



HAL
open science

Local null controllability of a fluid-rigid body interaction problem with Navier slip boundary conditions

Imene A Djebour

► **To cite this version:**

Imene A Djebour. Local null controllability of a fluid-rigid body interaction problem with Navier slip boundary conditions. 2020. hal-02454567v1

HAL Id: hal-02454567

<https://hal.science/hal-02454567v1>

Preprint submitted on 24 Jan 2020 (v1), last revised 31 May 2021 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Local null controllability of a fluid-rigid body interaction problem with Navier slip boundary conditions

Imene Aicha Djebour¹

¹Université de Lorraine, CNRS, Inria, IECL, F-54000 Nancy

January 24, 2020

Abstract

The aim of this work is to show the local null controllability of a fluid-solid interaction system by using a distributed control located in the fluid. The fluid is modeled by the incompressible Navier-Stokes system with Navier slip boundary conditions and the rigid body is governed by the Newton laws. Our main result yields that we can drive the velocities of the fluid and of the structure to 0 and we can control exactly the position of the rigid body, provided that its shape is not a disk. One important ingredient consists in a new Carleman estimate for a linear fluid-rigid body system with Navier boundary conditions.

Keywords: Navier-Stokes system, Navier slip boundary conditions, Null controllability, fluid-solid interaction system.

2010 Mathematics Subject Classification. 35Q30, 93C20, 93B05.

Contents

1	Introduction	1
2	Preliminaries	5
3	Change of variables	8
4	Carleman estimate for the Laplacian problem with Navier slip boundary conditions	10
5	Carleman estimate for the linearized system	18
6	Null controllability for the linearized system	26
7	Fixed point	30

1 Introduction

Let Ω be a bounded, non empty open subset of \mathbb{R}^2 with a regular boundary. We assume that Ω contains a rigid body and an incompressible viscous fluid. At each time $t > 0$, the domain of the rigid body is denoted by $\mathcal{S}(t) \subset \Omega$ that is assumed to be compact with non empty interior and regular. The fluid domain is denoted by $\mathcal{F}(t) = \Omega \setminus \mathcal{S}(t)$, and is assumed to be connected.

We consider the following system describing the evolution of the fluid which is governed by the incompressible Navier-Stokes system

$$\begin{cases} \partial_t U + (U \cdot \nabla)U - \nabla \cdot \mathbb{T}(U, P) = v^* \mathbf{1}_{\mathcal{O}} & \text{in } (0, T), \text{ in } \mathcal{F}(t), \\ \nabla \cdot U = 0 & \text{in } (0, T), \text{ in } \mathcal{F}(t). \end{cases} \quad (1.1)$$

In the above system, we have denoted by U the fluid velocity, P the fluid pressure and by v^* the control acting on the system through $\mathcal{O} \subset \mathbb{R}^2$, where \mathcal{O} is a non empty open subset such that $\overline{\mathcal{O}} \subset \mathcal{F}(t)$.

The Cauchy stress tensor $\mathbb{T}(U, P)$ is defined by

$$\mathbb{T}(U, P) = -PI_3 + 2\nu D(U), \quad D(U)_{i,j} = \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right),$$

where ν is the viscosity of the fluid. We denote for each time t , the position of the structure by $h(t) \in \mathbb{R}^2$ and by $R_{\theta(t)}$ the rotation matrix of angle θ of the solid defined by

$$R_{\theta(t)} = \begin{pmatrix} \cos \theta(t) & -\sin \theta(t) \\ \sin \theta(t) & \cos \theta(t) \end{pmatrix}.$$

Then, the flow of the structure is given by $X_{\mathcal{S}}(t, \cdot) : \mathcal{S} \rightarrow \mathcal{S}(t)$ where

$$X_{\mathcal{S}}(t, y) = h(t) + R_{\theta(t)} y, \quad t \in (0, T), \quad y \in \mathcal{S}, \quad (1.2)$$

where \mathcal{S} is a fixed subset of \mathbb{R}^2 , non empty, compact with a regular boundary.

We notice that $X_{\mathcal{S}}(t, \cdot)$ is invertible, we denote its inverse by $Y_{\mathcal{S}}(t, \cdot) : \mathcal{S}(t) \rightarrow \mathcal{S}$ where

$$Y_{\mathcal{S}}(t, x) = R_{\theta(t)}^{-1}(x - h(t)), \quad t \in (0, T), \quad x \in \mathcal{S}(t).$$

Thus, the Eulerian velocity of the structure is given by

$$U_{\mathcal{S}}(t, x) = h'(t) + R'_{\theta(t)} R_{\theta(t)}^{-1}(x - h(t)), \quad t \in (0, T), \quad x \in \mathcal{S}(t).$$

We denote by a^\perp , the vector $\begin{pmatrix} -a_2 \\ a_1 \end{pmatrix}$, for any $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^2$. We notice that $R'_{\theta(t)} R_{\theta(t)}^{-1}$ is a skew-symmetric matrix, then the Eulerian velocity of the structure writes

$$U_{\mathcal{S}}(t, x) = h'(t) + \omega(t)(x - h(t))^\perp,$$

where $\omega(t) = \theta'(t)$ represents the angular velocity of the rigid body.

We denote by $\mathcal{S}_{h,\theta}$ the set

$$\mathcal{S}_{h,\theta} = h + R_\theta \mathcal{S},$$

and we define the corresponding fluid domain

$$\mathcal{F}_{h,\theta} = \Omega \setminus \mathcal{S}_{h,\theta},$$

for any $h \in \mathbb{R}^2$, $\theta \in \mathbb{R}$. Then, with these notations, we have

$$\mathcal{S}(t) = \mathcal{S}_{h(t),\theta(t)}, \quad \mathcal{F}(t) = \mathcal{F}_{h(t),\theta(t)}.$$

We point out that the fluid domain is depending on the displacement of the solid structure, consequently, it depends on time.

We denote by \hat{n} the outward unit normal to $\partial\mathcal{F}(t)$, where $\partial\mathcal{F}(t) = \partial\Omega \cup \partial\mathcal{S}(t)$.

The motion of the structure is governed by the balance equations for linear and angular momenta

$$\begin{cases} mh''(t) = - \int_{\partial\mathcal{S}(t)} \mathbb{T}(U, P) \hat{n} \, d\Gamma & t \in (0, T), \\ J\omega'(t) = - \int_{\partial\mathcal{S}(t)} (x - h(t))^\perp \cdot \mathbb{T}(U, P) \hat{n} \, d\Gamma & t \in (0, T). \end{cases} \quad (1.3)$$

We complete (1.1) and (1.3) by the Navier slip boundary conditions. In order to write these boundary conditions, we need to introduce some notations. We denote by $a_{\hat{n}}$ and $a_{\hat{\tau}}$ the normal and the tangential parts of $a \in \mathbb{R}^2$:

$$a_{\hat{n}} = (a \cdot \hat{n}) \hat{n}, \quad a_{\hat{\tau}} = a - a_{\hat{n}}.$$

Then, the boundary conditions write as follows

$$\begin{cases} U_{\hat{n}} = 0 & \text{in } (0, T), \text{ on } \partial\Omega, \\ [2\nu D(U) \hat{n} + \beta_\Omega U]_{\hat{\tau}} = 0 & \text{in } (0, T), \text{ on } \partial\Omega, \\ (U - U_S)_{\hat{n}} = 0 & \text{in } (0, T), \text{ on } \partial\mathcal{S}(t), \\ [2\nu D(U) \hat{n} + \beta_S (U - U_S)]_{\hat{\tau}} = 0 & \text{in } (0, T), \text{ on } \partial\mathcal{S}(t), \end{cases} \quad (1.4)$$

where $\beta_\Omega \geq 0$ and $\beta_S \geq 0$ are the friction coefficients.

Let $h^0, \tilde{\ell}^0 \in \mathbb{R}^2, \theta^0, \omega^0 \in \mathbb{R}$ and $u^0 \in [H^1(\mathcal{F}_{h^0, \theta^0})]^2$. We furnish the following initial conditions

$$U(0, x) = u^0(x), \quad x \in \mathcal{F}_{h^0, \theta^0}, \quad h'(0) = \tilde{\ell}^0, \quad \omega(0) = \omega^0, \quad h(0) = h^0, \quad \theta(0) = \theta^0, \quad (1.5)$$

such that the following compatibility conditions are satisfied

$$\begin{cases} \nabla \cdot u^0 = 0 & \text{in } \mathcal{F}_{h^0, \theta^0}, \\ u_{\hat{n}}^0 = 0 & \text{on } \partial\Omega, \\ (u^0 - u_S^0)_{\hat{n}} = 0 & \text{on } \partial\mathcal{S}_{h^0, \theta^0}, \end{cases} \quad (1.6)$$

where $u_S^0(x) = \tilde{\ell}^0 + \omega^0(x - h^0)^\perp$. Without loss of generality, we assume that the center of gravity of \mathcal{S} is at the origin. Then, $h(t)$ will be the position of the center of mass of the rigid body $\mathcal{S}(t)$.

Our main objective in this paper is to look for a control v^* acting on \mathcal{O} such that for any $(h_T, \theta_T) \in \mathbb{R}^2 \times \mathbb{R}$ with

$$\mathcal{S}_{h_T, \theta_T} \subset \Omega \setminus \overline{\mathcal{O}}, \quad (1.7)$$

we get that $h(T) = h_T, \theta(T) = \theta_T$ and the velocities of the fluid and of the rigid body are equal to 0 at time T .

The main result of this paper is stated below:

Theorem 1.1. *Assume that \mathcal{S} is not a disk and let (h_T, θ_T) that satisfies (1.7). Then, there exists $\varepsilon > 0$ such that for any $(u^0, h^0, \tilde{\ell}^0, \omega^0, \theta^0)$ that satisfies (1.6) and*

$$\|u^0\|_{[H^1(\mathcal{F}_{h^0, \theta^0})]^2} + |h^0 - h_T| + |\tilde{\ell}^0| + |\omega^0| + |\theta^0 - \theta_T| \leq \varepsilon,$$

there exists a control $v^* \in L^2(0, T; [L^2(\mathcal{O})]^2)$ such that

$$U(T, \cdot) = 0 \text{ in } \mathcal{F}_{h_T, \theta_T}, \quad h(T) = h_T, \quad h'(T) = 0, \quad \omega(T) = 0, \quad \theta(T) = \theta_T.$$

Without loss of generality, we can always assume that

$$h_T = 0, \quad \theta_T = 0, \quad \text{and thus } \mathcal{S}_{h_T, \theta_T} = \mathcal{S}, \quad \mathcal{F}_{h_T, \theta_T} = \mathcal{F}.$$

Remark 1.2. 1. Let us note that if \mathcal{S} is a disk, the boundary conditions (1.4) on $\partial\mathcal{S}(t)$ write

$$\begin{cases} (U(t, \cdot) - h'(t))_{\widehat{n}} = 0 & t \in (0, T), \text{ on } \partial\mathcal{S}(t), \\ [2\nu D(U)\widehat{n} + \beta_{\mathcal{S}}(U - U_{\mathcal{S}})]_{\widehat{\tau}} = 0 & \text{in } (0, T), \text{ on } \partial\mathcal{S}(t), \end{cases}$$

and we see that the angular velocity only appears on the second boundary condition (and if $\beta_{\mathcal{S}} > 0$) and unhappily it seems difficult to derive Carleman estimates that allow us to control ω in that case. That is why we exclude in Theorem 1.1 the case of a disk.

2. Theorem 1.1 is still valid in dimension three, but in this case we should assume that \mathcal{S} is not rotationally symmetric. This geometrical assumption is sufficient to prove the Carleman inequality obtained in Section 5. More precisely, using this geometrical hypothesis, for any $z \in [L^2(\Omega)]^3$, $\ell \in \mathbb{R}^3$, $k \in \mathbb{R}^3$ such that

$$\nabla \cdot z = 0 \text{ in } \mathcal{F}, \quad z \cdot n = (\ell + k \times y) \cdot n 1_{\partial\mathcal{S}} \text{ on } \partial\mathcal{F},$$

we have

$$\|z\|_{[L^2(\mathcal{F})]^3} \geq C(|\ell| + |k|),$$

where C is a positive constant independent of z , ℓ and k . This argument is proved in [22, Lemma 2.2].

Several works were devoted to the study of fluid-rigid body interaction systems, in particular, when the fluid is governed by the Navier-Stokes system. Existence results concerning this kind of systems with Dirichlet boundary conditions were considered in [5], [7], [8],[24], [25], [15], [19], [23] etc. For the case of the Navier slip boundary conditions (1.4), the existence of weak solutions is proved in [13] and the existence of strong solutions is obtained in [26]. In [26] and [14], the authors proved that collisions can occur in final time between the rigid body and the domain cavity with some assumptions on the solid geometry.

Concerning the controllability, let us mention [10] and [21], where the authors obtained the local exact controllability of the 2D or 3D Navier-Stokes equations with Dirichlet boundary conditions considering distributed controls. The controllability of the Navier Stokes system with nonlinear Navier boundary conditions was studied in [17]. In [18], the authors established the local controllability with $N - 1$ scalar controls. Concerning the controllability results of fluid-structure systems with Dirichlet boundary conditions, in dimension 2, we mention the paper [3], where the authors proved the null controllability in velocity and the exact controllability for the position of the rigid body assuming some geometric properties for the solid and provided that the initial conditions are small enough, more precisely a condition of smallness on the H^3 norm of the initial fluid velocity is needed. The authors used the Kakutani's fixed point theorem to deduce the null controllability of the nonlinear system. We have also the paper [20] where the authors considered the structure of a rigid ball, their result relies on semigroup theory. In the latest paper, only an assumption on the H^1 norm of the initial fluid velocity is needed. In dimension 3, we mention [2], the same result was proved without any assumptions on the solid geometry while a condition of smallness on the H^2 norm of the initial fluid velocity is needed. We also mention [22], where the authors considered the interaction between a viscous and incompressible fluid modeled by the Boussinesq system and a rigid body with arbitrary shape, they proved null controllability of the associated system.

In this paper, we prove the local null controllability of the system (1.1), (1.3), (1.4), (1.5), that is the case of the Navier slip boundary conditions in the presence of the structure. We follow the same method as [20]: we use a change of variables to write our system in a fixed domain and use a fixed point argument to reduce our problem to the null controllability of a linear fluid-rigid body system, that is coupling the Stokes system with ODE for the structure velocity. To do this we derive a Carleman estimates for the corresponding system.

One of the main difficulties to obtain such an estimate is to manage the boundary conditions and more precisely to obtain estimates of the rigid velocity with the good weights. An important step for this calculation is a Carleman estimates for the Laplacian equation with divergence free condition and Navier slip boundary conditions, which is given in section 4.

The outline of this paper is as follows: in section 2, we give some preliminaries. We emphasize that one of the main difficulties in this problem is that we are dealing with a coupled system set on a non cylindrical domain. Then, in section 3, we remap the problem into an equivalent system given in a fixed geometry. In section 5,

we establish a new Carleman inequality. In section 6, we prove the null controllability of the linearized system. Finally, in section 7, we prove Theorem 1.1 and deduce the null controllability of the system by applying a fixed-point argument.

2 Preliminaries

In this section, we prove some regularity results of an associated linearized problem. We consider the following linear system

$$\begin{cases} \partial_t \bar{w} - \nabla \cdot \mathbb{T}(\bar{w}, \bar{\pi}) = F_1 & \text{in } (0, T), \text{ in } \mathcal{F}, \\ \nabla \cdot \bar{w} = 0 & \text{in } (0, T), \text{ in } \mathcal{F}, \\ m \ell'_{\bar{w}}(t) = - \int_{\partial \mathcal{S}} \mathbb{T}(\bar{w}, \bar{\pi}) n \, d\Gamma + F_2 & t \in (0, T), \\ J(k'_{\bar{w}}(t)) = - \int_{\partial \mathcal{S}} y^\perp \cdot \mathbb{T}(\bar{w}, \bar{\pi}) n \, d\Gamma + F_3 & t \in (0, T), \end{cases} \quad (2.1)$$

with the boundary conditions

$$\begin{cases} \bar{w}_n = 0 & \text{in } (0, T), \text{ on } \partial \Omega, \\ [2\nu D(\bar{w})n + \beta_\Omega \bar{w}]_\tau = 0 & \text{in } (0, T), \text{ on } \partial \Omega, \\ (\bar{w} - \bar{w}_\mathcal{S})_n = 0 & \text{in } (0, T), \text{ on } \partial \mathcal{S}, \\ [2\nu D(\bar{w})n + \beta_\mathcal{S}(\bar{w} - \bar{w}_\mathcal{S})]_\tau = 0 & \text{in } (0, T), \text{ on } \partial \mathcal{S}, \end{cases} \quad (2.2)$$

where $\bar{w}_\mathcal{S}(y) = \ell_{\bar{w}} + k_{\bar{w}} y^\perp$, completed with the initial conditions

$$\bar{w}(0, \cdot) = w^0, \quad \text{in } \mathcal{F}, \quad \ell_{\bar{w}}(0) = \ell_w^0, \quad k_{\bar{w}}(0) = k_w^0. \quad (2.3)$$

We have the following regularity result for the system (2.1), (2.2) and (2.3) which is proved in [26].

Theorem 2.1. *Let $T > 0$. Suppose that $F_1 \in L^2(0, T; [L^2(\mathcal{F})]^2)$, $F_2 \in [L^2(0, T)]^2$ and $F_3 \in L^2(0, T)$ are given functions and $w^0 \in [H^1(\mathcal{F})]^2$ such that*

$$\nabla \cdot w^0 = 0, \quad \text{in } \mathcal{F}, \quad w_n^0 = 0, \quad \text{on } \partial \Omega, \quad w_n^0(y) = (\ell_w^0 + k_w^0 y^\perp)_n, \quad y \in \partial \mathcal{S}.$$

Then, there exists a unique solution to problem (2.1), (2.2) and (2.3) such that

$$\begin{aligned} \bar{w} &\in L^2(0, T; [H^2(\mathcal{F})]^2) \cap C([0, T]; [H^1(\mathcal{F})]^2) \cap H^1(0, T; [L^2(\mathcal{F})]^2), \quad \bar{\pi} \in L^2(0, T; H^1(\mathcal{F})/\mathbb{R}), \\ (\ell_{\bar{w}}, k_{\bar{w}}) &\in [H^1(0, T)]^2 \times H^1(0, T). \end{aligned}$$

Moreover, it satisfies the following estimate

$$\begin{aligned} &\|\bar{w}\|_{L^2(0, T; [H^2(\mathcal{F})]^2) \cap C([0, T]; [H^1(\mathcal{F})]^2) \cap H^1(0, T; [L^2(\mathcal{F})]^2)} + \|\bar{\pi}\|_{L^2(0, T; H^1(\mathcal{F})/\mathbb{R})} + \|\ell_{\bar{w}}\|_{[H^1(0, T)]^2} + \|k_{\bar{w}}\|_{H^1(0, T)} \\ &\leq C \left(\|F_1\|_{L^2(0, T; [L^2(\mathcal{F})]^2)} + \|F_2\|_{[L^2(0, T)]^2} + \|F_3\|_{L^2(0, T)} + \|w^0\|_{[H^1(\mathcal{F})]^2} + |\ell_w^0| + |k_w^0| \right). \end{aligned} \quad (2.4)$$

The proof of the above theorem is based on semigroup theory. For the sake of completeness, we just recall the main ideas of the proof.

We note that w^0 and \bar{w} are extended by $\ell_w^0 + k_w^0 y^\perp$ and $\ell_{\bar{w}} + k_{\bar{w}} y^\perp$ on \mathcal{S} respectively. Let define the following Hilbert spaces

$$\begin{aligned} \mathbb{H} &= \{w \in [L^2(\Omega)]^2 \mid \nabla \cdot w = 0, \quad w_n = 0, \text{ on } \partial \Omega, \quad D(w) = 0 \text{ in } \mathcal{S}\}, \\ \mathbb{V} &= \{w \in \mathbb{H} \mid w|_{\mathcal{F}} \in [H^1(\mathcal{F})]^2\}. \end{aligned}$$

We notice that the condition $D(w) = 0$ on \mathcal{S} is equivalent to $w = w_\mathcal{S} \in \mathcal{R}$ on \mathcal{S} where

$$\mathcal{R} = \{v \in \mathbb{R}^2 \mid \text{there exist } \ell_v \in \mathbb{R}^2, \quad k_v \in \mathbb{R} \text{ such that } v(y) = \ell_v + k_v y^\perp\}.$$

For $w, v \in \mathbb{H}$, we define the inner product on \mathbb{H} by

$$\langle w, v \rangle = \int_{\mathcal{F}} w \cdot v \, dy + m \ell_v \cdot \ell_w + J k_v k_w.$$

Let define also the orthogonal projector $\mathbb{P} : [L^2(\Omega)]^2 \rightarrow \mathbb{H}$.

