_n'|\partial_x\psi|^2 + \frac{1}{2}\int_\Omega F_n''\partial_x|\psi|^2 + \int_\Omega F_n\partial_x|\partial_y\psi|^2 + \int_\Omega F_n'|\partial_y\psi|^2\\ &= 2\int_\Omega F_n'|\partial_x\psi|^2 - \frac{1}{2}\int_\Omega F_n'''|\psi|^2\\ &= 2\int_\Omega f_n|\partial_x\psi|^2 - \frac{1}{2}\int_\Omega f_n''|\psi|^2\,,\end{aligned}$$

keeping in mind the properties of F_n and $\psi \in \mathsf{D}(A) \cap \mathsf{D}(H)$. For brevity, here we have stopped to write the measures of integration in the integrals.

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Since f_n is non-negative, we immediately see that $\dot{b}$ is bounded from below. Explicitly,

$$\dot{b} \geq -\frac{\|f_n''\|_\infty}{2} = -\frac{\|f_1''\|_\infty}{2n^2}\,,$$

so the lower bound actually tends to 0 as $n \to \infty$.

Since $\dot{b}[\psi] = (\psi, \dot{B}\psi)$, where

$$\dot{B} := -2\partial_x f_n(x)\partial_x - \frac{1}{2}f_n''(x)\,, \qquad \mathsf{D}(\dot{B}) := \mathsf{D}(A) \cap \mathsf{D}(H)\,,$$

is an obviously symmetric below bounded operator in $L^2(\Omega)$, it follows that $\dot{b}$ is closable (see, e.g., [22, Thm. VI.1.2.7]). The closure b satisfies

$$b[\psi] = 2\int_\Omega f_n|\partial_x\psi|^2 - \frac{1}{2}\int_\Omega f_n''|\psi|^2\,, \qquad \mathsf{D}(b) = \left\{\psi \in L^2(\Omega) \,\middle|\, \sqrt{f_n}\,\partial_x\psi \in L^2(\Omega)\right\}\,.$$

By the representation theorem, we have

$$B = -2\partial_x f_n(x)\partial_x - \frac{1}{2}f_n''(x)\,, \qquad \mathsf{D}(B) = \left\{\psi \in \mathsf{D}(b) \,\middle|\, \partial_x(f_n\partial_x\psi) \in L^2(\Omega)\right\}\,.$$

It is evident that $\mathsf{D}(H) \subset \mathsf{D}(B)$.

Summing up, in this subsection we have established property (c).

D. The second commutator

Here we follow the same lines as in the previous section. Let $\psi \in \mathsf{D}(A) \cap \mathsf{D}(H)$ and compute

$$\dot{c}[\psi] = 2\Re(-2\partial_x f_n(x)\partial_x\psi - \frac{1}{2}f_n''(x)\psi, F_n\partial_x\psi + \tfrac{1}{2}F_n'\psi)\,.$$

First consider

$$\begin{aligned}-4\Re\int_\Omega(\partial_x f_n(x)\partial_x\psi)F_n\overline{\partial_x\psi} &= -4\int_\Omega f_n'(x)F_n(x)|\partial_x\psi|^2 - 2\int_\Omega f_n(x)F_n(x)\partial_x|\partial_x\psi|^2\\ &= 2\int_\Omega f_n^2(x)|\partial_x\psi|^2 - 2\int_\Omega f_n'(x)F_n(x)|\partial_x\psi|^2.\end{aligned}$$

Then

$$\begin{aligned}-2\Re\int_\Omega(\partial_x f_n(x)\partial_x\psi)F_n'\overline{\psi} &= -\int_\Omega f_n'(x)F_n'(x)\partial_x|\psi|^2 - 2\Re\int_\Omega f_n(x)F_n'(x)(\partial_x^2\psi)\overline{\psi}\\ &= \int_\Omega f_n'(x)f_n(x)\partial_x|\psi|^2 + 2\int_\Omega f_n^2(x)|\partial_x\psi|^2\\ &= -\int_\Omega (f_n'(x)f_n(x))'|\psi|^2 + 2\int_\Omega f_n^2(x)|\partial_x\psi|^2.\end{aligned}$$

We also have

$$-\Re\int_\Omega f_n''(x)F_n(x)\psi\partial_x\overline{\psi} = -\frac{1}{2}\int_\Omega f_n''(x)F_n(x)\partial_x|\psi|^2 = \frac{1}{2}\int_\Omega (f_n''(x)F_n(x))'|\psi|^2$$

Finally we get

$$\begin{aligned}\dot{c}[\psi] = 4\int_\Omega f_n^2(x)|\partial_x\psi|^2 &- 2\int_\Omega f_n'(x)F_n(x)|\partial_x\psi|^2\\ &- \int_\Omega \big(f_n''(x)f_n(x) + f_n'(x)^2 - \frac{1}{2}f_n'''(x)F_n(x)\big)|\psi|^2.\end{aligned} \tag{3.11}$$

By Proposition 2, $\dot{c}$ is continuous in the graph norm associated with H and so extends continuously to the form c defined again by the equation (3.11) on $\mathsf{D}(H)$. Then it defines a bounded map $C \in \mathscr{B}(\mathsf{D}(H), \mathsf{D}(H)^*)$ and the statement (d) is proved.

