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# Characterisation of the pressure term in the incompressible Navier–Stokes equations on the whole space

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## Abstract

We characterise the pressure term in the incompressible 2D and 3D Navier–Stokes equations for solutions defined on the whole space.

**Keywords :** Navier–Stokes equations, pressure term, Leray projection, extended Galilean invariance, suitable solutions

**AMS classification :** 35Q30, 76D05.

## Introduction

In the context of the Cauchy initial value problem for Navier–Stokes equations on  $\mathbb{R}^d$  (with  $d = 2$  or  $d = 3$ )

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{u} = 0, & \mathbf{u}(0, \cdot) = \mathbf{u}_0 \end{cases}$$

an important problem is to propose a formula for the gradient of the pressure, which is an auxiliary unknown (usually interpreted as a Lagrange multiplier for the constraint of incompressibility).

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As we shall not assume differentiability of  $\mathbf{u}$  in our computations, it is better to write the equations as

$$(NS) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \nabla p + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}(0, \cdot) = \mathbf{u}_0 \end{cases}$$

Taking the Laplacian of equations (NS), since we have for a vector field  $\mathbf{w}$  the identity

$$-\Delta \mathbf{w} = \nabla \wedge (\nabla \wedge \mathbf{w}) - \nabla (\nabla \cdot \mathbf{w})$$

we get the equations

$$\partial_t \Delta u = \Delta^2 u + \nabla \wedge (\nabla \wedge (\nabla \cdot (\mathbf{u} \otimes \mathbf{u} - \mathbb{F})))$$

and

$$0 = -\Delta \nabla p - \nabla (\nabla \cdot (\nabla \cdot (\mathbf{u} \otimes \mathbf{u} - \mathbb{F}))) = -\Delta \nabla p - \nabla \left( \sum_{1 \leq i, j \leq d} \partial_i \partial_j (u_i u_j - F_{i,j}) \right).$$

Thus, the rotational-free unknown  $\nabla p$  obeys a Poisson equation. If  $G_d$  is the fundamental solution of the operator  $-\Delta$  :

$$G_2 = \frac{1}{2\pi} \ln\left(\frac{1}{|x|}\right), \quad G_3 = \frac{1}{4\pi|x|}$$

(which satisfies  $-\Delta G_d = \delta$ ), we *formally* have

$$\nabla p = G_d * \nabla \left( \sum_{1 \leq i, j \leq d} \partial_i \partial_j (u_i u_j - F_{i,j}) \right) + H \quad (1)$$

with  $\Delta H = 0$ . In the litterature, one usually finds the assumption that  $\nabla p$  vanishes at infinity and this is read as  $H = 0$ . Equivalently, this is read as

$$\partial_t \mathbf{u} = G_d * \nabla \wedge (\nabla \wedge \partial_t \mathbf{u});$$

the operator

$$\mathbb{P} = G_d * \nabla \wedge (\nabla \wedge \cdot)$$

is called the Leray projection operator and the decomposition (*when justified*)

$$\mathbf{w} = \mathbb{P} \mathbf{w} + G_d * \nabla (\nabla \cdot \mathbf{w})$$

the Hodge decomposition of the vector field  $\mathbf{w}$ .

Hence, an important issue when dealing with the Navier–Stokes equations is to study whether in formula (1) the first half of the right-hand term is well-defined, and if so which values the second half (the harmonic part  $H$ ) may have.

In order to give some meaning to the formal convolution  $G_d * \nabla \partial_i \partial_j (u_i u_j)$  or to  $(\nabla \partial_i \partial_j G_d) * (u_i u_j)$ , we should require  $u_i$  to be locally  $L_t^2 L_x^2$  (in order to define  $u_i u_j$  as a distribution) and to have small increase at infinity, since the distribution  $\nabla \partial_i \partial_j G$  has small decay at infinity (it belongs to  $L^1 \cap L^\infty$  far from the origin and is  $O(|x|^{-(d+1)})$ ). Thus, we will focus on solutions  $\mathbf{u}$  that belong to  $L^2((0, T), L^2(\mathbb{R}^d, w_{d+1} dx))$  where

$$w_\gamma(x) = (1 + |x|)^{-\gamma}.$$

We shall recall various examples (from recent or older litterature) of solutions belonging to the space  $L^2((0, T), L^2(\mathbb{R}^d, w_\gamma dx))$  (with  $\gamma \in \{d, d+1\}$ ). As  $F_{i,j}$  plays a role similar to  $u_i u_j$ , we shall assume that  $\mathbb{F} \in L^1((0, T), L^1(\mathbb{R}^d, w_\gamma dx))$ .

## 1 Main results.

First, we precise the meaning of  $\nabla p$  in equations (NS) :

**Lemma 1.1** *Consider the dimension  $d \in \{2, 3\}$  and  $\gamma \geq 0$ . Let  $0 < T < +\infty$ . Let  $\mathbb{F}$  be a tensor  $\mathbb{F}(t, x) = (F_{i,j}(t, x))_{1 \leq i, j \leq d}$  such that  $\mathbb{F}$  belongs to  $L^1((0, T), L^1(\mathbb{R}^d, w_\gamma dx))$ , and let  $\mathbf{u}$  be a vector field  $\mathbf{u}(t, x) = (u_i(t, x))_{1 \leq i \leq d}$  such that  $\mathbf{u}$  belongs to  $L^2((0, T), L^2(\mathbb{R}^d, w_\gamma dx))$  and  $\nabla \cdot \mathbf{u} = 0$ . Define the distribution  $S$  by*

$$\mathbf{S} = \Delta \mathbf{u} - \nabla \cdot (\mathbf{u} \otimes \mathbf{u} - \mathbb{F}) - \partial_t \mathbf{u}.$$

*Then the following assertions are equivalent :*

(A)  $S$  is curl-free :  $\nabla \wedge \mathbf{S} = 0$ .

(B) There exists a distribution  $p \in \mathcal{D}'((0, T) \times \mathbb{R}^d)$  such that  $S = \nabla p$ .

