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0 Abstract and basic information

In this paper we consider the heat equation with Neumann, Robin and mixed boundary conditions (with coefficients on the boundary which depend on the space variable). The main results concern the behaviour of the cost of null controllability with respect to the diffusion coefficient when the control acts in the interior. First, we prove that if we almost have Dirichlet boundary conditions in the part of the boundary in which the flux of the transport enters the cost of the controllability decays. Next, we show some examples of Neumann and mixed boundary conditions in which for any time $T > 0$ the cost explodes exponentially as the diffusion coefficient vanishes. Finally, we study some systems in which the control is located on the whole domain.

Key words: Heat equation, singular limits, spectral decomposition, transport equation, uniform controllability

AMS subject classification: 35B25, 35K05, 93B05, 93C20

Abbreviated title: Cost of a transport-diffusion control problem

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1 Introduction

1.1 Main results

Let $\Omega \subset \mathbb{R}^d$ a domain (bounded connected open set), $\Gamma \subset \partial\Omega$ a relative open subset and $\Gamma^* := \partial\Omega \setminus \Gamma$. We consider a mesurable function $a(x, \varepsilon)$ defined in $\Gamma \times (0, \varepsilon_0)$ for some $\varepsilon_0 > 0$. We study the control problem given by:

$$\begin{cases} y_t - \varepsilon\Delta y + \partial_{x_1} y = 1_\omega f, & \text{in } Q, \\ \partial_n y + ay = 0, & \text{on } \Sigma_N, \\ y = 0, & \text{on } \Sigma_D, \\ y(0, \cdot) = y^0, & \text{on } \Omega, \end{cases} \tag{1.1}$$

for $\varepsilon \in (0, \varepsilon_0)$. As usual, n denotes the outward normal vector on $\partial\Omega$, $Q := (0, T) \times \Omega$, $Q_\omega := (0, T) \times \omega$ and $\Sigma := (0, T) \times \partial\Omega$. In addition, we define $\Sigma_N := (0, T) \times \Gamma$ and $\Sigma_D := (0, T) \times \Gamma^*$. The null controllability of (1.1) is already proved in [12] and [10] when $a \in L^\infty(\Gamma)$; however, with a gross estimation on how the cost of the controllability behaves when the diffusion coefficient $\varepsilon > 0$ vanishes. In this paper we give more precise bounds with respect to ε about the cost. We study the cost of the control with the usual norms; that is:

$$K(\Omega, \omega, T, \varepsilon) := \sup_{y^0 \in L^2(\Omega)\setminus\{0\}} \inf_{f \in S(y^0)} \frac{\|f\|_{L^2(Q_\omega)}}{\|y^0\|_{L^2(\Omega)}},$$

for

$$S(y^0) := \{f \in L^2(Q_\omega) : \text{ the solution of (1.1) satisfies } y(T, \cdot) = 0\}.$$

The cost also depends on a and Γ, although we do not write it explicitly.

This paper follows a well-established research line which inquires about the cost of the controllability of systems with a small diffusion and a transport term. The first of such control problem to be analysed was the heat equation in dimension 1 with Dirichlet boundary conditions in [8]. Afterwards, the same problem but in any dimension and with any speed belonging to $W^{1,+\infty}(\mathbb{R}^+ \times \Omega)$ was studied in [18]. More recently, better approximations of the optimal time in which the cost of the control decays have been given: the lower bound was improved in [24] through complex analysis and properties of the entire functions and the upper bound was improved in [13, 23] (in the first one through complex analysis and in the second one by transforming the original equation into the pure heat equation). As for similar results, work has been done in the KdV equation (see [15, 16, 2, 3]), in the Burgers equation (see [14]), in the Stokes system (see [1]) and in an artificial advection-diffusion problem (see [5, 6]). Finally, the

current study of (1.1) is a contribution to the literature (and in particular the case $a = 0$ and $\Gamma = \emptyset$ is an open problem proposed in [18, Remark 3]).

As in the previously cited papers, we study the adjoint problem, which is given by:

$$\begin{cases} -\varphi_t - \varepsilon\Delta\varphi - \partial_{x_1}\varphi = 0, & \text{in } Q, \\ \varepsilon\partial_n\varphi + (\varepsilon a + n_1)\varphi = 0, & \text{on } \Sigma_N, \\ \varphi = 0, & \text{on } \Sigma_D, \\ \varphi(T,\cdot) = \varphi^T, & \text{on } \Omega. \end{cases} \tag{1.2}$$

Indeed, we have the classical equality given by the Hilbert Uniqueness Method (see, for instance, [25, 21]):

$$K(\Omega,\omega,T,\varepsilon) = \sup_{\varphi^T \in L^2(\Omega)\setminus\{0\}} \frac{\|\varphi(0,\cdot)\|_{L^2(\Omega)}}{\|\varphi\|_{L^2(Q_\omega)}}, \tag{1.3}$$

where φ denotes the solution of (1.2).

As a novelty with respect to the literature, we obtain some properties of (1.2) (and in particular the Carleman inequality) by studying the solutions of:

$$\begin{cases} -\phi_t - \varepsilon\Delta\phi + \frac{1}{4\varepsilon}\phi = 0, & \text{in } Q, \\ \varepsilon\partial_n\phi + \left(\varepsilon a + \frac{n_1}{2}\right)\phi = 0, & \text{on } \Sigma_N, \\ \phi = 0, & \text{on } \Sigma_D, \\ \phi(T,\cdot) = \phi^T, & \text{on } \Omega. \end{cases} \tag{1.4}$$

The advantage of studying the solutions of (1.4) is that its elliptic part is self-adjoint.

Remark 1.1. We have obtained system (1.4) by considering the functional:

$$w \mapsto e^{(2\varepsilon)^{-1}x_1} w, \tag{1.5}$$

which is an homeomorphism from the solutions of (1.2) with initial value in $L^2(\Omega)$ to the solutions of (1.4) with initial value in $L^2(\Omega)$. Furthermore, this functional is useful to translate information about the solutions of (1.4) to information about the solutions of (1.2), for instance about their regularity.

We now define the operator:

$$\mathcal{A}^\varepsilon u := -\varepsilon\Delta u - \partial_{x_1} u, \tag{1.6}$$

and the domain:

$$D(\mathcal{A}^\varepsilon) := \left\{ w \in H^1(\Omega) : \mathcal{A}^\varepsilon w \in L^2(\Omega),\ \varepsilon\partial_n w + (\varepsilon a + n_1)w = 0 \text{ on } \Gamma,\ w = 0 \text{ on } \Gamma^* \right\}. \quad (1.7)$$

We recall that $-\Delta w \in L^2(\Omega)$ and $w \in H^1(\Omega)$ implies that $\partial_n w$ is well-defined as an element of $H^{-1/2}(\Gamma)$. With the purpose of understanding the solutions of (1.2) we study the spectral problem:

$$\begin{cases} -\varepsilon\Delta u - \partial_{x_1} u = \lambda u, & \text{in } \Omega, \\ \varepsilon\partial_n u + (\varepsilon a + n_1)\, u = 0, & \text{on } \Gamma, \\ u = 0, & \text{on } \Gamma^*. \end{cases} \quad (1.8)$$

In a similar way, with the purpose of understanding the solutions of (1.8) we study the spectral problem:

$$\begin{cases} -\Delta v = \tilde{\lambda} v, & \text{in } \Omega, \\ \varepsilon\partial_n v + \left(\varepsilon a + \frac{n_1}{2}\right) v = 0, & \text{on } \Gamma, \\ v = 0, & \text{on } \Gamma^*, \end{cases} \quad (1.9)$$

and we define:

$$D(\Delta) := \left\{ w \in H^1(\Omega) : -\Delta w \in L^2(\Omega),\ \varepsilon\partial_n w + \left(\varepsilon a + \frac{n_1}{2}\right) w = 0 \text{ on } \Gamma,\ w = 0 \text{ on } \Gamma^* \right\}.$$

We can relate the solutions of (1.8) and (1.9) with the functional (1.5):

Remark 1.2. We have that $(v, \tilde{\lambda})$ is a solution of (1.9) if and only if:

$$(u, \lambda) := \left(v e^{-(2\varepsilon)^{-1} x_1}, \varepsilon\tilde{\lambda} + \frac{1}{4\varepsilon} \right),$$

is a solution of (1.8). Similarly, (u, λ) is a solution of (1.8) if and only if:

$$\left(u e^{(2\varepsilon)^{-1} x_1}, \frac{\lambda}{\varepsilon} - \frac{1}{4\varepsilon^2} \right)$$

is a solution of (1.9).

We consider $(1.9)_1$ instead of $-\varepsilon\Delta v + (4\varepsilon)^{-1} v = \lambda v$ in order to isolate the "part" of the eigenvalue that is caused by the diffusion. This is useful for instance to prove Lemma 2.4 and 3.4 below.

From now on we need the following technical hypothesis:

Hypothesis 1.1. We suppose that for all $\varepsilon \in (0, \varepsilon_0)$ there is $s > 3/2$ such that $\mathcal{D}(\Delta) \subset H^s(\Omega)$ continuously.

Remark 1.3. The hypothesis 1.1 include all the cases in which $a \in W^{1/2,\infty}(\Gamma)$ and:

$$\partial\Gamma \cap \partial\Gamma^* = \emptyset, \tag{1.10}$$

which in particular includes segments. The reason is that if we have (1.10) next to each component of $\partial\Omega$ we ave either Dirichlet boundary condition or Robin boundary conditions, so we can use the well-known regularity results. In addition, there is no reason to believe that Hypothesis 1.1 is false if $a \in W^{1/2,\infty}(\Gamma)$, but this is still an open problem.

Remark 1.4. It can be checked directly that the laplacian is self-adjoint in $D(\Delta)$ if we have Hypothesis 1.10 $a \in W^{1/2,\infty}(\Gamma)$. Indeed, if $w \in D(\Delta^*)$ we see first that $-\Delta w \in L^2(\Omega)$ by considering that $\mathcal{D}(\Omega) \subset D(\Delta)$. Afterwards, we see that w equals the solution of $-\Delta v = -\Delta w$, $(1.9)_2$ and $(1.9)_3$. Thereby, $L^2(\Omega)$ has a spectral decomposition (for each ε) by the solutions of (1.9).

