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Abstract

In this note, we prove some non-asymptotic concentration inequalities for functionals, called innovations, of inhomogeneous Neymann-Scott point processes, a particular class of spatial point process models. Innovation is a functional built from the counting measure minus its integral compensator. The result is then applied to obtain almost sure rate of convergence for such functionals.

Keywords: Spatial point processes; Almost sure convergence rate; Deviation inequalities; Campbell's Theorem.

1. Introduction

Spatial point patterns are datasets containing random locations of some event of interest. These datasets arise in many scientific fields such as biology, epidemiology, seismology, hydrology. Spatial point processes are the stochastic models generating such data. We refer to Stoyan et al. (1995) or Møller and Waagepetersen (2004) for an overview on spatial point processes. These references cover practical as well as theoretical aspects. The reference model is the spatial Poisson point process which models random locations of points without any interaction between points. Several alternative models exist in the literature. In this paper we are interested in the class

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of Neymann-Scott point process (NSPP for short), a class of models widely used in the literature to produce attractive patterns. Roughly speaking, a NSPP can be seen as a two-level model. Generate first, a set of cluster centers from a (homogeneous or inhomogeneous) Poisson point process. Such points are also sometimes called *mothers*. Then, generate several (homogeneous or inhomogeneous) Poisson point processes, each being concentrated similarly around its *mother*. The resulting pattern formed by these offspring points is a NSPP. Due to this simple interpretation, this model has gained a lot of attention.

Let $\mathbf{X}$ be a spatial point process defined on $\mathbb{R}^d$ which we view as a locally finite random set. We thus assume that $\mathbf{X}$ is simple which prevents two points to occur at the same location (see e.g. Daley and Vere-Jones (2008)). We assume that $\mathbf{X}$ has an intensity measure $\rho_{\mathbf{X}}$. By Campbell's Theorem (Møller and Waagepetersen, 2004), $\rho_{\mathbf{X}}$ is characterized by the fact that for any function $h : \mathbb{R}^d \to \mathbb{R}$ such that $h\rho_{\mathbf{X}} \in L^1(\mathbb{R}^d)$:

$$\mathrm{E} \sum_{u \in \mathbf{X}} h(u) = \int_{\mathbb{R}^d} h(u) \rho_{\mathbf{X}}(u) \mathrm{d}u. \tag{1}$$

To prove asymptotic properties for example for estimators of a parametric form of the intensity function, one is often led to study functionals derived from Campbell's formula: these functionals are called 'innovations' in the literature (Baddeley et al., 2005; Coeurjolly, 2015). They are defined on a given domain W, with Lebesgue measure $|W|$, by

$$I_W(\mathbf{X}; \varphi) = \sum_{u \in \mathbf{X} \cap W} \varphi(u) - \int_W \varphi(u) \rho_{\mathbf{X}}(u) \mathrm{d}u, \tag{2}$$

for some locally integrable function $\varphi : \mathbb{R}^d \to \mathbb{R}$. Such functionals play an important role in statistics as well as in model validation. For instance, if $\mathbf{X}$ is stationary and $\varphi = 1$, $I_W(\mathbf{X}, 1) = N(W) - \rho|W|$, so $|W|^{-1} I_W(\mathbf{X}, 1) = \hat{\rho} - \rho$ measures the departure from the non-parametric estimator of ρ to ρ. For inhomogeneous point processes, such functionals are also used to define the methodology which estimates a parametric form of the intensity function and to prove asymptotic results, see e.g.Schoenberg (2005); Waagepetersen (2007); Guan and Loh (2007).

In an increasing domain asymptotic framework, i.e. if $\mathbf{X}$ is observed in an increasing sequence of bounded domains $(W_n)_{n\geq 1}$, properties are well-understood for very large classes of point process models (which embrace NSPP): for instance $|W_n|^{-1} I_{W_n}(\mathbf{X};\varphi)$ converges almost surely to 0, $|W_n|^{-1/2} I_{W_n}(\mathbf{X};\varphi)$ tends to a normal distribution, etc. Concentration inequalities for such functionals have already been considered by Pemantle and Peres (2014) or Coeurjolly (2015) for determinantal point processes and Gibbs point processes respectively. In this paper, we exploit general inequalities for inhomogeneous Poisson point processes obtained by Reynaud-Bouret (2003) (see also Ané and Ledoux (2000); Wu (2000); Breton et al. (2007); Bachmann and Peccati (2016) for more general functionals) to derive non-asymptotic concentration inequalities for inhomogeneous NSPP.