**Theorem 1** *Consider the dimension  $d \in \{2, 3\}$ . Let  $0 < T < +\infty$ . Let  $\mathbb{F}$  be a tensor  $\mathbb{F}(t, x) = (F_{i,j}(t, x))_{1 \leq i, j \leq d}$  such that  $\mathbb{F}$  belongs to  $L^1((0, T), L^1(\mathbb{R}^d, w_{d+1} dx))$ . Let  $\mathbf{u}$  be a solution of the following problem*

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \mathbf{S} + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{u} = 0, \quad \nabla \wedge \mathbf{S} = 0, \end{cases} \quad (2)$$

*such that :  $\mathbf{u}$  belongs to  $L^2((0, T), L_{w_{d+1}}^2(\mathbb{R}^d))$ , and  $\mathbf{S}$  belongs to  $\mathcal{D}'((0, T) \times \mathbb{R}^d)$ .*

Let us choose  $\varphi \in \mathcal{D}(\mathbb{R}^d)$  such that  $\varphi(x) = 1$  on a neighborhood of 0 and define

$$A_{i,j,\varphi} = (1 - \varphi)\partial_i\partial_j G_d.$$

Then, there exists  $g(t) \in L^1((0, T))$  such that

$$\mathbf{S} = \nabla p_\varphi + \partial_t g$$

with

$$\begin{aligned} p_\varphi &= \sum_{i,j} (\varphi\partial_i\partial_j G_d) * (u_i u_j - F_{i,j}) \\ &\quad + \sum_{i,j} \int (A_{i,j,\varphi}(x-y) - A_{i,j,\varphi}(-y))(u_i(t,y)u_j(t,y) - F_{i,j}(t,y)) dy. \end{aligned}$$

Moreover,

- $\nabla p_\varphi$  does not depend on the choice of  $\varphi$  : if we change  $\varphi$  in  $\psi$ , then

$$p_\varphi(t, x) - p_\psi(t, x) = \sum_{i,j} \int (A_{i,j,\psi}(-y) - A_{i,j,\varphi}(-y))(u_i(t,y)u_j(t,y) - F_{i,j}(t,y)) dy.$$

- $\nabla p_\varphi$  is the unique solution of the Poisson problem

$$\Delta \mathbf{w} = -\nabla(\nabla \cdot (\nabla \cdot (\mathbf{u} \otimes \mathbf{u} - \mathbb{F})))$$

with

$$\lim_{\tau \rightarrow +\infty} e^{\tau \Delta} \mathbf{w} = 0 \text{ in } \mathcal{D}'.$$

- if  $\mathbb{F}$  belongs more precisely to  $L^1((0, T), L^1_{w_d}(\mathbb{R}^d))$  and  $\mathbf{u}$  belongs to  $L^2((0, T), L^2_{w_d}(\mathbb{R}^d))$ , then  $g = 0$  and  $\nabla p_\varphi = \nabla p_0$  where

$$p_0 = \sum_{i,j} (\varphi\partial_i\partial_j G_d) * (u_i u_j - F_{i,j}) + \sum_{i,j} ((1 - \varphi)\partial_i\partial_j G_d) * (u_i u_j - F_{i,j}).$$

( $p_0$  does not actually depend on  $\varphi$  and could have been defined as  $p_0 = \sum_{i,j} (\partial_i\partial_j G_d) * (u_i u_j - F_{i,j})$ .)

When  $\mathbb{F} = 0$ , the case  $g \neq 0$  can easily be reduced to a change of referential, due to the extended Galilean invariance of the Navier–Stokes equations :

**Theorem 2** Consider the dimension  $d \in \{2, 3\}$ . Let  $0 < T < +\infty$ . Let  $\mathbf{u}$  be a solution of the following problem

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \mathbf{S} \\ \nabla \cdot \mathbf{u} = 0, \quad \nabla \wedge \mathbf{S} = 0, \quad \mathbf{u}(0, x) = \mathbf{u}_0(x) \end{cases} \quad (3)$$

such that :  $\mathbf{u}$  belongs to  $L^2((0, T), L^2_{w_{d+1}}(\mathbb{R}^d))$ , and  $\mathbf{S}$  belongs to  $\mathcal{D}'((0, T) \times \mathbb{R}^d)$ .

Let us choose  $\varphi \in \mathcal{D}(\mathbb{R}^d)$  such that  $\varphi(x) = 1$  on a neighborhood of 0 and define

$$A_{i,j,\varphi} = (1 - \varphi) \partial_i \partial_j G_d.$$

We decompose  $\mathbf{S}$  into

$$\mathbf{S} = \nabla p_\varphi + \partial_i g$$

with

$$\begin{aligned} p_\varphi &= \sum_{i,j} (\varphi \partial_i \partial_j G_d) * (u_i u_j) \\ &+ \sum_{i,j} \int (A_{i,j,\varphi}(x - y) - A_{i,j,\varphi}(-y)) (u_i(t, y) u_j(t, y)) dy \end{aligned}$$

and

$$g(t) \in L^1((0, T)).$$

Let us define

$$E(t) = \int_0^t g(\lambda) d\lambda$$

and

$$\mathbf{w}(t, x) = \mathbf{u}(t, x - E(t)) + g(t).$$

Then,  $\mathbf{w}$  is a solution of the Navier–Stokes problem

$$\begin{cases} \partial_t \mathbf{w} = \Delta \mathbf{w} - \nabla \cdot (\mathbf{w} \otimes \mathbf{w}) - \nabla q_\varphi \\ \nabla \cdot \mathbf{w} = 0, \quad \mathbf{w}(0, x) = \mathbf{u}_0(x) \\ q_\varphi = \sum_{i,j} (\varphi \partial_i \partial_j G_d) * (w_i w_j) + \sum_{i,j} \int (A_{i,j,\varphi}(x - y) - A_{i,j,\varphi}(-y)) (w_i(t, y) w_j(t, y)) dy \end{cases} \quad (4)$$

## 2 Curl-free vector fields.

In this section we prove Lemma 1.1 with simple arguments :

**Proof.** We take a partition of unity on  $(0, T)$

$$\sum_{j \in \mathbb{Z}} \omega_j = 1$$

with  $\omega_j$  supported in  $(2^{j-2}T, 2^jT)$  for  $j < 0$ , in  $(T/4, 3T/4)$  for  $j = 0$  and in  $(T - 2^{-j}T, T - 2^{-(j+2)}T)$  for  $j > 1$ . We define

$$V_j = -\omega_j \mathbf{u} + \int_0^t \omega_j \Delta \mathbf{u} - \omega_j \nabla \cdot (\mathbf{u} \otimes \mathbf{u} - \mathbb{F}) + (\partial_t \omega_j) \mathbf{u} \, ds.$$

Then  $V_j$  is a sum of the form  $A + \Delta B + \nabla \cdot C + D$  with  $A, B, C$  and  $D$  in  $L^1((0, T), L^1(\mathbb{R}^d, w_{d+1} dx))$ ; thus, by Fubini's theorem, we may see it as a time-dependent tempered distribution. Moreover,  $\partial_t V_j = \omega_j \mathbf{S}$ ,  $V_j$  is equal to 0 for  $t$  in a neighbourhood of 0, and  $\nabla \wedge V_j = 0$ . Moreover,  $\mathbf{S} = \sum_{j \in \mathbb{Z}} \partial_t V_j$ .

