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R. Beerens[1], A. Bisoffi[2], L. Zaccarian[3], H. Nijmeijer[1], W.P.M.H. Heemels[1], N. van de Wouw[1,4]

***Abstract*—We present a reset control approach to achieve setpoint regulation of a motion system with a Proportional-Integral-Derivative (PID) based controller, subject to Coulomb friction and a velocity-weakening (Stribeck) contribution. While classical PID control results in persistent oscillations (hunting), the proposed reset mechanism induces asymptotic stability of the setpoint, and significant overshoot reduction. Moreover, robustness to unknown Coulomb friction levels, and unknown Stribeck contributions is guaranteed. The closed-loop dynamics are formulated in a hybrid systems framework, using a novel hybrid description of the Coulomb friction element, and asymptotic stability of the setpoint is proven accordingly. The working principle of the controller is demonstrated experimentally on a motion stage of an electron microscope, showing superior performance over classical PID control.**

I. Introduction

Friction is a performance-limiting factor in many high-precision motion systems, as it limits the achievable positioning accuracy and settling times. Many different control techniques for frictional motion systems exist in the literature. A branch of control solutions relies on developing as-accurate-as-possible friction models, used for online compensation in a control loop, see, e.g., [5], [16], [24]. These model-based friction compensation methods are typically prone to model mismatches due to, e.g., unreliable friction measurements, or time-varying or uncertain friction characteristics. These techniques, therefore, may suffer from over- or undercompensation of friction, thereby resulting in loss of stability of the setpoint [31], and thus limiting the achievable positioning accuracy. Adaptive control methods (see, e.g., [3], [12]) provide some robustness to time-varying friction characteristics, but model mismatches (and the associated performance limitations) still remain. Non-model-based control schemes have also been proposed, examples of which are impulsive control (see, e.g., [28], [36]), dithering-based techniques (see, eg., [22]), sliding-mode control (see, e.g., [7]), or switched control [27]. These non-model-based controllers, however, employ high-frequency control signals, risking excitation of high-frequency dynamics. Moreover, tuning and implementation of such controllers is not straightforward.

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[1] Eindhoven University of Technology, Department of Mechanical Engineering, P.O. Box 513, 5600MB Eindhoven, The Netherlands, {r.beerens; m.heemels; h.nijmeijer; n.v.d.wouw}@tue.nl

[2] ENTEG and the J.C. Willems Center for Systems and Control, Univ. of Groningen, 9747 AG Groningen, The Netherlands, a.bisoffi@rug.nl

[3]CNRS, LAAS, Univ. de Toulouse, 31400 Toulouse, France, and University of Trento, 38122 Trento, Italy, zaccarian@laas.fr

[4]University of Minnesota, Civil, Environmental & Geo-Engineering Department, MN 55455, USA.

Despite the availability of a wide range of (nonlinear) control techniques for frictional systems, linear controllers are still used in the vast majority of industrial motion systems due to the existence of intuitive design and tuning tools. In industry, the classical proportional-integral-derivative (PID) controller is commonly used for motion systems with friction. In particular, integral action is capable of compensating for unknown static friction, due to the build up of control force from integrating the position error. However, PID control suffers from two distinct performance limitations when applied to frictional motion systems. First, the use of a classical PID controller in combination with static (Coulomb) friction results in long settling times (see, e.g., [11, Remark 3]) adversely affecting the machine throughput. This limitation has been addressed in [8], where a reset integrator is proposed that significantly improves the transient performance and decreases settling times for motion systems with Coulomb friction. Second, a PID-controlled motion system suffering from friction including the velocity-weakening (i.e., Stribeck) effect does not achieve stability of the setpoint, so that the achievable positioning accuracy is limited. More specifically, while the integrator action compensates for the static part of the friction, overcompensation of friction occurs as the velocity increases, due to the velocity-weakening effect. As a result, the system overshoots the setpoint and ends up in persistent stick-slip oscillations (called *hunting*), see, e.g., [4], [20], compromising stability of the setpoint.

In this paper, we address the setpoint stabilization problem of a PID-controlled motion system with Stribeck friction. In particular, we propose a reset integral controller that achieves asymptotic stability of the setpoint, despite the presence of *unknown* static (Coulomb) friction, and an *unknown* velocity-weakening (Stribeck) effect in the friction characteristic. By building upon the well-known PID controller, we aim at lowering the threshold for control practitioners to use non-linear control strategies in the industry. The proposed reset enhancements can be used as an augmentation of any classical, loop-shaped PID controller.

Reset and hybrid controllers have been an active field of research in the past decades. Their development started with the Clegg integrator [14] and the first order reset element [21]. Since then, reset controllers have mainly been used to improve the performance of *linear* motion systems, see, e.g., [1], [25]. Specific examples are the hybrid integrator-gain system [15], improving the tracking performance while minimizing the high-frequency content in the control signal. Overshoot reduction of linear systems using hybrid control is presented, e.g., in [9], [39]. Analysis and design tools for reset controllers are presented in [26], [38] and in the recent overviews [6], [30]. In the context of frictional systems, reset control has been applied in [8], where transient performance of PID-based motion systems with Coulomb friction is improved. Reset controllers have already been applied to improve *performance* of motion systems, but, to the best of our knowledge, not yet for *stabilization* of nonlinear frictional motion systems.

The contributions of this paper are as follows. The first one is the design of a novel reset controller for systems with Stribeck friction, aiming at asymptotically stabilizing a constant setpoint. The second contribution is the development of a hybrid formulation of the closed-loop system, where the Stribeck friction element is captured by a hybrid bisimulation model (in the sense of [37, Def. 2.5]), instead of the commonly used set-valued force law (see, e.g., [2, Sec. 1.3]). The latter model builds upon our previous work in [10], where we now include the Stribeck effect and a radically different two-phase resetting law. The third contribution is a proof of asymptotic stability, and the fourth contribution is an experimental demonstration of the effectiveness of the proposed controller on an industrial high-precision positioning system.

The paper is organized as follows. In Section II, we present our reset PID controller design. In Section III, we formulate the reset closed loop as a hybrid system, state the main stability result, and exploit intrinsic robustness properties to obtain a suitable experimental implementation. In Section IV, we experimentally validate the proposed reset controller on a high-accuracy industrial positioning system. The second part of the paper is devoted to nontrivial derivations necessary to prove the main stability result. In Section V, we establish boundedness of solutions and semiglobal dwell-time properties, which lead to building the hybrid bisimulation model. With it, we prove our main result in Section VI.

Notation: Given $x \in \mathbb{R}^n$, $|x|$ is its Euclidean norm. $\mathbb{B}$ is the closed unit ball, of appropriate dimensions, in the Euclidean norm. $\operatorname{sign}(\cdot)$ denotes the classical sign function, i.e., $\operatorname{sign}(y) := y/|y|$ for $y \neq 0$ and $\operatorname{sign}(0) := 0$. $\operatorname{Sign}(\cdot)$ (with an upper-case S) denotes the *set-valued* sign function, i.e., $\operatorname{Sign}(y) := \{\operatorname{sign}(y)\}$ for $y \neq 0$, and $\operatorname{Sign}(y) := [-1, 1]$ for $y = 0$. For $c > 0$, the deadzone function $y \mapsto \mathrm{dz}_c(y)$ is defined as: $\mathrm{dz}_c(y) := 0$ if $|y| \leq c$, $\mathrm{dz}_c(y) := y - c\operatorname{sign}(y)$ if $|y| > c$. For column vectors $x_1 \in \mathbb{R}^{d_1}, \ldots, x_m \in \mathbb{R}^{d_m}$, the notation $(x_1, \ldots, x_m)$ is equivalent to $[x_1^\top \ldots x_m^\top]^\top$. $e_3 := (0, 0, 1)$ is the third unit vector generating $\mathbb{R}^3$. $\wedge$, $\vee$, $\implies$ denote the logical conjunction, disjunction, implication.

For a hybrid solution ψ [18, Def. 2.6] with hybrid time domain $\operatorname{dom}\psi$ [18, Def. 2.3], the function $j(\cdot)$ is defined as $j(t) := \min_{(t,k)\in \operatorname{dom}\psi} k$. Function $j(\cdot)$ depends on the specific solution ψ that it addresses, but with a slight abuse of notation we use a unified symbol $j(\cdot)$ because the solution under consideration is always clear from the context. A hybrid solution is maximal if it cannot be extended [18, Def. 2.7], and is complete if its domain is unbounded (in the t- or j-direction) [18, p. 30]. For a hybrid system $\mathcal{H}$ and a set S, $\psi \in \mathcal{S}_\mathcal{H}(x)$ (respectively, $\psi \in \mathcal{S}_\mathcal{H}(S)$) means that ψ is a maximal solution to $\mathcal{H}$ with $\psi(0, 0) = x$ (respectively, $\psi(0, 0) \in S$), and $\mathcal{S}_\mathcal{H}$ is the set of all maximal solutions to $\mathcal{H}$.

II. System description and controller design

A single-degree-of-freedom mass m sliding on a horizontal plane with position z_1 and velocity z_2 is subject to a control input $\bar{u}$ and a friction force belonging to a set $\Psi(z_2)$, governed by the dynamics

$$\dot{z}_1 = z_2, \quad \dot{z}_2 \in \frac{1}{m}\left(\Psi(z_2) + \bar{u}\right). \tag{1}$$

The friction characteristic is modeled by the next set-valued mapping of the velocity:

$$z_2 \rightrightarrows \Psi(z_2) := -\bar{F}_s \operatorname{Sign}(z_2) - \alpha z_2 + \bar{f}(z_2), \tag{2}$$

where $\bar{F}_s$ is the static friction, αz_2 the viscous friction contribution (with $\alpha \geq 0$ the viscous friction coefficient), and $\bar{f}$ a nonlinear velocity-dependent friction contribution, encompassing the Stribeck effect.

For a reference position $r \in \mathbb{R}$, our goal is formulated next.

Problem 1. *Design a* reset *PID controller for $\bar{u}$ in* (1)-(2) *that globally asymptotically stabilizes the setpoint $(z_1, z_2) = (r, 0)$, in the presence of an* unknown *static friction $\bar{F}_s$ and an* unknown *velocity-dependent friction contribution $\bar{f}$.*

The need for integrator action in Problem 1 is motivated by the fact that it is able to compensate for an *unknown* static friction force $\bar{F}_s$. However, due to overcompensation of friction in the subsequent slip phase (caused by the velocity-weakening effect), persistent oscillations emerge so that asymptotic stability of the setpoint is not achieved with a *classical* PID controller. Enhancing the classical PID controller with resets instead results in asymptotic stability of the setpoint, as we will show in this paper.

A. Classical PID controller

Consider a *classical* PID controller for input $\bar{u}$ in (1), i.e.,

$$\bar{u} = -\bar{k}_p(z_1 - r) - \bar{k}_d z_2 - \bar{k}_i z_3, \quad \dot{z}_3 = z_1 - r, \tag{3}$$

where z_3 is the PID controller state, and $\bar{k}_p$, $\bar{k}_d$, $\bar{k}_i$ represent the proportional, derivative, and integral gains, respectively. As in [8], [11], we use mass-normalized parameters and shifted state variables that facilitate later the construction of Lyapunov functions for the stability analysis:

$$k_p := \tfrac{\bar{k}_p}{m},\ k_d := \tfrac{\bar{k}_d + \alpha}{m},\ k_i := \tfrac{\bar{k}_i}{m},\ F_s := \tfrac{\bar{F}_s}{m},\ f := \tfrac{\bar{f}}{m}, \tag{4}$$

$$\hat{x} := \begin{bmatrix} \hat{\sigma} \\ \hat{\phi} \\ \hat{v} \end{bmatrix} := \begin{bmatrix} -k_i(z_1 - r) \\ -k_p(z_1 - r) - k_i z_3 \\ z_2 \end{bmatrix}. \tag{5}$$

Using (4) and (5), model (1)-(3) corresponds to

$$\begin{aligned} \dot{\hat{x}} = \begin{bmatrix} \dot{\hat{\sigma}} \\ \dot{\hat{\phi}} \\ \dot{\hat{v}} \end{bmatrix} &\in \begin{bmatrix} -k_i \hat{v} \\ \hat{\sigma} - k_p \hat{v} \\ \hat{\phi} - k_d \hat{v} - F_s \operatorname{Sign}(\hat{v}) + f(\hat{v}) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -k_i \\ 1 & 0 & -k_p \\ 0 & 1 & -k_d \end{bmatrix} \begin{bmatrix} \hat{\sigma} \\ \hat{\phi} \\ \hat{v} \end{bmatrix} - e_3(F_s \operatorname{Sign}(\hat{v}) - f(\hat{v})) \\ &=: A\hat{x} - e_3(F_s \operatorname{Sign}(\hat{v}) - f(\hat{v})) =: \hat{\mathcal{F}}_x(\hat{x}). \end{aligned} \tag{6}$$

Note that $\hat{\sigma}$ is a generalized position error, and $\hat{\phi}$ is the controller state encompassing the proportional and integral control actions.

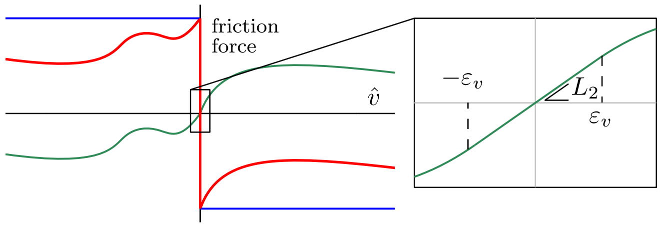

Fig. 1. Example of a friction force satisfying Assumption 1. Total friction (—), static contribution F_s (—), velocity-dependent contribution f (—).

Let us now adopt the following assumptions on the velocity-dependent friction characteristic f and the controller gains.

Assumption 1. *Function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the following:*

(i) $|f(\hat{v})| \leq F_s$ *for all* $\hat{v}$*;*
(ii) $\hat{v} f(\hat{v}) \geq 0$ *for all* $\hat{v}$*;*
(iii) f *is globally Lipschitz with Lipschitz constant* $L > 0$*;*
(iv) for some $\varepsilon_v > 0$ *and* $L_2 \in (k_d, L]$, $f(\hat{v}) = L_2 \hat{v}$ *for all* $|\hat{v}| \leq \varepsilon_v$*.*

A possible f satisfying Assumption 1 is depicted in Fig. 1. Items *(i)-(iii)* are clearly not restrictive for typical friction laws. For item *(iv)*, we emphasize that ε_v can be selected arbitrarily small. As a result, item *(iv)* is hardly restrictive.

In the new coordinates $\hat{x}$, a solution is said to be in a *stick* or *slip* phase when it belongs, respectively, to the sets

$$\mathcal{E}_{\text{stick}} := \{\hat{x} \in \mathbb{R}^3 \colon \hat{v} = 0, |\hat{\phi}| \leq F_s\},\ \mathcal{E}_{\text{slip}} := \mathbb{R}^3 \backslash \mathcal{E}_{\text{stick}}. \quad (7)$$

Indeed, from Assumption 1, when $\hat{v} = 0$, until $|\hat{\phi}| < F_s$, the only possible evolution in (6) is with $\dot{\hat{v}} = 0$ (a stick phase).

Assumption 2. *The control gains* k_p*,* k_d*,* k_i *satisfy* $k_p > 0$*,* $k_i > 0$*,* $k_p k_d > k_i$*.*

The selection of gains as in Assumption 2 is equivalent to the origin being globally asymptotically stable for the closed-loop (6) in the frictionless case (i.e., $F_s = 0$ and $f(\hat{v}) = 0$ for all $\hat{v}$) by the Routh-Hurwitz criterion, and is therefore not restrictive.

The next lemma splits the differential inclusion (6) into three cases that simplify its analysis.

Lemma 1. *Consider model* (6) *under Assumptions 1-2 and the initial conditions in Table I. The following hold.*
(i) For each initial condition $\hat{x}_0 \in \mathbb{R}^3$*, there exists a unique solution* $\hat{x}$ *to* (6) *with* $\hat{x}(0) = \hat{x}_0$*, which is also complete.*
(ii) For each initial condition $\hat{x}_0 = (\hat{\sigma}_0, \hat{\phi}_0, \hat{v}_0)$ *satisfying* (8)*, there exists* $T > 0$ *such that the unique solution* $\hat{x}$ *to* (6) *with* $\hat{x}(0) = \hat{x}_0$ *coincides over* $[0, T]$ *with the unique solution* $\tilde{x}$ *to*

$$\dot{\tilde{x}} = A\tilde{x} - e_3(F_s - f(\tilde{v})), \quad \tilde{x}(0) = \hat{x}_0, \quad (11)$$

which satisfies $\tilde{v}(t) > 0$ *for all* $t \in (0, T]$*.*
(iii) For each initial condition $\hat{x}_0 = (\hat{\sigma}_0, \hat{\phi}_0, \hat{v}_0)$ *satisfying* (9)*, there exists* $T > 0$ *such that the unique solution* $\hat{x}$ *to* (6) *with* $\hat{x}(0) = \hat{x}_0$ *coincides over* $[0, T]$ *with the unique solution* $\tilde{x}$ *to*

$$\dot{\tilde{x}} := \begin{bmatrix} \dot{\tilde{\sigma}} \\ \dot{\tilde{\phi}} \\ \dot{\tilde{v}} \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{\sigma} \\ 0 \end{bmatrix}, \quad \tilde{x}(0) = \hat{x}_0, \quad (12)$$

which satisfies $\tilde{v}(t) = 0$ *for all* $t \in [0, T]$*.*
(iv) For each $\hat{x}_0 = (\hat{\sigma}_0, \hat{\phi}_0, \hat{v}_0)$ *satisfying* (10)*, there exists* $T > 0$ *such that the unique solution* $\hat{x}$ *to* (6) *with* $\hat{x}(0) = \hat{x}_0$ *coincides over* $[0, T]$ *with the unique solution* $\tilde{x}$ *to*

$$\dot{\tilde{x}} = A\tilde{x} - e_3(-F_s - f(\tilde{v})), \quad \tilde{x}(0) = \hat{x}_0, \quad (13)$$

which satisfies $\tilde{v}(t) < 0$ *for all* $t \in (0, T]$*.*

TABLE I
INITIAL CONDITIONS CONSIDERED IN LEMMA 1.

$(\hat{v}_0 > 0) \vee (\hat{v}_0 = 0 \wedge \hat{\phi}_0 > F_s)$ $\vee (\hat{v}_0 = 0 \wedge \hat{\phi}_0 = F_s \wedge \hat{\sigma}_0 > 0)$	(8)
$(\hat{v}_0 = 0 \wedge \hat{\sigma}_0 > 0 \wedge \hat{\phi}_0 \in [-F_s, F_s))$ $\vee (\hat{v}_0 = 0 \wedge \hat{\sigma}_0 = 0 \wedge \hat{\phi}_0 \in [-F_s, F_s])$ $\vee (\hat{v}_0 = 0 \wedge \hat{\sigma}_0 < 0 \wedge \hat{\phi}_0 \in (-F_s, F_s])$	(9)
$(\hat{v}_0 < 0) \vee (\hat{v}_0 = 0 \wedge \hat{\phi}_0 < -F_s)$ $\vee (\hat{v}_0 = 0 \wedge \hat{\phi}_0 = -F_s \wedge \hat{\sigma}_0 < 0)$	(10)

Proof. Let us prove each item separately.

Item (i). As for completeness of solutions from each $\hat{x}_0 \in \mathbb{R}^3$, note first that the set-valued mapping $\hat{\mathcal{F}}_x$ is outer semicontinuous, locally bounded, and such that, for each $\hat{x} \in \mathbb{R}^3$, $\hat{\mathcal{F}}(\hat{x})$ is nonempty and convex. Then, results such as [18, Prop. 6.10] guarantee completeness of maximal solutions because no finite escape times can occur for (6).

We prove then uniqueness of complete solutions from $\hat{x}_0$. With L in Assumption 1(iii), define $f_L(\hat{v}) := L\hat{v} - f(\hat{v})$ and note that f_L is nondecreasing. Indeed, for $\hat{v}_1 < \hat{v}_2$, $-L(\hat{v}_2 - \hat{v}_1) \leq f(\hat{v}_2) - f(\hat{v}_1) \leq L(\hat{v}_2 - \hat{v}_1)$ from Assumption 1(iii), hence $L\hat{v}_1 - f(\hat{v}_1) \leq L\hat{v}_2 - f(\hat{v}_2)$, so that $\hat{v}_1 < \hat{v}_2$ implies $f_L(\hat{v}_1) := L\hat{v}_1 - f(\hat{v}_1) \leq L\hat{v}_2 - f(\hat{v}_2) =: f_L(\hat{v}_2)$. By defining

$$\Psi_L(\hat{v}) := F_s \operatorname{Sign}(\hat{v}) + f_L(\hat{v}), \quad (14)$$

(6) is equivalently rewritten as

$$\begin{aligned} \dot{\hat{x}} &\in \begin{bmatrix} 0 & 0 & -k_i \\ 1 & 0 & -k_p \\ 0 & 1 & L - k_d \end{bmatrix} \hat{x} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Psi_L(\hat{v}) \\ &=: A_L \hat{x} - e_3 \Psi_L(\hat{v}). \end{aligned} \quad (15)$$

Consider two complete solutions $\hat{x}_a = (\hat{\sigma}_a, \hat{\phi}_a, \hat{v}_a)$ and $\hat{x}_b = (\hat{\sigma}_b, \hat{\phi}_b, \hat{v}_b)$ with the same initial condition $\hat{x}_0$, i.e., $\hat{x}_a(0) = \hat{x}_b(0) = \hat{x}_0$, and we show now that $\hat{x}_a(t) = \hat{x}_b(t)$ for all $t \geq 0$. Define $\eta = (\eta_1, \eta_2, \eta_3) := \hat{x}_a - \hat{x}_b$, so that $\eta(0) = 0$. The solutions $\hat{x}_a$ and $\hat{x}_b$ are complete, so we have by (15) that for almost all $t \geq 0$, $\dot{\eta}(t) \in A_L \eta(t) - e_3 \left(\Psi_L(\hat{v}_a(t)) - \Psi_L(\hat{v}_b(t))\right)$. With λ denoting the maximum singular value of A_L, we have for almost all $t \geq 0$,

$$\begin{aligned} &\tfrac{d}{dt} \tfrac{|\eta(t)|^2}{2} \in \eta(t)^\top A_L \eta(t) + \eta_3(t) \left(\Psi_L(\hat{v}_b(t)) - \Psi_L(\hat{v}_a(t))\right) \\ &\quad \implies \tfrac{d}{dt} \tfrac{|\eta(t)|^2}{2} \leq \lambda |\eta(t)|^2 + N(t), \end{aligned} \quad (16)$$

where

$$N(t) := \max_{\substack{f_b \in \Psi_L(\hat{v}_a(t) - \eta_3(t)) \\ f_a \in \Psi_L(\hat{v}_a(t))}} \eta_3(t) \left(f_b - f_a\right). \quad (17)$$

By (14), note that $N(t)$ in (17) can be rewritten equivalently as

$$N(t) = \max_{\substack{f'_b \in F_s \operatorname{Sign}(\hat{v}_a(t)-\eta_3(t)) \\ f'_a \in F_s \operatorname{Sign}(\hat{v}_a(t))}} \eta_3(t)\Big(f'_b - f'_a + f_L\big(\hat{v}_a(t) - \eta_3(t)\big) - f_L\big(\hat{v}_a(t)\big)\Big).$$

Whether $\hat{v}_a(t)$ and $\hat{v}_a(t)-\eta_3(t)$ are positive, zero, or negative, inspection of all cases reveals that $N(t) \le 0$ for all $t \ge 0$ because we established above that f_L is nondecreasing. As a result, (16) satisfies $\frac{d}{dt}\frac{|\eta(t)|^2}{2} \le \lambda|\eta(t)|^2$, for almost all $t \ge 0$. Then, $\eta(0) = 0$ implies $\eta(t) = 0$ for all $t \ge 0$ by standard comparison theorems (e.g., [23, Lem. 3.4]).

