Degree-based goodness-of-fit tests for heterogeneous random graph models : independent and exchangeable cases

Sarah Ouadah, Stéphane Robin, Pierre Latouche

Degree-based goodness-of-fit tests for heterogeneous random graph models : independent and exchangeable cases

Sarah Ouadah[1,*], Stéphane Robin[1], Pierre Latouche[2]

([1]) UMR MIA-Paris, INRA, AgroParisTech, Université Paris-Saclay, 75005 Paris, France

([2]) Université de Paris, MAP5, CNRS, 75006 Paris, France

July 29, 2019

Abstract

The degrees are a classical and relevant way to study the topology of a network. They can be used to assess the goodness-of-fit for a given random graph model. In this paper we introduce goodness-of-fit tests for two classes of models. First, we consider the case of independent graph models such as the heterogeneous Erdös-Rényi model in which the edges have different connection probabilities. Second, we consider a generic model for exchangeable random graphs called the W-graph. The stochastic block model and the expected degree distribution model fall within this framework. We prove the asymptotic normality of the degree mean square under these independent and exchangeable models and derive formal tests. We study the power of the proposed tests and we prove the asymptotic normality under specific sparsity regimes. The tests are illustrated on real networks from social sciences and ecology, and their performances are assessed via a simulation study.

Keywords: random graphs; graphon; goodness-of-fit; degree variance; W-graph.

1 Introduction

Interaction networks are used in many fields such as biology, sociology, ecology, economics or energy to describe the interactions existing between a set of individuals or entities. Formally, an interaction network can be viewed as a graph, the nodes of which being the individuals, and an edge between two nodes being present if these two individuals interact. Characterizing the general organization of such a network, namely its topology, can help in understanding the behavior of the system as a whole.

In the last decades, the distribution of the degrees (i.e. the number of connections of each node) has appeared as a simple and relevant way to study the topology of a network,

see Snijders (1981) and Barabási and Albert (1999). The degree distribution can also be used to infer complex graph models such as in Bickel et al. (2011). From a more descriptive view-point, a very imbalanced distribution may reveal a network whose edges highly concentrate around few nodes, whereas a multi-modal distribution may reveal the existence of clusters of nodes as observed by Channarond et al. (2012). However, in practice, assessing the significance of such patterns remains an open problem.
The variance of the degrees has been considered since the earliest statistical studies of networks, for instance by Snijders (1981). The first idea was simply to compare its empirical value to the expected one under a null random graph model, typically the Erdös-Rényi (ER) model introduced by Erdös and Rényi (1959), where each degree has a binomial distribution. Because the ER model is rarely a reasonable model to be tested, we define a generalized version of the degree variance statistic, which we name the degree mean square statistic. This statistic generalizes the degree variance in the sense that it measures the discrepancy between the observed degrees and their expected values under several heterogeneous models we define hereafter.
For a given random graph under a specific model M_0, the degree mean square statistic is defined by

$$W_{\theta_0} = \frac{1}{n}\sum_i (D_i - \mathbb{E}_{\theta_0} D_i)^2,$$

where D_i stands for the degree of node i and $\mathbb{E}_{\theta_0} D_i$ for its expected value under a given model with parameter θ_0. More specifically, in the following we will consider independent models parametrized with a probability matrix $\mathbf{p}$ and exchangeable models parametrized with a function Φ. Although these models not only differ in terms of parameter, for the sake of clarity, the corresponding quantities will be simply indexed with $\mathbf{p}$ and Φ, respectively. We propose goodness-of-fit tests for several random graph models, by showing the asymptotic normality of this statistic W_{θ_0} under null hypothesis and their alternatives. In addition, because large networks are often sparse, we study under which sparsity regime the asymptotic distributions derived before still hold.

The notations and the main models considered are the following. We consider an undirected graph $\mathcal{G} = (\{1, \dots n\}, \mathcal{E})$ with no self loop, that is the connection of a node to itself, and denote Y the corresponding $n \times n$ adjacency matrix. Thus, the entry Y_{ij} of Y is 1 if $(i, j) \in \mathcal{E}$, and 0 otherwise. Because $\mathcal{G}$ is undirected with no self loop, we have $Y_{ij} = Y_{ji}, \forall i \neq j$ and $Y_{ii} = 0$, for all i's. We further define D_i the degree of node i by $D_i = \sum_{j \neq i} Y_{ij}$. In terms of random graph models, we consider two cases: the independent case and the exchangeable one. In the independent case, $ER(p)$ refers to the Erdös-Rényi model, according to which all edges (Y_{ij}) are independent Bernoulli variables with same probability p to exist. $HER(\mathbf{p})$ stands for the heterogeneous Erdös-Rényi model where edges are independent with respective probability p_{ij} to exist. The $n \times n$ matrix $\mathbf{p}$ has entries p_{ij}, it is symmetric with null diagonal. In the exchangeable case, we consider a generic model for exchangeable random graphs called the W-graph introduced in Lovász and Szegedy (2006) and Diaconis and Janson (2008). It is based on a *graphon* function $\Phi : [0, 1]^2 \mapsto [0, 1]$ and denoted by $EG(\Phi)$. An unobserved coordinate $U_i \sim \mathcal{U}[0, 1]$ is associated with each node $i (1 \leq i \leq n)$ and edges are drawn independently conditional

the U_i's as $Y_{ij}|U_i, U_j \sim \mathcal{B}[\Phi(U_i, U_j)]$. The stochastic block model (SBM, introduced by Holland and Leinhardt (1979) and further studied by Nowicki and Snijders (2001), and the expected degree distribution (EDD) model, defined by Chung and Lu (2002), fall within this framework.

Goodness-of-fit tests of the models we consider have received little attention until recently. Cerqueira et al. (2017) propose a goodness-of-fit test for the $HER(\mathbf{p})$ model andMaugis et al. (2017) for the $EG(\Phi)$ model, both when independent and identically distributed (i.i.d.) copies of the graph are available. Lei (2016) and Bickel and Sarkar (2016) derived goodness-of-fit tests for the number of communities in stochastic block models by showing the asymptotic behavior of the largest singular value of a residual adjacency matrix. Their respective null models are $ER(p)$ in Bickel and Sarkar (2016) and an SBM with K communities in Lei (2016). Yang et al. (2014) proposed a test statistic for the goodness-of-fit of a given graphon function and used a Monte-Carlo sampling to approximate its null distribution. More recently, Gao and Lafferty (2017b) proved the asymptotic normality of subgraph counts to test the $ER(p)$ model against an SBM.

The paper is organized as follows. Section 2 is devoted to independent graph models and Section 3 to the the exchangeable ones. The performances of the proposed tests are assessed via a simulation study in Section 4. More specifically, the asymptotic distribution of the degree mean square statistic under models $HER(\mathbf{p})$ and $EG(\Phi)$ is derived Sections 2.1 and 3.1, respectively. The asymptotic normality under some specific sparsity regimes is studied in Sections 2.3 and 3.3. In Section 2.2, we establish a test for the null hypothesis stating that $\mathcal{G}$ arises from $HER(\mathbf{p}^0)$ and give its power. The last part of this section is devoted to the illustration of the HER goodness-of-fit test on some examples. In the same manner, Section 3.2 deals with the EG model and its extensions, meaning the SBM and EDD model.

2 Independent random graph models

We consider the heterogeneous Erdös-Rényi model $HER(\mathbf{p})$, in which the edges are independent and have different respective probabilities to exist: $Y_{ij} \sim \mathcal{B}[p_{ij}]$.

The asymptotic framework is the following.

Assumption 1 *In the non-sparse setting, we consider an infinite matrix $\mathbf{P}$, the elements of which are all in the interval $[c, 1-c]$ for some arbitrarily small constant $c \in (0, 1/2)$. For the HER model, then we build a sequence of matrices $\mathbf{p}^n$ made of the first n rows and columns of $\mathbf{P}$. Finally, we consider a sequence of independent graphs $\mathcal{G}^n = (\{1, \dots n\}, \mathcal{E}^n)$, with increasing size n and respective probability matrices $\mathbf{p}^n$.*

All quantities computed on $\mathcal{G}^n$ should therefore be indexed by n as well. But for the sake of clarity, we will drop the index n in $\mathbf{p}^n$ in the rest of the paper.

2.1 Asymptotic normality

We consider a goodness-of-fit test for the $HER(\mathbf{p}^0)$ model. For a given random graph with a matrix $\mathbf{p}^0$ of connection probabilities, we consider the following degree mean square statistic:

$$W_{\mathbf{p}^0} = \frac{1}{n}\sum_i (D_i - \mu_i^0)^2,$$

where $D_i = \sum_{j\neq i} Y_{ij}$ and μ_i^0 stand for the expected degree of node i under $HER(\mathbf{p}^0)$, namely $\mu_i^0 = \sum_{j\neq i} p_{ij}^0$.

We establish the asymptotic normality of $W_{\mathbf{p}^0}$ under model $HER(\mathbf{p})$. The proof relies on projections of $W_{\mathbf{p}^0}$ on suitable spaces and the Lindeberg-Lévy Theorem (see e.g. Theorem 7.2, p.42 in Billingsley (1968)) which is recalled below. We derive all projections involved in the Hoeffding decomposition (see, e.g., Chapter 11 in van der Vaart (1998)) to easily calculate the moments of $W_{\mathbf{p}^0}$. As for the asymptotic normality, we decompose $W_{\mathbf{p}^0}$ into the sum of its Hájek projection (see, e.g., Chapter 11 in van der Vaart (1998)) to which we apply the Lindeberg-Lévy Theorem, and a negligible term. A similar strategy has already been used for graph studies, for instance in Bloznelis (2005) to prove the asymptotic normality of the variance degree under model $ER(p)$ and in Nowicki and Wierman (1988) to prove the one of subgraph counts in random graphs.

Theorem 1 (Lindeberg-Lévy Theorem in Billingsley (1968)) *Let $(X_{nu})_{1\leq u\leq k_n}$ be a triangular array of independent random variables with means 0 and finite variances $(\sigma_{nu}^2)_{1\leq u\leq k_n}$. Let $B_n^2 = \sum_{u=1}^{k_n}\sigma_{nu}^2$. If the Lindeberg condition*

$$A_n^2(\epsilon)/B_n^2 \to 0, \quad \text{as } n\to\infty, \quad \text{for each } \epsilon>0, \quad \text{where} \quad A_n^2(\epsilon) = \sum_{u=1}^{k_n}\int_{\{|x_{nu}|>\epsilon B_n\}} x_{nu}^2 dP \quad (1)$$

is satisfied then

$$\frac{1}{B_n}\sum_{u=1}^{k_n} X_{nu} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

Remark 1 *Let consider the case of binary random variables X_{nu} with mean 0. More specifically, set $X_{nu} = a_{nu}Z_{nu}$, $a_{nu}\in\mathbb{R}$, where Z_{nu} are centered Bernoulli variables, that is to say Z_{nu} takes value $1-p_{nu}$ with probability p_{nu} and value $-p_{nu}$ with probability $1-p_{nu}$. Because $|X_{nu}|\leq a_{nu}$, the realization of the event $|X_{nu}|\geq \epsilon B_n$ in the definition of $A_n^2(\epsilon)$ in (1) is controlled by $|a_{nu}|\geq \epsilon B_n$. Therefore, all X_{nu} for which $|a_{nu}|<\epsilon B_n$ do not contribute to $A_n^2(\epsilon)$. If this holds for all X_{nu}, then the Lindeberg condition is directly satisfied. If not, only the X_{nu} for which it does not hold have to be considered in the calculation of $A_n^2(\epsilon)$ and, because $|Z_{nu}|\leq 1$, their contribution is upper-bounded by their variance $\sigma_{nu}^2 = a_{nu}^2 p_{nu}(1-p_{nu})$. In the forthcoming theorems proofs, we will verify the Lindeberg condition using this observation.*

Theorem 2 *Under model $HER(\mathbf{p})$ and Assumption 1, the statistic $W_{\mathbf{p}^0}$ is asymptotically normal:*

$$(W_{\mathbf{p}^0} - \mathbb{E}_{\mathbf{p}} W_{\mathbf{p}^0})/\mathbb{S}_{\mathbf{p}} W_{\mathbf{p}^0} \xrightarrow{D} \mathcal{N}(0,1),$$

where $\mathbb{S}$ denotes the standard deviation and

$$\mathbb{E}_{\mathbf{p}} W_{\mathbf{p}^0} = \frac{2}{n}\left(\sum_{1\le i<j\le n} (\sigma_{ij}^2 + \delta_{ij}^2) + \sum_{1\le i<j<k\le n} (\delta_{ij}\delta_{ik} + \delta_{ij}\delta_{jk} + \delta_{ik}\delta_{jk}) \right), \tag{2}$$

where $\sigma_{ij}^2 = p_{ij}(1-p_{ij})$ and $\delta_{ij} = p_{ij} - p_{ij}^0$. Moreover

$$\begin{aligned} \mathbb{V}_{\mathbf{p}} W_{\mathbf{p}^0} &= \frac{4}{n^2}\Big(\sum_{1\le i<j\le n} \sigma_{ij}^2 (1 - 2p_{ij} + \Delta_i + \Delta_j)^2 \\ &\quad + \sum_{1\le i<j<k\le n} \left(\sigma_{ij}^2\sigma_{ik}^2 + \sigma_{ij}^2\sigma_{jk}^2 + \sigma_{ik}^2\sigma_{jk}^2\right) \Big), \end{aligned} \tag{3}$$

with $\Delta_i = \sum_{j\neq i} \delta_{ij}$.