We choose  $\Phi \in \mathcal{S}(\mathbb{R}^d)$  such that the Fourier transform of  $\Phi$  is compactly supported and is equal to 1 in the neighbourhood of 0. Then  $\Phi * V_j$  is well-defined and  $\nabla \wedge (\Phi * V_j) = 0$ . We define

$$X_j = \Phi * V_j \text{ and } Y_j = V_j - X_j.$$

We have

$$Y_j = \nabla \left( \frac{1}{\Delta} \nabla \cdot Y_j \right)$$

and (due to Poincaré's lemma)

$$X_j = \nabla \left( \int_0^1 x \cdot X_j(t, \lambda x) d\lambda \right)$$

We find  $\mathbf{S} = \nabla p$  with

$$p = \partial_t \sum_{j \in \mathbb{Z}} \left( \int_0^1 x \cdot X_j(t, \lambda x) d\lambda + \frac{1}{\Delta} \nabla \cdot Y_j \right).$$

◇

### 3 The Poisson problem $-\Delta U = \partial_k \partial_i \partial_j h$

We first consider a simple Poisson problem :

**Proposition 3.1** *Let  $h \in L^1(\mathbb{R}^d, (1 + |x|)^{-(d+1)} dx)$  then*

$$U = U_1 + U_2 = (\partial_k(\varphi \partial_i \partial_j G_d)) * h + \partial_k((1 - \varphi) \partial_i \partial_j G_d) * h.$$

*is a distribution such that  $U_2$  belongs to  $L^1(\mathbb{R}^d, (1 + |x|)^{-(d+1)} dx)$  and  $U$  is a solution of the problem*

$$-\Delta U = \partial_k \partial_i \partial_j h. \quad (5)$$

*More precisely,  $U$  is the unique solution in  $\mathcal{S}'$  such that  $\lim_{\tau \rightarrow 0} e^{\tau \Delta} U = 0$  in  $\mathcal{S}'$ .*

**Proof.** We may write  $\partial_j G_d$  as

$$\partial_j G_d = - \int_0^{+\infty} \partial_j W_t dt$$

where  $W_t(x)$  is the heat kernel  $W_t(x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$ , so that on  $\mathbb{R}^d \setminus \{0\}$ , we have

$$\partial_j G_d = c_d \frac{x_j}{|x|^d} \text{ with } c_d = \frac{1}{2(4\pi)^{d/2}} \int_0^{+\infty} e^{-\frac{1}{4u}} \frac{du}{u^{\frac{d+2}{2}}}$$

The first part defining  $U$ ,  $U_1 = (\partial_k(\varphi \partial_i \partial_j G_d)) * h$ , is well defined, since  $\partial_k(\varphi \partial_i \partial_j G_d)$  is a compactly supported distribution. To control  $U_2$ , we write

$$\begin{aligned} & \int \int \frac{1}{(1 + |x|)^{d+1}} |\partial_k(1 - \varphi) \partial_i \partial_j G_d(x - y)| |h(y)| dy dx \\ & \leq \int \int \frac{1}{(1 + |x|)^{d+1}} \frac{C}{(1 + |x - y|)^{d+1}} |h(y)| dy dx \\ & \leq C' \int \frac{1}{(1 + |y|)^{d+1}} |h(y)| dy \end{aligned}$$

since

$$\begin{aligned} & \int \frac{1}{(1 + |x|)^{d+1}} \frac{1}{(1 + |x - y|)^{d+1}} dx \\ & \leq \int_{|x| > \frac{|y|}{2}} \frac{1}{(1 + |x|)^{d+1}} \frac{1}{(1 + |x - y|)^{d+1}} dx + \int_{|x - y| > \frac{|y|}{2}} \frac{1}{(1 + |x|)^{d+1}} \frac{1}{(1 + |x - y|)^{d+1}} dx \\ & \leq \frac{2^{d+1}}{(1 + |y|)^{d+1}} \int \frac{1}{(1 + |x - y|)^{d+1}} dx + \frac{2^{d+1}}{(1 + |y|)^{d+1}} \int \frac{1}{(1 + |x|)^{d+1}} dx \\ & \leq C \frac{1}{(1 + |y|)^{d+1}}. \end{aligned}$$



Now that we know that  $U$  is well defined, we may compute  $-\Delta U$ .  $-\Delta U_1$  is equal to

$$(-\Delta \partial_k(\varphi \partial_i \partial_j G_d)) * h = \partial_k(\varphi \partial_i \partial_j h) - \partial_k((\Delta \varphi) \partial_i \partial_j G_d) * h - 2 \sum_{1 \leq l \leq d} \partial_k((\partial_l \varphi) \partial_l \partial_i \partial_j G_d) * h.$$

For computing  $-\Delta U_2$ , we see that we can differentiate under the integration sign and find

$$-\Delta U_2 = \partial_k((1 - \varphi) \partial_i \partial_j h) + \partial_k((\Delta \varphi) \partial_i \partial_j G_d) * h + 2 \sum_{1 \leq l \leq d} \partial_k((\partial_l \varphi) \partial_l \partial_i \partial_j G_d) * h.$$

Thus,  $U$  is a solution of the Poisson problem.

Computing  $e^{\tau \Delta} U$ , we find that

$$e^{\tau \Delta} U = (e^{\tau \Delta} \partial_k \partial_i \partial_j G_d) * h$$

and thus

$$|e^{\tau \Delta} U(x)| \leq C \int \frac{1}{(\sqrt{\tau} + |x - y|)^{d+1}} |h(y)| dy.$$

By the dominated convergence theorem, we get that  $\lim_{\tau \rightarrow 0} e^{\tau \Delta} U = 0$  in  $L^1(\mathbb{R}^d, (1 + |x|)^{-(d+1)} dx)$ . If  $V$  is another solution of the same Poisson problem with  $V \in \mathcal{S}'$  and  $\lim_{\tau \rightarrow 0} e^{\tau \Delta} V = 0$  in  $\mathcal{S}'$ , then  $\Delta(U - V) = 0$  and  $U - V \in \mathcal{S}'$ , so that  $U - V$  is a polynomial; with the assumption that  $\lim_{\tau \rightarrow 0} e^{\tau \Delta}(U - V) = 0$ , we find that this polynomial is equal to 0.  $\diamond$

If we have better integrability on  $h$ , then of course we have better integrability of  $U_2$ . For instance, we have :

**Proposition 3.2** *Let  $h \in L^1(\mathbb{R}^d, (1 + |x|)^{-d} dx)$  then*

$$U_2 = \partial_k((1 - \varphi) \partial_i \partial_j G_d) * h.$$

*belongs to  $L^1(\mathbb{R}^d, (1 + |x|)^{-d})$ .*

**Proof.**

We write

$$\begin{aligned} & \int \int \frac{1}{(1 + |x|)^d} |\partial_k((1 - \varphi) \partial_i \partial_j G_d(x - y))| |h(y)| dy dx \\ & \leq C \int \int \frac{1}{(1 + |x|)^d} \frac{1}{(1 + |x - y|)^{d+1}} |h(y)| dy dx. \end{aligned}$$

