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# Study of the cost of the controllability of second order parabolic equations with a small diffusion and a transport term

Jon Asier Bárcena-Petisco\*<sup>†</sup>

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## 0 Abstract and basic information

In this paper we consider the heat equation with Neumann, Robin and mixed boundary conditions (with coefficients on the boundary which depend on the space variable). The main results concern the behaviour of the cost of the null controllability with respect to the diffusion coefficient when the control acts in the interior. First, we prove that if we almost have Dirichlet boundary conditions in the part of the boundary in which the flux of the transport enters, the cost of the controllability decays for a time  $T$  sufficiently large. Next, we show some examples of Neumann and mixed boundary conditions in which for any time  $T > 0$  the cost explodes exponentially as the diffusion coefficient vanishes. Finally, we study the cost of the problem with Neumann boundary conditions when the control is localized in the whole domain.

**Key words:** Heat equation, singular limits, spectral decomposition, transport equation, uniform controllability

**AMS subject classification:** 35B25, 35K05, 93B05, 93C20

**Abbreviated title:** Cost of a transport-diffusion control problem

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**Declarations of interest:** None.

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# 1 Introduction

Let  $\Omega$  be a domain (a bounded connected open set of  $\mathbb{R}^d$ ),  $\Gamma \subset \partial\Omega$  be a relative open subset and  $\Gamma^* := \partial\Omega \setminus \Gamma$ . We consider in this paper a measurable function  $a(x, \varepsilon)$  defined in  $\Gamma \times (0, \varepsilon_0)$  for some  $\varepsilon_0 > 0$  such that  $a(\cdot, \varepsilon) \in L^\infty(\Omega)$  for all  $\varepsilon \in (0, \varepsilon_0)$ . We study the control problem given by:

$$\begin{cases} y_t - \varepsilon \Delta y + \partial_{x_1} y = 1_\omega f, & \text{in } Q, \\ \partial_n y + ay = 0, & \text{on } \Sigma_N, \\ y = 0, & \text{on } \Sigma_D, \\ y(0, \cdot) = y^0, & \text{on } \Omega, \end{cases} \quad (1.1)$$

for  $\varepsilon \in (0, \varepsilon_0)$ . As usual,  $n$  denotes the outward normal vector on  $\partial\Omega$ ,  $Q := (0, T) \times \Omega$ ,  $Q_\omega := (0, T) \times \omega$  and  $\Sigma := (0, T) \times \partial\Omega$ . In addition, we define  $\Sigma_N := (0, T) \times \Gamma$  and  $\Sigma_D := (0, T) \times \Gamma^*$ . The null controllability of (1.1) is already proved in [14] and [12] (for  $\Gamma = \partial\Omega$  and a function  $a(t, x) \in L^\infty(\Sigma)$ , though the proof can be easily adapted for proving the controllability of (1.1)). However, following the proofs we get a gross estimate on how the cost of the controllability behaves when the diffusion coefficient  $\varepsilon > 0$  vanishes. Having more precise bounds with respect to  $\varepsilon$  of the cost is thus an open question and in this paper we answer it in some important cases. We study the cost of the null controllability with the usual norms; that is:

$$K(\Omega, \omega, T, \varepsilon) := \sup_{y^0 \in L^2(\Omega) \setminus \{0\}} \inf_{f \in S(y^0)} \frac{\|f\|_{L^2(Q_\omega)}}{\|y^0\|_{L^2(\Omega)}},$$

for:

$$S(y^0) := \{f \in L^2(Q_\omega) : \text{the solution of (1.1) satisfies } y(T, \cdot) = 0\}.$$

The cost also depends on  $a$  and  $\Gamma$ , although we do not expose it explicitly.

This paper follows a well-established research line which inquires about the cost of the null controllability of systems with a small diffusion and a transport term. The first of such control problems to be analysed was the heat equation in dimension one with Dirichlet boundary conditions in [10]. Afterwards, the same problem but in any dimension and with any speed belonging to  $W^{1,\infty}(\mathbb{R}^+ \times \Omega)$  was studied in [19]. More recently, better approximations of the optimal time in which the cost of the control decays have been given: the lower bound was improved in [26] through complex analysis and properties of the entire functions, and the upper bound was improved in [15, 25] (in the first one through complex analysis and, in the second one, by transforming the original equation into the pure heat equation). As for similar results, several results have been obtained for the KdV equation (see [17, 18, 3, 4]), the Burgers equation (see [16]), the Stokes system (see [2]), an artificial advection-diffusion problem (see [7, 8]) and a

fourth-order parabolic equation (see [5, 27]). Finally, the current study of (1.1) is a contribution to the literature as it seeks to understand the evolution of the cost of the null controllability of the transport-diffusion equation with a large variety of boundary conditions when the diffusion vanishes (and, in particular, the case  $a = 0$  and  $\Gamma = \emptyset$  was an open problem proposed in [19, Remark 3]).

As in the previously cited papers, we study the adjoint problem, which is given by:

$$\begin{cases} -\varphi_t - \varepsilon \Delta \varphi - \partial_{x_1} \varphi = 0, & \text{in } Q, \\ \varepsilon \partial_n \varphi + (\varepsilon a + n_1) \varphi = 0, & \text{on } \Sigma_N, \\ \varphi = 0, & \text{on } \Sigma_D, \\ \varphi(T, \cdot) = \varphi^T, & \text{on } \Omega. \end{cases} \quad (1.2)$$

Indeed, we have the classical equality given by the Hilbert Uniqueness Method (see, for instance, [28, 23]):

$$K(\Omega, \omega, T, \varepsilon) = \sup_{\varphi^T \in L^2(\Omega) \setminus \{0\}} \frac{\|\varphi(0, \cdot)\|_{L^2(\Omega)}}{\|\varphi\|_{L^2(Q_\omega)}}, \quad (1.3)$$

where  $\varphi$  denotes the solution of (1.2).

As a novelty with respect to the literature, we obtain some properties of (1.2) (and, in particular, the Carleman inequality) by studying the solutions of a system whose elliptic operator is self-adjoint. Indeed, considering the map:

$$w \mapsto e^{(2\varepsilon)^{-1}x_1} w, \quad (1.4)$$

we get the system:

$$\begin{cases} -\phi_t - \varepsilon \Delta \phi + \frac{1}{4\varepsilon} \phi = 0, & \text{in } Q, \\ \varepsilon \partial_n \phi + \left(\varepsilon a + \frac{n_1}{2}\right) \phi = 0, & \text{on } \Sigma_N, \\ \phi = 0, & \text{on } \Sigma_D, \\ \phi(T, \cdot) = \phi^T, & \text{on } \Omega. \end{cases} \quad (1.5)$$

The motivation for (1.4) is to have no first-order term in the elliptic operator, as it is a necessary condition for a elliptic operator to be self-adjoint.

*Remark 1.1.* The map (1.4) is an homeomorphism from the solutions of (1.2) with initial value in  $L^2(\Omega)$  to the solutions of (1.5) with initial value in  $L^2(\Omega)$ . That homeomorphism is useful to translate information about the solutions of (1.5) to information about the solutions of (1.2), for instance about their regularity.

With the purpose of understanding the solutions of (1.2) we also study the spectral problem:

$$\begin{cases} -\varepsilon\Delta u - \partial_{x_1}u = \lambda u, & \text{in } \Omega, \\ \varepsilon\partial_n u + (\varepsilon a + n_1)u = 0, & \text{on } \Gamma, \\ u = 0, & \text{on } \Gamma^*. \end{cases} \quad (1.6)$$

In that sense, with the purpose of understanding the solutions of (1.6) we study the spectral problem:

$$\begin{cases} -\Delta v = \tilde{\lambda}v, & \text{in } \Omega, \\ \varepsilon\partial_n v + (\varepsilon a + \frac{n_1}{2})v = 0, & \text{on } \Gamma, \\ v = 0, & \text{on } \Gamma^*, \end{cases} \quad (1.7)$$

and we define:

$$D(\Delta_\varepsilon) := \left\{ w \in H^1(\Omega) : -\Delta w \in L^2(\Omega), \varepsilon\partial_n w + \left(\varepsilon a + \frac{n_1}{2}\right)w = 0 \text{ on } \Gamma, w = 0 \text{ on } \Gamma^* \right\}.$$

We recall that  $-\Delta w \in L^2(\Omega)$  and  $w \in H^1(\Omega)$  imply that  $\partial_n w$  is well-defined as an element of  $H^{-1/2}(\partial\Omega)$ . We have considered (1.7)<sub>1</sub> instead of  $-\varepsilon\Delta v + (4\varepsilon)^{-1}v = \lambda v$  in order to isolate the ‘‘part’’ of the eigenvalue that is caused by the diffusion, which is the meaning of  $\tilde{\lambda}$ . This is useful for instance in Lemmas 2.4 and 3.6 below.

*Remark 1.2.* We can relate the solutions of (1.6) and (1.7) with the map (1.4). Indeed, we have that  $(v, \tilde{\lambda})$  is a solution of (1.7) if and only if:

$$\left( v e^{-(2\varepsilon)^{-1}x_1}, \varepsilon\tilde{\lambda} + \frac{1}{4\varepsilon} \right),$$

is a solution of (1.6). Similarly,  $(u, \lambda)$  is a solution of (1.6) if and only if:

$$\left( u e^{(2\varepsilon)^{-1}x_1}, \frac{\lambda}{\varepsilon} - \frac{1}{4\varepsilon^2} \right),$$

is a solution of (1.7).

*Remark 1.3.* It can be checked directly that the Laplacian is self-adjoint in  $D(\Delta_\varepsilon)$  for any Lipschitz domain  $\Omega$ . Indeed, if  $w \in D(\Delta_\varepsilon^*)$  we first see that  $-\Delta w \in L^2(\Omega)$  by considering that  $\mathcal{D}(\Omega) \subset D(\Delta)$ . Afterwards, we see that  $w$  is a solution by transposition in  $L^2(\Omega)$  of  $-\Delta v = -\Delta w$ , (1.7)<sub>2</sub> and (1.7)<sub>3</sub>, a solution which is unique and belongs to  $D(\Delta_\varepsilon)$ . Thereby,  $L^2(\Omega)$  has a spectral decomposition (for each  $\varepsilon$ ) by the solutions of (1.7).

*Remark 1.4.* We can also study the problem with null initial force associated to (1.1). Indeed, the map  $w \mapsto e^{-(2\varepsilon)^{-1}x_1}w$  transforms (1.1) into a symmetric version of (1.5) (backwards in time and with opposite transport). Thus, the direct and adjoint systems have similar nature and decay properties.

*Definition 1.5.* We denote by  $(\lambda_m^\varepsilon)_{m \in \mathbb{N}}$  the eigenvalues of (1.6) and by  $(\tilde{\lambda}_m^\varepsilon)_{m \in \mathbb{N}}$  the eigenvalues of (1.7). In addition, for each eigenvalue of (1.7) an eigenfunction  $v_m^\varepsilon$  is fixed, to which we impose that  $\|v_m^\varepsilon\|_{L^2(\Omega)} = 1$ . Finally, we denote  $u_m^\varepsilon := v_m^\varepsilon e^{-(2\varepsilon)^{-1}x_1}$ .

*Remark 1.6.* Since  $\overline{\text{span}\{v_m\}} = L^2(\Omega)$  (see Remark 1.3) we have that  $\overline{\text{span}\{u_m\}} = L^2(\Omega)$ .