The system (2.1), (2.2) and (2.3) can be reduced to the following form

$$\bar{w}' = A\bar{w} + F, \quad \bar{w}(0) = w^0, \quad (2.5)$$

where the operator A is defined by

$$Aw = \begin{cases} \nu \Delta w & \text{in } \mathcal{F}, \\ -\frac{2\nu}{m} \int_{\partial\mathcal{S}} D(w)n \, d\Gamma - \left(\frac{2\nu}{J} \left[\int_{\partial\mathcal{S}} y^\perp \cdot D(w)n \, d\Gamma \right] \right) y^\perp & \text{on } \partial\mathcal{S}, \end{cases} \quad (2.6)$$

$$A = \mathbb{P}A, \quad \mathcal{D}(A) = \mathcal{D}(\mathcal{A}) = \{w \in \mathbb{V}, w|_{\mathcal{F}} \in [H^2(\mathcal{F})]^2, [2\nu D(w)n + \beta_\Omega w]_\tau = 0, \text{ on } \partial\Omega, \\ [2\nu D(w)n + \beta_{\mathcal{S}}(w - w_{\mathcal{S}})]_\tau = 0, \text{ on } \partial\mathcal{S}\},$$

and

$$F = \mathbb{P} \left(F_1 1_{\mathcal{F}} + \left(\frac{F_2}{m} + \frac{F_3 y^\perp}{J} \right) 1_{\mathcal{S}} \right).$$

In [26, Lemma 3.1], it is proved that the operator A is self-adjoint and it generates a semigroup of contractions on \mathbb{H} . Thus, we deduce Theorem 2.1 (see [26, Proposition 3.3]).

We note here that since A is a self-adjoint operator, then for any $w \in \mathcal{D}(A)$, we have

$$\|w\|_{\mathcal{D}((-A)^{1/2})}^2 = \langle w, w \rangle + \langle w, Aw \rangle.$$

We also need some regularity results on the linear system (2.1), (2.2), (2.3).

Let $w^0 \in [H^3(\mathcal{F})]^2$, $\pi^0 \in H^2(\mathcal{F})$, $(\ell_w^0, k_w^0) \in \mathbb{R}^3$ and we set $w^1 \in [H^1(\mathcal{F})]^2$, $(\ell_w^1, k_w^1) \in \mathbb{R}^3$ such that

$$w^1 = \nabla \cdot \mathbb{T}(w^0, \pi^0) + F_1(0, \cdot)$$

$$\ell_w^1 = -\frac{1}{m} \int_{\partial\mathcal{S}} \mathbb{T}(w^0, \pi^0)n \, d\Gamma + \frac{1}{m} F_2(0),$$

and

$$k_w^1 = -\frac{1}{J} \int_{\partial\mathcal{S}} y^\perp \cdot \mathbb{T}(w^0, \pi^0)n \, d\Gamma + \frac{1}{J} F_3(0).$$

Moreover, we suppose that π^0 satisfies the following system

$$\begin{cases} \Delta \pi^0 = \nabla \cdot F_1(0, \cdot) & \text{in } \mathcal{F}, \\ \frac{\partial \pi^0}{\partial n} = (\nu \Delta w^0 + F_1(0, \cdot)) \cdot n - (\ell_w^1 + k_w^1 y^\perp) \cdot n 1_{\partial\mathcal{S}} & \text{on } \partial\mathcal{F}. \end{cases}$$

By the Lax-Milgram theorem, the above system admits a unique solution $\pi^0 \in H^1(\mathcal{F})/\mathbb{R}$, such that

$$\|\nabla \pi^0\|_{[L^2(\mathcal{F})]^2} + |\ell_w^1| + |k_w^1| \leq C \left(\|w^0\|_{[H^3(\mathcal{F})]^2} + \|F_1\|_{L^2(0,T;[H^2(\mathcal{F})]^2) \cap H^1(0,T;[L^2(\mathcal{F})]^2)} \right. \\ \left. + \|F_2\|_{[H^1(0,T)]^2} + \|F_3\|_{H^1(0,T)} \right).$$

Since $\partial\mathcal{F}$ is sufficiently regular, we get that

$$(\nu\Delta w^0 + F_1(0, \cdot)) \cdot n - (\ell_w^1 + k_w^1 y^\perp) \cdot n|_{\partial\mathcal{S}} \in H^{1/2}(\partial\mathcal{F}).$$

Then, using classical elliptic estimate of the Neumann system, we obtain

$$\begin{aligned} \|\pi^0\|_{H^2(\mathcal{F})} + |\ell_w^1| + |k_w^1| &\leq C \left(\|w^0\|_{[H^3(\mathcal{F})]^2} + \|F_1\|_{L^2(0,T;[H^2(\mathcal{F})]^2) \cap H^1(0,T;[L^2(\mathcal{F})]^2)} \right. \\ &\quad \left. + \|F_2\|_{[H^1(0,T)]^2} + \|F_3\|_{H^1(0,T)} \right). \end{aligned}$$

We set the compatibility conditions

$$\nabla \cdot w^1 = 0, \quad \text{in } \mathcal{F}, \quad w_n^1 = 0, \quad \text{on } \partial\Omega, \quad w_n^1(y) = (\ell_w^1 + k_w^1 y^\perp)_n, \quad y \in \partial\mathcal{S}. \quad (2.7)$$

Proposition 2.2. *Let $T > 0$. Suppose that $F_1 \in L^2(0, T; [H^2(\mathcal{F})]^2) \cap H^1(0, T; [L^2(\mathcal{F})]^2)$, $F_2 \in [H^1(0, T)]^2$, $F_3 \in H^1(0, T)$ and let (w^0, ℓ_w^0, k_w^0) such that the compatibility conditions (2.7) hold. Then, there exists a unique solution to problem (2.1), (2.2) and (2.3) such that*

$$\begin{aligned} \bar{w} &\in L^2(0, T; [H^4(\mathcal{F})]^2) \cap H^2(0, T; [L^2(\mathcal{F})]^2), \quad \bar{\pi} \in L^2(0, T; H^3(\mathcal{F})) \cap H^1(0, T; H^1(\mathcal{F})), \\ (\ell_{\bar{w}}, k_{\bar{w}}) &\in [H^2(0, T)]^2 \times H^2(0, T). \end{aligned}$$

Moreover, it satisfies the following estimate

$$\begin{aligned} \|\bar{w}\|_{L^2(0,T;[H^4(\mathcal{F})]^2) \cap H^2(0,T;[L^2(\mathcal{F})]^2)} + \|\bar{\pi}\|_{L^2(0,T;H^3(\mathcal{F})) \cap H^1(0,T;H^1(\mathcal{F}))} + \|\ell_{\bar{w}}\|_{[H^2(0,T)]^2} + \|k_{\bar{w}}\|_{H^2(0,T)} \\ \leq C \left(\|F_1\|_{L^2(0,T;[H^2(\mathcal{F})]^2) \cap H^1(0,T;[L^2(\mathcal{F})]^2)} + \|F_2\|_{[H^1(0,T)]^2} + \|F_3\|_{H^1(0,T)} \right. \\ \left. + \|w^0\|_{[H^3(\mathcal{F})]^2} + |\ell_w^0| + |k_w^0| \right). \quad (2.8) \end{aligned}$$

Proof. We differentiate the system (2.1), (2.2). we get

$$\begin{cases} \partial_{tt}\bar{w} - \nabla \cdot \mathbb{T}(\partial_t\bar{w}, \partial_t\bar{\pi}) = \partial_t F_1 & \text{in } (0, T), \quad \text{in } \mathcal{F}, \\ \nabla \cdot (\partial_t\bar{w}) = 0 & \text{in } (0, T), \quad \text{in } \mathcal{F}, \\ m\ell_{\bar{w}}''(t) = - \int_{\partial\mathcal{S}} \mathbb{T}(\partial_t\bar{w}, \partial_t\bar{\pi})n \, d\Gamma + \partial_t F_2 & t \in (0, T), \\ J(k_{\bar{w}}''(t)) = - \int_{\partial\mathcal{S}} y^\perp \cdot \mathbb{T}(\partial_t\bar{w}, \partial_t\bar{\pi})n \, d\Gamma + \partial_t F_3 & t \in (0, T), \end{cases} \quad (2.9)$$

with the boundary conditions

$$\begin{cases} \partial_t\bar{w}_n = 0 & \text{in } (0, T), \quad \text{on } \partial\Omega, \\ [2\nu D(\partial_t\bar{w})n + \beta_\Omega \partial_t\bar{w}]_\tau = 0 & \text{in } (0, T), \quad \text{on } \partial\Omega, \\ (\partial_t\bar{w} - \bar{w}'_S)_n = 0 & \text{in } (0, T), \quad \text{on } \partial\mathcal{S}, \\ [2\nu D(\partial_t\bar{w})n + \beta_S(\partial_t\bar{w} - \bar{w}'_S)]_\tau = 0 & \text{in } (0, T), \quad \text{on } \partial\mathcal{S}, \end{cases} \quad (2.10)$$

where $\bar{w}'_S = \ell_{\bar{w}}' + k_{\bar{w}}' y^\perp$ with the initial conditions

$$\partial_t\bar{w}(0, \cdot) = w^1, \quad \text{in } \mathcal{F}, \quad \ell_{\bar{w}}'(0) = \ell_w^1, \quad k_{\bar{w}}'(0) = k_w^1, \quad \ell_{\bar{w}}(0) = \ell_w^0, \quad k_{\bar{w}}(0) = k_w^0. \quad (2.11)$$

Since (2.7) is satisfied, we can apply Theorem 2.1, we get

$$\begin{aligned} \|\partial_t\bar{w}\|_{L^2(0,T;[H^2(\mathcal{F})]^2) \cap H^1(0,T;[L^2(\mathcal{F})]^2)} + \|\partial_t\bar{\pi}\|_{L^2(0,T;H^1(\mathcal{F}))} + \|\ell_{\bar{w}}'\|_{[H^1(0,T)]^2} + \|k_{\bar{w}}'\|_{H^1(0,T)} \\ \leq C \left(\|F_1\|_{L^2(0,T;[H^2(\mathcal{F})]^2) \cap H^1(0,T;[L^2(\mathcal{F})]^2)} + \|F_2\|_{[H^1(0,T)]^2} + \|F_3\|_{H^1(0,T)} + \|w^0\|_{[H^3(\mathcal{F})]^2} \right). \quad (2.12) \end{aligned}$$

We obtain from (2.12) that $\ell_{\bar{w}} \in [H^2(0, T)]^2$ and $k_{\bar{w}} \in H^2(0, T)$. Then, using the regularity results for the unstationary Stokes system with Navier boundary conditions proved in [16], combined with (2.12) and (2.4), we get

$$\begin{aligned} & \|\bar{w}\|_{L^2(0, T; [H^4(\mathcal{F})]^2) \cap H^2(0, T; [L^2(\mathcal{F})]^2)} + \|\bar{\pi}\|_{L^2(0, T; H^3(\mathcal{F})) \cap H^1(0, T; H^1(\mathcal{F}))} + \|\ell_{\bar{w}}\|_{[H^2(0, T)]^2} + \|k_{\bar{w}}\|_{H^2(0, T)} \\ & \leq C \left(\|F_1\|_{L^2(0, T; [H^2(\mathcal{F})]^2) \cap H^1(0, T; [L^2(\mathcal{F})]^2)} + \|F_2\|_{[H^1(0, T)]^2} + \|F_3\|_{H^1(0, T)} \right. \\ & \qquad \qquad \qquad \left. + \|w^0\|_{[H^3(\mathcal{F})]^2} + |\ell_w^0| + |k_w^0| \right). \end{aligned}$$

Then, we obtain (2.8). \square

3 Change of variables

To treat the free boundary problem (1.1), (1.3), (1.4), (1.5), we consider an equivalent system written in a fixed domain using a change of variables that was already introduced in [24]. In fact, we construct an extension of the structure flow (1.2) over Ω by a regular and incompressible flow. First, we need to control the distance between the structure and the boundary $\partial(\Omega \setminus \bar{\mathcal{O}})$.

The condition (1.7) implies that there exists $d > 0$ such that

$$d(\partial(\Omega \setminus \bar{\mathcal{O}}), \mathcal{S}) = d.$$

Let $t \in [0, T]$. We have for a fixed $y \in \mathcal{S}$

$$d(y, \mathcal{S}(t)) \leq |h(t) + R_{\theta(t)}y - y|.$$

Then, we get

$$d(\mathcal{S}(t), \mathcal{S}) \leq C (|h(t)| + |R_{\theta(t)} - I_2|). \quad (3.1)$$

If

$$|h(t)| + |R_{\theta(t)} - I_2| \leq \frac{d}{2C}, \quad (3.2)$$

we get

$$d(\mathcal{S}(t), \mathcal{S}) \leq \frac{d}{2}.$$

Thus, we obtain

$$d(\partial(\Omega \setminus \bar{\mathcal{O}}), \mathcal{S}(t)) \geq \frac{d}{2}, \quad t \in [0, T].$$

In other words, we only assume that no collision occurs between the structure and the boundary $\partial(\Omega \setminus \bar{\mathcal{O}})$ at time T . In fact, if the initial data are small enough, then the displacement of the structure remains small, then (3.2) is satisfied. Thus, no contact can occur between the solid and the boundary for any $t \in [0, T]$.

Following [24], we can construct a change of variables X and Y with the following properties

- For any $t \in [0, T]$, X and Y are C^∞ diffeomorphisms from Ω into itself,
- The function X is invertible of inverse Y ,
- In a neighborhood of $\mathcal{S} = \mathcal{S}(T)$, $X(t, y) = X_{\mathcal{S}}(t, y) = h(t) + R_{\theta(t)}y$,
- In a neighborhood of $\partial\Omega$ and of \mathcal{O} , $X(t, y) = y$,
- $\det \nabla X(t, y) = 1$, for all $y \in \Omega$,

- In a neighborhood of \mathcal{S} , $\nabla X(t, y) = R_{\theta(t)}$ and $\nabla Y(t, X(t, y)) = R_{\theta(t)}^{-1}$.

Moreover, we have

$$\|X\|_{H^2(0,T;[C^2(\bar{\Omega})]^2)} + \|Y\|_{H^2(0,T;[C^2(\bar{\Omega})]^2)} \leq C(\|h\|_{[H^2(0,T)]^2} + \|\theta\|_{H^2(0,T)}), \quad (3.3)$$

where C depends on T .

Now, we set

$$u(t, y) = \text{Cof}(\nabla X(t, y))^* U(t, X(t, y)), \quad P(t, y) = p(t, X(t, y)).$$

Then, we have

$$u_{\mathcal{S}}(t, y) = R_{\theta(t)}^{-1}(h'(t) + \omega(t)(x - h(t))^\perp) = \ell(t) + \omega(t)y^\perp,$$

where $\ell(t) = R_{\theta(t)}^{-1}h'(t)$. We transform the system (1.1), (1.3), (1.4), (1.5) by using this change of variables. Calculations of this type are already done in [26, Lemma 2.1]. Thus, the system (1.1), (1.3), (1.4), (1.5) is equivalent to

$$\begin{cases} \partial_t u - \nu \mathcal{L}u + \mathcal{M}u + \mathcal{N}u + \mathcal{G}p = v^* 1_{\mathcal{O}} & \text{in } (0, T), \text{ in } \mathcal{F}, \\ \nabla \cdot u = 0 & \text{in } (0, T), \text{ in } \mathcal{F}, \end{cases} \quad (3.4)$$

$$\begin{cases} m\ell'(t) = - \int_{\partial\mathcal{S}} \mathbb{T}(u, p)n \, d\Gamma - m\omega\ell(t)^\perp & t \in (0, T), \\ J\omega'(t) = - \int_{\partial\mathcal{S}} y^\perp \cdot \mathbb{T}(u, p)n \, d\Gamma & t \in (0, T), \end{cases} \quad (3.5)$$

$$\begin{cases} u_n = 0 & \text{in } (0, T), \text{ on } \partial\Omega, \\ [2\nu D(u)n + \beta_\Omega u]_\tau = 0 & \text{in } (0, T), \text{ on } \partial\Omega, \\ (u - u_{\mathcal{S}})_n = 0 & \text{in } (0, T), \text{ on } \partial\mathcal{S}, \\ [2\nu D(u)n + \beta_{\mathcal{S}}(u - u_{\mathcal{S}})]_\tau = 0 & \text{in } (0, T), \text{ on } \partial\mathcal{S}, \end{cases} \quad (3.6)$$

where n and τ respectively stand for the normal and the tangential vectors on $\partial\mathcal{F}$, with

$$\begin{aligned} [\mathcal{L}u]_i &= \sum_{j,k} \frac{\partial}{\partial y_j} \left(g^{jk} \frac{\partial u_i}{\partial y_k} \right) + 2 \sum_{j,k,l} g^{kl} \Gamma_{jk}^i \frac{\partial u_j}{\partial y_l} + \sum_{j,k,l} \left(\frac{\partial}{\partial y_k} (g^{kl} \Gamma_{jl}^i) + \sum_m g^{kl} \Gamma_{jl}^m \Gamma_{km}^i \right) u_j, \\ [\mathcal{M}u]_i &= \sum_j \frac{\partial Y_j}{\partial t} \frac{\partial u_i}{\partial y_j} + \sum_{j,k} \left(\Gamma_{jk}^i \frac{\partial Y_k}{\partial t} + \frac{\partial Y_i}{\partial x_k} \frac{\partial^2 X_k}{\partial t \partial y_j} \right) u_j, \\ [\mathcal{N}u]_i &= \sum_j u_j \frac{\partial u_i}{\partial y_j} + \sum_{j,k} \Gamma_{jk}^i u_j u_k, \\ [\mathcal{G}p]_i &= \sum_j g^{ij} \frac{\partial p}{\partial y_j}, \end{aligned}$$

where

$$g^{ij} = \sum_k \frac{\partial Y_i}{\partial x_k} \frac{\partial Y_j}{\partial x_k}, \quad g_{ij} = \sum_k \frac{\partial X_i}{\partial y_k} \frac{\partial X_j}{\partial y_k},$$

and

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl} \left(\frac{\partial g_{il}}{\partial y_j} + \frac{\partial g_{jl}}{\partial y_i} + \frac{\partial g_{ij}}{\partial y_l} \right).$$

Finally, we set the initial conditions for $y \in \mathcal{F}$

$$u(0, y) = \text{Cof}(\nabla X(0, y))^* u^0(X(0, y)) = u_0(y), \quad \ell(0) = R_{\theta^0}^{-1} \tilde{\ell}^0 = \ell^0, \quad \omega(0) = \omega^0. \quad (3.7)$$

4 Carleman estimate for the Laplacian problem with Navier slip boundary conditions

We prove first, a Carleman inequality for the Stokes problem with non-homogeneous Navier boundary conditions. From [4, Lemma 1.1], we can construct a function $\eta \in C^2(\overline{\mathcal{F}})$ such that

$$\eta > 0 \text{ in } \mathcal{F}, \quad \eta = 0 \text{ on } \partial\mathcal{F}, \quad |\nabla\eta| > 0 \text{ in } \overline{\mathcal{F}} \setminus \overline{\mathcal{O}_\eta} \text{ where } \mathcal{O}_\eta \subset\subset \mathcal{O}, \quad \nabla\eta \cdot n < 0, \text{ on } \partial\mathcal{F}. \quad (4.1)$$

Let $\lambda > 0$ and let take $\alpha = e^{\lambda\eta}$. We have the following proposition.