Item (ii). The proof of this item and the following ones is based on the proof of [11, Claim 1]. We only consider $\hat{v}_0 = 0$, $\hat{\phi}_0 > F_s$ because the other cases are handled similarly. From (11) we have $\dot{\tilde{v}} = \tilde{\phi} - k_d\tilde{v} - F_s + f(\tilde{v})$ with $\tilde{v}_0 = 0$, $\tilde{\phi}_0 > F_s$ so that $\dot{\tilde{v}}(0) > 0$. Hence, there exists $T > 0$ such that for all $t \in (0, T]$, $\tilde{v}(t) > 0$ and $F_s \operatorname{Sign}(\tilde{v}(t)) = \{F_s\}$. Therefore, this unique solution $\tilde{x}$ to (11) substituted in (6) satisfies indeed $\dot{\tilde{x}}(t) \in \hat{\mathcal{F}}_x(\tilde{x}(t))$ for almost all $t \in [0, T]$.

Item (iii). We only consider $\hat{v}_0 = 0$, $\hat{\sigma}_0 > 0$, $\hat{\phi}_0 \in [-F_s, F_s)$ because the other cases are handled similarly. The explicit solution to (12) is then $\tilde{\sigma}(t) = \hat{\sigma}_0 > 0$, $\tilde{\phi}(t) = \hat{\phi}_0 + \hat{\sigma}_0 t$, $\tilde{v}(t) = 0$ on the interval $[0, T] := [0, \frac{F_s - \hat{\phi}_0}{\hat{\sigma}_0}]$. This unique solution $\tilde{x}$ to (12) substituted in (6) satisfies indeed $\dot{\tilde{x}}(t) \in \hat{\mathcal{F}}_x(\tilde{x}(t))$ for almost all $t \in [0, T]$ because for all $t \in [0, T]$ a value of $\operatorname{Sign}(0)$ can be selected such that $0 \in \hat{\phi}_0 + \hat{\sigma}_0 t - F_s \operatorname{Sign}(0)$.

Item (iv). This item is proven as item (ii). □

B. Reset controller design

In order to solve Problem 1, we replace the integrator in (3) and (6) with a *reset* integrator. The integrator performs two types of resets whose design is best explained in the *original* coordinates z (instead of $\hat{x}$). The key mechanism of these resets is to enforce that the integrator control force (given by $\bar{k}_i z_3$) *always points in the direction of the setpoint*, namely

$$z_3(z_1 - r) \ge 0, \tag{18}$$

which imposes an initialization constraint on the integrator state z_3 and is then satisfied along all hybrid solutions of the resulting closed loop. Due to the phase lag associated with a *linear* integrator, property (18) cannot be achieved with a classical PID controller, see, e.g., [33, §1.3].

To obtain well-defined reset conditions ensuring (18), we augment the PID controller dynamics with an extra boolean state $\hat{b} \in \{-1, 1\}$, characterizing whether the mass moves *towards* the setpoint ($\hat{b} = 1$), or *away from* the setpoint ($\hat{b} = -1$, typically occurring after an overshoot of the position error). More precisely, $\hat{b}$ always satisfies

$$\hat{b} z_2(z_1 - r) \le 0, \tag{19}$$

along the hybrid solutions. To ensure (19) (and also (18)) our two types of resets are triggered by a zero crossing of each one of the two factors in (19). The first reset is triggered by the zero-crossing of the position error $z_1 - r$ (marking the start of an overshoot of the position error) and is given by

$$(z_1 - r = 0 \wedge \hat{b} = 1) \implies (z_3^+ = -z_3, \quad \hat{b}^+ = -\hat{b}). \tag{20a}$$

Besides the fact that the reset in (20a) is required to obtain stability of the setpoint, it also induces significant overshoot reduction, as illustrated in Section II-C.

The second reset yields a change of the integrator state z_3 to zero, when the velocity z_2 hits zero *after an overshoot*, i.e.,

$$(z_2 = 0 \wedge \hat{b} = -1) \implies (z_3^+ = 0, \quad \hat{b}^+ = -\hat{b}). \tag{20b}$$

The reset in (20b) is required to obtain asymptotic stability of the setpoint. Indeed, if it were absent, this would not allow the integrator state z_3 to decrease in absolute value, since (18) forces z_3 and $z_1 - r$ in (3) to always have the same sign (and $\dot{z}_3 = z_1 - r$ from (3)). A (sufficiently) large initial condition for z_3 would then hinder asymptotic stability of the setpoint. In summary, the resulting closed-loop system with the proposed reset PID controller is given by (1)-(3), with the resetting laws (20).

C. Illustrative example

We will illustrate the working principle of the proposed reset controller by means of a simulation example, using a numerical time-stepping scheme [2, Chap. 10].

First consider system (1)-(3), where only a *classical* PID controller (3) is employed. The mass m is unitary, the static friction is $\bar{F}_s = 0.981$ N, the viscous friction coefficient α is zero, and the velocity-dependent friction contribution is

$$\bar{f}(z_2) = \begin{cases} L_2 z_2, & |z_2| \le \varepsilon_v \\ (\bar{F}_s - \bar{F}_c)\kappa z_2/(1 + \kappa|z_2|)^{-1}, & |z_2| > \varepsilon_v, \end{cases}$$

with Coulomb friction level $\bar{F}_c = \bar{F}_s/3$, $\kappa = 20$ s/m the Stribeck shape parameter, $L_2 = 12.8$ Ns/m, and $\varepsilon_v = 10^{-3}$ m/s, satisfying Assumption 1. We take $\bar{k}_p = 18$ N/m, $\bar{k}_d = 2$ Ns/m, and $\bar{k}_i = 30$ N/(ms), satisfying Assumption 2. The constant setpoint is $r = 0$, and the initial conditions are $z_1(0) = -0.05$ m, $z_2(0) = 0$ m/s, $z_3(0) = 0$ ms. The position response is presented in the top plot of Fig. 2 (– –), where persistent oscillations (hunting) are evident.

Now consider the *reset* closed loop (1)-(3), (20). The reset controller achieves, first, asymptotic stability of the setpoint $(z_1, z_2) = (r, 0)$ (as we will prove later on), and, second, a significant overshoot reduction as compared to the classical PID response, see the top plot of Fig. 2 (—). Controller resets according to (20a) (i.e., at a zero-crossing of the position error) and according to (20b) (i.e., when the velocity hits zero after the previous reset has occurred) are indicated in the insets. The arising (discontinuous) control force is presented in the middle plot of Fig. 2.

The bottom plot of Fig. 2 is an anticipation for the specific property, established in the next section, that the state $\hat{\phi}$ in (5) never becomes zero when the reset mechanism is active, whereas it keeps crossing zero for the classical PID (the logarithm of $|\hat{\phi}|$ goes to $-\infty$). Notice that $\hat{\phi}$ is reset according to (20b) at increasingly smaller values ($\hat{\phi}^+ = -k_p(z_1 - r)$) as the state approaches the settling condition $z_1 - r = 0$ and

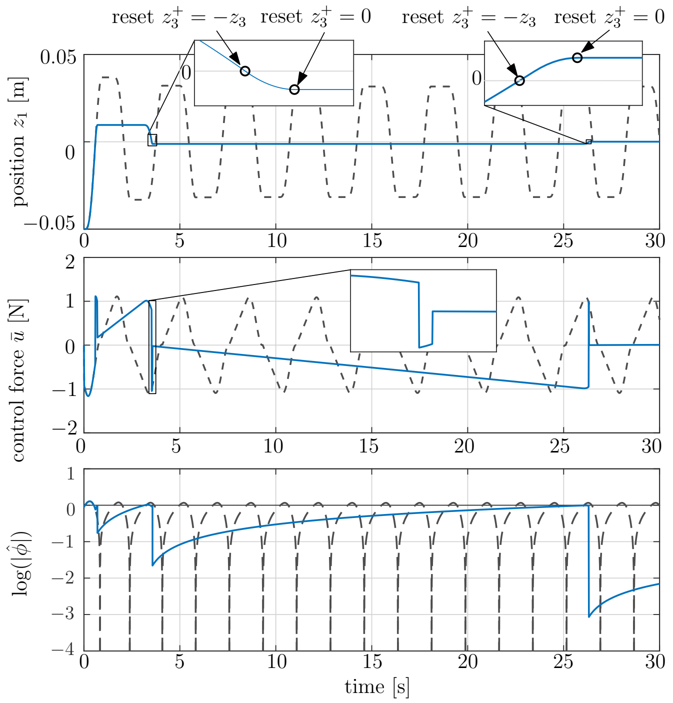

Fig. 2. Simulated response of the position z_1 (top), the control force $\bar{u}$ (middle), and the absolute value of state $\hat{\phi}$ in (5) in logarithmic scale (bottom) for the classical (– –) and reset (—) PID control schemes.

$z_2 = 0$, which is to be expected due to homogeneity of the reset law. Nevertheless, $\hat{\phi}$ never reaches zero (as rigorously established in Proposition 2 of the next section).

III. Main Result

A. Hybrid model formulation and stability theorem

To state our main result, we write the reset closed loop (1)-(3), (20) using the hybrid formalism of [18]. The resulting hybrid system, denoted by $\hat{\mathcal{H}}$, has an augmented state vector $\hat{\xi}$ ranging in a constrained set comprising a correct initialization of the logic variable $\hat{b}$ and the continuous controller state $\hat{\phi}$:

$$\hat{\xi} := (\hat{x}, \hat{b}) := (\hat{\sigma}, \hat{\phi}, \hat{v}, \hat{b}) \in \hat{\Xi} \tag{21a}$$

$$\hat{\Xi} := \{(\hat{x}, \hat{b}) \in \mathbb{R}^3 \times \{-1, 1\} \colon \hat{b}\hat{v}\hat{\sigma} \geq 0,\ \hat{\sigma}\hat{\phi} \geq \tfrac{k_p}{k_i}\hat{\sigma}^2,\ \hat{b}\hat{v}\hat{\phi} \geq 0\}.$$

In $\hat{\Xi}$, the first constraint (inherited from (19)) imposes that $\hat{b}\hat{v}$ and $\hat{\sigma}$ never have opposite signs, while the second constraint (inherited from (18)) imposes that $\hat{\sigma}$ and $\hat{\phi}$ never have opposite signs. With these two constraints in place, one should impose that also $\hat{b}\hat{v}$ and $\hat{\phi}$ never have opposite signs, as ensured by the third constraint characterizing $\hat{\Xi}$. [1]

More specifically, using (4) and (5) to represent (1)-(3), the corresponding closed-loop model (6) augmented with the resets (20) follows the hybrid dynamics

$$\hat{\mathcal{H}}\colon \begin{cases} \dot{\hat{\xi}} \in \hat{\mathcal{F}}(\hat{\xi}), & \hat{\xi} \in \hat{\mathcal{C}} := \hat{\Xi} \quad (21b) \\ \hat{\xi}^+ = \begin{cases} \hat{g}_\sigma(\hat{\xi}), & \text{if } \hat{\xi} \in \hat{\mathcal{D}}_\sigma \\ \hat{g}_v(\hat{\xi}), & \text{if } \hat{\xi} \in \hat{\mathcal{D}}_v, \end{cases} & \hat{\xi} \in \hat{\mathcal{D}} := \hat{\mathcal{D}}_\sigma \cup \hat{\mathcal{D}}_v \quad (21c) \end{cases}$$

[1]Note that the first two constraints in $\hat{\Xi}$ do not imply $\hat{b}\hat{v}\hat{\phi} \geq 0$, because with $\hat{\sigma} = 0$ the first two constrains are satisfied for any (even opposite and nonzero) selections of $\hat{b}\hat{v}$ and $\hat{\phi}$.

Herein, the flow map is given by

$$\hat{\mathcal{F}}(\hat{\xi}) := \begin{bmatrix} -k_i\hat{v} \\ \hat{\sigma} - k_p\hat{v} \\ \hat{\phi} - k_d\hat{v} - F_s\,\mathrm{Sign}(\hat{v}) + f(\hat{v}) \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\mathcal{F}}_x(\hat{x}) \\ 0 \end{bmatrix}, \tag{21d}$$

and the jump maps and jump sets are given by

$$\hat{g}_\sigma(\hat{\xi}) := \begin{bmatrix} \hat{\sigma} \\ -\hat{\phi} \\ \hat{v} \\ -\hat{b} \end{bmatrix}, \quad \hat{g}_v(\hat{\xi}) := \begin{bmatrix} \hat{\sigma} \\ \frac{k_p}{k_i}\hat{\sigma} \\ \hat{v} \\ -\hat{b} \end{bmatrix}, \tag{21e}$$

$$\hat{\mathcal{D}}_\sigma := \{\hat{\xi} \in \hat{\Xi} : \hat{\sigma} = 0, \hat{b} = 1\}, \tag{21f}$$

$$\hat{\mathcal{D}}_v := \{\hat{\xi} \in \hat{\Xi} : \hat{v} = 0, \hat{b} = -1\}, \tag{21g}$$

where we emphasize that $\hat{\mathcal{D}}_\sigma$ and $\hat{\mathcal{D}}_v$ are disjoint, because they correspond to the two different values of $\hat{b}$. $\hat{g}_\sigma$ and $\hat{\mathcal{D}}_\sigma$ correspond to the resetting mechanism in (20a), and $\hat{g}_v$ and $\hat{\mathcal{D}}_v$ to that in (20b).

Based on formulation (21) of the hybrid closed loop (1)-(3), (20), we focus for stability of the setpoint on the compact set defined by all possible equilibria of the flow map (21d):

$$\hat{\mathcal{A}} := \{\hat{\xi} \in \hat{\Xi} \colon \hat{\sigma} = 0, |\hat{\phi}| \leq F_s, \hat{v} = 0\}. \tag{22}$$

Our main result, proven in Section VI-C is stated next.

Theorem 1. *Under Assumptions 1-2, the set $\hat{\mathcal{A}}$ in (22) is globally asymptotically stable (GAS) for $\hat{\mathcal{H}}$ in (21).*

Let us now discuss two distinct features of our reset PID controller (in Proposition 1 and 2 below) that are instrumental for practical implementation of the controller. First, the set $\hat{\Xi}$ in (21a), where solutions are allowed to evolve, is only a subset of $\mathbb{R}^3 \times \{-1, 1\}$ (just as a bouncing ball [18, Ex. 1.1] evolving only in the half-space above the ground). Despite this fact (and just as in a bouncing ball), maximal solutions are complete because $\hat{\mathcal{D}}_\sigma$ and $\hat{\mathcal{D}}_v$ lay at the boundary of $\hat{\Xi}$ and allow solutions to jump, so that they can be indefinitely extended, as formalized next.

Proposition 1. *Hybrid system* (21) *satisfies the hybrid basic conditions of [18, Assumption 6.5]. Moreover, under Assumptions 1-2, all maximal solutions are complete.*

Proof. Verifying the hybrid basic conditions of [18, Assumption 6.5] is straightforward from closedness of sets $\hat{\mathcal{C}}$, $\hat{\mathcal{D}}_\sigma$ and $\hat{\mathcal{D}}_v$, and the regularity properties of $\hat{\mathcal{F}}$, $\hat{g}_\sigma$ and $\hat{g}_v$.

To prove completeness of maximal solutions, first we show that for each $\hat{\xi}_0 = (\hat{\sigma}_0, \hat{\phi}_0, \hat{v}_0, \hat{b}_0) \in \hat{\mathcal{C}} \cup \hat{\mathcal{D}}$ there exists a nontrivial solution $\hat{\xi}$ to $\hat{\mathcal{H}}$ starting from $\hat{\xi}_0$ (i.e., $\mathrm{dom}\,\hat{\xi}$ contains at least one point different from $(0, 0)$). For convenience, we rephrase the conditions in $\hat{\mathcal{C}} = \hat{\Xi}$ in (21a) as

$$h_1(\hat{\xi}) := \hat{b}\hat{v}\hat{\sigma} \geq 0,$$
$$h_2(\hat{\xi}) := \hat{\sigma}\hat{\phi} - \tfrac{k_p}{k_i}\hat{\sigma}^2 \geq 0,$$
$$h_3(\hat{\xi}) := \hat{b}\hat{v}\hat{\phi} \geq 0.$$

We divide into the cases $\hat{b}_0 = 1$ and $\hat{b}_0 = -1$.

For $\hat{b}_0 = 1$, a nontrivial solution exists for $\hat{\xi}_0 \in \hat{\mathcal{D}}_\sigma$, where $\hat{\sigma}_0 = 0$. We then need to show that for each $\hat{\xi}_0 \in \hat{\mathcal{C}} \backslash \hat{\mathcal{D}}_\sigma$, there exists a nontrivial flowing solution (i.e., an absolutely continuous function $\hat{\xi}$: $[0,T] \to \mathbb{R}^4$ with $T > 0$ satisfying $\dot{\hat{\xi}}(t) \in \hat{\mathcal{F}}(\hat{\xi}(t))$ for almost all $t \in [0,T]$, such that $\hat{\xi}(0) = \hat{\xi}_0$ and $\hat{\xi}(t) \in \hat{\mathcal{C}}$ for all $t \in (0,T]$). We then list all possible cases for $\hat{\xi}_0 \in \hat{\mathcal{C}} \backslash \hat{\mathcal{D}}_\sigma$, and show that there exists a nontrivial flowing solution starting from each of these cases.

1) $\hat{\sigma}_0 > 0$, $\hat{v}_0 > 0$, $\hat{\phi}_0 > \frac{k_p}{k_i}\hat{\sigma}_0$ or $\hat{\sigma}_0 < 0$, $\hat{v}_0 < 0$, $\hat{\phi}_0 < \frac{k_p}{k_i}\hat{\sigma}_0$: a nontrivial flowing solution defined as above exists because these points belong to the interior of $\hat{\mathcal{C}}$.

2) $\hat{\sigma}_0 > 0$, $\hat{v}_0 > 0$, $\hat{\phi}_0 = \frac{k_p}{k_i}\hat{\sigma}_0$ or $\hat{\sigma}_0 < 0$, $\hat{v}_0 < 0$, $\hat{\phi}_0 = \frac{k_p}{k_i}\hat{\sigma}_0$: in the former case, we need to verify that the corresponding flowing solution belongs to $\hat{\mathcal{C}}$ in (21b). Since $\hat{\sigma}_0 > 0$, $\hat{v}_0 > 0$, and $\hat{\phi}_0 > 0$, it holds $h_1(\hat{\xi}(t)) > 0$ and $h_3(\hat{\xi}(t)) > 0$ for $t \in [0,T]$ with $T > 0$. Since $\hat{\phi}_0 = \frac{k_p}{k_i}\hat{\sigma}_0$, $h_2(\hat{\xi}(0)) = 0$ and it is sufficient to verify that $\dot{h}_2(\hat{\xi}(0)) > 0$ to conclude the existence of a nontrivial flowing solution. Indeed, $\dot{h}_2(\hat{\xi}(0)) = -k_i\hat{v}_0\hat{\phi}_0 + \hat{\sigma}_0(\hat{\sigma}_0 - k_p\hat{v}_0) - 2\frac{k_p}{k_i}\hat{\sigma}_0(-k_i\hat{v}_0) = \hat{\sigma}_0^2 > 0$. The latter case follows analogously.

3) $\hat{\sigma}_0 > 0$, $\hat{v}_0 = 0$, $\hat{\phi}_0 > \frac{k_p}{k_i}\hat{\sigma}_0$ or $\hat{\sigma}_0 < 0$, $\hat{v}_0 = 0$, $\hat{\phi}_0 < \frac{k_p}{k_i}\hat{\sigma}_0$: in the former case, $\hat{\sigma}_0 > 0$, $\hat{v}_0 = 0$, $\hat{\phi}_0 > \frac{k_p}{k_i}\hat{\sigma}_0 > 0$ can only correspond to an initial condition in (8) or (9) in Lemma 1, which both give rise to $\hat{v}(t) \geq 0$ for all $t \in [0,T]$ by Lemma 1, items (ii) and (iii). Then, it holds $h_1(\hat{\xi}(t)) \geq 0$ and $h_3(\hat{\xi}(t)) \geq 0$ for $t \in [0,T]$ with $T > 0$ (by shrinking $T > 0$ if needed). Moreover, $h_2(\hat{\xi}(0)) > 0$ and a nontrivial flowing solution exists. The latter case follows analogously.

4) $\hat{\sigma}_0 > 0$, $\hat{v}_0 = 0$, $\hat{\phi}_0 = \frac{k_p}{k_i}\hat{\sigma}_0$ or $\hat{\sigma}_0 < 0$, $\hat{v}_0 = 0$, $\hat{\phi}_0 = \frac{k_p}{k_i}\hat{\sigma}_0$: similar to item 3) above. In particular, $h_2(\hat{\xi}(0)) = 0$ and $\dot{h}_2(\hat{\xi}(0)) = -k_i\hat{v}_0\hat{\phi}_0 + \hat{\sigma}_0(\hat{\sigma}_0 - k_p\hat{v}_0) - 2\frac{k_p}{k_i}\hat{\sigma}_0(-k_i\hat{v}_0) = \hat{\sigma}_0^2 > 0$, so it also holds $h_2(\hat{\xi}(t)) \geq 0$ for $t \in [0,T]$ with $T > 0$.

For $\hat{b}_0 = -1$, a nontrivial solution exists for $\hat{\xi}_0 \in \hat{\mathcal{D}}_v$, where $\hat{v}_0 = 0$. We then list all possible cases for $\hat{\xi}_0 \in \hat{\mathcal{C}} \backslash \hat{\mathcal{D}}_v$, and show that there exists a nontrivial flowing solution from each of theses cases.

1) $\hat{v}_0 > 0$, $\hat{\sigma}_0 < 0$, $\hat{\phi}_0 < \frac{k_p}{k_i}\hat{\sigma}_0$ or $\hat{v}_0 < 0$, $\hat{\sigma}_0 > 0$, $\hat{\phi}_0 > \frac{k_p}{k_i}\hat{\sigma}_0$: a nontrivial flowing solution exists because these points belong to the interior of $\hat{\mathcal{C}}$.