Remark 1.5. We can also study the problem with null initial force associated to (1.1). Indeed, the functional $w \mapsto e^{-(2\varepsilon)^{-1}x_1}w$ transforms (1.1) into a symmetric version of (1.4) (time backwards and with opposite transport). Thus, the direct and adjoint systems have a similar nature and similar decay properties.

Definition 1.6. We denote by $(\tilde{\lambda}_m^\varepsilon)_{m\in\mathbb{N}}$ the eigenvalues of (1.9) and by $(\lambda_m^\varepsilon)_{m\in\mathbb{N}}$ the eigenvalues of (1.8). In addition, for each eigenvalue of (1.9) an eigenfunction v_m^ε is fixed, to which we impose that $\|v_m^\varepsilon\|_{L^2(\Omega)} = 1$. Finally, we denote $u_m^\varepsilon := v_m^\varepsilon e^{-(2\varepsilon)^{-1}x_1}$. The ε might be omitted to avoid excessive redundancy if clear.

Remark 1.7. Since $\overline{\text{span}\{v_m\}} = L^2(\Omega)$ (see Remark 1.4) we have that $\overline{\text{span}\{u_m\}} = L^2(\Omega)$.

The idea of using spectral decomposition is not new in Control Theory (see, for instance, [7]). Indeed, for the heat and the Stokes context alone, there are many documents which deal with eigenfunctions of the elliptic operator for proving the existence of some control (see, for example, [20, 22, 4]), for estimating the cost of the control (see, for instance, [9]), and for giving negative answer to the existence of a control (see, for example, [22]). As for a system with small diffusion and a transport term, a spectral decomposition indirectly appears in [8, 24], when the authors get lower bounds for the optimal time T_0 in which the cost of the control decays exponentially with ε. In addition, a spectral decomposition has been used in [1] to get the dissipation estimate in a transport-diffusion Stokes system. Indeed, in this paper we follow the phylosophy of [1] of using as much information as possible about the spectral decomposition, with the contribution that now while proving the Carleman we work directly in the symmetrized system; that is, in (1.4).

Coming back to (1.2), what determines if the solutions of (1.2) decays is $\tilde{\lambda}_0^\varepsilon$. We recall that by Rayleigh principle we have the equality:

$$\tilde{\lambda}_0^\varepsilon = \min\left\{\int_\Omega |\nabla v|^2 + \int_\Gamma \left(a + \frac{n_1}{2\varepsilon}\right)|v|^2 : v \in H^1(\Omega), \|v\|_{L^2(\Omega)} = 1, v = 0 \text{ on } \Gamma^*\right\}. \tag{1.11}$$

Looking at (1.11) we can do a division of cases depending on whether the following inequality is satisfies: :

$$(a + (2\varepsilon)^{-1} n_1) 1_\Gamma \geq 0. \tag{1.12}$$

If (1.12) is satisfied, we have that $\tilde{\lambda}_1^\varepsilon \geq 0$, which implies that the solutions of (1.2) decay exponentially on ε^{-1} (see Lemma 2.4 below). In fact, we have the following theorem:

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^d$ be a C^2 domain, $\omega \subset \Omega$ be a subdomain and $\varepsilon_0 > 0$ be so that (a, ε, Γ) satisfies that $a \in L^\infty(\Omega)$, (1.12) and Hypothesis 1.1. Then, there are $T, c, C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have that:*

$$K(\Omega, \omega, T, \varepsilon) \leq C e^{-c\varepsilon^{-1}}. \tag{1.13}$$

There are several important examples that satisfy the hypotheses of Theorem 1.8:

- Dirichlet boundary conditions ($\Gamma = \emptyset$). This controllability result is already proved in [8, 18]. However, the proof is different in the sense that on this paper we prove the decay with the spectral decomposition (instead of with a comparison theorem or an Agmon inequality) and the proof of the Carleman presents some differences.

- Segments in which we have boundary conditions on the left edge and Neumann conditions on the right edge. The physical meaning of those boundary conditions is that the function is almost null near the left edge and almost constant near the right edge.

- The following control problem:

$$\begin{cases} y_t - \varepsilon\Delta y + \partial_{x_1} y = 1_\omega f, & \text{in } Q, \\ -\varepsilon\partial_n y + \frac{n_1}{2} y = 0, & \text{on } \Sigma, \\ y(0, \cdot) = y^0, & \text{on } \Omega. \end{cases} \tag{1.14}$$

  This control problem is (1.1) with $a = -\frac{n_1}{2\varepsilon}$ and $\Gamma = \partial\Omega$, so in a sense it is an edge case among those which satisfy (1.12). It is symmetric with its adjoint system and it can represent a physical situation in which we almost have Dirichlet boundary conditions.

- Any system satisfying (1.10) and $a \geq (2\varepsilon)^{-1} n_1$; that is, in which we almost have Dirichlet boundary conditions.

The alternative situation to (1.12) is:

$$\left(a + \frac{n_1}{2\varepsilon}\right) 1_\Gamma \not\geq 0. \tag{1.15}$$

Under the hypothesis 1.15 we might have negative eigenvalues for (1.9) and the cost of the control might explode.

Let us get an estimate of the lower bound of λ_0^ε when $a \geq 0$; that is, when system (1.1) has a mixed of Dirichlet and Robin boundary conditions with a positive value a.

Proposition 1.9. *Let $\Omega \subset \mathbb{R}^d$ a Lipschitz domain and $a \geq 0$. Then, we have that:*

$$\tilde{\lambda}_0^\varepsilon \geq -\frac{1}{4\varepsilon^2}. \tag{1.16}$$

Proof. Let $v \in H^1(\Omega)$ such that $\|v\|_{L^2(\Omega)} = 1$ and such that $v = 0$ on Γ^*. We find from the Green formula and an appropriate Cauchy-Schwarz inequality that:

$$\int_\Gamma \frac{n_1}{2\varepsilon}|v|^2 = \int_{\partial\Omega} \frac{n_1}{2\varepsilon}|v|^2 = \int_\Omega \frac{v\partial_{x_1} v}{\varepsilon} \geq -\int_\Omega |\partial_{x_1} v|^2 - \frac{1}{4\varepsilon^2}\int_\Omega |v|^2.$$

Thus, if we have $a \geq 0$ and $v \in H^1(\Omega)$ such that $\|v\|_{L^2(\Omega)} = 1$ and such that $v = 0$ on Γ^*, we find that:

$$\int_\Omega |\nabla v|^2 + \int_\Gamma \left(a + \frac{n_1}{2\varepsilon}\right)|v|^2 \geq \int_\Omega |\nabla v|^2 - \int_\Omega |\partial_{x_1} v|^2 - \frac{1}{4\varepsilon^2} \geq -\frac{1}{4\varepsilon^2}. \tag{1.17}$$

Finally, we get (1.16) from (1.11) and (1.17). □

Remark 1.10. In order to have an equality in (1.16), if we follow the proof of Proposition 1.9, we need that $a = 0$, that v just depends on x_1 and that $\partial_{x_1} v = -(2\varepsilon)^{-1} v$, i.e. we need that $v = e^{-(2\varepsilon)^{-1} x_1}$ (and so $\Gamma = \partial\Omega$). This situation corresponds to the case of pure Neumann boundary conditions, which is studied in Section 3.1. In any other situation in which $a \geq 0$ the inequality (1.16) is strict.

In this paper we study some subcases of (1.15):

- We consider (1.1) with pure Neumann boundary conditions; that is, $a = 0$ and $\Gamma = \partial\Omega$. We remark that under those boundary conditions (1.1) is given by:

$$\begin{cases} y_t - \varepsilon\Delta y + \partial_{x_1} y = 1_\omega f, & \text{in } Q, \\ \partial_n y = 0, & \text{on } \Sigma, \\ y(0,\cdot) = y^0, & \text{on } \Omega. \end{cases} \tag{1.18}$$

The study of (1.18) is an open problem proposed in [18, Remark 3] and it represents that the mass gets in and out only tangentially. Let us denote:

$$\pi_1(x) := x_1, \quad p_l := \inf \pi_1(\Omega), \quad p_r := \sup \pi_1(\Omega). \tag{1.19}$$

We prove in this paper the following result:

Theorem 1.11. *Let $h > 0$, $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, and $\omega \subset \Omega$ be an open subset such that:*

$$\pi_1(\omega) \subset (p_l + h, p_r). \tag{1.20}$$

Then, for all $T > 0$ there is $c > 0$ such that for all $\varepsilon > 0$ the cost of the control of problem (1.18) *explodes exponentially; that is, such that:*

$$K(\Omega, \omega, T, \varepsilon) \geq ce^{c\varepsilon^{-1}}. \tag{1.21}$$

- We consider the segment $\Omega := (-L, 0)$, $a = 0$ (Neumann boundary conditions) and $\Gamma := \{-L\}$. In this situation the control problem (1.1) is given by:

$$\begin{cases} y_t - \varepsilon \partial_{xx} y + \partial_x y = f 1_\omega, & \text{in } Q, \\ \partial_x y(\cdot, -L) = 0, & \text{on } (0, T), \\ y(\cdot, 0) = 0, & \text{on } (0, T), \\ y(0, \cdot) = y^0, & \text{on } (-L, 0). \end{cases} \tag{1.22}$$

The control problem (1.22) describes for instance a valley in which the density of animals is constant near the left edge and it goes to 0 near the right edge (for instance if there is a cliff). The intuition may suggest that as ε gets smaller the cost of the control decreases, as in the Dirichlet case. However, it is just the opposite, the fact that there is a flux coming from the left dominates and the cost of the control actually increases when the random movement vanishes. In particular, we prove that:

Theorem 1.12. *Let $L, h > 0$, $\Omega := (-L, 0)$ and $\omega \subset (-L + h, 0)$ be an open subset. Then, for all $T > 0$ there is $c > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the cost the control of* (1.22) *is bounded superiorly by:*

$$K(\Omega, \omega, T, \varepsilon) \geq ce^{c\varepsilon^{-1}}. \tag{1.23}$$

- We get a lower bound of the cost of the controllability when $\omega = \Omega$; that is, in which the control domain is the whole domain. In particular, we prove that:

Theorem 1.13. *Let Ω be a Lipschitz domain, $T > 0$ and (Γ, a) be such that $a \in L^\infty(\Gamma)$, Hypothesis 1.1 is satisfies and:*

$$\lambda_0^\varepsilon \to 0. \tag{1.24}$$

Then, we have the bound:

$$\liminf_{\varepsilon \to 0} K(\Omega, \Omega, T, \varepsilon) \geq \frac{1}{\sqrt{T}}. \tag{1.25}$$

The conditions stated in Theorem 1.13 include the control problem (1.18) (see Remark 1.10) and (1.22) (see (3.15)).