Proposition 4.1. *Let \mathcal{F} and \mathcal{O} be two open sets such that $\mathcal{O} \subset \mathcal{F}$. Suppose that the friction coefficient β is a non negative constant, then there exist $C = C(\mathcal{F}, \mathcal{O}) > 0$, s_1 and λ_1 where $s_1 = s_1(\mathcal{F}, \mathcal{O})$, $\lambda_1 = \lambda_1(\mathcal{F}, \mathcal{O})$, such that the solution $\psi \in [H^2(\mathcal{F})]^2$ of the system*

$$\begin{cases} -\Delta\psi = f & \text{in } \mathcal{F}, \\ \nabla \cdot \psi = 0 & \text{in } \mathcal{F}, \\ \psi_n = a_n & \text{on } \partial\mathcal{F}, \\ [2D(\psi)n + \beta\psi]_\tau = b & \text{on } \partial\mathcal{F}, \end{cases} \quad (4.2)$$

satisfies the inequality

$$\begin{aligned} & s^2\lambda^2 \int_{\mathcal{F}} e^{2s\alpha} \alpha^2 |\nabla\psi|^2 \, dy + s^4\lambda^4 \int_{\mathcal{F}} e^{2s\alpha} \alpha^4 |\psi|^2 \, dy + \beta s^3\lambda^2 e^{2s} \int_{\partial\mathcal{F}} |\psi_\tau|^2 \, d\Gamma \\ & \leq C \left(s^4\lambda^4 \int_{\mathcal{O}} e^{2s\alpha} \alpha^4 |\psi|^2 \, dy + s \int_{\mathcal{F}} e^{2s\alpha} \alpha |f|^2 \, dy + s^3\lambda^2 e^{2s} \int_{\partial\mathcal{F}} |\nabla_\tau(a \cdot n)|^2 \, d\Gamma \right. \\ & \quad \left. + s^4\lambda^2 e^{2s} \|a_n\|_{[H^{1/2}(\partial\mathcal{F})]^2}^2 + s^3\lambda^2 e^{2s} \int_{\partial\mathcal{F}} |b|^2 \, d\Gamma \right). \end{aligned} \quad (4.3)$$

for any $s \geq s_1$ and $\lambda \geq \lambda_1$, where $f \in [L^2(\mathcal{F})]^2$, $b \in [H^{1/2}(\partial\mathcal{F})]^2$, $a \in [H^{3/2}(\partial\mathcal{F})]^2$.

Proof. The proof is inspired from [17] where in our case, we need to take into account the non homogeneous Navier slip boundary conditions.

Step 1: Let $w = e^{s\alpha}\psi$. The first equation of the system (4.2) becomes

$$-\Delta w - s^2\lambda^2\alpha^2 |\nabla\eta|^2 w + 2s\lambda\alpha \nabla w \nabla\eta + s\lambda^2\alpha |\nabla\eta|^2 w + s\lambda\alpha \Delta\eta w = e^{s\alpha} f, \quad (4.4)$$

We write $\Delta w = \nabla \cdot (\nabla w + (\nabla w)^*) - \nabla(\nabla \cdot w)$. Using that $\nabla \cdot w = s\lambda\alpha \nabla\eta \cdot w$, we get

$$-\Delta w = -\nabla \cdot (\nabla w + (\nabla w)^*) + s\lambda\alpha (\nabla w)^* \nabla\eta + s\lambda\alpha \nabla^2 \eta w + s\lambda^2\alpha (\nabla\eta \cdot w) \nabla\eta.$$

Then, (4.4) can be written as

$$\begin{aligned} & -\nabla \cdot (\nabla w + (\nabla w)^*) + s\lambda\alpha (\nabla w)^* \nabla\eta - s^2\lambda^2\alpha^2 |\nabla\eta|^2 w + 2s\lambda\alpha \nabla w \nabla\eta + s\lambda^2\alpha |\nabla\eta|^2 w + s\lambda\alpha \Delta\eta w \\ & \quad + s\lambda\alpha \nabla^2 \eta w + s\lambda^2\alpha (\nabla\eta \cdot w) \nabla\eta = e^{s\alpha} f. \end{aligned} \quad (4.5)$$

We multiply (4.5) by $\alpha^{1/2}$, then (4.5) is equivalent to

$$Mw + Nw = \tilde{g}, \quad (4.6)$$

where

$$Mw = -\alpha^{1/2} \nabla \cdot (\nabla w + (\nabla w)^*) - s^2\lambda^2\alpha^{5/2} |\nabla\eta|^2 w + s\lambda\alpha^{3/2} (\nabla w)^* \nabla\eta, \quad (4.7)$$

$$Nw = 2s\lambda\alpha^{3/2} \nabla w \nabla\eta + 4s\lambda^2\alpha^{3/2} |\nabla\eta|^2 w, \quad (4.8)$$

and

$$\tilde{g} = \alpha^{1/2} e^{s\alpha} f + 3s\lambda^2 \alpha^{3/2} |\nabla\eta|^2 w - s\lambda \alpha^{3/2} \Delta\eta w - s\lambda \alpha^{3/2} \nabla^2 \eta w - s\lambda^2 \alpha^{3/2} (\nabla\eta \cdot w) \nabla\eta. \quad (4.9)$$

Multiplying (4.6) by its self, we notice that we only need to consider the terms $\sum_{i,j} \langle (Mw)_i, (Nw)_j \rangle$. First, we have

$$\begin{aligned} \langle (Mw)_1, (Nw)_1 \rangle &= -2s\lambda \int_{\mathcal{F}} \alpha^2 \nabla \cdot (\nabla w + (\nabla w)^*) \cdot \nabla w \nabla\eta \, dy \\ &= -2s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n) (\nabla w + (\nabla w)^*) n \cdot (\nabla w n) \, d\Gamma + 2s\lambda \int_{\mathcal{F}} \alpha^2 (\nabla w \nabla^2 \eta) : (\nabla w + (\nabla w)^*) \, dy \\ &\quad + 4s\lambda^2 \int_{\mathcal{F}} \alpha^2 (\nabla w + (\nabla w)^*) \nabla\eta \cdot \nabla w \nabla\eta \, dy + 2s\lambda \sum_{i,j,k} \int_{\mathcal{F}} \alpha^2 (\partial_j w_i + \partial_i w_j) \partial_{k_j}^2 w_i \partial_k \eta \, dy. \end{aligned} \quad (4.10)$$

We set

$$\begin{aligned} A &= 4s\lambda^2 \int_{\mathcal{F}} \alpha^2 (\nabla w + (\nabla w)^*) \nabla\eta \cdot \nabla w \nabla\eta \, dy = 4s\lambda^2 \int_{\mathcal{F}} \alpha^2 |\nabla w \nabla\eta|^2 \, dy \\ &\quad + 4s\lambda^2 \int_{\mathcal{F}} \alpha^2 ((\nabla w)^* \nabla\eta) \cdot (\nabla w \nabla\eta) \, dy = A_1 + A_2. \end{aligned} \quad (4.11)$$

We obtain

$$\begin{aligned} A_2 &= 4s\lambda^2 \int_{\partial\mathcal{F}} |\nabla\eta|^2 (\nabla w n) \cdot n (w \cdot n) \, d\Gamma - 4s\lambda^2 \sum_{i,j,k} \int_{\mathcal{F}} \partial_i (\alpha^2 \partial_k \eta \partial_j \eta) \partial_j w_i w_k \, dy \\ &\quad - 4s^2 \lambda^3 \int_{\mathcal{F}} \alpha^2 \nabla\eta \cdot \nabla (\alpha \nabla\eta \cdot w) (\nabla\eta \cdot w) \, dy, \end{aligned} \quad (4.12)$$

where we have used that $\nabla \cdot w = s\lambda \alpha \nabla\eta \cdot w$. An integration by parts for the last term gives

$$\begin{aligned} A_2 &= 4s\lambda^2 \int_{\partial\mathcal{F}} \alpha^2 |\nabla\eta|^2 (\nabla w n) \cdot n (w \cdot n) \, d\Gamma - 4s\lambda^2 \sum_{i,j,k} \int_{\mathcal{F}} \partial_i (\alpha^2 \partial_k \eta \partial_j \eta) \partial_j w_i w_k \, dy \\ &\quad + 2s^2 \lambda^3 \int_{\mathcal{F}} \alpha^3 \Delta\eta |\nabla\eta \cdot w|^2 \, dy + 2s^2 \lambda^4 \int_{\mathcal{F}} |\nabla\eta|^2 \alpha^3 |\nabla\eta \cdot w|^2 \, dy - 2s^2 \lambda^3 \int_{\partial\mathcal{F}} (\nabla\eta \cdot n) |\nabla\eta \cdot w|^2 \, d\Gamma. \end{aligned} \quad (4.13)$$

On the other hand, we have

$$\begin{aligned} B &= 2s\lambda \sum_{i,j,k} \int_{\mathcal{F}} \alpha^2 (\partial_j w_i + \partial_i w_j) \partial_{k_j}^2 w_i \partial_k \eta \, dy \\ &= s\lambda \int_{\mathcal{F}} \alpha^2 \nabla\eta \cdot \nabla |\nabla w|^2 \, dy + 2s\lambda \sum_{i,j,k} \int_{\mathcal{F}} \alpha^2 \partial_i w_j \partial_{k_j}^2 w_i \partial_k \eta \, dy = B_1 + B_2. \end{aligned} \quad (4.14)$$

An integration by parts for the terms B_1 and B_2 , gives

$$B_1 = s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n) |\nabla w|^2 \, d\Gamma - s\lambda \int_{\mathcal{F}} \alpha^2 \Delta\eta |\nabla w|^2 \, dy - 2s\lambda^2 \int_{\mathcal{F}} |\nabla\eta|^2 \alpha^2 |\nabla w|^2 \, dy, \quad (4.15)$$

$$\begin{aligned} B_2 &= 2s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n) \nabla w : (\nabla w)^* \, d\Gamma - 2s\lambda \int_{\mathcal{F}} \Delta\eta \alpha^2 \nabla w : (\nabla w)^* \, dy \\ &\quad - 4s\lambda^2 \int_{\mathcal{F}} \alpha^2 |\nabla\eta|^2 \nabla w : (\nabla w)^* \, dy - 2s\lambda \sum_{i,j,k} \int_{\mathcal{F}} \alpha^2 \partial_{k_i}^2 w_j \partial_j w_i \partial_k \eta \, dy = B_{21} + B_{22} + B_{23} + B_{24}. \end{aligned} \quad (4.16)$$

We make an integration by parts for B_{24} , we get

$$\begin{aligned}
B_{24} = & -2s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)(\nabla wn) \cdot ((\nabla w)^* n) \, d\Gamma + 2s\lambda \int_{\mathcal{F}} \alpha^2 (\nabla^2 \eta \nabla w) : \nabla w \, dy \\
& + 4s\lambda^2 \int_{\mathcal{F}} \alpha^2 ((\nabla w)^* \nabla \eta) \cdot ((\nabla w) \nabla \eta) \, dy + 2s^2 \lambda^2 \int_{\mathcal{F}} \alpha^3 (\nabla^2 \eta w) \cdot (\nabla w \nabla \eta) \, dy \\
& + \sum_{i,j,k} s^2 \lambda^3 \int_{\mathcal{F}} \alpha^3 \partial_k \eta \partial_j \eta \partial_i \eta (\partial_k (w_i w_j)) \, dy + 2s^2 \lambda^2 \int_{\mathcal{F}} \alpha^3 (\nabla w \nabla \eta) \cdot ((\nabla w)^* \nabla \eta) \, dy. \quad (4.17)
\end{aligned}$$

We notice that the third term in (4.17) corresponds to A_2 , while the fifth term in (4.17) denoted by B_{245} gives

$$\begin{aligned}
B_{245} = & s^2 \lambda^3 \sum_{i,j,k} \int_{\mathcal{F}} \alpha^3 \partial_k \eta \partial_j \eta \partial_i \eta \partial_k (w_i w_j) \, dy = - \sum_{i,j,k} s^2 \lambda^3 \int_{\mathcal{F}} \partial_k (\alpha^3 \partial_k \eta \partial_j \eta \partial_i \eta) (w_i w_j) \, dy \\
& + s^2 \lambda^3 \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)^3 |w \cdot n|^2 \, d\Gamma,
\end{aligned}$$

we note that

$$- \sum_{i,j,k} s^2 \lambda^3 \int_{\mathcal{F}} \partial_k (\alpha^3 \partial_k \eta \partial_j \eta \partial_i \eta) (w_i w_j) \, dy \geq -Cs^2 \lambda^3 (1 + \lambda) \int_{\mathcal{F}} \alpha^4 |w|^2 \, dy.$$

We treat the sixth term in (4.17), we obtain

$$\begin{aligned}
B_{246} = & 2s^2 \lambda^2 \int_{\mathcal{F}} \alpha^3 (\nabla w \nabla \eta) \cdot ((\nabla w)^* \nabla \eta) \, dy = -2s^2 \lambda^2 \int_{\mathcal{F}} \alpha^3 \partial_i (\partial_j \eta \partial_k \eta) \partial_j w_i w_k \, dy \\
& - 3s^2 \lambda^3 \sum_{i,j,k} \int_{\mathcal{F}} \alpha^3 \partial_k \eta \partial_j \eta \partial_i \eta \partial_k (w_i w_j) \, dy - s^3 \lambda^3 \int_{\mathcal{F}} \alpha^2 \nabla \eta \cdot \nabla |\alpha \nabla \eta \cdot w|^2 \, dy \\
& + 2s^2 \lambda^2 \int_{\partial\mathcal{F}} |\nabla \eta|^2 ((\nabla w)^* n) \cdot n (w \cdot n) \, d\Gamma = C_1 - 3B_{245} + C_3 + C_4. \quad (4.18)
\end{aligned}$$

$$C_3 = s^3 \lambda^3 \int_{\mathcal{F}} \alpha^4 \Delta \eta |w \cdot \nabla \eta|^2 \, dy + 2s^3 \lambda^4 \int_{\mathcal{F}} \alpha^4 |\nabla \eta|^2 |w \cdot \nabla \eta|^2 \, dy - s^3 \lambda^3 \int_{\partial\mathcal{F}} (\nabla \eta \cdot n)^3 |w \cdot n|^2 \, d\Gamma. \quad (4.19)$$

Thus, we obtain

$$\begin{aligned}
\langle (Mw)_1, (Nw)_1 \rangle \geq & -2s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)(\nabla w + (\nabla w)^*) n \cdot (\nabla wn) \, d\Gamma + s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n) |\nabla w|^2 \, d\Gamma \\
& + 2s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n) \nabla w : (\nabla w)^* \, d\Gamma - 2s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)(\nabla wn) \cdot ((\nabla w)^* n) \, d\Gamma \\
& + 8s\lambda^2 \int_{\partial\mathcal{F}} |\nabla \eta|^2 (\nabla wn) \cdot n (w \cdot n) \, d\Gamma + 2s^2 \lambda^2 \int_{\partial\mathcal{F}} |\nabla \eta|^2 ((\nabla w)^* n) \cdot n (w \cdot n) \, d\Gamma \\
& - 6s^2 \lambda^3 \int_{\partial\mathcal{F}} (\nabla\eta \cdot n) |\nabla \eta \cdot w|^2 \, d\Gamma - s^3 \lambda^3 \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)^3 |w \cdot n|^2 \, d\Gamma \\
& + 2s^3 \lambda^4 \int_{\mathcal{F}} |\nabla \eta|^2 \alpha^4 |w \cdot \nabla \eta|^2 \, dy - 4s\lambda^2 \int_{\mathcal{F}} \alpha^2 |\nabla \eta|^2 \nabla w : (\nabla w)^* \, dy - 2s\lambda^2 \int_{\mathcal{F}} |\nabla \eta|^2 \alpha^2 |\nabla w|^2 \, dy \\
& - \varepsilon \left(s\lambda^2 \int_{\mathcal{F}} \alpha^2 |\nabla w|^2 \, dy + s^3 \lambda^4 \int_{\mathcal{F}} \alpha^4 |w|^2 \, dy \right), \quad (4.20)
\end{aligned}$$

for all $\varepsilon > 0$ and for $s > 1, \lambda > 1$. We get also

$$\begin{aligned}
\langle (Mw)_1, (Nw)_2 \rangle &= -4s\lambda^2 \int_{\mathcal{F}} |\nabla\eta|^2 \alpha^2 [\nabla \cdot (\nabla w + (\nabla w)^*)] \cdot w \, dy \\
&= -4s\lambda^2 \int_{\partial\mathcal{F}} |\nabla\eta|^2 ((\nabla w) + (\nabla w)^*)n \cdot w \, d\Gamma + 8s\lambda^2 \int_{\mathcal{F}} \alpha^2 (\nabla^2 \eta \nabla \eta) (\nabla w + (\nabla w)^*) \cdot w \, dy \\
&\quad + 8s\lambda^3 \int_{\mathcal{F}} |\nabla\eta|^2 \alpha^2 (\nabla \eta (\nabla w + (\nabla w)^*)) \cdot w \, dy + 4s\lambda^2 \int_{\mathcal{F}} |\nabla\eta|^2 \alpha^2 |\nabla w|^2 \, dy \\
&\quad + 4s\lambda^2 \int_{\mathcal{F}} |\nabla\eta|^2 \alpha^2 \nabla w : (\nabla w)^* \, dy. \quad (4.21)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\langle (Mw)_2, (Nw)_1 \rangle &= -s^3\lambda^3 \int_{\mathcal{F}} |\nabla\eta|^2 \alpha^4 \nabla \eta \cdot \nabla |w|^2 \, dy = -s^3\lambda^3 \int_{\partial\mathcal{F}} (\nabla \eta \cdot n)^3 |w|^2 \, d\Gamma \\
&\quad + s^3\lambda^3 \int_{\mathcal{F}} \nabla \cdot (|\nabla\eta|^2 \nabla \eta) \alpha^4 |w|^2 \, dy + 4s^3\lambda^4 \int_{\mathcal{F}} |\nabla\eta|^4 \alpha^4 |w|^2 \, dy. \quad (4.22)
\end{aligned}$$

We get

$$\langle (Mw)_2, (Mw)_2 \rangle = -4s^3\lambda^4 \int_{\mathcal{F}} |\nabla\eta|^4 \alpha^4 |w|^2 \, dy. \quad (4.23)$$

We obtain

$$\begin{aligned}
\langle (Mw)_3, (Nw)_1 \rangle &= B_{246} \geq -3s^2\lambda^3 \int_{\partial\mathcal{F}} (\nabla \eta \cdot n)^3 |w \cdot n|^2 \, d\Gamma + 2s^2\lambda^2 \int_{\partial\mathcal{F}} |\nabla\eta|^2 ((\nabla w)^* n) \cdot n(w \cdot n) \, d\Gamma \\
&\quad + 2s^3\lambda^4 \int_{\mathcal{F}} |\nabla\eta|^2 \alpha^4 |w \cdot \nabla\eta|^2 \, dy - s^3\lambda^3 \int_{\partial\mathcal{F}} (\nabla \eta \cdot n)^3 |w \cdot n|^2 \, d\Gamma \\
&\quad - \varepsilon \left(s\lambda^2 \int_{\mathcal{F}} \alpha^2 |\nabla w|^2 \, dy + s^3\lambda^4 \int_{\mathcal{F}} \alpha^4 |w|^2 \, dy \right). \quad (4.24)
\end{aligned}$$

We have also

$$\begin{aligned}
\langle (Mw)_3, (Nw)_2 \rangle &= 4s^2\lambda^3 \int_{\mathcal{F}} |\nabla\eta|^2 \alpha^3 (\nabla w)^* \nabla \eta \cdot w \, dy = 4s^2\lambda^3 \int_{\partial\mathcal{F}} (\nabla \eta \cdot n)^3 |w \cdot n|^2 \, d\Gamma \\
&\quad - 4s^2\lambda^3 \int_{\mathcal{F}} (\nabla (|\nabla\eta|^2 \alpha^3 \nabla \eta) \cdot w) \cdot w \, dy - 4s^3\lambda^4 \int_{\mathcal{F}} |\nabla\eta|^2 \alpha^4 |w \cdot \nabla\eta|^2 \, dy. \quad (4.25)
\end{aligned}$$