2) $\hat{v}_0 > 0$, $\hat{\sigma}_0 < 0$, $\hat{\phi}_0 = \frac{k_p}{k_i}\hat{\sigma}_0$ or $\hat{v}_0 < 0$, $\hat{\sigma}_0 > 0$, $\hat{\phi}_0 = \frac{k_p}{k_i}\hat{\sigma}_0$: in the former case, since $\hat{v}_0 > 0$, $\hat{\sigma}_0 < 0$, and $\hat{\phi}_0 < 0$, it holds that $h_1(\hat{\xi}(t)) > 0$ and $h_3(\hat{\xi}(t)) > 0$ for $t \in [0,T]$ with $T > 0$. Since $\hat{\phi}_0 = \frac{k_p}{k_i}\hat{\sigma}_0$, $h_2(\hat{\xi}(0)) = 0$ and it is sufficient to verify that $\dot{h}_2(\hat{\xi}(0)) > 0$ to conclude the existence of a nontrivial flowing solution. Indeed, $\dot{h}_2(\hat{\xi}(0)) = \hat{\sigma}_0^2 > 0$. The latter case follows analogously.

3) $\hat{v}_0 > 0$, $\hat{\sigma}_0 = 0$, $\hat{\phi}_0 < 0$ or $\hat{v}_0 < 0$, $\hat{\sigma}_0 = 0$, $\hat{\phi}_0 > 0$: in the former case, since $\hat{v}_0 > 0$ and $\hat{\phi}_0 < 0$, it holds that $h_3(\hat{\xi}(t)) > 0$ for $t \in [0,T]$ with $T > 0$. Moreover, from $\dot{\hat{\sigma}} = -k_i\hat{v}$ and $\hat{v}_0 > 0$, we have that $\hat{\sigma}(t) < 0$ for all $t \in (0,T]$ (shrink T if necessary), so that $h_1(\hat{\xi}(t)) \geq 0$ for all $t \in [0,T]$ with $T > 0$. Since $\hat{\sigma}_0 = 0$, $h_2(\hat{\xi}(0)) = 0$ and it is sufficient to verify that $\dot{h}_2(\hat{\xi}(0)) > 0$ to conclude the existence of a nontrivial flowing solution. Indeed, $\dot{h}_2(\hat{\xi}(0)) = -k_i\hat{v}_0\hat{\phi}_0 > 0$. The latter case follows analogously.

4) $\hat{v}_0 > 0$, $\hat{\sigma}_0 = 0$, $\hat{\phi}_0 = 0$ or $\hat{v}_0 < 0$, $\hat{\sigma}_0 = 0$, $\hat{\phi}_0 = 0$: in the former case, $h_1(\hat{\xi}(0)) = h_2(\hat{\xi}(0)) = h_3(\hat{\xi}(0)) = 0$. From $\dot{\hat{\sigma}} = -k_i\hat{v}$, $\dot{\hat{\phi}} = \hat{\sigma} - k_p\hat{v}$ and $\hat{\sigma}_0 = 0$, $\hat{v}_0 > 0$, we have that $\hat{\sigma}(t) < 0$ and $\hat{\phi}(t) < 0$ for all $t \in (0,T]$, so that $h_1(\hat{\xi}(t)) > 0$ and $h_3(\hat{\xi}(t)) > 0$ for all $t \in [0,T]$ with $T > 0$. As for h_2, we take into account that $\hat{\sigma}(t) < 0$ for all $t \in (0,T]$, so we can consider, instead of h_2, the simplified constraint $\tilde{h}_2(\hat{\xi}) := \hat{\phi} - \frac{k_p}{k_i}\hat{\sigma} \leq 0$. $\dot{\tilde{h}}_2(\hat{\xi}) := \hat{\sigma} - k_p\hat{v} - \frac{k_p}{k_i}(-k_i\hat{v}) = \hat{\sigma}$ so that $\dot{\tilde{h}}_2(\hat{\xi}(0)) = 0$, and $\ddot{\tilde{h}}_2(\hat{\xi}) = -k_i\hat{v}$ so that $\ddot{\tilde{h}}_2(\hat{\xi}(0)) < 0$, as we needed to prove. The latter case follows analogously.

Second, we show that solutions are complete through [17, Thm. S3]. $\hat{\mathcal{H}}$ satisfies the Basic Assumptions of [17, p. 43]. [18, Thm. S3, case *(b)*] cannot occur because the flow map is a linear system with bounded input. [18, Thm. S3, case *(c)*] cannot occur because $\hat{g}_\sigma(\hat{\mathcal{D}}_\sigma) \cup \hat{g}_v(\hat{\mathcal{D}}_v) \subset \hat{\mathcal{C}} \cup \hat{\mathcal{D}}$ (as it can be verified through (21e), (21f), (21g)). Then only [18, Thm. S3, case *(a)*] remains, i.e., each solution $\hat{\xi}$ is complete. $\square$

A relevant property enjoyed by the solutions of (21) is that the transformed controller state $\hat{\phi}$ never reaches zero, unless it is initialized at zero or reaches the attractor $\hat{\mathcal{A}}$ in finite time. This fact, useful in Section IV, was illustrated in Section II-C by the bottom plot of Fig. 2 and is formalized next.

Proposition 2. *For $\hat{\mathcal{H}}$ in* (21)*, all solutions $\hat{\xi}$ starting in*

$$\hat{\Xi}_0 := \{\hat{\xi} \in \hat{\Xi} : \ \hat{\phi} \neq 0\} \tag{23}$$

and never reaching $\hat{\mathcal{A}}$, satisfy $\hat{\phi}(t,j) \neq 0$ for all $(t,j) \in \operatorname{dom}\hat{\xi}$.

Proof. The proof amounts to showing that no solution evolving in $\hat{\Xi}_0$ can reach a point where $\hat{\phi} = 0$ after either jumping or flowing, unless it reaches $\hat{\mathcal{A}}$.

Consider solutions flowing in $\hat{\mathcal{C}} := \hat{\Xi}$. If a solution reaches $\hat{\phi} = 0$ while flowing in $\hat{\mathcal{C}}$, there necessarily exists a reverse solution starting at $\hat{\xi}_0 = (\hat{\sigma}_0, \phi_0, \hat{v}_0, \hat{b}_0) = (0, 0, \hat{v}_0, \hat{b}_0) \in \hat{\Xi}$ (with $\hat{\sigma}_0 = 0$ because of constraint $\hat{\sigma}\hat{\phi} \geq \frac{k_p}{k_i}\hat{\sigma}^2$ and $\hat{v}_0 \neq 0$ otherwise the solution would be in $\hat{\mathcal{A}}$, which is ruled out by assumption) and flowing in backward time according to $-\hat{\mathcal{F}}(\hat{\xi})$ (in (21d)) while remaining in $\hat{\Xi}$. However such a reverse solution does not exist as we show next for $\hat{v}_0 > 0$ (the case $\hat{v}_0 < 0$ is analogous). Since $\hat{v}_0 > 0$, $\hat{v}$ remains positive for a small enough backward time interval and the backward dynamics $\dot{\hat{\sigma}} = k_i\hat{v} > 0$ implies that $\hat{\sigma}$ is also positive in that interval. Hence, constraint $\hat{\sigma}\hat{\phi} \geq \frac{k_p}{k_i}\hat{\sigma}^2$ in (21a) becomes $h(\hat{\xi}) := \hat{\phi} - \frac{k_p}{k_i}\hat{\sigma} \geq 0$ for all such (sufficiently small) times. Let us note that $h(\hat{\xi}_0) = 0$ and that in backward time $\dot{h}(\hat{\xi}) = -\hat{\sigma} + k_p\hat{v} - \frac{k_p}{k_i}(k_i\hat{v}) = -\hat{\sigma}$, which is strictly negative for all such (sufficiently small) *nonzero* times. Then, $h(\hat{\xi})$ would become negative and the candidate solution would not remain in $\hat{\Xi}$, therefore its existence is ruled out.

Bearing in mind that solutions cannot reach $\hat{\phi} = 0$ while flowing, unless they reach $\hat{\mathcal{A}}$, we consider then jumps in (21e). No jump from $\hat{\Xi}_0 \cap \hat{\mathcal{D}}_v$ can give $\hat{\phi}^+ = \frac{k_p}{k_i}\hat{\sigma} = 0$, otherwise from the condition $\hat{v} = 0$ in $\hat{\mathcal{D}}_v$ we would obtain $\hat{\xi}^+ \in \hat{\mathcal{A}}$, which is ruled out by assumption. For jumps from $\hat{\Xi}_0 \cap \hat{\mathcal{D}}_v$, the conclusion is obvious since $\hat{\phi}^+ = -\hat{\phi}$. $\square$

B. Robustness properties and experimental implementation

The regularity properties established in Proposition 1, together with the fact that set $\hat{\mathcal{A}}$ in (22) is compact, enable applying the robustness results in [18, Ch. 7]. In particular, the GAS result of Theorem 1 implies robust uniform global stability and uniform global attractivity of $\hat{\mathcal{A}}$. Among other things, the semiglobal practical robustness of stability established in [18, Lemma 7.20] reveals that one should expect a graceful performance degradation in the presence of uncertainties and unmodeled phenomena. Robustness results are especially relevant in view of the next proposition, which provides insight about the behavior of solutions to (21).

Proposition 3. *Each solution $\hat{\xi}$ to* (21) *is such that:*
(i) if it reaches $\hat{\mathcal{A}}$ in finite time, then it remains in $\hat{\mathcal{A}}$ forever (namely, $\hat{\mathcal{A}}$ is strongly forward invariant [18, Def. 6.25]);
(ii) if it never reaches $\hat{\mathcal{A}}$ (namely, $\hat{\xi}(t,j) \notin \hat{\mathcal{A}}$ for all $(t,j) \in \operatorname{dom}(\hat{\xi})$), then it evolves forever in the t direction (namely, $\sup_t \operatorname{dom}\hat{\xi} = +\infty$).

Proof. Item (i) follows[2] by inspecting all possible solutions starting in $\hat{\mathcal{A}}$, which may flow in $\hat{\mathcal{C}}$ or jump from $\hat{\mathcal{D}}_\sigma$ or $\hat{\mathcal{D}}_v$. When flowing in $\hat{\mathcal{C}} \cap \hat{\mathcal{A}}$, Lemma 1(iii) guarantees that $\hat{\sigma}$, $\hat{\phi}$, and $\hat{v}$ stay constant. Across jumps we have $\hat{g}_\sigma(\hat{\mathcal{A}}) \subset \hat{\mathcal{A}}$; $\hat{g}_v(\hat{\mathcal{A}}) \subset \hat{\mathcal{A}}$, which proves item (i). Proving item (ii) requires nontrivial derivations and is done at the end of Section V-B. □

Remark 1. An important consequence of item (ii) Proposition 3 is that no Zeno solutions emerge from model (21) as long as solutions are not in $\hat{\mathcal{A}}$. The absence of Zeno solutions is key to well representing the core continuous-time behavior of the plant. Just as in a bouncing ball [18, Ex. 1.1], however, Zeno solutions emerge inside $\hat{\mathcal{A}}$, and it is expected that frequent and ineffective controller resets occur in practical implementations (due to measurement noise) *when* the closed loop evolution gets close to $\hat{\mathcal{A}}$. To avoid ineffective resets, it is then reasonable and advisable to disable the controller resets whenever the velocity $\hat{v}$ and position error $\hat{\sigma}$ are small enough. In particular, resets should be disabled after resetting from $\hat{\mathcal{D}}_v$ because map $\hat{g}_v$ in (21e) ensures that $\hat{\phi}$ is reset to a small value too whenever $\hat{\sigma}$ is small. A small value of $\hat{\phi}$ yields a small value of the control force, compared to the friction force, which generates robustness against other force disturbances. ⌟

Remark 2. Due to the regularity properties of the hybrid model, we expect solutions to remain close to nominal ones in the presence of perturbations (as in noisy environments). The presence of measurements noise may hinder the detection of the zero crossings of $\hat{\sigma}$ (for jumping from $\hat{\mathcal{D}}_\sigma$) or the zero crossing of $\hat{v}$ (for jumping from $\hat{\mathcal{D}}_v$). An elegant and effective solution for the robust detection of zero crossing stems from Proposition 2 combined with the observations in Remark 1, ensuring that the resetting mechanism is only active outside $\hat{\mathcal{A}}$. In particular, Proposition 2 ensures that as long as we pick initial conditions in $\hat{\Xi}_0$ (that is, from (23), we do not initialize $\hat{\phi} = -k_p(z_1 - r) - k_i z_3$ at zero[3]), $\hat{\phi}$ never reaches zero. Then, exploiting the inequalities characterizing $\hat{\Xi}$ in (21a), we discuss below that solutions starting in $\hat{\Xi}_0$ remain unchanged if the zero-measure sets $\hat{\mathcal{D}}_\sigma$ and $\hat{\mathcal{D}}_v$ are exchanged for the sets

$$\bar{\mathcal{D}}_\sigma := \{\hat{\xi} : \hat{\sigma}\hat{\phi} \leq 0, \hat{b} = 1\} \tag{24}$$

$$\bar{\mathcal{D}}_v := \{\hat{\xi} : \hat{v}\hat{\phi} \geq 0, \hat{b} = -1\}, \tag{25}$$

which satisfy $\bar{\mathcal{D}}_\sigma \cap \hat{\Xi}_0 = \hat{\mathcal{D}}_\sigma \cap \hat{\Xi}_0$ and $\bar{\mathcal{D}}_v \cap \hat{\Xi}_0 = \hat{\mathcal{D}}_v \cap \hat{\Xi}_0$. Since $\hat{\phi}$ is never zero during the transient from Proposition 2, the conditions (24), (25) are effective at robustly detecting the zero crossings of $\hat{\sigma}$ and $\hat{v}$, respectively. In fact, a reset condition similar to (25) has already been successfully used in [8] to robustly detect a zero crossing of the velocity. ⌟

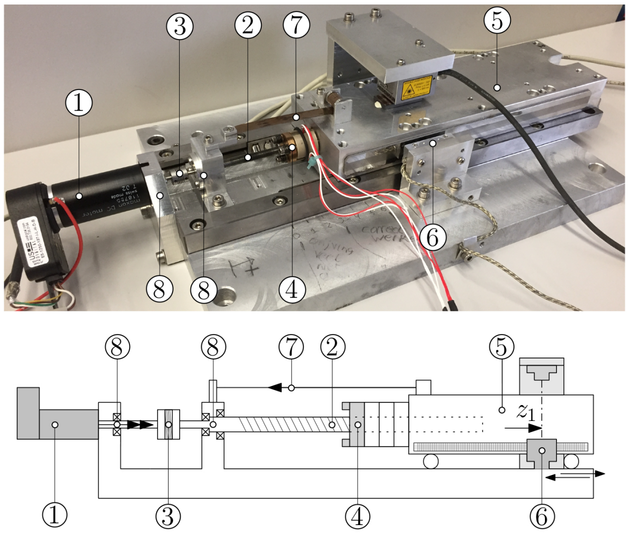

Fig. 3. Experimental setup of a nano-positioning motion stage, representative for a sample manipulation stage in an electron microscope [35].

IV. Experimental case study

We demonstrate the working principle and the effectiveness of the proposed reset controller on an industrial high-precision motion platform consisting of a sample manipulation stage of an electron microscope [35]. First, we show that employing a *classical* (linear) PID controller indeed leads to persistent oscillations (hunting), as pointed out in Section I. Second, we implement the proposed reset controller using the sets (24), (25) introduced in Remark 2 to robustly detect the zero crossings of the position error and velocity. We illustrate 1) the asymptotic stability properties of the reset controller in the presence of friction with unknown static and velocity-dependent contributions (including the Stribeck effect) as established in Theorem 1, and 2) that the overshoot is reduced with respect to the classical PID controller.

A. Experimental setup

The considered experimental setup is shown in Fig. 3. It consists of a Maxon RE25 DC servo motor 1 connected to

[2]Note that item (i) of Proposition 3 is also implied by the stability of $\hat{\mathcal{A}}$ established in Theorem 1, but since this item is instrumental to proving Theorem 1 in Section VI-C, we pursue a different proof to avoid circularity.

[3]When starting the controller with a nonzero position error $z_1 - r \neq 0$ (which is typically the case), the requirement $\hat{\phi} \neq 0$ is easily ensured by initializing the integrator state z_3 at zero.

a spindle 2 via a coupling 3 that is stiff in the rotational direction, while being flexible in the translational direction. The spindle drives a nut 4, transforming the rotary motion of the spindle to a translational motion of the attached carriage 5, with a ratio of $7.96 \cdot 10^{-5}$ m/rad. The position of the carriage is measured by a linear Renishaw encoder 6 with a resolution of 1 nm (and a peak noise level of 4 nm). The carriage is connected to the fixed world with a leaf spring 7, eliminating backlash in the spindle-nut connection. The desired position accuracy to be achieved is 10 nm, as specified by the manufacturer.

For frequencies up to 200 Hz, the system dynamics can be well described by (1), for which Theorem 1 applies when using our reset PID controller. In this case, z_1 represents the position of the carriage. The mass $m = 172.6$ kg represents the transformed inertia of the motor and the spindle (with an *equivalent* mass of 171 kg), plus the mass of the carriage (1.6 kg). The friction force is mainly induced by the bearings supporting the motor axis and the spindle (see 8 in Fig. 3), by the contact between the spindle and the nut, and, to a lesser extent, by the contact between the carriage and the guidance. The contact between the spindle and the nut is lubricated, which induces the Stribeck effect. Since the system is rigid and behaves like a single mass for frequencies up to 200 Hz, these friction forces can be summed up to provide a single net friction characteristic Ψ in (1).

Remark 3. The experimental setup is the same as the setup used in [8, Sec. 5], where Coulomb and viscous friction was dominantly present. For the experiments in this paper, a different carriage position and spindle orientation, and different lubrication conditions result in a significant Stribeck effect instead, as illustrated by the experiments. ⌟

B. Classical PID

Experiments with a classical PID controller (3) have been performed, with controller gains $\bar{k}_p = 10^7$ N/m, $\bar{k}_d = 2 \cdot 10^3$ Ns/m, and $\bar{k}_i = 10^8$ N/(ms), satisfying Assumption 2. Indeed, from (4) the conditions in Assumption 2 are equivalent to $\bar{k}_p > 0$, $\bar{k}_i > 0$, and $\frac{\bar{k}_p(\bar{k}_d+\alpha)}{m} > \bar{k}_i$. The last holds because $\alpha > 0$ and the gains above satisfy $\frac{\bar{k}_p \bar{k}_d}{m} > \bar{k}_i$. The position response and the corresponding control force are visualized in the top and middle plots of Fig. 4 for three different experiments. Persistent oscillations, and thus the lack of stability of the setpoint, are clearly visible, and confirm the presence of a significant Stribeck effect. The bottom plot of Fig. 4 shows that the controller state $\hat{\phi}$ keeps crossing zero (its logarithm becomes negatively unbounded), see also the dashed curve of the lower plot of Fig. 2.

C. Reset PID

We now employ the proposed reset controller, with the same controller gains as for the classical PID case. We use the reset conditions in (24), (25) to robustly detect zero crossings of the position error and the velocity, which are equivalent to the next conditions in the physical coordinates z

$$\bar{\mathcal{D}}_\sigma = \{(z, \hat{b}) \colon \bar{k}_i(z_1 - r)(\bar{k}_p(z_1 - r) + \bar{k}_i z_3) \leq 0,\ \hat{b} = 1\}, \tag{26a}$$

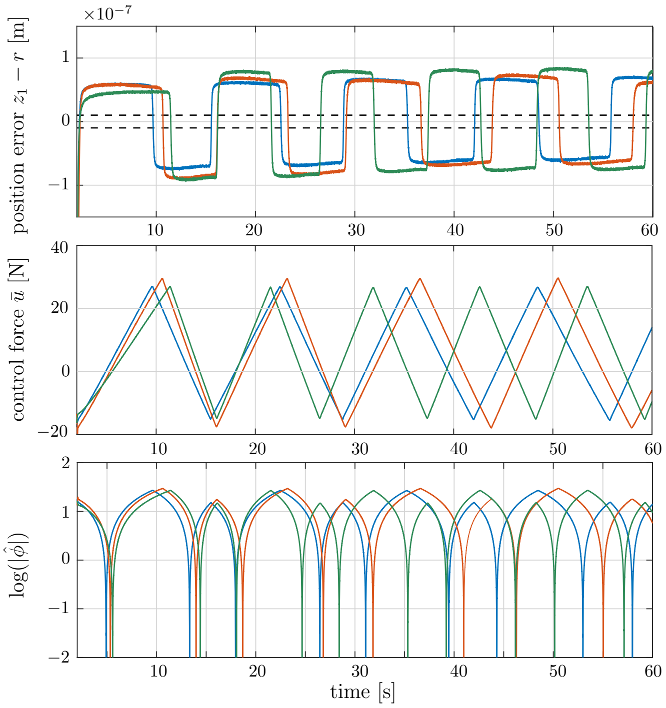

Fig. 4. Responses of position (top), control force (middle) and logarithm of $|\hat{\phi}|$ (bottom) for three experiments with a classical PID controller. The desired accuracy band ((– –) in the top plot) is clearly not achieved with the classical PID controller. The bottom plot shows that $\hat{\phi}$ keeps crossing zero.

$$\bar{\mathcal{D}}_v = \{(z, \hat{b}) \colon z_2(\bar{k}_p(z_1 - r) + \bar{k}_i z_3) \leq 0, \hat{b} = -1\}. \tag{26b}$$

We emphasize that these sets are independent of the mass m, thereby resulting in a simplified implementation. To avoid ineffective resets triggered by measurement noise according to Remark 1, a stopping criterion is used that disables resets when the evolution is close to the setpoint. Specifically, resets are disabled whenever the position error is within the desired accuracy band of 10 nm (i.e., $|z_1 - r| \leq 10$ nm) *after a reset from* $\bar{\mathcal{D}}_v$, because having a low integral control force compared to the static friction yields robustness to other force disturbances.

Consider Fig. 5, reporting in the top and middle plots the position error and control force for three experiments with the proposed reset controller. For comparison purposes we enable the controller resets when the PI control force $\hat{\phi}$ and the position error $\hat{\sigma}$ have the same sign (see (21a)) after the first zero crossing of the position error. The activation times are indicated by the vertical dashed lines. From the top plot we observe that, using the reset enhancements, the system settles within the desired accuracy band of 10 nm after only two resets, the first one from $\bar{\mathcal{D}}_\sigma$ and the second one from $\bar{\mathcal{D}}_v$. The corresponding control force, displayed in the middle subplot, is discontinuous due to the controller resets, as highlighted in the inset. Instead, the classical PID controller does not result in the desired accuracy (cf. Fig. 4). We also emphasize that the controller resets from $\bar{\mathcal{D}}_\sigma$ suppress overshoot.