The idea of using a spectral decomposition is not new in Control Theory (see, for instance, [9]). Indeed, for the heat equation alone, there are many documents which deal with eigenfunctions of the elliptic operator for proving the existence of some control (see, for example, [22] for Dirichlet boundary conditions and [1] for a coupled heat equation system with Robin boundary conditions) and for estimating the cost of the control (see, for instance, [11]). Similarly, for the Stokes system it has been used to prove the existence of a control (see, for example, [6]) and for providing a negative answer to the existence of a control (see, for instance, [24]). As for a system with a small diffusion and a transport term, a spectral decomposition indirectly appears in [10, 26], when the authors get lower bounds for the optimal time  $T_0$  in which the cost of the control decays exponentially with  $\varepsilon$ . In addition, a spectral decomposition has been used in [2] to get the dissipation estimate in a transport-diffusion Stokes system. Indeed, in this paper we follow the philosophy of [2] of using as much information as possible about the spectral decomposition, with the contribution that now while proving the Carleman we work directly in the symmetrized system; that is, in (1.5).

Coming back to (1.2), what determines if the solutions of (1.2) decays is  $\tilde{\lambda}_0^\varepsilon$ . We recall that by Rayleigh principle we have the equality:

$$\tilde{\lambda}_0^\varepsilon = \min \left\{ \int_{\Omega} |\nabla v|^2 dx + \int_{\Gamma} \left( a + \frac{n_1}{2\varepsilon} \right) |v|^2 dx : v \in H^1(\Omega), \|v\|_{L^2(\Omega)} = 1, v = 0 \text{ on } \Gamma^* \right\}. \quad (1.8)$$

Looking at (1.8) we can do a division of cases depending on whether the following inequality is satisfied:

$$(a + (2\varepsilon)^{-1}n_1)1_{\Gamma} \geq 0. \quad (1.9)$$

If (1.9) is satisfied, we have that  $\tilde{\lambda}_0^\varepsilon \geq 0$ , so the solutions of (1.2) decay exponentially on  $\varepsilon^{-1}$  (see Lemma 2.4 below). In addition, we have the following controllability result:

**Theorem 1.7.** *Let  $\Omega$  be a  $C^2$  domain,  $\omega \subset \Omega$  be a subdomain and assume that  $(\Gamma, a)$  satisfies (1.9). Then, there are  $T_0, c, C > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  and  $T \geq T_0$  we have that:*

$$K(\Omega, \omega, T, \varepsilon) \leq C e^{-c\varepsilon^{-1}}. \quad (1.10)$$

The proof of Theorem 1.7 is given in Section 2.

There are several important examples that satisfy the hypotheses of Theorem 1.7:

- Dirichlet boundary conditions ( $\Gamma = \emptyset$ ). This controllability result is already proved in [10, 19]. However, our proof is different since we prove the decay with the spectral decomposition (instead of with a comparison theorem or an Agmon inequality) and we prove the Carleman inequality for (1.5) instead of (1.2).
- Segments in which we have Dirichlet boundary conditions on the left edge and Neumann conditions on the right edge. The physical meaning of those boundary conditions is that the function is almost null near the left edge and almost constant near the right edge.
- Any system in which  $a \geq 0$  and  $n_1 1_\Gamma \geq 0$ ; that is, in which we have Dirichlet boundary conditions on the part of the boundary in which the flux of the transport enters and either Dirichlet or Robin with a positive coefficient on the other part of the boundary.
- Any system in which we almost have Dirichlet boundary conditions on the part of the boundary in which the flux of the transport enters and either Dirichlet or Robin with a coefficient whose negative part is not too large on the other part of the boundary.

The alternative to (1.9) is:

$$\left(a + \frac{n_1}{2\varepsilon}\right) 1_\Gamma \not\geq 0. \quad (1.11)$$

The order relation in (1.11) is understood as elements of  $L^\infty(\partial\Omega)$ , not as something that happens at almost every point. Under the hypothesis (1.11) we might have eigenvalues close to 0, null or negative for (1.6), so we may have no decay and the cost of the control might explode. In particular, in this paper we study some subcases of (1.11):

- We consider (1.1) with pure Neumann boundary conditions; that is,  $a = 0$  and  $\Gamma = \partial\Omega$ . We remark that under those boundary conditions (1.1) is given by:

$$\begin{cases} y_t - \varepsilon\Delta y + \partial_{x_1} y = 1_\omega f, & \text{in } Q, \\ \partial_n y = 0, & \text{on } \Sigma, \\ y(0, \cdot) = y^0, & \text{on } \Omega. \end{cases} \quad (1.12)$$

The study of the cost of the null controllability of (1.12) is an open problem proposed in [19, Remark 3], and it represents that the mass gets in and out only tangentially. Let us denote:

$$\pi_1(x) := x_1, \quad p_l := \inf \pi_1(\Omega), \quad p_r := \sup \pi_1(\Omega). \quad (1.13)$$

We prove in this paper that the cost of the control of (1.12) explodes exponentially:

**Theorem 1.8.** *Let  $h > 0$ ,  $\Omega$  be a domain, and  $\omega \subset \Omega$  be an open subset such that:*

$$\pi_1(\omega) \subset (p_l + h, p_r). \quad (1.14)$$

*Then, for all  $T > 0$  there is  $c > 0$  such that for all  $\varepsilon > 0$ :*

$$K(\Omega, \omega, T, \varepsilon) \geq ce^{\varepsilon^{-1}}, \quad (1.15)$$

*for  $K$  the cost of the control problem (1.12).*

- We consider a segment with Neumann boundary conditions on the left edge of the segment and Dirichlet boundary conditions on its right edge; that is,  $\Omega := (-L, 0)$ ,  $\Gamma := \{-L\}$  and  $a = 0$ . In this situation the control problem (1.1) is given by:

$$\begin{cases} y_t - \varepsilon \partial_{xx} y + \partial_x y = f1_\omega, & \text{in } Q, \\ \partial_x y(\cdot, -L) = 0, & \text{on } (0, T), \\ y(\cdot, 0) = 0, & \text{on } (0, T), \\ y(0, \cdot) = y^0, & \text{on } (-L, 0). \end{cases} \quad (1.16)$$

The control problem (1.16) describes for instance a valley in which the density of animals is constant near the left edge and it goes to 0 near the right one (for instance if there is a cliff). The intuition may suggest that as  $\varepsilon$  gets smaller the cost of the control decreases, as in the Dirichlet case. However, it is just the opposite, the fact that there is a flux coming from the left dominates and the cost of the control to 0 actually increases when the random movement's coefficient vanishes. In particular, we prove that:

**Theorem 1.9.** *Let  $L, h > 0$ ,  $\Omega = (-L, 0)$  and  $\omega \subset (-L + h, 0)$  be an open subset. Then, for all  $T > 0$  there are  $c, \varepsilon_0 > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  we have the estimate:*

$$K(\Omega, \omega, T, \varepsilon) \geq ce^{\varepsilon^{-1}}, \quad (1.17)$$

*for  $K$  the cost of the control problem (1.16).*

- We get a lower bound of the cost of the controllability when  $\omega = \Omega$ ; that is, in which the control domain is the whole domain. In particular, we prove that:

**Theorem 1.10.** *Let  $\Omega$  be a Lipschitz domain,  $T > 0$  and  $(\Gamma, a)$  satisfying:*

$$\lambda_0^\varepsilon \rightarrow 0. \quad (1.18)$$

*Then, we have the bound:*

$$\liminf_{\varepsilon \rightarrow 0} K(\Omega, \Omega, T, \varepsilon) \geq \frac{1}{\sqrt{T}}. \quad (1.19)$$



The condition stated in Theorem 1.10 includes the control problem (1.12) and (1.16) as it is shown below in Remark 3.2 and (3.17) respectively.

- We study, assuming  $\omega = \Omega$  and (1.18), if the cost of the control is at least  $O(T^{-1/2})$  when  $\varepsilon \rightarrow 0$ . In particular, for Neumann boundary conditions we prove that:

**Theorem 1.11.** *Let  $\Omega$  be a  $C^2$  domain. Then, there is  $C > 0$  such that for all  $\varepsilon > 0$  and  $T > 0$ :*

$$K(\Omega, \Omega, T, \varepsilon) \leq C \left( \frac{1}{\sqrt{T}} + \frac{1}{\sqrt{\varepsilon}} \right), \quad (1.20)$$

for  $K$  the cost of the control problem (1.12).

In addition, if we are in a segment we prove that:

**Theorem 1.12.** *Let  $L > 0$  and  $\Omega = (-L, 0)$ . We have for all  $T \geq 4L$ :*

$$\limsup_{\varepsilon \rightarrow 0} K(\Omega, \Omega, T, \varepsilon) \leq \frac{4}{\sqrt{T}}, \quad (1.21)$$

for  $K$  the cost of the control problem (1.12).

*Remark 1.13.* The difficulty when proving Theorems 1.11 and 1.12 is that  $f$  is required to be in  $L^2(Q)$ . If we just look for controls in  $L^2(0, T; H^{-2}(\Omega))$ , then it suffices to consider  $f = 0$  on  $(0, t)$  for  $t > 0$  a small value, and then consider the affine function which joins  $y(t, 0)$  and 0.

*Remark 1.14.* All the previously mentioned subcases have in common that  $a \geq 0$ . Consequently, as proved in the introduction of Section 3, the solutions of (1.2) do not explode. Thus, those properties of the control problems are caused by the transport term interacting with those specific boundary conditions.

As for the results stated for subcases of (1.11), Theorem 1.8 is similar to the result presented in [2] for the Stokes equation in dimension 3. As for the other results, the most surprising one is Theorem 1.9, as Dirichlet boundary conditions on the right edge may mislead us in believing that we have exponential decay on the cost of the control.

The rest of the paper is organized as follows: in Section 2 we prove Theorem 1.7, in Section 3 we study the case (1.11) and in Section 4 we do some remarks and present some open problems.