Then, we get

$$\begin{aligned}
\langle (Mw), (Nw) \rangle &\geq -2s\lambda \int_{\partial\mathcal{F}} (\nabla \eta \cdot n) (\nabla w + (\nabla w)^*)n \cdot (\nabla w n) \, d\Gamma + s\lambda \int_{\partial\mathcal{F}} (\nabla \eta \cdot n) |\nabla w|^2 \, d\Gamma \\
&\quad + 2s\lambda \int_{\partial\mathcal{F}} (\nabla \eta \cdot n) \nabla w : (\nabla w)^* \, d\Gamma - 2s\lambda \int_{\partial\mathcal{F}} (\nabla \eta \cdot n) (\nabla w n) \cdot ((\nabla w)^* n) \, d\Gamma \\
&\quad - 4s\lambda^2 \int_{\partial\mathcal{F}} |\nabla\eta|^2 (\nabla w n + (\nabla w)^* n)_\tau \cdot w_\tau \, d\Gamma + 4s^2\lambda^2 \int_{\partial\mathcal{F}} |\nabla\eta|^2 ((\nabla w)^* n) \cdot n(w \cdot n) \, d\Gamma \\
&\quad - 2s^3\lambda^3 \int_{\partial\mathcal{F}} (\nabla \eta \cdot n)^3 |w \cdot n|^2 \, d\Gamma - s^3\lambda^3 \int_{\partial\mathcal{F}} (\nabla \eta \cdot n)^3 |w|^2 \, d\Gamma \\
&\quad + 2s\lambda^2 \int_{\mathcal{F}} |\nabla\eta|^2 \alpha^2 |\nabla w|^2 \, dy - 5s^2\lambda^3 \int_{\partial\mathcal{F}} (\nabla \eta \cdot n)^3 |w \cdot n|^2 \, d\Gamma \\
&\quad - \varepsilon \left(s\lambda^2 \int_{\mathcal{F}} \alpha^2 |\nabla w|^2 \, dy + s^3\lambda^4 \int_{\mathcal{F}} \alpha^4 |w|^2 \, dy \right), \quad (4.26)
\end{aligned}$$

where we have used

$$\begin{aligned} -4s\lambda^2 \int_{\partial\mathcal{F}} |\nabla\eta|^2 ((\nabla w) + (\nabla w)^*)n \cdot w \, d\Gamma + 8s\lambda^2 \int_{\partial\mathcal{F}} |\nabla\eta|^2 (\nabla w)^*n \cdot n(w \cdot n) \, d\Gamma \\ = -4s\lambda^2 \int_{\partial\mathcal{F}} |\nabla\eta|^2 (((\nabla w) + (\nabla w)^*)n)_\tau \cdot w_\tau \, d\Gamma. \end{aligned}$$

Step 2: We derive a Carleman estimate for $\tilde{w} = e^{s\tilde{\alpha}}\psi$ with $\tilde{\alpha} = e^{-\lambda\eta}$, the calculus will be analogous and we will get the same terms up to a sign. We obtain

$$\begin{aligned} \langle (\tilde{M}\tilde{w}), (\tilde{N}\tilde{w}) \rangle \geq & 2s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)(\nabla\tilde{w} + (\nabla\tilde{w})^*)n \cdot (\nabla\tilde{w}n) \, d\Gamma - s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n) |\nabla\tilde{w}|^2 \, d\Gamma \\ & - 2s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)\nabla\tilde{w} : (\nabla\tilde{w})^* \, d\Gamma + 2s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)(\nabla\tilde{w}n) \cdot ((\nabla\tilde{w})^*n) \, d\Gamma \\ & - 4s\lambda^2 \int_{\partial\mathcal{F}} |\nabla\eta|^2 (\nabla\tilde{w}n + (\nabla\tilde{w})^*n)_\tau \cdot \tilde{w}_\tau \, d\Gamma + 4s^2\lambda^2 \int_{\partial\mathcal{F}} |\nabla\eta|^2 ((\nabla\tilde{w})^*n) \cdot n(\tilde{w} \cdot n) \, d\Gamma \\ & + 2s^3\lambda^3 \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)^3 |\tilde{w} \cdot n|^2 \, d\Gamma + s^3\lambda^3 \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)^3 |\tilde{w}|^2 \, d\Gamma \\ & + 2s\lambda^2 \int_{\mathcal{F}} |\nabla\eta|^2 \tilde{\alpha}^2 |\nabla\tilde{w}|^2 \, dy + 5s^2\lambda^3 \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)^3 |\tilde{w} \cdot n|^2 \, d\Gamma \\ & - \varepsilon \left(s\lambda^2 \int_{\mathcal{F}} \tilde{\alpha}^2 |\nabla\tilde{w}|^2 \, dy + s^3\lambda^4 \int_{\mathcal{F}} \tilde{\alpha}^4 |\tilde{w}|^2 \, dy \right). \quad (4.27) \end{aligned}$$

Step 3: We deal with the surface integrals. We note that on the boundary $\partial\mathcal{F}$, we have

$$\nabla w_i = e^s(\nabla\psi + s\lambda\nabla\eta\psi_i), \quad \nabla\tilde{w}_i = e^s(\nabla\psi - s\lambda\nabla\eta\psi_i), \quad \text{on } \partial\mathcal{F}. \quad (4.28)$$

Then,

$$\nabla wn = e^s(\nabla\psi n + s\lambda(\nabla\eta \cdot n)\psi), \quad \nabla\tilde{w}n = e^s(\nabla\psi n - s\lambda(\nabla\eta \cdot n)\psi), \quad \text{on } \partial\mathcal{F}, \quad (4.29)$$

and

$$(\nabla w)^*n = e^s((\nabla\psi)^*n + s\lambda(\psi \cdot n)\nabla\eta), \quad (\nabla\tilde{w})^*n = e^s((\nabla\psi)^*n - s\lambda(\psi \cdot n)\nabla\eta), \quad \text{on } \partial\mathcal{F}. \quad (4.30)$$

The boundary terms in (4.26) are reduced to

$$\begin{aligned} S = & -2s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)(\nabla w + (\nabla w)^*)n \cdot (\nabla wn) \, d\Gamma + s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n) |\nabla w|^2 \, d\Gamma \\ & + 2s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)\nabla w : (\nabla w)^* \, d\Gamma - 2s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)(\nabla wn) \cdot ((\nabla w)^*n) \, d\Gamma \\ & - 4s\lambda^2 \int_{\partial\mathcal{F}} |\nabla\eta|^2 (\nabla wn + (\nabla w)^*n)_\tau \cdot w_\tau \, d\Gamma + 4s^2\lambda^2 \int_{\partial\mathcal{F}} |\nabla\eta|^2 ((\nabla w)^*n) \cdot n(w \cdot n) \, d\Gamma \\ & - 2s^3\lambda^3 \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)^3 |w \cdot n|^2 \, d\Gamma - s^3\lambda^3 \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)^3 |w|^2 \, d\Gamma - 5s^2\lambda^3 \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)^3 |w \cdot n|^2 \, d\Gamma. \end{aligned}$$

The boundary terms in (4.27) write

$$\begin{aligned} \tilde{S} = & 2s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)(\nabla\tilde{w} + (\nabla\tilde{w})^*)n \cdot (\nabla\tilde{w}n) \, d\Gamma - s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n) |\nabla\tilde{w}|^2 \, d\Gamma \\ & - 2s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)\nabla\tilde{w} : (\nabla\tilde{w})^* \, d\Gamma + 2s\lambda \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)(\nabla\tilde{w}n) \cdot ((\nabla\tilde{w})^*n) \, d\Gamma \\ & - 4s\lambda^2 \int_{\partial\mathcal{F}} |\nabla\eta|^2 (\nabla\tilde{w}n + (\nabla\tilde{w})^*n)_\tau \cdot \tilde{w}_\tau \, d\Gamma + 4s^2\lambda^2 \int_{\partial\mathcal{F}} |\nabla\eta|^2 ((\nabla\tilde{w})^*n) \cdot n(\tilde{w} \cdot n) \, d\Gamma \\ & + 2s^3\lambda^3 \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)^3 |\tilde{w} \cdot n|^2 \, d\Gamma + s^3\lambda^3 \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)^3 |\tilde{w}|^2 \, d\Gamma + 5s^2\lambda^3 \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)^3 |\tilde{w} \cdot n|^2 \, d\Gamma. \end{aligned}$$

Using that $\alpha = \tilde{\alpha} = 1$ and $w = \tilde{w}$ on $\partial\mathcal{F}$, the boundary terms are reduced to

$$\begin{aligned}
S + \tilde{S} &= -2s\lambda \left(\int_{\partial\mathcal{F}} (\nabla\eta \cdot n)(\nabla w + (\nabla w)^*)n \cdot (\nabla wn) \, d\Gamma - \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)(\nabla\tilde{w} + (\nabla\tilde{w})^*)n \cdot (\nabla\tilde{w}n) \, d\Gamma \right) \\
&\quad + s\lambda \left(\int_{\partial\mathcal{F}} (\nabla\eta \cdot n)|\nabla w|^2 \, d\Gamma - \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)|\nabla\tilde{w}|^2 \, d\Gamma \right) \\
&\quad + 2s\lambda \left(\int_{\partial\mathcal{F}} (\nabla\eta \cdot n)\nabla w : (\nabla w)^* \, d\Gamma - \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)\nabla\tilde{w} : (\nabla\tilde{w})^* \, d\Gamma \right) \\
&\quad - 2s\lambda \left(\int_{\partial\mathcal{F}} (\nabla\eta \cdot n)(\nabla wn) \cdot ((\nabla w)^*n) \, d\Gamma - \int_{\partial\mathcal{F}} (\nabla\eta \cdot n)(\nabla\tilde{w}n) \cdot ((\nabla\tilde{w})^*n) \, d\Gamma \right) \\
&\quad - 4s\lambda^2 \left(\int_{\partial\mathcal{F}} |\nabla\eta|^2 (\nabla wn + (\nabla w)^*n)_\tau \cdot w_\tau \, d\Gamma + \int_{\partial\mathcal{F}} |\nabla\eta|^2 (\nabla\tilde{w}n + (\nabla\tilde{w})^*n)_\tau \cdot \tilde{w}_\tau \, d\Gamma \right) \\
&\quad + 4s^2\lambda^2 \left(\int_{\partial\mathcal{F}} |\nabla\eta|^2 ((\nabla w)^*n) \cdot n(w \cdot n) \, d\Gamma + \int_{\partial\mathcal{F}} |\nabla\eta|^2 ((\nabla\tilde{w})^*n) \cdot n(\tilde{w} \cdot n) \, d\Gamma \right) = \sum_{i=1}^6 I_i.
\end{aligned}$$

Using (4.28), (4.29), (4.30) and (4.2)₄, we get

$$\begin{aligned}
I_1 + I_4 &= -8s^2\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 (\nabla\psi n + (\nabla\psi)^*n) \cdot \psi \, d\Gamma - 8s^2\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 (\nabla\psi n \cdot n)(\psi \cdot n) \, d\Gamma, \\
I_2 &= 4s^2\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 \nabla\psi n \cdot \psi \, d\Gamma, \\
I_3 &= 8s^2\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 (\nabla\psi)^*n \cdot \psi \, d\Gamma, \\
I_5 &= 8\beta s\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 |\psi_\tau|^2 \, d\Gamma - 8s\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 b \cdot \psi \, d\Gamma, \\
I_6 &= 8s^2\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 (\nabla\psi n \cdot n)(\psi \cdot n) \, d\Gamma.
\end{aligned}$$

Then, we get

$$\begin{aligned}
I_1 + I_4 &= -8s^2\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 (\nabla\psi n + (\nabla\psi)^*n)_\tau \cdot \psi_\tau \, d\Gamma - 24s^2\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 (\nabla\psi n \cdot n)(\psi \cdot n) \, d\Gamma \\
&= 8\beta s^2\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 |\psi_\tau|^2 \, d\Gamma - 8s^2\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 b \cdot \psi \, d\Gamma - 24s^2\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 (\nabla\psi n \cdot n)(\psi \cdot n) \, d\Gamma,
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= 4s^2\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 (\nabla\psi n)_\tau \cdot \psi_\tau \, d\Gamma + 4s^2\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 (\nabla\psi n) \cdot n(\psi \cdot n) \, d\Gamma \\
&= -4s^2\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 (\nabla\psi)^*n_\tau \cdot \psi_\tau \, d\Gamma - 4s^2\lambda^2\beta \int_{\partial\mathcal{F}} e^{2s} |\psi_\tau|^2 \, d\Gamma \\
&\quad + 4s^2\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 b_\tau \cdot \psi_\tau \, d\Gamma + 4s^2\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 (\nabla\psi n) \cdot n(\psi \cdot n) \, d\Gamma.
\end{aligned}$$

We notice that

$$-4s^2\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 [(\nabla\psi)^*n]_\tau \cdot \psi_\tau \, d\Gamma = -4s^2\lambda^2 \int_{\partial\mathcal{F}} e^{2s} |\nabla\eta|^2 [\nabla_\tau(a \cdot n) \cdot \psi_\tau - (\nabla n \tau \cdot \psi)(\psi \cdot \tau)] \, d\Gamma.$$

Using the inequality in [12, Theorem II.4.1] with $r = 2, q = 2$, we obtain

$$s^2 \lambda^2 \int_{\partial \mathcal{F}} |\psi|^2 d\Gamma \leq C \left(s \lambda \int_{\mathcal{F}} |\nabla \psi|^2 dy + s^3 \lambda^3 \int_{\mathcal{F}} |\psi|^2 dy \right).$$

Applying the same arguments, we get for I_3

$$\begin{aligned} I_3 &= 8s^2 \lambda^2 \int_{\partial \mathcal{F}} e^{2s} |\nabla \eta|^2 ((\nabla \psi)^* n)_\tau \cdot \psi_\tau d\Gamma + 8s^2 \lambda^2 \int_{\partial \mathcal{F}} e^{2s} |\nabla \eta|^2 (\nabla \psi n) \cdot n(\psi \cdot n) d\Gamma \\ &= 8s^2 \lambda^2 \int_{\partial \mathcal{F}} e^{2s} |\nabla \eta|^2 [\nabla_\tau(a \cdot n) \cdot \psi_\tau - (\nabla n \tau \cdot \psi)(\psi \cdot \tau)] d\Gamma + 8s^2 \lambda^2 \int_{\partial \mathcal{F}} e^{2s} |\nabla \eta|^2 (\nabla \psi n) \cdot n(\psi \cdot n) d\Gamma. \end{aligned}$$

Then, combining all these inequalities, we get

$$\begin{aligned} & s \lambda^2 \int_{\mathcal{F}} e^{2s\alpha} \alpha^2 |\nabla \eta|^2 |\nabla \psi|^2 dy + s^2 \lambda^2 \beta e^{2s} \int_{\partial \mathcal{F}} |\psi_\tau|^2 d\Gamma \\ & \leq \varepsilon s^3 \lambda^4 \int_{\mathcal{F}} e^{2s\alpha} \alpha^4 |\psi|^2 dy + \varepsilon s \lambda^2 \int_{\mathcal{F}} e^{2s\alpha} \alpha^2 |\nabla \psi|^2 dy + C \left(\int_{\mathcal{F}} e^{2s\alpha} \alpha |f|^2 dy + s^2 \lambda^2 e^{2s} \int_{\partial \mathcal{F}} |\nabla(a \cdot n)\tau|^2 d\Gamma \right. \\ & \quad \left. + s^2 \lambda^2 e^{2s} \int_{\partial \mathcal{F}} |b|^2 d\Gamma \right) + 4s^2 \lambda^2 e^{2s} \int_{\partial \mathcal{F}} |\nabla \eta|^2 (\nabla \psi n \cdot n)(\psi \cdot n) d\Gamma. \end{aligned}$$

Using the fact that $|\nabla \eta| > 0$ on $\mathcal{F} \setminus \mathcal{O}_\eta$, we obtain

$$\begin{aligned} & s \lambda^2 \int_{\mathcal{F} \setminus \mathcal{O}_\eta} e^{2s\alpha} \alpha^2 |\nabla \psi|^2 dy + s^2 \lambda^2 \beta e^{2s} \int_{\partial \mathcal{F}} |\psi_\tau|^2 d\Gamma \\ & \leq \varepsilon s^3 \lambda^4 \int_{\mathcal{F}} e^{2s\alpha} \alpha^4 |\psi|^2 dy + \varepsilon s \lambda^2 \int_{\mathcal{F}} e^{2s\alpha} \alpha^2 |\nabla \psi|^2 dy + C \left(\int_{\mathcal{F}} e^{2s\alpha} \alpha |f|^2 dy + s^2 \lambda^2 \int_{\partial \mathcal{F}} e^{2s} |\nabla(a \cdot n)\tau|^2 d\Gamma \right. \\ & \quad \left. + s^2 \lambda^2 e^{2s} \int_{\partial \mathcal{F}} |b|^2 d\Gamma \right) + 4s^2 \lambda^2 e^{2s} \int_{\partial \mathcal{F}} |\nabla \eta|^2 (\nabla \psi n \cdot n)(\psi \cdot n) d\Gamma. \end{aligned}$$

We add the term $s \lambda^2 \int_{\mathcal{O}_\eta} e^{2s\alpha} \alpha^2 |\nabla \psi|^2 dy$ in the both sides of the last equation, to get

$$\begin{aligned} & s \lambda^2 \int_{\mathcal{F}} e^{2s\alpha} \alpha^2 |\nabla \psi|^2 dy + s^2 \lambda^2 \beta \int_{\partial \mathcal{F}} e^{2s} |\psi_\tau|^2 d\Gamma \\ & \leq \varepsilon s^3 \lambda^4 \int_{\mathcal{F}} e^{2s\alpha} \alpha^4 |\psi|^2 dy + \varepsilon s \lambda^2 \int_{\mathcal{F}} e^{2s\alpha} \alpha^2 |\nabla \psi|^2 dy + C \left(\int_{\mathcal{F}} e^{2s\alpha} \alpha |f|^2 dy + s^2 \lambda^2 e^{2s} \int_{\partial \mathcal{F}} |\nabla(a \cdot n)\tau|^2 d\Gamma \right. \\ & \quad \left. + s^2 \lambda^2 e^{2s} \int_{\partial \mathcal{F}} |b|^2 d\Gamma \right) + 4s^2 \lambda^2 e^{2s} \int_{\partial \mathcal{F}} |\nabla \eta|^2 (\nabla \psi n \cdot n)(\psi \cdot n) d\Gamma. \end{aligned}$$

We use the following inequality that is proved in [6, Lemma 3]

$$s^3 \lambda^4 \int_{\mathcal{F}} e^{2s\alpha} \alpha^4 |\psi|^2 dy \leq C \left(s \lambda^2 \int_{\mathcal{F}} e^{2s\alpha} \alpha^2 |\nabla \psi|^2 dy + s^3 \lambda^4 \int_{\mathcal{O}_\eta} e^{2s\alpha} \alpha^4 |\psi|^2 dy \right). \quad (4.31)$$

Then, we get for sufficiently large s and λ

$$\begin{aligned} & s \lambda^2 \int_{\mathcal{F}} e^{2s\alpha} \alpha^2 |\nabla \psi|^2 dy + s^3 \lambda^4 \int_{\mathcal{F}} e^{2s\alpha} \alpha^4 |\psi|^2 dy + s^2 \lambda^2 \beta e^{2s} \int_{\partial \mathcal{F}} |\psi_\tau|^2 d\Gamma \\ & \leq C \left(s^3 \lambda^4 \int_{\mathcal{O}_\eta} e^{2s\alpha} \alpha^4 |\psi|^2 dy + s \lambda^2 \int_{\mathcal{O}_\eta} e^{2s\alpha} \alpha^2 |\nabla \psi|^2 dy + \int_{\mathcal{F}} e^{2s\alpha} \alpha |f|^2 dy + s^2 \lambda^2 e^{2s} \int_{\partial \mathcal{F}} |\nabla(a \cdot n)\tau|^2 d\Gamma \right. \\ & \quad \left. + s^2 \lambda^2 e^{2s} \int_{\partial \mathcal{F}} |b|^2 d\Gamma \right) + 4s^2 \lambda^2 e^{2s} \int_{\partial \mathcal{F}} |\nabla \eta|^2 (\nabla \psi n \cdot n)(\psi \cdot n) d\Gamma. \quad (4.32) \end{aligned}$$