For all three experiments, the desired accuracy is achieved after the first reset from $\bar{\mathcal{D}}_v$. According to Remark 1, the resets are then deactivated (see the vertical dotted lines in

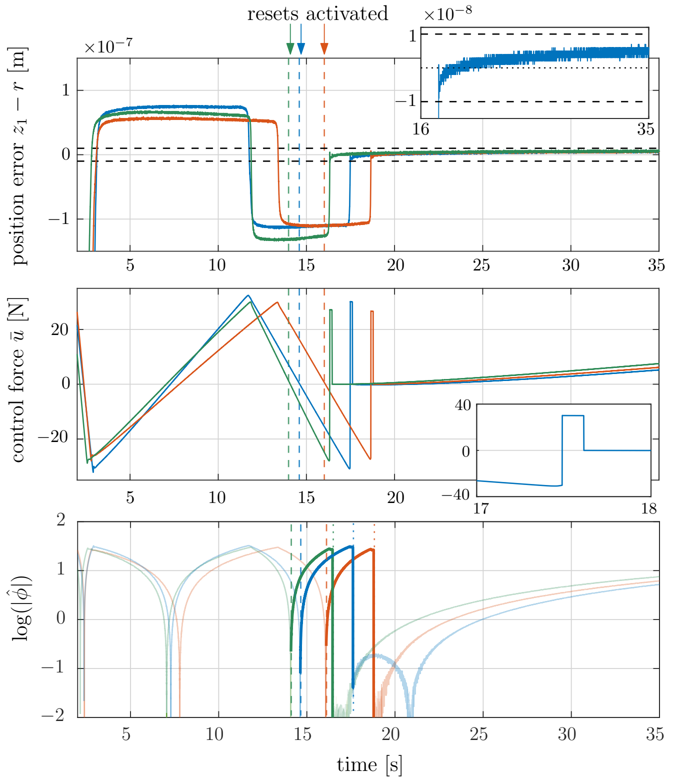

Fig. 5. Responses of position (top), control force (middle) and logarithm of $|\hat{\phi}|$ (bottom) for three experiments with the *reset* PID controller. The bottom plot shows that $\hat{\phi}$ never becomes zero when using resets.

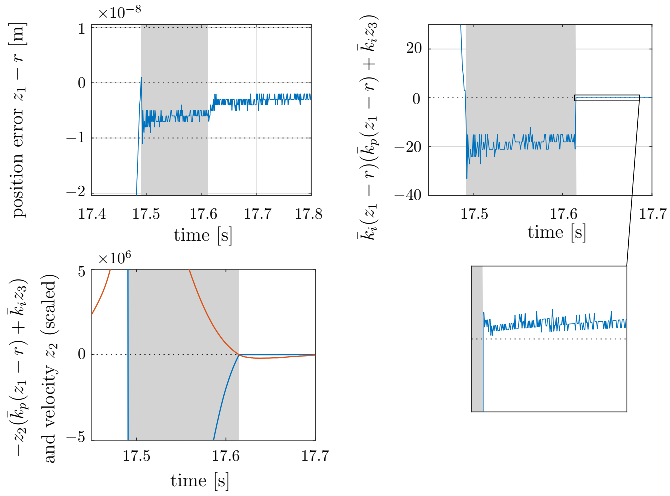

Fig. 6. Zoomed view of a position response (top left), and controller reset conditions (top right and lower left, (—)). The lower left subplot also depicts the velocity signal (—).

the bottom plot). Then, the reset PID is active in the time intervals between the dashed and dotted vertical lines reported in the bottom plot and those intervals correspond to the darker strokes in that same plot. We note, as indicated in Remark 2, that the reset conditions in the jump sets $\bar{\mathcal{D}}_\sigma$ and $\bar{\mathcal{D}}_v$ correctly trigger the controller resets despite the presence of measurement noise. Indeed, as established in Proposition 2, $\hat{\phi}$ never becomes zero while the resets are active (cf. the simulation results in the bottom plot of Fig. 2).

Let us now analyze the response at the nanometer scale. Consider the position error response as a result of the controller resets in more detail, using Fig. 6. In this figure, a time interval where $\hat{b} = -1$ is indicated in gray; its boundaries then indicate two reset instants. Similarly, the white areas correspond to intervals where $\hat{b} = 1$. First, consider the upper left subplot, which shows a zoomed view of the position error of the blue response of Fig. 5. As soon as the error crosses zero at about 17.5 s, a controller reset from $\bar{\mathcal{D}}_\sigma$ is triggered, which toggles the sign of z_3. As a result of stiffness-like effects in the friction characteristic (see [8, Sec. 5], [5, Sec. 2.1]) combined with the sudden (large) change of the control force, a "jump" of the position error is observed, which prevents the system from actually overshooting the setpoint. Despite this unmodeled effect, the hysteresis mechanism embedded in $\hat{b}$ prevents an immediate reset from happening again, thus illustrating the robustness properties discussed in Section III-B. Later, at about 17.6 s, a reset from $\bar{\mathcal{D}}_v$ occurs, which resets z_3 to zero. Once again, due to the stiffness effects, a "jump" of the position error occurs (but lower in magnitude, due to the smaller discontinuity in the control force as compared to the previous reset from $\bar{\mathcal{D}}_\sigma$). We then observe that the position error crosses zero slowly as a result of frictional creep effects (see [8, Sec. 5.4] and [32]), see the inset in the top subplot of Fig. 5. However, the position error remains well within the desired accuracy band of 10 nm, so further resets are disabled according to our stopping criterion.

Next, we analyze the reset conditions in (26a) and (26b) depicted in the upper right and lower left plots of Fig. 6 as a function of time for the blue response in Fig. 5. From the upper right plot, it is evident that indeed a reset from $\bar{\mathcal{D}}_\sigma$ in (26a) occurs at about 17.5 s when $\hat{b} = 1$ and $\bar{k}_i(z_1 - r)(\bar{k}_p(z_1 - r) + \bar{k}_i z_3) \leq 0$, which is satisfied as soon as the position error crosses zero (see also Fig. 5). Because overshoot is prevented due to the frictional stiffness effects, the reset condition $\bar{k}_i(z_1 - r)(\bar{k}_p(z_1 - r) + \bar{k}_i z_3) \leq 0$ remains true after the reset. However, $\hat{b} = -1$ prevents further resets, which shows that the proposed reset controller exhibits further robustness characteristics with respect to such small-scale frictional effects. Consider then the lower left plot, and recall that a reset from $\bar{\mathcal{D}}_v$ in (26b) should occur whenever $\hat{b} = -1$ (satisfied because of the occurrence of the previous reset from $\bar{\mathcal{D}}_\sigma$), and when the velocity hits zero. Detecting the latter is successfully done by evaluating the inequality $-z_2(\bar{k}_p(z_1 - r) + \bar{k}_i z_3) \geq 0$ (see also (25) and Remark 2), even though the velocity signal experiences some lag due to the online, noise-reducing low-pass filtering. Since the error $z_1 - r$ is now within the desired accuracy band, the stopping criterion prevents further resets.

In summary, the use of the proposed reset control strategy results in a high setpoint-accuracy, in contrast to the use of a classical PID controller, which results in persistent oscillations. Moreover, overshoot is suppressed, and the controller reset conditions rely only on measurable signals, causing the controller to reset at the correct instants, despite the presence of measurement noise.

V. Semiglobal properties and bisimulation model

In this section we establish a few important stepping stones towards proving Theorem 1. We first show in Section V-A that solutions to (21) are uniformly globally bounded, which enables proving a semiglobal dwell-time property of solutions in Section V-B. Finally, we define a semiglobal bisimulation model (in the sense of [37, Def. 2.5]) in Section V-C. This model allows proving Theorem 1 in the next Section VI.

A. Uniform global boundedness

Consider the discontinuous Lyapunov-like function

$$W(\hat{\xi}) = \begin{bmatrix}\hat{\sigma}\\ \hat{v}\end{bmatrix}^\top \begin{bmatrix}\frac{k_d}{k_i} & -1\\ -1 & k_p\end{bmatrix} \begin{bmatrix}\hat{\sigma}\\ \hat{v}\end{bmatrix} + \min_{F\in F_s \operatorname{Sign}(\hat{v})} (\hat{b}\hat{\phi} - F)^2, \quad (27)$$

which was used (with $\hat{b}=1$) in [11, Eq. (13)] and [8, Eq. (14)] to prove global attractivity in the case of only Coulomb friction.

Due to its discontinuity, W cannot be used to establish stability, but can be used to prove boundedness of solutions to (21). In particular, for W in (27) it holds that the matrix $\begin{bmatrix}\frac{k_d}{k_i} & -1\\ -1 & k_p\end{bmatrix}$ is positive definite (by Assumption 2), and that [4] for $\hat{b} \in \{-1,1\}$, $\frac{\hat{\phi}^2}{2} - F_s^2 \le \min_{F\in F_s \operatorname{Sign}(\hat{v})}(\hat{b}\hat{\phi} - F)^2 \le 2\hat{\phi}^2 + 2F_s^2$. By these inequalities, we construct the bounds

$$W(\hat{\xi}) \le \bar{c}_W |\hat{x}|^2 + 2F_s^2, \quad |\hat{x}|^2 \le \underline{c}_W W(\hat{\xi}) + \underline{c}_W F_s^2, \quad (28)$$

for some scalars $\bar{c}_W \ge 1$, $\underline{c}_W \ge 1$. Bounds (28) show that boundedness of $W(\hat{\xi})$ is equivalent to boundedness of $|\hat{x}|$.

In the presence of Coulomb friction, function W was shown to enjoy useful non-increase properties in [8], [11]. These properties were key to proving global attractivity. However, these non-increase properties are destroyed here due to the velocity-weakening (Stribeck) contribution f in (21d), which was not considered in [8], [11]. In particular, by defining

$$c_3 := 2(k_p k_d - k_i) > 0 \quad (29)$$

($c_3 > 0$ by Assumption 2), the next lemma provides some useful characterization of the increase/decrease properties of W. Its proof is given in Appendix A.

Lemma 2. *Under Assumptions 1-2, W in (27) with c_3 in (29) enjoys the following properties along dynamics (21).*

1) *For each $p \in \{\sigma, v\}$, we have*

$$W(g_p(\hat{\xi})) - W(\hat{\xi}) \le 0 \quad \forall \hat{\xi} \in \mathcal{D}_p. \quad (30)$$

2) *For all $\hat{\xi} = (\hat{\sigma},\hat{\phi},\hat{v},\hat{b}) \in \mathcal{S}_{\hat{\mathcal{H}}}$ and each flowing interval $I^j := \{t\colon (t,j)\in \operatorname{dom}\hat{\xi}\}$ with $\hat{b}(t_j,j) = -1$,*

$$W(\hat{\xi}(t_2,j)) - W(\hat{\xi}(t_1,j)) \le \int_{t_1}^{t_2} -c_3 \hat{v}(t,j)^2 dt, \quad (31)$$

for all $t_1, t_2 \in I^j$ with $t_1 \le t_2$.

3) *There exists a scalar $\bar{W} > 0$ such that each solution $\hat{\xi} = (\hat{\sigma},\hat{\phi},\hat{v},\hat{b}) \in \mathcal{S}_{\hat{\mathcal{H}}}$ satisfying $\hat{\xi}(t_j, j-1) \in \hat{\mathcal{D}}_v$, jumping to $\hat{\xi}(t_j,j) = \hat{g}_v(\hat{\xi}(t_j,j-1))$ and then flowing up to $\hat{\xi}(t_{j+1},j) \in \hat{\mathcal{D}}_\sigma$ satisfies:*

$$W(\hat{\xi}(t_j,j)) \ge \bar{W} \implies W(\hat{\xi}(t_{j+1},j)) \le W(\hat{\xi}(t_j,j)). \quad (32)$$

While not being suitable for proving attractivity, function W in (27) and Lemma 2 are useful to prove in the next proposition that solutions to (21) are bounded.

Proposition 4. *Under Assumptions 1-2, for each compact set $\mathcal{K}$, there exists $M > 0$ such that each solution $\hat{\xi} \in \mathcal{S}_{\hat{\mathcal{H}}}(\mathcal{K})$ satisfies $\hat{\xi}(t,j) \in M\mathbb{B}$ for all $(t,j) \in \operatorname{dom}\hat{\xi}$.*

Proof. Consider dynamics (21) and notice that the state $\hat{b}$ is bounded because it evolves in a bounded set. Focusing the attention on the remaining states $\hat{x} = (\hat{\sigma},\hat{\phi},\hat{v})$, their flow obeys the (flow) dynamics in (6) where A is Hurwitz due to Assumption 2, and the term multiplying e_3 is bounded by F_s, due to Assumption 1. In particular, from standard bounded-input bounded-output (BIBO) results for linear systems, there exist scalars $k_A \ge 1$ and $h_A > 0$ such that any solution $\hat{\xi} = (\hat{x},\hat{b})$ satisfies [5]

$$|\hat{x}(t,j)|^2 \le k_A |\hat{x}(t_j,j)|^2 + h_A, \quad \forall t \in [t_j, t_{j+1}], \quad (33)$$

where $t_0 = 0$, t_j (with $j \ge 1$) denotes a jump time, and possibly $t_{j+1} = +\infty$ with the last flowing interval being open and unbounded. Consider now a solution to (21) which may: a) flow forever (i.e., experiences no jumps), in which case bound (33) with $j=0$ provides the desired global bound; b) exhibit one jump only, in which case the desired global bound is obtained by concatenating twice bound (33); c) flow and/or jump multiple times, in which case the solution alternately jumps from $\hat{\mathcal{D}}_\sigma$ and $\hat{\mathcal{D}}_v$ (due to the toggling nature of $\hat{b}$). Hence, the solution jumps from $\hat{\mathcal{D}}_v$ at either t_1 or (at most) at t_2. Consider the scenario of a first jump happening from $\hat{\mathcal{D}}_\sigma$ at time $(t_1,0)$, which leads to $|\hat{x}(t_1,1)|^2 = |\hat{x}(t_1,0)|^2$ due to $\hat{g}_\sigma$ in (21e), and then a second jump from $\hat{\mathcal{D}}_v$ at time $(t_2,1)$, which leads to $|\hat{x}(t_2,2)|^2 \le |\hat{x}(t_2,1)|^2$ due to $\hat{g}_v$ in (21e) and $\hat{\mathcal{D}}_v$ in (21g) (indeed, $|\hat{\phi}(t_2,2)| = \frac{k_p}{k_i}|\hat{\sigma}(t_2,1)| \le |\hat{\phi}(t_2,1)|$ from constraint $\hat{\sigma}\hat{\phi} \ge \frac{k_p}{k_i}\hat{\sigma}^2 \ge 0$ in $\hat{\mathcal{D}}_v$, which is equivalent to $|\hat{\sigma}||\hat{\phi}| \ge \frac{k_p}{k_i}|\hat{\sigma}|^2$). For this described scenario, concatenating bounds yields

$$\max_{(t,j)\in \operatorname{dom}\hat{\xi}, t+j \le t_2+2} |\hat{x}(t,j)|^2 \le \bar{k}_A |\hat{x}(0,0)|^2 + \bar{h}_A, \quad (34)$$

where we used $\bar{k}_A := k_A^2 \ge k_A \ge 1$, $\bar{h}_A := h_A(1+k_A) \ge h_A$. This described scenario can be viewed as the worst-case-scenario, because bound (34) also applies to the other scenario where the jump from $\hat{\mathcal{D}}_\sigma$ does not occur and the jump from $\hat{\mathcal{D}}_v$ occurs at t_1, because $\bar{k}_A \ge k_A$ and $\bar{h}_A \ge h_A$. Then, we can consider only this described worst-case-scenario without loss of generality. Inequality (34) hence establishes a uniform bound for all solutions, until a first jump from $\hat{\mathcal{D}}_v$.

To complete the proof we must establish a uniform bound on solutions performing a jump from $\hat{\xi}(t_2,1) \in \hat{\mathcal{D}}_v$. To this end, we use bounds (28) with (33) to arrive at

$$W(\hat{\xi}(t,j)) \le k_W W(\hat{\xi}(t_j,j)) + h_W, \quad \forall t \in [t_j, t_{j+1}], \quad (35)$$

[4] The derivation of the next inequalities is as follows. $\frac{\hat{\phi}^2}{2} - F_s^2 \le \mathrm{dz}^2_{F_s}(\hat{\phi}) = \mathrm{dz}^2_{F_s}(\hat{b}\hat{\phi}) = \min_{F\in[-F_s,F_s]}(\hat{b}\hat{\phi} - F)^2 \le \min_{F\in F_s \operatorname{Sign}(\hat{v})}(\hat{b}\hat{\phi} - F)^2 \le \min_{F=F_s \operatorname{sign}(\hat{v})}(\hat{b}\hat{\phi} - F)^2 \le 2\hat{\phi}^2 + 2F_s^2$.

[5] Note that classical BIBO bounds involve the norm not squared, but those easily extend to (33) by using $(k|x_0| + h)^2 \le 2k^2|x_0|^2 + 2h^2$.

along any flowing solution, where $k_W := \bar{c}_W \underline{c}_W k_A \geq 1$ (since $\bar{c}_W \geq 1$, $\underline{c}_W \geq 1$, and $k_A \geq 1$) and $h_W := \bar{c}_W(k_A \underline{c}_W F_s^2 + h_A) + 2F_s^2 > 0$.

We are now ready to complete bound (34) beyond hybrid time $(t_2, 2)$. We can focus on solutions exhibiting infinitely many jumps without loss of generality, by noting that the analysis also applies to solutions that eventually stop jumping, because the last bound established below in (38)-(39) will hold on the last (unbounded) flowing interval. Given any such solution $\hat{\xi}$ that keeps exhibiting jumps, denote

$$W_0 := W(\hat{\xi}(t_2, 2)) \leq \bar{c}_W(\bar{k}_A|\hat{x}(0,0)|^2 + \bar{h}_A) + 2F_s^2, \quad (36)$$

where we combined (34) and (28). Due to the toggling nature of $\hat{b}$ in dynamics (21), jumps must occur alternatively from $\hat{\mathcal{D}}_v$ at times $(t_2, 1)$, $(t_4, 3)$ and so on (i.e., at jump times $t_2, t_4, \ldots$ with even indices), and from $\hat{\mathcal{D}}_\sigma$ at jump times with odd indices. We proceed by induction. Assume that at time $(t_{2i}, 2i)$ (after a jump from $\hat{\mathcal{D}}_v$) we have

$$W(\hat{\xi}(t_{2i}, 2i)) \leq \max\{k_W \bar{W} + h_W, W_0\}, \quad (37)$$

which is true for $i = 1$ (the base case of induction), because of (36). As for the induction step, (35) yields for $j = 2i$

$$W(\hat{\xi}(t, 2i)) \leq k_W W(\hat{\xi}(t_{2i}, 2i)) + h_W, \ \forall t \in [t_{2i}, t_{2i+1}]. \quad (38)$$

We obtain that $W(\hat{\xi}(t_{2i+1}, 2i)) \leq \max\{k_W \bar{W} + h_W, W(\hat{\xi}(t_{2i}, 2i))\}$ because for $W(\hat{\xi}(t_{2i}, 2i)) < \bar{W}$, it holds that $W(\hat{\xi}(t_{2i+1}, 2i)) \leq k_W \bar{W} + h_W$ (by (38)), and for $W(\hat{\xi}(t_{2i}, 2i)) \geq \bar{W}$, it holds that $W(\hat{\xi}(t_{2i+1}, 2i)) \leq W(\hat{\xi}(t_{2i}, 2i))$ (by (32) in Lemma 2). Then, $W(\hat{\xi}(t_{2i+1}, 2i)) \leq \max\{k_W \bar{W} + h_W, W(\hat{\xi}(t_{2i}, 2i))\}$ can be propagated to the subsequent time interval using the nonincreasing properties of W established in (30) and (31) of Lemma 2, as follows:

$$\begin{aligned} W(\hat{\xi}(t, 2i+1)) \leq \max\{k_W \bar{W} + h_W, W(\hat{\xi}(t_{2i}, 2i))\}, \\ \forall t \in [t_{2i+1}, t_{2(i+1)}]. \end{aligned} \quad (39)$$

Finally, using again the nonincrease property in (30) and bound (37) for $j = 2i$, we obtain

$$\begin{aligned} W(\hat{\xi}(t_{2(i+1)}, 2(i+1))) &\leq \max\{k_W \bar{W} + h_W, W(\hat{\xi}(t_{2i}, 2i))\} \\ &\leq \max\{k_W \bar{W} + h_W, W_0\}, \end{aligned}$$

which corresponds to (37), completes the induction proof, and establishes then that (37) holds for all $i \geq 1$.

Summarizing, we combine bounds (38) and (39) (and then use $k_W \geq 1$, $h_W > 0$, (37), and finally (36)) to obtain for all $(t, j) \in \operatorname{dom} \hat{\xi}$ with $t + j \geq t_2 + 2$,

$$\begin{aligned} W(\hat{\xi}(t, j)) \leq \max\{&k_W(k_W \bar{W} + h_W) + h_W, \\ &k_W\big(\bar{c}_W(\bar{k}_A|\hat{x}(0,0)|^2 + \bar{h}_A) + 2F_s^2\big) + h_W\}. \end{aligned}$$

In other words, W remains uniformly bounded, so does $\hat{x}$ (by (28)), and $\hat{\xi}$ (since $\hat{b}$ evolves in $\{-1, 1\}$), and the proof of uniform boundedness of solutions is completed. ☐

B. Semiglobal dwell time

We establish now a second useful property of solutions of $\hat{\mathcal{H}}$, whose stick-to-slip transitions must occur at instants of time separated by a guaranteed dwell-time. This peculiar dwell time is uniform in any compact set of initial conditions, therefore it is semiglobal.

To formalize our dwell-time result, define the sets

$$\begin{aligned} \hat{\mathcal{S}}_1 &:= \{\hat{\xi} \in \hat{\Xi} : \hat{\phi} \geq F_s, \hat{v} = 0, \hat{b} = 1\}, \\ \hat{\mathcal{S}}_{-1} &:= \{\hat{\xi} \in \hat{\Xi} : \hat{\phi} \leq -F_s, \hat{v} = 0, \hat{b} = 1\}, \\ \hat{\mathcal{S}}_0 &:= \{\hat{\xi} \in \hat{\Xi} : \hat{\phi} = \tfrac{k_p}{k_i}\hat{\sigma}, |\hat{\phi}| < F_s, \hat{v} = 0, \hat{b} = 1\}. \end{aligned} \quad (40)$$

The first two intuitively associated with stick-to-slip transitions, see also (7) and the third one completing the image of $\hat{\mathcal{D}}_v$ through $\hat{g}_v$. We show in the next proposition that any solution visiting these sets enjoys a uniform semiglobal dwell time before its velocity changes sign, unless it reaches the attractor $\hat{\mathcal{A}}$, where it will remain due to Proposition 3(i).

Proposition 5. *Let Assumptions 1-2 hold. For each compact set $\mathcal{K}$, there exists $\delta(\mathcal{K}) > 0$ such that each solution $\hat{\xi} = (\hat{\sigma}, \hat{\phi}, \hat{v}, \hat{b}) \in \mathcal{S}_{\hat{\mathcal{H}}}(\mathcal{K})$ with $\hat{\xi}(t, j) \in \hat{\mathcal{S}}_1 \cup \hat{\mathcal{S}}_{-1} \cup \hat{\mathcal{S}}_0$, satisfies* either
(i) $\hat{\xi}(t', j') \in \hat{\mathcal{A}}$ for some $t' \in [t, t + \delta(\mathcal{K})]$, or
(ii) if (i) does not hold, then for each $\tau \in [t, t + \delta(\mathcal{K})]$ we have $(\tau, j(\tau)) \in \operatorname{dom} \hat{\xi}$ and

$$\begin{aligned} \hat{\xi}(t, j) \in \hat{\mathcal{S}}_1 &\implies \hat{v}(\tau, j(\tau)) \geq 0, \\ \hat{\xi}(t, j) \in \hat{\mathcal{S}}_{-1} &\implies \hat{v}(\tau, j(\tau)) \leq 0, \end{aligned}$$

for all such $\tau \in [t, t + \delta(\mathcal{K})]$.