## 2 Proof of Theorem 1.7

In this section we prove Theorem 1.7. In particular, in Section 2.1 we prove the decay of the solutions of (1.2), in Section 2.2 we prove a Carleman inequality for the solutions of (1.5), and in Section 2.3 we end the proof of Theorem 1.7. For this section we use the following notation:

*Definition 2.1.* Let  $\Omega \subset \mathbb{R}^d$  a non-empty set. We define:

$$\text{diam}_{x_1}(\Omega) := \sup \left\{ x_1 : \left( \{x_1\} \times \mathbb{R}^{d-1} \right) \cap \Omega \neq \emptyset \right\} - \inf \left\{ x_1 : \left( \{x_1\} \times \mathbb{R}^{d-1} \right) \cap \Omega \neq \emptyset \right\}.$$

## 2.1 Decay of the solutions of (1.2) for a sufficiently large time

We denote  $(\tilde{\lambda}_m^0)_{m \in \mathbb{N}}$  the eigenvalues of the system:

$$\begin{cases} -\Delta v = \tilde{\lambda}^0 v, & \text{in } \Omega, \\ \partial_n v = 0, & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

It is easy to prove that all eigenvalues of (2.1) are positive. In addition, we recall the Weyl asymptotics law for the Neumann system (see, for instance, [20]):

**Lemma 2.2** (Weyl asymptotic). *Let  $\Omega$  be a Lipschitz domain and let  $N(\gamma)$  be the number of eigenvalues (counting repetitions) of (2.1) that belong to  $(-\infty, \gamma]$ . We have:*

$$\lim_{\gamma \rightarrow \infty} \frac{N(\gamma)}{\gamma^{d/2}} = \frac{\text{Vol}(B(0, 1)) \text{Vol}(\Omega)}{(2\pi)^d},$$

for  $d$  the dimension of  $\Omega$ . In particular, there is  $C > 0$  such that for all  $\gamma \in \mathbb{R}$ :

$$N(\gamma) \leq C(1 + |\gamma|^{d/2}). \quad (2.2)$$

To continue with, we can compare  $\tilde{\lambda}_m^\varepsilon$  (see Definition 1.5) with  $\tilde{\lambda}_m^0$  by using the min-max variational principle and obtain the following result:

**Lemma 2.3.** *Let  $\Omega$  be a Lipschitz domain,  $\varepsilon > 0$  and assume that  $(\Gamma, a)$  satisfies (1.9). Then, for all  $m \in \mathbb{N}$  we have the following estimate:*

$$\tilde{\lambda}_m^\varepsilon \geq \tilde{\lambda}_m^0.$$

Let us now study the solutions of (1.2):

**Lemma 2.4.** *Let  $\Omega$  be a Lipschitz domain,  $\varepsilon > 0$  and assume that  $(\Gamma, a)$  satisfies (1.9). Then, for all  $\varphi^T \in L^2(\Omega)$  the solution of (1.2) satisfies in  $C^0([0, T]; L^2(\Omega))$ :*

$$\varphi(t, x) = \sum_{m \in \mathbb{N}} \int_{\Omega} \varphi^T(z) e^{(2\varepsilon)^{-1} z_1} v_m^\varepsilon(z) dz v_m^\varepsilon(x) e^{-(2\varepsilon)^{-1} x_1} \exp \left[ - \left( \varepsilon \tilde{\lambda}_m^\varepsilon + \frac{1}{4\varepsilon} \right) (T - t) \right]. \quad (2.3)$$

In particular, the series in the right-hand side of (2.3) is absolutely convergent in  $L^2(\Omega)$  for all  $t < T$ . Moreover, for all  $T_0 > 0$  and  $\delta > 0$  there is  $C > 0$  which does not depend on  $\Gamma, a$  nor  $\varepsilon$  such that for all  $T \geq T_0$ ,  $\varphi^T \in L^2(\Omega)$  and  $t \in [0, T - T_0]$  we have the estimate:

$$\|\varphi(t, \cdot)\|_{L^2(\Omega)} \leq C \|\varphi^T\|_{L^2(\Omega)} \exp \left( \frac{2 \text{diam}_{x_1}(\Omega) + \delta - (T - t)}{4\varepsilon} \right). \quad (2.4)$$

*Proof.* We can easily prove (2.3) for  $t = T$  using that  $v_m^\varepsilon$  is an orthonormal basis in  $L^2(\Omega)$ . Indeed, if  $\varphi^T \in L^2(\Omega)$  we have that:

$$\varphi^T(x) = (\varphi^T(x)e^{(2\varepsilon)^{-1}x_1})e^{-(2\varepsilon)^{-1}x_1} = \sum_{m \in \mathbb{N}} \int_{\Omega} \varphi^T(z)e^{(2\varepsilon)^{-1}z_1} v_m^\varepsilon(z) dz v_m^\varepsilon(x) e^{-(2\varepsilon)^{-1}x_1}.$$

We prove (2.3) and (2.4) simultaneously for all  $t \in [0, T)$ . We obtain from Remark 1.2 that (2.3) is true in  $\text{span}\{u_m\}$ . Since the left hand side of (2.3) belongs to  $C^0([0, T]; L^2(\Omega))$  (see [12] and [21, Chapter III]) and since  $\text{span}\{u_m\}_{m \in \mathbb{N}}$  is dense in  $L^2(\Omega)$  (see Remark 1.6), it is enough to prove that the right-hand side is an endomorphism on  $L^2(\Omega)$  viewed as a map of pre-image  $\varphi^T$  and time  $t \in [0, T)$  fixed. For that purpose, using the triangle inequality and that  $\|v_m\|_{L^2(\Omega)} = 1$ , we get the following:

$$\begin{aligned} & \left\| \sum_{m \in \mathbb{N}} \int_{\Omega} (\varphi^T(z)e^{(2\varepsilon)^{-1}z_1}) v_m^\varepsilon(z) dz v_m^\varepsilon(x) e^{-(2\varepsilon)^{-1}x_1} \exp \left[ - \left( \varepsilon \tilde{\lambda}_m^\varepsilon + \frac{1}{4\varepsilon} \right) (T-t) \right] \right\|_{L^2(\Omega, dx)} \\ & \leq \|\varphi^T\|_{L^2(\Omega)} \exp \left( \frac{2 \text{diam}_{x_1}(\Omega) - (T-t)}{4\varepsilon} \right) \sum_{m \in \mathbb{N}} e^{-\varepsilon \tilde{\lambda}_m^\varepsilon (T-t)}. \end{aligned} \quad (2.5)$$

Thus, the only thing left is to bound  $\sum_{m \in \mathbb{N}} e^{-\varepsilon \tilde{\lambda}_m^\varepsilon (T-t)}$ . We obtain from Lemmas 2.3 and 2.2 and the positivity of the eigenvalues of (2.1) the estimate:

$$\begin{aligned} \sum_{m \in \mathbb{N}} e^{-\varepsilon \tilde{\lambda}_m^\varepsilon (T-t)} & \leq \sum_{m \in \mathbb{N}} e^{-\varepsilon \tilde{\lambda}_m^0 (T-t)} \leq \sum_{m \geq 1} N(m) e^{-\varepsilon(m-1)(T-t)} \\ & \leq C \left( 1 + \sum_{m \geq 2} (1 + m^{d/2}) e^{-\varepsilon(m-1)(T-t)} \right) \leq C \left( 1 + \int_0^{+\infty} (1 + (z+2)^{d/2}) e^{-\varepsilon z(T-t)} dz \right) \\ & \leq C \left( 1 + \int_0^{+\infty} (1 + z^{d/2}) e^{-\varepsilon z(T-t)} dz \right) \leq C \left( 1 + \frac{1}{\varepsilon(T-t)} + \frac{1}{\varepsilon^{(d+2)/2} (T-t)^{(d+2)/2}} \right). \end{aligned} \quad (2.6)$$

So, from (2.6),  $T-t \geq T_0$ , and  $z^{-k/2} e^{-z^{-1}} \in L^\infty(\mathbb{R}^+)$  for all  $k \in \mathbb{R}^+$ , we obtain that:

$$\sum_{m \in \mathbb{N}} e^{-\varepsilon \tilde{\lambda}_m^\varepsilon (T-t)} \leq C e^{\delta \varepsilon^{-1}}. \quad (2.7)$$

Consequently, we end the proof by combining (2.5) and (2.7).  $\square$

*Remark 2.5.* Lemma 2.4 can be generalized to the case in which  $\tilde{\lambda}_0^\varepsilon \geq 0$  and in which we have some polynomial bound for  $|\{i : \tilde{\lambda}_i^\varepsilon \leq m\}|$  that depends on  $m$  and uniform for all  $\varepsilon \in (0, \varepsilon_0)$ .

## 2.2 A Carleman inequality for the solutions of (1.5)

We prove in this section a Carleman inequality for (1.5) when  $\Omega$  is  $C^2$ . First of all, we consider  $\eta$  a  $C^2(\bar{\Omega})$  function satisfying:

$$\eta = 0 \text{ on } \partial\Omega, \quad \eta \geq 0 \text{ on } \Omega, \quad \|\eta\|_\infty = 1, \quad \inf_{\Omega \setminus \omega_0} |\nabla \eta| = \delta > 0, \quad (2.8)$$

for  $\omega_0 \subset\subset \omega$  an open non-empty set. The existence of such a function if  $\Omega$  is a  $C^2$  domain is proved, for instance, in [14]. With that auxiliary function in mind, we define, for some  $\tilde{T} > 0$ , the following weights in  $\tilde{Q} := (0, \tilde{T}) \times \Omega$ :

$$\alpha_\pm(t, x) := \frac{e^{8\lambda} - e^{\lambda(6 \pm \eta(x))}}{t(\tilde{T} - t)}, \quad \xi_\pm(t, x) := \frac{e^{\lambda(6 \pm \eta(x))}}{t(\tilde{T} - t)}, \quad (2.9)$$

Weights of this kind first appeared in [14], but this version is borrowed from [13] with the adaptations for equations in which we do not have Dirichlet boundary conditions proposed in [12]. We remark that we have the usual bounds:

$$|\partial_{x_i} \alpha_\pm| = |\partial_{x_i} \xi_\pm| \leq C\lambda \xi_\pm, \quad |\partial_t \alpha_\pm| + |\partial_t \xi_\pm| \leq C\tilde{T} \xi_\pm^2, \quad |\partial_{tt} \alpha_\pm| \leq C\xi_\pm^2(1 + \tilde{T}^2 \xi_\pm). \quad (2.10)$$

We prove in this section the following result:

**Proposition 2.6.** *Let  $\Omega$  be a  $C^2$  domain,  $\omega \subset \Omega$  be a subdomain and assume that  $(\Gamma, a)$  satisfies (1.9). Then, there is  $C > 0$  such that if  $\lambda \geq C$ ,  $\tau \geq C(\tilde{T} + \tilde{T}^2)\varepsilon^{-1}$  and  $\phi^{\tilde{T}} \in L^2(\Omega)$  we have that:*

$$\begin{aligned} \tau^3 \lambda^4 \iint_{\tilde{Q}} e^{-2\tau\alpha_+} \xi_+^3 |\phi|^2 + \tau \lambda^2 \iint_{\tilde{Q}} e^{-2\tau\alpha_+} \xi_+^3 |\nabla \phi|^2 + \tau^2 \lambda^2 \iint_{\tilde{\Sigma}_N} e^{-2\tau\alpha_+} \xi_+^2 \left(a + \frac{n_1}{2\varepsilon}\right) |\phi|^2 \\ \leq C\tau^3 \lambda^4 \iint_{\tilde{Q}_\omega} e^{-2\tau\alpha_+} \xi_+^3 |\phi|^2 \end{aligned} \quad (2.11)$$

for  $\phi$  the solution of (1.5) with  $\tilde{T}$  instead of  $T$ ,  $\tilde{\Sigma}_N := (0, \tilde{T}) \times \Gamma$  and  $\tilde{Q}_\omega := (0, \tilde{T}) \times \omega$ .

In the statement of Proposition 2.6 and in its proof we omit the infinitesimals  $dt$  and  $dx$  as they can be deduced by looking at the integration domain. In the proof of Proposition 2.6 we denote by  $O(\mathcal{G}(\varepsilon, \tau, \lambda, \psi))$  a generic function for which there is  $C > 0$  depending only on  $\Omega$ ,  $\omega$  and  $\omega_0$  and which satisfies:

$$|O(\mathcal{G}(\varepsilon, \tau, \lambda, \psi))| \leq C\mathcal{G}(\varepsilon, \tau, \lambda, \psi),$$

if  $\varepsilon \in (0, \varepsilon_0)$ ,  $\lambda \geq C$ ,  $\tau \geq C(\tilde{T} + \tilde{T}^2)\varepsilon^{-1}$  and if  $\psi$  is a regular function. In that definition  $\mathcal{G}$  is a positive operator.