We recall that

$$\nabla \times \psi = \frac{\partial \psi_2}{\partial y_1} - \frac{\partial \psi_1}{\partial y_2}.$$

Since ψ is divergence free, we have that

$$\Delta \psi = -\nabla \times (\nabla \times \psi).$$

We have used the fact that for any scalar function $a : \mathbb{R}^2 \rightarrow \mathbb{R}$, we have

$$\nabla \times a = \begin{pmatrix} \partial_2 a \\ -\partial_1 a \end{pmatrix}.$$

We recall the Green formula

$$\int_{\mathcal{F}} \Delta \psi \cdot \widehat{v} \, dy = - \int_{\mathcal{F}} (\nabla \times \psi) \cdot (\nabla \times \widehat{v}) \, dy - \int_{\partial \mathcal{F}} (\widehat{v} \cdot \tau) (\nabla \times \psi) \, d\Gamma, \quad \forall \widehat{v} \in [H^1(\mathcal{F})]^2,$$

with $\tau = \begin{pmatrix} -n_2 \\ n_1 \end{pmatrix}$. Then, we obtain

$$\begin{aligned} \int_{\partial \mathcal{F}} |\nabla \eta|^2 (\nabla \psi n \cdot n) (\psi \cdot n) \, d\Gamma &= \int_{\partial \mathcal{F}} |\nabla \eta|^2 \nabla \psi n \cdot \psi_n \, d\Gamma \\ &= \int_{\mathcal{F}} \nabla \psi : \nabla \widehat{v} \, dy - \int_{\mathcal{F}} (\nabla \times \psi) \cdot (\nabla \times \widehat{v}) \, dy - \int_{\partial \mathcal{F}} (\widehat{v} \cdot \tau) (\nabla \times \psi) \, d\Gamma, \end{aligned}$$

where \widehat{v} is a $[H^1(\mathcal{F})]^2$ such that $\widehat{v} = |\nabla \eta|^2 \psi_n$ on $\partial \mathcal{F}$ and

$$\|\widehat{v}\|_{[H^1(\mathcal{F})]^2} \leq C \|\psi_n\|_{[H^{1/2}(\partial \mathcal{F})]^2}.$$

Thus, the last term in the right hand side of the inequality (4.32) gives

$$s^2 \lambda^2 e^{2s} \int_{\partial \mathcal{F}} |\nabla \eta|^2 (\nabla \psi n \cdot n) (\psi \cdot n) \, d\Gamma \leq \varepsilon s \lambda^2 e^{2s} \int_{\mathcal{F}} |\nabla \psi|^2 \, dy + C s^3 \lambda^2 e^{2s} \|\psi_n\|_{[H^{1/2}(\partial \mathcal{F})]^2}^2.$$

Then, we obtain

$$\begin{aligned} s \lambda^2 \int_{\mathcal{F}} e^{2s\alpha} \alpha^2 |\nabla \psi|^2 \, dy + s^3 \lambda^4 \int_{\mathcal{F}} e^{2s\alpha} \alpha^4 |\psi|^2 \, dy + s^2 \lambda^2 \beta e^{2s} \int_{\partial \mathcal{F}} |\psi_\tau|^2 \, d\Gamma \\ \leq C \left(s^3 \lambda^4 \int_{\mathcal{O}_\eta} e^{2s\alpha} \alpha^4 |\psi|^2 \, dy + s \lambda^2 \int_{\mathcal{O}_\eta} e^{2s\alpha} \alpha^2 |\nabla \psi|^2 \, dy + \int_{\mathcal{F}} e^{2s\alpha} \alpha |f|^2 \, dy + s^2 \lambda^2 e^{2s} \int_{\partial \mathcal{F}} |\nabla(a \cdot n) \tau|^2 \, d\Gamma \right. \\ \left. + s^3 \lambda^2 e^{2s} \|\psi_n\|_{[H^{1/2}(\partial \mathcal{F})]^2}^2 + s^2 \lambda^2 e^{2s} \int_{\partial \mathcal{F}} |b|^2 \, d\Gamma \right). \end{aligned}$$

To adsorb the second term of the right hand side, we proceed like [11, inequality (1.62)] which shows that the integral of $e^{2s\alpha} \alpha^2 |\nabla \psi|^2$ over \mathcal{O}_η can be estimated by $e^{2s\alpha} \alpha^4 |\psi|^2$ over a larger set \mathcal{O} .

Indeed, we define $\theta \in C_0^2(\overline{\mathcal{O}})$ such that $\theta \equiv 1$ in \mathcal{O}_η and $0 \leq \theta \leq 1$. We obtain

$$s \lambda^2 \int_{\mathcal{O}_\eta} e^{2s\alpha} \alpha^2 |\nabla \psi|^2 \, dy \leq s \lambda^2 \int_{\mathcal{O}} e^{2s\alpha} \theta \alpha^2 |\nabla \psi|^2 \, dy,$$

whence

$$\begin{aligned} s \lambda^2 \int_{\mathcal{O}} e^{2s\alpha} \theta \alpha^2 |\nabla \psi|^2 \, dy &= -s \lambda^2 \int_{\mathcal{O}} e^{2s\alpha} \theta \alpha^2 \Delta \psi \cdot \psi \, dy - s \lambda^2 \int_{\mathcal{O}} e^{2s\alpha} \alpha^2 (\nabla \theta \cdot \nabla) \psi \cdot \psi \, dy \\ &\quad - 2s \lambda^3 \int_{\mathcal{O}} e^{2s\alpha} \theta \alpha^2 (\nabla \eta \cdot \nabla) \psi \cdot \psi \, dy - 2s^2 \lambda^3 \int_{\mathcal{O}} e^{2s\alpha} \theta \alpha^3 \nabla \eta \nabla \psi \cdot \psi \, dy \\ &\leq C \left(\int_{\mathcal{O}} e^{2s\alpha} |\Delta \psi|^2 \, dy + s^3 \lambda^4 \int_{\mathcal{O}} e^{2s\alpha} \alpha^4 |\psi|^2 \, dy + \varepsilon s \lambda^2 \int_{\mathcal{O}} e^{2s\alpha} \theta \alpha^2 |\nabla \psi|^2 \, dy \right). \end{aligned}$$

Thus, we get

$$\begin{aligned}
& s\lambda^2 \int_{\mathcal{F}} e^{2s\alpha} \alpha^2 |\nabla\psi|^2 dy + s^3\lambda^4 \int_{\mathcal{F}} e^{2s\alpha} \alpha^4 |\psi|^2 d\Gamma + s^2\lambda^2\beta e^{2s} \int_{\partial\mathcal{F}} |\psi_\tau|^2 d\Gamma \\
& \leq C \left(s^3\lambda^4 \int_{\mathcal{O}} e^{2s\alpha} \alpha^4 |\psi|^2 dy + \int_{\mathcal{F}} e^{2s\alpha} \alpha |f|^2 dy + s^2\lambda^2 e^{2s} \int_{\partial\mathcal{F}} |\nabla(a \cdot n)_\tau|^2 d\Gamma \right. \\
& \quad \left. + s^3\lambda^2 e^{2s} \|\psi_n\|_{[H^{1/2}(\partial\mathcal{F})]^2}^2 + s^2\lambda^2 e^{2s} \int_{\partial\mathcal{F}} |b|^2 d\Gamma \right),
\end{aligned}$$

for λ and s sufficiently large. Thus, we obtain (4.3). \square

5 Carleman estimate for the linearized system

We consider the following adjoint system

$$\left\{ \begin{array}{l} -\partial_t v - \nabla \cdot \mathbb{T}(v, q) = F_1 \quad \text{in } (0, T), \quad \text{in } \mathcal{F}, \\ \nabla \cdot v = 0 \quad \text{in } (0, T), \quad \text{in } \mathcal{F}, \\ m\ell'_v(t) = \int_{\partial\mathcal{S}} \mathbb{T}(v, q)n d\Gamma + F_2 \quad t \in (0, T), \\ J(k'_v(t)) = \int_{\partial\mathcal{S}} y^\perp \cdot \mathbb{T}(v, q)n d\Gamma + F_3 \quad t \in (0, T), \end{array} \right. \quad (5.1)$$

with the boundary conditions

$$\left\{ \begin{array}{l} v_n = 0 \quad \text{in } (0, T), \quad \text{on } \partial\Omega, \\ [2\nu D(v)n + \beta_\Omega v]_\tau = 0 \quad \text{in } (0, T), \quad \text{on } \partial\Omega, \\ (v - v_S)_n = 0 \quad \text{in } (0, T), \quad \text{on } \partial\mathcal{S}, \\ [2\nu D(v)n + \beta_S(v - v_S)]_\tau = 0 \quad \text{in } (0, T), \quad \text{on } \partial\mathcal{S}, \end{array} \right. \quad (5.2)$$

where $v_S(y) = \ell_v + k_v y^\perp$, completed with the initial condition

$$v(T, \cdot) = v_T, \quad \text{in } \mathcal{F}, \quad \ell_v(T) = \ell_T, \quad k_v(T) = k_T. \quad (5.3)$$

Let $\eta \in C^2(\overline{\mathcal{F}})$ which verifies (4.1) with $\mathcal{O}_\eta \subset\subset \mathcal{O}$ a non empty open set.

Let $\lambda > 0$ and

$$\beta(t, y) = \frac{e^{\lambda(2N+2)\|\eta\|_{L^\infty(\Omega)}} - e^{\lambda(2N\|\eta\|_{L^\infty(\Omega)} + \eta(y))}}{t^N(T-t)^N}, \quad (5.4)$$

$$\widehat{\beta}(t) = \max_{y \in \overline{\mathcal{F}}} \beta(t, y) = \frac{e^{\lambda(2N+2)\|\eta\|_{L^\infty(\Omega)}} - e^{\lambda 2N\|\eta\|_{L^\infty(\Omega)}}}{t^N(T-t)^N}, \quad \beta^*(t) = \min_{y \in \overline{\mathcal{F}}} \beta(t, y),$$

$$\xi(t, y) = \frac{e^{\lambda(2N\|\eta\|_{L^\infty(\Omega)} + \eta(y))}}{t^N(T-t)^N}, \quad \xi^*(t) = \min_{y \in \overline{\mathcal{F}}} \xi(t, y) = \frac{e^{\lambda 2N\|\eta\|_{L^\infty(\Omega)}}}{t^N(T-t)^N}, \quad \widehat{\xi}(t) = \max_{y \in \overline{\mathcal{F}}} \xi(t, y). \quad (5.5)$$

with $N > 0$ a sufficiently large integer.

Proposition 5.1. *Assume that \mathcal{S} is not a disk.*

There exist $C = C(\Omega, \mathcal{O})$ and $\overline{C} = \overline{C}(\Omega, \mathcal{O})$ such that for any $F_1 \in L^2(0, T; [L^2(\mathcal{F})]^2)$, $F_2 \in [L^2(0, T)]^2$ and $F_3 \in L^2(0, T)$ and for any $v_T \in [L^2(\mathcal{F})]^2$, $\ell_T \in \mathbb{R}^2$ and $k_T \in \mathbb{R}$, the solution (v, ℓ_v, k_v) of the system (5.1), (5.2)

and (5.3) verifies the inequality

$$\begin{aligned}
& s^3 \lambda^4 \int_0^T \int_{\mathcal{F}} e^{-5s\widehat{\beta}} (\xi^*)^3 |\nabla v|^2 \, dydt + s^4 \lambda^4 \int_0^T \int_{\mathcal{F}} e^{-5s\widehat{\beta}} (\xi^*)^4 |v|^2 \, dydt \\
& + s^4 \lambda^4 \int_0^T e^{-5s\widehat{\beta}} (\xi^*)^4 \left(|\ell_v(t)|^2 + |k_v(t)|^2 \right) dt \leq \overline{C} \left(s^5 \lambda^6 \int_0^T \int_{\mathcal{O}} e^{-2s\widehat{\beta}^* - 3s\widehat{\beta}} (\widehat{\xi})^5 |v|^2 \, dydt + \int_0^T \int_{\mathcal{F}} e^{-3s\widehat{\beta}} |F_1|^2 \, dydt \right. \\
& \quad \left. + \int_0^T e^{-3s\widehat{\beta}} |F_2|^2 \, dt + \int_0^T e^{-3s\widehat{\beta}} |F_3|^2 \, dt \right). \quad (5.6)
\end{aligned}$$

for all $\lambda \geq C$ and $s \geq C(T^N + T^{2N})$.

Proof. Step 1: Decomposition of the solution

Let $\rho(t) = e^{-\frac{3}{2}s\widehat{\beta}(t)}$ and let us write

$$\rho v = \varphi + z, \quad \rho \ell_v = \ell_\varphi + \ell_z, \quad \rho k_v = k_\varphi + k_z, \quad \rho q = q_\varphi + q_z, \quad (5.7)$$

where

$$\left\{ \begin{array}{l} -\partial_t \varphi - \nabla \cdot \mathbb{T}(\varphi, q_\varphi) = -\rho' v \quad \text{in } (0, T), \quad \text{in } \mathcal{F}, \\ \nabla \cdot \varphi = 0 \quad \text{in } (0, T), \quad \text{in } \mathcal{F}, \\ m \ell'_\varphi(t) = \int_{\partial \mathcal{S}} \mathbb{T}(\varphi, q_\varphi) n \, d\Gamma + m \rho' \ell_v \quad t \in (0, T), \\ J(k'_\varphi(t)) = \int_{\partial \mathcal{S}} y^\perp \cdot \mathbb{T}(\varphi, q_\varphi) n \, d\Gamma + J \rho' k_v \quad t \in (0, T), \end{array} \right. \quad (5.8)$$

with the boundary conditions

$$\left\{ \begin{array}{l} \varphi_n = 0 \quad \text{in } (0, T), \quad \text{on } \partial \Omega, \\ [2\nu D(\varphi) n + \beta_\Omega \varphi]_\tau = 0 \quad \text{in } (0, T), \quad \text{on } \partial \Omega, \\ (\varphi - \varphi_S)_n = 0 \quad \text{in } (0, T), \quad \text{on } \partial \mathcal{S}, \\ [2\nu D(\varphi) n + \beta_S (\varphi - \varphi_S)]_\tau = 0 \quad \text{in } (0, T), \quad \text{on } \partial \mathcal{S}, \end{array} \right. \quad (5.9)$$

where $\varphi_S(y) = \ell_\varphi + k_\varphi y^\perp$, with

$$\varphi(T, \cdot) = 0, \quad \text{in } \mathcal{F}, \quad \ell_\varphi(T) = 0, \quad k_\varphi(T) = 0.$$

and

$$\left\{ \begin{array}{l} -\partial_t z - \nabla \cdot \mathbb{T}(z, q_z) = \rho F_1 \quad \text{in } (0, T), \quad \text{in } \mathcal{F}, \\ \nabla \cdot z = 0 \quad \text{in } (0, T), \quad \text{in } \mathcal{F}, \\ m \ell'_z(t) = \int_{\partial \mathcal{S}} \mathbb{T}(z, q_z) n \, d\Gamma + \rho F_2 \quad t \in (0, T), \\ J(k'_z(t)) = \int_{\partial \mathcal{S}} y^\perp \cdot \mathbb{T}(z, q_z) n \, d\Gamma + \rho F_3 \quad t \in (0, T), \end{array} \right. \quad (5.10)$$

with the boundary conditions

$$\left\{ \begin{array}{l} z_n = 0 \quad \text{in } (0, T), \quad \text{on } \partial \Omega, \\ [2\nu D(z) n + \beta_\Omega z]_\tau = 0 \quad \text{in } (0, T), \quad \text{on } \partial \Omega, \\ (z - z_S)_n = 0 \quad \text{in } (0, T), \quad \text{on } \partial \mathcal{S}, \\ [2\nu D(z) n + \beta_S (z - z_S)]_\tau = 0 \quad \text{in } (0, T), \quad \text{on } \partial \mathcal{S}, \end{array} \right.$$

where $z_S(y) = \ell_z + k_z y^\perp$, completed with the initial condition

$$z(T, \cdot) = 0, \quad \text{in } \mathcal{F}, \quad \ell_z(T) = 0, \quad k_z(T) = 0.$$

Using Theorem 2.1, we have

$$\begin{aligned} & \|z\|_{H^1(0,T;[L^2(\mathcal{F})]^2)} + \|z\|_{L^2(0,T;[H^2(\mathcal{F})]^2)} + \|\ell_z\|_{[H^1(0,T)]^2} + \|k_z\|_{H^1(0,T)} \\ & \leq C \left(\|\rho F_1\|_{L^2(0,T;[L^2(\mathcal{F})]^2)} + \|\rho F_2\|_{[L^2(0,T)]^2} + \|\rho F_3\|_{L^2(0,T)} \right). \end{aligned} \quad (5.11)$$

Step 2: We apply the curl operator to the first equation of (5.8), to get

$$-\partial_t(\nabla \times \varphi) - \nu \Delta(\nabla \times \varphi) = -\rho' \nabla \times v \quad \text{in } \mathcal{F}. \quad (5.12)$$

We obtain a heat equation. We recall that

$$\nabla \times \varphi = \frac{\partial \varphi_2}{\partial y_1} - \frac{\partial \varphi_1}{\partial y_2}.$$

We apply the inequality (19) of [2] replacing ψ by $\nabla \times \varphi$. We note that in our case the domain \mathcal{F} does not depend on time, then the inequality (19) of [2] holds true without terms involving the flow \hat{u} . We get

$$\begin{aligned} & s\lambda^2 \int_{\mathcal{F}} e^{-2s\beta} \xi |\nabla(\nabla \times \varphi)|^2 dy + s^3 \lambda^4 \int_{\mathcal{F}} e^{-2s\beta} \xi^3 |\nabla \times \varphi|^2 dy + s^3 \lambda^3 \int_{\partial\mathcal{F}} e^{-2s\hat{\beta}} (\xi^*)^3 |\nabla \times \varphi|^2 d\Gamma \\ & \leq C \left(s^3 \lambda^4 \int_{\mathcal{O}} e^{-2s\beta} \xi^3 |\nabla \times \varphi|^2 dy + s\lambda \int_{\partial\mathcal{F}} e^{-2s\hat{\beta}} \xi^* |\nabla(\nabla \times \varphi)\tau|^2 d\Gamma \right. \\ & \quad \left. + s^{-1} \lambda^{-1} \int_0^T \int_{\partial\mathcal{F}} e^{-2s\hat{\beta}} (\xi^*)^{-1} |\nabla \times \partial_t \varphi|^2 d\Gamma dt + \int_0^T \int_{\mathcal{F}} e^{-2s\beta} (\rho')^2 |\nabla \times v|^2 dy dt \right). \end{aligned} \quad (5.13)$$