To the end of proving Proposition 5, we state the following lemma, where L_2 is defined in Assumption 1(iv), and whose proof is given in Appendix B.

Lemma 3. *Let Assumptions 1-2 hold.*
(a) For each $M > 0$, there exists $\delta_0(M) > 0$ such that for each initial condition $\tilde{x}_0 = (\tilde{\sigma}_0, \tilde{\phi}_0, 0) \in M\mathbb{B}$, the unique solution $\tilde{x}$ (with $\tilde{x}(0) = \tilde{x}_0$) to (11) *coincides over* $[0, \delta_0(M)]$ *with the unique solution $\check{x}$ (with $\check{x}(0) = \tilde{x}_0$) to*

$$\dot{\check{x}} = A\check{x} - e_3(F_s - L_2\check{v}). \quad (41)$$

(b) There exists $\delta_1 > 0$ such that for each initial condition $\check{x}_0 = (\check{\sigma}_0, \check{\phi}_0, 0)$ with

$$\check{\sigma}_0 \geq 0, \check{\phi}_0 \geq F_s, \begin{bmatrix} \check{\sigma}_0 \\ \check{\phi}_0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ F_s \end{bmatrix} \quad (42)$$

($\check{\sigma}_0 \leq 0, \check{\phi}_0 \leq -F_s, \begin{bmatrix} \check{\sigma}_0 \\ \check{\phi}_0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ -F_s \end{bmatrix}$, respectively), the unique solution $\check{x}$ (with $\check{x}(0) = \check{x}_0$) to (41) *satisfies for all* $t \in (0, \delta_1]$, *$\check{v}(t) > 0$ and $\check{\phi}(t) > F_s$ ($\check{v}(t) < 0$ and $\check{\phi}(t) < -F_s$, respectively).*

Proof of Proposition 5. Consider first the case $\hat{\xi}(t, j) \in \hat{\mathcal{S}}_1$.

If $\hat{\xi}(t, j) = (0, F_s, 0, 1) \in \hat{\mathcal{S}}_1$, $\hat{\xi}(t, j) = (0, F_s, 0, 1) \in \hat{\mathcal{A}}$, and the solution satisfies case *(i)* of the lemma. We consider then $\hat{\xi}(t, j) \neq (0, F_s, 0, 1)$ in the rest of the proof.

By Proposition 4, for each compact set $\mathcal{K}$, there exists $M > 0$ such that for all $(t, j) \in \operatorname{dom} \hat{\xi}$ when $\hat{\xi}(t, j) \in \hat{\mathcal{S}}_1$, $\hat{\xi}(t, j) \in \hat{\mathcal{S}}_1 \cap M\mathbb{B}$. Define $\delta'(\mathcal{K}) := \min\{\delta_0(M), \delta_1\} > 0$, with $\delta_0(M)$ and δ_1 as in Lemma 3.

Evolution with only flow.
Suppose $\hat{\xi} = (\hat{x}, \hat{b})$ with $\hat{\xi}(t,j) \in \hat{\mathcal{S}}_1 \backslash \{(0, F_s, 0, 1)\} \cap M\mathbb{B}$ flows on $[t, t+\delta'(\mathcal{K})]$.

Since $\hat{\xi}(t,j) \in \hat{\mathcal{S}}_1 \backslash \{(0, F_s, 0, 1)\} \cap M\mathbb{B}$, it holds that $\hat{x}(t,j) = (\hat{\sigma}(t,j), \hat{\phi}(t,j), 0) \in M\mathbb{B}$. Then, Lemma 3(a) ensures that the unique solution $\tilde{x}$ (with $\tilde{x}(t) = \hat{x}(t,j)$) to (11) coincides over the interval $[t, t+\delta'(\mathcal{K})]$ with the unique solution $\breve{x}$ (with $\breve{x}(t) = \hat{x}(t,j)$) to (41), which is such that $\breve{v}(\tau) > 0$ and $\breve{\phi}(\tau) > F_s$ for all $\tau \in (t, t+\delta'(\mathcal{K})]$ by Lemma 3(b) because $\breve{x}(t) = \hat{x}(t,j)$ satisfies (42) (by combining conditions $\hat{\phi} \geq F_s$ and $\hat{\sigma}\hat{\phi} \geq \frac{k_p}{k_i}\hat{\sigma}^2 \geq 0$ in $\hat{\mathcal{S}}_1$).

Since $\hat{\xi}$ flows according to (21d), its component $\hat{x}$ satisfies (6). Solutions to (6) are unique by Lemma 1(i). Since $\tilde{x}$ satisfies the conditions in (8) for all $\tau \in [t, t+\delta'(\mathcal{K})]$, the component $\hat{x}$ of $\hat{\xi}$ must coincide with $\tilde{x}$ on the interval $[t, t+\delta'(\mathcal{K})]$. Hence, $(\tau, j(\tau)) \in \operatorname{dom} \hat{\xi}$, $\hat{v}(\tau, j(\tau)) \geq 0$ and $\hat{\phi}(\tau, j(\tau)) \geq F_s$ for all $\tau \in [t, t+\delta'(\mathcal{K})]$, so the solution $\hat{\xi}$ satisfies case *(ii)* of the proposition.

Evolution with flow and jumps.
The only other possible evolution of $\hat{\xi}$ entails a jump from $\hat{\mathcal{D}}_\sigma$ for some $\tau_1 \in [t, t+\delta'(\mathcal{K}))$ such that $\hat{\sigma}(\tau_1, j) = 0$ (the solution $\hat{\xi}$ cannot jump from $\hat{\mathcal{D}}_v$ due to $\hat{b}(t,j) = 1$ and $\dot{\hat{b}} = 0$ in (21d)). Since $[t, \tau_1] \subset [t, t+\delta'(\mathcal{K})]$, we know from "Evolution with only flow" above that $\hat{v}(\tau_1, j) \geq 0$ and $\hat{\phi}(\tau_1, j) \geq F_s$ if $\hat{\xi}$ flows in $\hat{\mathcal{C}}$ before jumping from $\hat{\mathcal{D}}_\sigma$. Then, by $\hat{g}_\sigma$ in (21e), $\hat{\sigma}(\tau_1, j+1) = \hat{\sigma}(\tau_1, j) = 0$, $\hat{\phi}(\tau_1, j+1) = -\hat{\phi}(\tau_1, j) \leq -F_s$, $\hat{v}(\tau_1, j+1) = \hat{v}(\tau_1, j) \geq 0$, $\hat{b}(\tau_1, j+1) = -\hat{b}(\tau_1, j) = -1$. Define τ_2 as the time $\tau_2 \geq \tau_1$ such that

$$\hat{v}(\tau, j+1) > 0 \text{ for all } \tau \in (\tau_1, \tau_2), \text{ and } \hat{v}(\tau_2, j+1) = 0. \quad (43)$$

Note that $\tau_2 = \tau_1$ is not excluded. The solution $\hat{\xi}$ can only flow on (τ_1, τ_2) since, with $\hat{b}(\tau_1, j+1) = -1$, jumps can only occur from $\hat{\mathcal{D}}_v$ where $\hat{v}$ has to be 0. Moreover, from (43), for all $\tau \in [\tau_1, \tau_2]$

$$\begin{aligned}
\hat{\sigma}(\tau, j+1) &= \hat{\sigma}(\tau_1, j+1) + \int_{\tau_1}^{\tau} -k_i \hat{v}(\tilde{\tau}, j+1) d\tilde{\tau} \leq 0 \\
\hat{\phi}(\tau, j+1) &= \hat{\phi}(\tau_1, j+1) + \int_{\tau_1}^{\tau} (\hat{\sigma}(\tilde{\tau}, j+1) - k_p \hat{v}(\tilde{\tau}, j+1)) d\tilde{\tau} \\
&\leq \hat{\phi}(\tau_1, j+1) \leq -F_s,
\end{aligned}$$

hence

$$\begin{aligned}
&\hat{v}(\tau_2, j+1) = 0, \ \hat{\sigma}(\tau_2, j+1) \leq 0, \\
&\hat{\phi}(\tau_2, j+1) \leq -F_s, \begin{bmatrix} \hat{\sigma}(\tau_2, j+1) \\ \hat{\phi}(\tau_2, j+1) \end{bmatrix} \neq \begin{bmatrix} 0 \\ -F_s \end{bmatrix}
\end{aligned} \quad (44)$$

where the solution satisfies case *(i)* of the proposition in case $\begin{bmatrix} \hat{\sigma}(\tau_2, j+1) \\ \hat{\phi}(\tau_2, j+1) \end{bmatrix} = \begin{bmatrix} 0 \\ -F_s \end{bmatrix}$.

We rule out the possibility that $\hat{\xi}$ flows from (44) at $(\tau_2, j+1)$. Indeed, if $\hat{\xi}$ flowed, there exist $T > 0$ by Lemma 1(iv) such that the component $\hat{x}$ of $\hat{\xi}$ coincides over $[\tau_2, \tau_2 + T]$ with the unique solution $\tilde{x}$ to (13) with $\tilde{x}(\tau_2) = \hat{x}(\tau_2, j+1)$, which satisfies $\tilde{v}(\tau) < 0$ for all $\tau \in (\tau_2, \tau_2 + T]$. Such a flowing evolution, however, is not possible because the condition $\hat{b}\hat{v}\hat{\phi} \geq 0$ would be violated on $(\tau_2, \tau_2 + T]$ (shrink T if needed) since $\hat{b}(\tau_2, j+1) = -1$. Then, completeness of maximal solutions in Proposition 1 concludes that the only possible evolution from (44) at $(\tau_2, j+1)$ is a jump from $\hat{\mathcal{D}}_v$.

Now consider two cases for $\hat{\sigma}(\tau_2, j+1)$ in (44) by defining

$$\hat{\sigma}_{\text{th}} := \frac{F_s}{2} \frac{k_i}{k_p} > 0 \text{ and } \delta'' := \frac{F_s}{2\hat{\sigma}_{\text{th}}} = \frac{k_p}{k_i} > 0, \quad (45)$$

thanks to Assumption 2.

Evolution with flow and jumps: $\hat{\sigma}(\tau_2, j+1) \in [-\hat{\sigma}_{\text{th}}, 0]$.
By $\hat{g}_v$ in (21e), $\hat{\sigma}(\tau_2, j+2) = \hat{\sigma}(\tau_2, j+1) \in [-\hat{\sigma}_{\text{th}}, 0]$, $\hat{\phi}(\tau_2, j+2) = \frac{k_p}{k_i}\hat{\sigma}(\tau_2, j+1) \in \left[-\frac{F_s}{2}, 0\right]$ and $\hat{b}(\tau_2, j+2) = 1$. If $\hat{\sigma}(\tau_2, j+2) = 0$, then the solution satisfies case *(i)* of the proposition. Otherwise, no jump can occur over $[\tau_2, \tau_2 + \delta'')$ with δ'' in (45), and $\hat{v}(\tau, j+2) = 0$ for all $\tau \in [\tau_2, \tau_2 + \delta'']$ by Lemma 1(iii). Then, $(\tau, j(\tau)) \in \operatorname{dom} \hat{\xi}$ and $\hat{v}(\tau, j(\tau)) \geq 0$ for all $\tau \in [t, \tau_2 + \delta'']$ (with $\tau_2 \geq t$ from before), so the solution satisfies case *(ii)* of the proposition.

Evolution with flow and jumps: $\hat{\sigma}(\tau_2, j+1) \in (-\infty, -\hat{\sigma}_{\text{th}})$.
Recall that $\hat{\sigma}(\tau_1, j+1) = 0$ and note that for all $\tau \in [\tau_1, \tau_2]$,

$$|\dot{\hat{\sigma}}(\tau, j+1)| \leq |\dot{\hat{x}}(\tau, j+1)| \leq |A|M + F_s,$$

from (21d), Assumption 1, and Proposition 4. Hence, from $\hat{\sigma}(\tau_2, j+1) = \hat{\sigma}(\tau_1, j+1) + \int_{\tau_1}^{\tau_2} \dot{\hat{\sigma}}(\tau, j+1) d\tau = \int_{\tau_1}^{\tau_2} \dot{\hat{\sigma}}(\tau, j+1) d\tau$, we have

$$|\hat{\sigma}(\tau_2, j+1)| \leq (|A|M + F_s)(\tau_2 - \tau_1). \quad (46)$$

Since $|\hat{\sigma}(\tau_2, j+1)| \geq \hat{\sigma}_{\text{th}}$, (46) implies

$$\begin{aligned}
&(|A|M + F_s)(\tau_2 - \tau_1) \geq \hat{\sigma}_{\text{th}} \\
&\qquad \iff \tau_2 - \tau_1 > \frac{\hat{\sigma}_{\text{th}}}{|A|M + F_s} =: \delta'''(\mathcal{K}) > 0.
\end{aligned}$$

Then, $(\tau, j(\tau)) \in \operatorname{dom} \hat{\xi}$ and $\hat{v}(\tau, j(\tau)) \geq 0$ for all $\tau \in [t, \tau_1 + \delta'''(\mathcal{K})]$ (with $\tau_1 \geq t$ from before), so the solution satisfies case *(ii)* of the proposition.

The proof of the case $\hat{\xi}(t,j) \in \hat{\mathcal{S}}_1$ is completed by selecting $\delta(\mathcal{K}) := \min\{\delta'(\mathcal{K}), \delta'', \delta'''(\mathcal{K})\} > 0$. The case $\hat{\xi}(t,j) \in \hat{\mathcal{S}}_{-1}$ follows parallel arguments and is omitted.

Consider now the case $\hat{\xi}(t,j) \in \hat{\mathcal{S}}_0$, which is only sketched because the proof is similar in nature to the previous one but simpler. In this case two things may happen: either $|\hat{\phi}| = \frac{k_p}{k_i}|\hat{\sigma}|$ is smaller than $\frac{F_s}{2}$ and then the solution must remain in a stick phase from where it cannot jump (because jumps only from $\hat{\mathcal{D}}_\sigma$ are allowed with $\hat{b} = 1$, and these jumps would bring the solution to $\hat{\mathcal{A}}$, which is ruled out by assumption); or otherwise $|\hat{\phi}| = \frac{k_p}{k_i}|\hat{\sigma}|$ is not smaller than $\frac{F_s}{2}$, which implies that no jump can happen before some uniform amount of time because $|\hat{\sigma}|$ is bounded away from zero and $\dot{\sigma}$ is bounded. □

Based on the previous results we are now ready to complete the missing proof of item (ii) of Proposition 3.

Proof of item (ii) of Proposition 3. The proof uses Propositions 1 and 5. In particular, each solution starts in some compact set $\mathcal{K}$ and after any jump from $\hat{\mathcal{D}}_v$ it lands in the set $\hat{\mathcal{S}}_1 \cup \hat{\mathcal{S}}_{-1} \cup \hat{\mathcal{S}}_0$. From this set, Proposition 5 implies that it flows for some uniform time interval $\delta(\mathcal{K})$ (unless it reaches $\hat{\mathcal{A}}$ and nothing needs to be proven). Due to the hysteresis mechanism enforced by the toggling $\hat{b}$, jumps are alternating from $\hat{\mathcal{D}}_v$ and $\hat{\mathcal{D}}_\sigma$ and the guaranteed flow $\delta(\mathcal{K})$ after each jump from $\hat{\mathcal{D}}_v$ implies that these solutions (which are complete due to Proposition 1) flow forever. Similarly, any solution performing a finite number of jumps, must flow forever due to Proposition 1. □

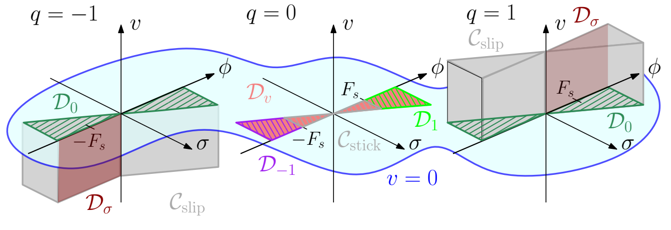

Fig. 7. Projections to the (σ, ϕ, v) space of the flow and jump sets in (47f), indicating the sector condition $\sigma\phi \geq \frac{k_p}{k_i}\sigma^2$.

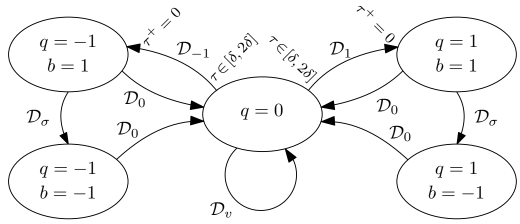

Fig. 8. Hybrid-automaton illustration of (47).

C. Semiglobal bisimulation model

Based on the results of Section V-B and inspired by the proof given in [10] for the case of only Coulomb friction, we now introduce a hybrid model being semiglobally bisimilar to (21), in the sense of [37, Def. 2.5] (see also [29]). This model is the key tool used in Section VI to prove Theorem 1. More specifically, recalling the (arbitrarily large) compact set $\mathcal{K}$ discussed in Section V-B (see Proposition 5), the bisimulation model is parametric in $\delta > 0$ (where parameter δ captures the $\delta(\mathcal{K})$ of Section V-B) and from Proposition 5 we can prove that its outputs are semiglobally coincident with the solutions to (21). This bisimilarity property allows proving Theorem 1, because for each $\delta > 0$, the bisimulation model admits an intuitive and elegant Lyapunov function certifying asymptotic stability. Based on the hybrid description of Coulomb friction presented in [10], we now present the bisimulation model $\mathcal{H}_\delta$ parameterized by $\delta > 0$. The overall state of $\mathcal{H}_\delta$ is

$$\begin{aligned} &\xi := (\sigma, \phi, v, b, q, \tau) \in \Xi, \\ &\Xi := \big\{\xi \in \mathbb{R}^3 \times \{-1, 1\} \times \{-1, 0, 1\} \times [0, 2\delta] : \\ &\qquad qv \geq 0, bq\sigma \geq 0, \sigma\phi \geq \tfrac{k_p}{k_i}\sigma^2, bq\phi \geq 0\big\}. \end{aligned} \tag{47a}$$

With respect to the state $\hat{\xi}$ of $\hat{\mathcal{H}}$ in (21), we add the logical state $q \in \{-1, 0, 1\}$ (whose sign is never opposite to the sign of v due to the constraints in Ξ), and the timer τ, ranging in the compact set $[0, 2\delta]$. The constrained dynamics of $\mathcal{H}_\delta$ are

$$\mathcal{H}_\delta : \begin{cases} \dot{\xi} = \mathcal{F}(\xi), & \xi \in \mathcal{C}_{\text{slip}} \cup \mathcal{C}_{\text{stick}} \qquad (47b) \\ \xi^+ \in \mathcal{G}(\xi), & \xi \in \displaystyle\bigcup_{p \in \{\sigma, v, 0, 1, -1\}} \mathcal{D}_p. \qquad (47c) \end{cases}$$

The flow and jump maps $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{H}_\delta$ are defined as

$$\mathcal{F}(\xi) := \begin{bmatrix} -k_i v \\ \sigma - k_p v \\ -k_d v + |q|\phi - q(F_s - |f(v)|) \\ 0 \\ 0 \\ 1 - \mathrm{dz}_1(\tau/\delta) \end{bmatrix}, \tag{47d}$$

$$\mathcal{G}(\xi) := \bigcup_{p \in \{\sigma, v, 0, 1, -1\} :\, \xi \in \mathcal{D}_p} \{g_p(\xi)\}, \tag{47e}$$

$$\begin{aligned} g_\sigma(\xi) &:= \begin{bmatrix} \sigma & -\phi & v & -b & q & \tau \end{bmatrix}^\top, \\ g_v(\xi) &:= \begin{bmatrix} \sigma & \frac{k_p}{k_i}\sigma & v & -b & q & \tau \end{bmatrix}^\top, \\ g_0(\xi) &:= \begin{bmatrix} \sigma & \phi & v & b & 0 & \tau \end{bmatrix}^\top, \\ g_1(\xi) &:= \begin{bmatrix} \sigma & \phi & v & b & 1 & 0 \end{bmatrix}^\top, \\ g_{-1}(\xi) &:= \begin{bmatrix} \sigma & \phi & v & b & -1 & 0 \end{bmatrix}^\top. \end{aligned}$$

The flow and jump sets of $\mathcal{H}_\delta$ are defined as

$$\begin{aligned} \mathcal{C}_{\text{slip}} &:= \{\xi \in \Xi \colon |q| = 1\}, \\ \mathcal{C}_{\text{stick}} &:= \{\xi \in \Xi \colon v = 0, |\phi| \leq F_s, q = 0\}, \\ \mathcal{D}_\sigma &:= \{\xi \in \Xi \colon \sigma = 0, b = 1, |q| = 1\}, \\ \mathcal{D}_v &:= \{\xi \in \Xi \colon v = 0, b = -1, q = 0\}, \\ \mathcal{D}_0 &:= \{\xi \in \Xi \colon v = 0, |q| = 1\}, \\ \mathcal{D}_1 &:= \{\xi \in \Xi \colon v = 0, \phi \geq F_s, b = 1, q = 0, \tau \in [\delta, 2\delta]\}, \\ \mathcal{D}_{-1} &:= \{\xi \in \Xi \colon v = 0, \phi \leq -F_s, b = 1, q = 0, \tau \in [\delta, 2\delta]\}, \end{aligned} \tag{47f}$$

and are visualized in Fig. 7. Based on (47f), we define

$$\mathcal{C} := \mathcal{C}_{\text{slip}} \cup \mathcal{C}_{\text{stick}}, \;\; \mathcal{D} := \mathcal{D}_\sigma \cup \mathcal{D}_v \cup \mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_{-1}. \tag{47g}$$

A hybrid automaton corresponding to $\mathcal{H}_\delta$ is in Fig. 8.

We establish in Proposition 6 below that $\mathcal{H}_\delta$ in (47) captures all solutions to the original closed-loop model $\hat{\mathcal{H}}$ in (21) in a semiglobal fashion, which verifies the semiglobal bisimulation between the two models $\mathcal{H}_\delta$ and $\hat{\mathcal{H}}$. Importantly, the next proposition allows extending semiglobally the stability properties of $\mathcal{H}_\delta$ to $\hat{\mathcal{H}}$. For a hybrid solution ψ, we use in the proposition the notation $j(t) := \min_{(t,k) \in \operatorname{dom} \psi} k$. With a slight abuse of notation we use a unified symbol $j(\cdot)$ because the solution under consideration is always clear from the context.

Proposition 6. *Let Assumptions 1-2 hold. For each compact set $\mathcal{K}$ and the corresponding $\delta(\mathcal{K}) > 0$ characterized in Proposition 5, for each solution $\hat{\xi} = (\hat{\sigma}, \hat{\phi}, \hat{v}, \hat{b})$ to $\hat{\mathcal{H}}$ with $\hat{\xi}(0,0) = \hat{\xi}_0 \in \mathcal{K}$, there exist q_0, τ_0 and a solution $\xi = (\sigma, \phi, v, b, q, \tau)$ to $\mathcal{H}_{\delta(\mathcal{K})}$ starting at $\xi(0,0) = (\hat{\xi}_0, q_0, \tau_0)$, such that*

$$\begin{aligned} &\hat{\sigma}(t, j(t)) = \sigma(t, j(t)),\, \hat{\phi}(t, j(t)) = \phi(t, j(t)), \\ &\hat{v}(t, j(t)) = v(t, j(t)),\, \hat{b}(t, j(t)) = b(t, j(t)), \end{aligned} \tag{48}$$

for all $t \geq 0$ such that $\hat{\xi}(t, j(t)) \notin \hat{\mathcal{A}}$.