For  $|y| < 1$ , we have

$$\int \frac{1}{(1+|x|)^{d+1}} \frac{1}{(1+|x-y|)^{d+1}} dx \leq \int \frac{1}{(1+|x|)^{d+1}} dx \leq C$$

and for  $|y| > 1$ , as the real number  $\int_{|x| < \frac{1}{2}} \frac{1}{|x|^{d-1}} \frac{1}{|x - \frac{y}{|y|}|^2} dx$  is finite and does not depend on  $y$ , we can write

$$\begin{aligned} & \int \frac{1}{(1+|x|)^d} \frac{1}{(1+|x-y|)^{d+1}} dx \\ & \leq \int_{|x| > \frac{|y|}{2}} \frac{1}{(1+|x|)^d} \frac{1}{(1+|x-y|)^{d+1}} dx + \int_{|x| < \frac{|y|}{2}} \frac{1}{(1+|x|)^d} \frac{1}{(1+|x-y|)^{d+1}} dx \\ & \leq \frac{2^d}{(1+|y|)^d} \int \frac{1}{(1+|x-y|)^{d+1}} dx + \frac{2^{d-1}}{(1+|y|)^{d-1}} \int_{|x| < \frac{|y|}{2}} \frac{1}{|x|^{d-1}} \frac{1}{|x-y|^2} dx \\ & \leq C \frac{1}{(1+|y|)^d} + C \frac{1}{(1+|y|)^{d-1}} \frac{1}{|y|} \int_{|x| < \frac{1}{2}} \frac{1}{|x|^{d-1}} \frac{1}{|x - \frac{y}{|y|}|^2} dx \\ & \leq C' \frac{1}{(1+|y|)^d}. \end{aligned}$$

This concludes the proof.  $\diamond$

## 4 The Poisson problem $-\Delta V = \partial_i \partial_j h$

**Proposition 4.1** *Let  $h \in L^1((1+|x|)^{-d-1}dx)$  and  $A_\varphi = (1-\varphi)\partial_i \partial_j G_d$  then*

$$V = V_1 + V_2 = (\varphi \partial_i \partial_j G_d) * h + \int (A_\varphi(x-y) - A_\varphi(-y)) h(y) dy$$

*is a distribution such that  $V_2$  belongs to  $L^1((1+|x|)^{-\gamma})$ , for  $\gamma > d+1$ , and  $V$  is a solution of the problem*

$$-\Delta V = \partial_i \partial_j h. \quad (6)$$

**Proof.** We know that  $V_1$  is well defined since  $\varphi \partial_i \partial_j G_d$  is a supported compactly distribution, and we will verify that  $V_2$  is well defined.

We have

$$\begin{aligned} & \int \int \frac{1}{(1+|x|)^\gamma} |A_\varphi(x-y) - A_\varphi(-y)| dx \\ & \leq C \|A_\varphi\|_{L^\infty} \int \frac{1}{(1+|x|)^\gamma} dx \end{aligned}$$

For  $|y| > 1$ , we have by the mean value inequality

$$\int_{|x| < \frac{|y|}{2}} \frac{1}{(1 + |x|)^\gamma} |A_\varphi(x - y) - A_\varphi(-y)| dx \leq C \frac{1}{|y|^{d+1}} \int_{|x| < \frac{|y|}{2}} \frac{|x|}{(1 + |x|)^\gamma} dx$$

and we can control the other part as follows

$$\int_{|x| > \frac{|y|}{2}} \frac{1}{(1 + |x|)^\gamma} |A_\varphi(-y)| dx \leq C \frac{1}{|y|^d} \int_{|x| > \frac{|y|}{2}} \frac{1}{(|x|)^\gamma} \leq C \frac{1}{|y|^\gamma}$$

and for  $\varepsilon > 0$  such that  $\gamma - \varepsilon \geq d + 1$ , we have

$$\begin{aligned} \int_{|x| > \frac{|y|}{2}} \frac{1}{(1 + |x|)^\gamma} |A_\varphi(x - y)| dx &\leq C \int_{|x| > \frac{|y|}{2}} \frac{1}{|x|^\gamma} \frac{1}{(1 + |x - y|)^d} dx \\ &\leq C \int_{|x| > \frac{|y|}{2}} \frac{1}{|x|^\gamma} \frac{1}{|x - y|^{d-\varepsilon}} dx \\ &\leq C \frac{1}{|y|^{d+1}} \int_{|x| > \frac{1}{2}} \frac{1}{|x|^\gamma} \frac{1}{|x - \frac{y}{|y|}|^{d-\varepsilon}} dx \\ &\leq C' \frac{1}{|y|^d}. \end{aligned}$$

◇

## 5 Proof of Theorems 1 and 2.

We may now prove Theorem 1 :

**Proof.** Taking the divergence of

$$\partial_t \mathbf{u} = \Delta \mathbf{u} - \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \mathbf{S} + \nabla \cdot \mathbb{F},$$

we obtain

$$-\sum_{i,j} \partial_i \partial_j (u_i u_j) + \sum_{i,j} \partial_i \partial_j F_{i,j} - \nabla \cdot \mathbf{S} = 0$$

and

$$-\Delta \mathbf{S} = \nabla \left( \sum_{i,j} \partial_i \partial_j (u_i u_j - F_{i,j}) \right).$$

We write  $h_{i,j} = u_i u_j - F_{i,j}$ , and  $A_\varphi = (1 - \varphi) \partial_i \partial_j G_d$ . By Proposition 4.1, we can define

$$p_\varphi = \sum_{i,j} (\varphi \partial_i \partial_j G_d) * h_{i,j} + \sum_{i,j} \int (A_{i,j,\varphi}(x - y) - A_{i,j,\varphi}(-y)) h_{i,j}(y) dy$$

and

$$U = U_1 + U_2 = \nabla \sum_{i,j} (\varphi \partial_i \partial_j G_d) * h_{i,j} + \nabla \sum_{i,j} ((1 - \varphi) \partial_i \partial_j G_d) * h_{i,j} = \nabla p_\varphi.$$

Let  $\tilde{U} = \mathbf{S} - U$ . First, we remark that  $\Delta U = \Delta \mathbf{S}$  so that  $\Delta \tilde{U} = 0$ , hence  $\tilde{U}$  is harmonic in the space variable.