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E. The Mourre estimate

Finally, we are concerned with the essential condition (e). We rewrite the restriction of B as follows

$$\begin{aligned} B \restriction \mathsf{D}(H) &= Hf_n + f_nH + 2f_n\partial_y^2 + \frac{1}{2}f_n'' \\ &= 2E + \underbrace{(H-E)f_n + f_n(H-E)}_{B_1} + 2\underbrace{f_n\partial_y^2}_{B_2} + \underbrace{2E(f_n-1) + \frac{1}{2}f_n''}_{B_3} . \end{aligned} \tag{3.12}$$

Now we look at the individual terms and try to eventually estimate $P_\delta B P_\delta$ from below by a positive multiple of P_δ plus a compact operator sandwiched between the projections P_δ's.

1. *Operator B_1*

For every $\varphi \in L^2(\Omega)$, we have

$$\begin{aligned} |(\varphi, P_\delta B_1 P_\delta \varphi)| &\leq \|P_\delta\varphi\|^2 \left(\|P_\delta(H-E)f_nP_\delta\| + \|P_\delta f_n(H-E)P_\delta\|\right) \\ &\leq 2\|P_\delta\varphi\|^2\|P_\delta(H-E)\| \\ &\leq 2\delta\|P_\delta\varphi\|^2 . \end{aligned}$$

Here we have used the spectral theorem at the last estimate. Hence, this term can be made negligible by choosing δ small and we shall estimate it as

$$P_\delta B_1 P_\delta \geq -2\delta P_\delta .$$

2. *Operator B_2*

We demonstrate our approach on $T^+ := P_\delta f_n^+ \partial_y^2 P_\delta$; the operator $T^- := P_\delta f_n^- \partial_y^2 P_\delta$ can be handled in a similar way. At the same time, let us suppose that $E_l < E < E_{l+1}$.

Let H^+ be the self-adjoint realisation of the Laplacian in $L^2(\Omega)$, subject to the Dirichlet boundary conditions on $\mathbb{R}\times\{d\}$ and the Neumann boundary condition on $\mathbb{R}\times\{0\}$. Let $\{\psi_k\}_{k\in\mathbb{N}^*}$ be the eigenfunctions of the one-dimensional Laplacian in $L^2((0,d))$, subject to the Neumann boundary condition at 0 and the Dirichlet boundary condition at d. We define

$$\Pi_k^+\varphi(x,y) := \psi_k(y)\big(\psi_k, \varphi(x,\cdot)\big)_{L^2((0,d))} ,$$

the projection on the kth transverse mode of H^+. We have

$$T^+ = P_\delta(-\sum_{k=1}^{l} E_k f_n^+ \Pi_k^+ + R^+)P_\delta \tag{3.13}$$

with

$$R^+ := \sum_{k\geq l+1} -E_k P_\delta f_n^+ \Pi_k^+ P_\delta . \tag{3.14}$$

Note that the operator R^+ is not compact. Denote by $h_k^+ = -\partial_x^2 \otimes 1 + E_k$ the restriction of H^+ on $\Pi_k^+ L^2(\Omega)$. Let $Z := E + i\eta$ with $\eta > 0$. We have for any $m \in \mathbb{N}^*$,

$$(h_k^+ - Z)^m f_n^+ \Pi_k^+ = \Pi_k^+ (H^+ - Z)^m f_n^+ = \Pi_k^+ (H - Z)^m f_n^+$$

on the domain of the right-hand side. Now let us choose $\eta := \delta$. If $k \geq l+1$, then

$$\|(h_k^+ - Z)^{-m}\Pi_k^+\| \leq (E_k - E)^{-m} .$$

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At the same time, if $k \geq l+1$, we have

$$\begin{aligned} P_\delta \Pi_k^+ f_n^+ P_\delta &= P_\delta (h_k^+ - Z)^{-m} \Pi_k^+ (H-Z)^m f_n^+ P_\delta \\ &= P_\delta (h_k^+ - Z)^{-m} \Pi_k^+ f_n^+ (H-Z)^m P_\delta + P_\delta (h_k^+ - Z)^{-m} \Pi_k^+ [(H-Z)^m, f_n^+] P_\delta \,. \end{aligned} \tag{3.15}$$

The first term on the right-hand side of the second line of (3.15) can be estimated as

$$\|P_\delta (h_k^+ - Z)^{-m} \Pi_k^+ f_n^+ (H-Z)^m P_\delta\| \leq C(E_k - E)^{-m} \delta^m. \tag{3.16}$$