For proving Proposition 2.6 we adapt the proofs of [19] and [12]. In particular, many of the operations here are inspired in [19, Proposition 1], though we use the technique proposed in [12] to treat the boundary terms. One new difficulty that we have to face is that the solutions of (1.5) may not belong to  $L^2(0, T; H^2(\Omega))$  (indeed, we can just guarantee that  $\partial_n \phi = -\left(a(x, \varepsilon) + \frac{n_1}{2\varepsilon}\right) \phi \in L^2(\tilde{\Sigma}_N)$ ), but as  $\Omega$  is  $C^2$  we can handle it with a density argument.

*Proof.* As in [12], we consider the following change of variables:

$$\psi^\pm := e^{-\tau\alpha_\pm} \phi. \quad (2.12)$$

Since  $\eta = 0$  on  $\partial\Omega$  we have the following equalities on  $\tilde{\Sigma} := (0, \tilde{T}) \times \partial\Omega$ :

$$\xi_+ = \xi_-, \quad \alpha_+ = \alpha_-, \quad \psi^+ = \psi^-, \quad \psi_t^+ = \psi_t^-, \quad \nabla_{\text{tg}} \psi^+ = \nabla_{\text{tg}} \psi^-, \quad (2.13)$$

for  $\nabla_{\text{tg}} \psi := \nabla \psi - (\partial_n \psi)n$ . In particular, we can omit the sign when we are working on  $\tilde{\Sigma}$  to improve the clarity of the proof.

We next consider the equalities:

$$L_1^\pm \psi^\pm + L_2^\pm \psi^\pm = L_3^\pm \psi^\pm, \quad (2.14)$$

for:

$$\begin{cases} L_1^\pm \psi^\pm := -2\varepsilon\tau\lambda^2 |\nabla\eta|^2 \xi_\pm \psi^\pm \mp 2\varepsilon\tau\lambda \xi_\pm \nabla\eta \cdot \nabla\psi^\pm + \psi_t^\pm, \\ L_2^\pm \psi^\pm := \varepsilon\tau^2\lambda^2 |\nabla\eta|^2 \xi_\pm^2 \psi^\pm + \varepsilon\Delta\psi^\pm + \tau\partial_t(\alpha_\pm)\psi^\pm + \frac{1}{4\varepsilon}\psi^\pm, \\ L_3^\pm \psi^\pm := \pm\varepsilon\tau\lambda\Delta\eta \xi_\pm \psi^\pm - \varepsilon\tau\lambda^2 |\nabla\eta|^2 \xi_\pm \psi^\pm. \end{cases} \quad (2.15)$$

As usual, we denote  $(L_i^\pm \psi)_j$  the  $j$ -th term of  $L_i^\pm \psi$ . Moreover, we are going to calculate the product  $(L_1^\pm \psi^\pm, L_2^\pm \psi^\pm)_{L^2(\tilde{Q})}$ . To do so, we first compute it for generic functions  $\phi \in C^\infty(\tilde{Q})$ , as we then use a density argument.

To begin with, we have that, for  $\lambda \geq C$ ,  $\tau \geq C\tilde{T}^2$  (see (2.8) for the definition of  $\delta$ ):

$$\begin{aligned} & ((L_1^\pm \psi^\pm)_1 + (L_1^\pm \psi^\pm)_2, (L_2^\pm \psi^\pm)_1)_{L^2(\tilde{Q})} \\ &= \varepsilon^2\tau^3\lambda^4 \iint_{\tilde{Q}} |\nabla\eta|^4 \xi_\pm^3 |\psi^\pm|^2 + O\left(\varepsilon^2\tau^2\lambda^3 \iint_{\tilde{Q}} \xi_\pm^3 |\psi^\pm|^2\right) \mp \varepsilon^2\tau^3\lambda^3 \iint_{\tilde{\Sigma}} |\nabla\eta|^2 \xi^3 \partial_n \eta |\psi|^2 \\ &\geq \frac{3\delta^4}{4} \varepsilon^2\tau^3\lambda^4 \iint_{\tilde{Q}} \xi_\pm^3 |\psi^\pm|^2 - \delta^4 \varepsilon^2\tau^3\lambda^4 \iint_{\tilde{Q}_{\omega_0}} \xi_\pm^3 |\psi^\pm|^2 \mp \varepsilon^2\tau^3\lambda^3 \iint_{\tilde{\Sigma}} |\nabla\eta|^2 \xi^3 \partial_n \eta |\psi|^2. \end{aligned} \quad (2.16)$$

As for the third term in  $L_1^\pm \psi^\pm$ , we can integrate by parts in time and get, taking into account (2.10)<sub>2</sub>:

$$((L_1^\pm \psi^\pm)_3, (L_2^\pm \psi^\pm)_1)_{L^2(\tilde{Q})} = O\left(\varepsilon\tilde{T}\tau^2\lambda^2 \iint_{\tilde{Q}} \xi_\pm^3 |\psi^\pm|^2\right). \quad (2.17)$$

Summing up, thanks to (2.16), (2.17) and (2.8)<sub>4</sub>, we obtain for  $\lambda \geq C$  and  $\tau \geq C(\tilde{T} + \tilde{T}^2)\varepsilon^{-1}$  the estimate:

$$\sum_{i \in \{+, -\}} (L_1^i \psi^i, (L_2^i \psi^i)_1)_{L^2(\tilde{Q})} \geq \sum_{i \in \{+, -\}} \frac{\delta^4}{2} \varepsilon^2 \tau^3 \lambda^4 \iint_{\tilde{Q}} \xi_i^3 |\psi^i|^2 - \delta^4 \varepsilon^2 \tau^3 \lambda^4 \iint_{\tilde{Q}_{\omega_0}} \xi_i^3 |\psi^i|^2. \quad (2.18)$$

To continue with we find, integrating by parts and with the Cauchy-Schwarz inequality, that:

$$\begin{aligned} ((L_1^\pm \psi^\pm)_1, (L_2^\pm \psi^\pm)_2)_{L^2(\tilde{Q})} &= 2\varepsilon^2 \tau \lambda^2 \iint_{\tilde{Q}} |\nabla \eta|^2 \xi_\pm |\nabla \psi^\pm|^2 - 2\varepsilon^2 \tau \lambda^2 \iint_{\tilde{\Sigma}} |\nabla \eta|^2 \xi (\partial_n \psi^\pm) \psi \\ &\quad + O\left(\varepsilon^2 \tau^2 \lambda^4 \iint_{\tilde{Q}} \xi_\pm^2 |\psi^\pm|^2 + \varepsilon^2 \lambda^2 \iint_{\tilde{Q}} |\nabla \psi^\pm|^2\right). \end{aligned} \quad (2.19)$$

Next, considering (2.8)<sub>1</sub>, we obtain integrating by parts that:

$$\begin{aligned} ((L_1^\pm \psi^\pm)_2, (L_2^\pm \psi^\pm)_2)_{L^2(\tilde{Q})} &= \mp 2\varepsilon^2 \tau \lambda \iint_{\tilde{\Sigma}} \partial_n \eta \xi |\partial_n \psi^\pm|^2 \\ &\quad + O\left(\varepsilon^2 \tau \lambda \iint_{\tilde{Q}} \xi_\pm |\nabla \psi^\pm|^2\right) + 2\varepsilon^2 \tau \lambda^2 \iint_{\tilde{Q}} \xi_\pm |\nabla \eta \cdot \nabla \psi^\pm|^2 \pm \varepsilon^2 \tau \lambda \iint_{\tilde{Q}} \xi_\pm \nabla \eta \cdot \nabla |\nabla \psi^\pm|^2 \\ &= \mp 2\varepsilon^2 \tau \lambda \iint_{\tilde{\Sigma}} \partial_n \eta \xi |\partial_n \psi^\pm|^2 + O\left(\varepsilon^2 \tau \lambda \iint_{\tilde{Q}} \xi_\pm |\nabla \psi^\pm|^2\right) + 2\varepsilon^2 \tau \lambda^2 \iint_{\tilde{Q}} \xi_\pm |\nabla \eta \cdot \nabla \psi^\pm|^2 \\ &\quad - \varepsilon^2 \tau \lambda^2 \iint_{\tilde{Q}} |\nabla \eta|^2 \xi_\pm |\nabla \psi^\pm|^2 + O\left(\varepsilon^2 \tau \lambda \iint_{\tilde{Q}} \xi_\pm |\nabla \psi^\pm|^2\right) \pm \varepsilon^2 \tau \lambda \iint_{\tilde{\Sigma}} \partial_n \eta \xi (|\nabla_{\text{tg}} \psi|^2 + |\partial_n \psi^\pm|^2). \end{aligned} \quad (2.20)$$

To continue with, we have that:

$$((L_1^\pm \psi^\pm)_3, (L_2^\pm \psi^\pm)_2)_{L^2(\tilde{Q})} = \varepsilon \iint_{\tilde{\Sigma}} \psi_t \partial_n \psi^\pm, \quad (2.21)$$

as  $\psi(0, \cdot) = \psi(\tilde{T}, \cdot) = 0$ .

Let us compute the boundary terms involving  $\partial_n \psi^\pm$  that appear in (2.19)-(2.21). On the one hand, from (2.12) we find that:

$$\begin{aligned} -2\varepsilon^2 \tau \lambda^2 \iint_{\tilde{\Sigma}} |\nabla \eta|^2 \xi (\partial_n \psi^\pm) \psi &= \mp 2\varepsilon^2 \tau^2 \lambda^3 \iint_{\tilde{\Sigma}} |\nabla \eta|^2 \partial_n \eta \xi^2 |\psi|^2 \\ &\quad - 2\varepsilon^2 \tau \lambda^2 \int_0^T \langle \partial_n \phi, |\nabla \eta|^2 \xi e^{-2\tau\alpha} \phi \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}. \end{aligned} \quad (2.22)$$

In addition, we obtain from (2.8)<sub>1</sub> that:

$$\begin{aligned} \mp \varepsilon^2 \tau \lambda \iint_{\tilde{\Sigma}} \partial_n \eta \xi |\partial_n \psi^\pm|^2 &= -2\varepsilon^2 \tau^2 \lambda^2 \int_0^T \langle \partial_n \phi, e^{-2\tau\alpha} |\nabla \eta|^2 \xi^2 \phi \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} \\ &\quad \mp \varepsilon^2 \tau^3 \lambda^3 \iint_{\tilde{\Sigma}} (\partial_n \eta)^3 \xi^3 |\psi|^2 \mp \varepsilon^2 \tau \lambda \iint_{\tilde{\Sigma}} \partial_n \eta \xi e^{-2\tau\alpha} |\partial_n \phi|^2. \end{aligned} \quad (2.23)$$

Finally, we have that:

$$\varepsilon \iint_{\tilde{\Sigma}} \psi_t \partial_n \psi^\pm = \pm \varepsilon \tau \lambda \iint_{\tilde{\Sigma}} \partial_n \eta \xi \psi \psi_t + \varepsilon \int_0^T \langle \partial_n \phi, e^{-\tau \alpha} \psi_t \rangle_{H^{-1/2}(\partial \Omega), H^{1/2}(\partial \Omega)}. \quad (2.24)$$