Section 2 presents some additional notation and the general definition of inhomogeneous NSPP. We establish our concentration inequality in Section 3. Finally, Section 4 proposes an application of this inequality. We obtain the almost sure rate of convergence for innovation-type functionals in an increasing domain framework. In particular, we show that for the Matérn cluster process and the Thomas process which are the two most well-used examples of (inhomogeneous) NSPP (see Section 2), $(|W_n| \log |W_n|)^{-1/2} I_{W_n}(\mathbf{X};\varphi)$ converges almost surely to 0 as $n \to \infty$.

2. Spatial point processes and NSPP

Let $\mathbf{X}$ be a spatial point process defined on $\mathbb{R}^d$ and let o be the origin in $\mathbb{R}^d$. In this note, we focus on inhomogeneous Neymann-Scott point processes which belong to the class of Cox processes, a class which is defined as follows (see e.g. Møller and Waagepetersen (2004)): let $\mathbf{\Lambda} = \{\Lambda(u)\}_{u\in\mathbb{R}^d}$ be a nonnegative locally integrable random field defined on $\mathbb{R}^d$, if $\mathbf{X}$ conditionally on $\mathbf{\Lambda}$ is an inhomogeneous Poisson point process on $\mathbb{R}^d$, then $\mathbf{X}$ is said to be a Cox process with latent random intensity $\mathbf{\Lambda}$. An inhomogeneous NSPP is a Cox process driven by a random field $\mathbf{\Lambda}$ of the form

$$\mathbf{\Lambda}(u) = \ell(u) \sum_{c \in \mathbf{C}} k(u,c) \tag{3}$$

where ℓ is a locally nonnegative integrable function, where $\mathbf{C}$, the process generating the *mothers*, is an inhomogeneous Poisson point process defined on $\mathbb{R}^d$ with intensity function $\rho_{\mathbf{C}}$ (assumed to be locally integrable) and where $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+$ is a kernel density function, in the sense that for any $u \in \mathbb{R}^d$, $k(u, \cdot)$ is a density on $\mathbb{R}^d$.

Using Campbell's Theorem applied to $\mathbf{C}$, we obtain the intensity for $\mathbf{X}$

$$\rho_{\mathbf{X}}(u) = \mathrm{E}\,\mathbf{\Lambda}(u) = \ell(u) \int_{\mathbb{R}^d} k(u,c)\rho_{\mathbf{C}}(c)\mathrm{d}c. \tag{4}$$

Let us discuss two situations. On the one hand, usually, see e.g. Waagepetersen (2007), $\mathbf{C}$ is assumed to be homogeneous with intensity parameter $\rho_C > 0$. In this case $\rho_{\mathbf{X}}(u) = \rho_{\mathbf{C}}\,\ell(u)$. One usually models $\rho_{\mathbf{X}}$ and let $\ell(u) = \rho_{\mathbf{X}}(u)/\rho_{\mathbf{C}}$. It is also commonly assumed that the kernel k is invariant by translation, i.e. $k(u,c) = k(u-c,o) = k(u-c)$ (with a slight abuse of notation) which makes $\mathbf{X}$ second-order reweighted stationary, i.e. the pair correlation function of $\mathbf{X}$ is invariant by translation. Two well-known examples of such processes are the inhomogeneous Matérn cluster point process and the inhomogeneous Thomas process respectively defined with k the uniform kernel $k(u,c) = (\omega_d R^d)^{-1}\mathbf{1}(\|c-u\| \leq R)$ for some finite $R > 0$ (where ω_d is the volume of the d-dimensional unit ball) and the d-dimensional Gaussian kernel with variance $\sigma^2\mathbf{I}_d$, for $\sigma > 0$. On the other hand and more recently, Mrkvička and Soubeyrand (2017) model the cluster centers by an inhomogeneous Poisson point process and take $\ell = 1$. In this situation, if the kernel k is invariant by translation, the intensity of $\mathbf{X}$ writes $\rho_{\mathbf{X}}(u) = \rho_{\mathbf{C}} * k(u)$ where $*$ stands for the convolution product. Our main result will be general enough to embrace both types of inhomogeneity described above.