Proof. First note that strong forward invariance of $\hat{\mathcal{A}}$ as per Proposition 3(i) implies that for any solution $\hat{\xi}$, property $\hat{\xi}(t, j(t)) \notin \hat{\mathcal{A}}$ implies $\hat{\xi}(s, j(s)) \notin \hat{\mathcal{A}}$ for all $s \leq t$. Hence, the semiglobal dwell time conclusions of Proposition 5 apply for the considered time instants t in (48).

It is apparent that the timer τ (i) does not affect the flow or jump maps of components (σ, ϕ, v, b, q) in (47d) and (47e); (ii) it may inhibit jumps *only* from $\mathcal{D}_1$ or $\mathcal{D}_{-1}$, see (47g) and the graphical representation in Fig. 8. Due to this reason, we begin by selecting $\tau_0 = \delta(\mathcal{K})$, so that no jumps are inhibited at $(0, 0)$. In fact, sets $\mathcal{D}_1$ and $\mathcal{D}_{-1}$ are suitable liftings to higher dimensional spaces (involving the extra variables q and τ) of, respectively, the sets $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_{-1}$ defined in (40).

As a consequence, we may prove the bisimulation property (48) without focusing on the timer τ, because the fact that $\hat{\xi} = (\hat{\sigma}, \hat{\phi}, \hat{v}, \hat{b})$ and the components (σ, ϕ, v, b) of a solution ξ coincide over a time interval implies, by the semiglobal dwell time of $\hat{\xi}$ in Proposition 5, that the condition on τ enforced in $\mathcal{D}_1$ and $\mathcal{D}_{-1}$ is always satisfied (since the velocity $\hat{v}$ will not change its sign for a time interval of length at least $\delta(\mathcal{K})$). This is done in the next lemma, whose proof amounts to checking all the possible (nonunique) evolutions of $\hat{\mathcal{H}}$ and of $\mathcal{H}_{\delta(\mathcal{K})}$, and is given in Appendix C.

Lemma 4. *Under Assumptions 1-2, for each solution* $\hat{\xi} = (\hat{\sigma}, \hat{\phi}, \hat{v}, \hat{b})$ *to* $\hat{\mathcal{H}}$ *with* $\hat{\xi}(0,0) = \hat{\xi}_0 \in \mathcal{K}$, *there exists* q_0 *such that some solution* ξ *to* $\mathcal{H}_{\delta(\mathcal{K})}$ *with* $\mathcal{D}_1$ *and* $\mathcal{D}_{-1}$ *replaced by*

$$\bar{\mathcal{D}}_1 := \{\xi \in \Xi \colon v = 0, \phi \geq F_s, b = 1, q = 0\}, \tag{49a}$$

$$\bar{\mathcal{D}}_{-1} := \{\xi \in \Xi \colon v = 0, \phi \leq -F_s, b = 1, q = 0\}, \tag{49b}$$

(namely, without any τ*-induced jump inhibition), starting at* $\xi(0,0) = (\hat{\xi}_0, q_0, \delta(\mathcal{K}))$ *satisfies* (48) *for all* $t \geq 0$ *such that* $\hat{\xi}(t, j(t)) \notin \hat{\mathcal{A}}$.

The solution ξ characterized in Lemma 4 never reaches $\bar{\mathcal{D}}_1$ or $\bar{\mathcal{D}}_{-1}$ with $\tau < \delta(\mathcal{K})$, otherwise the solution $\hat{\xi}$ would belong to $\hat{\mathcal{S}}_1$ or $\hat{\mathcal{S}}_{-1}$ in (40), contradicting Proposition 5. Thus, ξ is also a solution to $\mathcal{H}_{\delta(\mathcal{K})}$ and this completes the proof. □

VI. Stability Analysis

For the bisimulation model $\mathcal{H}_\delta$ of Section V-C, we construct in Section VI-A a weak Lyapunov function V. Based on V, GAS of $\mathcal{H}_\delta$ is proven in the subsequent Section VI-B. Finally, in Section VI-C, the semiglobal bisimulation result of Proposition 6 is used to prove Theorem 1.

A. Lipschitz Lyapunov function for the bisimulation model

To prove suitable stability properties of $\mathcal{H}_\delta$ in (47), we introduce the following lifting of the attractor $\hat{\mathcal{A}}$ in (22) as

$$\mathcal{A} := \{\xi \in \Xi : \sigma = v = 0, \phi \in F_s \operatorname{Sign}(bq)\}, \tag{50}$$

where the extra variables q and τ can be selected arbitrarily within the set Ξ.

The advantage of introducing $\mathcal{H}_\delta$ resides in the next locally Lipschitz Lyapunov function

$$\begin{aligned} V(\xi) := &\begin{bmatrix}\sigma \\ v\end{bmatrix}^\top \begin{bmatrix}\frac{k_d}{k_i} & -1 \\ -1 & k_p\end{bmatrix} \begin{bmatrix}\sigma \\ v\end{bmatrix} + |q|(\phi - bqF_s)^2 \\ &+ (1-|q|)\mathrm{dz}^2_{F_s}(\phi) + 2\frac{k_p}{k_i}F_s\big(bq\sigma + (1-|q|)|\sigma|\big), \end{aligned} \tag{51}$$

where the first three terms can be seen as a smooth version of the discontinuous Lyapunov-like function (27) and the last nonsmooth nonnegative term ensures a desirable non-increase property along dynamics (47). To deal with the nonsmooth (but Lipschitz) expression $|\sigma|$ in the last term, we use the Clarke generalized gradient $\partial V(y)$ of V at y (see [13, Ch. 2]).

The next proposition establishes useful properties required of a hybrid Lyapunov function, that is, positive definiteness with respect to $\mathcal{A}$ and radial unboundedness, non-increase along flow in $\mathcal{C}$, and non-increase across jumps from $\mathcal{D}$. These properties establish what we could not prove in Lemma 2 for function W in (27), where (31) was only guaranteed when flowing with $\hat{b} = -1$.

Proposition 7. *Under Assumptions 1-2, the Lyapunov function* V *in* (51) *satisfies the next properties along dynamics* (47).

(i) V *is positive definite with respect to* $\mathcal{A}$ *in* $\mathcal{C} \cup \mathcal{D}$ *and radially unbounded relative to* $\mathcal{C} \cup \mathcal{D}$.

(ii) With $c_3 > 0$ *in* (29), *we have*

$$V^\circ(\xi) := \max_{\nu \in \partial V(\xi)} \langle \nu, \mathcal{F}(\xi)\rangle \leq -c_3 v^2 \leq 0, \ \forall \xi \in \mathcal{C}. \tag{52}$$

(iii) For each $p \in \{\sigma, v, 1, -1, 0\}$, *we have*

$$\Delta V_p(\xi) := V(g_p(\xi)) - V(\xi) \leq 0, \qquad \forall \xi \in \mathcal{D}_p. \tag{53}$$

Proof. We prove the lemma item by item.

Item (i). Positive definiteness with respect to $\mathcal{A}$ in $\mathcal{C} \cup \mathcal{D}$ follows by verifying that for each $\xi \in \mathcal{C} \cup \mathcal{D}$, $V(\xi) \geq 0$ and $V(\xi) = 0$ if and only if $\xi \in \mathcal{A}$. To see this, for each $\xi \in \mathcal{C} \cup \mathcal{D}$, V is a sum of nonnegative terms in (51) since the 2×2 matrix is positive definite from Assumption 2, and $bq\sigma \geq 0$ in $\mathcal{C} \cup \mathcal{D}$ (see Ξ in (47a)). Moreover, for each $\xi \in \mathcal{C} \cup \mathcal{D}$, $V(\xi) = 0$ if and only if $\xi \in \mathcal{A}$ because $\xi \in \mathcal{A}$ implies that $V(\xi) = |q|(\phi - bqF_s)^2 = 0$ and $V(\xi) = 0$ implies that all the nonnegative terms of the sum in (51) must be zero, hence $\sigma = v = 0$ and for $|q| = 1$, $\phi = bqF_s$ and for $q = 0$, $\phi \in [-F_s, F_s]$, and the last two cases imply together $\phi \in F_s \operatorname{Sign}(bq)$. Radial unboundedness must be checked only in the σ, v and ϕ components because b, q and τ are bounded in $\mathcal{C} \cup \mathcal{D} \subset \Xi$. To this end, non-negativity of the last two terms in (51) and positive definiteness of $\begin{bmatrix}\frac{k_d}{k_i} & -1 \\ -1 & k_p\end{bmatrix}$ (from Assumption 2) show the result.

Item (ii) For the derivation of V°, we use $\frac{d}{d\phi}\left(\mathrm{dz}^2_{F_s}(\phi)\right) = 2\mathrm{dz}_{F_s}(\phi)$, and $\partial(|\sigma|) = \operatorname{Sign}(\sigma)$. From (47d),

$$\begin{aligned} V^\circ(\xi) &= 2\frac{k_d}{k_i}\sigma\dot{\sigma} - 2v\dot{\sigma} - 2\sigma\dot{v} + 2k_p v\dot{v} + 2|q|(\phi - bqF_s)\dot{\phi} \\ &\quad + 2(1-|q|)\mathrm{dz}_{F_s}(\phi)\dot{\phi} + 2\frac{k_p}{k_i}F_s bq\dot{\sigma} \\ &\quad + \max_{\varsigma \in \operatorname{Sign}(\sigma)}\left(2\frac{k_p}{k_i}F_s(1-|q|)\varsigma\dot{\sigma}\right) \\ &= 2\frac{k_d}{k_i}\sigma(-k_i v) - 2v(-k_i v) - 2\sigma\big(-k_d v + |q|\phi - q(F_s \\ &\quad - |f(v)|)\big) + 2k_p v\big(-k_d v + |q|\phi - q(F_s - |f(v)|)\big) \\ &\quad + 2|q|(\phi - bqF_s)(\sigma - k_p v) + 2(1-|q|)\mathrm{dz}_{F_s}(\phi)(\sigma - k_p v) \\ &\quad + 2\frac{k_p}{k_i}F_s bq(-k_i v) + \max_{\varsigma \in \operatorname{Sign}(\sigma)}\left(2\frac{k_p}{k_i}F_s(1-|q|)\varsigma(-k_i v)\right), \end{aligned}$$

where the deadzone term is zero because $|q| = 1$ in $\mathcal{C}_{\text{slip}}$, and $q = 0$ and $|\phi| \leq F_s$ in $\mathcal{C}_{\text{stick}}$. Similarly, the term in the maximum is zero because because $|q| = 1$ in $\mathcal{C}_{\text{slip}}$, and $q = 0$ and $v = 0$ in $\mathcal{C}_{\text{stick}}$. Since $|q|q = q$ for $\xi \in \Xi$, some computations yield

$$\begin{aligned} V^\circ(\xi) &= -2c_3 v^2 + 2q\sigma(F_s - |f(v)|) - 2F_s bq\sigma \\ &\qquad - 2k_p qv(F_s - |f(v)|) \\ &\leq -2c_3 v^2 + 2q\sigma(F_s - |f(v)|) - 2F_s bq\sigma \leq -2c_3 v^2 \leq 0 \end{aligned}$$

where the first inequality follows from $qv \geq 0$ in $\mathcal{C}$ and $F_s - |f(v)| \geq 0$ for all v by Assumption 1*(i)*, and the second inequality follows from $bq\sigma \geq 0$ in $\mathcal{C}$ and $2q\sigma(F_s - |f(v)|) - 2F_s bq\sigma \leq 2|q||\sigma|(F_s - |f(v)|) - 2F_s|q||\sigma| = -2|q||\sigma||f(v)| \leq 0$.

Item (iii). In (53), we address separately each p corresponding to a jump from $\mathcal{D}_p$ with jump map g_p.

(*Jump* $p = \sigma$) For each $\xi \in \mathcal{D}_\sigma$, $|q| = |q^+| = 1$ and $\sigma = 0$, so

$$\begin{aligned}\Delta V_\sigma(\xi) &= (\phi^+ - b^+ qF_s)^2 - (\phi - bqF_s)^2 \\ &= (-\phi + bqF_s)^2 - (\phi - bqF_s)^2 = 0.\end{aligned}$$

(*Jump* $p = v$) For each $\xi \in \mathcal{D}_v$, $q = q^+ = 0$, so

$$\Delta V_v(\xi) = \mathrm{dz}^2_{F_s}(\phi^+) - \mathrm{dz}^2_{F_s}(\phi) = \mathrm{dz}^2_{F_s}(|\phi^+|) - \mathrm{dz}^2_{F_s}(|\phi|) \leq 0$$

because $|\phi^+| = \frac{k_p}{k_i}|\sigma| \leq |\phi|$ from constraint $\sigma\phi \geq \frac{k_p}{k_i}\sigma^2 \geq 0$ in $\mathcal{D}_v$, which is equivalent to $|\sigma||\phi| \geq \frac{k_p}{k_i}|\sigma|^2$.

(*Jump* $p \in \{1, -1\}$) For each $\xi \in \mathcal{D}_{-1}$ or $\xi \in \mathcal{D}_1$, $b = b^+ = 1$, $q = 0$ and $|q^+| = 1$, so

$$\begin{aligned}\Delta V_i(\xi) &= (\phi - bq^+ F_s)^2 - \mathrm{dz}^2_{F_s}(\phi) + 2\frac{k_p}{k_i}F_s bq^+\sigma - 2\frac{k_p}{k_i}F_s|\sigma| \\ &\leq (\phi - q^+ F_s)^2 - \mathrm{dz}^2_{F_s}(\phi) = 0.\end{aligned}$$

where the inequality holds since $bq^+\sigma \leq |\sigma|$ and the last equality holds since $q^+\phi \geq F_s$.

(*Jump* $p = 0$) For each $\xi \in \mathcal{D}_0$, $|q| = 1$ and $q^+ = 0$, so

$$\begin{aligned}\Delta V_0(\xi) &= \mathrm{dz}^2_{F_s}(\phi) - (\phi - bqF_s)^2 + 2\frac{k_p}{k_i}F_s|\sigma| - 2\frac{k_p}{k_i}F_s bq\sigma \\ &= \mathrm{dz}^2_{F_s}(\phi) - (\phi - bqF_s)^2 \leq 0,\end{aligned}$$

where the last equality holds since $bq\sigma = |\sigma|$ (by $bq\sigma \geq 0$, $|b| = 1$, and $|q| = 1$ in $\mathcal{D}_0$) and the inequality holds since $(\phi - bqF_s)^2 \geq \mathrm{dz}^2_{F_s}(\phi)$. $\square$

B. Global asymptotic stability of the bisimulation model

Proposition 7 of the previous section shows that function V in (51) is a weak Lyapunov function certifying stability of $\mathcal{A}$ in (50) for $\mathcal{H}_\delta$. To establish global attractivity (thus, global asymptotic stability), we exploit the hybrid invariance principle in [34, Thm. 1] in the next proposition.

Proposition 8. *Under Assumptions 1-2, for each $\delta > 0$, the set $\mathcal{A}$ in* (50) *is globally asymptotically stable for $\mathcal{H}_\delta$ in* (47)*.*

Proof. The proof is based on [34, Thm. 1]. The set $\mathcal{A}$ in (50) is compact and $\mathcal{H}_\delta$ in (47) satisfies the hybrid basic conditions in [18, Assumption 6.5]. We check the other assumptions of [34, Thm. 1] below.

(i) $\mathcal{G}(\mathcal{D} \cap \mathcal{A}) \subset \mathcal{A}$ for $\mathcal{G}$ in (47c). Indeed, $g_\sigma(\mathcal{D}_\sigma \cap \mathcal{A}) \subset g_\sigma(\mathcal{A}) \subset \mathcal{A}$, $g_v(\mathcal{D}_v \cap \mathcal{A}) \subset \mathcal{A}$, $g_0(\mathcal{D}_0 \cap \mathcal{A}) \subset g_0(\mathcal{A}) \subset \mathcal{A}$, $g_1(\mathcal{D}_1 \cap \mathcal{A}) \subset \mathcal{A}$, and $g_{-1}(\mathcal{D}_{-1} \cap \mathcal{A}) \subset \mathcal{A}$.

(ii) Conditions on V. The Lyapunov function V satisfies $\mathcal{C} \cup \mathcal{D} \subset \operatorname{dom} V$, is continuous in $\mathcal{C} \cup \mathcal{D}$ and locally Lipschitz near each point in $\mathcal{C}$, and is positive definite with respect to $\mathcal{A}$ in $\mathcal{C} \cup \mathcal{D}$ and radially unbounded relative to $\mathcal{C} \cup \mathcal{D}$ by Proposition 7, item (i). The Lyapunov nonincrease conditions have been established in Proposition 7, items (ii)-(iii).

(iii) No complete solution keeps V constant and nonzero.

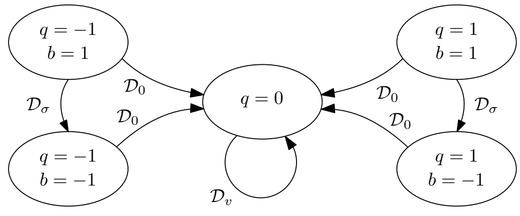

Fig. 9. The auxiliary version of the hybrid automaton in Fig. 8 used in the proof of Proposition 8.

We preliminarily show that the dwell time enforced by the timer τ in $\mathcal{H}_\delta$ and the logical variables imply that complete solutions exhibit an infinite amount of flow. To this end, on the automaton of Fig. 8, we need to remove the jumps from $\mathcal{D}_1$ or $\mathcal{D}_{-1}$ (which are only enabled if $\tau \geq \delta$). The remaining jumps are those in Fig. 9, revealing that when $\tau < \delta$, after at most two jumps it must hold that $q = 0$. From $q = 0$, the only possible jump is from $\mathcal{D}_v$ (where $b = -1$), which maps to $b = 1$, so that at most one jump from $\mathcal{D}_v$ is possible. In summary, at most three jumps can happen when $\tau < \delta$ and the solution would not be complete. This proves that all complete solutions exhibit an infinite amount of flow.

Suppose now by contradiction that there exists a complete solution ξ_{bad} to $\mathcal{H}_\delta$ that keeps V constant and nonzero. Being complete, this solution exhibits an infinite amount of flow, which should happen outside $\mathcal{A}$ (otherwise V would be zero). Moreover, ξ_{bad} must start with a zero initial velocity v, which should remain zero all along the solution, because v remains constant across any possible jump and any flowing solution from $v \neq 0$ will cause a decrease of V from item (ii) of Proposition 7.

Such a flowing solution with $v \equiv 0$ cannot flow in $\mathcal{C}_{\text{slip}} \backslash \mathcal{A}$. Indeed, $f(v) = L_2 v$ for all $|v| \leq \epsilon_v$ by Assumption 1(iv). We have then from (47d) that the first three components of $\mathcal{F}$ are

$$\begin{aligned}&\text{for } q = 1: & \begin{bmatrix} -k_i v \\ \sigma - k_p v \\ -k_d v + \phi - F_s + L_2 v \end{bmatrix} &=: A_{L_2}\begin{bmatrix} \sigma \\ \phi - F_s \\ v \end{bmatrix}, \\ &\text{for } q = -1: & \begin{bmatrix} -k_i v \\ \sigma - k_p v \\ -k_d v + \phi + F_s + L_2 v \end{bmatrix} &=: A_{L_2}\begin{bmatrix} \sigma \\ \phi + F_s \\ v \end{bmatrix},\end{aligned}$$

with $A_{L_2} := \begin{bmatrix} 0 & 0 & -k_i \\ 1 & 0 & -k_p \\ 0 & 1 & -k_d + L_2 \end{bmatrix}$. Since the pair $(\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, A_{L_2})$ is observable, the only solutions (σ, ϕ, v) compatible with $v \equiv 0$ are constant and correspond to the points where $v = 0$ and $\begin{bmatrix} q \\ b \\ \sigma \\ \phi \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ F_s \end{bmatrix}$ and $\begin{bmatrix} q \\ b \\ \sigma \\ \phi \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -F_s \end{bmatrix}$ where the value of b is imposed by the constraint $bq\phi \geq 0$ in $\mathcal{C}_{\text{slip}}$. By (50), both points belong to $\mathcal{A}$ so ξ_{bad} cannot evolve there.

We conclude by showing that ξ_{bad} cannot flow indefinitely in $\mathcal{C}_{\text{stick}} \backslash \mathcal{A}$. Indeed, the first three components of $\mathcal{F}$ in (47d) are $(0, \sigma, 0)$, with σ being *nonzero* (otherwise, ξ_{bad} would be in $\mathcal{A}$). With such indefinite flowing, ϕ would grow unbounded and this contradicts $|\phi| \leq F_s$ (required in $\mathcal{C}_{\text{stick}} \backslash \mathcal{A}$). In particular, any such ξ_{bad} must eventually reach a point where $(v, \sigma, \phi, b) = (0, \sigma, \operatorname{sign}(\sigma)F_s, 1)$ (possibly after a jump from $\mathcal{D}_v$), from where it must jump from $\mathcal{D}_1$ or $\mathcal{D}_{-1}$ to a point where $|q^+| = 1$, $\sigma^+ = \sigma$ is nonzero, and $b^+ = 1$. Any subsequent flow (which must happen because an infinite amount of flow occurs), must occur in $\mathcal{C}_{\text{slip}}\backslash\mathcal{A}$ and is ruled out by the previous analysis. Hence, the proof is complete. $\square$

C. Proof of Theorem 1

We are now able to prove Theorem 1, because the semiglobal bisimilarity properties of Proposition 6 allow extending the stability results of Proposition 8 to system $\hat{\mathcal{H}}$ in (21), provided solutions are bounded as per Proposition 4.