Summing up, if we consider (2.19)-(2.24) we get that:

$$\begin{aligned} \sum_{i \in \{+, -\}} (L_1^i \psi^i, (L_2^i \psi^i)_2)_{L^2(\tilde{Q})} &\geq \sum_{i \in \{+, -\}} O \left( \varepsilon^2 \tau^2 \lambda^4 \iint_{\tilde{Q}} \xi_i^2 |\psi^i|^2 \right) \\ &\quad + \sum_{i \in \{+, -\}} \frac{\delta^2}{2} \varepsilon^2 \tau \lambda^2 \iint_{\tilde{Q}} \xi_i |\nabla \psi^i|^2 - \delta^2 \varepsilon^2 \tau \lambda^2 \iint_{\tilde{Q}_{\omega_0}} \xi_i |\nabla \psi^i|^2 \\ -4\varepsilon^2 \lambda^2 \int_0^T \langle \partial_n \phi, e^{-2\tau \alpha} |\nabla \eta|^2 (\tau \xi + \tau^2 \xi^2) \phi \rangle_{H^{-1/2}(\partial \Omega), H^{1/2}(\partial \Omega)} &+ 2\varepsilon \int_0^T \langle \partial_n \phi, e^{-\tau \alpha} \psi_t \rangle_{H^{-1/2}(\partial \Omega), H^{1/2}(\partial \Omega)}. \end{aligned} \quad (2.25)$$

Next, we obtain from (2.10)<sub>2</sub> that:

$$((L_1^\pm \psi^\pm)_1, (L_2^\pm \psi^\pm)_3)_{L^2(\tilde{Q})} = O \left( \varepsilon \tau^2 \lambda^2 \tilde{T} \iint_{\tilde{Q}} \xi_\pm^3 |\psi^\pm|^2 \right). \quad (2.26)$$

In addition, by integrating by parts we have that:

$$((L_1^\pm \psi^\pm)_2, (L_2^\pm \psi^\pm)_3)_{L^2(\tilde{Q})} = O \left( \varepsilon \tau^2 \lambda^2 \tilde{T} \iint_{\tilde{Q}} \xi_\pm^3 |\psi^\pm|^2 \right) \mp \varepsilon \tau^2 \lambda \iint_{\tilde{\Sigma}} \xi \partial_t \alpha \partial_n \eta |\psi|^2. \quad (2.27)$$

Finally, from (2.10)<sub>3</sub> we find that:

$$((L_1^\pm \psi^\pm)_3, (L_2^\pm \psi^\pm)_3)_{L^2(\tilde{Q})} = -\frac{\tau}{2} \iint_{\tilde{Q}} \partial_{tt}(\alpha_\pm) |\psi^\pm|^2 = O \left( \tau \iint_{\tilde{Q}} \xi_\pm^2 (1 + \tilde{T}^2 \xi_\pm) |\psi^\pm|^2 \right). \quad (2.28)$$

Summing up, combining (2.26)-(2.28) we obtain that:

$$\sum_{i \in \{+, -\}} (L_1^i \psi^i, (L_2^i \psi^i)_3)_{L^2(\tilde{Q})} = O \left( \varepsilon^2 \tau^3 \lambda^2 \iint_{\tilde{Q}} \xi_\pm^3 |\psi^\pm|^2 \right). \quad (2.29)$$

To continue with, we have that:

$$((L_1^\pm \psi^\pm)_1, (L_2^\pm \psi^\pm)_4)_{L^2(\tilde{Q})} = O \left( \tau \lambda^2 \iint_{\tilde{Q}} \xi_\pm |\psi^\pm|^2 \right). \quad (2.30)$$

Next, we have that, after integrating by parts:

$$((L_1^\pm \psi^\pm)_2, (L_2^\pm \psi^\pm)_4)_{L^2(\tilde{Q})} = O \left( \tau \lambda^2 \iint_{\tilde{Q}} \xi_\pm |\psi^\pm|^2 \right) \pm \frac{\tau \lambda}{4} \iint_{\tilde{\Sigma}} \xi \partial_n \eta |\psi|^2. \quad (2.31)$$

Finally, we obtain from  $\psi^\pm(0, \cdot) = \psi^\pm(T, \cdot) = 0$  that:

$$((L_2^\pm \psi^\pm)_3, (L_2^\pm \psi^\pm)_4)_{L^2(\tilde{Q})} = 0. \quad (2.32)$$

Summing up the results obtained in (2.30)-(2.32), we get that:

$$\sum_{i \in \{+, -\}} (L_1^i \psi^i, (L_2^i \psi^i)_4)_{L^2(\tilde{Q})} = O\left(\tau \lambda^2 \iint_{\tilde{Q}} \xi_\pm |\psi^\pm|^2\right). \quad (2.33)$$

So, if we add (2.18), (2.25), (2.29) and (2.33), we get, after absorptions, for  $\lambda \geq C$  and  $\tau \geq C(\tilde{T} + \tilde{T}^2)\varepsilon^{-1}$ :

$$\begin{aligned} & 2 \sum_{i \in \{+, -\}} (L_1^i \psi^i, L_2^i \psi^i) + 2\delta^4 \varepsilon^2 \tau^3 \lambda^4 \iint_{\tilde{Q}_{\omega_0}} \xi_i^3 |\psi^i|^2 + 2\delta^2 \varepsilon^2 \tau \lambda^2 \iint_{\tilde{Q}_{\omega_0}} \xi_i |\nabla \psi^i|^2 \\ & \geq \sum_{i \in \{+, -\}} \frac{\delta^4}{4} \varepsilon^2 \tau^3 \lambda^4 \iint_{\tilde{Q}} \xi_i^3 |\psi^i|^2 + \frac{\delta^2}{2} \varepsilon^2 \tau \lambda^2 \iint_{\tilde{Q}} \xi_i |\nabla \psi^i|^2 \\ & - 4\varepsilon^2 \lambda^2 \int_0^T \langle \partial_n \phi, e^{-2\tau\alpha} |\nabla \eta|^2 (\tau\xi + \tau^2 \xi^2) \phi \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} + 2\varepsilon \int_0^T \langle \partial_n \phi, e^{-\tau\alpha} \psi_t \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}. \end{aligned} \quad (2.34)$$

Using the density of  $C^\infty(\bar{Q})$  in  $H^1(0, T; D(\Delta))$  we obtain that (2.34) is true for all solutions of (1.5) such that  $\phi^{\tilde{T}} \in \mathcal{D}(\Omega)$ . So, from now on  $\phi$  denotes any solution of (1.5) such that  $\phi^{\tilde{T}} \in \mathcal{D}(\Omega)$ .

Considering (1.5)<sub>2</sub>, (1.5)<sub>3</sub>, that  $a$  is independent of the time variable and that  $\psi^\pm(0, \cdot) = \psi^\pm(\tilde{T}, \cdot) = 0$  we find that:

$$\begin{aligned} & -4\varepsilon^2 \lambda^2 \int_0^T \langle \partial_n \phi, e^{-2\tau\alpha} |\nabla \eta|^2 (\tau\xi + \tau^2 \xi^2) \phi \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} + 2\varepsilon \int_0^T \langle \partial_n \phi, e^{-\tau\alpha} \psi_t \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} \\ & = 4\varepsilon^2 \lambda^2 \iint_{\tilde{\Sigma}_N} |\nabla \eta|^2 (\tau\xi + \tau^2 \xi^2) \left(a + \frac{n_1}{2\varepsilon}\right) |\psi|^2 + 2\varepsilon \iint_{\tilde{\Sigma}_N} \left(a + \frac{n_1}{2\varepsilon}\right) \psi \psi_t \\ & = 4\varepsilon^2 \lambda^2 \iint_{\tilde{\Sigma}_N} |\nabla \eta|^2 (\tau\xi + \tau^2 \xi^2) \left(a + \frac{n_1}{2\varepsilon}\right) |\psi|^2. \end{aligned} \quad (2.35)$$

Consequently, from (2.34) and (2.35) we obtain for all solutions of (1.5) such that  $\phi^{\tilde{T}} \in \mathcal{D}(\Omega)$  the estimate:

$$\begin{aligned} & 2 \sum_{i \in \{+, -\}} (L_1^i \psi^i, L_2^i \psi^i) + 2\delta^4 \varepsilon^2 \tau^3 \lambda^4 \iint_{\tilde{Q}_{\omega_0}} \xi_i^3 |\psi^i|^2 + 2\delta^2 \varepsilon^2 \tau \lambda^2 \iint_{\tilde{Q}_{\omega_0}} \xi_i |\nabla \psi^i|^2 \\ & \geq \sum_{i \in \{+, -\}} \frac{\delta^4}{2} \varepsilon^2 \tau^3 \lambda^4 \iint_{\tilde{Q}} \xi_i^3 |\psi^i|^2 + \frac{\delta^2}{2} \varepsilon^2 \tau \lambda^2 \iint_{\tilde{Q}} \xi_i |\nabla \psi^i|^2 \\ & \quad + 4\varepsilon^2 \lambda^2 \iint_{\tilde{\Sigma}_N} |\nabla \eta|^2 (\tau\xi + \tau^2 \xi^2) \left(a + \frac{n_1}{2\varepsilon}\right) |\psi|^2. \end{aligned} \quad (2.36)$$



We recall that  $\inf_{\partial\Omega} |\nabla\eta| > 0$  thanks to (2.8)<sub>4</sub>. Thus, using (1.9) it is classical to obtain (2.11) from (2.36) (see, for instance, [14] and [12]) for all solutions of (1.5) such that  $\phi^{\tilde{T}} \in \mathcal{D}(\Omega)$ . Afterwards, we get (2.11) for all solutions of (1.5) such that  $\phi^{\tilde{T}} \in L^2(\Omega)$  by density.  $\square$

### 2.3 Conclusion of the proof of Theorem 1.7

Let  $T \geq 1$ . Considering Proposition 2.6 for  $\tilde{T} = 1$  and Remark 1.1 we obtain for all  $\varphi^T \in L^2(\Omega)$ ,  $\lambda \geq C$  and  $\tau \geq C\varepsilon^{-1}$  the estimate:

$$\begin{aligned} \tau^3 \lambda^4 \int_0^1 \int_{\Omega} e^{-2\tau\alpha_+(t,x) - \varepsilon^{-1}x_1} \xi_+^3(t,x) |\varphi(T-1+t,x)|^2 dx dt \\ \leq C \tau^3 \lambda^4 \int_0^1 \int_{\Omega} e^{-2\tau\alpha_+(t,x) - \varepsilon^{-1}x_1} \xi_+^3(t,x) |\varphi(T-1+t,x)|^2 dx dt, \end{aligned} \quad (2.37)$$

for  $\varphi$  the solution of (1.2). So, fixing  $\lambda$  large enough, and  $\tau = \tau_0\varepsilon^{-1}$  for  $\tau_0$  large enough, we have that (2.37) implies that:

$$\|\varphi\|_{L^2((T-2/3, T-1/3) \times \Omega)} \leq C e^{C\varepsilon^{-1}} \|\varphi\|_{L^2((T-1, T) \times \omega)} \leq C e^{C\varepsilon^{-1}} \|\varphi\|_{L^2(Q_\omega)}. \quad (2.38)$$

Moreover, from Lemma 2.4 taking  $T_0 = \delta = 1$  we find a constant  $C > 0$  such that for all  $t' \in (T-2/3, T-1/3)$  we have that:

$$\|\varphi(0, \cdot)\|_{L^2(\Omega)} \leq C \exp\left[\frac{2 \operatorname{diam}_{x_1}(\Omega) + 1 - t'}{4\varepsilon}\right] \|\varphi(t', \cdot)\|_{L^2(\Omega)} \leq C \exp\left[\frac{C-T}{4\varepsilon}\right] \|\varphi(t', \cdot)\|_{L^2(\Omega)}. \quad (2.39)$$

So, combining (2.38) and (2.39), we get that:

$$\|\varphi(0, \cdot)\|_{L^2(\Omega)} \leq C \exp\left[\frac{C-T}{4\varepsilon}\right] \|\varphi\|_{L^2((T-2/3, T-1/3) \times \Omega)} \leq C \exp\left[\frac{C-T}{4\varepsilon}\right] \|\varphi\|_{L^2(Q_\omega)}. \quad (2.40)$$

In particular, if  $T$  is sufficiently large, from (1.3) and (2.40) we obtain (1.10).