- We study if $\omega = \Omega$ and if we have (1.24), the cost of the control is at least $O(T^{-1/2})$ when $\varepsilon \to 0$. In particular, for Neumann boundary conditions we prove that:

Theorem 1.14. *Let Ω be a C^2 domain. Then, there is $C > 0$ such that for all $\varepsilon > 0$ and $T > 0$:*

$$K(\Omega, \Omega, T, \varepsilon) \leq C \left(\frac{1}{\sqrt{T}} + \frac{1}{\sqrt{\varepsilon}} \right), \tag{1.26}$$

for K the cost of the control problem (1.18)*.*

In addition, if we are in a segment we prove that:

Theorem 1.15. *Let $L > 0$ and $\Omega := (-L, 0)$. We have for all $T \geq 4L$:*

$$\limsup_{\varepsilon \to 0} K(\Omega, \Omega, T, \varepsilon) \leq \frac{4}{\sqrt{T}}, \tag{1.27}$$

for K the cost of the control problem (1.18)*.*

Remark 1.16. The difficulty when proving Theorem 1.14 and Theorem 1.15 is that f is required to be in $L^2(Q)$. If we just need having $f \in L^2(0, T; H^{-2}(\Omega))$, then it suffices to consider the affine function which joins y^0 and 0.

As for the results stated for subcases of (1.15), Theorem 1.11 is similar to the result presented in [1] for the Stokes equation in dimension 3. As for the others, the most surprising is Theorem 1.12 as one can think that having Dirichlet boundary conditions in the right-hand side is a sufficient condition to have exponential decay on the cost of the control.

The rest of the paper is organized as follows: in Section 2 we prove Theorem 1.8, in Section 3 we study the case (1.15) and in Section 4 we do some remarks and present some open problems.

2 Proof of Theorem 1.8

The section is split in three parts: in the first one we prove the decay of the solutions of (1.2), in the second one we prove a Carleman inequality for the solutions of (1.4), and in the third one we end the proof of Theorem 1.8. For this section we use the following definition:

Definition 2.1. Let $\Omega \subset \mathbb{R}^d$ a non-empty set. We define:

$$\operatorname{diam}_{x_1}(\Omega) := \sup\left\{x_1 : \left(\{x_1\} \times \mathbb{R}^{d-1}\right) \cap \Omega \neq \emptyset\right\} - \inf\left\{x_1 : \left(\{x_1\} \times \mathbb{R}^{d-1}\right) \cap \Omega \neq \emptyset\right\}.$$

2.1 Decay of the solutions of (1.2) for a sufficiently large time

We denote by $(\tilde{\lambda}_m^0)_{m\in\mathbb{N}}$ the eigenvalues of the system:

$$\begin{cases} -\Delta v = \tilde{\lambda}^0 v & \text{in } \Omega, \\ \partial_n v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

It is easy to prove that all eigenvalues of (2.1) are positive. In addition, we recall the Weyl asymptotics law for the Neumann system (see, for instance, [19]):

Lemma 2.2 (Weyl asymptotic)**.** *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and let $N(\gamma)$ be the number of eigenvalues (counting repetitions) of (2.1) that belong to $(-\infty, \gamma]$. We have:*

$$\lim_{\gamma\to\infty} \frac{N(\gamma)}{\gamma^{d/2}} = \frac{\operatorname{Vol}(B(0,1))\operatorname{Vol}(\Omega)}{(2\pi)^d}.$$

In particular, there is $C > 0$ such that, for all $\gamma \in \mathbb{R}$:

$$N(\gamma) \leq C(1 + |\gamma|^{d/2}). \tag{2.2}$$

In addition, we recall the following spectral result which follows from the min-max variational principle:

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and assume that (a, Γ) satisfies that $a \in L^\infty(\Gamma)$ and (1.12). Then, for all $m \in \mathbb{N}$ we have the following estimate:*

$$\tilde{\lambda}_m^\varepsilon \geq \tilde{\lambda}_m^0.$$

Let us now study the solutions of (1.2):

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and assume that (a, Γ) satisfies that $a \in L^\infty(\Gamma)$ and (1.12). Then, for all $\varphi^T \in L^2(\Omega)$ the solution of (1.2) satisfies for all $t \in [0, T]$:*

$$\varphi(t,x) = \sum_{m\in\mathbb{N}} \int_\Omega \varphi^T(z) e^{(2\varepsilon)^{-1}z_1} v_m^\varepsilon(z)dz v_m^\varepsilon(x) e^{-(2\varepsilon)^{-1}x_1} \exp\left[-\left(\varepsilon\tilde{\lambda}_m^\varepsilon + \frac{1}{4\varepsilon}\right)(T-t)\right]. \tag{2.3}$$

In particular, the series in the right-hand side of (2.3) *is absolutely convergent in* $L^2(\Omega)$ *for all* $t < T$. *Moreover, for all* $T_0 > 0$ *and* $\delta > 0$ *there is* $C > 0$ *such that for all* $T \geq T_0$, $\varphi^T \in L^2(\Omega)$ *and* $t \in [0, T - T_0]$ *we have the estimate:*

$$\|\varphi(t,\cdot)\|_{L^2(\Omega)} \leq C\|\varphi^T\|_{L^2(\Omega)} \exp\left(\frac{2\,\mathrm{diam}_{x_1}(\Omega) + \delta - (T-t)}{4\varepsilon}\right). \tag{2.4}$$

Proof. We can easily prove (2.3) for $t = T$ using that v_m^ε is an orthonormal basis in $L^2(\Omega)$. Indeed, if $\varphi^T \in L^2(\Omega)$ we have that:

$$\varphi^T(x) = (\varphi^T(x)e^{(2\varepsilon)^{-1}x_1})e^{-(2\varepsilon)^{-1}x_1} = \sum_{m\in\mathbb{N}} \int_\Omega \varphi^T(z)e^{(2\varepsilon)^{-1}z_1} v_m^\varepsilon(z)dz v_m^\varepsilon(x)e^{-(2\varepsilon)^{-1}x_1}.$$

We prove (2.3) and (2.4) simultaneously for $t \in [0, T)$. We obtain from Remark 1.2 that (2.3) is true in $\mathrm{span}\{u_m\}$. Since the left hand side of (2.3) belongs to $C^0([0,T]; L^2(\Omega))$ (see [10]) and since $\mathrm{span}\{u_m\}_{m\in\mathbb{N}}$ is dense in $L^2(\Omega)$ (see Remark 1.7), it is enough to prove that the right-hand side is an endomorphism on $L^2(\Omega)$. For that purpose using the triangle inequality and that $\|v_m\|_{L^2(\Omega)} = 1$ we get that:

$$\begin{aligned}\left\| \sum_{m\in\mathbb{N}} \int_\Omega (\varphi^T(z)e^{(2\varepsilon)^{-1}z_1}) v_m^\varepsilon(z)dz v_m^\varepsilon(x)e^{-(2\varepsilon)^{-1}x_1} \exp\left[-\left(\varepsilon\tilde{\lambda}_m^\varepsilon + \frac{1}{4\varepsilon}\right)(T-t)\right]\right\|_{L^2(\Omega,dx)} \\ \leq \|\varphi^T\|_{L^2(\Omega)} \exp\left(\frac{2\,\mathrm{diam}_{x_1}(\Omega) - (T-t)}{4\varepsilon}\right) \sum_{m\in\mathbb{N}} e^{-\varepsilon\tilde{\lambda}_m^\varepsilon(T-t)}. \end{aligned} \tag{2.5}$$

Thus, the only thing left is to bound $\sum_m e^{-\varepsilon\tilde{\lambda}_m^\varepsilon(T-t)}$. We obtain from Lemma 2.3, Lemma 2.2 and the positivity of the eigenvalues of (2.1) the estimate:

$$\begin{aligned}\sum_{m\in\mathbb{N}} e^{-\varepsilon\tilde{\lambda}_m^\varepsilon(T-t)} &\leq \sum_{m\in\mathbb{N}} e^{-\varepsilon\tilde{\lambda}_m^0(T-t)} \leq \sum_{m\geq 1} N(m)e^{-\varepsilon(m-1)(T-t)} \\ &\leq C\left(1 + \sum_{m\geq 2}(1+m^{d/2})e^{-\varepsilon(m-1)(T-t)}\right) \leq C\left(1 + \int_0^{+\infty}(1+(z+2)^{d/2})e^{-\varepsilon z(T-t)}dz\right) \\ &\leq C\left(1 + \int_0^{+\infty}(1+z^{d/2})e^{-\varepsilon z(T-t)}dz\right) = C\left(1 + \frac{1}{\varepsilon(T-t)} + \frac{1}{\varepsilon^{(d+2)/2}(T-t)^{(d+2)/2}}\right). \end{aligned} \tag{2.6}$$

So, from (2.6), $T - t \geq T_0$, and $z^{-k/2}e^{-z^{-1}} \in L^\infty(\mathbb{R}^+)$ for all $k \in \mathbb{R}^+$, we obtain that:

$$\sum_{m\in\mathbb{N}} e^{-\varepsilon\tilde{\lambda}_m^\varepsilon(T-t)} \leq Ce^{\delta\varepsilon^{-1}}. \tag{2.7}$$

Consequently, we end the proof by combining (2.5) and (2.7). □

Remark 2.5. Lemma 2.4 can be generalized to the case in which $\tilde{\lambda}_0^\varepsilon \geq 0$ and in which we have some polynomial bound for $|\{i : \tilde{\lambda}_i^\varepsilon \leq m\}|$ that depends on m and uniform for all $\varepsilon \in (0, \varepsilon_0)$.