Proof. Let us begin with the calculation of $W_{\mathbf{p}^0}$ moments. We first observe that,

$$\begin{aligned} nW_{\mathbf{p}^0} &= \sum_i (D_i - \mu_i + \mu_i - \mu_i^0)^2 = \sum_i \left(\sum_{j\neq i} \widetilde{Y}_{ij} + \delta_{ij} \right)^2 \\ &= 2 \sum_{1\le i<j\le n} (\widetilde{Y}_{ij} + \delta_{ij})^2 \\ &\quad + 2 \sum_{1\le i<j<k\le n} (\widetilde{Y}_{ij} + \delta_{ij})(\widetilde{Y}_{ik} + \delta_{ik}) + (\widetilde{Y}_{ij} + \delta_{ij})(\widetilde{Y}_{jk} + \delta_{jk}) + (\widetilde{Y}_{ik} + \delta_{ik})(\widetilde{Y}_{jk} + \delta_{jk}), \end{aligned}$$

where $\widetilde{Y}_{ij} = Y_{ij} - p_{ij}$ and $\mu_i = \sum_{j\neq i} p_{ij}$. Then, we write the Hoeffding decomposition of $W_{\mathbf{p}^0}$:

$$W_{\mathbf{p}^0} = P_\emptyset W_{\mathbf{p}^0} + \sum_{1\le i<j\le n} P_{\{ij\}} W_{\mathbf{p}^0} + \sum_{1\le i<j<k\le n} \left(P_{\{ij,ik\}} W_{\mathbf{p}^0} + P_{\{ij,jk\}} W_{\mathbf{p}^0} + P_{\{ik,kj\}} W_{\mathbf{p}^0} \right), \tag{4}$$

where

$$\begin{aligned} P_\emptyset W_{\mathbf{p}^0} &= \mathbb{E}_{\mathbf{p}} W_{\mathbf{p}^0}, \\ P_{\{ij\}} W_{\mathbf{p}^0} &= \mathbb{E}_{\mathbf{p}}(W_{\mathbf{p}^0} | Y_{ij}) - \mathbb{E}_{\mathbf{p}} W_{\mathbf{p}^0}, \\ P_{\{ij,ik\}} W_{\mathbf{p}^0} &= \mathbb{E}_{\mathbf{p}}(W_{\mathbf{p}^0} | Y_{ij}, Y_{ik}) - \mathbb{E}_{\mathbf{p}}(W_{\mathbf{p}^0} | Y_{ij}) - \mathbb{E}_{\mathbf{p}}(W_{\mathbf{p}^0} | Y_{ik}) + \mathbb{E}_{\mathbf{p}} W_{\mathbf{p}^0}. \end{aligned}$$

Combining the definitions above with the expression (4) of $W_{\mathbf{p}^0}$, we obtain that,

$$\begin{aligned} P_\emptyset W_{\mathbf{p}^0} &= \frac{2}{n} \sum_{1\le i<j\le n} (\sigma_{ij}^2 + \delta_{ij}^2) + \frac{2}{n} \sum_{1\le i<j<k\le n} \delta_{ij}\delta_{ik} + \delta_{ij}\delta_{jk} + \delta_{ik}\delta_{jk}, \\ P_{\{ij\}} W_{\mathbf{p}^0} &= \frac{2}{n} \widetilde{Y}_{ij} \left(1 + \Delta_i + \Delta_j\right) - \sigma_{ij}^2 = \frac{2}{n} \widetilde{Y}_{ij} \left(1 - 2p_{ij} + (\Delta_i + \Delta_j)\right), \qquad (5) \\ P_{\{ij,ik\}} W_{\mathbf{p}^0} &= \frac{2}{n} \widetilde{Y}_{ij} \widetilde{Y}_{ik}. \qquad (6) \end{aligned}$$

Observe now that,

$$n\mathbb{E}_{\mathbf{p}}W_{\mathbf{p}^0} = 2\sum_{1\le i<j\le n}(\sigma_{ij}^2+\delta_{ij}^2)+2\sum_{1\le i<j<k\le n}\delta_{ij}\delta_{ik}+\delta_{ij}\delta_{jk}+\delta_{ik}\delta_{jk}.$$

Because the $\widetilde{Y}_{ij}$ are independent with zero mean, the projections are all orthogonal with each other, which gives

$$\begin{aligned} n^2\mathbb{V}_{\mathbf{p}}W_{\mathbf{p}^0} &= n^2\sum_{1\le i<j\le n}\mathbb{V}_{\mathbf{p}}(P_{\{ij\}}W_{\mathbf{p}^0}) \\ &\quad +n^2\sum_{1\le i<j<k\le n}\left(\mathbb{V}_{\mathbf{p}}(P_{\{ij,ik\}}W_{\mathbf{p}^0})+\mathbb{V}_{\mathbf{p}}(P_{\{ij,jk\}}W_{\mathbf{p}^0})+\mathbb{V}_{\mathbf{p}}(P_{\{ik,jk\}}W_{\mathbf{p}^0})\right) \\ &= 4\sum_{1\le i<j\le n}\sigma_{ij}^2(1-2p_{ij}+\Delta_i+\Delta_j)^2+4\sum_{1\le i<j<k\le n}\left(\sigma_{ij}^2\sigma_{ik}^2+\sigma_{ij}^2\sigma_{jk}^2+\sigma_{ik}^2\sigma_{jk}^2\right). \end{aligned}$$

We now turn to the asymptotic normality of $W_{\mathbf{p}^0}$. Let decompose $W_{\mathbf{p}^0}$ as follows.

$$W_{\mathbf{p}^0}-\mathbb{E}_{\mathbf{p}}W_{\mathbf{p}^0} = W^*_{\mathbf{p}^0}-\mathbb{E}_{\mathbf{p}}W_{\mathbf{p}^0}+W_{\mathbf{p}^0}-W^*_{\mathbf{p}^0},$$

where $W^*_{\mathbf{p}^0} = P_\emptyset W_{\mathbf{p}^0}+\sum_{1\le i<j\le n}P_{\{ij\}}W_{\mathbf{p}^0}$ is the Hájek projection of $W_{\mathbf{p}^0}$, which corresponds to the first two terms of the Hoeffding's decomposition. We will show that $W^*_{\mathbf{p}^0}-\mathbb{E}_{\mathbf{p}}W_{\mathbf{p}^0}$ is asymptotically normal and that $W_{\mathbf{p}^0}-W^*_{\mathbf{p}^0}$ is a negligible term.
Let consider $W^*_{\mathbf{p}^0}-\mathbb{E}_{\mathbf{p}}W_{\mathbf{p}^0}=\sum_{1\le i<j\le n}P_{\{ij\}}W_{\mathbf{p}^0}$ and apply Theorem 1 to the projections $P_{\{ij\}}W_{\mathbf{p}^0}$ which stand for the X_{nu}. We first observe that these projections are each proportional to the $\widetilde{Y}_{ij}$ which are all independent centered Bernoulli variables. We may now use Remark 1. We denote the a_{nu} by $a_{n\{ij\}}$, the explicit expression of which is given in (5). We observe that, under Assumption 1, $a_{n\{ij\}}=\Theta(1)$ and $B_n^2=\mathbb{V}_{\mathbf{p}}\left(W^*_{\mathbf{p}^0}-\mathbb{E}_{\mathbf{p}}W_{\mathbf{p}^0}\right)=\Theta(n^2)$. It implies that the Lindeberg condition is fulfilled because, for any ϵ, each a_{nu} becomes smaller than ϵB_n when n goes to infinity. Now by considering (4) the Hoeffding decomposition of $W_{\mathbf{p}^0}$, we see that

$$W_{\mathbf{p}^0}-W^*_{\mathbf{p}^0}=\sum_{1\le i<j<k\le n}\left(P_{\{ij,ik\}}W_{\mathbf{p}^0}+P_{\{ij,jk\}}W_{\mathbf{p}^0}+P_{\{ik,kj\}}W_{\mathbf{p}^0}\right).$$

Then we observe that $a_{n\{ij,ik\}}$ given in (6) is $\Theta(n^{-1})$ and therefore that $\mathbb{V}_{\mathbf{p}}\left(W_{\mathbf{p}^0}-W^*_{\mathbf{p}^0}\right)=\Theta(n)$. We conclude to the asymptotic normality of $W_{\mathbf{p}^0}$ by combining the one of $W^*_{\mathbf{p}^0}-\mathbb{E}\mathbf{p}W_{\mathbf{p}^0}$ and the fact that $\mathbb{V}_{\mathbf{p}}\left(W_{\mathbf{p}^0}-W^*_{\mathbf{p}^0}\right)/\mathbb{V}_{\mathbf{p}}W^*_{\mathbf{p}^0}\to 0$ as $n\to\infty$. ■

Plug-in version of the test. In many situations, $\mathbf{p}^0$ is actually unknown and one needs to resort to an estimate $\widehat{\mathbf{p}}^0$. There is no hope to get a precise estimate when n increases if no restriction is imposed to $\mathbf{p}^0$. When a vector of covariates $x_{ij}\in\mathbb{R}^d$ is available for

each pair of nodes, one natural way to impose such a restriction is to assume that p_{ij}^0 has a logistic form, that is $\text{logit}(p_{ij}^0) = x_{ij}^\intercal \beta$, where β is the vector of regression coefficients and $\text{logit}(u) = \log(u/(1-u))$. A plug-in version of the proposed test can be obtained by fitting the logistic model to the observed edges to get an estimate $\widehat{\beta}$, which provides us with $\widehat{\mathbf{p}}^0$, which in turn provides us with a plug-in version $W_{\widehat{\mathbf{p}}^0}$ of the test statistic.
The simulation study presented in Section 4 shows that $W_{\widehat{\mathbf{p}}^0}$ behaves well for large graphs. A possible strategy to understand the asymptotic behavior of $W_{\widehat{\mathbf{p}}^0}$ would be to control the difference between $W_{\mathbf{p}^0}$ and $W_{\widehat{\mathbf{p}}^0}$. Indeed, denoting $\widehat{\mu}_i^0 = \sum_{j \neq i} \widehat{p}_{ij}^0$ and $\Delta_i = \widehat{\mu}_i^0 - \mu_i^0$, $W_{\widehat{\mathbf{p}}^0}$ can be decomposed as

$$W_{\widehat{\mathbf{p}}^0} := \frac{1}{n} \sum_i \left(D_i - \widehat{\mu}_i^0\right)^2 = W_{\mathbf{p}^0} - \frac{2}{n} \sum_i \left(D_i - \mu_i^0\right) \Delta_i + \frac{1}{n} \sum_i \Delta_i^2. \tag{7}$$

If the $\widehat{p}_{ij}^0$ result from a parametric estimation based on the $O(n^2)$ edges, we expect the estimation error $|p_{ij}^0 - \widehat{p}_{ij}^0|$ to be $O_P(n^{-1})$, which makes the last term of (7) negligible. Still, the joint dependence structure of the D_i and Δ_i is quite intricate, which makes the control of the second term of (7) not straightforward. In Section 2.2, we present a specific case where we prove the asymptotically normality of the plug-in version of the test.

Degree variance test

We consider the following statistic which is the empirical degree variance for the test of $H_0 = ER$ versus $H_1 = HER(\mathbf{p})$.

$$V = \frac{1}{n} \sum_i \left(D_i - \overline{D}\right)^2,$$

where $\overline{D} = (1/n) \sum_j D_j$. The variance of the degrees has been naturally considered earlier in statistical studies of networks. Hagberg (2003) derives the exact moments of the degree variance and suggests to use a Gamma distribution as in Hagberg (2000). Snijders (1981) also gives the first two moments of the degree variance, but conditionally to the total number of edges. To our knowledge the first and only proof of the asymptotic normality of the degree variance under the ER model is given in a technical report from Bloznelis (2005). Here, we establish the asymptotic normality of V under model $HER(\mathbf{p})$ and obtain the ER version as a consequence.

Corollary 1 *Under model $HER(\mathbf{p})$ and Assumption 1, the degree variance is asymptotically normal:*

$$\left(V - \mathbb{E}_{\mathbf{p}} V\right) / \mathbb{S}_{\mathbf{p}} V \xrightarrow{D} \mathcal{N}(0,1),$$

with

$$\begin{aligned} \mathbb{E}_{\mathbf{p}} V &= \frac{2(n-2)}{n^2} \sum_{1 \leq i < j \leq n} p_{ij} + \frac{2(n-4)}{n^2} \sum_{1 \leq i < j < k \leq n} \{p_{ij}p_{ik} + p_{ij}p_{jk} + p_{ik}p_{jk}\} \\ &\quad - \frac{8}{n^2} \sum_{1 \leq i < j < k < l \leq n} \{p_{ij}p_{k\ell} + p_{ik}p_{j\ell} + p_{i\ell}p_{jk}\}, \end{aligned}$$

and

$$\begin{aligned}\mathbb{V}_{\mathbf{p}}V &= \frac{1}{4n^4}\sum_{1\le i<j\le n}\sigma_{ij}^2\left(4(n-2)+4(n-4)\sum_{k\notin(i,j)}(p_{i,k}+p_{j,k})-16\sum_{k<l\notin(i,j)}p_{k\ell}\right)^2\\ &+\frac{1}{n^4}\sum_{1\le i<j<k\le n}4(n-4)^2\left\{\sigma_{ij}^2\sigma_{ik}^2+\sigma_{ij}^2\sigma_{jk}^2+\sigma_{ik}^2\sigma_{jk}^2\right\}\\ &+\frac{1}{n^4}\sum_{1\le i<j<k<l\le n}64\left\{\sigma_{ij}^2\sigma_{k\ell}^2+\sigma_{ik}^2\sigma_{j\ell}^2+\sigma_{i\ell}^2\sigma_{jk}^2\right\}.\end{aligned}$$

The proof follows the line of this of Theorem 2 and is given in Appendix A.1.
Note that the asymptotic normality of the degree variance under model $ER(p)$ is a straightforward application of Corollary 1 to the case where all p_{ij} are equal to p. We have,

$$\left(V-\mathbb{E}_pV\right)/\mathbb{S}_pV\xrightarrow{D}\mathcal{N}(0,1),$$

where $\mathbb{E}_pV=n^{-1}(n-1)(n-2)pq$ and $\mathbb{V}_pV=n^{-3}2(n-1)(n-2)^2pq\left(1+(n-6)pq\right)$, as given in Hagberg (2000).