## 3 Some control problems in which we have (1.11)

In this section we study some subcases of (1.11). In particular, in Section 3.1 we prove Theorem 1.8, in Section 3.2 we prove Theorem 1.9, and in Section 3.3 we prove Theorems 1.10, 1.11 and 1.12. We first prove, though, that as stated in Remark 1.14, if  $a \geq 0$  we have that  $\lambda_0^\varepsilon \geq 0$ :

**Proposition 3.1.** *Let  $\Omega$  be a Lipschitz domain and  $a \geq 0$ . Then, we have that:*

$$\lambda_0^\varepsilon \geq 0. \quad (3.1)$$

*Proof.* Let  $v \in H^1(\Omega)$  such that  $\|v\|_{L^2(\Omega)} = 1$  and such that  $v = 0$  on  $\Gamma^*$ . We find from the Green formula and an appropriate Cauchy-Schwarz inequality that:

$$\int_{\Gamma} \frac{n_1}{2\varepsilon} |v|^2 dx = \int_{\partial\Omega} \frac{n_1}{2\varepsilon} |v|^2 dx = \int_{\Omega} \frac{v \partial_{x_1} v}{\varepsilon} dx \geq - \int_{\Omega} |\partial_{x_1} v|^2 dx - \frac{1}{4\varepsilon^2} \int_{\Omega} |v|^2 dx.$$

Thus, if we have  $a \geq 0$  and  $v \in H^1(\Omega)$  such that  $\|v\|_{L^2(\Omega)} = 1$  and such that  $v = 0$  on  $\Gamma^*$ , we find that:

$$\int_{\Omega} |\nabla v|^2 dx + \int_{\Gamma} \left( a + \frac{n_1}{2\varepsilon} \right) |v|^2 dx \geq \int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} |\partial_{x_1} v|^2 dx - \frac{1}{4\varepsilon^2} \geq -\frac{1}{4\varepsilon^2}. \quad (3.2)$$

Finally, we get (3.1) from (1.8), (3.2) and Remark 1.2.  $\square$

*Remark 3.2.* In order to have an equality in (3.1), by a careful reading of Proposition 3.1 we realize that we need that  $a = 0$ , that  $v$  just depends on  $x_1$  and that  $\partial_{x_1} v = -(2\varepsilon)^{-1}v$ ; i.e. we need that  $v = e^{-(2\varepsilon)^{-1}x_1}$  (and thus  $\Gamma = \partial\Omega$ ). This situation corresponds to the case of pure Neumann boundary conditions, which is studied in Section 3.1. In any other situation in which  $a \geq 0$  the inequality (3.1) is strict.

### 3.1 Proof of Theorem 1.8

In order to prove (1.15) we consider that the adjoint system of (1.12) is given by (see (1.2)):

$$\begin{cases} -\varphi_t - \varepsilon \Delta \varphi - \partial_{x_1} \varphi = 0, & \text{in } Q, \\ \varepsilon \partial_n \varphi + n_1 \varphi = 0, & \text{on } \Sigma, \\ \varphi(T, \cdot) = \varphi^T, & \text{on } \Omega. \end{cases} \quad (3.3)$$

In particular, we use (1.3) for  $\bar{\varphi}(x) = e^{-x_1 \varepsilon^{-1}}$ , which is a (steady) solution of (3.3) (see Remarks 3.2 and 1.2). On the one hand, we remark that (see (1.13) for the notation):

$$\|\bar{\varphi}(0, \cdot)\|_{L^2(\Omega)} \geq |\pi_1^{-1}(p_l, p_l + h/2) \cap \Omega|^{1/2} \exp\left(\frac{-p_l - h/2}{\varepsilon}\right). \quad (3.4)$$

Because  $\Omega$  is an open set and because of (1.13) we have that:

$$|\pi_1^{-1}(p_l, p_l + h/2) \cap \Omega| > 0.$$

On the other hand, (1.14) implies the estimate:

$$\|\bar{\varphi}\|_{L^2(Q_\omega)} \leq T^{1/2} |\omega|^{1/2} \|e^{-x_1 \varepsilon^{-1}}\|_{L^\infty(\pi_1^{-1}(p_l + h, p_r) \cap \omega)} \leq T^{1/2} |\omega|^{1/2} \exp\left(\frac{-p_l - h}{\varepsilon}\right). \quad (3.5)$$

Consequently, combining (1.3), (3.4) and (3.5) we find that:

$$K(\Omega, \omega, T, \varepsilon) \geq \frac{\|\bar{\varphi}(0, \cdot)\|_{L^2(\Omega)}}{\|\bar{\varphi}\|_{L^2(Q_\omega)}} \geq \frac{|\pi_1^{-1}(p_l, p_l + h/2) \cap \Omega|^{1/2}}{T^{1/2} |\omega|^{1/2}} \exp\left(\frac{h}{2\varepsilon}\right),$$

which implies (1.15).

### 3.2 Proof of Theorem 1.9

We first remark that the adjoint system of (1.16) is given by (see (1.2)):

$$\begin{cases} -\varphi_t - \varepsilon \partial_{xx} \varphi - \partial_x \varphi = 0, & \text{in } Q, \\ \varepsilon \partial_x \varphi(\cdot, -L) + \varphi(\cdot, -L) = 0, & \text{on } (0, T), \\ \varphi(\cdot, 0) = 0, & \text{on } (0, T), \\ \varphi(T, \cdot) = \varphi^T, & \text{on } (-L, 0). \end{cases} \quad (3.6)$$

In addition, the spectral problem (1.7) can be written as:

$$\begin{cases} -v'' = \tilde{\lambda} v, & \text{in } (-L, 0), \\ 2\varepsilon v'(-L) + v(-L) = 0, \\ v(0) = 0. \end{cases} \quad (3.7)$$

In order to prove Theorem 1.9, we first prove the following lemma:

**Lemma 3.3.** *Let  $L > 0$ . Then, there is  $\varepsilon_0 > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  there is a unique  $r_\varepsilon > 0$  such that there are non-trivial solutions of (3.7) for  $\tilde{\lambda} = -r_\varepsilon^2$ . In fact, we have:*

$$r_\varepsilon \in \left(0, \frac{1}{2\varepsilon}\right), \quad (3.8)$$

and the non-trivial solutions are those proportional to  $\sinh(-r_\varepsilon x)$ . In addition, we have the limit:

$$\frac{1}{2\varepsilon} - r_\varepsilon \rightarrow 0. \quad (3.9)$$

*Remark 3.4.* From Proposition 3.1 and Remark 3.2 we already know that all the eigenvalues are strictly bigger than  $-(2\varepsilon)^{-2}$ ; thus, it is clear that  $r_\varepsilon$  must satisfy (3.8).

*Proof of Lemma 3.3.* The structure of the proof is the following: first we give an equivalent condition for  $-r_\varepsilon^2$  to be an eigenvalue, second we show that (3.7) has a unique strictly negative eigenvalue, and finally we prove (3.9).

*Step 1: an equivalent condition.* First, we recall that when  $\tilde{\lambda} = -r^2 < 0$ , the solutions of (3.7)<sub>1</sub> are given by:

$$Ae^{rx} + Be^{-rx} : A, B \in \mathbb{R}.$$

Moreover, the boundary conditions of (3.7) imply the system:

$$\begin{cases} (2\varepsilon r e^{-rL} + e^{-rL}) A + (-2\varepsilon r e^{rL} + e^{rL}) B = 0, \\ A + B = 0. \end{cases} \quad (3.10)$$

Thus, (3.10) has a non-trivial solution if and only if:

$$e^{-rL} + 2\varepsilon r e^{-rL} = e^{rL} - 2\varepsilon r e^{rL};$$

that is, if and only if:

$$\frac{1 + 2\varepsilon r}{1 - 2\varepsilon r} = e^{2rL}. \quad (3.11)$$

In addition, from (3.10)<sub>2</sub> we obtain that the associated eigenfunctions are those proportional to  $\sinh(-r_\varepsilon x)$ . Finally, from (3.11) we have that all the possible positive roots are in  $(0, (2\varepsilon)^{-1})$ , so we just have to prove existence and uniqueness in that interval.

*Step 2: the proof of the fact that (3.11) has a unique solution in  $(0, (2\varepsilon)^{-1})$ .* Let us denote:

$$g_1(r) := \frac{1 + 2\varepsilon r}{1 - 2\varepsilon r} = -1 + \frac{2}{1 - 2\varepsilon r}, \quad g_2(r) := e^{2rL}.$$

We have the equality:

$$g_1(0) = g_2(0) = 1. \quad (3.12)$$

In addition, the derivative of the functions are given by:

$$g_1'(r) = \frac{4\varepsilon}{(1 - 2\varepsilon r)^2}, \quad g_2'(r) := 2Le^{2rL}. \quad (3.13)$$

Consequently, combining (3.12) and (3.13) we obtain a constant  $c > 0$  such that for  $\varepsilon$  small enough,  $g_2 > g_1$  in  $(0, c)$ . Moreover, since  $\lim_{r \uparrow (2\varepsilon)^{-1}} g_1(r) = +\infty$  and  $g_2((2\varepsilon)^{-1}) \in \mathbb{R}$ , we have at least one root of (3.11) in  $(0, (2\varepsilon)^{-1})$ . In order to show the uniqueness of the root, we define:

$$g_3(r) := \frac{g_2'(r)}{g_1'(r)} = \frac{L}{2\varepsilon} (1 - 2\varepsilon r)^2 e^{2rL}.$$

Since in  $(0, (2\varepsilon)^{-1})$  the function  $g_3$  has the same sign as  $2L(1 - 2\varepsilon r) - 4\varepsilon$ , we have  $g_3'(r) \leq 0$  if and only if:

$$r \geq \frac{2L - 4\varepsilon}{4L\varepsilon} = \frac{1}{2\varepsilon} - \frac{1}{L}.$$

So, since  $g_3(0) = \frac{L}{4\varepsilon}$ , for  $\varepsilon$  small enough there is a unique:

$$\bar{r}_\varepsilon \in \left( \frac{1}{2\varepsilon} - \frac{1}{L}, \frac{1}{2\varepsilon} \right)$$

such that  $g_3(\bar{r}_\varepsilon) = 1$ ,  $g_3 > 1$  in  $[0, \bar{r}_\varepsilon]$  and  $g_3$  is strictly decreasing in  $[\bar{r}_\varepsilon, (2\varepsilon)^{-1}]$ . This implies that  $g_2'(r) > g_1'(r)$  in  $[0, \bar{r}_\varepsilon)$  and  $g_2'(r) < g_1'(r)$  in  $(\bar{r}_\varepsilon, (2\varepsilon)^{-1})$ . Consequently, (3.11) has a unique root in  $(0, (2\varepsilon)^{-1})$  which, to be more precise, belongs to  $[\bar{r}_\varepsilon, (2\varepsilon)^{-1}]$  and which we denote from now on by  $r_\varepsilon$ .