2.2 A Carleman inequality for the solutions of (1.4)

We consider the usual Fursikov-Imanuvilov weights. First of all, we consider η a $C^2(\overline{\Omega})$ function satisfying:

$$\eta = 0 \text{ on } \partial\Omega, \quad \eta \geq 0 \text{ on } \Omega, \quad \|\eta\|_\infty = 1, \quad \inf_{\Omega\setminus\omega_0} |\nabla\eta| = \delta > 0, \tag{2.8}$$

for $\omega_0 \subset\subset \omega$ an open non-empty set. The existence of such a function if Ω is a C^2 domain is proved for instance in [12]. With that auxiliary function in mind, we define, for some $\tilde{T} > 0$, the following weights in $\tilde{Q} := (0, \tilde{T}) \times \Omega$:

$$\begin{aligned} &\alpha_\pm(t,x) := \frac{e^{8\lambda} - e^{\lambda(\pm\eta(x)+6)}}{t(\tilde{T}-t)}, \quad \xi_\pm(t,x) := \frac{e^{\lambda(\pm\eta(x)+6)}}{t(\tilde{T}-t)}, \\ &\alpha^*_\pm(t) = \max_{x\in\overline{\Omega}} \alpha_\pm(t,x), \qquad \xi^*_\pm(t) = \min_{x\in\overline{\Omega}} \xi_\pm(t,x). \end{aligned} \tag{2.9}$$

Weights of this kind first appeared in [12], but this version is borrowed from [11] with the adaptations for non-Dirichlet situations proposed in [10]. We remark that we have the usual bounds:

$$|\partial_{x_i}\alpha_\pm| = |\partial_{x_i}\xi_\pm| \leq C\lambda\xi_\pm, \quad |\partial_t\alpha_\pm| \leq C\tilde{T}\xi^2_\pm, \quad |\partial^2_{t^2}\alpha_\pm| \leq C\xi^2_\pm(1+\tilde{T}^2\xi_\pm). \tag{2.10}$$

We prove in this section the following result:

Proposition 2.6. *Let $\Omega \subset \mathbb{R}^d$ be a C^2 domain, $\omega \subset \Omega$ be a subdomain and $\varepsilon_0 > 0$ be so that (a, ε, Γ) satisfies $a \in L^\infty(\Gamma)$, (1.12) and Hypothesis 1.1. Then, there is $C > 0$ such that if $\lambda \geq C$, $\tau \geq (\tilde{T} + \tilde{T}^2)\varepsilon^{-1}$ and $\phi^T \in L^2(\Omega)$ we have that:*

$$\begin{aligned} \tau^3\lambda^4 \iint_{\tilde{Q}} e^{-2s\alpha}\xi^3|\phi|^2 + \tau\lambda^2 \iint_{\tilde{Q}} e^{-2s\alpha}\xi^3|\nabla\phi|^2 + \tau^2\lambda^2 \iint_{\tilde{\Sigma}_N} e^{-2\tau\alpha}\xi^2 \left(a + \frac{n_1}{2\varepsilon}\right)|\phi|^2 \\ \leq C\tau^3\lambda^4 \iint_{\tilde{Q}_\omega} e^{-2s\alpha}\xi^3|\phi|^2 \end{aligned} \tag{2.11}$$

for ϕ the solution of (1.4) with $\tilde{T}$ instead of T, and $\tilde{\Sigma}_N := (0, \tilde{T}) \times \Gamma$.

In the statement of Proposition 2.6 and in its proof we omit the infinitesimals dt and dx as they can deduced by looking at the integration domain.

For proving Proposition 2.6 we adapt the proofs of [18] and [10]. In the proof of Proposition 2.6 we denote by $O(g(\varepsilon, \tau, \lambda, \phi))$ a generic function for which there is $C > 0$ depending only on Ω, ω and ω_0 and which satisfies:

$$|O(g(\varepsilon, \tau, \lambda, \phi))| \leq Cg(\varepsilon, \tau, \lambda, \phi).$$

if $\lambda \geq C$, and $\tau \geq (\tilde{T} + \tilde{T}^2)\varepsilon^{-1}$. In that definition g is a positive function.

Proof. By density arguments it suffices to prove (2.35) for $\phi^0 \in \mathcal{D}(\Omega)$. Thus, we can suppose that $\phi \in H^1(0,T;D(\Delta))$. As in [10], we now consider the following change of variables:

$$\psi^\pm := e^{-\tau\alpha_\pm}\phi. \tag{2.12}$$

We remark that, $\psi^\pm$ have Dirichlet boundary conditions on $\tilde{\Sigma}_D := (0,\tilde{T})\times\Gamma^*$. In addition, since $\eta = 0$ on $\tilde{\Sigma} := (0,\tilde{T})\times\partial\Omega$ we have the following equalities on $\tilde{\Sigma}$:

$$\xi_+ = \xi_-, \quad \alpha_+ = \alpha_-, \quad \psi^+ = \psi^-, \quad \psi_t^+ = \psi_t^-, \quad \partial_{\mathrm{tg}}\psi^+ = \partial_{\mathrm{tg}}\psi^-. \tag{2.13}$$

In particular, we might omit the sign when we are working on Σ to avoid this redundancy. In that sense, we have that:

$$\partial_n\psi^\pm = \pm\tau\lambda\xi\partial_n\eta\psi - \left(a + \frac{n_1}{2\varepsilon}\right)\psi, \quad \text{ on } \tilde{\Sigma}_N. \tag{2.14}$$

We next consider the equalities:

$$L_1^\pm\psi^\pm + L_2^\pm\psi^\pm = L_3^\pm\psi^\pm, \tag{2.15}$$

for:

$$\begin{cases} L_1^\pm\psi^\pm := & -2\varepsilon\tau\lambda^2|\nabla\eta|^2\xi_\pm\psi^\pm \mp 2\varepsilon\tau\lambda\xi_\pm\nabla\eta\cdot\nabla\psi^\pm + \psi_t^\pm, \\ L_2^\pm\psi^\pm := & \varepsilon\tau^2\lambda^2|\nabla\eta|^2\xi_\pm^2\psi^\pm + \varepsilon\Delta\psi^\pm + \tau\partial_t(\alpha_\pm)\psi^\pm + \frac{1}{4\varepsilon}\psi^\pm, \\ L_3^\pm\psi^\pm := & \pm\varepsilon\tau\Delta\eta^\pm\xi_\pm\psi^\pm - \varepsilon\tau\lambda^2|\nabla\eta|^2\xi_\pm\psi^\pm. \end{cases} \tag{2.16}$$

As usual, we denote $(L_i^\pm\psi)_j$ the j term of $L_i^\pm\psi$. Moreover, we are going to calculate the product $(L_1^\pm\psi^\pm, L_2^\pm\psi^\pm)_{L^2(\tilde{Q})}$. The operations here are inspired in [18, Proposition 1], though we use the technique proposed in [10] to treat the boundary terms.

To begin with, we have that, for $\lambda \geq C$, $\tau \geq C\tilde{T}^2$ (see (2.8) for the definition of δ):

$$\begin{aligned} &((L_1^\pm\psi^\pm)_1 + (L_1^\pm\psi^\pm)_2, (L_2^\pm\psi^\pm)_1)_{L^2(\tilde{Q})} \\ &\quad = \varepsilon^2\tau^3\lambda^4\iint_{\tilde{Q}}|\nabla\eta|^4\xi_\pm^3|\psi^\pm|^2 + O\left(\varepsilon^2 s^2\lambda^3\iint_{\tilde{Q}}\xi_\pm^3|\psi^\pm|^2\right) \mp \varepsilon^2\tau^3\lambda^3\iint_{\tilde{\Sigma}_N}|\nabla\eta|^2\xi^3\partial_n\eta|\psi|^2 \\ &\quad \geq \frac{3\delta^4}{4}\varepsilon^2\tau^3\lambda^4\iint_{\tilde{Q}}\xi_\pm^3|\psi^\pm|^2 - \delta^4\varepsilon^2\tau^3\lambda^4\iint_{\tilde{Q}_{\omega_0}}\xi_\pm^3|\psi^\pm|^2 \mp \varepsilon^2\tau^3\lambda^3\iint_{\tilde{\Sigma}_N}|\nabla\eta|^2\xi^3\partial_n\eta|\psi|^2. \end{aligned} \tag{2.17}$$

As for the third term in $L_1^\pm\psi^\pm$, we can integrate by parts in time and get, taking into account (2.10):

$$((L_1^\pm\psi^\pm)_3, (L_2^\pm\psi^\pm)_1)_{L^2(\tilde{Q})} = O\left(\varepsilon\tilde{T}\tau^2\lambda^2\iint_{\tilde{Q}}\xi_\pm^3|\psi^\pm|^2\right). \tag{2.18}$$

Summing up, thanks to (2.17), (2.18), (2.8) and (2.13), we obtain for $\lambda \geq C$ and $\tau \geq C(\tilde{T} + \tilde{T}^2)\varepsilon^{-1}$ the estimate:

$$\sum_{i\in\{+,-\}} (L_1^i\psi^i, (L_2^i\psi^i)_1)_{L^2(\tilde{Q})} \geq \sum_{i\in\{+,-\}} \frac{\delta^4}{2}\varepsilon^2\tau^3\lambda^4 \iint_{\tilde{Q}} \xi_i^3|\psi^i|^2 - \delta^4\varepsilon^2\tau^3\lambda^4 \iint_{Q_{\omega_0}} \xi_i^3|\psi^i|^2. \quad (2.19)$$

To continue with, we have that, integrating by parts, with Cauchy-Schwarz inequality:

$$\begin{aligned}((L_1^\pm\psi^\pm)_1, (L_2^\pm\psi^\pm)_2)_{L^2(\tilde{Q})} &= 2\varepsilon^2\tau\lambda^2 \iint_{\tilde{Q}} |\nabla\eta|^2\xi_\pm|\nabla\psi^\pm|^2 - 2\varepsilon^2\tau\lambda^2 \iint_{\tilde{\Sigma}_N} |\nabla\eta|^2\xi(\partial_n\psi^\pm)\psi \\ &\quad + O\left(\varepsilon^2\tau^2\lambda^4 \iint_{\tilde{Q}} \xi_\pm^2|\psi^\pm|^2 + \varepsilon^2\lambda^2 \iint_{\tilde{Q}} |\nabla\psi^\pm|^2\right). \quad (2.20)\end{aligned}$$