Arguing as [2, pp.6-7], we treat the local term appearing in the right hand side of (5.13), we obtain

$$\begin{aligned} & s\lambda^2 \int_0^T \int_{\mathcal{F}} e^{-2s\beta} \xi |\nabla(\nabla \times \varphi)|^2 dy dt + s^3 \lambda^4 \int_0^T \int_{\mathcal{F}} e^{-2s\beta} \xi^3 |\nabla \times \varphi|^2 dy dt \\ & \quad + s^3 \lambda^3 \int_0^T \int_{\partial\mathcal{F}} e^{-2s\hat{\beta}} (\xi^*)^3 |\nabla \times \varphi|^2 d\Gamma dt \leq C \left(s^5 \lambda^6 \int_0^T \int_{\mathcal{O}} e^{-2s\beta} \xi^5 |\varphi|^2 dy dt \right. \\ & \quad \left. + s\lambda \int_0^T \int_{\partial\mathcal{F}} e^{-2s\hat{\beta}} \xi^* |\nabla(\nabla \times \varphi)\tau|^2 d\Gamma dt + s^{-1} \lambda^{-1} \int_0^T \int_{\partial\mathcal{F}} e^{-2s\hat{\beta}} (\xi^*)^{-1} |\nabla \times \partial_t \varphi|^2 d\Gamma dt \right. \\ & \quad \left. + \int_0^T \int_{\mathcal{F}} e^{-2s\beta} (\rho')^2 |\nabla \times v|^2 dy dt \right), \end{aligned} \quad (5.14)$$

for $\lambda \geq C$ and $s \geq C(T^N + T^{2N})$. We notice that φ satisfies the following problem

$$\begin{cases} \Delta \varphi = -\nabla \times (\nabla \times \varphi) & \text{in } \mathcal{F}, \\ \nabla \cdot \varphi = 0 & \text{in } \mathcal{F}, \end{cases}$$

with the boundary conditions (5.9). Applying the Carleman inequality proved in Proposition 4.1, we obtain

$$\begin{aligned} & s^2 \lambda^2 \int_{\Omega} e^{2se^{\lambda\eta}} e^{2\lambda\eta} |\nabla \varphi|^2 dy + s^4 \lambda^4 \int_{\Omega} e^{2se^{\lambda\eta}} e^{4\lambda\eta} |\varphi|^2 dy \\ & \leq C \left(s^4 \lambda^4 \int_{\mathcal{O}} e^{2se^{\lambda\eta}} e^{4\lambda\eta} |\varphi|^2 dy + s \int_{\mathcal{F}} e^{2se^{\lambda\eta}} e^{\lambda\eta} |\nabla \times (\nabla \times \varphi)|^2 dy \right. \\ & \quad \left. + \beta_S s^3 \lambda^2 e^{2s} \int_{\partial\mathcal{S}} |(\varphi_S)_\tau|^2 d\Gamma + s^4 \lambda^2 e^{2s} \|\varphi_S \cdot n\|_{H^{3/2}(\partial\mathcal{S})}^2 \right), \end{aligned} \quad (5.15)$$

where we have used that $a = \varphi_S 1_{\partial S}$ and $b = \beta_S(\varphi_S)_\tau 1_{\partial S}$. We replace s in (5.15) by $\frac{s e^{2N\lambda\|\eta\|_{L^\infty(\Omega)}}}{t^N(T-t)^N}$. Multiplying (5.15) by

$$e^{-2s \frac{e^{(2N+2)\lambda\|\eta\|_{L^\infty(\Omega)}}}{t^N(T-t)^N}},$$

and integrating over $(0, T)$, we get

$$\begin{aligned} s^2 \lambda^2 \int_0^T \int_{\mathcal{F}} e^{-2s\beta} \xi^2 |\nabla \varphi|^2 \, dydt + s^4 \lambda^4 \int_0^T \int_{\mathcal{F}} e^{-2s\beta} \xi^4 |\varphi|^2 \, dydt &\leq C \left(s^4 \lambda^4 \int_0^T \int_{\mathcal{O}} e^{-2s\beta} \xi^3 |\varphi|^2 \, dydt \right. \\ &+ s \int_0^T \int_{\mathcal{F}} e^{-2s\beta} \xi |\nabla \times (\nabla \times \varphi)|^2 \, dydt + \beta_S s^3 \lambda^2 \int_0^T e^{-2s\hat{\beta}}(\xi^*)^3 \int_{\partial S} |(\varphi_S)_\tau|^2 \, d\Gamma \, dt \\ &\left. + s^4 \lambda^2 \int_0^T e^{-2s\hat{\beta}}(\xi^*)^4 \|\varphi_S \cdot n\|_{H^{3/2}(\partial S)}^2 \, dt \right). \end{aligned} \quad (5.16)$$

Applying the estimates obtained in [1, Theorem 2.2], we get

$$\|\varphi\|_{[H^1(\mathcal{F})]^2}^2 \leq C \left(\|\varphi\|_{[L^2(\mathcal{F})]^2}^2 + \|\nabla \cdot \varphi\|_{L^2(\mathcal{F})}^2 + \|\nabla \times \varphi\|_{[L^2(\mathcal{F})]^2}^2 + \|\varphi_S \cdot n\|_{H^{1/2}(\partial S)}^2 \right). \quad (5.17)$$

Then, we multiply (5.17) by $s^3 \lambda^4 e^{-2s\hat{\beta}}(\xi^*)^3$, we get

$$\begin{aligned} s^3 \lambda^4 \int_0^T \int_{\mathcal{F}} e^{-2s\hat{\beta}}(\xi^*)^3 |\nabla \varphi|^2 \, dydt &\leq C \left(s^3 \lambda^4 \int_0^T \int_{\mathcal{F}} e^{-2s\hat{\beta}}(\xi^*)^3 |\varphi|^2 \, dydt \right. \\ &\left. + s^3 \lambda^4 \int_0^T \int_{\mathcal{F}} e^{-2s\hat{\beta}}(\xi^*)^3 |\nabla \times \varphi|^2 \, dydt + s^3 \lambda^4 \int_0^T e^{-2s\hat{\beta}}(\xi^*)^3 \|\varphi_S \cdot n\|_{H^{1/2}(\partial S)}^2 \, dt \right). \end{aligned} \quad (5.18)$$

Adding (5.18), (5.16) and (5.14), we deduce

$$\begin{aligned} s^3 \lambda^4 \int_0^T \int_{\mathcal{F}} e^{-2s\hat{\beta}}(\xi^*)^3 |\nabla \varphi|^2 \, dydt + s^4 \lambda^4 \int_0^T \int_{\mathcal{F}} e^{-2s\beta} \xi^4 |\varphi|^2 \, dydt \\ + s^3 \lambda^3 \int_0^T \int_{\partial \mathcal{F}} e^{-2s\hat{\beta}}(\xi^*)^3 |\nabla \times \varphi|^2 \, d\Gamma dt &\leq C \left(s^5 \lambda^6 \int_0^T \int_{\mathcal{O}} e^{-2s\beta} \xi^5 |\varphi|^2 \, dydt \right. \\ + s\lambda \int_0^T \int_{\partial \mathcal{F}} e^{-2s\hat{\beta}} \xi^* |\nabla(\nabla \times \varphi)\tau|^2 \, d\Gamma dt + s^{-1} \lambda^{-1} \int_0^T \int_{\partial \mathcal{F}} e^{-2s\hat{\beta}}(\xi^*)^{-1} |\nabla \times \varphi_t|^2 \, d\Gamma dt \\ + \beta_S s^3 \lambda^2 \int_0^T e^{-2s\hat{\beta}}(\xi^*)^3 \int_{\partial S} |(\varphi_S)_\tau|^2 \, d\Gamma \, dt + s^4 \lambda^2 \int_0^T e^{-2s\hat{\beta}}(\xi^*)^4 \|\varphi_S \cdot n\|_{H^{3/2}(\partial S)}^2 \, dt \\ \left. + \int_0^T \int_{\mathcal{F}} e^{-2s\beta}(\rho')^2 |\nabla \times v|^2 \, dydt \right). \end{aligned} \quad (5.19)$$

Taking (s, λ) large enough, the fifth term in the right hand side of (5.19) can be transported to the left side. Indeed, since φ_S is rigid, from [22, Lemma 2.2], we have

$$\int_{\mathcal{F}} |\varphi(t, \cdot)|^2 \, dy \geq C \|\varphi_S(t) \cdot n\|_{H^{3/2}(\partial S)}^2, \quad (5.20)$$

for any shape of the body \mathcal{S} . Moreover, we have the following relation

$$(\nabla \times \varphi) \tau = \nabla \varphi n - (\nabla \varphi)^* n, \quad \text{on } \partial \mathcal{F}, \quad (5.21)$$

where $\tau = \begin{pmatrix} -n_2 \\ n_1 \end{pmatrix}$. Multiplying (5.21) by τ , we get

$$\nabla \times \varphi = (\nabla \varphi n) \cdot \tau - ((\nabla \varphi)^* n) \cdot \tau, \quad \text{on } \partial \mathcal{F}. \quad (5.22)$$

In the other hand, we have

$$((\nabla \varphi)^* n) \cdot \tau = \sum_{i,j} \partial_i \varphi_j n_j \tau_i = \sum_{i,j} \partial_i (\varphi_j n_j) \tau_i - \sum_{i,j} \partial_i n_j \varphi_j \tau_i = \nabla (\varphi \cdot n) \cdot \tau - \nabla n \tau \cdot \varphi, \quad \text{on } \partial \mathcal{F}. \quad (5.23)$$

Using the boundary conditions (5.9), we can write

$$\beta_S (\varphi_S)_\tau = \nu (\nabla \varphi n + (\nabla \varphi)^* n)_\tau + \beta_S \varphi_\tau, \quad \text{on } \partial \mathcal{S}. \quad (5.24)$$

Using (5.24), (5.23) and (5.22), we get

$$\beta_S \int_{\partial \mathcal{S}} |(\varphi_S)_\tau|^2 d\Gamma \leq C \left(\int_{\partial \mathcal{F}} |\nabla \times \varphi|^2 d\Gamma + \int_{\partial \mathcal{F}} |\varphi|^2 d\Gamma + \|\varphi_S \cdot n\|_{H^{3/2}(\partial \mathcal{S})}^2 \right). \quad (5.25)$$

Multiplying (5.25) by $s^3 \lambda^3 e^{-2s\hat{\beta}}(\xi^*)^3$ and integrating over $(0, T)$, we obtain

$$s^3 \lambda^3 \int_0^T \int_{\partial \mathcal{S}} e^{-2s\hat{\beta}}(\xi^*)^3 |(\varphi_S)_\tau|^2 d\Gamma dt \leq C \left(s^3 \lambda^3 \int_0^T \int_{\partial \mathcal{F}} e^{-2s\hat{\beta}}(\xi^*)^3 |\nabla \times \varphi|^2 d\Gamma dt + s^3 \lambda^3 \int_0^T \int_{\partial \mathcal{F}} e^{-2s\hat{\beta}}(\xi^*)^3 |\varphi|^2 d\Gamma dt + s^3 \lambda^3 \int_0^T e^{-2s\hat{\beta}}(\xi^*)^3 \|\varphi_S \cdot n\|_{H^{3/2}(\partial \mathcal{S})}^2 dt \right). \quad (5.26)$$

Using the inequality in [12, Theorem II.4.1] with $r = 2, q = 2$, we obtain

$$s^3 \lambda^3 \int_{\partial \mathcal{F}} e^{-2s\hat{\beta}}(\xi^*)^3 |\varphi|^2 d\Gamma \leq C \left(s^3 \lambda^3 \int_{\mathcal{F}} e^{-2s\hat{\beta}}(\xi^*)^3 |\nabla \varphi|^2 dy + s^3 \lambda^3 \int_{\mathcal{F}} e^{-2s\hat{\beta}}(\xi^*)^3 |\varphi|^2 dy \right). \quad (5.27)$$

Thus, adding (5.27), (5.26), (5.19) and using (5.20), we get

$$s^3 \lambda^4 \int_0^T \int_{\mathcal{F}} e^{-2s\hat{\beta}}(\xi^*)^3 |\nabla \varphi|^2 dy dt + s^4 \lambda^4 \int_0^T \int_{\mathcal{F}} e^{-2s\hat{\beta}} \xi^4 |\varphi|^2 dy dt + s^4 \lambda^4 \int_0^T e^{-2s\hat{\beta}}(\xi^*)^4 |\ell_\varphi(t)|^2 dt + s^3 \lambda^3 \int_0^T e^{-2s\hat{\beta}}(\xi^*)^3 |k_\varphi(t)|^2 dt \leq C \left(s^5 \lambda^6 \int_0^T \int_{\mathcal{O}} e^{-2s\hat{\beta}} \xi^5 |\varphi|^2 dy dt + s \lambda \int_0^T \int_{\partial \mathcal{F}} e^{-2s\hat{\beta}} \xi^* |\nabla (\nabla \times \varphi) \tau|^2 d\Gamma dt + s^{-1} \lambda^{-1} \int_0^T \int_{\partial \mathcal{F}} e^{-2s\hat{\beta}}(\xi^*)^{-1} |\nabla \times \varphi_t|^2 d\Gamma dt + \int_0^T \int_{\mathcal{F}} e^{-2s\hat{\beta}}(\rho')^2 |\nabla \times v|^2 dy dt \right). \quad (5.28)$$

We emphasize here that to absorb the boundary terms appearing in the right hand side of (5.28), we need to impose that \mathcal{S} is not a disk to have sufficient weights to manipulate the angular velocity k_φ . In fact, if \mathcal{S} is not a disk, from [22, Lemma 2.2], we have

$$\int_{\mathcal{F}} |\varphi(t, \cdot)|^2 dy \geq C |k_\varphi(t)|^2. \quad (5.29)$$

Let us deal with the last term in the right hand side of (5.28). Noticing that $|\rho'| \leq C s \rho (\xi^*)^{1+1/N}$, we get

$$\int_0^T \int_{\mathcal{F}} e^{-2s\hat{\beta}}(\rho')^2 |\nabla \times v|^2 dy dt = \int_0^T \int_{\mathcal{F}} e^{-2s\hat{\beta}}(\rho')^2 (\rho)^{-2} |\nabla \times \rho v|^2 dy dt \leq C s^2 \int_0^T \int_{\mathcal{F}} e^{-2s\hat{\beta}}(\xi^*)^3 (|\nabla \times \varphi|^2 + |\nabla \times z|^2) dy dt, \quad (5.30)$$

$$\begin{aligned} \|\zeta_1' \varphi\|_{H^{5/18}(0,T;[L^2(\mathcal{F})]^2)}^2 &\leq C \left(s^4 \lambda \int_0^T \int_{\mathcal{F}} e^{-2s\widehat{\beta}}(\xi^*)^4 |\varphi|^2 dydt \right. \\ &\quad \left. + s^{2/5} \lambda \int_0^T \int_{\mathcal{F}} e^{-2s\widehat{\beta}}(\xi^*)^{2/5+36/5N} |\partial_t \varphi|^2 dydt + s^{12/5} \lambda \int_0^T \int_{\mathcal{F}} e^{-2s\widehat{\beta}}(\xi^*)^{12/5+46/5N} |\varphi|^2 dydt \right), \end{aligned} \quad (5.35)$$

$$\begin{aligned} \|\zeta_1' \ell_\varphi\|_{[H^{5/18}(0,T)]^2}^2 &\leq C \left(s^4 \lambda \int_0^T e^{-2s\widehat{\beta}}(\xi^*)^4 |\ell_\varphi|^2 dt \right. \\ &\quad \left. + s^{2/5} \lambda \int_0^T e^{-2s\widehat{\beta}}(\xi^*)^{2/5+36/5N} |\ell_\varphi'|^2 dt + s^{32/9} \lambda \int_0^T e^{-2s\widehat{\beta}}(\xi^*)^{32/9+23/9N} |\ell_\varphi|^2 dt \right), \end{aligned} \quad (5.36)$$

and

$$\begin{aligned} \|\zeta_1' k_\varphi\|_{H^{5/18}(0,T)}^2 &\leq C \left(s^4 \lambda \int_0^T e^{-2s\widehat{\beta}}(\xi^*)^4 |k_\varphi|^2 dt \right. \\ &\quad \left. + s^{2/5} \lambda \int_0^T e^{-2s\widehat{\beta}}(\xi^*)^{2/5+36/5N} |k_\varphi'|^2 dt + s^{32/9} \lambda \int_0^T e^{-2s\widehat{\beta}}(\xi^*)^{32/9+23/9N} |k_\varphi|^2 dt \right). \end{aligned} \quad (5.37)$$

Taking $N \geq 12$, we note that the second term in the right hand side of each inequality (5.35), (5.36) and (5.37) can be absorbed by the left hand side of (5.33).

On the other hand, we notice that $|\zeta_1 \rho'| \leq C s^{3/2} \lambda^{1/2} e^{-s\widehat{\beta}}(\xi^*)^{3/2+1/N} \rho$ and we get as for (5.34)

$$\begin{aligned} \int_0^T |\zeta_1 \rho'|^2 \|v\|_{[H^{5/9}(\mathcal{F})]^2}^2 dt &\leq C \int_0^T s^3 \lambda e^{-2s\widehat{\beta}}(\xi^*)^{3+2/N} \|\rho v\|_{[H^{5/9}(\mathcal{F})]^2}^2 dt \\ &\leq \varepsilon s^4 \lambda \int_0^T \int_{\mathcal{F}} e^{-2s\widehat{\beta}}(\xi^*)^4 |\rho v|^2 dydt + C s^{11/5} \lambda \int_0^T \int_{\mathcal{F}} e^{-2s\widehat{\beta}}(\xi^*)^3 |\nabla(\rho v)|^2 dydt. \end{aligned}$$

Using the decomposition (5.7) and the regularity estimate (5.11), we deduce from the above inequality

$$\begin{aligned} \|\zeta_1 \rho' v\|_{L^2(0,T;[H^{5/9}(\mathcal{F})]^2)}^2 &\leq \varepsilon s^4 \lambda \int_0^T \int_{\mathcal{F}} e^{-2s\widehat{\beta}}(\xi^*)^4 |\varphi|^2 dydt + C \left(s^{11/5} \lambda \int_0^T \int_{\mathcal{F}} e^{-2s\widehat{\beta}}(\xi^*)^3 |\nabla \varphi|^2 dydt \right. \\ &\quad \left. + \int_0^T \int_{\mathcal{F}} e^{-3s\widehat{\beta}} |F_1|^2 dydt + \int_0^T e^{-3s\widehat{\beta}} |F_2|^2 dt + \int_0^T e^{-3s\widehat{\beta}} |F_3|^2 dt \right). \end{aligned} \quad (5.38)$$

Since $\rho' = -\frac{3}{2} s(\widehat{\beta})' \rho$, we have

$$\begin{aligned} \|\zeta_1 \rho' v\|_{H^1(0,T;[L^2(\mathcal{F})]^2)}^2 &\leq C s^2 \left(\left\| \zeta_1(\widehat{\beta})' \rho v \right\|_{L^2(0,T;[L^2(\mathcal{F})]^2)}^2 + \left\| (\zeta_1(\widehat{\beta})')' \rho v \right\|_{L^2(0,T;[L^2(\mathcal{F})]^2)}^2 \right. \\ &\quad \left. + \left\| \zeta_1(\widehat{\beta})' \partial_t(\rho v) \right\|_{L^2(0,T;[L^2(\mathcal{F})]^2)}^2 \right) \\ &\leq C \left(s^3 \lambda \int_0^T \int_{\mathcal{F}} e^{-2s\widehat{\beta}}(\xi^*)^{3+2/N} |\partial_t(\rho v)|^2 dydt + s^5 \lambda \int_0^T \int_{\mathcal{F}} e^{-2s\widehat{\beta}}(\xi^*)^{5+4/N} |\rho v|^2 dydt \right), \end{aligned}$$

where we have used that

$$\left| \zeta_1(\widehat{\beta})' \right|^2 \leq C s \lambda e^{-2s\widehat{\beta}}(\xi^*)^{3+2/N},$$

and

$$\left| (\zeta_1(\widehat{\beta})')' \right|^2 \leq C s^3 \lambda e^{-2s\widehat{\beta}}(\xi^*)^{5+4/N}.$$

Using interpolation arguments and the Young inequality, we find as for (5.35)

$$\begin{aligned} \|\zeta_1 \rho' v\|_{H^{5/18}(0,T;[L^2(\mathcal{F})]^2)}^2 &\leq C \left(s^{2/5} \lambda \int_0^T \int_{\mathcal{F}} e^{-2s\widehat{\beta}} (\xi^*)^{2/5+36/(5N)} |\partial_t(\rho v)|^2 dy dt \right. \\ &\quad \left. + s^4 \lambda \int_0^T \int_{\mathcal{F}} e^{-2s\widehat{\beta}} (\xi^*)^4 |\rho v|^2 dy dt + s^{12/5} \lambda \int_0^T \int_{\mathcal{F}} e^{-2s\widehat{\beta}} (\xi^*)^{12/5+46/(5N)} |\rho v|^2 dy dt \right). \end{aligned} \quad (5.39)$$