*Step 3: the proof of (3.9).* In order to prove (3.9), we first consider that for  $\varepsilon$  small enough:

$$r_\varepsilon \geq \bar{r}_\varepsilon \geq \frac{1}{2\varepsilon} - \frac{1}{L} \geq \frac{1}{4\varepsilon}. \quad (3.14)$$

Moreover, from  $g_1(r_\varepsilon) = g_2(r_\varepsilon)$ , we obtain that:

$$\frac{1/\varepsilon}{\frac{1}{2\varepsilon} - r_\varepsilon} = e^{2r_\varepsilon L} + 1,$$

which implies the equality:

$$\frac{1}{2\varepsilon} - r_\varepsilon = \frac{1}{\varepsilon(e^{2r_\varepsilon L} + 1)}.$$

Using (3.14), we get for  $\varepsilon$  small enough that:

$$\frac{1}{2\varepsilon} - r_\varepsilon \leq \frac{1}{\varepsilon e^{L(2\varepsilon)^{-1}}}. \quad (3.15)$$

Consequently, we obtain (3.9) from (3.15) and (3.8).  $\square$

We now end the proof of Theorem 1.9. Remark 1.2 implies that:

$$(u^\varepsilon(x), \lambda_0^\varepsilon) := \left( \sinh(-r_\varepsilon x) e^{-(2\varepsilon)^{-1}x}, -\varepsilon r_\varepsilon^2 + \frac{1}{4\varepsilon} \right) \quad (3.16)$$

is a solution of (1.6). Hence  $\hat{\varphi}(t, x) = u^\varepsilon(x) e^{\lambda_0^\varepsilon(t-T)}$  is a solution of (3.6). Moreover, we obtain from  $\varepsilon r_\varepsilon < 1/2$  (see (3.8)) and (3.9) the limit:

$$\lambda_0^\varepsilon = \varepsilon \left( \frac{1}{2\varepsilon} - r_\varepsilon \right) \left( \frac{1}{2\varepsilon} + r_\varepsilon \right) = \left( \frac{1}{2\varepsilon} - r_\varepsilon \right) \left( \frac{1}{2} + \varepsilon r_\varepsilon \right) \rightarrow 0. \quad (3.17)$$

We prove (1.17) with the help of (1.3). On the one hand, for  $x \in (-L, -L + h/2)$  and  $\varepsilon$  small enough we have the bound:

$$u^\varepsilon(x) = \sinh(-r_\varepsilon x) e^{-(2\varepsilon)^{-1}x} \geq \frac{1}{4} e^{-(r_\varepsilon + (2\varepsilon)^{-1})x} \geq \frac{1}{4} e^{(r_\varepsilon + (2\varepsilon)^{-1})(L-h/2)}.$$

Moreover, using (3.17) we find for  $\varepsilon$  small enough that  $e^{-\lambda_0^\varepsilon T} \geq 1/2$ . So, we get the bound:

$$\|\hat{\varphi}(0, \cdot)\|_{L^2(-L, 0)} \geq \|\hat{\varphi}(0, \cdot)\|_{L^2(-L, -L+h/2)} \geq \frac{\sqrt{2h}}{16} e^{(r_\varepsilon + (2\varepsilon)^{-1})(L-h/2)}. \quad (3.18)$$

On the other hand, from  $\omega \subset (-L + h, 0)$  we obtain the estimate:

$$\|\hat{\varphi}\|_{L^2(Q_\omega)} \leq \sqrt{T(L-h)} e^{(r_\varepsilon + (2\varepsilon)^{-1})(L-h)}. \quad (3.19)$$

To prove (3.19) we have used that:

$$\|e^{\lambda_0^\varepsilon(t-T)}\|_{L^\infty(0, T)} \leq 1,$$

and that in  $(-L + h, 0)$ :

$$\sinh(-r_\varepsilon x) e^{-(2\varepsilon)^{-1}x} \leq \frac{e^{-(r_\varepsilon + (2\varepsilon)^{-1})x}}{2} \leq \frac{e^{(r_\varepsilon + (2\varepsilon)^{-1})(L-h)}}{2}.$$

Finally, combining (1.3), (3.18), (3.19) and that  $r_\varepsilon \geq 0$ , we get (1.17).

*Remark 3.5.* The key idea of the proof is that  $\lambda_0^\varepsilon \rightarrow 0$ , but we also need to know how the eigenfunctions are distributed in  $\Omega$ .

### 3.3 Control results when the control domain is $\Omega$

In this section we first prove Theorem 1.10, we then prove Theorem 1.11 and we finally prove Theorem 1.12.

*Proof of Theorem 1.10.* From Definition 1.5 we obtain that  $\varphi^\varepsilon(t, x) = u_0^\varepsilon(x)e^{\lambda_0^\varepsilon(T-t)}$  is a solution of (1.2) (see Definition 1.5 for the notation). Furthermore, (1.18) implies that:

$$\int_0^T e^{-2\lambda_0^\varepsilon(T-t)} dt \rightarrow T.$$

Consequently, we have the limit:

$$\frac{\|\varphi^\varepsilon(0, \cdot)\|_{L^2(\Omega)}^2}{\iint_Q |\varphi^\varepsilon|^2 dxdt} = \frac{\|u_0^\varepsilon e^{-\lambda_0^\varepsilon T}\|_{L^2(\Omega)}^2}{\iint_Q |u_0^\varepsilon e^{-\lambda_0^\varepsilon(T-t)}|^2 dxdt} = \frac{e^{-2\lambda_0^\varepsilon T}}{\int_0^T e^{-2\lambda_0^\varepsilon(T-t)} dt} \rightarrow \frac{1}{T},$$

which together with (1.3) implies (1.19).  $\square$

*Proof of Theorem 1.11.* Let  $\varphi$  be a solution of (3.3) and  $\chi \leq 1$  be a regular positive cut-off function whose value is 1 in  $[0, 1/3]$  and 0 in  $[2/3, 1]$ . Recalling that the adjoint system of (1.12) is given by (3.3) we find that:

$$\begin{aligned} \frac{1}{2} \int_\Omega |\varphi(0, x)|^2 dx &= -\frac{1}{2} \iint_Q \partial_t \left( \chi \left( \frac{t}{T} \right) |\varphi|^2 \right) dxdt \\ &= -\frac{1}{2T} \iint_Q \chi' \left( \frac{t}{T} \right) |\varphi|^2 dxdt + \iint_Q \chi \left( \frac{t}{T} \right) (\varepsilon \Delta \varphi + \partial_{x_1} \varphi) \varphi dxdt \\ &= -\frac{1}{2T} \iint_Q \chi' \left( \frac{t}{T} \right) |\varphi|^2 dxdt - \varepsilon \iint_Q \chi \left( \frac{t}{T} \right) |\nabla \varphi|^2 dxdt \\ &\quad - \iint_\Sigma \chi \left( \frac{t}{T} \right) \frac{n_1}{2} \varphi^2 dxdt \leq C \left( \frac{1}{T} + \frac{1}{\varepsilon} \right) \iint_Q |\varphi|^2 dxdt. \end{aligned} \quad (3.20)$$

For the last inequality in (3.20) we have used classical estimates on the trace term. In particular, we have used the following estimate:

$$\left| \iint_\Sigma \chi \left( \frac{t}{T} \right) \frac{n_1}{2} \varphi^2 dxdt \right| \leq \varepsilon \iint_Q \chi \left( \frac{t}{T} \right) |\nabla \varphi|^2 dxdt + \frac{C}{\varepsilon} \iint_Q |\varphi|^2 dxdt.$$

Thus, we obtain (1.20) from (1.3) and (3.20).  $\square$

We now prove Theorem 1.12 thanks to an explicit computation of the solutions of (3.3). For that purpose, we remark that the spectral problem (1.7) is given when  $\Gamma = \partial\Omega$  and  $a = 0$  by:

$$\begin{cases} -v'' = \tilde{\lambda}v, & \text{in } (-L, 0), \\ 2\varepsilon v' + v = 0, & \text{on } \{-L, 0\}. \end{cases} \quad (3.21)$$

It can be proved easily that the only negative eigenvalue is  $\tilde{\lambda}_0^\varepsilon = -\frac{1}{4\varepsilon^2}$  and that the associated eigenfunction is:

$$v_0^\varepsilon(x) = \frac{e^{-(2\varepsilon)^{-1}x}}{\|e^{-(2\varepsilon)^{-1}z}\|_{L^2(\Omega, dz)}}.$$

In addition, it can be proved for  $m \in \mathbb{N}_*$  that:

$$\tilde{\lambda}_m = \left(\frac{2\pi m}{L}\right)^2,$$

and its associated eigenfunction is:

$$v_m^\varepsilon(x) = \frac{2\varepsilon\sqrt{\tilde{\lambda}_m}\cos\left(\sqrt{\tilde{\lambda}_m}x\right) + \sin\left(\sqrt{\tilde{\lambda}_m}x\right)}{\|2\varepsilon\sqrt{\tilde{\lambda}_m}\cos\left(\sqrt{\tilde{\lambda}_m}z\right) + \sin\left(\sqrt{\tilde{\lambda}_m}z\right)\|_{L^2(\Omega, dz)}}.$$

Consequently, Remark 1.2 implies that the spectral decomposition by the elliptic operator associated to the adjoint system is given by  $u_m^\varepsilon(x) = v_m^\varepsilon(x)e^{-(2\varepsilon)^{-1}x}$ , and their associated eigenvalues are given by  $\lambda_m^\varepsilon = \varepsilon\tilde{\lambda}_m + (4\varepsilon)^{-1}$ . In particular, we have that:

$$u_0^\varepsilon(x) = \frac{e^{-\varepsilon^{-1}x}}{\|e^{-(2\varepsilon)^{-1}z}\|_{L^2(\Omega, dz)}}, \quad \lambda_0 = 0.$$

In addition, we have the following result:

**Lemma 3.6.** *Let  $L > 0$  and  $\Omega := (-L, 0)$ . Then, for all  $\varphi^T \in L^2(\Omega)$  the solution of (3.3) satisfies in  $C^0([0, T]; L^2(\Omega))$ :*

$$\begin{aligned} \varphi(t, x) &= \frac{\int_\Omega \varphi^T(z) dz}{\|e^{-(2\varepsilon)^{-1}z}\|_{L^2(\Omega, dz)}^2} e^{-\varepsilon^{-1}x} \\ &+ \sum_{m \in \mathbb{N}_*} \int_\Omega \varphi^T(z) e^{(2\varepsilon)^{-1}z} v_m^\varepsilon(z) dz v_m^\varepsilon(x) e^{-(2\varepsilon)^{-1}x} \exp\left[-\left(\varepsilon\tilde{\lambda}_m + \frac{1}{4\varepsilon}\right)(T-t)\right]. \end{aligned} \quad (3.22)$$