2.2 Test and power

We now study the test of $H_0=HER(\mathbf{p}^0)$ versus $H_1=HER(\mathbf{p})$. The next Corollaries provide the null distribution of the test statistic $W_{\mathbf{p}^0}$ and the power of the associate test.

Corollary 2 *Under model $HER(\mathbf{p}^0)$ and Assumption 1, the statistic $W_{\mathbf{p}^0}$ is asymptotically normal with moments:*

$$\mathbb{E}_{\mathbf{p^0}}W_{\mathbf{p}^0}=\frac{2}{n}\sum_{1\le i<j\le n}{\sigma_{ij}^0}^2, \tag{8}$$

$$\mathbb{V}_{\mathbf{p^0}}W_{\mathbf{p}^0}=\frac{1}{n^2}\left(4\sum_{1\le i<j\le n}{\sigma_{ij}^0}^2(1-2p_{ij}^0)^2+\sum_{1\le i<j<k\le n}\left({\sigma_{ij}^0}^2{\sigma_{ik}^0}^2+{\sigma_{ij}^0}^2{\sigma_{jk}^0}^2+{\sigma_{ik}^0}^2{\sigma_{jk}^0}^2\right)\right), \tag{9}$$

where ${\sigma_{ij}^0}^2=p_{ij}^0(1-p_{ij}^0)$.

This is a direct consequence of Theorem 2 in the special case of the $HER(\mathbf{p}^0)$ model for which all δ_{ij}'s are zero ($\delta_{ij}=p_{ij}-p_{ij}^0$).

A formal test with asymptotic level α can be constructed based on Corollary 2, which rejects H_0 as soon as $W_{\mathbf{p}^0}$ exceeds $\mathbb{E}_{\mathbf{p^0}}W_{\mathbf{p}^0}+t_{1-\alpha}\mathbb{S}_{HER(\mathbf{p}^0)}W_{\mathbf{p}^0}$, where $t_{1-\alpha}$ stands for the $1-\alpha$ quantile of the standard Gaussian distribution. The power of this test is given by the following Corollary.

Corollary 3 *The asymptotic power of the test for $H_0 = HER(\mathbf{p}^0)$ versus $H_1 = HER(\mathbf{p})$ is*

$$\pi(\mathbf{p}) = 1 - \Phi\left(\left(\mathbb{E}_{\mathbf{p}^0} W_{\mathbf{p}^0} + t_{1-\alpha} \mathbb{S}_{\mathbf{p}^0} W_{\mathbf{p}^0} - \mathbb{E}_{\mathbf{p}} W_{\mathbf{p}^0} \right) / \mathbb{S}_{\mathbf{p}} W_{\mathbf{p}^0} \right), \tag{10}$$

where Φ stands for the cumulative distribution function [cdf] of the standard normal distribution and $t_{1-\alpha} = \Phi^{-1}(1-\alpha)$.

The following corollary gives a sufficient condition on the departure between $\mathbf{p}$ and $\mathbf{p}^0$ to ensure that the proposed test is asymptotically powerful.

Corollary 4 *For probability matrices $\mathbf{p}^0$ and $\mathbf{p}$, define*

$$\begin{aligned} \Delta_n(\mathbf{p}^0, \mathbf{p}) &:= \mathbb{E}_{\mathbf{p}} W_{\mathbf{p}^0} - \mathbb{E}_{\mathbf{p}^0} W_{\mathbf{p}^0} \\ &= \frac{2}{n} \left(\sum_{1 \le i < j \le n} \delta_{ij}^2 + \sum_{1 \le i < j < k \le n} (\delta_{ij}\delta_{ik} + \delta_{ij}\delta_{jk} + \delta_{ik}\delta_{jk}) + \sum_{1 \le i < j \le n} (\sigma_{ij}^2 - {\sigma_{ij}^0}^2) \right). \end{aligned}$$

If $\Delta_n(\mathbf{p}^0, \mathbf{p}) = \Theta(n^\alpha)$ is positive and $\alpha > 1/2$, then under Assumption 1, the test $H_0 = HER(\mathbf{p}^0)$ versus $H_1 = HER(\mathbf{p})$ is asymptotically powerful.

Proof. It is sufficient to prove that the argument of the cdf Φ in (10) tends to minus infinity as n increases. From (3) and (9), we have that under Assumption 1, $\mathbb{S}_{\mathbf{p}} W_{\mathbf{p}^0} = \Theta(n^{1/2})$ and $\mathbb{S}_{\mathbf{p}^0} W_{\mathbf{p}^0} = \Theta(n^{1/2})$. As a consequence,when $\Delta_n(\mathbf{p}^0, \mathbf{p}) > 0$ and $\alpha > 1/2$, the negative argument of Φ in (10) goes to infinity at rate $n^{\alpha - 1/2}$, which concludes the proof. ■

Note that, when $\Delta_n(\mathbf{p}^0, \mathbf{p}) < 0$ the same corollary holds for the test which rejects H_0 as soon as $W_{\mathbf{p}^0} < \mathbb{E}_{\mathbf{p}^0} W_{\mathbf{p}^0} + t_\alpha \mathbb{S}_{HER(\mathbf{p}^0)} W_{\mathbf{p}^0}$, where t_α stands for the α quantile of the standard Gaussian distribution.

Degree variance test

We now consider the use of the statistic V for the test of $H_0 = ER$ versus $H_1 = HER(\mathbf{p})$. Because the probability is unknown in practice, we consider the following test statistic using a plug-in version of the moments, namely

$$\left(V - \mathbb{E}_{\widehat{p}} V \right) / \mathbb{S}_{\widehat{p}} V,$$

where $\widehat{p} = [n(n-1)]^{-1} \sum_{i \neq j} Y_{ij}$.
The asymptotic power $\pi(\mathbf{p}) = \mathbb{P}_{\mathbf{p}}\{V > t_\alpha\}$ of the considered test, with nominal level $\alpha > 0$, is

$$\pi(\mathbf{p}) = 1 - \Phi\left(\left(\mathbb{E}_{\bar{p}} V + t_{1-\alpha} \mathbb{S}_{\bar{p}} V - \mathbb{E}_{\mathbf{p}} V \right) / \mathbb{S}_{\mathbf{p}} V \right),$$

where $\overline{p} = [n(n-1)]^{-1} \sum_{i \neq j} p_{ij}$. This results from the asymptotic normality of $(V - \mathbb{E}_{\bar{p}} V)/\mathbb{S}_{\bar{p}} V$ under the $HER(\mathbf{p})$ model. Actually, the asymptotic distribution of the test based on $(V - \mathbb{E}_{\bar{p}} V)/\mathbb{S}_{\bar{p}} V$ is the same as the one of the test based on the statistic $(V - \mathbb{E}_{\widehat{p}} V)/\mathbb{S}_{ER(\widehat{p})} V$ (see Lemma 2 in Appendix A.2), and we have shown that under model ER, $(V - \mathbb{E}_{\widehat{p}} V)/\mathbb{S}_{ER(\widehat{p})} V$ is asymptotically normal (see Lemma 1 in Appendix A.2).

Remark 2 *The $ER(p)$ model corresponds to $HER(\mathbf{p}^0)$ where the matrix $\mathbf{p}^0$ has all entries equal to p. In this case, the test statistic $W_{\mathbf{p}^0}$ can be viewed as the theoretical version of the empirical variance statistic V studied in Section 2.1 as*

$$W_{\mathbf{p}^0} = \frac{1}{n}\sum_i \left(D_i - (n-1)p\right)^2 .$$

Because as $\widehat{p}$ is an average over $\Theta(n^2)$ edges, we have that $(\widehat{p}-p)^2 = \Theta_P(n^{-2})$ so $W_{\mathbf{p}^0} - V = (n-1)^2(\widehat{p}-p)^2 = \Theta_P(1)$. Combined with arguments similar to these of Corollary 1 and Lemma 2, this implies that, under the ER model, the tests based on V and $W_{\mathbf{p}^0}$ are asymptotically equivalent.

Illustration

We illustrate the use of the proposed test on the following series of networks.

Karate network: it describes the friendships between a subset of $n = 34$ members of a karate club at a university in the US, observed from 1970 to 1972 and was originally studied by Zachary (1977). The network is made of four known groups characterized by a node qualitative descriptor.

Ecological networks: this consists in two ecological networks first introduced in Vacher et al. (2008) and further studied in Mariadassou et al. (2010). Each of these networks describe the interaction between a series of $n = 51$ trees and $n = 154$ fungi, respectively. In the tree network, two trees interact if they share at least one common fungal parasite. As for the fungal network, two fungi are linked if they are hosted by at least one common tree species. Three quantitative edge descriptors are available characterizing the genetic, geographic, and taxonomic distances between the tree species.

Political blogs network: this consists in a set of $n = 196$ French political blogs studied in Latouche et al. (2011). Two blogs are connected if one contains an hyperlink to the other.

Each node is associated with a political party from the left wing to the right wing and the status of the writer is also given (political analyst or not).

CKM: this data set was created by Burt (1987) from the data originally collected by Coleman et al. (1966). The network we considered characterizes the friendship relationships among $n = 219$ physicians, each physician being asked to name three friends.

The physicians were also asked to answer to a series of questions regarding their profession, corresponding to node covariates. Note that we imputed the missing values in the data set using the missMDA R package of Josse and Husson (2016).

Faux Dixon High network: this network characterizes the (directed) friendship between $n = 248$ students. It results from a simulation based upon an exponential random graph model fit, see Handcock et al. (2008), to data from one school community from the AdHealth Study, Wave I of Resnick et al. (1997).

Node covariates are provided, namely the grade, sex, and race of each student.

AdHealth 67: this data set is related to the Faux Dixon network described previously. However, it was constructed from the original data of the AdHealth study, and not simulated from any random graph model. The AdHealth study was conducted using in-school questionnaires, from 1994 to 1995. Students were asked to designate their friends and to answer to a series of questions. Results were collected in schools from 84 communities. In our study, we considered a network associated to school community 67 which characterizes the undirected friendship relationships between $n = 530$ students.

Nodes covariates are the same as the one of the Faux Dixon network.

For some networks, only node descriptors x_i and x_j are available and building an *edge* descriptor x_{ij} from *node* descriptors is not straightforward as depicted in Hunter et al. (2008). In these examples, the node descriptors are all qualitative. For each category of each node descriptor, we build binary edge descriptors indicating if both node belong to the same category, or if at least one on the two belong to it. The precise definition of the edge covariates for each dataset is explained in Latouche et al. (2018).

We fist applied the degree variance test to each of these networks to check if their topology is similar to the one of an ER network. As expected, their topology are far too heterogeneous to fit an $ER(p)$ model, and the null hypothesis is rejected for each one of them.

The question is then to know if the available covariates on edges are sufficient to explain the heterogeneity of the network, at least in terms of degrees. To address this question, for each network separately, we fitted a logistic regression model $\text{logit}(p^0_{ij}) = x_{ij}^\intercal \beta$, which provided us with an estimate $\widehat{\mathbf{p}}^0$ of the connection probability matrix $\mathbf{p}^0$. We then applied the degree mean square test to check if the considered covariates are sufficient to explain the heterogeneity of the network.

The results given in Table 1 show the ability of the proposed test to detect a departure from the degrees predicted by the covariates. Indeed, the null hypothesis is rejected for all networks except for the CKM and Karate networks. As for the ecological networks, these results are consistent with these from Mariadassou et al. (2010), who detected a residual heterogeneity in the valued versions of these networks after correction for these covariates.

2.3 Case of sparse graphs

We discuss the validity of Theorem 2 when considering sparse graphs. Sparsity can be defined in two ways. Either each connection probability vanishes as n grows, or the frac-

Table 1: Degree mean square HER test. TestStat $= (W_{\widehat{\mathbf{p}}^0} - \mathbb{E}_{\widehat{\mathbf{p}}^0})/\mathbb{S}_{\widehat{\mathbf{p}}^0}$.

Network	n	mean$(\widehat{p}^0_{ij})$	st-dev$(\widehat{p}^0_{ij})$	$W_{\widehat{\mathbf{p}}^0}$	$\mathbb{E}_{\widehat{\mathbf{p}}^0} W_{\widehat{\mathbf{p}}^0}$	$\mathbb{S}_{\widehat{\mathbf{p}}^0} W_{\widehat{\mathbf{p}}^0}$	TestStat
Karate	34	0.135	0.149	3.84	3.22	0.88	0.71
Trees	51	0.553	0.2	140.23	10.66	2.11	61.55
Fungis	154	0.226	0.021	592.12	26.82	3.06	184.55
Blogs	196	0.075	0.112	84.82	11.05	1.2	61.5
CKM	219	0.015	0.035	3.16	3	0.32	0.5
Faux Dixon	248	0.02	0.037	11.34	4.41	0.43	16.05
AdHealth	530	0.007	0.008	8.77	3.43	0.24	22.27

tion of non-zero connection probabilities decreases as n grows. The following Proposition deals with a combination of both scenarios.