Next, considering that $\eta = 0$ on $\partial\Omega$, integrating by parts formally (assuming that $\psi^\pm \in C^\infty(\overline{Q})$):

$$\begin{aligned}&((L_1^\pm\psi^\pm)_2, (L_2^\pm\psi^\pm)_2)_{L^2(\tilde{Q})} = \mp 2\varepsilon^2\tau\lambda \iint_{\tilde{\Sigma}} \partial_n\eta\xi|\partial_n\psi^\pm|^2 \\ &\quad + O\left(\varepsilon^2\tau\lambda \iint_{\tilde{Q}} \xi_\pm|\nabla\psi^\pm|^2\right) + 2\varepsilon^2\tau\lambda^2 \iint_{\tilde{Q}} \xi_\pm|\nabla\eta\cdot\nabla\psi^\pm|^2 \pm \varepsilon^2\tau\lambda \iint_{\tilde{Q}} \xi_\pm\nabla\eta\cdot\nabla|\nabla\psi^\pm|^2 \\ &\quad = \mp 2\varepsilon^2\tau\lambda \iint_{\tilde{\Sigma}} \partial_n\eta\xi|\partial_n\psi^\pm|^2 + O\left(\varepsilon^2\tau\lambda \iint_{\tilde{Q}} \xi_\pm|\nabla\psi^\pm|^2\right) + 2\varepsilon^2\tau\lambda^2 \iint_{\tilde{Q}} \xi_\pm|\nabla\eta\cdot\nabla\psi^\pm|^2 \\ &- \varepsilon^2\tau\lambda^2 \iint_{\tilde{Q}} |\nabla\eta|^2\xi_\pm|\nabla\psi^\pm|^2 + O\left(\varepsilon^2\tau\lambda \iint_{\tilde{Q}} \xi_\pm|\nabla\psi^\pm|^2\right) \pm \varepsilon^2\tau\lambda \iint_{\tilde{\Sigma}} \partial_n\eta\xi(|\partial_{\mathrm{tg}}\psi|^2 + |\partial_n\psi^\pm|^2). \quad (2.21)\end{aligned}$$

Actually, for all $\psi^\pm \in L^2(0,T;D(A))$ the left-hand side of (2.21) equals the right-hand side of (2.21) because of Hypothesis 1.1 and because $C^\infty(\overline{\tilde{Q}})$ is dense in $L^2(0,T;D(A))$. To continue with, we find from $(2.13)_4$ and (2.14) that:

$$\begin{aligned}((L_1^\pm\psi^\pm)_3, (L_2^\pm\psi^\pm)_2)_{L^2(\tilde{Q})} &= \varepsilon \iint_{\tilde{\Sigma}_N} \psi_t^\pm\partial_n\psi^\pm \\ &= -\varepsilon \iint_{\tilde{\Sigma}_N} \left(a + \frac{n_1}{2\varepsilon}\right)\psi_t\psi \pm \varepsilon\tau\lambda \iint_{\tilde{\Sigma}_N} \xi\partial_n\eta\psi_t\psi = \pm\varepsilon\tau\lambda \iint_{\tilde{\Sigma}_N} \xi\partial_n\eta\psi_t\psi. \quad (2.22)\end{aligned}$$

We have used the assumption a is independent of the time variable and that $\psi(0,\cdot) = \psi(\tilde{T},\cdot) = 0$. Let us compute the boundary terms involving $\partial_n\psi^\pm$ that appear in (2.21). On the one hand, we have that:

$$\mp\varepsilon^2\tau\lambda \iint_{\tilde{\Sigma}_D} \partial_n\eta\xi|\partial_n\psi^\pm|^2 = \mp\varepsilon^2\tau\lambda \iint_{\tilde{\Sigma}_D} \partial_n\eta\xi|e^{-s\alpha}\partial_n\phi|^2. \quad (2.23)$$

On the other hand, we obtain from (2.14) the equality:

$$\begin{aligned}\mp\varepsilon^2\tau\lambda \iint_{\tilde{\Sigma}_N} \partial_n\eta\xi|\partial_n\psi^\pm|^2 &= 2\varepsilon^2\tau^2\lambda^2 \iint_{\tilde{\Sigma}_N} (\partial_n\eta)^2\xi^2\left(a + \frac{n_1}{2\varepsilon}\right)|\psi|^2 \\ &\quad \mp\varepsilon^2\tau^3\lambda^3 \iint_{\tilde{\Sigma}_N} (\partial_n\eta)^3\xi^3|\psi|^2 \mp \varepsilon^2\tau\lambda \iint_{\tilde{\Sigma}} \partial_n\eta\xi\left(a + \frac{n_1}{2\varepsilon}\right)^2|\psi|^2. \quad (2.24)\end{aligned}$$

Summing up, if we consider (2.19)-(2.24) and $(2.8)_4$ we get that:

$$\sum_{i\in\{+,-\}}(L_1^i\psi^i,(L_2^i\psi^i)_2)_{L^2(\tilde{Q})}\geq\sum_{i\in\{+,-\}}O\left(\varepsilon^2\tau^2\lambda^4\iint_{\tilde{Q}}\xi_i^2|\psi^i|^2\right)$$
$$+\sum_{i\in\{+,-\}}\frac{\delta^2}{2}\varepsilon^2\tau\lambda^2\iint_{\tilde{Q}}\xi_i|\nabla\psi^i|^2-\delta^2\varepsilon^2\tau\lambda^2\iint_{\tilde{Q}_{\omega_0}}\xi_i|\nabla\psi^i|^2$$
$$+4\varepsilon^2\tau^2\lambda^2\iint_{\tilde{\Sigma}_N}(\partial_n\eta)^2\xi^2\left(a+\frac{n_1}{2\varepsilon}\right)|\psi|^2. \quad (2.25)$$

Next, we have that, because of (2.10):

$$((L_1^\pm\psi^\pm)_1,(L_2^\pm\psi^\pm)_3)_{L^2(\tilde{Q})}=O\left(\varepsilon\tau^2\lambda^2\tilde{T}\iint_{\tilde{Q}}\xi_\pm^3|\psi^\pm|^2\right). \quad (2.26)$$

In addition, thanks to an integration by parts we have that:

$$((L_1^\pm\psi^\pm)_2,(L_2^\pm\psi^\pm)_3)_{L^2(\tilde{Q})}=O\left(\varepsilon\tau^2\lambda^2\tilde{T}\iint_{\tilde{Q}}\xi_\pm^3|\psi^\pm|^2\right)\mp\varepsilon\tau^2\lambda\iint_{\tilde{\Sigma}_N}\xi\partial_t\alpha\partial_n\eta\psi^2. \quad (2.27)$$

Finally, from $(2.10)_3$ we find that:

$$((L_1^\pm\psi^\pm)_3,(L_2^\pm\psi^\pm)_3)_{L^2(\tilde{Q})}=-\frac{\tau}{2}\iint_{\tilde{Q}}\partial_t^2(\alpha_\pm)|\psi^\pm|^2=O\left(\tau\iint_{\tilde{Q}}\xi_\pm^2(1+\tilde{T}^2\xi_\pm)|\psi^\pm|^2\right). \quad (2.28)$$

Summing up, considering that $\lambda\geq C$ and that $\tau\geq C(\tilde{T}+\tilde{T}^2)\varepsilon^{-1}$, we obtain that:

$$\sum_{i\in\{+,-\}}(L_1^i\psi^i,(L_2^i\psi^i)_3)_{L^2(\tilde{Q})}=O\left(\varepsilon^2\tau^3\lambda^2\iint_{\tilde{Q}}\xi_\pm^3|\psi^\pm|^2\right). \quad (2.29)$$

To continue with, we have that:

$$((L_1^\pm\psi^\pm)_1,(L_2^\pm\psi^\pm)_4)_{L^2(\tilde{Q})}=O\left(\tau\lambda^2\iint_{\tilde{Q}}\xi_\pm|\psi^\pm|^2\right). \quad (2.30)$$

Next, we have that, after integrating by parts:

$$((L_1^\pm\psi^\pm)_2,(L_2^\pm\psi^\pm)_4)_{L^2(\tilde{Q})}=O\left(\tau\lambda^2\iint_{\tilde{Q}}\xi_\pm|\psi^\pm|^2\right)\mp\frac{\tau\lambda}{4}\iint_{\tilde{\Sigma}_N}\xi\partial_n\eta|\psi|^2. \quad (2.31)$$

Finally, we obtain from $\psi^\pm(0,\cdot)=\psi^\pm(T,\cdot)=0$ that:

$$((L_2^\pm\psi^\pm)_3,(L_2^\pm\psi^\pm)_4)_{L^2(\tilde{Q})}=0. \quad (2.32)$$

Summing up the results obtained in (2.30)-(2.32), we get that:

$$\sum_{i\in\{+,-\}}(L_1^i\psi^i,(L_2^i\psi^i)_4)_{L^2(\tilde{Q})}=O\left(\tau\lambda^2\iint_{\tilde{Q}}\xi_\pm|\psi^\pm|^2\right). \quad (2.33)$$

So, if we add (2.20), (2.25), (2.29) and (2.33), we get, after absorptions, for $\lambda \geq C$ and $\tau \geq C(\tilde{T}+\tilde{T}^2)\varepsilon^{-1}$:

$$2\sum_{i\in\{+,-\}}(L_1^i\psi^i, L_2^i\psi^i) + 2\delta^4\varepsilon^2\tau^3\lambda^4\iint_{\tilde{Q}_{\omega_0}}\xi_i^3|\psi^i|^2 + 2\delta^2\varepsilon^2\tau\lambda^2\iint_{\tilde{Q}_{\omega_0}}\xi_i|\nabla\psi^i|^2$$
$$\geq \sum_{i\in\{+,-\}}\frac{\delta^4}{2}\varepsilon^2\tau^3\lambda^4\iint_{\tilde{Q}}\xi_i^3|\psi^i|^2 + \frac{\delta^2}{2}\varepsilon^2\tau\lambda^2\iint_{\tilde{Q}}\xi_i|\nabla\psi^i|^2$$
$$+ 4\varepsilon^2\tau^2\lambda^2\iint_{\tilde{\Sigma}_N}(\partial_n\eta)^2\xi^2\left(a+\frac{n_1}{2\varepsilon}\right)|\psi|^2. \quad (2.34)$$