On the other hand, for a test function  $\alpha \in \mathcal{D}(\mathbb{R})$  such that  $\alpha(t) = 0$  for all  $|t| \geq \varepsilon$ , and a test function  $\beta \in \mathcal{D}(\mathbb{R}^3)$ , and for  $t \in (\varepsilon, T - \varepsilon)$ , we have

$$\begin{aligned} \tilde{U}(t) *_{t,x} (\alpha \otimes \beta) &= (\mathbf{u} * (-\partial_t \alpha \otimes \beta + \alpha \otimes \Delta \beta) + (-\mathbf{u} \otimes \mathbf{u} + \mathbb{F}) \cdot * (\alpha \otimes \nabla \beta))(t, \cdot) \\ &\quad - \sum_{i,j} ((h_{ij}) * (\nabla (\varphi \partial_i \partial_j G_d) * (\alpha \otimes \beta)))(t, \cdot) - (U_2 * (\alpha \otimes \beta))(t, \cdot). \end{aligned}$$

By Proposition 3.1, we conclude that  $\tilde{U} * (\alpha \otimes \beta)(t, \cdot)$  belongs to the space  $L^1(\mathbb{R}^d, (1 + |x|)^{-d-1})$ . Thus, it is a tempered distribution; as it is harmonic, it must be polynomial. The integrability in  $L^1(\mathbb{R}^d, (1 + |x|)^{-d-1})$  implies that this polynomial is constant.

If  $\mathbb{F}$  belongs more precisely to  $L^1((0, T), L^1_{w_d}(\mathbb{R}^d))$  and  $\mathbf{u}$  belongs to  $L^2((0, T), L^2_{w_d}(\mathbb{R}^d))$ , we find that this polynomial belongs to  $L^1(\mathbb{R}^d, w_d dx)$ , hence is equal to 0.

Then, using the identity approximation  $\Phi_\varepsilon = \frac{1}{\varepsilon^4} \alpha(\frac{t}{\varepsilon}) \beta(\frac{x}{\varepsilon})$  and letting  $\varepsilon$  go to 0, we obtain a similar result for  $\tilde{U}$ . Thus  $\mathbf{S} = \nabla p_\varphi + f(t)$ , with  $f(t) = 0$  if  $\mathbb{F}$  belongs to  $L^1((0, T), L^1_{w_d}(\mathbb{R}^d))$  and  $\mathbf{u}$  belongs to  $L^2((0, T), L^2_{w_d}(\mathbb{R}^d))$ .

As  $f$  does not depend on  $x$ , we may take a function  $\beta \in \mathcal{D}(\mathbb{R}^d)$  with  $\int \beta dx = 1$  and write  $f = f * \beta$ ; we find that

$$f(t) = \partial_t (\mathbf{u}_0 * \beta - \mathbf{u} * \beta) + \int_0^t \mathbf{u} * \Delta \beta - (\mathbf{u} \otimes \mathbf{u} - \mathbb{F}) \cdot * \nabla \beta - p_\varphi * \nabla \beta ds = \partial_t g.$$

As  $\partial_t \partial_j g = \partial_j f = 0$  and  $\partial_j g(0, \cdot) = 0$ , we find that  $g$  depends only on  $t$ ; moreover, the formula giving  $g$  proves that  $g \in L^1((0, T))$ .  $\diamond$

The proof of Theorem 2 is classical and the result is known as the extended Galilean invariance of the Navier–Stokes equations :

**Proof.** Let us suppose that

$$\partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p_\varphi - \frac{d}{dt} g(t),$$

with  $g \in L^1((0, T))$ . We define

$$E(t) = \int_0^t g(\lambda) d\lambda \text{ and } \mathbf{w} = \mathbf{u}(t, x - E(t)) + g(t).$$

We have

$$\begin{aligned}
\partial_t \mathbf{w} &= \partial_t \mathbf{u}(t, x - E(t)) - g(t) \cdot \nabla \mathbf{u}(t, x - E(t)) + \frac{d}{dt} g(t) \\
&= \Delta \mathbf{u}(t, x - E(t)) - [(\mathbf{u} \cdot \nabla) \mathbf{u}](t, x - E(t)) - \nabla p_\varphi(t, x - E(t)) - \frac{d}{dt} g(t) \\
&\quad - g(t) \cdot \nabla \mathbf{u}(t, x - E(t)) + \frac{d}{dt} g(t) \\
&= \Delta \mathbf{w} - (\mathbf{w} \cdot \nabla) \mathbf{w} - \nabla p_\varphi(t, x - E(t)).
\end{aligned}$$

If we define  $q_\varphi(t, x) = p_\varphi(t, x - E(t))$ , we find that we have

$$q_\varphi = \sum_{i,j} (\varphi \partial_i \partial_j G_d) * (w_i w_j) + \sum_{i,j} \int (A_{i,j,\varphi}(x-y) - A_{i,j,\varphi}(-y)) (w_i(t, y) w_j(t, y)) dy.$$

The theorem is proved.  $\diamond$

## 6 Applications

A consequence of Proposition 3.1 is that we may define the Leray projection operator on the divergence of tensors that belong to  $L^1((0, T), L^1(\mathbb{R}^d, w_{d+1} dx))$ :

**Definition 6.1** *Let  $\mathbb{H} \in L^1((0, T), L^1(\mathbb{R}^d, w_{d+1} dx))$  and  $\mathbf{w} = \nabla \cdot \mathbb{H}$ . The Leray projection  $\mathbb{P}(\mathbf{w})$  of  $\mathbf{w}$  on solenoidal vector fields is defined by*

$$\mathbb{P} \mathbf{w} = \mathbf{w} - \nabla p_\varphi$$

where  $\nabla p_\varphi$  is the unique solution of

$$-\Delta \nabla p = \nabla(\nabla \cdot \mathbf{w})$$

such that

$$\lim_{\tau \rightarrow +\infty} e^{\tau \Delta} \nabla p = 0.$$

A special form of the Navier–Stokes equations is then given by

$$(MNS) \quad \partial_t \mathbf{u} = \Delta \mathbf{u} - \mathbb{P} \nabla \cdot (\mathbf{u} \otimes \mathbf{u} - \mathbb{F}), \quad \mathbf{u}(0, \cdot) = \mathbf{u}_0.$$

This leads to the integro-differential equation

$$u = e^{t\Delta} \mathbf{u}_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (\mathbf{u} \otimes \mathbf{u} - \mathbb{F}) ds.$$

The kernel of the convolution operator  $e^{(t-s)\Delta}\mathbb{P}\nabla\cdot$  is called the Oseen kernel; its study is the core of the method of mild solutions of Kato and Fujita [12]. Thus, we will call equations (MNS) a mild formulation of the Navier–Stokes equations.