*In particular, the series in the right-hand side of (3.22) is absolutely convergent in  $L^2(\Omega)$  for all  $t < T$ . Moreover, for all  $T_0 > 0$  and  $\delta > 0$  there is  $C > 0$  such that for all  $\varphi^T \in L^2(\Omega)$ ,  $T \geq T_0$  and  $t \in [0, T - T_0]$  we have the estimate:*

$$\begin{aligned} &\left\| \sum_{m \in \mathbb{N}_*} \int_\Omega \varphi^T(z) e^{(2\varepsilon)^{-1}z} v_m^\varepsilon(z) dz v_m^\varepsilon(x) e^{-(2\varepsilon)^{-1}x} \exp\left[-\left(\varepsilon\tilde{\lambda}_m + \frac{1}{4\varepsilon}\right)(T-t)\right] \right\|_{L^2(\Omega, dx)} \\ &\leq C \|\varphi^T\|_{L^2(\Omega)} \exp\left(\frac{2L + \delta - (T-t)}{4\varepsilon}\right). \end{aligned} \quad (3.23)$$

The proof of Lemma 3.6 is analogous to that of Lemma 2.4. In addition, we remark that, for all  $m \in \mathbb{N}$  and  $\bar{t} \in [t, T]$ :

$$\begin{aligned}
& \left( \int_{\Omega} \varphi^T(z) e^{(2\varepsilon)^{-1}z} v_m^\varepsilon(z) dz \right) v_m^\varepsilon(x) e^{-(2\varepsilon)^{-1}x} \exp \left[ - \left( \varepsilon \tilde{\lambda}_m + \frac{1}{4\varepsilon} \right) (T - t) \right] \\
&= \left( \int_{\Omega} \varphi^T(z) e^{(2\varepsilon)^{-1}z} v_m^\varepsilon(z) dz \right) \exp \left[ - \left( \varepsilon \tilde{\lambda}_m + \frac{1}{4\varepsilon} \right) (T - \bar{t}) \right] v_m^\varepsilon(x) e^{-(2\varepsilon)^{-1}x} \exp \left[ - \left( \varepsilon \tilde{\lambda}_m + \frac{1}{4\varepsilon} \right) (\bar{t} - t) \right] \\
&= \left( \int_{\Omega} \left\{ \sum_{r \in \mathbb{N}} \left( \int_{\Omega} \varphi^T(z) e^{(2\varepsilon)^{-1}z} v_r^\varepsilon(z) dz \right) v_r^\varepsilon(\tilde{z}) e^{-(2\varepsilon)^{-1}\tilde{z}} \exp \left[ - \left( \varepsilon \tilde{\lambda}_r + \frac{1}{4\varepsilon} \right) (T - \bar{t}) \right] \right\} \right. \\
&\quad \left. e^{(2\varepsilon)^{-1}\tilde{z}} v_m^\varepsilon(\tilde{z}) d\tilde{z} \right) v_m^\varepsilon(x) e^{-(2\varepsilon)^{-1}x} \exp \left[ - \left( \varepsilon \tilde{\lambda}_m + \frac{1}{4\varepsilon} \right) (\bar{t} - t) \right] \\
&= \left( \int_{\Omega} \varphi(\bar{t}, z) e^{(2\varepsilon)^{-1}z} v_m^\varepsilon(z) dz \right) v_m^\varepsilon(x) e^{-(2\varepsilon)^{-1}x} \exp \left[ - \left( \varepsilon \tilde{\lambda}_m + \frac{1}{4\varepsilon} \right) (\bar{t} - t) \right]. \quad (3.24)
\end{aligned}$$

We have used, in the second equality of (3.24), that the eigenfunctions of (3.21) form an orthonormal set of functions and Fubini's Theorem (it can be proved as in Lemma 2.4 that the series is convergent in  $L^2(\Omega)$ ). A consequence of (3.24) and Lemma 3.6 is the following:

**Corollary 3.7.** *Let  $L > 0$  and  $\Omega := (-L, 0)$ . Then, for all  $T_0 > 0$  and  $\delta > 0$  there is  $C > 0$  such that for all  $\varphi^T \in L^2(\Omega)$ ,  $T \geq T_0$ ,  $\bar{t} \in [T_0, T]$  and  $t \in [0, \bar{t} - T_0]$  we have the estimate:*

$$\begin{aligned}
& \left\| \sum_{m \in \mathbb{N}_*} \int_{\Omega} \varphi^T(z) e^{(2\varepsilon)^{-1}z} v_m(z) dz v_m(x) e^{-(2\varepsilon)^{-1}x} \exp \left[ - \left( \varepsilon \tilde{\lambda}_m + \frac{1}{4\varepsilon} \right) (T - t) \right] \right\|_{L^2(\Omega, dx)} \\
& \leq C \|\varphi(\bar{t}, \cdot)\|_{L^2(\Omega, dx)} \exp \left( \frac{2L + \delta - (\bar{t} - t)}{4\varepsilon} \right). \quad (3.25)
\end{aligned}$$

In particular, we obtain that for all  $T_0 > 0$  and  $\delta > 0$  there is  $C > 0$  such that for all  $\varphi^T \in L^2(\Omega)$ ,  $T \geq T_0$  and  $\bar{t} \in [T_0, T]$ :

$$\begin{aligned}
& \left\| \sum_{m \in \mathbb{N}_*} \int_{\Omega} \varphi^T(z) e^{(2\varepsilon)^{-1}z} v_m(z) dz v_m(x) e^{-(2\varepsilon)^{-1}x} \exp \left[ - \left( \varepsilon \tilde{\lambda}_m + \frac{1}{4\varepsilon} \right) (T - t) \right] \right\|_{L^2(\Omega, dx)} \\
& \leq \frac{C}{T - \bar{t}} \int_{\bar{t}}^T \|\varphi(s, \cdot)\|_{L^2(\Omega)} ds \exp \left( \frac{2L + \delta - (\bar{t} - t)}{4\varepsilon} \right). \quad (3.26)
\end{aligned}$$

*Proof of Theorem 1.12.* In order to prove Theorem 1.12 we split the solutions of system (3.3) into two parts with the help of Lemma 3.6. Indeed, we define:

$$\varphi^1(x) := \frac{\int_{\Omega} \varphi^T(z) dz e^{-\varepsilon x}}{\|e^{-(2\varepsilon)^{-1}z}\|_{L^2(\Omega, dz)}^2},$$

and:

$$\varphi^2(t, x) := \sum_{m \in \mathbb{N}_*} \int_{\Omega} \varphi^T(z) e^{(2\varepsilon)^{-1}z} v_m^\varepsilon(z) dz v_m^\varepsilon(x) e^{-(2\varepsilon)^{-1}x} \exp \left[ - \left( \varepsilon \tilde{\lambda}_m + \frac{1}{4\varepsilon} \right) (T - t) \right].$$



We remark that:

$$\varphi(t, x) = \varphi^1(x) + \varphi^2(t, x), \quad \text{in } Q.$$

Next, we estimate  $\varphi^1$  and  $\varphi^2(0, \cdot)$  with respect to  $\|\varphi\|_{L^2(Q)}$ :

- If  $\varphi^1 = 0$  the estimate is trivial. Consequently, we suppose from now on that  $\varphi_1 \neq 0$ . We denote:

$$\mathcal{N} := \left\{ t : \varphi^2(t, \cdot) \in B_{L^2(\Omega)} \left( -\varphi^1, \frac{\|\varphi^1\|_{L^2(\Omega)}}{2} \right) \right\},$$

for:

$$B_X(w, r) := \{v \in X : \|v - w\|_X < r\}.$$

We have that:

$$|\mathcal{N}| < 3L, \tag{3.27}$$

since otherwise there are some  $t_1$  and  $t_2$  such that  $t_2 - t_1 \geq 3L$  and:

$$\varphi^2(t_1, \cdot), \varphi^2(t_2, \cdot) \in B_{L^2(\Omega)} \left( -\varphi^1, \frac{\|\varphi^1\|_{L^2(\Omega)}}{2} \right),$$

but by (3.23) this is impossible. Indeed, the system is autonomous and if  $t_2 \in \mathcal{N}$  we have the estimate:

$$\|\varphi(t_2, \cdot)\|_{L^2(\Omega)} \leq \frac{\|\varphi^1\|_{L^2(\Omega)}}{2},$$

so by (3.25) for  $\bar{t} = t_2$  and  $\delta = L/2$  we have that  $t_1 \notin \mathcal{N}$  for  $\varepsilon$  small enough. In addition, we remark that in  $[0, T] \setminus \mathcal{N}$  we have that:

$$\|\varphi(t, \cdot)\|_{L^2(\Omega)} \geq \frac{\|\varphi^1\|_{L^2(\Omega)}}{2}. \tag{3.28}$$

Thus, we obtain from (3.27) and (3.28) that for all  $\varphi^T \in L^2(\Omega)$  and  $T > 4L$ :

$$\frac{\|\varphi^1\|_{L^2(\Omega)}}{\|\varphi\|_{L^2(Q)}} \leq \frac{\|\varphi^1\|_{L^2(\Omega)}}{\|\varphi\|_{L^2(\left((0, T) \setminus \mathcal{N}\right) \times \Omega)}} \leq \frac{2}{\sqrt{T - 3L}} \leq \frac{2}{\sqrt{T/4}} = \frac{4}{\sqrt{T}}. \tag{3.29}$$

- As a direct consequence of (3.26) for  $\bar{t} = T - L$  we have that, independently of  $\varphi^T$ :

$$\frac{\|\varphi^2(0, \cdot)\|_{L^2(\Omega)}}{\|\varphi\|_{L^2(Q)}} \leq \frac{Ce^{-c\varepsilon^{-1}} \|\varphi\|_{L^2(\left((T-L, T) \times \Omega\right)}}}{\|\varphi\|_{L^2(Q)}} \leq Ce^{-c\varepsilon^{-1}}. \tag{3.30}$$

Thus, combining (3.29) and (3.30) we obtain (1.21) for  $\varepsilon$  small enough.  $\square$

## 4 Further comments and open problems

We remark the following:

- We obtain similar results for the boundary controllability.
- Under the hypothesis (1.9) if  $T$  is small the cost of the null controllability of (1.1) explodes exponentially when  $\varepsilon$  goes to 0, which can be proved for instance as in [19] and [2].

We now present some problems that remain open:

- Get precise estimates of the cost of the control problem (1.1) for a large time and a small diffusion coefficient when  $a$  also depends on the time variable.
- Determine if Theorem 1.12 is still true for all domains.
- Analyse under the hypothesis (1.11) when there is some  $y^0$  such that the norm of the minimal norm control which takes  $y^0$  to 0 increases exponentially and when, for all  $y^0$  and for  $\varepsilon$  small enough, the cost is bounded by a constant (or, even better, when for all  $y^0$  the associated control does not decay with  $\varepsilon$  and it is just a problem of uniformity).
- Study the cost of the controllability in the context of Section 3.3 when  $(-L, -L + h) \subset \omega$ .
- Determine under the hypothesis (1.11) if for all domain  $\Omega$ , subdomain  $\omega \subset \Omega$ ,  $T > 0$  and  $a \in L^\infty(\Gamma \times (0, \varepsilon_0))$  there is a constant  $C > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  we have that:

$$K(\Omega, \omega, T, \varepsilon) \leq C e^{C\varepsilon^{-1}}. \quad (4.1)$$

Indeed, following the techniques in [14] and considering the equation (1.5) we can easily see that:

$$K(\Omega, \omega, T, \varepsilon) \leq C e^{C\varepsilon^{-4/3}};$$

however, (4.1) would be more coherent with the literature (see [19]).

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