We get also

$$\begin{aligned} \|\zeta_1 \rho' \ell_v\|_{[H^{5/18}(0,T)]^2}^2 &\leq C \left(s^4 \lambda \int_0^T e^{-2s\widehat{\beta}} (\xi^*)^4 |\rho \ell_v|^2 dt \right. \\ &\quad \left. + s^{2/5} \lambda \int_0^T e^{-2s\widehat{\beta}} (\xi^*)^{2/5+36/5N} |(\rho \ell_v)'|^2 dt + s^{32/9} \lambda \int_0^T e^{-2s\widehat{\beta}} (\xi^*)^{32/9+23/9N} |\rho \ell_v|^2 dt \right), \end{aligned} \quad (5.40)$$

and

$$\begin{aligned} \|\zeta_1 \rho' k_v\|_{[H^{5/18}(0,T)]^2}^2 &\leq C \left(s^4 \lambda \int_0^T e^{-2s\widehat{\beta}} (\xi^*)^4 |\rho k_v|^2 dt \right. \\ &\quad \left. + s^{2/5} \lambda \int_0^T e^{-2s\widehat{\beta}} (\xi^*)^{2/5+36/5N} |(\rho k_v)'|^2 dt + s^{32/9} \lambda \int_0^T e^{-2s\widehat{\beta}} (\xi^*)^{32/9+23/9N} |\rho k_v|^2 dt \right). \end{aligned} \quad (5.41)$$

Now using that $23/9 > 5/2$, we have from the trace theorem

$$s \lambda \int_0^T \int_{\partial\mathcal{F}} e^{-2s\widehat{\beta}} \xi^* |\nabla(\nabla \times \varphi) \tau|^2 d\Gamma dt \leq C \|\zeta_1 \varphi\|_{L^2(0,T;[H^{23/9}(\mathcal{F})]^2)}^2.$$

From (5.7), (5.33), (5.34), (5.35), (5.36), (5.37), (5.38), (5.39), (5.40) and (5.41), we deduce

$$\begin{aligned} s \lambda \int_0^T \int_{\partial\mathcal{F}} e^{-2s\widehat{\beta}} \xi^* |\nabla(\nabla \times \varphi) \tau|^2 d\Gamma dt &\leq C s^4 \lambda \left(\int_0^T \int_{\mathcal{F}} e^{-2s\widehat{\beta}} \xi^4 |\varphi|^2 dy dt \right. \\ &\quad \left. + \int_0^T \int_{\mathcal{F}} e^{-2s\widehat{\beta}} (\xi^*)^3 |\nabla \varphi|^2 dy dt + \int_0^T e^{-2s\widehat{\beta}} (\xi^*)^4 (|\ell_\varphi(t)|^2 + |k_\varphi(t)|^2) dt \right. \\ &\quad \left. + \int_0^T \int_{\mathcal{F}} e^{-3s\widehat{\beta}} |F_1|^2 dy dt + \int_0^T e^{-3s\widehat{\beta}} |F_2|^2 dt + \int_0^T e^{-3s\widehat{\beta}} |F_3|^2 dt \right). \end{aligned} \quad (5.42)$$

To estimate the second term in (5.32), we take $\zeta_2(t) = s^{-1/2} \lambda^{-1/2} e^{-s\beta(t)} (\xi^*)^{-1/2}(t)$ and we follow again the arguments in [2], we obtain

$$\begin{aligned} \|\zeta_2 \varphi_t\|_{L^2(0,T;[H^{14/9}(\mathcal{F})]^2)} &\leq C \left(\|\zeta_2'' \varphi\|_{L^2(0,T;[L^2(\mathcal{F})]^2)} + \|\zeta_2'' \ell_\varphi\|_{[L^2(0,T)]^2} + \|\zeta_2'' k_\varphi\|_{L^2(0,T)} + \|(\zeta_2 \rho')' v\|_{L^2(0,T;[L^2(\mathcal{F})]^2)} \right. \\ &\quad \left. + \|(\zeta_2 \rho')' \ell_v\|_{[L^2(0,T)]^2} + \|(\zeta_2 \rho')' k_v\|_{L^2(0,T)} + \|\rho F_1\|_{L^2(0,T;[L^2(\mathcal{F})]^2)} + \|\rho F_2\|_{[L^2(0,T)]^2} + \|\rho F_3\|_{L^2(0,T)} \right). \end{aligned}$$

We note that

$$|\zeta_2''| \leq C s^{3/2} \lambda^{-1/2} (\xi^*)^{3/2+2/N} e^{-s\widehat{\beta}},$$

and

$$\|(\zeta_2 \rho')'\| \leq C s^{3/2} \lambda^{-1/2} (\xi^*)^{3/2+2/N} e^{-s\widehat{\beta}} \rho.$$

Using the trace theorem, we have

$$s^{-1} \lambda^{-1} \int_0^T \int_{\partial\mathcal{F}} e^{-2s\widehat{\beta}} (\xi^*)^{-1} |\nabla \times \varphi_t|^2 d\Gamma dt \leq C \|\zeta_2 \varphi_t\|_{L^2(0,T;[H^{14/9}(\mathcal{F})]^2)}^2.$$

Thus, for $N \geq 4$, $\lambda \geq C$ and $s \geq C(T^N + T^{2N})$, we obtain

$$\begin{aligned} s^{-1}\lambda^{-1} \int_0^T \int_{\partial\mathcal{F}} e^{-2s\widehat{\beta}(\xi^*)^{-1}} |\nabla \times \varphi_t|^2 d\Gamma dt &\leq Cs^4\lambda \left(\int_0^T \int_{\partial\mathcal{F}} e^{-2s\widehat{\beta}(\xi^*)^4} |\varphi|^2 dydt \right. \\ &+ \int_0^T e^{-2s\widehat{\beta}(\xi^*)^4} (|\ell_\varphi(t)|^2 + |k_\varphi(t)|^2) dt + \int_0^T \int_{\mathcal{F}} e^{-3s\widehat{\beta}} |F_1|^2 dydt \\ &\left. + \int_0^T e^{-3s\widehat{\beta}} |F_2|^2 dt + \int_0^T e^{-3s\widehat{\beta}} |F_3|^2 dt \right). \end{aligned} \quad (5.43)$$

Combining (5.31), (5.42), (5.43) and (5.7), we get finally (5.6) for $N \geq 12$, $\lambda \geq C$ and $s \geq C(T^N + T^{2N})$. \square

6 Null controllability for the linearized system

In this section, we prove the null controllability of the linear system

$$\begin{cases} \partial_t \bar{u} - \nabla \cdot \mathbb{T}(\bar{u}, \pi) = v^* 1_{\mathcal{O}} + F_1 & \text{in } (0, T), \text{ in } \mathcal{F}, \\ \nabla \cdot \bar{u} = 0 & \text{in } (0, T), \text{ in } \mathcal{F}, \end{cases} \quad (6.1)$$

$$\begin{cases} m\bar{h}''(t) = - \int_{\partial\mathcal{S}} \mathbb{T}(\bar{u}, \pi)n d\Gamma + F_2 & t \in (0, T), \\ J\bar{\theta}''(t) = - \int_{\partial\mathcal{S}} y^\perp \cdot \mathbb{T}(\bar{u}, \pi)n d\Gamma + F_3 & t \in (0, T), \end{cases} \quad (6.2)$$

$$\begin{cases} \bar{u}_n = 0 & \text{in } (0, T), \text{ on } \partial\Omega, \\ [2\nu D(\bar{u})n + \beta_\Omega \bar{u}]_\tau = 0 & \text{in } (0, T), \text{ on } \partial\Omega, \\ (\bar{u} - \bar{u}_S)_n = 0 & \text{in } (0, T), \text{ on } \partial\mathcal{S}, \\ [2\nu D(\bar{u})n + \beta_S (\bar{u} - \bar{u}_S)]_\tau = 0 & \text{in } (0, T), \text{ on } \partial\mathcal{S}, \end{cases} \quad (6.3)$$

with the initial conditions

$$\bar{u}(0) = u_0, \quad \bar{h}'(0) = \ell^0, \quad \bar{\theta}'(0) = \omega^0, \quad \bar{h}(0) = h^0, \quad \bar{\theta}(0) = \theta^0. \quad (6.4)$$

The system (6.1),(6.2), (6.3), (6.4) can be written as

$$\begin{cases} Z'(t) = AZ'(t) + Bv^* + F, \\ a'(t) = CZ(t), \\ Z(0) = Z^0, \\ a(0) = a^0, \end{cases} \quad (6.5)$$

where A is defined as in section 2 and

$$B = \mathbb{P}(v^* 1_{\mathcal{O}}), \quad F = \mathbb{P} \left(F_1 1_{\mathcal{F}} + \left(\frac{F_2}{m} + \frac{F_3 y^\perp}{J} \right) 1_{\mathcal{S}} \right).$$

The vector a is defined by $a = (\bar{h}, \bar{\theta})$ and we define the operator \mathcal{C} for $Z \in \mathbb{H}$ as

$$\mathcal{C}Z = (\ell_{\bar{u}}, k_{\bar{u}}), \quad \text{if } Z = \ell_{\bar{u}} + k_{\bar{u}} y^\perp, \text{ in } \mathcal{S},$$

where $\bar{h}' = \ell_{\bar{u}}$, $\bar{\theta}' = k_{\bar{u}}$. The equation $Z(0) = Z^0$ corresponds to the initial conditions $\bar{u}(0) = u_0$, $\bar{h}'(0) = \ell^0$, $\bar{\theta}'(0) = \omega^0$ and the vector a^0 corresponds to $a^0 = (h^0, \theta^0)$.

Now, let us fix $\lambda \geq C$, $s \geq C(T^N + T^{2N})$ and let consider

$$\rho_1(t) = \begin{cases} s^2 \lambda^2 e^{-\frac{5}{2}s\widehat{\beta}(T/2)} (\xi^*(T/2))^2 & t \in (0, T/2), \\ s^2 \lambda^2 e^{-\frac{5}{2}s\widehat{\beta}(t)} (\xi^*(t))^2 & t \in (T/2, T), \end{cases} \quad (6.6)$$

$$\rho_2(t) = \begin{cases} e^{-\frac{3}{2}s\widehat{\beta}(T/2)} & t \in (0, T/2), \\ e^{-\frac{3}{2}s\widehat{\beta}(t)} & t \in (T/2, T), \end{cases} \quad (6.7)$$

$$\rho_3(t) = \begin{cases} s^{5/2}\lambda^3 e^{-s\beta^*(T/2) - \frac{3}{2}s\widehat{\beta}(T/2)} (\widehat{\xi}(T/2))^{5/2} & t \in (0, T/2), \\ s^{5/2}\lambda^3 e^{-s\beta^*(t) - \frac{3}{2}s\widehat{\beta}(t)} (\widehat{\xi}(t))^{5/2} & t \in (T/2, T), \end{cases} \quad (6.8)$$

and

$$\rho_4(t) = \begin{cases} e^{-\frac{11}{8}s\widehat{\beta}(T/2)} & t \in (0, T/2), \\ e^{-\frac{11}{8}s\widehat{\beta}(t)} & t \in (T/2, T). \end{cases} \quad (6.9)$$

We notice that ρ_i are continuous positive functions such that $\rho_i(T) = 0$.

Let define the following spaces

$$\mathcal{H} = \left\{ f \in L^2(0, T; \mathbb{H}) \mid \frac{f}{\rho_1} \in L^2(0, T; \mathbb{H}) \right\},$$

$$\mathcal{Z} = \left\{ z \in L^2(0, T; \mathbb{H}) \mid \frac{z}{\rho_2} \in L^2(0, T; \mathbb{H}) \right\},$$

$$\mathcal{U} = \left\{ v^* \in L^2(0, T; [L^2(\mathcal{O})]^2) \mid \frac{v^*}{\rho_3} \in L^2(0, T; [L^2(\mathcal{O})]^2) \right\}.$$

Now, we can state the null controllability of the linearized system (6.5)

Proposition 6.1. *There exists a linear bounded operator $E_T : \mathbb{H} \times \mathbb{R}^3 \times \mathcal{F} \rightarrow \mathcal{U}$ such that for any $(Z^0, a^0, F) \in \mathbb{H} \times \mathbb{R}^3 \times \mathcal{F}$, the control $v^* = E_T(Z^0, a^0, F)$ is such that the solution (Z, a) of (6.5) satisfies $Z \in \mathcal{Z}$ and $a(T) = 0$. Moreover, if $Z^0 \in \mathcal{D}((-A)^{1/2})$, then*

$$\frac{Z}{\rho_4} \in L^2(0, T; \mathcal{D}(A)) \cap C([0, T]; \mathcal{D}((-A)^{1/2})) \cap H^1(0, T; \mathbb{H})$$

and we have the estimate

$$\left\| \frac{Z}{\rho_4} \right\|_{L^2(0, T; \mathcal{D}(A)) \cap C([0, T]; \mathcal{D}((-A)^{1/2})) \cap H^1(0, T; \mathbb{H})} \leq C \left(\|F\|_{\mathcal{H}} + |a^0| + \|Z^0\|_{\mathcal{D}((-A)^{1/2})} \right). \quad (6.10)$$

Proof. The second part of Proposition 6.1 comes from the fact that

$$\frac{\rho_i}{\rho_4} \in L^\infty(0, T) \text{ for } i = 1, 3, \quad \left| \frac{\rho'_4 \rho_2}{(\rho_4)^2} \right| \leq \widehat{C} e^{-\frac{1}{8}s\widehat{\beta}(t)} \leq C, \quad t \in [0, T].$$

Then, using [20, Corollary 4.3], we get

$$\frac{Z}{\rho_4} \in L^2(0, T; \mathcal{D}(A)) \cap C([0, T]; \mathcal{D}((-A)^{1/2})) \cap H^1(0, T; \mathbb{H}),$$

such that (6.10) is satisfied.

Let us prove the first part. The proof is similar to [20, Theorem 4.4]. The adjoint system associated to the linear system (6.1), (6.2) and (6.3) can be written as

$$\begin{cases} -\partial_t v - \nabla \cdot \mathbb{T}(v, q) = \gamma^1 & \text{in } (0, T), \text{ in } \mathcal{F}, \\ \nabla \cdot v = 0 & \text{in } (0, T), \text{ in } \mathcal{F}, \\ m\ell'_v(t) = \int_{\partial\mathcal{S}} \mathbb{T}(v, q)n \, d\Gamma + \ell_{\gamma^1} + \ell & t \in (0, T), \\ J(k'_v(t)) = \int_{\partial\mathcal{S}} y^\perp \cdot \mathbb{T}(v, q)n \, d\Gamma + k_{\gamma^1} + k & t \in (0, T), \end{cases} \quad (6.11)$$

$$\begin{cases} v_n = 0 & \text{in } (0, T), \text{ on } \partial\Omega, \\ [2\nu D(v)n + \beta_\Omega v]_\tau = 0 & \text{in } (0, T), \text{ on } \partial\Omega, \\ (v - v_S)_n = 0 & \text{in } (0, T), \text{ on } \partial\mathcal{S}, \\ [2\nu D(v)n + \beta_S(v - v_S)]_\tau = 0 & \text{in } (0, T), \text{ on } \partial\mathcal{S}, \end{cases} \quad (6.12)$$

where $v_S(y) = \ell_v + k_v y^\perp$, with

$$v(T, \cdot) = 0, \quad \text{in } \mathcal{F}, \quad \ell_v(T) = 0, \quad k_v(T) = 0.$$

Here $\gamma^2 = (\ell, k) \in \mathbb{R}^3$ and $\gamma^1 \in L^2(0, T; \mathbb{H})$. Following the arguments of [20, Theorem 4.1], we show that the null controllability of the system (6.1), (6.2), (6.4) is equivalent to show the following observability inequality

$$|\gamma^2|^2 + \|v(0)\|_{\mathbb{H}}^2 + \int_0^T \|\rho_1 v\|_{\mathbb{H}}^2 dt \leq C \left(\int_0^T \|\rho_2 \gamma^1\|_{\mathbb{H}}^2 dt + \int_0^T \int_{\mathcal{O}} |\rho_3 v|^2 dy dt \right). \quad (6.13)$$

We set

$$\rho_i^*(t) = \begin{cases} \rho_i(T-t) & t \in (0, T/2), \\ \rho_i(t) & t \in (T/2, T). \end{cases} \quad (6.14)$$

Then, (5.6) implies

$$\int_0^T \|\rho_1^* v\|_{\mathbb{H}}^2 dt \leq C \left(\int_0^T \|\rho_2^*(\gamma^1 + \mathcal{C}^* \gamma^2)\|_{\mathbb{H}}^2 dt + \int_0^T \int_{\mathcal{O}} |\rho_3^* v|^2 dy dt \right). \quad (6.15)$$

Next, we argue as [3, Proposition 4]. Let consider a non negative function $\bar{\eta} \in C^1([0, T])$ such that

$$0 \leq \bar{\eta} \leq 1 \text{ in } [0, T], \quad \bar{\eta} = 1 \text{ in } [0, T/2], \quad \bar{\eta} = 0 \text{ in } [3T/4, T].$$

Then $(\bar{\eta}v, \bar{\eta}\ell_v, \bar{\eta}k_v)$ satisfies the energy estimates

$$\begin{aligned} & \|\bar{\eta}v\|_{L^2(0, T; [H^2(\mathcal{F})]^2) \cap C([0, T]; [H^1(\mathcal{F})]^1) \cap H^1(0, T; [L^2(\mathcal{F})]^2)} + \|\bar{\eta}\ell_v\|_{[H^1(0, T)]^2} + \|\bar{\eta}k_v\|_{H^1(0, T)} \\ & \leq C \left(\|\bar{\eta}'v\|_{L^2(0, T; [L^2(\mathcal{F})]^2)} + \|\bar{\eta}'\ell_v\|_{[L^2(0, T)]^2} + \|\bar{\eta}'k_v\|_{L^2(0, T)} + \|\bar{\eta}(\gamma^1 + \mathcal{C}^* \gamma^2)\|_{L^2(0, T; \mathbb{H})}^2 \right). \end{aligned}$$

Whence

$$\begin{aligned} & \|v\|_{L^2(0, T/2; [H^2(\mathcal{F})]^2) \cap C([0, T/2]; [H^1(\mathcal{F})]^2) \cap H^1(0, T/2; [L^2(\mathcal{F})]^2)} + \|\ell_v\|_{[H^1(0, T/2)]^2} + \|k_v\|_{H^1(0, T/2)} \\ & \leq C \left(\|v\|_{L^2(T/2, 3T/4; [L^2(\mathcal{F})]^2)} + \|\ell_v\|_{[L^2(T/2, 3T/4)]^2} + \|k_v\|_{L^2(T/2, 3T/4)} + \|(\gamma^1 + \mathcal{C}^* \gamma^2)\|_{L^2(0, 3T/4; \mathbb{H})}^2 \right). \quad (6.16) \end{aligned}$$

Using (6.15), we have

$$\begin{aligned} & \|v\|_{L^2(T/2, 3T/4; [L^2(\mathcal{F})]^2)} + \|\ell_v\|_{[L^2(T/2, 3T/4)]^2} + \|k_v\|_{L^2(T/2, 3T/4)} \leq C \int_0^T \|\rho_1^* v\|_{\mathbb{H}}^2 dt \\ & \leq C \left(\int_0^T \|\rho_2^*(\gamma^1 + \mathcal{C}^* \gamma^2)\|_{\mathbb{H}}^2 dt + \int_0^T \int_{\mathcal{O}} |\rho_3^* v|^2 dy dt \right), \end{aligned}$$

where we have used that the function $\frac{1}{\rho_1^*}$ is bounded in $[T/2, 3T/4]$.