Proposition 1 *Consider the HER($\mathbf{p}$) model, when $p_{ij} = p^*_{ij} n^{-a}$, $a > 0$, p^*_{ij} following Assumption 1 and a fraction $1 - n^{-b}$, $b \geq 0$, of p_{ij}'s is set to zero. The p^0_{ij}'s satisfy the same assumptions. Then, provided that $a + b < 2$, the statistic $W_{\mathbf{p}^0}$ is asymptotically normal.*

Proof. We will show that $W^*_{\mathbf{p}^0} - \mathbb{E}_{\mathbf{p}} W_{\mathbf{p}^0}$ is asymptotically normal then that $W_{\mathbf{p}^0} - W^*_{\mathbf{p}^0}$ is a negligible term. The projections $P_{\{ij\}} W_{\mathbf{p}^0}$ involved in $W^*_{\mathbf{p}^0} - \mathbb{E}_{\mathbf{p}} W_{\mathbf{p}^0}$ still stand for the X_{nu} and $a_{n\{ij\}}$ expressed in (5) stand for a_{nu} (notation of Remark 1). Since under Assumption 1 $\Delta_i = \Theta(n^{1-a-b})$, we see that $a_{n\{ij\}} = \Theta(n^{-(a+b)})$ if $a + b < 1$ and $\Theta(n^{-1})$ if $a + b > 1$. Therefore, we have $\mathbb{V}_{\mathbf{p}} P_{\{ij\}} V = \Theta\left(n^{-3a-2b}\right)$ if $a + b < 1$ and $\Theta\left(n^{-a-2}\right)$ if $a + b > 1$. Combining this with the number of non-zero terms which equals $\Theta(n^{2-b})$, we get that $B_n^2 = \Theta\left(n^{2-3(a+b)}\right)$ if $a + b < 1$ and $\Theta\left(n^{-(a+b)}\right)$ if $a + b > 1$. Comparing $A_n^2(\epsilon)$ with B_n^2, we see that the Lindeberg condition is fulfilled for $a + b < 2$.
Now we consider $W_{\mathbf{p}^0} - W^*_{\mathbf{p}^0}$ as the sum of the projections $P_{\{ij,ik\}} W_{\mathbf{p}^0}$. The $a_{n\{ij,ik\}}$ given in (6) equal $\Theta(n^{-1})$, thus $\mathbb{V}_{\mathbf{p}} P_{\{ij,ik\}} W_{\mathbf{p}^0} = \Theta\left(n^{-2a-2}\right)$. Since the number of non-zero terms in the sum is $\Theta(n^{3-2b})$, we have therefore $\mathbb{V}_{\mathbf{p}}\left(W_{\mathbf{p}^0} - W^*_{\mathbf{p}^0}\right) = \Theta(n^{1-2(a+b)})$.
We conclude to the asymptotic normality of $W_{\mathbf{p}^0}$ by combining the one of $W^*_{\mathbf{p}^0} - \mathbb{E}_{\mathbf{p}} W_{\mathbf{p}^0}$ under condition $a + b < 2$ and the fact that $\mathbb{V}_{\mathbf{p}}\left(W_{\mathbf{p}^0} - W^*_{\mathbf{p}^0}\right) / \mathbb{V}_{\mathbf{p}} W^*_{\mathbf{p}^0} \to 0$ as $n \to \infty$ under the same condition. ■

Remark 3 *The condition $a + b < 2$ ensures that, although the density of the graph goes to zero, the number of edges still goes to infinity as n grows.*

We now extend Corollary 1 for the degree variance to sparse graphs, considering a setting similar to this of Proposition 1.

Corollary 5 *Consider the HER($\mathbf{p}$) model, with exactly the same conditions as in Proposition 1. Then, provided that $a + b < 2$, the V statistic is asymptotically normal.*

The proof follows the line of this of Proposition 1 and is given in Appendix A.3.

3 Exchangeable random graph models

We consider $EG(\Phi)$ a generic model for exchangeable random graphs based on a *graphon* function $\Phi : [0,1]^2 \mapsto [0,1]$ and commonly called the W-graph introduced in Lovász and Szegedy (2006) and Diaconis and Janson (2008). Under $EG(\Phi)$, a coordinate $U_i \sim \mathcal{U}[0,1]$ is associated with each node $i (1 \leq i \leq n)$ and edges are drawn independently conditional the U_i's as

$$Y_{ij}|U_i, U_j \sim \mathcal{B}[\Phi(U_i, U_j)].$$

Many statistical models such as the expected degree-corrected SBM, see Dasgupta et al. (2004); Karrer and Newman (2011), and the random Rash model, see Rasch (1960), fall into this framework. In this paper, we focus on the stochastic block model (SBM) and the expected degree distribution (EDD) model.

SBM. The SBM introduced in Holland and Leinhardt (1979) and Nowicki and Snijders (2001) consists in a mixture model for random graph as pointed out by Daudin et al. (2008), in which a discrete variable $Z_i \in \{1, \dots K\}$ is associated with each node and edges are drawn conditionally as $Y_{ij}|Z_i, Z_j \sim \mathcal{B}[\pi_{Z_i,Z_j}]$, where $[\pi_{k\ell}]_{k,\ell}$ stands for the so-called connectivity matrix. Indeed, SBM corresponds to a W-graph with block-wise constant graphon function, see Latouche and Robin (2016).

EDD. The EDD model is an exchangeable version of the expected degree sequence model studied in Chung and Lu (2002) and of the configuration model from Newman (2003). Under these two models, the degree of each node is fixed which makes them non exchangeable. Under the EDD, an expected degree K_i (not necessarily integer) is first drawn independently and identically for each node from some distribution G and the edges are drawn independently conditional on the K_i as $Y_{ij}|K_i, K_j \sim \mathcal{B}[K_iK_j/\kappa]$, so $\mathbb{E}(D_i|K_i) \propto K_i$. EDD corresponds to a W-graph with product-form graphon function: $\Phi(u,v) = g(u)g(v)$, taking $g(u) = G^{-1}(u)/\sqrt{\kappa}$. Young and Scheinerman (2007) consider a specific case of this model.

3.1 Asymptotic normality

We propose a goodness-of-fit test for the W-graph model. For a given graphon Φ^0, we consider the following degree mean square statistic.

$$W_{\Phi^0} = \frac{1}{n}\sum_i (D_i - (n-1)\phi_1^0)^2,$$

where ϕ_1^0 stands for the marginal probability for any given edge to exist, namely $\phi_1^0 = \int\int \Phi^0(u,v)\mathrm{d}u\mathrm{d}v$. We establish the asymptotic normality of W_{Φ^0} under model $EG(\Phi)$. The proof relies on a central limit theorem for acyclic patterns from Bickel et al. (2011), which is recalled hereafter.

Let us consider a fixed pattern R (i.e. a given graph as displayed in Figure 1) with m nodes and set of edges $\mathcal{E}_R$. Let us consider a random graph $\mathcal{G}_R$ with m nodes generated

by $EG(\Phi)$. We define $P(R)$ and its empirical version $\hat{P}(R)$ computed on a graph $\mathcal{G}$ with n nodes as follows.

$$P(R) = \mathbb{P}\left(\mathcal{G}_R = R\right), \quad \text{and} \quad \hat{P}(R) = \binom{n}{m}^{-1} N(R)^{-1} \sum_{\mathcal{G}_S \subset \mathcal{G}} \mathbb{1}\left(\mathcal{G}_S \sim R\right), \tag{11}$$

where $\sim$ stands for the isomorphic relation and $N(R)$ is the number of graphs isomorphic to R. Let us denote ϕ_j the probability P of pattern R_j given in Figure 1 as defined in Bickel et al. (2011): $\phi_j = P(R_j)$.

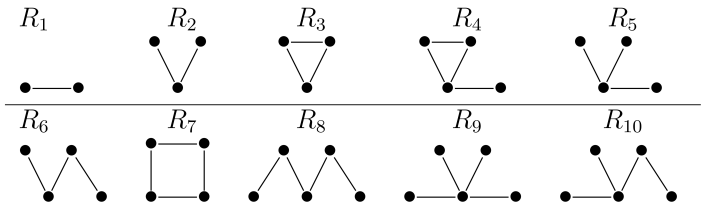

Figure 1: Definition of the patterns R_1 to R_{10} involved in the calculation of the moment of the W statistics.

Theorem 3 (Bickel et al. (2011)) *Consider a set of fixed patterns* $\mathbf{R} = (R_1, \ldots, R_k)$ *with respective sizes* $m_j \leq m$ *and* $\int\int \left(\Phi(u,v)/\phi_1\right)^{2|\mathcal{E}_{R_j}|} du dv < \infty$ *(*$\Phi = \Phi(n)$ *and* $\phi_1 = \phi_1(n)$*). Suppose that* $(n-1)\phi_1$ *is of order* $n^{1-2/p}$ *or higher. Then,*

$$\sqrt{n}\left((\tilde{P}(R_1), \ldots, \tilde{P}(R_k)) - (\mathbb{E}\tilde{P}(R_1), \ldots, \mathbb{E}\tilde{P}(R_k))\right) \xrightarrow{D} \mathcal{N}(\mathbf{0}, \mathbf{\Sigma_R}),$$

where $\tilde{P}(R_j) = \hat{\phi}_1^{-|\mathcal{E}_{R_j}|}\hat{P}(R_j)$ *with* $\hat{\phi}_1 = \sum_i D_i/[n(n-1)]$*. We further have* $\hat{\phi}_1/\phi_1 \to^P 1$*.*

Theorem 4 *Under model* $EG(\Phi)$*, the statistic* W_{Φ^0} *is asymptotically normal :*

$$(W_{\Phi^0} - \mathbb{E}_\Phi W_{\Phi^0})/\mathbb{S}_\Phi W_{\Phi^0} \xrightarrow{D} \mathcal{N}(0,1),$$

with moments

$$\mathbb{E}_\Phi W_{\Phi^0} = n^{-1}\left\{n(n-1)^2(\phi_1^0)^2 + [1 - 2(n-1)\phi_1^0]n_1\phi_1 + n_2\phi_2\right\}, \tag{12}$$

$$\begin{aligned}\mathbb{V}_\Phi W_{\Phi^0} = n^{-2}\Bigg\{ & 4[1-2(n-1)\phi_1^0]^2\left(\frac{n_1}{2}\phi_1 + n_2\phi_2 + \frac{n_3}{4}\phi_1^2 - \frac{n_1^2}{4}\phi_1^2\right) \\ & + 8[1-2(n-1)\phi_1^0]\left[\frac{n_2}{2}(2\phi_2+\phi_3) + \frac{n_3}{2}(\phi_5+2\phi_6) + \frac{n_4}{2}\phi_1\phi_2 - \frac{n_1 n_2}{4}\phi_1\phi_2\right] \\ & + 4\left[\frac{n_2}{6}(3\phi_2+6\phi_3) + \frac{n_3}{2}(4\phi_4+2\phi_5+2\phi_6+\phi_7)\right. \\ & \left. + \frac{n_4}{4}(4\phi_8+\phi_9+4\phi_{10}) + \left(\frac{n_5}{5} - \frac{n_2^2}{4}\right)\phi_2^2\right]\Bigg\}, \end{aligned} \tag{13}$$

where $n_j = \prod_{k=0}^{j}(n-k)$ *and* $\phi_j = P(R_j)$ *defined just above.*

Proof. The proof relies on the fact that the statistic W_{Φ^0} is a linear combination of the $\hat{P}(R_j)$ of three particular patterns R_j to which we will apply Theorem 3. Let us begin with the calculation of the moments of W_{Φ^0}. First observe that,

$$\begin{aligned}\sum_i [D_i - (n-1)\phi_1^0]^2 &= n(n-1)^2(\phi_1^0)^2 + 2[1-2(n-1)\phi_1^0]\sum_{i<j} Y_{ij} \\ &\quad +2\sum_{1\le i<j<k\le n} Y_{ij}Y_{ik} + Y_{ji}Y_{jk} + Y_{ki}Y_{kj} \\ &= n(n-1)^2(\phi_1^0)^2 + 2[1-2(n-1)\phi_1^0]M_1 + 2M_2,\end{aligned}$$

where

$$M_1 = \sum_{1\le i<j\le n} Y_{ij}, \qquad M_2 = \sum_{1\le i<j<k\le n} Y_{ij}Y_{ik} + Y_{ij}Y_{jk} + Y_{ik}Y_{jk}.$$

Then, we see that,

$$\mathbb{E}_\Phi M_1 = \frac{n_1}{2}\phi_1 \quad \text{and} \quad \mathbb{E}_\Phi M_2 = \frac{n_2}{2}\phi_2,$$

which gives $\mathbb{E}_\Phi W_{\Phi^0}$.
Next, we calculate the three forthcoming expectations (calculation details are given in Appendix A.4):

$$\begin{aligned}\mathbb{E}_\Phi(M_1^2) &= \frac{n_1}{2}\phi_1 + n_2\phi_2 + \frac{1}{4}n_3(\phi_1)^2, \\ \mathbb{E}_\Phi(M_1M_2) &= \frac{n_2}{2}(2\phi_2+\phi_3) + \frac{n_3}{2}(\phi_5+2\phi_6) + \frac{n_4}{4}\phi_1\phi_2 \\ \mathbb{E}_\Phi(M_2^2) &= \frac{n_2}{6}(3\phi_2+6\phi_3) + \frac{n_3}{2}(4\phi_4+2\phi_5+2\phi_6+\phi_7) + \frac{n_4}{4}(4\phi_8+4\phi_{10}+\phi_9) + \frac{n_5}{4}\phi_2^2,\end{aligned}$$

which give $\mathbb{V}_\Phi W_{\Phi^0}$.