First, define

$$\hat{\mathcal{A}}_6 := \{(\hat{\sigma}, \hat{\phi}, \hat{v}, \hat{b}, q, \tau) \colon \hat{\sigma} = \hat{v} = 0, |\hat{\phi}| \leq F_s, \\ \hat{b} \in \{-1, 1\}, q \in \{-1, 0, 1\}, \tau \in [0, 2\delta]\},$$

which extends $\hat{\mathcal{A}} \subset \mathbb{R}^4$ in (22) to the new directions q and τ, so that $\hat{\mathcal{A}}_6 \subset \mathbb{R}^6$. It holds that $\hat{\mathcal{A}}_6 \supset \mathcal{A}$ with $\mathcal{A}$ in (50). Then, for each $\xi = (\hat{\xi}, q, \tau) \in \Xi$,

$$|\xi|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} |\xi - y| \geq \inf_{y\in\hat{\mathcal{A}}_6} |\xi - y| = \inf_{y\in\hat{\mathcal{A}}_6} |(\hat{\xi}, q, \tau) - y| = |\hat{\xi}|_{\hat{\mathcal{A}}}. \tag{54}$$

We need to show stability and global attractivity of $\hat{\mathcal{A}}$, where the latter entails by [18, Def. 7.1] that for each solution $\hat{\xi}$ to $\hat{\mathcal{H}}$, $\hat{\xi}$ is bounded and satisfies

$$\lim_{t+j\to\infty} |\hat{\xi}(t, j)|_{\hat{\mathcal{A}}} = 0, \tag{55}$$

since maximal solutions are complete by Proposition 1. Boundedness of solutions is guaranteed by Proposition 4. Proposition 6 guarantees that for each compact set $\mathcal{K}$ and the corresponding $\delta(\mathcal{K}) > 0$, each solution $\hat{\xi}$ to $\hat{\mathcal{H}}$ with $\hat{\xi}(0, 0) \in \mathcal{K}$ coincides with the (x, b) components of some solution ξ to $\mathcal{H}_{\delta(\mathcal{K})}$ for all $t \geq 0$ such that $\hat{\xi}(t, j(t)) \notin \hat{\mathcal{A}}$, i.e., from (48), for any such t it holds that $\xi(t, j(t)) = \big(\hat{\xi}(t, j(t)), q(t, j(t)), \tau(t, j(t))\big)$ for all $t \geq 0$ such that $\hat{\xi}(t, j(t)) \notin \hat{\mathcal{A}}$. Then, (54) implies that

$$|\xi(t, j(t))|_{\mathcal{A}} \geq |\hat{\xi}(t, j(t))|_{\hat{\mathcal{A}}} \tag{56}$$

for all $t \geq 0$ such that $\hat{\xi}(t, j(t)) \notin \hat{\mathcal{A}}$. If there exists $t' \geq 0$ such that $\hat{\xi}(t', j(t')) \in \hat{\mathcal{A}}$, then (55) is proven by Proposition 3(i). If instead $\hat{\xi}(t, j(t)) \notin \hat{\mathcal{A}}$ for all t in the domain of $\hat{\xi}$, then $\sup_t \hat{\xi} = +\infty$ by Proposition 3(ii) and then $\sup_t \xi = +\infty$ as well by (48). Moreover, Proposition 8 implies $\lim_{t\to\infty} |\xi(t, j(t))|_{\mathcal{A}} = 0$ and then (56) also implies $\lim_{t\to\infty} |\hat{\xi}(t, j(t))|_{\hat{\mathcal{A}}} = 0$, which also proves (55), namely global attractivity of $\hat{\mathcal{A}}$.

Since $\mathcal{A}$ is compact and both $\mathcal{H}_\delta$ and $\hat{\mathcal{H}}$ satisfy the hybrid basic conditions [18, Assumption 6.5], global asymptotic stability of $\mathcal{A}$ for $\mathcal{H}_\delta$ in Proposition 8 implies uniform global stability and uniform global attractivity by [18, Thm. 7.12]. Hence, $\hat{\mathcal{A}}$ is uniformly globally attractive. Since $\hat{\mathcal{A}}$ is also strongly forward invariant by Proposition 3(i), then $\hat{\mathcal{A}}$ is stable by [18, Prop. 7.5], which together with its global attractivity implies its global asymptotic stability.

VII. Conclusions

We proposed a novel reset integrator control strategy for motion systems with unknown Coulomb and velocity-dependent friction (including the Stribeck effect) that achieves global asymptotic stability of the setpoint, and reduces overshoot with respect to the classical PID controller. The closed-loop system dynamics is formulated as a hybrid system, using a hybrid description of the Coulomb friction element, and global asymptotic stability of the setpoint is proven. The working principle and effectiveness of the controller are experimentally demonstrated in a case study on a high-precision positioning application.

Appendix A
Proof of Lemma 2

We prove the lemma item by item.

Proof of item 1. For all $\hat{\xi} \in \hat{\mathcal{D}}_\sigma$, we have

$$W(g_\sigma(\hat{\xi})) - W(\hat{\xi}) = \\ \min_{F\in F_s \operatorname{Sign}(\hat{v})} ((\hat{b}\hat{\phi})^+ - F)^2 - \min_{F\in F_s \operatorname{Sign}(\hat{v})} (\hat{b}\hat{\phi} - F)^2 = 0,$$

because $(\hat{b}\hat{\phi})^+ = \hat{b}\hat{\phi}$. For all $\hat{\xi} \in \hat{\mathcal{D}}_v$, we have

$$W(g_v(\hat{\xi})) - W(\hat{\xi}) = \\ \min_{F\in F_s \operatorname{Sign}(0)} ((\hat{b}\hat{\phi})^+ - F)^2 - \min_{F\in F_s \operatorname{Sign}(0)} (\hat{b}\hat{\phi} - F)^2 \\ = \min_{F\in[-F_s, F_s]} (\tfrac{k_p}{k_i}\hat{\sigma} - F)^2 - \min_{F\in[-F_s, F_s]} (-\hat{\phi} - F)^2 \\ = \mathrm{dz}^2_{F_s}(\tfrac{k_p}{k_i}\hat{\sigma}) - \mathrm{dz}^2_{F_s}(-\hat{\phi}) = \mathrm{dz}^2_{F_s}(\tfrac{k_p}{k_i}\hat{\sigma}) - \mathrm{dz}^2_{F_s}(\hat{\phi}) \leq 0,$$

because $|\hat{\phi}| \geq \frac{k_p}{k_i}|\hat{\sigma}|$ due to the fact that $\hat{\sigma}\hat{\phi} \geq \frac{k_p}{k_i}\hat{\sigma}^2$ in $\hat{\mathcal{D}}_v$.

Proof of item 2. By Lemma 1, for each initial condition, the component $\hat{x}$ of the (unique) flowing solution $\hat{\xi}$ coincides with the unique solution $\tilde{x}$ to one of (11)-(13) on a finite time interval with length $T > 0$. Because such unique solution to (11), (12), (13) has respectively $\tilde{v}$ positive, zero, negative over such interval with length T by Lemma 1, it can be shown respectively that for all t in such interval

$$W\Big(\begin{bmatrix}\tilde{x}(t)\\-1\end{bmatrix}\Big) = W_1(\tilde{x}(t)), \quad W\Big(\begin{bmatrix}\tilde{x}(t)\\-1\end{bmatrix}\Big) = W_0(\tilde{x}(t)), \\ W\Big(\begin{bmatrix}\tilde{x}(t)\\-1\end{bmatrix}\Big) = W_{-1}(\tilde{x}(t)),$$

with

$$W_1(\tilde{x}) := \begin{bmatrix}\tilde{\sigma}\\\tilde{v}\end{bmatrix}^\top \begin{bmatrix}\frac{k_d}{k_i} & -1\\-1 & k_p\end{bmatrix} \begin{bmatrix}\tilde{\sigma}\\\tilde{v}\end{bmatrix} + (-\tilde{\phi} - F_s)^2 \tag{57}$$

$$W_0(\tilde{x}) := \tfrac{k_d}{k_i}\tilde{\sigma}^2 \tag{58}$$

$$W_{-1}(\tilde{x}) := \begin{bmatrix}\tilde{\sigma}\\\tilde{v}\end{bmatrix}^\top \begin{bmatrix}\frac{k_d}{k_i} & -1\\-1 & k_p\end{bmatrix} \begin{bmatrix}\tilde{\sigma}\\\tilde{v}\end{bmatrix} + (-\tilde{\phi} + F_s)^2, \tag{59}$$

in the same way as [11, Claim 1, item 2)].

In the rest of the proof we consider W_1 (or W_0 or W_{-1}, respectively) instead of W during the flow of $\hat{\xi}$ only when the component $\hat{x}$ of the solution $\hat{\xi}$ coincides with the solution $\tilde{x}$ to (11) (or (12) or (13), respectively). So, we can exploit the conditions satisfied by $\hat{\xi}$ while flowing in the flow set $\hat{\mathcal{C}}$ in (21b), in particular $\hat{b}\hat{\sigma}\hat{v} \geq 0$. For $\hat{b} = -1$ we have then $\tilde{\sigma}(t)\tilde{v}(t) \leq 0$. When considering the solution $\tilde{x}$ to (11) (resp. (13)), it holds $\tilde{v}(t) > 0$ (resp. $\tilde{v}(t) < 0$), so $\hat{\sigma}(t)\hat{v}(t) \leq 0$ implies $\tilde{\sigma}(t) \leq 0$ (resp. $\tilde{\sigma}(t) \geq 0$). We use these conditions for the bounds in the following (60). Some computations yield the derivative of W_1 along solutions to (11), of W_0 along solutions to (12), and of W_{-1} along solutions to (13), respectively, as

$$\begin{aligned}
&\tfrac{d}{dt}W_1(\tilde{x}(t)) = -c_3\tilde{v}(t)^2 + 2\tilde{\sigma}(t)(F_s - f(\tilde{v}(t)))\\
&\quad - 2k_p\tilde{v}(t)(F_s - f(\tilde{v}(t))) + 2F_s\tilde{\sigma}(t) - 2F_s k_p\tilde{v}(t)\\
&\le -c_3\tilde{v}(t)^2\\
&\tfrac{d}{dt}W_0(\tilde{x}(t)) = 0 \le -c_3\tilde{v}(t)^2\\
&\tfrac{d}{dt}W_{-1}(\tilde{x}(t)) = -c_3\tilde{v}(t)^2 - 2\tilde{\sigma}(t)(F_s + f(\tilde{v}(t)))\\
&\quad + 2k_p\tilde{v}(t)(F_s + f(\tilde{v}(t))) - 2F_s\tilde{\sigma}(t) + 2F_s k_p\tilde{v}(t)\\
&\le -c_3\tilde{v}(t)^2,
\end{aligned} \tag{60}$$

where the bounds were justified before (recall $|f(\tilde{v})| \le F_s$ for all $\tilde{v}$ by Assumption 1(i)) and, as for W_0, $\hat{v}$ is identically zero. We now use (60) together with the reasoning in [11, Sec. V.A] as follows: $\hat{\xi} \mapsto W(\hat{\xi})$ and $t \mapsto W(\hat{\xi}(t))$ are lower semicontinuous by the same argument as in [11, Sec. V.A]. Moreover, $W_1(\tilde{x}(\cdot))$, $W_0(\tilde{x}(\cdot))$, and $W_{-1}(\tilde{x}(\cdot))$ are differentiable, thus $W\left(\left[\begin{smallmatrix}\hat{x}(\cdot)\\-1\end{smallmatrix}\right]\right)$ is at least differentiable from the right. The lower right Dini derivative coincides with the right derivative, and the right derivative is upper bounded on each interval with length T by $-c_3\hat{v}(t,j)^2$ from (60). This allows to invoke [19, Thm. 9] in the same way as in [11, Fact 1], which leads to (31).

Proof of item 3. Consider an arbitrary solution $\hat{\xi} = (\hat{\sigma},\hat{\phi},\hat{v},\hat{b}) = (\hat{x},\hat{b}) \in \mathcal{S}_{\hat{\mathcal{H}}}$ satisfying $\hat{\xi}(t_j, j-1) \in \hat{\mathcal{D}}_v$ in (21g), jumping to $\hat{\xi}(t_j,j) = \hat{g}_v(\hat{\xi}(t_j,j-1))$ in (21e) and then flowing up to $\hat{\xi}(t_{j+1},j) \in \hat{\mathcal{D}}_\sigma$ in (21f), so that $t_{j+1} > t_j$ by [18, Def. 2.6]. Moreover, $\hat{b}$ is constant and equal to 1 along this interval of flow $[t_j, t_{j+1}]$. Using the expression of $\hat{\mathcal{F}}_x(\hat{x})$ in (6), and the fact that matrix A therein is Hurwitz by Assumption 2, by linearity we may split the arising response in a homogeneous (or free, or zero-input) response $\hat{x}_h$ from initial condition

$$\hat{x}_0 := \hat{x}(t_j,j) = \hat{\sigma}(t_j,j)\begin{bmatrix}1\\ \frac{k_p}{k_i}\\ 0\end{bmatrix} \tag{61}$$

(by (21e) and (21g)), and a forced response $\hat{x}_f$ from a zero initial condition caused by the bounded input $e_3(F_s\,\mathrm{Sign}(\hat{v}) - f(\hat{v}))$ of maximum size F_s (by Assumption 1). Define

$$\hat{V}(\hat{x}) := \hat{x}^\top\hat{P}\hat{x} := \hat{x}^\top\begin{bmatrix}\frac{k_d}{k_i} & 0 & -1\\ 0 & 1 & 0\\ -1 & 0 & k_p\end{bmatrix}\hat{x},$$

with $\hat{P} > 0$ (by Assumption 2), and satisfying

$$A^\top\hat{P} + \hat{P}A = -\begin{bmatrix}0&0&0\\0&0&0\\0&0&c_3\end{bmatrix},$$

with $c_3 > 0$ in (29), which provides a weak Lyapunov function for $\dot{\hat{x}} = A\hat{x}$, due to observability of the pair $([\,0\ 0\ c_3\,], A)$. Then, since the solution flows at (t_j,j) and $t_{j+1} > t_j$, $\hat{V}$ is a weak Lyapunov function for $\dot{\hat{x}} = A\hat{x}$, and the pair $([\,0\ 0\ c_3\,], A)$ is observable, there exists $\eta \in (0,1)$ such that

$$\hat{V}(\hat{x}_h(t_{j+1})) = \eta^2\hat{V}(\hat{x}_0) = \eta^2\hat{\sigma}(t_j,j)^2\hat{V}\left(\begin{bmatrix}1\\ \frac{k_p}{k_i}\\ 0\end{bmatrix}\right), \tag{62}$$

because $V(\hat{x}_h(\cdot))$ would remain constant on $[t_j,t_{j+1}]$ only for $\hat{x}$ identically zero, which is excluded by the fact that we consider $|\hat{\sigma}(t_j,j)|$ sufficiently large through the selection of $\bar{W}$ (as argued below in the proof). On the other hand, from bounded-input bounded-output stability of dynamics (6), we have that

$$|\hat{x}(t_{j+1},j) - \hat{x}_h(t_{j+1})| = |\hat{x}_f(t_{j+1})| \le \hat{h}_A, \tag{63}$$

for some $\hat{h}_A > 0$ (cf. (33)). Consider now the homogeneous-of-degree-1 function

$$\hat{x} \mapsto \hat{U}(\hat{x}) := \sqrt{\hat{V}(\hat{x})},$$

which is globally Lipschitz (namely, $|\hat{U}(\hat{x}) - \hat{U}(\hat{x}_h)| \le L_{\hat{U}}|\hat{x} - \hat{x}_h|$ for all $\hat{x}$, $\hat{x}_h \in \mathbb{R}^3$ and some Lipschitz constant $L_{\hat{U}} > 0$) because its gradient is constant along rays starting at the origin. Using (62) and (63), we have

$$\begin{aligned}
\hat{U}(\hat{x}(t_{j+1},j)) &\le \hat{U}(\hat{x}_h(t_{j+1})) + L_{\hat{U}}|\hat{x}(t_{j+1},j) - \hat{x}_h(t_{j+1})|\\
&\le \eta|\hat{\sigma}(t_j,j)|\hat{U}\big((1,\tfrac{k_p}{k_i},0)\big) + L_{\hat{U}}\hat{h}_A,
\end{aligned}$$

with $\hat{U}\big((1,\frac{k_p}{k_i},0)\big) > 0$. As a consequence we have

$$\begin{aligned}
\hat{V}(\hat{x}(t_{j+1},j)) &\le \Big(\eta|\hat{\sigma}(t_j,j)|\hat{U}\big((1,\tfrac{k_p}{k_i},0)\big) + L_{\hat{U}}\hat{h}_A\Big)^2\\
&= \hat{V}(\hat{x}_0)\left(\eta + \frac{L_{\hat{U}}\hat{h}_A}{|\hat{\sigma}(t_j,j)|\hat{U}\big((1,\frac{k_p}{k_i},0)\big)}\right)^2.
\end{aligned} \tag{64}$$

For $\eta \in (0,1)$ it is possible to select $\tilde{\eta}_V \in (\eta,1)$ and $\sigma_{M1} > 0$ sufficiently large such that

$$\tilde{\eta}_V = \eta + \frac{L_{\hat{U}}\hat{h}_A}{\sigma_{M1}\hat{U}\big((1,\frac{k_p}{k_i},0)\big)}.$$

With this equation, we obtain for $\tilde{\eta}_V \in (\eta,1)$

$$|\hat{\sigma}(t_j,j)| \ge \sigma_{M1} \implies \hat{V}(\hat{x}(t_{j+1},j)) \le \tilde{\eta}_V^2\hat{V}(\hat{x}_0). \tag{65}$$

Consider now function W defined in (27) and relate it to $\hat{V}$ through

$$\hat{V}(\hat{x}_0) = \hat{\sigma}(t_j,j)^2\begin{bmatrix}1\\ \frac{k_p}{k_i}\end{bmatrix}^\top\begin{bmatrix}\frac{k_d}{k_i} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1\\ \frac{k_p}{k_i}\end{bmatrix} = \left(\frac{k_d}{k_i} + \frac{k_p^2}{k_i^2}\right)\hat{\sigma}(t_j,j)^2. \tag{66}$$

Introduce $\sigma_{M2} := \frac{k_i}{k_p}F_s\max\{1, \frac{1}{1-\sqrt{\tilde{\eta}_V}}\} > 0$ (recall $\tilde{\eta}_V < 1$ and positive). For $|\hat{\sigma}(t_j,j)| \ge \sigma_{M2}$, we have

$$\begin{aligned}
&W(\hat{\xi}(t_j,j)) = \frac{k_d}{k_i}\hat{\sigma}(t_j,j)^2 + \left(\frac{k_p}{k_i}|\hat{\sigma}(t_j,j)| - F_s\right)^2\\
&\ge \frac{k_d}{k_i}\left(|\hat{\sigma}(t_j,j)| - \frac{k_i}{k_p}F_s\right)^2 + \frac{k_p^2}{k_i^2}\left(|\hat{\sigma}(t_j,j)| - \frac{k_i}{k_p}F_s\right)^2\\
&= \left(\frac{k_d}{k_i} + \frac{k_p^2}{k_i^2}\right)\hat{\sigma}(t_j,j)^2\left(1 - \frac{k_iF_s}{k_p|\hat{\sigma}(t_j,j)|}\right)^2\\
&\ge \hat{V}(\hat{x}_0)\tilde{\eta}_V,
\end{aligned}$$

where, in the given order, the first equality follows from (61) and $\min_{F\in[-F_s,F_s]}(\hat{\phi}(t_j,j) - F)^2 = (|\hat{\phi}(t_j,j)| - F_s)^2$ for $|\hat{\sigma}(t_j,j)| \ge \sigma_{M2} \ge \frac{k_i}{k_p}F_s$, the first inequality follows from $|\hat{\sigma}(t_j,j)| \ge \sigma_{M2} \ge \frac{k_i}{k_p}F_s$, the second inequality follows from the expression of $\hat{V}(\hat{x}_0)$ in (66), and $|\hat{\sigma}(t_j,j)| \ge \sigma_{M2} \ge \frac{k_i}{k_p}F_s\frac{1}{1-\sqrt{\tilde{\eta}_V}}$. Then,

$$|\hat{\sigma}(t_j,j)| \ge \sigma_{M2} \implies W(\hat{\xi}(t_j,j)) \ge \tilde{\eta}_V\hat{V}(\hat{x}_0). \tag{67}$$

Finally, during flow with $\hat b = 1$, we have $\hat b \hat v \hat\sigma \geq 0$ and $\hat\sigma\hat\phi \geq \frac{k_p}{k_i}\hat\sigma^2$ (see (21b) and (21a)). Therefore, if $\hat\sigma(t_j, j) > 0$, we have from $\hat b\hat v\hat\sigma \geq 0$ that $\hat v(t_j, j) \geq 0$, and since $\hat\sigma$ or $\hat v$ cannot be negative on $[t_j, t_{j+1}]$ (by absolute continuity of flowing solutions, this would imply that $\hat\sigma$ and $\hat v$ have become simultaneously zero before that, and with $\hat\sigma = \hat v = 0$, $\hat x$ would belong to $\hat{\mathcal{A}}$ or necessarily jump to $\hat{\mathcal{A}}$, and remain there by Proposition 3(i)), then $\hat\sigma(t, j) \geq 0$ and $\hat v(t, j) \geq 0$ for all $t \in [t_j, t_{j+1}]$. Hence, from $\hat\sigma\hat\phi \geq \frac{k_p}{k_i}\hat\sigma^2$ and $\hat b\hat v\hat\phi \geq 0$, we have $\hat\phi(t, j) \geq 0$ for all $t \in [t_j, t_{j+1}]$. Similarly, for $\hat\sigma(t_j, j) < 0$ we have $\hat v(t, j) \leq 0$ and $\hat\phi(t, j) \leq 0$ for all $t \in [t_j, t_{j+1}]$. In both cases (namely with $\hat v(t_{j+1}, j) \geq 0$, $\hat\phi(t_{j+1}, j) \geq 0$ and with $\hat v(t_{j+1}, j) \leq 0$, $\hat\phi(t_{j+1}, j) \leq 0$), $\min_{F\in F_s \operatorname{Sign}(\hat v(t_{j+1},j))}(\hat\phi(t_{j+1}, j) - F)^2 \leq \hat\phi(t_{j+1}, j)^2 + F_s^2$, [6] so that

$$W(\hat\xi(t_{j+1}, j)) \leq \hat V(\hat x(t_{j+1}, j)) + F_s^2. \tag{68}$$

At (t_j, j) we have from (28) the bound

$$\begin{aligned} W(\hat\xi(t_j, j)) &\leq \bar c_W \left| \begin{bmatrix} \hat\sigma(t_j,j) \\ \frac{k_p}{k_i}\hat\sigma(t_j,j) \\ 0 \end{bmatrix} \right|^2 + 2F_s^2 \\ &= \bar c_W \Big(1 + \tfrac{k_p^2}{k_i^2}\Big)\hat\sigma(t_j, j)^2 + 2F_s^2. \end{aligned} \tag{69}$$

Take

$$\begin{aligned} &\sigma_M := \max\{\sigma_{M1}, \sigma_{M2}\} > 0, \\ &\bar W := \max\{\bar c_W \Big(1 + \tfrac{k_p^2}{k_i^2}\Big)\sigma_M^2 + 2F_s^2, \frac{F_s^2}{1 - \tilde\eta_V}\} > 0, \end{aligned} \tag{70}$$

which is well defined because $\tilde\eta_V$ satisfies $0 < \tilde\eta_V < 1$. The left and right entries in the selection of $\bar W$ allow, respectively, to prove the following two implications:

$$W(\hat\xi(t_j, j)) \geq \bar W \implies |\hat\sigma(t_j, j)| \geq \sigma_M \tag{71}$$

$$W(\hat\xi(t_j, j)) \geq \bar W \implies W(\hat\xi(t_j, j)) \geq \tilde\eta_V W(\hat\xi(t_j, j)) + F_s^2. \tag{72}$$

More specifically, (71) holds because, with $W(\hat\xi(t_j, j)) \geq \bar W$, the left term in (70) implies $W(\hat\xi(t_j, j)) \geq \bar c_W \Big(1 + \frac{k_p^2}{k_i^2}\Big)\sigma_M^2 + 2F_s^2$ and (71) follows by comparison with (69). Instead (72) is proven by using $W(\hat\xi(t_j, j)) \geq \bar W \geq \frac{F_s^2}{1-\tilde\eta_V}$ and rearranging. Finally, $W(\hat\xi(t_j, j)) \geq \bar W$ implies

$$\begin{aligned} W(\hat\xi(t_{j+1}, j)) &\overset{(68)}{\leq} \hat V(\hat x(t_{j+1}, j)) + F_s^2 \overset{(71),(65)}{\leq} \tilde\eta_V^2 \hat V(\hat x_0) + F_s^2 \\ &\overset{(71),(67)}{\leq} \tilde\eta_V W(\hat\xi(t_j, j)) + F_s^2 \overset{(72)}{\leq} W(\hat\xi(t_j, j)), \end{aligned}$$

as to be proven.