If we have (1.12), it is classical to obtain (2.35) from (2.34) (see, for instance, [12] and [10]) □

2.3 End of the proof of Theorem 1.8

Let $T \geq 1$. Considering Proposition 2.6 for $\tilde{T}=1$ and Remark 1.1 we obtain for all $\varphi^T \in L^2(\Omega)$, $\lambda \geq C$ and $\tau \geq C\varepsilon^{-1}$ the estimate:

$$\tau^3\lambda^4\int_0^1\int_\Omega e^{-2s\alpha+\varepsilon^{-1}x_1}\xi^3|\varphi(T-1+t,x)|^2dxdt$$
$$\leq C\tau^3\lambda^4\int_0^1\int_\Omega e^{-2s\alpha+\varepsilon^{-1}x_1}\xi^3|\varphi(T-1+t,x)|^2dxdt, \quad (2.35)$$

for φ the solution of (1.2). So, fixing λ large enough, and $\tau=\tau_0\varepsilon^{-1}$ for τ_0 large enough, we have that (2.35) implies that:

$$\|\varphi\|_{L^2((T-2/3,T-1/3)\times\Omega)} \leq Ce^{C\varepsilon^{-1}}\|\varphi\|_{L^2((T-1,T)\times\omega)} \leq Ce^{C\varepsilon^{-1}}\|\varphi\|_{L^2(Q_\omega)}. \quad (2.36)$$

Moreover, from Lemma 2.4 taking $T_0=\delta=1$ we find a constant $C>0$ such that for all $t'\in(T-2/3,T-1/3)$ we have that:

$$\|\varphi(0,\cdot)\|_{L^2(\Omega)} \leq C\exp\left[\frac{2\,\mathrm{diam}_{x_1}(\Omega)+1-t'}{4\varepsilon}\right]\|\varphi(t',\cdot)\|_{L^2(\Omega)}$$
$$\leq C\exp\left[\frac{C-T}{4\varepsilon}\right]\|\varphi(t',\cdot)\|_{L^2(\Omega)}. \quad (2.37)$$

So, combining (2.36) and (2.37), we get that, for any $T\geq 1$:

$$\|\varphi(0,\cdot)\|_{L^2(\Omega)} \leq C\exp\left[\frac{C-T}{4\varepsilon}\right]\|\varphi\|_{L^2((T-2/3,T-1/3)\times\Omega)} \leq C\exp\left[\frac{C-T}{4\varepsilon}\right]\|\varphi\|_{L^2(Q_\omega)}.$$

In particular, if T is sufficiently large, by (1.3) we obtain (1.13).

3 Some control problems in which we have (1.15)

In this section we study some subcases of (1.15). In particular, in Section 3.1 we prove Theorem 1.11, in Section 3.2 we prove Theorem 1.12, and in Section 3.3 we prove Theorems 1.13, 1.14 and 1.15.

3.1 Proof of Theorem 1.11

In order to prove (1.21) we consider that the adjoint system of (1.18) is given by (see (1.2)):

$$\begin{cases} -\varphi_t - \varepsilon\Delta\varphi - \partial_{x_1}\varphi = 0, & \text{in } Q, \\ \varepsilon\partial_n\varphi + n_1\varphi = 0, & \text{on } \Sigma, \\ \varphi(T,\cdot) = \varphi^T, & \text{on } \Omega. \end{cases} \tag{3.1}$$

In particular, we use (1.3) for $\varphi(x) := e^{-x_1\varepsilon^{-1}}$, which is a (steady) solution of (3.1) (see Remarks 1.10 and 1.2). On the one hand, we remark that (see (1.19) for the notation):

$$\|\varphi(0,\cdot)\|_{L^2(\Omega)} \geq |\pi_1^{-1}(p_l, p_l + h/2) \cap \Omega|^{1/2} \exp\left(\frac{-p_l - h/2}{\varepsilon}\right). \tag{3.2}$$

Because Ω is an open set and because of (1.19) we have that:

$$|\pi_1^{-1}(p_l, p_l + h/2) \cap \Omega| > 0.$$

On the other hand, (1.20) implies the estimate:

$$\|\varphi\|_{L^2(Q_\omega)} \leq T^{1/2}|\omega|^{1/2}\|e^{-x_1\varepsilon^{-1}}\|_{L^\infty(\pi_1^{-1}(p_l+h,p_r)\cap\omega)} \leq T^{1/2}|\omega|^{1/2}\exp\left(\frac{-p_l - h}{\varepsilon}\right). \tag{3.3}$$

Consequently, combining (1.3), (3.2) and (3.3) we find that:

$$K(\Omega,\omega,T,\varepsilon) \geq \frac{\|\varphi(0,\cdot)\|_{L^2(\Omega)}}{\|\varphi\|_{L^2(Q_\omega)}} \geq \frac{|\pi_1^{-1}(p_l, p_l + h/2) \cap \Omega|^{1/2}}{T^{1/2}|\omega|^{1/2}} \exp\left(\frac{h}{2\varepsilon}\right),$$

which implies (1.21).

3.2 Proof of Theorem 1.12

We first remark that the adjoint system of (1.22) is given by (see (1.2)):

$$\begin{cases} -\varphi_t - \varepsilon\partial_{xx}\varphi - \partial_x\varphi = 0, & \text{in } Q, \\ \varepsilon\partial_x\varphi(\cdot,-L) + \varphi(\cdot,-L) = 0, & \text{on } (0,T), \\ \varphi(\cdot,0) = 0, & \text{on } (0,T), \\ \varphi(T,\cdot) = \varphi^T, & \text{on } (-L,0). \end{cases} \tag{3.4}$$

In addition, the spectral problem (1.9) can be written as:

$$\begin{cases} -v'' = \tilde{\lambda} v, & \text{in } (-L, 0), \\ 2\varepsilon v'(-L) + v(-L) = 0, & \\ v(0) = 0. & \end{cases} \tag{3.5}$$

In order to prove Theorem 1.12, we first prove the following lemma:

Lemma 3.1. *Let $L > 0$. Then, there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there is a unique $r_\varepsilon > 0$ such that there are non-null solutions of (3.5) for $\tilde{\lambda} = -r_\varepsilon^2$. In fact, we have:*

$$r_\varepsilon \in \left(0, \frac{1}{2\varepsilon}\right), \tag{3.6}$$

and the non-null solutions are those proportional to $\sinh(-r_\varepsilon x)$. In addition, we have the limit:

$$\frac{1}{2\varepsilon} - r_\varepsilon \to 0. \tag{3.7}$$

Remark 3.2. From Proposition 1.9 and Remark 1.10 we already know that all the eigenvalues are strictly bigger than $-(2\varepsilon)^{-2}$; thus, it is clear that r_ε must satisfy (3.6).

Proof of Lemma 3.1. The structure of the proof is the following: first we give an equivalent condition for $-r_\varepsilon^2$ to be an eigenvalue, second we show that (3.5) has a unique strictly negative eigenvalue, and finally we prove (3.7).

Step 1: an equivalent condition. First, we recall that when $\tilde{\lambda} = -r^2 < 0$ the solutions of $(3.5)_1$ are given by:

$$Ae^{rx} + Be^{-rx} : \quad A, B \in \mathbb{R}.$$

Moreover, the boundary conditions of (3.5) imply the system:

$$\begin{cases} \left(2\varepsilon r e^{-rL} + e^{-rL}\right) A + \left(-2\varepsilon r e^{rL} + e^{rL}\right) B = 0, \\ A + B = 0. \end{cases} \tag{3.8}$$

Thus, (3.8) has a non-null solution if and only if:

$$e^{-rL} + 2\varepsilon r e^{-rL} = e^{rL} - 2\varepsilon r e^{rL};$$

that is, if and only if:

$$\frac{1 + 2\varepsilon r}{1 - 2\varepsilon r} = e^{2rL}. \tag{3.9}$$

In addition, from $(3.8)_2$ we obtain that the associated eigenfunctions are those proportional to $\sinh(-r_\varepsilon x)$. Finally, from (3.9) we have that all the possible positive roots are in $(0, (2\varepsilon)^{-1})$, so just have to prove existence and uniqueness in that interval.

Step 2: (3.9) *has a unique solution in* $(0,(2\varepsilon)^{-1})$. Let us denote:

$$g_1(r) := \frac{1+2\varepsilon r}{1-2\varepsilon r} = -1 + \frac{2}{1-2\varepsilon r}, \qquad g_2(r) := e^{2rL}.$$

We have the equality:

$$g_1(0) = g_2(0) = 1. \tag{3.10}$$

In addition, the derivative of the functions are given by:

$$g_1'(r) = \frac{4\varepsilon}{(1-2\varepsilon r)^2}, \qquad g_2'(r) := 2Le^{2rL}. \tag{3.11}$$

Consequently, combining (3.10) and (3.11) we obtain a constant $c > 0$ such that for ε small enough $g_2 > g_1$ on $(0,c)$. Moreover, since $g_1((2\varepsilon)^{-1}) = +\infty$ and since $g_2((2\varepsilon)^{-1}) < +\infty$, we have at least one root of (3.9) in $(0,(2\varepsilon)^{-1})$. In order to show the uniqueness of the root, we define:

$$g_3(r) := \frac{g_2'(r)}{g_1'(r)} = \frac{L}{2\varepsilon}(1-2\varepsilon r)^2 e^{2rL}.$$

Since in $(0,(2\varepsilon)^{-1})$ the function g_3' has the same sign as $2L(1-2\varepsilon r) - 4\varepsilon$, we have $g_3'(r) \leq 0$ if and only if:

$$r \geq \frac{2L-4\varepsilon}{4L\varepsilon} = \frac{1}{2\varepsilon} - \frac{1}{L}.$$

So, since $g_3(0) = \frac{L}{4\varepsilon}$, for ε small enough there is a unique:

$$\overline{r}_\varepsilon \in \left(\frac{1}{2\varepsilon} - \frac{1}{L}, \frac{1}{2\varepsilon}\right)$$

such that $g_3(\overline{r}_\varepsilon) = 1$, $g_3 > 1$ in $[0,\overline{r}_\varepsilon]$ and g_3 is strictly decreasing in $[\overline{r}_\varepsilon,(2\varepsilon)^{-1}]$. This implies that $g_2'(r) > g_1'(r)$ in $[0,\overline{r}_\varepsilon)$ and $g_2'(r) < g_1'(r)$ in $(\overline{r}_\varepsilon,(2\varepsilon)^{-1})$. Consequently, (3.9) has a unique root in $(0,(2\varepsilon)^{-1})$, which, to be more precise, belongs to $[\overline{r}_\varepsilon,(2\varepsilon)^{-1}]$ and which we denote from now on by r_ε.