We now turn to the asymptotic normality of W_{Φ^0}. By using definition (11) of $\hat{P}$ and the one of $\tilde{P}$ given in Theorem 3, we observe that,

$$\begin{aligned}nW_{\Phi^0} &= \sum_i [D_i - (n-1)\phi_1^0]^2 \\ &= n(n-1)^2(\phi_1^0)^2 + [1-2(n-1)\phi_1^0]\sum_{j\neq i} Y_{ij} + \sum_{k\neq j\neq i} Y_{ij}Y_{ik}(1-Y_{jk}) + \sum_{k\neq j\neq i} Y_{ij}Y_{ik}Y_{jk} \\ &= n(n-1)^2(\phi_1^0)^2 + [1-2(n-1)\phi_1^0]n_1\hat{P}(R_1) + \frac{1}{3}n_2\hat{P}(R_2) + n_2\hat{P}(R_3) \\ &= n(n-1)^2(\phi_1^0)^2 + [1-2(n-1)\phi_1^0]n_1\hat{\phi}_1\tilde{P}(R_1) + \frac{1}{3}n_2\hat{\phi}_1{}^2\tilde{P}(R_2) + n_2\hat{\phi}_1{}^3\tilde{P}(R_3), \quad (14)\end{aligned}$$

where R_1, R_2 and R_3 are depicted in Figure 1. Thus we obtain the following linear combination of $\tilde{P}(R_1)$, $\tilde{P}(R_2)$ and $\tilde{P}(R_3)$:

$$\begin{aligned}n^{-3/2}\left(W_{\Phi^0} - (n-1)^2(\phi_1^0)^2\right) &= \Theta(n^{1/2}) \times \hat{\phi}_1\tilde{P}(R_1) + \Theta(n^{1/2}) \times \hat{\phi}_1{}^2\tilde{P}(R_2) \\ &\quad +\Theta(n^{1/2}) \times \hat{\phi}_1{}^3\tilde{P}(R_3). \quad (15)\end{aligned}$$

Let us apply the asymptotic normality result of Theorem 3 to the right-hand side of Equation (15). Since $\hat{\phi}_1 \to^P \phi_1$ by Theorem 3, we conclude by the Slutsky's lemma. Note that condition $\int\int (\Phi(u,v)/\phi_1)^{2|\mathcal{E}_{R_j}|}\,\mathrm{d}u\mathrm{d}v < \infty, j = 2,3$ is fulfilled because $\Phi \leq 1$ and ϕ_1 are constants. ■

Remark 4 *The test statistics $W_{\mathbf{p}^0}$ in the independent case and W_{Φ^0} in the exchangeable case measure both the discrepancy between the observed degrees and their expected values under specifics models. Let us stress that the latent layer in the exchangeable case implies an additional variability of the degrees. The third term in Equation* (15) *is a consequence of this additional variability.*

Particular cases: SBM and EDD

Because SBM and EDD are special cases of the W-graph, all results above apply to them. Interestingly, for both models, the critical calculation of coefficients ϕ_1 to ϕ_{10} can be achieved exactly. Indeed, the calculation of the first two moments of pattern counts under SBM and EDD is explicitly addressed in Picard et al. (2008). In this reference, it is already observed that patterns 4 to 10 from Figure 1 need to be considered as 'super-patterns' (or 'super-motifs') of patterns 2 and 3 and that the variance of the count of a given pattern depends on the expected frequency of its super-patterns.
The formula of ϕ_j for SBM is explicitly in Picard et al. (2008). Denoting α_k the probability for any given node to belong to group k ($1 \leq k \leq K$), we have that

$$\phi_j = P(R_j) = \sum_{k_1}^{K} \cdots \sum_{k_{p_j}}^{K} \alpha_{k_1} \ldots \alpha_{k_{p_j}} \prod_{1 \leq u < v \leq p_j} \pi_{k_u k_v}^{m_{uv}^j}$$

where p_j stands for number of nodes in pattern R_j and m_{uv}^j is 1 if nodes u and v are connected in pattern R_j and 0 otherwise.
The EDD model is also studied in Picard et al. (2008) but needs to be adapted to the W-graph framework. For $\phi(u,v) = g(u)g(v)$, we have that

$$\phi_j = \prod_{u=1}^{p_j} g_{d_u^j}, \qquad \text{where} \quad g_k = \int_0^1 g^k(u)\mathrm{d}u$$

and d_u^j stands for the degree of node u within the pattern R_j. Some examples are

$$\phi_1 = g_1^2, \qquad \phi_2 = g_1^2 g_2, \qquad \phi_3 = g_2^3, \qquad \phi_4 = g_1 g_2^2 g_3, \qquad \phi_{10} = g_1^3 g_2 g_3.$$

Plug-in version of the test. In many situations, Φ^0 is unknown and one needs to resort to an estimate $\widehat{\Phi}^0$. The question is then to understand the asymptotic behaviour of $(W_{\widehat{\Phi}^0} - \mathbb{E}_\Phi W_{\widehat{\Phi}^0})/\mathbb{S}_\Phi W_{\widehat{\Phi}^0}$. A first strategy would consist in estimating Φ^0 from the data. Still, few results are available regarding the statistical properties of the graphon estimates.

More recently, Gao and Lafferty (2017a) considered a simpler statistic, the moments of which can be estimated via the empirical counts of the patterns R_1, R_2, R_3 (see Figure 1). They proved the asymptotic normality of its plug-in version in the degree-corrected SBM model. In our case, this would require to establish asymptotic results about quantities that combine patterns R_1 to R_{10}, in a particularly intricate manner.

3.2 Test and power

We now study the test of $H_0 = EG(\Phi^0)$ versus $H_1 = EG(\Phi)$. The next Corollaries provide the null distribution of the test statistic W_{Φ^0} and the power of the associated test. They are direct consequences of Theorem 4.

Corollary 6 *Under the model based on Φ^0 the statistic W_{Φ^0} is asymptotically normal with moments expressed as those of Theorem 4 with all ϕ_j replaced by ϕ_j^0.*

Recall that the particular terms $\delta_{ij} = p_{ij} - p_{ij}^0$ appear in the moments of $W_{\mathbf{p}^0}$ under model $HER(\mathbf{p})$ whereas it is not the case anymore under $HER(\mathbf{p}^0)$ (see Theorem 2 and Corollary 2 in sections 2.1 and 2.2). Notice that this simple measure of discrepancy between two alternative models is not visible in the moments of W_{Φ^0} but spread out all differences between ϕ_j and ϕ_j^0.

A formal test with asymptotic level α can be constructed based on Corollary 6, which reject H_0 as soon as W_{Φ^0} exceeds $\mathbb{E}_{\Phi^0} W_{\Phi^0} + t_{1-\alpha} \mathbb{S}_{\Phi^0} W_{\Phi^0}$. The expression of its power follows.

Corollary 7 *The asymptotic power of the considered test is*

$$\pi(\mathbf{p}) = 1 - \Phi\left(\left(\mathbb{E}_{\Phi^0} W_{\Phi^0} + t_{1-\alpha} \mathbb{S}_{\Phi^0} W_{\Phi^0} - \mathbb{E}_{\Phi} W_{\Phi^0} \right) / \mathbb{S}_{\Phi} W_{\Phi^0} \right). \tag{16}$$

Remark 5 *Let consider the test of $H_0 = ER$ versus $H_1 = EG(\Phi)$. This simply corresponds to the degree variance test based on the statistic V described in Section 2.2.*

The following corollary gives a sufficient condition on the departure between Φ and Φ^0 to ensure that the proposed test is asymptotically powerful.

Corollary 8 *For functions Φ^0 and Φ, define*

$$\Delta_n(\Phi^0, \Phi) := \mathbb{E}_{\Phi} W_{\Phi^0} - \mathbb{E}_{\Phi^0} W_{\Phi^0} = [1 - 2(n-1)\phi_1^0] n_1 (\phi_1 - \phi_1^0) + n_2 (\phi_2 - \phi_2^0).$$

If $\Delta_n(\Phi^0, \Phi) > 0$, then the test $H_0 = EG(\Phi^0)$ versus $H_1 = EG(\Phi)$ is asymptotically powerful.

Proof. The proof follows the line of Corollary 4. The expression of $\Delta_n(\Phi^0, \Phi)$ comes from (12). Because functions Φ and Φ^0 are fixed, we have that $\Delta_n(\Phi^0, \Phi) = \Theta(n^3)$. Furthermore, from (13), we have that $\mathbb{S}_{\Phi} W_{\Phi^0} = \Theta(n^{3/2})$ and $\mathbb{S}_{\Phi^0} W_{\Phi^0} = \Theta(n^{3/2})$. As a consequence, if $\Delta_n(\Phi^0, \Phi) > 0$, the negative argument of Φ in (10) goes to infinity at rate $n^{3/2}$, which concludes the proof. ■

Remark 6 *In Corollary 8,*
(i) the condition only depends on the relative frequencies of R_1 and R_2. If Φ and Φ^0 have the same ϕ_1 and ϕ_2 but differ in terms of, say ϕ_k ($k > 2$) the proposed test may no be able to detect the discrepancy.
(ii) observe that, if the function Φ depends on n (and is denoted Φ_n), the asymptotic power is still guaranteed as long as $\Delta_n(\Phi^0, \Phi_n) = \Theta(n^\alpha) > 0$ with $\alpha > 3/2$.

Illustration

As an illustration of the proposed test, we consider the networks described in Section 2.2. The question is to know if a fitted graphon is sufficient to explain the heterogeneity of a network, at least in terms of degrees. To address this question, for each network separately, we estimated a graphon function using the variational expectation maximization of Daudin et al. (2008) to provide estimates of the SBM model parameters and build the corresponding block-wise constant graphon function. The number of blocks was estimated using the model selection criterion considered in Daudin et al. (2008). This is implemented in the package `mixer` (available on the `https://cran.r-project.org/`). We then calculated the moments of the graphon and applied the degree mean square test to check if the fitted graphon is sufficient to explain the heterogeneity of the network. The results are given in Table 2.

Table 2: Degree mean square EG test for an SBM-graphon. TestStat $= (W_{\widehat{\Phi}^0} - \mathbb{E}_{\hat{\Phi}^0})/\mathbb{S}_{\hat{\Phi}^0}$.

Network	n	density	K	$W_{\widehat{\Phi}^0}$	$\mathbb{E}_{\hat{\Phi}^0}$	$\mathbb{S}_{\hat{\Phi}^0}$	TestStat
Karate	34	0.139	4	14.6	15.57	6.16	-0.16
Tree	51	0.54	5	163.14	162.84	17.31	0.02
Fungi	154	0.227	15	597.6	584.42	116.63	0.11
Blog	196	0.075	11	104.72	92.77	25.89	0.46
CKM	219	0.015	3	3.9	4.04	0.76	-0.18
FauxDixon	248	0.02	5	16.78	11.97	1.94	2.48
AdHealth	530	0.007	4	10.7	7.54	1.42	2.22

Using the normal approximation for the distribution of $W_{\widehat{\Phi}^0}$ under H_0, the $EG(\widehat{\Phi}^0)$ model is rejected for two of these networks: FauxDixon and AdHealth. The highest test statistic is observed for the FauxDixon network, which has actually been simulated under a model that does not belong to the class of $EG(\Phi)$.

3.3 Case of sparse graphs

The following theorem discusses the validity of Theorem 4 when considering sparse graphs, namely when $\phi_1 = \phi_1(n)$ vanishes as n grows with a rate we specify.