The mild formulation together with the local Leray energy inequality has been as well a key tool for extending Leray’s theory of weak solutions in  $L^2$  to the setting of weak solutions with infinite energy. We may propose a general definition of suitable Leray-type weak solutions :

**Definition 6.2 (Suitable Leray-type solution)**

Let  $\mathbb{F} \in L^2((0, T), L^2(\mathbb{R}^d, \frac{1}{(1+|x|)^{d+1}}))$  and  $\mathbf{u}_0 \in L^2(\mathbb{R}^d, \frac{1}{(1+|x|)^{d+1}})$  with  $\nabla \cdot \mathbf{u}_0 = 0$ . We consider the Navier–Stokes problem on  $(0, T) \times \mathbb{R}^d$  :

$$\begin{aligned}\partial_t \mathbf{u} &= \Delta \mathbf{u} - \mathbb{P}(\mathbf{u} \otimes \mathbf{u} - \mathbb{F}), \\ \nabla \cdot \mathbf{u} &= 0, \quad \mathbf{u}(0, \cdot) = \mathbf{u}_0.\end{aligned}$$

A suitable Leray-type solution  $\mathbf{u}$  of the Navier–Stokes equations is a vector field  $\mathbf{u}$  defined on  $(0, T) \times \mathbb{R}^d$  such that :

- $\mathbf{u}$  is locally  $L_t^2 H_x^1$  on  $(0, T) \times \mathbb{R}^d$
- $\sup_{0 < t < T} \int |\mathbf{u}(t, x)|^2 \frac{1}{(1+|x|)^{d+1}} dx < +\infty$
- $\iint_{(0, T) \times \mathbb{R}^d} |\nabla \otimes \mathbf{u}(t, x)|^2 \frac{1}{(1+|x|)^{d+1}} dx dt < +\infty$
- the application  $t \in [0, T) \mapsto \int \mathbf{u}(t, x) \cdot \mathbf{w}(x) dx$  is continuous for every smooth compactly supported vector field  $\mathbf{w}$
- for every compact subset  $K$  of  $\mathbb{R}^d$ ,  $\lim_{t \rightarrow 0} \int_K |\mathbf{u}(t, x) - \mathbf{u}_0(x)|^2 dx = 0$ .
- defining  $p_\varphi$  as (the) solution of  $-\Delta p_\varphi = \sum_{i,j} \partial_i \partial_j (u_i u_j - F_{i,j})$  given by Proposition 4.1,  $\mathbf{u}$  is suitable in the sense of Caffarelli, Kohn and Nirenberg : there exists a non-negative locally bounded Borel measure  $\mu$  on  $(0, T) \times \mathbb{R}^d$  such that

$$\partial_t \left( \frac{|\mathbf{u}|^2}{2} \right) = \Delta \left( \frac{|\mathbf{u}|^2}{2} \right) - |\nabla \otimes \mathbf{u}|^2 - \nabla \cdot \left( \left( \frac{|\mathbf{u}|^2}{2} + p_\varphi \right) \mathbf{u} \right) + \mathbf{u} \cdot (\nabla \cdot \mathbb{F}) - \mu$$

**Remarks :**

- a) With those hypotheses,  $p_\varphi$  belongs locally to  $L_{t,x}^{3/2}$  and  $\mathbf{u}$  belongs locally to  $L_{t,x}^3$  so that the distribution  $(\frac{|\mathbf{u}|^2}{2} + p_\varphi)\mathbf{u}$  is well-defined.
- b) Suitability is a local assumption. It has been introduced by Caffarelli, Kohn and Nirenberg in 1982 [6] to get estimates on partial regularity for

weak Leray solutions. If we consider a solution of the Navier–Stokes equations on a small domain with no specifications on the behaviour of  $\mathbf{u}$  at the boundary, the estimates on the pressure (and the Leray projection operator) are no longer available. However, Wolf described in 2017 [21] a local decomposition of the pressure into a term similar to the Leray projection of  $\nabla \cdot (\mathbf{u} \otimes \mathbf{u})$  and a harmonic term; he could generalize the notion of suitability to this new description of the pressure. On the equivalence of various notions of suitability, see the paper by Chamorro, Lemarié-Rieusset and Mayoufi [8]. c) The relationship between the system (NS) and its mild formulation (MNS) described in Theorem 1 has been described by Furioli, Lemarié-Rieusset and Terraneo in 2000 [13, 16] in the context of uniformly locally square integrable solutions. See the paper by Dubois [11], as well.

We list here a few examples to be found in the litterature :

1. Solutions in  $L^2$  : in 1934, Leray [18] studied the Navier–Stokes problem (NS) with an initial data  $\mathbf{u}_0 \in L^2$  and a forcing tensor  $\mathbb{F} \in L_t^2 L_x^2$ . He then obtained a solution  $\mathbf{u} \in L^\infty L^2 \cap L^2 \dot{H}^1$ . Remark that this solution is automatically a solution of the mild formulation of the Navier–Stokes equations (MNS). Leray’s construction by mollification provides suitable solutions.
2. Solutions in  $L_{\text{uloc}}^2$  : in 1999, Lemarié-Rieusset [15, 16] studied the Navier–Stokes problem (MNS) with an initial data  $\mathbf{u}_0 \in L_{\text{uloc}}^2$  (and, later in [17], a forcing tensor  $\mathbb{F} \in (L_t^2 L_x^2)_{\text{uloc}}$ ). He obtained (local in time) existence of a suitable solution  $\mathbf{u}$  on a small strip  $(0, T_0) \times \mathbb{R}^d$  such that

$$\sup_{x_0 \in \mathbb{R}^d} \sup_{0 < t < T_0} \int_{B(x_0, 1)} |\mathbf{u}(t, x)|^2 dx < +\infty$$

and

$$\sup_{x_0 \in \mathbb{R}^d} \int_0^{T_0} \int_{B(x_0, 1)} |\nabla \otimes \mathbf{u}(t, x)|^2 dx < +\infty.$$

Remark that we have  $\mathbf{u} \in L^2((0, T_0), L^2(\mathbb{R}^d, \frac{1}{(1+|x|)^{d+1}} dx))$  but  $\mathbf{u}$  does not belong to  $L^2((0, T_0), L^2(\mathbb{R}^d, \frac{1}{(1+|x|)^d} dx))$ ; thus, in this setting, problems (NS) and (MNS) are not equivalent.

Various reformulations of local Leray solutions in  $L_{\text{uloc}}^2$  have been provided, such as Kikuchi and Seregin in 2007 [14] or Bradshaw and Tsai in 2019 [4]. The formulas proposed for the pressure, however, are actually equivalent, as they all imply that  $\mathbf{u}$  is solution to the (MNS) problem.