Since ρ_1 is constant in $(0, T/2)$ and $\frac{1}{\rho_2}$ is bounded in $(0, 3T/4)$, the equation (6.16) gives

$$\begin{aligned} \|v(0)\|_{\mathbb{H}}^2 + \int_0^{T/2} \|\rho_1 v\|_{\mathbb{H}}^2 dt & \leq C \left(\int_0^{3T/4} \|\rho_2(\gamma^1 + \mathcal{C}^* \gamma^2)\|_{\mathbb{H}}^2 dt \right. \\ & \quad \left. + \int_0^T \|\rho_2^*(\gamma^1 + \mathcal{C}^* \gamma^2)\|_{\mathbb{H}}^2 dt + \int_0^T \int_{\mathcal{O}} |\rho_3^* v|^2 dy dt \right). \end{aligned}$$

Thus, using that $\rho_1^* = \rho_1$ in $(T/2, T)$, the estimate (6.15) and using the fact that $\rho_i^* \leq \rho_i$ for $i = 1, 2, 3$ in $[0, T]$, we get

$$\|v(0)\|_{\mathbb{H}}^2 + \int_0^T \|\rho_1 v\|_{\mathbb{H}}^2 dt \leq C \left(\int_0^T \|\rho_2(\gamma^1 + \mathcal{C}^* \gamma^2)\|_{\mathbb{H}}^2 dt + \int_0^T \int_{\mathcal{O}} |\rho_3 v|^2 dy dt \right).$$

It remains to prove

$$|\gamma^2|^2 \leq C \left(\int_0^T \|\rho_2 \gamma^1\|_{\mathbb{H}}^2 dt + \int_0^T \int_{\mathcal{O}} |\rho_3 v|^2 dy dt \right). \quad (6.17)$$

The result follows by using a contradiction argument. In fact, assume that (6.17) is false. Then there exists a sequence $(\gamma_\kappa^1, \gamma_\kappa^2)_\kappa \in L^2(0, T; \mathbb{H}) \times \mathbb{R}^3$ such that

$$|\gamma_\kappa^2| = 1, \quad (6.18)$$

and

$$\int_0^T \|\rho_2 \gamma_\kappa^1\|_{\mathbb{H}}^2 dy dt + \int_0^T \int_{\mathcal{O}} |\rho_3 v_\kappa|^2 dy dt \rightarrow 0. \quad (6.19)$$

Thus, for $\varepsilon > 0$

$$\gamma_\kappa^1 \rightharpoonup 0 \text{ in } L^2(0, T - \varepsilon; \mathbb{H}),$$

and

$$\gamma_\kappa^2 \rightarrow \gamma^2 \text{ in } \mathbb{R}^3.$$

On the other hand, the solution $(v_\kappa)_\kappa$ of the system (6.11), (6.12) associated with $(\gamma_\kappa^1, \gamma_\kappa^2)$ verifies the estimate (arguing always like [3, Proposition 4] taking $\bar{\eta} = 1$ in $[0, T - \varepsilon]$ and $\bar{\eta} = 0$ in $[T - \frac{\varepsilon}{2}, T]$)

$$\begin{aligned} & \|v_\kappa\|_{L^2(0, T - \varepsilon; [H^2(\mathcal{F})]^2) \cap H^1(0, T - \varepsilon; [L^2(\mathcal{F})]^2)} + \|q_\kappa\|_{L^2(0, T - \varepsilon; H^1(\mathcal{F}))} + \|(\ell v)_\kappa\|_{[H^1(0, T - \varepsilon)]^2}^2 + \|(k v)_\kappa\|_{H^1(0, T - \varepsilon)}^2 \\ & \leq C \left(\int_0^T \|\rho_2(\gamma_\kappa^1 + \mathcal{C}^* \gamma_\kappa^2)\|_{\mathbb{H}}^2 dt + \int_0^T \int_{\mathcal{O}} |\rho_3 v_\kappa|^2 dy dt \right). \end{aligned}$$

From the above inequality, we have

$$v_\kappa \rightharpoonup v \text{ in } L^2(0, T - \varepsilon; \mathcal{D}(A)) \cap H^1(0, T - \varepsilon; \mathbb{H}),$$

and

$$q_\kappa \rightharpoonup q \text{ in } L^2(0, T - \varepsilon; H^1(\mathcal{F})).$$

Therefore, the couple (v, q) satisfies the system

$$\left\{ \begin{array}{l} -\partial_t v - \nabla \cdot \mathbb{T}(v, q) = 0 \quad \text{in } (0, T - \varepsilon), \quad \text{in } \mathcal{F}, \\ \nabla \cdot v = 0 \quad \text{in } (0, T - \varepsilon), \quad \text{in } \mathcal{F}, \\ m \ell'_v(t) = \int_{\partial \mathcal{S}} \mathbb{T}(v, q) n d\Gamma + \ell \quad t \in (0, T - \varepsilon), \\ J(k'_v(t)) = \int_{\partial \mathcal{S}} y^\perp \cdot \mathbb{T}(v, q) n d\Gamma + k \quad t \in (0, T - \varepsilon), \\ \left\{ \begin{array}{l} v_n = 0 \quad \text{in } (0, T - \varepsilon), \quad \text{on } \partial \Omega, \\ [2\nu D(v)n + \beta_\Omega v]_\tau = 0 \quad \text{in } (0, T - \varepsilon), \quad \text{on } \partial \Omega, \\ (v - v_S)_n = 0 \quad \text{in } (0, T - \varepsilon), \quad \text{on } \partial \mathcal{S}, \\ [2\nu D(v)n + \beta_S(v - v_S)]_\tau = 0 \quad \text{in } (0, T - \varepsilon), \quad \text{on } \partial \mathcal{S}. \end{array} \right. \end{array} \right.$$

Moreover, from (6.19), we have $v = 0$ in $(0, T - \varepsilon) \times \mathcal{O}$. Then, using the unique continuity property of the Stokes system (see for instance [9]), we get

$$v = \nabla q = 0 \text{ in } (0, T - \varepsilon) \times \mathcal{F}.$$

The boundary conditions read to

$$(\ell_v + k_v y^\perp)_n = 0, \quad \beta_S(\ell_v + k_v y^\perp)_\tau = 0, \quad y \in \partial\mathcal{S}.$$

Since \mathcal{S} is not a disk, we get that $\ell_v = 0$ and $k_v = 0$ in $(0, T - \varepsilon)$. Then, we obtain in particular that $\gamma^2 = (\ell, k) = 0$ from the equations of the structure motion which contradicts (6.18). \square

7 Fixed point

In this section, we prove Theorem 1.1 by applying a fixed-point argument. For this purpose, we follow the same steps as [20]. First, we give some estimates on the terms appearing in the system (3.4), (3.5) and (3.6). We have the following lemma that is proved in [24].

Lemma 7.1. *Let X and Y satisfying the properties given in Section 3. We obtain for all $(\bar{u}, \pi) \in [H^2(\mathcal{F})]^2 \times H^1(\mathcal{F})$, the following estimates, for all $t \in [0, T]$*

$$\begin{aligned} \frac{1}{\rho_4(t)^2} \|(\mathcal{L} - \Delta)\bar{u}\|_{[L^2(\mathcal{F})]^2} &\leq C \left(\left\| \frac{\bar{h}'}{\rho_4} \right\|_{[L^\infty(0,T)]^2} + \left\| \frac{\bar{\theta}'}{\rho_4} \right\|_{[L^2(0,T)]^2} \right) \left\| \frac{\bar{u}}{\rho_4(t)} \right\|_{[H^2(\mathcal{F})]^2}, \\ \frac{1}{\rho_4(t)^2} \|\mathcal{N}\bar{u}\|_{[L^2(\mathcal{F})]^2} &\leq C \left(1 + \left\| \frac{\bar{h}'}{\rho_4} \right\|_{[L^2(0,T)]^2} + \left\| \frac{\bar{\theta}'}{\rho_4} \right\|_{[L^2(0,T)]^2} \right) \left\| \frac{\bar{u}}{\rho_4(t)} \right\|_{[H^1(\mathcal{F})]^2} \left\| \frac{\bar{u}}{\rho_4(t)} \right\|_{[L^2(\mathcal{F})]^2}, \\ \frac{1}{\rho_4(t)^2} \|\mathcal{M}\bar{u}\|_{[L^2(\mathcal{F})]^2} &\leq C \left(\left\| \frac{\bar{h}'}{\rho_4} \right\|_{[L^2(0,T)]^2} + \left\| \frac{\bar{\theta}'}{\rho_4} \right\|_{[L^2(0,T)]^2} \right) \left\| \frac{\bar{u}}{\rho_4(t)} \right\|_{[H^1(\mathcal{F})]^2}, \\ \frac{1}{\rho_4(t)^2} \|(\mathcal{G} - \nabla)\pi\|_{[L^2(\mathcal{F})]^2} &\leq C \left(\left\| \frac{\bar{h}'}{\rho_4} \right\|_{[L^\infty(0,T)]^2} + \left\| \frac{\bar{\theta}'}{\rho_4} \right\|_{[L^2(0,T)]^2} \right) \left\| \frac{\pi}{\rho_4(t)} \right\|_{[H^1(\mathcal{F})]^2}. \end{aligned}$$

We have also

Lemma 7.2. *Let X and Y satisfying the properties given in Section 3. We obtain for all $(\bar{u}, \pi) \in [H^2(\mathcal{F})]^2 \times H^1(\mathcal{F})$ the following estimates, for all $t \in [0, T]$*

$$\begin{aligned} \frac{1}{\rho_4(t)^2} \|(\mathcal{L}^{(1)} - \mathcal{L}^{(2)})\bar{u}\|_{[L^2(\mathcal{F})]^2} &\leq C \left(\left\| \frac{\bar{h}'^{(1)} - \bar{h}'^{(2)}}{\rho_4} \right\|_{[L^\infty(0,T)]^2} + \left\| \frac{\bar{\theta}'^{(1)} - \bar{\theta}'^{(2)}}{\rho_4} \right\|_{[L^2(0,T)]^2} \right) \left\| \frac{\bar{u}}{\rho_4} \right\|_{[H^2(\mathcal{F})]^2}, \\ \frac{1}{\rho_4(t)^2} \|(\mathcal{N}^{(1)} - \mathcal{N}^{(2)})\bar{u}\|_{[L^2(\mathcal{F})]^2} &\leq C \left(\left\| \frac{\bar{h}'^{(1)} - \bar{h}'^{(2)}}{\rho_4} \right\|_{[L^2(0,T)]^2} + \left\| \frac{\bar{\theta}'^{(1)} - \bar{\theta}'^{(2)}}{\rho_4} \right\|_{[L^2(0,T)]^2} \right) \left\| \frac{\bar{u}}{\rho_4(t)} \right\|_{[H^1(\mathcal{F})]^2} \left\| \frac{\bar{u}}{\rho_4(t)} \right\|_{[L^2(\mathcal{F})]^2}, \\ \frac{1}{\rho_4(t)^2} \|(\mathcal{M}^{(1)} - \mathcal{M}^{(2)})\bar{u}\|_{[L^2(\mathcal{F})]^2} &\leq C \left(\left\| \frac{\bar{h}'^{(1)} - \bar{h}'^{(2)}}{\rho_4} \right\|_{[L^2(0,T)]^2} + \left\| \frac{\bar{\theta}'^{(1)} - \bar{\theta}'^{(2)}}{\rho_4} \right\|_{[L^2(0,T)]^2} \right) \left\| \frac{\bar{u}}{\rho_4(t)} \right\|_{[H^1(\mathcal{F})]^2}, \\ \frac{1}{\rho_4(t)^2} \|(\mathcal{G}^{(1)} - \mathcal{G}^{(2)})\pi\|_{[L^2(\mathcal{F})]^2} &\leq C \left(\left\| \frac{\bar{h}'^{(1)} - \bar{h}'^{(2)}}{\rho_4} \right\|_{[L^\infty(0,T)]^2} + \left\| \frac{\bar{\theta}'^{(1)} - \bar{\theta}'^{(2)}}{\rho_4} \right\|_{[L^2(0,T)]^2} \right) \left\| \frac{\pi}{\rho_4(t)} \right\|_{[H^1(\mathcal{F})]^2}. \end{aligned}$$

Now, we are in position to prove Theorem 1.1.

Proof of Theorem 1.1. For all $r > 0$, let us set

$$\mathcal{K}_r = \left\{ F \in \mathcal{H} \mid \left\| \frac{F}{\rho_1} \right\|_{L^2(0,T;\mathbb{H})} \leq r \right\}.$$

Let $F \in \mathcal{K}_r$, and assume that

$$\|u^0\|_{[H^1(\mathcal{F})]^2} + |\tilde{\ell}^0| + |h^0| + |\omega^0| + |\theta^0| \leq r. \quad (7.1)$$

From Proposition 6.1, the solution $(\bar{u}, \pi, \bar{h}, \bar{\theta})$ of the linear system (6.1), (6.2), (6.4) with $v^* = E_T(Z^0, a^0, F)$ satisfies $\bar{h}(T) = 0$, $\bar{\theta}(T) = 0$ and

$$\begin{aligned} \frac{\bar{u}}{\rho_4} &\in L^2(0, T; [H^2(\mathcal{F})]^2) \cap H^1(0, T; [L^2(\mathcal{F})]^2), & \frac{\nabla \pi}{\rho_4} &\in L^2(0, T; L^2(\mathcal{F})), \\ \frac{\bar{h}'}{\rho_4} &\in [H^1(0, T)]^2, & \frac{\bar{\theta}'}{\rho_4} &\in H^1(0, T). \end{aligned}$$

Using (7.1) and (6.10), we get

$$\left\| \frac{\bar{u}}{\rho_4} \right\|_{L^2(0,T;\mathcal{D}(A)) \cap H^1(0,T;\mathbb{H}) \cap C(0,T;(\mathcal{D}(-A))^{1/2})} + \left\| \frac{\nabla \pi}{\rho_4} \right\|_{[L^2(\mathcal{F})]^2} \leq Cr. \quad (7.2)$$

Using the condition (7.2), we can construct the change of variables defined in Section 3.

We can thus, define the mapping $\Phi : \mathcal{K}_r \rightarrow \mathcal{K}_r$, that associates $F \in \mathcal{K}_r$, we set

$$\Phi(F) = \begin{cases} \nu(\mathcal{L} - \Delta)\bar{u} - \mathcal{M}\bar{u} - \mathcal{N}\bar{u} + (\nabla - \mathcal{G})\pi & \text{in } \mathcal{F}, \\ -mk_{\bar{u}}\ell_{\bar{u}}^{\perp} & \text{in } \mathcal{S}, \end{cases}$$

where $(\bar{u}, \pi, \bar{h}, \bar{\theta})$ is the solution of the linear system (6.1), (6.2) and (6.3). Combining Lemma 7.1, the estimate (7.2) and

$$\frac{(\rho_4)^2}{\rho_1} \in L^\infty(0, T),$$

we obtain

$$\left\| \frac{\Phi(F)}{\rho_1} \right\|_{L^2(0,T;[L^2(\mathcal{F})]^2)} \leq C(1+r)^{d+1}r^2.$$

Then, for r small enough, we get $\Phi(\mathcal{K}_r) \subset \mathcal{K}_r$. Similarly, using Lemma 7.2, we get that

$$\left\| \frac{\Phi(F^1) - \Phi(F^2)}{\rho_1} \right\|_{L^2(0,T;[L^2(\mathcal{F})]^2)} \leq C(1+r)^{d+1}r \left\| \frac{F^1 - F^2}{\rho_1} \right\|_{L^2(0,T;[L^2(\mathcal{F})]^2)}.$$

Thus for r small enough, we obtain that $\Phi|_{\mathcal{K}_r}$ is a contraction. Then Φ admits a fixed point associated to (u, p, h, θ) , the unique solution of (3.4), (3.5), (3.6), (3.7), which ends the proof of Theorem 1.1. \square

References

- [1] Chérif Amrouche and Nour El Houda Seloula. L^p -theory for vector potentials and Sobolev's inequalities for vector fields. *C. R. Math. Acad. Sci. Paris*, 349(9-10):529–534, 2011.
- [2] M. Boulakia and S. Guerrero. Local null controllability of a fluid-solid interaction problem in dimension 3. *J. Eur. Math. Soc. (JEMS)*, 15(3):825–856, 2013.

- [3] Muriel Boulakia and Axel Osses. Local null controllability of a two-dimensional fluid-structure interaction problem. *ESAIM Control Optim. Calc. Var.*, 14(1):1–42, 2008.
- [4] Dongho Chae, O. Yu. Imanuvilov, and Sang Moon Kim. Exact controllability for semilinear parabolic equations with Neumann boundary conditions. *J. Dynam. Control Systems*, 2(4):449–483, 1996.
- [5] Carlos Conca, Jorge San Martín H., and Marius Tucsnak. Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid. *Comm. Partial Differential Equations*, 25(5-6):1019–1042, 2000.
- [6] Jean-Michel Coron and Sergio Guerrero. Null controllability of the N -dimensional Stokes system with $N - 1$ scalar controls. *J. Differential Equations*, 246(7):2908–2921, 2009.
- [7] B. Desjardins and M. J. Esteban. Existence of weak solutions for the motion of rigid bodies in a viscous fluid. *Arch. Ration. Mech. Anal.*, 146(1):59–71, 1999.
- [8] B. Desjardins and M. J. Esteban. On weak solutions for fluid-rigid structure interaction: compressible and incompressible models. *Comm. Partial Differential Equations*, 25(7-8):1399–1413, 2000.
- [9] Caroline Fabre and Gilles Lebeau. Prolongement unique des solutions de l'équation de Stokes. *Comm. Partial Differential Equations*, 21(3-4):573–596, 1996.
- [10] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov, and J.-P. Puel. Local exact controllability of the Navier-Stokes system. *J. Math. Pures Appl. (9)*, 83(12):1501–1542, 2004.
- [11] Enrique Fernández-Cara and Sergio Guerrero. Global Carleman inequalities for parabolic systems and applications to controllability. *SIAM J. Control Optim.*, 45(4):1399–1446, 2006.
- [12] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations*. Springer Monographs in Mathematics. Springer, New York, second edition, 2011. Steady-state problems.
- [13] David Gérard-Varet and Matthieu Hillairet. Existence of weak solutions up to collision for viscous fluid-solid systems with slip. *Comm. Pure Appl. Math.*, 67(12):2022–2075, 2014.
- [14] David Gérard-Varet, Matthieu Hillairet, and Chao Wang. The influence of boundary conditions on the contact problem in a 3D Navier-Stokes flow. *J. Math. Pures Appl. (9)*, 103(1):1–38, 2015.
- [15] Céline Grandmont and Yvon Maday. Existence for an unsteady fluid-structure interaction problem. *M2AN Math. Model. Numer. Anal.*, 34(3):609–636, 2000.
- [16] Gerd Grubb and V. A. Solonnikov. Reduction of basic initial-boundary value problems for the Stokes equation to initial-boundary value problems for parabolic systems of pseudodifferential equations. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 163(Kraev. Zadachi Mat. Fiz. i Smezhn. Vopr. Teor. Funktsii 19):37–48, 187, 1987.
- [17] Sergio Guerrero. Local exact controllability to the trajectories of the Navier-Stokes system with nonlinear Navier-slip boundary conditions. *ESAIM Control Optim. Calc. Var.*, 12(3):484–544, 2006.
- [18] Sergio Guerrero and Cristhian Montoya. Local null controllability of the N -dimensional Navier-Stokes system with nonlinear Navier-slip boundary conditions and $N - 1$ scalar controls. *J. Math. Pures Appl. (9)*, 113:37–69, 2018.
- [19] Max D. Gunzburger, Hyung-Chun Lee, and Gregory A. Seregin. Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions. *J. Math. Fluid Mech.*, 2(3):219–266, 2000.
- [20] Oleg Imanuvilov and Takéo Takahashi. Exact controllability of a fluid-rigid body system. *J. Math. Pures Appl. (9)*, 87(4):408–437, 2007.

- [21] Oleg Yu. Imanuvilov. Remarks on exact controllability for the Navier-Stokes equations. *ESAIM Control Optim. Calc. Var.*, 6:39–72, 2001.
- [22] Arnab Roy and Takéo Takahashi. Local null controllability of a rigid body moving into a boussinesq flow. working paper or preprint, August 2017.
- [23] Jorge Alonso San Martín, Victor Starovoitov, and Marius Tucsnak. Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid. *Arch. Ration. Mech. Anal.*, 161(2):113–147, 2002.
- [24] Takéo Takahashi. Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain. *Adv. Differential Equations*, 8(12):1499–1532, 2003.
- [25] Takéo Takahashi and Marius Tucsnak. Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid. *J. Math. Fluid Mech.*, 6(1):53–77, 2004.
- [26] Chao Wang. Strong solutions for the fluid-solid systems in a 2-D domain. *Asymptot. Anal.*, 89(3-4):263–306, 2014.