Step 3: the proof of (3.7). In order to prove (3.7), we first consider that for ε small enough:

$$r_\varepsilon \geq \overline{r}_\varepsilon \geq \frac{1}{2\varepsilon} - \frac{1}{L} \geq \frac{1}{4\varepsilon}. \tag{3.12}$$

Moreover, from $g_1(r_\varepsilon) = g_2(r_\varepsilon)$, we obtain that:

$$\frac{1/\varepsilon}{\frac{1}{2\varepsilon} - r_\varepsilon} = e^{2r_\varepsilon L} + 1,$$

which implies the equality:

$$\frac{1}{2\varepsilon} - r_\varepsilon = \frac{1}{\varepsilon(e^{2r_\varepsilon L}+1)}.$$

Using (3.12), we get that for ε small enough:

$$\frac{1}{2\varepsilon} - r_\varepsilon \leq \frac{1}{\varepsilon e^{L(2\varepsilon)^{-1}}}. \tag{3.13}$$

Consequently, we obtain (3.7) from (3.13) and (3.6). □

We now end the proof of Theorem 1.12. Remark 1.2 implies that:

$$(u^\varepsilon, \lambda_0^\varepsilon) := \left(\sinh(-r_\varepsilon x)e^{-(2\varepsilon)^{-1}x}, -\varepsilon r_\varepsilon^2 + \frac{1}{4\varepsilon}\right) \tag{3.14}$$

is a solution of (1.8). Hence $\varphi(t,x) = u^\varepsilon(x)e^{\lambda_0^\varepsilon(t-T)}$ is a solution of (3.4). Moreover,we obtain from (3.6) and (3.7) (and from $\varepsilon r_\varepsilon < 1/2$) the limit:

$$\lambda_0^\varepsilon = \varepsilon\left(\frac{1}{2\varepsilon} - r_\varepsilon\right)\left(\frac{1}{2\varepsilon} + r_\varepsilon\right) = \left(\frac{1}{2\varepsilon} - r_\varepsilon\right)\left(\frac{1}{2} + \varepsilon r_\varepsilon\right) \to 0. \tag{3.15}$$

We prove (1.23) with the help of (1.3). On the one hand, for $x \in (-L, -L + h/2)$ and ε small enough we have the bound:

$$\sinh(-r_\varepsilon x)e^{-(2\varepsilon)^{-1}x} \geq \frac{1}{4}e^{-(r_\varepsilon+(2\varepsilon)^{-1})x} \geq \frac{1}{4}e^{(r_\varepsilon+(2\varepsilon)^{-1})(L-h/2)}.$$

Moreover, using (3.15) we find for ε small enough that $e^{-\lambda_0^\varepsilon T} \geq 1/2$. So, we get the bound:

$$\|\varphi(0,\cdot)\|_{L^2(-L,0)} \geq \|\varphi(0,\cdot)\|_{L^2(-L,-L+h/2)} \geq \frac{\sqrt{2h}}{16}e^{(r_\varepsilon+(2\varepsilon)^{-1})(L-h/2)}. \tag{3.16}$$

On the other hand, from $\omega \subset (-L + h, 0)$ we obtain the estimate:

$$\|\varphi\|_{L^2(Q_\omega)} \leq \sqrt{T(L-h)}e^{(r_\varepsilon+(2\varepsilon)^{-1})(L-h)}. \tag{3.17}$$

To prove (3.17) we have used that:

$$\|e^{\lambda_0^\varepsilon(t-T)}\|_{L^\infty(0,T)} \leq 1,$$

and that on $(-L + h, 0)$:

$$\sinh(-r_\varepsilon x)e^{-(2\varepsilon)^{-1}x} \leq e^{-(r_\varepsilon+(2\varepsilon)^{-1})x} \leq e^{(r_\varepsilon+(2\varepsilon)^{-1})(L-h)}.$$

Finally, combining (1.3), (3.16), (3.17) and that $r_\varepsilon \geq 0$, we get (1.23).

Remark 3.3. The key idea of the proof is that $\lambda_0^\varepsilon \to 0$, but we also need to know how the eigenfunctions are distributed on the whole domain.

3.3 Control results when the control domain is Ω

In this section we first prove Theorem 1.13, we then prove Theorem 1.14 and we finally prove Theorem 1.15.

Proof of Theorem 1.13. From Definition 1.6 we obtain that $\varphi^\varepsilon(t,x) := u_0^\varepsilon e^{\lambda_0^\varepsilon(T-t)}$ is a solution of (1.2) (see Definition 1.6 for the notation). Furthermore, (1.24) implies that:

$$\int_0^T e^{-2\lambda_0^\varepsilon(T-t)} \to T.$$

Consequently, we have the limit:

$$\frac{\|\varphi^\varepsilon(0,\cdot)\|^2_{L^2(\Omega)}}{\iint_Q |\varphi^\varepsilon|^2} = \frac{\|u_0^\varepsilon e^{-\lambda_0^\varepsilon T}\|^2_{L^2(\Omega)}}{\iint_Q |u_0^\varepsilon e^{-\lambda_0^\varepsilon(T-t)}|^2} = \frac{e^{-2\lambda_0^\varepsilon T}}{\int_0^T e^{-2\lambda_0^\varepsilon(T-t)}} \to \frac{1}{T},$$

which together with (1.3) implies (1.25). □

Proof of Theorem 1.14. Let φ be a solution of (3.1) and $\chi \leq 1$ be a regular positive cut-off function whose value is 1 in $[0,1/3]$ and 0 in $[2/3,1]$. Considering (3.1) we find that:

$$\begin{aligned}\frac{1}{2}\int_\Omega |\varphi(0,\cdot)|^2 &= -\frac{1}{2}\iint_Q \partial_t\left(\chi\left(\frac{\cdot}{T}\right)|\varphi|^2\right)\\ &= -\frac{1}{2T}\iint_Q \chi'\left(\frac{\cdot}{T}\right)|\varphi|^2 + \iint_Q (\varepsilon\Delta\varphi + \partial_{x_1}\varphi)\varphi\chi\left(\frac{\cdot}{T}\right)\\ &= -\frac{1}{2T}\iint_Q \chi'\left(\frac{\cdot}{T}\right)|\varphi|^2 - \varepsilon\iint_Q |\nabla\varphi|^2\chi\left(\frac{\cdot}{T}\right)\\ &\quad - \iint_\Sigma \frac{n_1}{2}\varphi^2\chi\left(\frac{\cdot}{T}\right) \leq C\left(\frac{1}{T}+\frac{1}{\varepsilon}\right)\iint_Q |\varphi|^2. \quad (3.18)\end{aligned}$$

For the last inequality in (3.18) we have used classical estimates on the trace term. In particular, we have used the following estimate:

$$\left|\iint_\Sigma \frac{n_1}{2}\varphi^2\right| \leq \varepsilon\iint_Q |\nabla\varphi|^2 + \frac{C}{\varepsilon}\iint_Q |\varphi|^2.$$

Thus, we obtain (1.26) from (1.3) and (3.18). □

We now prove Theorem 1.15 thanks to an explicit computation of the solutions of (3.1). For that purpose, we recall that the spectral problem (1.9) is given by:

$$\begin{cases} -\Delta v = \tilde{\lambda} v, & \text{in } (-L,0),\\ 2\varepsilon v' + v = 0, & \text{on } \{-L,0\}. \end{cases} \quad (3.19)$$

It can be proved easily that the only negative eigenvalue is $\tilde{\lambda}_0 = -\frac{1}{4\varepsilon^2}$ and that the associated eigenfunction is:

$$v_0 = \frac{e^{-(2\varepsilon)^{-1}x}}{\left\|e^{-(2\varepsilon)^{-1}x}\right\|_{L^2(\Omega)}}.$$

In addition, it can be proved that the other eigenvalues are, for $m \in \mathbb{N}_*$:

$$\tilde{\lambda}_m = \left(\frac{2\pi m}{L}\right)^2,$$

and their associated eigenfunctions are:

$$v_m = \frac{2\varepsilon\sqrt{\tilde{\lambda}_m}\cos\left(\sqrt{\tilde{\lambda}_m}x\right) + \sin\left(\sqrt{\tilde{\lambda}_m}x\right)}{\left\|2\varepsilon\sqrt{\tilde{\lambda}_m}\cos\left(\sqrt{\tilde{\lambda}_m}x\right) + \sin\left(\sqrt{\tilde{\lambda}_m}x\right)\right\|_{L^2(\Omega)}}.$$

Consequently, Remark 1.2 implies that the spectral decomposition of the adjoint system is given by $\lambda_0 = 0$ and $\lambda_m = \varepsilon\tilde{\lambda}_m + (4\varepsilon)^{-1}$ and the associated eigenfunctions are those proportional to $u_0 = \frac{e^{-(2\varepsilon)^{-1}x}}{\|e^{-(2\varepsilon)^{-1}x}\|_{L^2(\Omega)}}$ and to $u_m = v_m e^{-(2\varepsilon)^{-1}x}$. In addition, we have the following result:

Lemma 3.4. *Let $L > 0$ and $\Omega := (-L, 0)$. Then, for all $\varphi^T \in L^2(\Omega)$ the solution of (3.1) satisfies for all $t \in [0, T]$:*

$$\begin{aligned}\varphi(t,x) = {} & \frac{\int_\Omega \varphi^T(z)dz}{\|e^{-(2\varepsilon)^{-1}z}\|^2_{L^2(\Omega)}} e^{-\varepsilon^{-1}x} \\ & + \sum_{m\in\mathbb{N}_*} \int_\Omega \varphi^T(z) e^{(2\varepsilon)^{-1}z} v_m(z)dz v_m(x) e^{-(2\varepsilon)^{-1}x} \exp\left[-\left(\varepsilon\tilde{\lambda}_m + \frac{1}{4\varepsilon}\right)(T-t)\right]. \quad (3.20)\end{aligned}$$