In the case of dimension  $d = 2$ , Basson [1] proved in 2006 that the solution  $\mathbf{u}$  is indeed global (i.e.  $T_0 = T$ ) and that, moreover, the solution is unique.

3. Solutions in a weighted Lebesgue space : in 2019, Fernández-Dalgo and Lemarié–Rieusset [9] considered data  $\mathbf{u}_0 \in L^2(\mathbb{R}^3, w_\gamma dx)$  and  $\mathbb{F} \in L^2((0, +\infty), L^2(\mathbb{R}^3, w_\gamma dx))$  with  $0 < \gamma \leq 2$ . They proved (global in time) existence of a suitable solution  $\mathbf{u}$  such that, for all  $T_0 < +\infty$ ,

$$\sup_{0 < t < T_0} \int |\mathbf{u}(t, x)|^2 w_\gamma(x) dx < +\infty$$

and

$$\int_0^{T_0} \int |\nabla \otimes \mathbf{u}(t, x)|^2 w_\gamma(x) dx < +\infty.$$

[Of course, for such solutions, (NS) and (MNS) are equivalent.] They showed that, for  $\frac{4}{3} < \gamma \leq 2$ , this frame of work is well adapted to the study of discretely self-similar solutions with locally  $L^2$  initial value, providing a new proof of the results of Chae and Wolf in 2018 [7] and of Bradshaw and Tsai in 2019 [3].

4. Homogeneous Statistical Solutions : in 1977, Vishik and Fursikov [19] considered the (MNS) problem with a random initial value  $\mathbf{u}_0(\omega)$ . The statistics of the initial distributions were supposed to be invariant though translation of the arguments of  $\mathbf{u}_0$  : for every Borel subset  $B$  of  $L^2_{\text{loc}}(\mathbb{R}^3)$  and every  $x_0 \in \mathbb{R}^3$ ,

$$Pr(\mathbf{u}_0(\cdot - x_0) \in B) = Pr(\mathbf{u}_0 \in B).$$

Another assumption was that  $\mathbf{u}_0$  has a bounded mean energy density :

$$e_0 = \mathbb{E} \left( \frac{\int_{|x| \leq 1} |\mathbf{u}_0|^2 dx}{\int_{|x| \leq 1} dx} \right) < +\infty.$$

Then

$$Pr(\mathbf{u}_0 \in L^2 \text{ and } u \neq 0) = 0$$

while, for any  $\epsilon > 0$ ,

$$Pr\left(\int |\mathbf{u}_0|^2 \frac{1}{(1 + |x|)^{3+\epsilon}} dx < +\infty\right) = 1.$$

In [20], they constructed a solution  $\mathbf{u}(t, x, \omega)$  that solved the Navier–Stokes equation for almost every initial value  $\mathbf{u}_0(\omega)$ , and the solution belonged almost surely to  $L_t^\infty L_x^2(\frac{1}{(1+|x|)^{3+\epsilon}} dx)$  with  $\nabla \otimes \mathbf{u} \in L_t^2 L_x^2(\frac{1}{(1+|x|)^{3+\epsilon}} dx)$ .



In 2006, Basson [2] gave a precise description of the pressure in those equations (which is equivalent to our description through the Leray projection operator) and proved the suitability of the solutions.

## 7 The space $B_\gamma^2$ .

Instead of dealing with weighted Lebesgue spaces, one may deal with a kind of local Morrey space, the space  $B_\gamma^2$ .

**Definition 7.1** For  $\gamma \geq 0$ , define  $w_\gamma(x) = \frac{1}{(1+|x|)^\gamma}$  and  $L_{w_\gamma}^p = L^p(\mathbb{R}^d, w_\gamma(x) dx)$ .

For  $1 \leq p < +\infty$ , we denote  $B_\gamma^p$  the Banach space of all functions  $u \in L_{w_\gamma}^p$  such that :

$$\|u\|_{B_\gamma^p} = \sup_{R \geq 1} \left( \frac{1}{R^\gamma} \int_{B(0,R)} |u|^p dx \right)^{1/p} < +\infty.$$

Similarly,  $B_\gamma^p L^p(0, T)$  is the Banach space of all functions  $u \in (L_t^p L_x^p)_{\text{loc}}$  such that

$$\|u\|_{B_\gamma^p L^p(0,T)} = \sup \left( \frac{1}{R^\gamma} \int_0^T \int |u|^p \right)^{\frac{1}{p}} dx ds.$$

**Lemma 7.1** Let  $\gamma \geq 0$  and  $\gamma < \delta < +\infty$ , we have the continuous embedding  $L_{w_\gamma}^p \hookrightarrow B_{\gamma,0}^p \hookrightarrow B_\gamma^p \hookrightarrow L_{w_\delta}^p$ , where  $B_{\gamma,0}^p \subset B_\gamma^p$  is the subspace of all functions  $u \in B_\gamma^p$  such that  $\lim_{R \rightarrow +\infty} \frac{1}{R^\gamma} \int_{B(0,R)} |u(x)|^p dx = 0$ .

**Proof.** Let  $u \in L_{w_\gamma}^p$ . We verify easily that  $\|u\|_{B_\gamma^p} \leq 2^{\gamma/p} \|u\|_{L_{w_\gamma}^p}$  and we see that

$$\frac{1}{R^\gamma} \int_{|x| \leq R} |u|^p dx = \int_{|x| \leq R} \frac{|u|^p}{(1+|x|)^\gamma} \frac{(1+|x|)^\gamma}{R^\gamma} dx$$

converges to zero when  $R \rightarrow +\infty$  by dominated convergence, so  $L_{w_\gamma}^p \hookrightarrow B_{\gamma,0}^p$ . To demonstrate the other part, we estimate

$$\begin{aligned} \int \frac{|u|^p}{(1+|x|)^\delta} dx &= \int_{|x| \leq 1} \frac{|u|^p}{(1+|x|)^\delta} dx + \sum_{n \in \mathbb{N}} \int_{2^{n-1} \leq |x| \leq 2^n} \frac{|u|^p}{(1+|x|)^\delta} dx \\ &\leq \int_{|x| \leq 1} |u|^p dx + \sum_{n \in \mathbb{N}} \frac{1}{(1+2^{n-1})^\delta} \int_{2^{n-1} \leq |x| \leq 2^n} |u|^p dx \\ &\leq \int_{|x| \leq 1} |u|^p dx + c \sum_{n \in \mathbb{N}} \frac{1}{2^{\delta n}} \int_{2^{n-1} \leq |x| \leq 2^n} |u|^p dx \\ &\leq (1 + c \sum_{n \in \mathbb{N}} \frac{1}{2^{(\delta-\gamma)n}}) \sup_{R \geq 1} \frac{1}{R^\gamma} \int_{|x| \leq R} |u|^p dx, \end{aligned}$$

thus,  $B_\gamma^p \subset L_{w_\delta}^p$ .  $\diamond$

**Remark :** Similarly, for all  $\delta > \gamma$ ,  $B_\gamma^p L^p(0, T) \subset L^p((0, T), L_{w_\delta}^p)$ .

**Proposition 7.1** *The space  $B_\gamma^p$  can be obtained by interpolation,*

$$B_\gamma^p = [L^p, L_{w_\delta}^p]_{\frac{\gamma}{\delta}, \infty}$$

for all  $0 < \gamma < \delta < \infty$ , and the norms

$$\|\cdot\|_{B_\gamma^p} \quad \text{and} \quad \|\cdot\|_{[L^p, L_{w_\delta}^p]_{\frac{\gamma}{\delta}, \infty}}$$

are equivalent.