We now focus on innovations type functionals as defined by (2). As mentioned in the introduction, such functionals are centered by Campbell's Theorem. It is therefore worth understanding how they concentrate around zero. This is investigated in the next section.

3. Main result

In what follows, we let W be a bounded set of $\mathbb{R}^d$, φ be a bounded function. We denote by $\|\varphi\|_\infty = \sup_{u\in\mathbb{R}^d} |\varphi(u)|$. For a nonnegative function $\rho \in L^1(W)$, we let

$$\|\varphi\|_\rho = \int_W |\varphi(u)|\rho(u)\mathrm{d}u.$$

The main result of this note is now presented.

Theorem 1. *Let* $\mathbf{X}$ *be an inhomogeneous NSPP with random intensity function* $\mathbf{\Lambda}$ *given by* (3)*. Then, for any bounded measurable function* $\varphi : \mathbb{R}^d \to \mathbb{R}$ *and any* $t > 0$*, we have*

$$\mathrm{P}\left(I_W(\mathbf{X};\varphi) \geq a_1\sqrt{2t} + a_2(2t)^{3/4} + a_3\frac{t}{3}\right) \leq 3\,\mathrm{e}^{-t} \tag{5}$$

where the positive constants a_1, a_2 *and* a_3 *are given by*

$$a_1 = \sqrt{\|\psi(\varphi)^2\|_{\rho_{\mathbf{C}}}} + \sqrt{\|\varphi^2\|_{\rho_{\mathbf{X}}}}, \tag{6}$$

$$a_2 = \left(\|\psi(\varphi^2)^2\|_{\rho_{\mathbf{C}}}\right)^{1/4} \tag{7}$$

$$a_3 = \|\varphi\|_\infty + \|\psi(\varphi)\|_\infty + \sqrt{6\|\psi(\varphi^2)\|_\infty}. \tag{8}$$

where $\psi(\cdot)$ *is an operator acting on measurable functions on* W *defined for any function* h *by*

$$\psi(h)(\cdot) = \int_W h(u)\ell(u)k(u,\cdot)\mathrm{d}u.$$

In the same vein, we have for any $t > 0$

$$\mathrm{P}\left(|I_W(\mathbf{X};\varphi)| \geq a_1\sqrt{2t} + a_2(2t)^{3/4} + a_3\frac{t}{3}\right) \leq 6\,\mathrm{e}^{-t}. \tag{9}$$

Proof. Since $\mathbf{X} \mid \mathbf{\Lambda}$ is an inhomogeneous Poisson point process on W with intensity $\mathbf{\Lambda}$ almost surely bounded, we can apply Reynaud-Bouret (2003, Proposition 7) and obtain for any $t > 0$

$$\mathrm{P}\left(\sum_{u\in\mathbf{X}\cap W} \varphi(u) - \int_W \varphi(u)\Lambda(u)\mathrm{d}u \geq \sqrt{2t\|\varphi^2\|_\Lambda} + \frac{t}{3}\|\varphi\|_\infty \middle| \mathbf{\Lambda}\right) \leq \mathrm{e}^{-t}$$

which by denoting $\Delta(\mathbf{\Lambda};h) = \int_W h(u)\Lambda(u)\mathrm{d}u - \int_W h(u)\rho_{\mathbf{X}}(u)\mathrm{d}u$, for some bounded measurable function h, rewrites

$$\mathrm{P}\left(I_W(\mathbf{X};\varphi) \geq \Delta(\mathbf{\Lambda};\varphi) + \sqrt{2t\left(\|\varphi^2\|_{\rho_{\mathbf{X}}} + \Delta(\mathbf{\Lambda};\varphi^2)\right)} + \frac{t}{3}\|\varphi\|_\infty \middle| \mathbf{\Lambda}\right) \leq \mathrm{e}^{-t}. \tag{10}$$

Now, observe that $\Delta(\mathbf{\Lambda};h)$ can be rewritten in terms of the Poisson point process $\mathbf{C}$

$$\Delta(\mathbf{\Lambda};h) = \sum_{c\in\mathbf{C}} \psi(h)(c) - \int_{\mathbb{R}^d} \psi(h)(c)\rho_{\mathbf{C}}(c)\mathrm{d}c. \tag{11}$$