Proposition 2 *Under the model based on the graphon Φ such that ϕ_1 and ϕ_1^0 are of order $n^{-2/3}$ or higher, if $\iint \left(\Phi(u,v)/\phi_1(n)\right)^6 dudv < \infty$ then the statistic W_{Φ^0} is asymptotically normal.*

Proof. We apply Theorem 3 to a function of W_{Φ^0} which is a linear combination of $\tilde{P}(R_1)$, $\tilde{P}(R_2)$ and $\tilde{P}(R_3)$ involving the quantity $\hat{\phi_1}/\phi_1$, where R_1, R_2 and R_3 refer to the patterns from Figure 1. Equations (14)–(15) state that

$$\begin{aligned} n^{-3/2}\left(W_{\Phi^0} - (n-1)^2(\phi_1^0)^2\right) &= \Theta(\phi_1^0\phi_1) \times \Theta(n^{1/2}) \times \frac{\hat{\phi_1}}{\phi_1}\tilde{P}(R_1) \\ &\quad + \Theta(\phi_1^2) \times \Theta(n^{1/2}) \times \left(\frac{\hat{\phi_1}}{\phi_1}\right)^2 \tilde{P}(R_2) \\ &\quad + \Theta(\phi_1^3) \times \Theta(n^{1/2}) \times \left(\frac{\hat{\phi_1}}{\phi_1}\right)^3 \tilde{P}(R_3). \end{aligned}$$

The asymptotic normality of $\sqrt{n}\left(\tilde{P}(R_1), \tilde{P}(R_2), \tilde{P}(R_3)\right)$ holds under conditions : $\int\int \left(\Phi(u,v)/\phi_1\right)^{2|\mathcal{E}_{R_j}|} dudv < \infty$ with $|\mathcal{E}_{R_j}| \leq 3$ and ϕ_1 being of order $n^{-2/p}$ or higher with $p = 3$. Now, we observe that under the condition that ϕ_1 and ϕ_1^0 are of order $n^{-\alpha}$ for $0 < \alpha < 2/3$,

$$\begin{aligned} n^{-3/2+2\alpha}\left(W_{\Phi^0} - (n-1)^2(\phi_1^0)^2\right) &= \Theta(n^{1/2}) \times \frac{\hat{\phi_1}}{\phi_1}\tilde{P}(R_1) + \Theta(n^{1/2}) \times \left(\frac{\hat{\phi_1}}{\phi_1}\right)^2 \tilde{P}(R_2) \\ &\quad + \Theta(n^{1/2-\alpha}) \times \left(\frac{\hat{\phi_1}}{\phi_1}\right)^3 \tilde{P}(R_3). \end{aligned}$$

Since $\hat{\phi_1}/\phi_1 \to^P 1$ by Theorem 3, we conclude by applying the asymptotic normality result of the same theorem to the right-hand side of the equality above combined with the Slutsky's lemma. Note that the third term mentioned in Remark 4 is negligible. ■

4 Simulation study

We designed a simulation study to assess the performance of the tests described above. More specifically, our purpose is to evaluate the power of these tests for various graph sizes and densities (mean connectivities). We also aim at illustrating for which graph size the asymptotic normal approximation is accurate; we especially focus on this point in the sparse regime.

4.1 Simulation design

Design for the independent case. We designed our simulation so that to mimic the situation where an heterogeneous model $HER(\mathbf{p}^0)$ is considered, which still misses some heterogeneity. More specifically, each node i was associated with a vector of covariates $x_i \in \mathbb{R}^d$ (all values were drawn i.i.d. with standard Gaussian distribution and d was set to 3). Each edge (i, j) was then associated with the covariate vector $x_{ij} = \sqrt{\pi}|x_i - x_j|/2$ so that all x_{ij} are positive with mean 1. The edges were then drawn according a logistic model: $\text{logit}(p_{ij}) = a + x_{ij}^\intercal \beta_1$ where $\beta_1^\intercal = [\beta_0^\intercal \ \beta]^\intercal \in \mathbb{R}^d$, $\beta_0 \in \mathbb{R}^{d-1}$. The constant a was set to preserve the mean connectivity, denoted ρ^* in the sequel. The probability matrix $\mathbf{p}^0 = [p_{ij}^0]$ of the null model was defined according to the same logistic model, removing the last covariate, namely $\text{logit}(p_{ij}^0) = a_0 + x_{ij}^{0\intercal}\beta_0$, where x_{ij}^0 is x_{ij} deprived from its last coordinate. Hence, the discrepancy between the null hypothesis and the true model is measured by the coefficient β of the last covariate. All β_0's were set to 1 except β which ranged from 0 to 2. We also studied the behaviour of the plug-in version as defined in the paragraph 'Plug-in version of the test' at the end of Section 2.1.

Design for the exchangeable case. We designed a situation where a null block-wise constant graphon Φ^0, associated to a SBM model, is contaminated by an alternative graphon of the form considered in Latouche and Robin (2016). Thus, graphs were sampled from an $EG(\Phi)$ model where $\Phi(u, v) = \Phi^0(u, v)\rho\beta^2 u^{\beta-1}v^{\beta-1}$. Note that Φ induces a random graph model related to the degree corrected SBM model of Karrer and Newman (2011) which has received strong attention in the last five years. This model, by characterizing explicitly the degrees of the vertices, is often employed as an alternative to the standard SBM model. Note however that in its original form the degree corrected SBM model is not exchangeable since the degree parameters are fixed. Conversely, Φ induces an exchangeable model here since the degree terms $u^{\beta-1}$ and $v^{\beta-1}$ are random. For the null graphon Φ^0, we considered a SBM with 2 blocks, with the same proportions. Moreover, Φ^0 was given a product form such that $\Phi^0(u, v) = \eta_k\eta_\ell$ if u and v are in blocks k and ℓ, respectively. We set $\eta_1 = 0.4$ and $\eta_2 = 0.5$. In this simulation framework, the discrepancy between the null hypothesis and the true model is measured by the term β which ranges from 1 to 2 and controls the imbalance of the expected degrees of the nodes. The null graphon is retrieved when $\beta = 1$. Finally, the term ρ was set in order to obtain the desired mean connectivity ρ^*.

Note that is both designs, the density of the network is kept constant equal to ρ^* when going away from the null model. Therefore, the departure from H_0 detected by the tests is not due to a mean degree difference. In both designs, β measures the departure from the null model, although the its nominal values are not comparable from one design to another. 1 000 simulations were ran for each combination of the parameters (n, ρ^*, β).

Sparse graphs. For both tests, we considered sparse graphs in the setting described in Sections 2.3 and 3.3. We focused on the asymptotic normality of the degree mean square statistic under the null hypothesis. To this aim, we designed a reference null probability

matrix $\mathbf{p}^{0*}$ and a reference null graphon Φ^{0*} as described above. We then considered the two sparsity scenarios:

- vanishing connection probabilities: $p^0_{ij} = p^{0*}_{ij} n^{-a}$ and $\Phi(u,v) = n^{-a}\Phi^{0*}(u,v)$;
- sparse connection probabilities: $p^0_{ij} = p^{0*}_{ij}$ with probability n^{-b} and 0 otherwise.

The second scenario does not make sense for the $EG(\Phi)$ test. The mean connectivity ρ^* was set to 0.1. The density of the graphs therefore decrease as $\rho^* n^{-a}$ and $\rho^* n^{-b}$, respectively.

Criteria. For each parameter configuration, we computed the moments of the respective statistics and derived the theoretical power. Based on the replicates, we estimated the empirical power, and its plug-in version in the independent case. For the sparse setting, the proximity with the normal distribution was investigated plotting the empirical quantiles versus the theoretical Gaussian quantiles (QQ-plots).

4.2 Results

Power and asymptotic normality. The power curves of the degree mean square tests in the independent and exchangeable cases are given in s 2 and 3, respectively. As expected, the power increases with the departure β, the graph size n and the network density ρ^*. We remind that the departure parameter β can not be compared between the two figures. The binomial confidence interval around the theoretical power informs us about the convergence to the asymptotic normality. We observe that the empirical power (dots) falls within this interval showing that the normal approximation is accurate for reasonably large ($n > 100$) graphs. This does not hold for the empirical version of the HER test (triangles), which suggests that the cumulative effect of all the estimation errors $|\widehat{p}^0_{ij} - p^0_{ij}|$ on $W_{\widehat{\mathbf{p}}^0}$ vanishes later than the convergence of $W_{\mathbf{p}^0}$ to normality. The power of both tests also depends on the density of the graph; it is satisfying for $\rho^* \geq 1\%$ in the independent case and for $\rho^* \geq 3\%$, in the exchangeable case. As for the empirical version of the HER test, it becomes reasonable only when n reaches 300, whatever the density.

Sparse graphs. Figures 4 and 5 display the QQ-plots of the standardized $W_{\mathbf{p}^0}$ and W_{Φ^0} statistics under the vanishing probabilities scenario for graphs with several sizes. Remember that the larger the power a, the sparser the graph. We observe again that normality holds for the non sparse graphs ($a = 0$) even for $n = 100$, but the departure is visible for $n = 100$ as soon as $a \geq 0.4$. The same is observed for $n = 1\,000$, although a bit later ($a \geq 0.8$). For the largest graph ($n = 10\,000$), normality holds until $a \simeq 1.2 - 1.4$ but does not seem to be reached for higher sparsity regimes. As expected, in the very sparse regime, normality can only be relied on for very large graphs. Similar conclusions can be drawn for the sparse probabilities scenario, each distribution being slightly closer to normal.

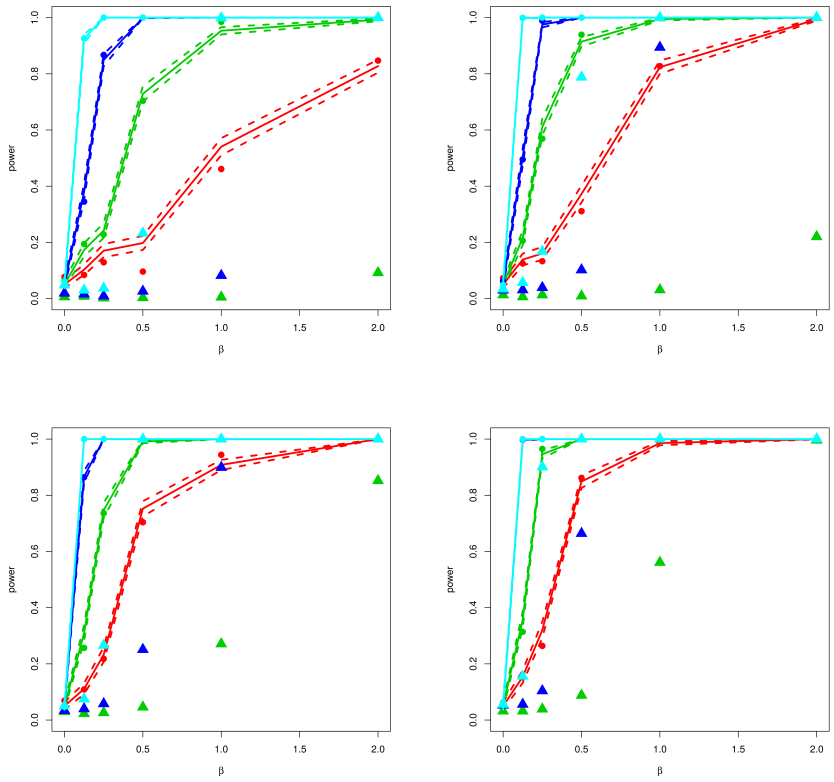

Figure 2: Power of the degree mean square test in the HER design, as a function of β the effect of the last covariate. Top left ($\log_{10}\rho^* = -2.5$), top right ($\log_{10}\rho^* = -2$), bottom left ($\log_{10}\rho^* = -1.5$), bottom right ($\log_{10}\rho^* = -1$). Color refers to the graph size: $n = 32$ (red), 100 (green), 316 (blue), 1 000 (cyan) (green, blue and cyan curves and points overlap in the last panels). Points = empirical power (average on 1 000 simulations): dots = $W_{\mathbf{p}^0}$ test, solid line = theoretical power, dashed line = binomial confidence interval for 1 000 simulations, triangles = $W_{\widehat{\mathbf{p}}^0}$ test (for $n \geq 100$).

Acknowledgements

This work has been partially funded by the research grant NGB (ANR-17-CE32-0011). We thank Pr Bloznelis for providing us with his report referred to as Bloznelis (2005).

References

Barabási, A. L. and R. Albert (1999). Emergence of scaling in random networks. *Science 286*, 509–512.

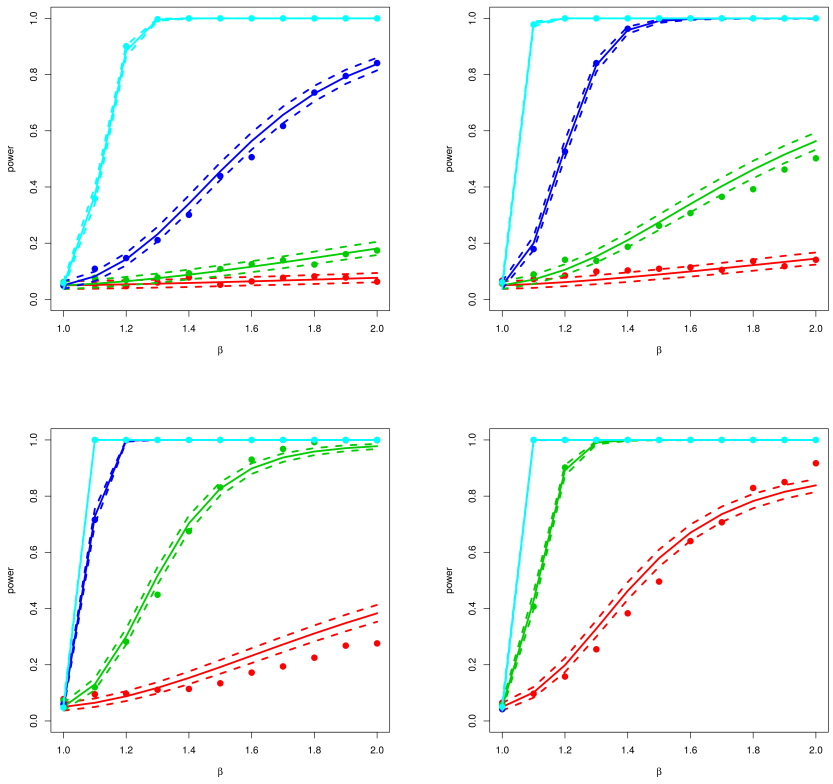

Figure 3: Power of the degree mean square test in the EG design, as a function of β which controls the degree imbalance. Same legend as in Figure 2.

Bickel, P. J., A. Chen, and E. Levina (2011). The method of moments and degree distributions for network models. *Ann. Stat. 39*(5), 2280–2301.