**Proof.** Let  $f \in B_\gamma^p$ . For  $A < 1$ , we write  $f_0 = 0$  and  $f_1 = f$ , then we have  $f = f_0 + f_1$  and  $\|f_1\|_{L_{w_\delta}^p} \leq CA^{\frac{\gamma}{\delta}-1} \|f\|_{B_\gamma^p}$ .

For  $A > 1$ , we let  $R = A^{\frac{\delta}{\gamma}} > 1$ . We write  $f_0 = f \mathbf{1}_{|x| \leq R}$  and  $f_1 = f \mathbf{1}_{|x| > R}$ , then

$$\|f_0\|_p \leq C \|f\|_{B_\gamma^p} R^{\frac{\gamma}{p}} = CA^{\frac{\gamma}{\delta}} \|f\|_{B_\gamma^p}$$

and

$$\begin{aligned} \|f_1\|_p^p &= \sum_{n \in \mathbb{N}} \int_{2^{n-1}R \leq |x| \leq 2^n R} \frac{|u|^p}{(1+|x|)^\delta} dx \\ &\leq CR^{\gamma-\delta} \sum_{n \in \mathbb{N}} \frac{1}{2^{(\delta-\gamma)n}} \|f\|_{B_\gamma^p}^p \\ &= CA^{(\frac{\gamma}{\delta}-1)p} \|f\|_{B_\gamma^p}^p \end{aligned}$$

Thus,  $B_\gamma^p \hookrightarrow [L^p, L_{w_\delta}^p]_{\frac{\gamma}{\delta}, \infty}$ .

Let  $f \in [L^p, L_{w_\delta}^p]_{\frac{\gamma}{\delta}, \infty}$ , then there exist  $c > 0$  such that for all  $A > 0$ , there exist  $f_0 \in L^p$  and  $f_1 \in L_{w_\delta}^p$  so that  $f = f_0 + f_1$ ,

$$\|f_0\|_p \leq cA^{\frac{\gamma}{\delta}} \quad \text{and} \quad \|f_1\|_{L_{w_\delta}^p} \leq cA^{\frac{\gamma}{\delta}-1}.$$

For  $j \in \mathbb{N}$  we take  $A = 2^{\frac{j\delta}{p}}$ , then

$$\begin{aligned}
& \frac{1}{2^{j\gamma}} \int_{|x| < 2^j} |f|^p dx \\
& \leq C \left( \frac{1}{2^{j\gamma}} \int_{|x| < 2^j} |f_0|^p dx + \frac{1}{2^{j\gamma}} \int_{|x| < 2^j} |f_1|^p dx \right) \\
& \leq C \left( \frac{1}{2^{j\gamma}} \|f_0\|_p^p + \frac{C}{2^{j\gamma}} \int_{|x| \leq 1} \frac{|f_1|^p}{(1+|x|)^\delta} dx + C \sum_{k=1}^j \frac{2^{k\delta}}{2^{j\gamma}} \int_{2^{k-1} < |x| < 2^k} \frac{|f_1|^p}{(1+|x|)^\delta} dx \right) \\
& \leq C \left( \frac{1}{2^{j\gamma}} \|f_0\|_p^p + C' 2^{j(\delta-\gamma)} \|f_1\|_{L_{w_\delta}^p}^p \right) \\
& \leq C''
\end{aligned}$$

which implies  $\sup_{j \in \mathbb{N}} \frac{1}{2^{j\gamma}} \int_{|x| < 2^j} |f|^p dx < +\infty$ , so  $\sup_{R \geq 1} \frac{1}{R^\gamma} \int_{|x| < R} |f|^p dx < +\infty$ .  $\diamond$

Thus, we can see that the local Morrey spaces  $B_\gamma^p$  are very close to the weighted Lebesgue spaces  $L_{w_\gamma}^p$ . Indeed, the methods and results of Fernández-Dalgo and Lemarié-Rieusset [9] can be easily extended to the setting of local Morrey spaces in dimension  $d = 2$  or  $d = 3$ : considering data  $\mathbf{u}_0 \in B_\gamma^2(\mathbb{R}^d)$  and  $\mathbb{F} \in (B_\gamma^2 L^2)(0, T)(\mathbb{R}^d)$  with  $0 < \gamma \leq 2$ , one gets (local in time) existence of a suitable solution  $\mathbf{u}$  for the (MNS) system on a small strip  $(0, T_0) \times \mathbb{R}^d$  such that  $\mathbf{u} \in L^\infty((0, T_0), B_\gamma^2)$  and  $\nabla \otimes \mathbf{u} \in (B_\gamma^2 L^2)(0, T_0)$ .

The case of  $\gamma = 2$  deserves some comments. In the case  $d = 3$ , the results is slightly more general than the results in [9], as the class  $B_2^2$  is larger than the space  $L_{w_2}^2$ . Equations in  $B_2^2$  have been very recently discussed by Bradshaw, Kukavica and Tsai [5]. The case  $d = 2$  is more intricate. Indeed, while the Leray projection operator is bounded on  $B_2^2(\mathbb{R}^3)$  (by interpolation with  $L^2$  and  $L_{w_\delta}^2$  with  $2 < \delta < 3$ , the Riesz transforms being bounded on  $L_{w_\delta}^2$  by the theory of Muckenhoupt weights), this is no longer the fact on  $B_2^2(\mathbb{R}^2)$ . Thus, one must be careful in the handling of the pressure. This has been done by Basson in his Ph. D. thesis in 2006 [1].

Local Morrey spaces  $B_d^2$  occur naturally in the setting of homogeneous statistical solutions. By using an ergodicity argument, Dostoglou [10] proved in 2001 that, under the assumptions of Vishik and Fursikov [19], we have

$$Pr(\mathbf{u}_0(., \omega) \in B_d^2(\mathbb{R}^d)) = 1.$$

Thus, the solutions of Vishik and Fursikov live in a smaller space than  $L_{w_{d+\epsilon}}^2$ .

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