Equation (11) suggests us to apply once more Reynaud-Bouret (2003, Proposition 7) to the point process $\mathbf{C}$. Doing this, we obtain for any bounded measurable function h and $t > 0$

$$\mathrm{P}\left(\Delta(\mathbf{\Lambda};h) \geq \sqrt{2t\|\psi(h)^2\|_{\rho_{\mathbf{C}}}} + \frac{t}{3}\|\psi(h)\|_\infty\right) \leq \mathrm{e}^{-t}. \tag{12}$$

Let $L_W(t;h) = \sqrt{2t\|\psi(h)^2\|_{\rho_{\mathbf{C}}}} + t\|\psi(h)\|_\infty/3$ and let us consider

$$F_1 = \{\Delta(\mathbf{\Lambda};\varphi) \leq L_W(t;\varphi)\} \text{ and } F_2 = \{\Delta(\mathbf{\Lambda};\varphi^2) \leq L_W(t;\varphi^2)\}.$$

Using for an event F, the notation $\bar{F}$ for the complementary event, we see that both events satisfy $\mathrm{P}(\bar{F_i}) \leq e^{-t}$. By combining (10) and (12) with $h = \varphi, \varphi^2$, we get

$$\begin{aligned}
&\mathrm{P}\left(I_W(\mathbf{X};\varphi) \geq L_W(t;\varphi) + \sqrt{2t(\|\varphi^2\|_{\rho_{\mathbf{X}}} + L_W(t;\varphi^2))} + \frac{t}{3}\|\varphi\|_\infty\right)\\
&= \mathrm{E}\left(\mathrm{P}\left(I_W(\mathbf{X};\varphi) \geq L_W(t;\varphi) + \sqrt{2t(\|\varphi^2\|_{\rho_{\mathbf{X}}} + L_W(t;\varphi^2))} + \frac{t}{3}\|\varphi\|_\infty \middle| \mathbf{\Lambda}\right)\right)\\
&= \mathrm{E}\left(\mathrm{P}\left(I_W(\mathbf{X};\varphi) \geq L_W(t;\varphi) + \sqrt{2t(\|\varphi^2\|_{\rho_{\mathbf{X}}} + L_W(t;\varphi^2))} + \frac{t}{3}\|\varphi\|_\infty, F_1\cap F_2 \middle| \mathbf{\Lambda}\right)\right)\\
&\quad + \mathrm{E}\left(\mathrm{P}\left(I_W(\mathbf{X};\varphi) \geq L_W(t;\varphi) + \sqrt{2t(\|\varphi^2\|_{\rho_{\mathbf{X}}} + L_W(t;\varphi^2))} + \frac{t}{3}\|\varphi\|_\infty, \overline{F_1\cap F_2} \middle| \mathbf{\Lambda}\right)\right)\\
&\leq \mathrm{E}\left(\mathrm{P}\left(I_W(\mathbf{X};\varphi) \geq \Delta(\mathbf{\Lambda};\varphi) + \sqrt{2t\left(\|\varphi^2\|_{\rho_{\mathbf{X}}} + \Delta(\mathbf{\Lambda};\varphi^2)\right)} + \frac{t}{3}\|\varphi\|_\infty \middle| \mathbf{\Lambda}\right)\right) + \mathrm{P}\left(\overline{F_1\cap F_2}\right)\\
&\leq 3\,\mathrm{e}^{-t}.
\end{aligned}$$

Equation (5) is obtained by noting that

$$L_W(t;\varphi) + \sqrt{2t(\|\varphi^2\|_{\rho_{\mathbf{X}}} + L_W(t;\varphi^2))} + \frac{t}{3}\|\varphi\|_\infty \leq a_1\sqrt{2t} + a_2\sqrt{2t\sqrt{2t}} + a_3\frac{t}{3}$$

where a_1, a_2 and a_3 are given by (6)-(8). Equation (9) is easily deduced since (5) can be applied to $-\varphi$. $\square$

Remark 2. We can simplify the terms appearing in the concentration inequality as follows. To do so, we note that

$$\|\psi(\varphi)^2\|_{\rho_{\mathbf{C}}} \leq \|\psi(|\varphi|)^2\|_{\rho_{\mathbf{C}}} \quad \text{and} \quad \|\psi(\varphi^2)^2\|_{\rho_{\mathbf{C}}} \leq \|\varphi\|_\infty^2 \|\psi(|\varphi|)^2\|_{\rho_{\mathbf{C}}}.$$