Bickel, P. J. and P. Sarkar (2016). Hypothesis testing for automated community detection in networks. *Journal of the Royal Statistical Society: Series B (Statistical Methodology) 78*(1), 253–273.

Billingsley, P. (1968). *Convergence of Probability Measures*. Wiley: New-York.

Bloznelis, M. (2005). Degree variance is asymptotically normal. Technical report, Vilnius university, Faculty of Mathematics and Informatics.

Burt, R. (1987). Social contagion and innovation: cohesion versus structural equivalence. *American Journal of Sociology 92*, 1287–1335.

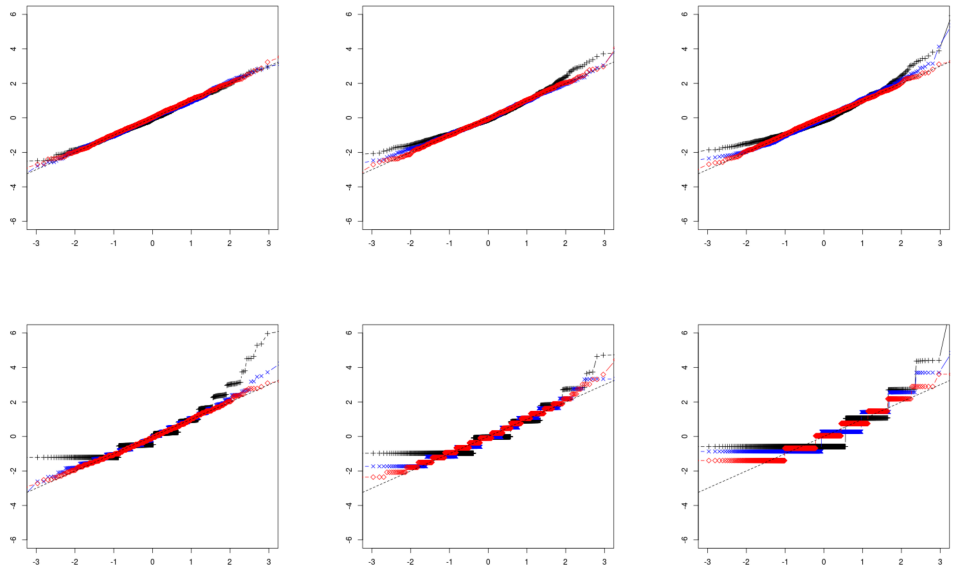

Figure 4: QQ-plots of the degree mean square statistics $W_{\mathbf{P}^0}$ in the HER design, for vanishing connection probabilities: $p_{ij} = p^*_{ij} n^{-a}$ and initial mean density $\rho^* = 0.1$. From top left to bottom right: $a = 0, 0.4, 0.8, 1.2, 1.4, 1.6$. Graph size $n = 100$ (+), 1 000 ($\times$) and 10 000 ($\diamond$).

Cerqueira, A., D. Fraiman, C. D. Vargas, and F. Leonardi (2017). A test of hypotheses for random graph distributions built from eeg data. *IEEE Transactions on Network Science and Engineering 4*(2), 75–82.

Channarond, A., J.-J. Daudin, and S. Robin (2012). Classification and estimation in the stochastic block model based on the empirical degrees. *Elec. J. Stat. 6*, 2574–601.

Chung, F. and L. Lu (2002). Connected components in random graphs with given expected degree sequences. *Annals of combinatorics 6*(2), 125–145.

Coleman, J., E. Katz, and H. Menzel (1966). Medical innovation: a diffusion study. indianapolis: the boobs-merrill company. *Behavioral Science 12*, 481–483.

Dasgupta, A., J. E. Hopcroft, and F. McSherry (2004). Spectral analysis of random graphs with skewed degree distributions. In *null*, pp. 602–610. IEEE.

Daudin, J.-J., F. Picard, and S. Robin (2008). A mixture model for random graphs. *Stat. Comput. 18*(2), 173–83.

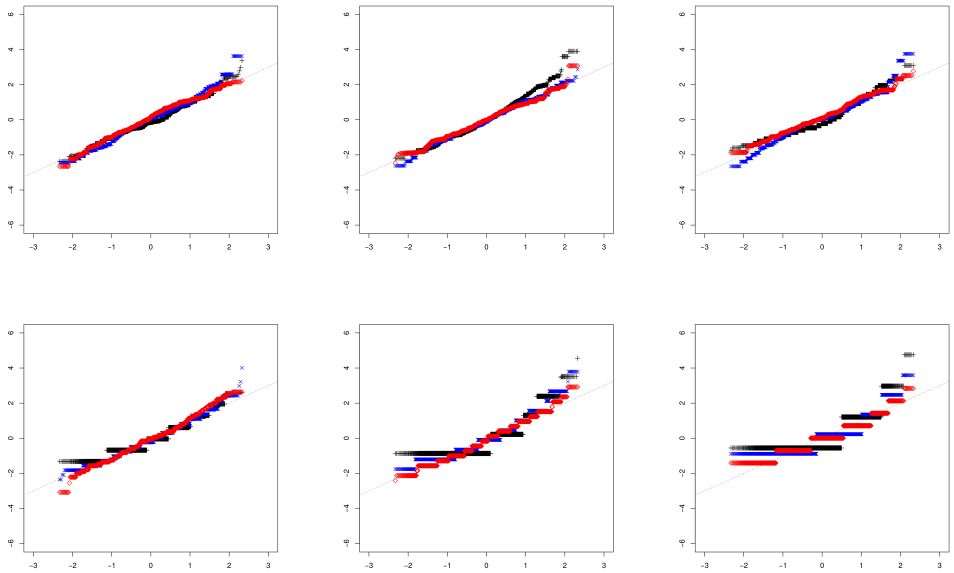

Figure 5: QQ-plots of the degree mean square statistics W_{Φ^0} in the EG design, for vanishing connection probabilities: $\Phi(u,v) = n^{-a}\Phi^{0*}(u,v)$ and initial mean density $\rho^* = 0.1$. Same legend as in Figure 4.

Diaconis, P. and S. Janson (2008). Graph limits and exchangeable random graphs. *Rend. Mat. Appl. 7*(28), 33–61.

Erdös, P. and A. Rényi (1959). On random graphs. *I Publicationes Mathematicae (Debrecen) 6*, 290–297.

Gao, C. and J. Lafferty (2017a). Testing for global network structure using small subgraph statistics. Technical Report 1710.00862, arXiv.

Gao, C. and J. Lafferty (2017b). Testing network structure using relations between small subgraph probabilities. *arXiv preprint arXiv:1704.06742*.

Hagberg, J. (2000). Centrality testing and the distribution of the degree variance in bernoulli graphs. Technical report, Department of Statistics, Stockholm University.

Hagberg, J. (2003). General moments of degrees in random graphs. *Stockholm University, Department of Statistics*.

Handcock, M., D. Hunter, C. Butss, S. Goodreau, and M. Morris (2008). Statnet: Software tools for the representation, visualization, analysis and simulation of network data. *Journal of Statistical Software 24*, 12–25.

Holland, P. W. and S. Leinhardt (1979). Structural sociometry. *Perspectives on social network research*, 63–83.

Hunter, D. R., S. M. Goodreau, and M. S. Handcock (2008). Goodness of fit of social network models. *Journal of the American Statistical Association 103*(481), 248–258.

Josse, J. and F. Husson (2016). missMDA: a package for handling missing values in multivariate data analysis. *Journal of Statistical Software 70*(1), 1–31.

Karrer, B. and M. E. J. Newman (2011). Stochastic blockmodels and community structure in networks. *Phys. Rev. E 83*, 016107.

Latouche, P., E. Birmelé, and C. Ambroise (2011). Overlapping stochastic block models with application to the French political blogosphere. *Ann. Appl. Stat. 5*(1), 309–336.

Latouche, P. and S. Robin (2016). Variational bayes model averaging for graphon functions and motif frequencies inference in W-graph models. *Statistics and Computing 26*, 1173–1185.

Latouche, P., S. Robin, and S. Ouadah (2018). Goodness of fit of logistic models for random graphs. *Journal of Computational and Graphical Statistics 27*(1), 98–109.

Lei, J. (2016). A goodness-of-fit test for stochastic block models. *The Annals of Statistics 44*(1), 401–424.

Lovász, L. and B. Szegedy (2006). Limits of dense graph sequences. *Journal of Combinatorial Theory, Series B 96*(6), 933 – 957.

Mariadassou, M., S. Robin, and C. Vacher (2010). Uncovering structure in valued graphs: a variational approach. *Ann. Appl. Statist. 4*(2), 715–42.

Maugis, P., C. E. Priebe, S. C. Olhede, and P. J. Wolfe (2017). Statistical inference for network samples using subgraph counts. *arXiv preprint arXiv:1701.00505*.

Newman, M. E. (2003). The structure and function of complex networks. *SIAM review 45*(2), 167–256.

Nowicki, K. and T. Snijders (2001). Estimation and prediction for stochastic blockstructures. *J. Amer. Statist. Ass. 96*, 1077–87.

Nowicki, K. and J. C. Wierman (1988). Subgraph counts in random graphs using incomplete u-statistics methods. *Discrete Math. 72*(1), 299–310.

Picard, F., J.-J. Daudin, M. Koskas, S. Schbath, and S. Robin (2008). Assessing the exceptionality of network motifs,. *J. Comput. Biol. 15*(1), 1–20.

Rasch, G. (1960). *Probabilistic Models for Some Intelligence and Attainment Tests.* Studies in mathematical psychology. Danmarks Paedagogiske Institut.

Resnick, M., P. S. Bearman, R. W. Blum, K. E. Bauman, K. M. Harris, J. Jones, J. Tabor, T. Beuhring, R. E. Sieving, M. Shew, et al. (1997). Protecting adolescents from harm: findings from the national longitudinal study on adolescent health. *Jama 278*(10), 823–832.

Snijders, T. A. B. (1981). The degree variance: An index of graph heterogeneity. *Social Networks 3*(3), 163–174.

Vacher, C., D. Piou, and M.-L. Desprez-Loustau (2008). Architecture of an antagonistic tree/fungus network: The asymmetric influence of past evolutionary history. *PLoS ONE 3*(3), 1740.

van der Vaart, A. W. (1998). *Asymptotic statistics*, Volume 3 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge.

Yang, J., C. Han, and E. Airoldi (2014). Nonparametric estimation and testing of exchangeable graph models. In *AISTATS*, pp. 1060–1067.

Young, S. J. and E. R. Scheinerman (2007). Random dot product graph models for social networks. In *International Workshop on Algorithms and Models for the Web-Graph*, pp. 138–149. Springer.

Zachary, W. (1977). An information flow model for conflict and fission in small groups. *Journal of Anthropological Research 33*, 452–473.

A Appendix

A.1 Proof of Corollary 1

Let express V as follows.

$$\begin{aligned} n^2 V &= \frac{1}{2}\sum_{i\neq j}(D_i - D_j)^2 \\ &= 2(n-2)\sum_{1\leq i<j\leq n} Y_{ij} \\ &\quad +2(n-4)\sum_{1\leq i<j<k\leq n}\{Y_{ij}Y_{ik}+Y_{ij}Y_{jk}+Y_{ik}Y_{jk}\} \\ &\quad -8\sum_{1\leq i<j<k<l\leq n}\{Y_{ij}Y_{k\ell}+Y_{ik}Y_{j\ell}+Y_{i\ell}Y_{jk}\}. \end{aligned} \tag{17}$$

Then we write the Hoeffding decomposition of V :

$$\begin{aligned} V &= P_\emptyset V + \sum_{1\leq i<j\leq n} P_{\{ij\}}V + \sum_{1\leq i<j<k\leq n}\left\{P_{\{ij,ik\}}V + P_{\{ij,jk\}}V + P_{\{ik,kj\}}V\right\} \\ &\quad + \sum_{1\leq i<j<k<l\leq n}\left\{P_{\{ij,kl\}}V + P_{\{ik,jl\}}V + P_{\{il,jk\}}V\right\}. \end{aligned} \tag{18}$$

Taking all projections with respect to $HER(\mathbf{p})$, we have

$$\begin{aligned} P_\emptyset V &= \frac{1}{2n^2}\left(4(n-2)\sum_{1\le i<j\le n} p_{ij} + 4(n-4)\sum_{1\le i<j<k\le n}\{p_{ij}p_{ik}+p_{ij}p_{jk}+p_{ik}p_{jk}\}\right) \\ &\quad -\frac{8}{n^2}\sum_{1\le i<j<k<l\le n}\{p_{ij}p_{k\ell}+p_{ik}p_{j\ell}+p_{i\ell}p_{jk}\}, \end{aligned}$$

which gives the expectation. The other projections provide the variance. We have,

$$P_{\{ij\}}V = \frac{1}{2n^2}\widetilde{Y}_{ij}\left(4(n-2)+4(n-4)\sum_{k\notin(i,j)}(p_{ik}+p_{jk}) - 16\sum_{k<l\notin(i,j)} p_{k\ell}\right), \tag{19}$$

$$P_{\{ij,ik\}}V = \frac{2(n-4)}{n^2}\widetilde{Y}_{ij}\widetilde{Y}_{ik}, \qquad \text{and} \qquad P_{\{ij,kl\}}V = -\frac{8}{n^2}\widetilde{Y}_{ij}\widetilde{Y}_{k\ell}. \tag{20}$$