In particular, the series in the right-hand side of (3.20) is absolutely convergent in $L^2(\Omega)$ for all $t < T$. Moreover, for all $T_0 > 0$ and $\delta > 0$ there is $C > 0$ such that for all $\varphi^T \in L^2(\Omega)$, $T \geq T_0$ and $t \in [0, T - T_0]$ we have the estimate:

$$\begin{aligned}&\left\|\sum_{m\in\mathbb{N}_*} \int_\Omega \varphi^T(z) e^{(2\varepsilon)^{-1}z} v_m(z)dz v_m(x) e^{-(2\varepsilon)^{-1}x} \exp\left[-\left(\varepsilon\tilde{\lambda}_m + \frac{1}{4\varepsilon}\right)(T-t)\right]\right\|_{L^2(\Omega,dx)} \\ &\qquad \leq C\|\varphi^T\|_{L^2(\Omega)} \exp\left(\frac{2L + \delta - (T-t)}{4\varepsilon}\right) \quad (3.21)\end{aligned}$$

The proof of Lemma 3.4 is analogous to the proof of Lemma 2.4. In addition, we remark that, for all $m \in \mathbb{N}$ and $\bar{t} \in (t, T]$:

$$\begin{aligned}&\left(\int_\Omega \varphi^T(z) e^{(2\varepsilon)^{-1}z} v_m(z)dz\right) v_m(x) e^{-(2\varepsilon)^{-1}x} \exp\left[-\left(\varepsilon\tilde{\lambda}_m + \frac{1}{4\varepsilon}\right)(T-t)\right] \\ &= \left(\int_\Omega \varphi^T(z) e^{(2\varepsilon)^{-1}z} v_m(z)dz\right) \exp\left[-\left(\varepsilon\tilde{\lambda}_m + \frac{1}{4\varepsilon}\right)(T-\bar{t})\right] v_m(x) e^{-(2\varepsilon)^{-1}x} \exp\left[-\left(\varepsilon\tilde{\lambda}_m + \frac{1}{4\varepsilon}\right)(\bar{t}-t)\right] \\ &= \Bigg(\int_\Omega \left\{\sum_{r\in\mathbb{N}} \left(\int_\Omega \varphi^T(z) e^{(2\varepsilon)^{-1}z} v_r(z)dz\right) v_r(\tilde{z}) e^{-(2\varepsilon)^{-1}\tilde{z}} \exp\left[-\left(\varepsilon\tilde{\lambda}_r + \frac{1}{4\varepsilon}\right)(T-\bar{t})\right]\right\} \\ &\qquad e^{(2\varepsilon)^{-1}\tilde{z}} v_m(\tilde{z})d\tilde{z}\Bigg) v_m(x) e^{-(2\varepsilon)^{-1}x} \exp\left[-\left(\varepsilon\tilde{\lambda}_m + \frac{1}{4\varepsilon}\right)(\bar{t}-t)\right] \\ &= \left(\int_\Omega \varphi(\bar{t}, z) e^{(2\varepsilon)^{-1}z} v_m(z)dz\right) v_m(x) e^{-(2\varepsilon)^{-1}x} \exp\left[-\left(\varepsilon\tilde{\lambda}_m + \frac{1}{4\varepsilon}\right)(\bar{t}-t)\right]. \quad (3.22)\end{aligned}$$

We have used in the second equality of (3.22) that the eigenfunctions of (3.19) form an orthonormal set of functions and Fubini's Theorem (it can be proved as in Lemma 2.4 that the series is convergent in $L^2(\Omega)$). A consequence of (3.22) is the following:

Corollary 3.5. *Let $L > 0$ and $\Omega := (-L, 0)$. Then, for all $T_0 > 0$ and $\delta > 0$ there is $C > 0$ such that for all $\varphi^T \in L^2(\Omega)$, $T \geq T_0$, $\overline{t} \in [T_0, T]$ and $t \in [0, \overline{t} - T_0]$ we have the estimate:*

$$\left\| \sum_{m \in \mathbb{N}_*} \int_\Omega \varphi^T(z) e^{(2\varepsilon)^{-1} z} v_m(z) dz v_m(x) e^{-(2\varepsilon)^{-1} x} \exp\left[- \left(\varepsilon \tilde{\lambda}_m + \frac{1}{4\varepsilon} \right) (T - t) \right] \right\|_{L^2(\Omega, dx)} \leq C \|\varphi(\overline{t}, \cdot)\|_{L^2(\Omega, dx)} \exp\left(\frac{2L + \delta - (\overline{t} - t)}{4\varepsilon} \right) \quad (3.23)$$

In particular, we obtain that for all $T_0 > 0$ and $\delta > 0$ there is $C > 0$ such that for all $\varphi^T \in L^2(\Omega)$, $T \geq T_0$ and $\overline{t} \in [T_0, T)$:

$$\left\| \sum_{m \in \mathbb{N}_*} \int_\Omega \varphi^T(z) e^{(2\varepsilon)^{-1} z} v_m(z) dz v_m(x) e^{-(2\varepsilon)^{-1} x} \exp\left[- \left(\varepsilon \tilde{\lambda}_m + \frac{1}{4\varepsilon} \right) (T - t) \right] \right\|_{L^2(\Omega, dx)} \leq \frac{C}{T - \overline{t}} \int_{\overline{t}}^T \|\varphi(s, \cdot)\|_{L^2(\Omega)} ds \exp\left(\frac{2L + \delta - (\overline{t} - t)}{4\varepsilon} \right). \quad (3.24)$$

Proof of Theorem 1.15. In order to proof Theorem 1.15 we split the solutions of system (3.1) into two parts with the help of Lemma 3.4. Indeed, we define:

$$\varphi_1(x) := \frac{\int_\Omega \varphi^T(z) dz e^{-\varepsilon x}}{\|e^{-(2\varepsilon)^{-1} z}\|^2_{L^2(\Omega)}}$$

and:

$$\varphi_2(t, x) := \sum_{m \in \mathbb{N}_*} \int_\Omega \varphi^T(z) e^{(2\varepsilon)^{-1} z} v_m(z) dz v_m(x) e^{-(2\varepsilon)^{-1} x} \exp\left[- \left(\varepsilon \tilde{\lambda}_m + \frac{1}{4\varepsilon} \right) (T - t) \right].$$

We remark that:

$$\varphi(t, x) = \varphi_1(x) + \varphi_2(t, x), \quad \text{in } Q.$$

In the following we estimate φ^1 and $\varphi^2(0, \cdot)$ with respect to $\|\varphi\|_{L^2(Q)}$:

- If $\varphi_1 = 0$ the estimate is trivial. Consequently, we suppose from now on that $\varphi_1 \neq 0$. We denote:

  $$\mathcal{N} := \left\{ t : \varphi_2(t, \cdot) \in B_{L^2(\Omega)} \left(-\varphi_1, \frac{\|\varphi_1\|_{L^2(\Omega)}}{2} \right) \right\},$$

  for

  $$B_X(w, r) := \{ v \in X : \|v - w\|_X < r \}.$$

We have that:

$$|\mathcal{N}| < 3L, \tag{3.25}$$

as otherwise there are some t_1 and t_2 such that $t_2 - t_1 \geq 3L$ and:

$$\varphi_2(t_1, \cdot), \varphi_2(t_2, \cdot) \in B_{L^2(\Omega)}\left(-\varphi_1, \frac{\|\varphi_1\|_{L^2(\Omega)}}{2}\right),$$

but by (3.21) this is impossible. Indeed, the system is autonomous and if $t_2 \in \mathcal{N}$ we have the estimate:

$$\|\varphi(t_2, \cdot)\|_{L^2(\Omega)} \leq \frac{\|\varphi_1\|_{L^2(\Omega)}}{2},$$

so $t_1 \notin \mathcal{N}$ for ε small enough. In addition, we remark that in $[0, T] \setminus \mathcal{N}$ we have that:

$$\|\varphi(t, \cdot)\|_{L^2(\Omega)} \geq \frac{\|\varphi^1\|_{L^2(\Omega)}}{2}. \tag{3.26}$$

Thus, we obtain from (3.25) and (3.26) that for all $\varphi^T \in L^2(\Omega)$ and $T > 4L$:

$$\frac{\|\varphi_1\|_{L^2(\Omega)}}{\|\varphi\|_{L^2(Q)}} \leq \frac{\|\varphi_1\|_{L^2(\Omega)}}{\|\varphi\|_{L^2(((0,T)\setminus\mathcal{N})\times\Omega)}} \leq \frac{2}{\sqrt{T - 3L}} \leq \frac{2}{\sqrt{T/4}} = \frac{4}{\sqrt{T}}. \tag{3.27}$$

- As a direct consequence of (3.24) for $\bar{t} = T - L$, we have that, independently of φ^T:

$$\frac{\|\varphi_2(0, \cdot)\|_{L^2(\Omega)}}{\|\varphi\|_{L^2(Q)}} \leq \frac{Ce^{-c\varepsilon^{-1}}\|\varphi\|_{L^2((T-L,T)\times\Omega)}}{\|\varphi\|_{L^2(Q)}} \leq Ce^{-c\varepsilon^{-1}}. \tag{3.28}$$

Thus, combining (3.27) and (3.28) we obtain (1.27) for ε small enough.

□

4 Further comments and open problems

We remark the following:

- We obtain similar results for the boundary controllability.
- Under the hypothesis (1.12) if T is small the cost of the null controllability of (1.1) goes to 0, which can be proved for instance as in [18] and [1].

We now present some problems that remain open:

- Get precise estimates of the cost of the control problem (1.1) when a also depends on the time variable.

- Determine if Theorem 1.15 is still true for all $\Omega \subset \mathbb{R}^d$ bounded domains.
- Under the hypothesis (1.15) to know when there is some y^0 such that the norm of the minimal norm control which takes y^0 to 0 increases exponentially and when for all y^0 and for ε small enough the cost is bounded by a constant; (or, even better, when for all y^0 the associated control does not decay with ε and it is just a problem of uniformity).
- Study the cost of the controllability in the context of Section 3.3 when $(-L, -L + h) \subset \omega$.

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