Similarly

$$\|\psi(\varphi)\|_\infty \leq \|\psi(|\varphi|)\|_\infty \quad \text{and} \quad \|\psi(\varphi^2)\|_\infty \leq \|\varphi\|_\infty \|\psi(|\varphi|)\|_\infty.$$

Using this and the fact that $\sqrt{ab} \leq (a+b)/2$ for $a, b \geq 0$, we have the following upper-bounds

$$\begin{aligned} a_1 &\leq \sqrt{\|\psi(|\varphi|)^2\|_{\rho_{\mathbf{C}}}} + \sqrt{\|\varphi^2\|_{\rho_{\mathbf{X}}}} \\ a_2\sqrt{2t\sqrt{2t}} &\leq \sqrt{t\|\varphi\|_\infty 2\sqrt{2t\|\psi(|\varphi|)^2\|_{\rho_{\mathbf{C}}}}} \leq \frac{t}{2}\|\varphi\|_\infty + \sqrt{2t}\sqrt{\|\psi(|\varphi|)^2\|_{\rho_{\mathbf{C}}}}, \\ a_3 &\leq \|\varphi\|_\infty + \|\psi(|\varphi|)\|_\infty + \sqrt{\|\varphi\|_\infty \times 6\|\psi(|\varphi|)\|_\infty} \leq \frac{3}{2}\|\varphi\|_\infty + 4\|\psi(|\varphi|)\|_\infty. \end{aligned}$$

This yields that $a_1\sqrt{2t} + a_2\sqrt{2t\sqrt{2t}} + a_3 t/3 \leq b_1\sqrt{2t} + b_2 t/3$ with

$$b_1 = 2\sqrt{\|\psi(|\varphi|)^2\|_{\rho_{\mathbf{C}}}} + \sqrt{\|\varphi^2\|_{\rho_{\mathbf{X}}}} \quad \text{and} \quad b_2 = 3\|\varphi\|_\infty + 4\|\psi(|\varphi|)\|_\infty$$

and we finally obtain the following simpler concentration inequalities

$$\mathrm{P}\left(I_W(\mathbf{X};\varphi) \geq b_1\sqrt{2t} + b_2\frac{t}{3}\right) \leq 3\,\mathrm{e}^{-t}, \quad \mathrm{P}\left(|I_W(\mathbf{X};\varphi)| \geq b_1\sqrt{2t} + b_2\frac{t}{3}\right) \leq 6\,\mathrm{e}^{-t}. \tag{13}$$

4. Almost sure behaviour of $I_W(\mathbf{X};\varphi)$

As an application of the previous inequality, we now assume some increasing domain asymptotic and establish an almost sure behaviour for innovation-type functionals.

Proposition 3. *Assume we observe* $\mathbf{X}$ *in* W_n *where* $(W_n)_{n\geq 1}$ *is an increasing sequence of convex bounded domains of* $\mathbb{R}^d$*, such that* $W_n \to \mathbb{R}^d$ *and such that there exists* $\beta > 0$ *such that* $\sum_{n\geq 1} |W_n|^{-\beta} < \infty$*. Let* $\mathbf{X}$ *be an inhomogeneous NSPP with kernel* k *that we assume symmetric and invariant by translation, i.e.* $k(u, c) = k(u - c, o) = k(u - c)$ *where* $u, c \in \mathbb{R}^d$ *and* o *is the origin in* $\mathbb{R}^d$*. Finally, assume that* $\max(\|\varphi\|_\infty, \|\ell\|_\infty, \|\rho_{\mathbf{C}}\|_\infty) < \infty$ *, then almost surely*

$$I_{W_n}(\mathbf{X}; \varphi) = \mathcal{O}_{a.s.}\left(\sqrt{|W_n| \log |W_n|}\right). \tag{14}$$

Remark 4. We say that $Y_n = \mathcal{O}_{a.s.}(a_n)$ for a sequence of random variables $(Y_n)_{n\geq 1}$ and a sequence of real number $(a_n)_{n\geq 1}$ if $Y_n = \mathcal{O}(a_n)$ with probability 1 (see e.g.(Serfling, 2009, p.92)). It means that there exists a set Ω_0 such that $\mathrm{P}(\Omega_0) = 1$ such that for each $\omega \in \Omega_0$, there exists a constant $B(\omega)$ such that $|Y_n(\omega)| \leq B(\omega) a_n$ for all n sufficiently large.