So,

$$n^4\mathbb{V}P_{\{ij\}}V = \sigma_{ij}^2\left(2(n-2)+2(n-4)\sum_{k\notin(i,j)}(p_{i,k}+p_{j,k}) - 8\sum_{k<l\notin(i,j)} p_{k\ell}\right)^2, \tag{21}$$

$$n^4\mathbb{V}P_{\{ij,ik\}}V = 4(n-4)^2\sigma_{ij}^2\sigma_{ik}^2, \qquad \text{and} \qquad n^4\mathbb{V}P_{\{ij,kl\}}V = 64\sigma_{ij}^2\sigma_{k\ell}^2, \tag{22}$$

and the variance of V follows by summing over all indexes.
As for the asymptotic normality, we consider $V-\mathbb{E}V = V^* - \mathbb{E}_{\mathbf{p}}V + V - V^*$, with $V^* = P_\emptyset V + \sum_{1\le i<j\le n} P_{\{ij\}}V$. In order to show that that $V^* - \mathbb{E}_{\mathbf{p}}V$ is asymptotically normal, we apply Theorem 1 to the projections $P_{\{ij\}}W_{\mathbf{p}^0}$ (which stand for the X_{nu}) by using Remark 1 and Assumption 1. The $a_{n\{ij\}} = \Theta(1)$ expressed in (19) stand for a_{nu}. Since $B_n^2 = \mathbb{V}_{\mathbf{p}}(V^* - \mathbb{E}_{\mathbf{p}}V) = \Theta(n^2)$, we conclude that the Lindeberg condition is fulfilled because, for any ϵ, each a_{nu} becomes smaller than ϵB_n when n goes to infinity. Now we consider $V - V^*$ as the linear combination of the projections $P_{\{ij,ik\}}V$ and $P_{\{ij,kl\}}V$. We notice that $a_{n\{ij,ik\}}$ and $a_{n\{ij,kl\}}$ given in (20) equal $\Theta(n^{-1})$ and $\Theta(n^{-2})$ respectively, and thus that $\mathbb{V}_{\mathbf{p}}(V-V^*) = \Theta(n)$. We conclude to the asymptotic normality of V by combining the one of $V^* - \mathbb{E}V$ and the fact that $\mathbb{V}_{\mathbf{p}}(V-V^*)/\mathbb{V}_{\mathbf{p}}V^* \to 0$ as $n\to\infty$.

A.2 Degree variance test power

Lemma 1 *Under model ER and Assumption 1, the degree variance is asymptotically normal:*

$$(V - \mathbb{E}_{\hat{p}}V)/\mathbb{S}_{\hat{p}}V \xrightarrow{D} \mathcal{N}(0,1).$$

Proof. The proof relies on the concentration of $\widehat{p}$ around p and on Slutsky's lemma (see, e.g., Theorem 4.4, p.27 in Billingsley Billingsley (1968)). First, write the statistic based on V as

$$\frac{V-\mathbb{E}_{\widehat{p}}V}{\mathbb{S}_{\widehat{p}}V} = \frac{\mathbb{S}_pV}{\mathbb{S}_{\widehat{p}}V}\left(\frac{V-\mathbb{E}_pV}{\mathbb{S}_pV} + \frac{\mathbb{E}_pV-\mathbb{E}_{\widehat{p}}V}{\mathbb{S}_pV}\right).$$

Then note that, under $ER(p)$, $(\widehat{p}-p)=\Theta_P(n^{-1})$, so $(\widehat{p}\widehat{q}-pq)=\Theta_{\mathbb{P}}(n^{-2})$, where $\widehat{q}$ stands for $1-\widehat{p}$. According to the moments given in Corollary 1, we have that $\mathbb{E}_pV=\Theta(n)pq$ and $\mathbb{V}_pV=\Theta(1)pq+\Theta(n)p^2q^2$. This entails that $\mathbb{E}_pV-\mathbb{E}_{\widehat{p}}V=\Theta_{\mathbb{P}}(n^{-1})$ and $\mathbb{V}_{\widehat{p}}V-\mathbb{V}_pV=\Theta_{\mathbb{P}}(n^{-2})$, so $\mathbb{S}_pV/\mathbb{S}_{\widehat{p}}V$ converges in probability to 1 and $(\mathbb{E}_pV-\mathbb{E}_{\widehat{p}}V)/\mathbb{S}_pV$ converges in probability to 0. The result then follows from Slutsky's lemma, used twice. ■

Lemma 2 *We have*

$$\left(V-\mathbb{E}_{\hat{p}}V\right)/\mathbb{S}_{\hat{p}}V-\left(V-\mathbb{E}_{\bar{p}}V\right)/\mathbb{S}_{\bar{p}}V\xrightarrow{\mathbb{P}}0,$$

where $\overline{p}=[n(n-1)]^{-1}\sum_{i\neq j}p_{ij}$.

The proof of this Lemma is similar to this of Lemma 1 and results from the concentration of $\widehat{p}$ around $\overline{p}$.

A.3 Proof of Corollary 5

The proof follows the line of this of Proposition 1 under Assumption 1. We begin with the asymptotic normality of $V^*-\mathbb{E}_{\mathbf{p}}V$. Since $\sum_{k\notin(i,j)}p_{ik}=\Theta(n^{1-a-b})$ and $\sum_{k<l\notin(i,j)}p_{k\ell}=\Theta(n^{2-a-b})$, we see that $a_{n\{ij\}}=\Theta(n^{-(a+b)})$ if $a+b<1$ and $\Theta(n^{-1})$ if $a+b>1$ ($a_{n\{ij\}}$ are given in assertion (19)). Therefore, we have $\mathbb{V}_{\mathbf{p}}P_{\{ij\}}V=\Theta\left(n^{-3a-2b}\right)$ if $a+b<1$ and $\Theta\left(n^{-a-2}\right)$ if $a+b>1$. Combining this with the number of non-zero terms which is $\Theta(n^{2-b})$, we get that $B_n^2=\Theta\left(n^{2-3(a+b)}\right)$ if $a+b<1$ and $\Theta\left(n^{-(a+b)}\right)$ if $a+b>1$. Comparing $A_n^2(\epsilon)$ with B_n^2, we see that the Lindeberg condition is fulfilled for $a+b<2$. Now we consider $V-V^*$ as the linear combination of the projections $P_{\{ij,ik\}}V$ and $P_{\{ij,kl\}}V$. We see that $a_{n\{ij,ik\}}=\Theta(n^{-1})$ and $a_{n\{ij,kl\}}=\Theta(n^{-2})$ ($a_{n\{ij,ik\}}$ and $a_{n\{ij,kl\}}$ are given in assertion (20)). Therefore, we have $\mathbb{V}_{\mathbf{p}}P_{\{ij,ik\}}V=\Theta\left(n^{-2a-2}\right)$ and $\mathbb{V}_{\mathbf{p}}P_{\{ij,kl\}}V=\Theta\left(n^{-2a-4}\right)$. Since the number of non-zero terms in the sums is $\Theta(n^{3-2b})$ and $\Theta(n^{4-2b})$ respectively, we have therefore $\mathbb{V}_{\mathbf{p}}\left(W_{\mathbf{p}^0}-W^*_{\mathbf{p}^0}\right)=\Theta(n^{-2(a+b)+1})$.
We conclude to the asymptotic normality of V by combining the one of $V^*-\mathbb{E}_{\mathbf{p}}V$ under condition $a+b<2$ and the fact that $\mathbb{V}_{\mathbf{p}}\left(V-V^*\right)/\mathbb{V}_{\mathbf{p}}V^*\to 0$ as $n\to\infty$.

A.4 Moments of W_{Φ^0} in the proof of Theorem 4

We have

$$\begin{aligned}
\mathbb{E}_\Phi(M_1^2) &= \mathbb{E}_\Phi\left(\sum_{i<j}{Y_{ij}}^2\right)+2\mathbb{E}_\Phi\left(\sum_{1\le i<j<k\le n}Y_{ij}Y_{ik}+Y_{ji}Y_{jk}+Y_{ki}Y_{kj}\right)\\
&\quad +2\mathbb{E}\Phi\left(\sum_{1\le i<j<k<l\le n}Y_{ij}Y_{k\ell}+Y_{ik}Y_{j\ell}+Y_{i\ell}Y_{jk}\right)\\
&= \frac{n_1}{2}\phi_1+n_2\phi_2+\frac{1}{4}n_3(\phi_1)^2,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_\Phi(M_1M_2) &= \binom{3}{2}\mathbb{E}_\Phi\left(\sum_{1\le i<j<k\le n} {Y_{ij}}^2Y_{ik}+{Y_{ij}}^2Y_{jk}+Y_{ij}Y_{ki}Y_{kj}\right)\\
&\quad+\binom{4}{1,1,2}\mathbb{E}_\Phi\left(\sum_{1\le i<j<k<l\le n} Y_{ij}Y_{ik}Y_{i\ell}+Y_{ij}Y_{ki}Y_{k\ell}+Y_{ij}Y_{\ell i}Y_{\ell k}\right)\\
&\quad+\binom{5}{2}\mathbb{E}_\Phi\left(\sum_{1\le i<j<k<l<m\le n} Y_{ij}Y_{k\ell}Y_{km}+Y_{ij}Y_{\ell k}Y_{\ell m}+Y_{ij}Y_{mk}Y_{\ell m}\right)\\
&= \binom{n}{2,1,n-3}(2\phi_2+\phi_3)+\binom{n}{1,1,2,n-4}(\phi_5+2\phi_6)+\binom{n}{2,3,n-5}(3\phi_1\phi_2)\\
&= \frac{n_2}{2}(2\phi_2+\phi_3)+\frac{n_3}{2}(\phi_5+2\phi_6)+\frac{n_4}{4}\phi_1\phi_2
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_\Phi(M_2^2) &= \sum_{1\le i<j<k\le n}\mathbb{E}_\Phi\Big({Y_{ij}}^2{Y_{ik}}^2+{Y_{ji}}^2{Y_{jk}}^2+{Y_{ki}}^2{Y_{kj}}^2\\
&\quad+2\left({Y_{ij}}^2Y_{ik}Y_{jk}+Y_{ij}{Y_{ik}}^2Y_{jk}+Y_{ij}Y_{ik}{Y_{kj}}^2\right)\Big)\\
&+\binom{4}{2,1,1}\sum_{1\le i<j<k<l\le n}\mathbb{E}_\Phi\Big(Y_{ij}Y_{ik}Y_{jk}Y_{j\ell}+Y_{ij}Y_{ik}Y_{kj}Y_{k\ell}+Y_{ij}Y_{ik}Y_{\ell j}Y_{\ell k}\\
&\quad+Y_{ji}{Y_{jk}}^2Y_{j\ell}+Y_{ji}{Y_{jk}}^2Y_{k\ell}+Y_{ji}Y_{jk}Y_{\ell j}Y_{\ell k}\\
&\quad+Y_{ki}{Y_{kj}}^2Y_{j\ell}+Y_{ki}{Y_{kj}}^2Y_{k\ell}+Y_{ki}Y_{kj}Y_{\ell j}Y_{\ell k}\Big)\\
&+\binom{5}{1,2,2}\sum_{1\le i<j<k<l<m\le n}\mathbb{E}_\Phi\Big(Y_{ij}Y_{ik}Y_{k\ell}Y_{km}+Y_{ij}Y_{ik}Y_{\ell k}Y_{\ell m}+Y_{ij}Y_{ik}Y_{mk}Y_{\ell m}\\
&\quad+Y_{ji}Y_{jk}Y_{k\ell}Y_{km}+Y_{ji}Y_{jk}Y_{\ell k}Y_{\ell m}+Y_{ji}Y_{jk}Y_{\ell k}Y_{\ell m}\\
&\quad+Y_{ki}Y_{kj}Y_{k\ell}Y_{km}+Y_{ki}Y_{kj}Y_{\ell k}Y_{\ell m}+Y_{ki}Y_{kj}Y_{mk}Y_{\ell m}\Big)\\
&+\binom{6}{3,3}\sum_{1\le i<j<k<l<m<u\le n}\mathbb{E}_\Phi\Big(Y_{ij}Y_{ik}Y_{\ell m}Y_{\ell u}+Y_{ij}Y_{ik}Y_{\ell m}Y_{mu}+Y_{ij}Y_{ik}Y_{\ell u}Y_{mu}\\
&\quad+Y_{ij}Y_{jk}Y_{\ell m}Y_{\ell u}+Y_{ij}Y_{jk}Y_{\ell m}Y_{mu}+Y_{ij}Y_{jk}Y_{\ell u}Y_{mu}\\
&\quad+Y_{ik}Y_{jk}Y_{\ell m}Y_{\ell u}+Y_{ik}Y_{jk}Y_{\ell m}Y_{mu}+Y_{ik}Y_{jk}Y_{\ell u}Y_{mu}\Big)\\
&= \binom{n}{3}(3\phi_2+6\phi_3)+\binom{n}{2,1,1,n-4}(4\phi_4+2\phi_5+2\phi_6+\phi_7)\\
&\quad+\binom{n}{1,2,2,n-5}(4\phi_8+4\phi_{10}+\phi_9)+\binom{n}{3,3,n-6}(9\phi_2^2)\\
&= \frac{n_2}{6}(3\phi_2+6\phi_3)+\frac{n_3}{2}(4\phi_4+2\phi_5+2\phi_6+\phi_7)+\frac{n_4}{4}(4\phi_8+4\phi_{10}+\phi_9)+\frac{n_5}{4}\phi_2^2.
\end{aligned}$$