Hereafter C denotes a generic strictly positive constant which does not depend on the index k and on δ (but depends of fixed $E_{l+1} - E$) and can change its value from line to line. If $m \geq 2$, we have

$$\begin{aligned} \Big\| \sum_{k \geq l+1} E_k P_\delta (h_k^+ - Z)^{-m} \Pi_k^+ f_n^+ (H-Z)^m P_\delta \Big\| &\leq C \sum_{k \geq l+1} E_k (E_k - E)^{-m} \delta^m \\ &\leq C \delta^m \,. \end{aligned} \tag{3.17}$$

Now we turn to estimating the second term on the right-hand side of the second line of (3.15). We choose $m := 2$. We could improve the bound to be obtained by choosing larger m, but with more complicated calculations. On the range of P_δ, we have

$$[(H-Z)^2, f_n^+] = 2[(H-Z), f_n^+](H-Z) + [(H-Z), [(H-Z), f_n^+]]$$

with

$$\begin{aligned} [(H-Z), f_n^+] &= -\big(\partial_x (f_n^+)' + (f_n^+)' \partial_x\big) \,, \\ [(H-Z), [(H-Z), f_n^+]] &= \partial_x^2 (f_n^+)'' + (f_n^+)'' \partial_x^2 + 2\partial_x (f_n^+)'' \partial_x \,. \end{aligned}$$

Noticing that the support of the derivative of f_n^+ is compact and not intersecting $\{x = 0\}$, we use Proposition 2 to obtain

$$\big\| [(H-Z)^2, f_n^+] P_\delta \big\| \leq C(n^{-1}\delta + n^{-2}) \,. \tag{3.18}$$

Consequently,

$$\Big\| \sum_{k \geq l+1} E_k P_\delta (h_k^+ - Z)^{-2} \Pi_k^+ [(H-Z)^2, f_n^+] P_\delta \Big\| \leq C(n^{-1}\delta + n^{-2}) \,. \tag{3.19}$$

Summing up, we have proved that, for δ small and n large,

$$\|R^+\| \leq C(n^{-1}\delta + \delta^2 + n^{-2}) \,. \tag{3.20}$$

When analyzing T^-, we consider H^- which is defined in the same manner as H^+ but with interchanged boundary conditions. The corresponding projections Π_k^- and the operator R^- are defined with an obvious modification of the formulae above. By using the same arguments as above, we get the same estimate (3.20) for R^-. Writing $R := R^+ + R^-$,

$$T^+ + T^- = P_\delta \left(-\sum_{k=1}^{l} E_k (\Pi_k^+ f_n^+ + \Pi_k^- f_n^-) + R \right) P_\delta.$$

But since $[\Pi_k^\pm, f_n^\pm] = 0$, then $\sum_{k=1}^l E_k \Pi_k^\pm f_n^\pm \leq E_l (\sum_{k=1}^l \Pi_k^\pm) f_n^\pm \leq E_l f_n^\pm$ and we conclude with the estimate

$$P_\delta B_2 P_\delta \geq -P_\delta (E_l + C\big(n^{-1}\delta + \delta^2 + n^{-2}\big)) P_\delta \tag{3.21}$$

valid in the form sense.

3. *Operator B_3*

The operator $P_\delta B_3 P_\delta$ is not small. However, since the function $g_n := 2E(f_n - 1) + \frac{1}{2} f_n''$ has a compact support, it follows that $g_n H^{-1}$ is a compact operator. This is seen form the fact that $\mathsf{R}(g_n H^{-1}) \subset H^1((-2n, 2n) \times (0, d))$ which is compactly embedded in $L^2((-2n, 2n) \times (0, d))$ by the Rellich-Kondrachov theorem (see, e.g., [19, Thm. 6.2]). Now

$$K_n := P_\delta B_3 P_\delta = P_\delta B_3 H^{-1} H P_\delta$$

is also a compact operator. Note that the presence of B_3 (its part $f_n - 1$) in (3.12) is the only obstruction to get a *strict* Mourre estimate (i.e. with $K = 0$).

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4. Conclusion

If $E_l < E < E_{l+1}$, it follows from the preceding subsections that, for δ small and n large, the Mourre estimate

$$P_\delta i[H,A]P_\delta \geq P_\delta\big(2(E-E_l-\delta) - C(n^{-1}\delta + \delta^2 + n^{-2}) + K_n\big)P_\delta \tag{3.22}$$

holds true, where K_n is a compact operator.

We have verified all the properties (a)–(e) required for the application of the abstract theorem of [10]. Since $\mathcal{T}$ is a discrete set, this concludes the proof that the singular continuous spectrum of H is empty.

In fact, our result gives more information. In particular, the limiting absorption principle holds for every energy $E \in \mathbb{R} \setminus \mathcal{T}$, see [10] or [11].

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