Proof. We apply Theorem 1 or more simply (13) with $t = t_n = \beta \log |W_n|$ and $b_1 = b_{1,n}$, $b_2 = b_{2,n}$

$$\mathrm{P}\left(|I_{W_n}(\mathbf{X}; \varphi)| \geq b_{1,n}\sqrt{2t_n} + b_{2,n}\frac{t_n}{3}\right) \leq \frac{6}{|W_n|^\beta}.$$

Since $\sum_{n\geq 1} |W_n|^{-\beta} < \infty$, $\sum_{n\geq 1} \mathrm{P}\left(|I_{W_n}(\mathbf{X}; \varphi)| \geq b_{1,n}\sqrt{2t_n} + b_{2,n}\frac{t_n}{3}\right) < \infty$, whereby we deduce from Borel-Cantelli's Lemma that

$$I_{W_n}(\mathbf{X}; \varphi) = \mathcal{O}_{a.s.}\left(b_{1,n}\sqrt{t_n} + b_{2,n}\frac{t_n}{3}\right).$$

Now, by the assumptions of Proposition 3 and from (4), we deduce that $\|\rho_{\mathbf{X}}\|_\infty = \mathcal{O}(1)$, $\|\varphi^2\|_{\rho_{\mathbf{X}}} = \mathcal{O}(|W_n|)$. Let us now bound $\|\psi(|\varphi|)\|_\infty$ and $\|\psi(|\varphi|)^2\|_{\rho_{\mathbf{C}}}$. First, since k is a density

$$\|\psi(|\varphi|)\|_\infty \leq \|\varphi\|_\infty \|\ell\|_\infty = \mathcal{O}(1).$$

Second, by denoting $r_n = \int_{W_n}\int_{W_n} k * k(v - u)\mathrm{d}u\mathrm{d}v$

$$\begin{aligned}\|\psi(|\varphi|)^2\|_{\rho_{\mathbf{C}}} &= \int_{\mathbb{R}^d} \psi(|\varphi|)^2(c)\rho_{\mathbf{C}}(c)\mathrm{d}c \\ &= \int_{\mathbb{R}^d}\int_{W_n}\int_{W_n} \varphi(u)\ell(u)\varphi(v)\ell(v)k(u-c)k(v-c)\mathrm{d}u\mathrm{d}v\mathrm{d}c \\ &\leq \|\varphi\|_\infty^2 \|\ell\|_\infty^2 \int_{W_n}\int_{W_n}\int_{\mathbb{R}^d} k(c')k(u - v + c')\mathrm{d}c'\mathrm{d}u\mathrm{d}v = \mathcal{O}(r_n).\end{aligned}$$

This yields that

$$
\begin{aligned}
b_{1,n}\sqrt{t_n} + b_{2,n}\frac{t_n}{3} &= \mathcal{O}\left(\sqrt{r_n \log|W_n|}\right) + \mathcal{O}\left(\sqrt{|W_n|\log|W_n|}\right) + \mathcal{O}\left(\log|W_n|\right) \\
&= \mathcal{O}\left(\sqrt{|W_n|\log|W_n|}\right).
\end{aligned}
$$

The latter equality ensues from the fact that $r_n/|W_n| \to 1$. To see this, let us denote $A_{-z} = \{x - z \text{ for } x \in A\}$. We note that using a change of variables

$$
\frac{r_n}{|W_n|} = \int_{\mathbb{R}^d} \frac{|W_n \cap (W_n)_{-z}|}{|W_n|}\, k * k(z)\mathrm{d}z.
$$

Now, by the assumptions made on the sequence $(W_n)_{n\geq 1}$ and by David (2008, Lemma A.2) (see also Heinrich and Klein (2011)), $\lim_{n\to\infty} |W_n \cap (W_n)_{-z}|/|W_n| = 1$ for any $z \in \mathbb{R}^d$. Hence, $r_n/|W_n|$ tends to $\int_{\mathbb{R}^d} k * k(z)\mathrm{d}z = 1$ since k is a density. □

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