Appendix B
Proof of Lemma 3

Let us prove each item separately.

Item (i). (11) can be written as

$$\dot{\tilde x} = A\tilde x - e_3\mathfrak{u}, \text{ with } |\mathfrak{u}| \leq 2F_s. \tag{73}$$

[6] For $\hat\phi(t_{j+1}, j) \geq 0$, $(\hat\phi(t_{j+1}, j) - F_s)^2 \leq \hat\phi(t_{j+1}, j)^2 + F_s^2$ and this gives the bound for the cases $\hat v(t_{j+1}, j) > 0$ and $\hat v(t_{j+1}, j) = 0$.

A is Hurwitz by Assumption 2 and bounded-input-bounded-output stability holds for (73). Then, for each $M > 0$ and $\tilde x_0 \in M\mathbb{B}$, there exist $\mathcal{M}(M)$ such that $|\tilde x(t)| \leq \mathcal{M}(M)$ for all $t \geq 0$. So, define

$$\delta_0(M) := \frac{\varepsilon_v}{|A|\mathcal{M}(M) + 2F_s} > 0, \tag{74}$$

which is indeed uniform over the initial condition $\hat x_0$. Then, (73) yields for $t \geq 0$

$$\begin{aligned} |\dot{\tilde v}(t)| &\leq |\dot{\tilde x}(t)| \leq |A||\tilde x(t)| + 2F_s \\ &\leq |A|\mathcal{M}(M) + 2F_s \leq \frac{\epsilon_v}{\delta_0(M)}. \end{aligned} \tag{75}$$

So, (75) and $\tilde v(0) = 0$ imply that as long as $t \in [0, \delta_0(M)]$, $|\tilde v(t)| \leq \varepsilon_v$. By Assumption 1(iv), (11) boils down to the differential equation in (41) and solutions with the same initial condition coincide over $[0, \delta_0(M)]$.

Item (ii). Define $\check\varphi := \check\phi - F_s$ and rewrite (41) as

$$\begin{aligned} \begin{bmatrix} \dot{\check\sigma} \\ \dot{\check\varphi} \\ \dot{\check v} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & -k_i \\ 1 & 0 & -k_p \\ 0 & 1 & -k_d + L_2 \end{bmatrix} \begin{bmatrix} \check\sigma \\ \check\varphi \\ \check v \end{bmatrix} \\ &=: A_{L_2}\check x. \end{aligned} \tag{76}$$

Expand the matrix exponential governing the solution to (76) from $\check x(0) = (\check\sigma_0, \check\phi_0, 0)$

$$\check\sigma(t) = \check\sigma_0\big(1 + \mathscr{O}(t^3)\big) + \check\varphi_0\big(-\tfrac{k_i t^2}{2} + \mathscr{O}(t^3)\big) \tag{77a}$$

$$\check\varphi(t) = \check\sigma_0\big(t + \mathscr{O}(t^3)\big) + \check\varphi_0\big(1 - \tfrac{k_p t^2}{2} + \mathscr{O}(t^3)\big) \tag{77b}$$

$$\check v(t) = \check\sigma_0\big(\tfrac{t^2}{2} + \mathscr{O}(t^3)\big) + \check\varphi_0\big(t - \tfrac{(k_d - L_2)t^2}{2} + \mathscr{O}(t^3)\big), \tag{77c}$$

where $\mathscr{O}(t^3)$ denotes the terms of order t^3 or higher in the Taylor expansion, and $\check\varphi_0 \geq 0$ since $\check\phi_0 \geq F_s$ by (42). Based on (77b)-(77c), note that

$$\begin{aligned} &\exists\delta_a > 0\colon \forall t \in (0, \delta_a] \quad t + \mathscr{O}(t^3) > 0 \\ &\exists\delta_b > 0\colon \forall t \in (0, \delta_b] \quad 1 - \tfrac{k_p t^2}{2} + \mathscr{O}(t^3) > 0 \\ &\exists\delta_c > 0\colon \forall t \in (0, \delta_c] \quad \tfrac{t^2}{2} + \mathscr{O}(t^3) > 0 \\ &\exists\delta_d > 0\colon \forall t \in (0, \delta_d] \quad t - \tfrac{(k_d - L_2)t^2}{2} + \mathscr{O}(t^3) > 0, \end{aligned}$$

where $\delta_a, \dots, \delta_d$ do *not* depend on the initial condition $\check\sigma_0$, $\check\varphi_0$. Take $\delta_1 := \min\{\delta_a, \delta_b, \delta_c, \delta_d\} > 0$. Then, for $t \in (0, \delta_1]$, $\check v(t) > 0$ and $\check\varphi(t) > 0$ (since in (77) at least one among $\check\sigma_0$ and $\check\varphi_0$ is strictly positive and both are nonnegative by (42)), i.e., $\check v(t) > 0$ and $\check\phi(t) > F_s$.

Appendix C
Proof of Lemma 4

For each solution $\hat\xi$ to $\hat{\mathcal{H}}$ with $\hat\xi(0, 0) = \hat\xi_0 \in \mathcal{K}$, it is sufficient to construct a suitable hybrid arc q to obtain a hybrid arc τ (with $\operatorname{dom}\tau = \operatorname{dom} q$, $\tau(0, 0) = \delta(\mathcal{K})$, and evolving according to flow and jump maps) and a solution $(\hat\xi, q, \tau)$ to $\mathcal{H}_{\delta(\mathcal{K})}$ for all $t \geq 0$ such that $\hat\xi(t, j(t)) \notin \hat{\mathcal{A}}$, modulo a reparametrization of the jump counter of $\hat\xi$ (yielding possibly different $j_\xi(t)$ and $j_{\hat\xi}(t)$ for the same $t \geq 0$). Note that the hybrid arc τ follows immediately from the hybrid arc q since, with the jump sets $\bar{\mathcal{D}}_1$ and $\bar{\mathcal{D}}_{-1}$ in (49a)-(49b), no other constraints on τ are imposed by flow or jump sets.

Hence, if we construct such hybrid arc q in the rest of the proof, (48) holds by construction.

Each solution $\hat{\xi}$ to (21) can only flow in $\hat{\mathcal{C}}$, jump from $\hat{\mathcal{D}}_\sigma$ or jump from $\hat{\mathcal{D}}_v$, and in each of these three cases the definition of hybrid solution in [18, Def. 2.6 and p. 124] implies the following. If $\hat{\xi}$ flows in $\hat{\mathcal{C}}$, for each $j \in \mathbb{Z}_{\geq 0}$ (i.e., the set of nonnegative integers) such that $I^j := \{t\colon (t,j) \in \operatorname{dom}\hat{\xi}\}$ has nonempty interior,

$$\left.\begin{aligned} &\hat{b}(t,j)\hat{v}(t,j)\hat{\sigma}(t,j) \geq 0 \\ &\hat{\sigma}(t,j)\hat{\phi}(t,j) \geq \tfrac{k_p}{k_i}\hat{\sigma}(t,j)^2 \\ &\hat{b}(t,j)\hat{v}(t,j)\hat{\phi}(t,j) \geq 0 \end{aligned}\right\} \text{ for all } t \in I^j; \tag{79a}$$

$$\dot{\hat{\xi}}(t,j) \in \hat{\mathcal{F}}(\hat{\xi}(t,j)) \text{ for almost all } t \in I^j. \tag{79b}$$

If $\hat{\xi}$ jumps from $\hat{\mathcal{D}}_\sigma$, for each $(t,j) \in \operatorname{dom}\hat{\xi}$ such that $(t,j+1) \in \operatorname{dom}\hat{\xi}$,

$$\hat{\sigma}(t,j) = 0, \hat{b}(t,j) = 1, \hat{v}(t,j)\hat{\phi}(t,j) \geq 0; \tag{80a}$$

$$\begin{aligned} &\hat{\sigma}(t,j+1) = \hat{\sigma}(t,j), \hat{\phi}(t,j+1) = -\hat{\phi}(t,j), \\ &\hat{v}(t,j+1) = \hat{v}(t,j), \hat{b}(t,j+1) = -\hat{b}(t,j). \end{aligned} \tag{80b}$$

If $\hat{\xi}$ jumps from $\hat{\mathcal{D}}_v$, for each $(t,j) \in \operatorname{dom}\hat{\xi}$ such that $(t,j+1) \in \operatorname{dom}\hat{\xi}$,

$$\hat{v}(t,j) = 0, \hat{\sigma}(t,j)\hat{\phi}(t,j) \geq \tfrac{k_p}{k_i}\hat{\sigma}(t,j)^2, \hat{b}(t,j) = -1; \tag{81a}$$

$$\begin{aligned} &\hat{\sigma}(t,j+1) = \hat{\sigma}(t,j),\ \hat{\phi}(t,j+1) = \tfrac{k_p}{k_i}\hat{\sigma}(t,j), \\ &\hat{v}(t,j+1) = \hat{v}(t,j),\ \hat{b}(t,j+1) = -\hat{b}(t,j). \end{aligned} \tag{81b}$$

Let us then consider the construction of the suitable hybrid signal q starting from time $(0,0)$ and separately in these three cases (79), (80), (81).

Suppose $\hat{\xi}$ flows in $\hat{\mathcal{C}}$ on the interval $I^0 =: [t_0, t_1] = [0, t_1]$ with $t_1 > 0$.[7] Note that for each $\hat{\xi} \in \hat{\Xi}$, $\hat{\mathcal{F}}(\hat{\xi}) = \begin{bmatrix} \hat{\mathcal{F}}_x(\hat{x}) \\ 0 \end{bmatrix}$, and the evolution according to $\hat{\mathcal{F}}_x$ is determined in Lemma 1(ii)-(iv). For convenience, we report the cases (8)-(10) here as

$$\begin{aligned} S_1 := \{\hat{x} \in \mathbb{R}^3\colon &(\hat{v} > 0) \vee (\hat{v} = 0 \wedge \hat{\phi} > F_s) \\ &\vee (\hat{v} = 0 \wedge \hat{\phi} = F_s \wedge \hat{\sigma} > 0) \end{aligned} \tag{82}$$

$$\begin{aligned} S_0 := \{\hat{x} \in \mathbb{R}^3\colon &(\hat{v} = 0 \wedge \hat{\sigma} > 0 \wedge \hat{\phi} \in [-F_s, F_s)) \\ &\vee (\hat{v} = 0 \wedge \hat{\sigma} = 0 \wedge \hat{\phi} \in [-F_s, F_s]) \\ &\vee (\hat{v} = 0 \wedge \hat{\sigma} < 0 \wedge \hat{\phi} \in (-F_s, F_s])\} \end{aligned} \tag{83}$$

$$\begin{aligned} S_{-1} := \{\hat{x} \in \mathbb{R}^3\colon &(\hat{v} < 0) \vee (\hat{v} = 0 \wedge \hat{\phi} < -F_s) \\ &\vee (\hat{v} = 0 \wedge \hat{\phi} = -F_s \wedge \hat{\sigma} < 0)\}. \end{aligned} \tag{84}$$

Note that S_1, S_0, S_{-1} form a partition of $\mathbb{R}^3$ (i.e., $\cup_{i\in\{1,0,-1\}} S_i = \mathbb{R}^3$ and $S_i \cap S_k = \emptyset$ for each $i,k \in \{1,0,-1\}$ with $i \neq k$). For $\hat{\xi}(0,0) = (\hat{x}(0,0), \hat{b}(0,0))$, assign $q(0,0)$ as 1, 0, -1 if $\hat{x}(0,0)$ belongs respectively to S_1, S_0, S_{-1}. Consider $t_1^\star$ as the smallest time in $(0, t_1]$ ($t_1^\star > 0$ by Lemma 1) such that

$$t_1^\star = t_1, \hat{x}(t,0) \in S_{q(0,0)}\ \forall t \in [0, t_1^\star], \text{ or} \tag{85a}$$

$$t_1^\star < t_1, \hat{x}(t,0) \in S_{q(0,0)}\ \forall t \in [0, t_1^\star), \hat{x}(t_1^\star, 0) \notin S_{q(0,0)}. \tag{85b}$$

Note that no other cases than (85a)-(85b) need considering since solutions are locally absolutely continuous during flow by [18, Def. 2.4], the solutions need to hit the set $\{\hat{x} \in \mathbb{R}^3\colon \hat{v} = 0\}$ to traverse from S_i to S_k (with $i,k \in \{1,0,-1\}$ and $i \neq k$), and the intersections of the sets S_{-1}, S_0, S_1 with $\{\hat{x} \in \mathbb{R}^3\colon \hat{v} = 0\}$ are as in Fig. 10.

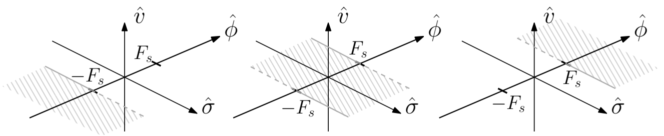

Fig. 10. Intersections of the sets S_{-1}, S_0, S_1 with $\{\hat{x} \in \mathbb{R}^3\colon \hat{v} = 0\}$. Solid and dashed lines at the boundary of each set mean respectively that those points belong and do not belong to that set.

Define $q(t,0) = q(0,0)$ for all $t \in [0, t_1^\star]$. We show now that, under (79),

$$(\hat{\xi}(t,0), q(t,0), \tau(t,0)) \in \mathcal{C}_{\text{slip}} \cup \mathcal{C}_{\text{stick}} \text{ for all } t \in [0, t_1^\star] \tag{86a}$$

$$\begin{bmatrix} \dot{\hat{\xi}}(t,0) \\ \dot{q}(t,0) \\ \dot{\tau}(t,0) \end{bmatrix} = \mathcal{F}\left(\begin{bmatrix} \hat{\xi}(t,0) \\ q(t,0) \\ \tau(t,0) \end{bmatrix}\right) \text{ for almost all } t \in [0, t_1^\star]. \tag{86b}$$

Indeed, consider separately the cases $q(0,0)$ equal to 1, 0, -1 and note that by the definition of $t_1^\star$ in (85), they imply respectively that $v(t,0)$ is nonnegative, zero, nonpositive for all $t \in [0, t_1^\star]$. As for $q(0,0) = 1$, we have that for all $t \in [0, t_1^\star]$, $q(t,0) = 1$ by our construction, $\hat{v}(t,0) \geq 0$ by the definition of $t_1^\star$ in (85), $\hat{b}(t,0)q(t,0)\hat{\sigma}(t,0) \geq 0$, $\hat{\sigma}(t,0)\hat{\phi}(t,0) \geq \frac{k_p}{k_i}\hat{\sigma}(t,0)^2$, and $\hat{b}(t,0)q(t,0)\hat{\phi}(t,0) \geq 0$ by (79a) and the first two relationships (note that from $\hat{x}(t,0) \in S_1$ for all $t \in [0, t_1^\star)$ in (85) and Lemma 1(ii), $\hat{v}$ cannot be identically zero on a nonempty time interval contained in $[0, t_1^\star)$). Then, for all $t \in [0, t_1^\star]$, $(\hat{\xi}(t,0), q(t,0), \tau(t,0)) \in \mathcal{C}_{\text{slip}}$ in (47f), so (86a) holds true. Moreover, (79b) and Lemma 1(ii) yield that for almost all $t \in [0, t_1^\star]$

$$\begin{aligned} &\dot{\hat{x}}(t,0) = A\hat{x}(t,0) - e_3(F_s - f(\hat{v}(t,0))) \\ &\dot{\hat{b}}(t,0) = 0, \end{aligned}$$

so that for almost all $t \in [0, t_1^\star]$

$$\begin{bmatrix} \dot{\hat{\xi}}(t,0) \\ \dot{q}(t,0) \\ \dot{\tau}(t,0) \end{bmatrix} = \begin{bmatrix} -k_i\hat{v}(t,0) \\ \hat{\sigma}(t,0) - k_p\hat{v}(t,0) \\ -k_d\hat{v}(t,0) + \hat{\phi}(t,0) - (F_s - f(\hat{v}(t,0))) \\ 0 \\ 0 \\ 1 - \mathrm{dz}_1(\tau(t,0)/\delta(\mathcal{K})) \end{bmatrix} = \mathcal{F}\left(\begin{bmatrix} \hat{\xi}(t,0) \\ q(t,0) \\ \tau(t,0) \end{bmatrix}\right)$$

and (86b) holds true as well. As for $q(0,0) = 0$, we have that for all $t \in [0, t_1^\star]$, $q(t,0) = 0$ by our construction, $\hat{v}(t,0) = 0$ and $|\hat{\phi}(t,0)| \leq F_s$ by the definition of $t_1^\star$ in (85), $\hat{\sigma}(t,0)\hat{\phi}(t,0) \geq \frac{k_p}{k_i}\hat{\sigma}(t,0)^2$ by (79a). Then, for all $t \in [0, t_1^\star]$, $(\hat{\xi}(t,0), q(t,0), \tau(t,0)) \in \mathcal{C}_{\text{stick}}$ in (47f), so (86a) holds true. Moreover, (79b) and Lemma 1(iii) yield that for almost all $t \in [0, t_1^\star]$,

$$\begin{aligned} &\dot{\hat{x}}(t,0) = \begin{bmatrix} 0 \\ \hat{\sigma}(t,0) \\ 0 \end{bmatrix} \\ &\dot{\hat{b}}(t,0) = 0, \end{aligned}$$

[7] We consider t_1 finite, but the reasoning in this part of the proof readily extends to the case $t_1 = +\infty$, i.e., when $\hat{\xi}$ only flows.

so that for almost all $t \in [0, t_1^\star]$,

$$\begin{bmatrix} \dot{\hat{\xi}}(t,0) \\ \dot{q}(t,0) \\ \dot{\tau}(t,0) \end{bmatrix} = \mathcal{F}\Big(\begin{bmatrix} \hat{\xi}(t,0) \\ q(t,0) \\ \tau(t,0) \end{bmatrix} \Big)$$

and (86b) holds true as well. As for $q(0,0) = -1$, we follow similar steps to $q(0,0) = 1$.

We now show that given $q(t_1^\star, 0)$ and $\hat{x}(t_1^\star, 0)$ and if $t_1^\star < t_1$ as in (85b) (this analysis is indeed not needed if (85a) holds), we can select q so that $(\hat{\xi}, q, \tau)$ jumps from $\bar{\mathcal{D}}_1$, $\mathcal{D}_0$ or $\bar{\mathcal{D}}_{-1}$ (see (49a), (47f), (49b)). Consider all the following possible cases, whereby we note that, e.g., $q(t_1^\star, 0) = 1$ and $\hat{x}(t_1^\star, 0) \in S_1$ is not a case to consider by the definition of $t_1^\star$ in (85b).

As for $q(t_1^\star, 0) = 1$ and $\hat{x}(t_1^\star, 0) \in S_0$, $\begin{bmatrix} \hat{\xi}(t_1^\star,0) \\ q(t_1^\star,0) \\ \tau(t_1^\star,0) \end{bmatrix} \in \mathcal{D}_0$ in (47f) since for all $t \in [0, t_1^\star)$, $\hat{x}(t, 0) \in S_1$, hence we can deduce $\hat{b}(t,0)\hat{\sigma}(t,0) \geq 0$, $\hat{\sigma}(t,0)\hat{\phi}(t,0) \geq \frac{k_p}{k_i}\hat{\sigma}(t,0)^2$ and $\hat{b}(t,0)\hat{\phi}(t,0) \geq 0$ from (79a). Moreover,

$$\begin{bmatrix} \xi(t_1^\star,1) \\ q(t_1^\star,1) \\ \tau(t_1^\star,1) \end{bmatrix} = g_0\Big(\begin{bmatrix} \hat{\xi}(t_1^\star,0) \\ q(t_1^\star,0) \\ \tau(t_1^\star,0) \end{bmatrix} \Big) = \begin{bmatrix} \hat{\xi}(t_1^\star,0) \\ 0 \\ \tau(t_1^\star,0) \end{bmatrix},$$

where $q(t_1^\star, 1) = 0$ allows repeating the reasoning presented for a flow on $[0, t_1^\star]$. We also note that both $(t_1^\star, 0)$ and $(t_1^\star, 1)$ belong to $\operatorname{dom} \xi = \operatorname{dom} q$ whereas $(t_1^\star, 1)$ does *not* belong in general to $\operatorname{dom} \hat{\xi}$, and this corresponds to the necessary reparameterization of the jump counter of $\hat{\xi}$ mentioned at the beginning of the proof of the lemma. Parallel arguments hold in the case $q(t_1^\star, 0) = -1$ and $\hat{x}(t_1^\star, 0) \in S_0$.

As for $q(t_1^\star, 0) = 0$ and $\hat{x}(t_1^\star, 0) \in S_1$, the definition of $t_1^\star$ in (85b), $\hat{x}(t_1^\star, 0) \in S_1$ and $q(t_1^\star, 0) = 0$, and the local absolute continuity of hybrid solutions along flow [18, Def. 2.4] imply that $\hat{v}(t_1^\star, 0) = 0$ and $\hat{\phi}(t_1^\star, 0) = F_s$. The latter implies $\hat{\sigma}(t_1^\star, 0) \geq 0$ from $\hat{\sigma}(t_1^\star, 0)\hat{\phi}(t_1^\star, 0) \geq \frac{k_p}{k_i}\hat{\sigma}(t_1^\star, 0)^2 \geq 0$ in (79a). Moreover, $\hat{x}(t_1^\star, 0) \in S_1$ implies $\hat{b}(t_1^\star, 0) = 1$ from the fact that $\hat{\xi}$ flows on $[0, t_1]$ with $t_1 > t_1^\star$, and the condition $\hat{b}\hat{v}\hat{\phi} \geq 0$ in (79a) (since $\hat{\phi}(t_1^\star, 0) = F_s$ and $\hat{v}(t, 0) > 0$ for all $t \in (t_1^\star, t_1^\star + T']$ for some $T' > 0$ by Lemma 1(ii), the condition $\hat{b}\hat{v}\hat{\phi} \geq 0$ gives $\hat{b} \geq 0$). We have then $\begin{bmatrix} \hat{\xi}(t_1^\star,0) \\ q(t_1^\star,0) \\ \tau(t_1^\star,0) \end{bmatrix} \in \bar{\mathcal{D}}_1$ in (49a) since $q(t_1^\star, 0) = 0$ in this case, $\hat{v}(t_1^\star, 0) = 0$ and $\hat{\phi}(t_1^\star, 0) = F_s$ (as just motivated), $\hat{\sigma}(t_1^\star, 0)\hat{\phi}(t_1^\star, 0) \geq \frac{k_p}{k_i}\hat{\sigma}(t_1^\star, 0)^2$ (by (79a)) and $\hat{b}(t_1^\star, 0) = 1$ (as just motivated). Moreover,

$$\begin{bmatrix} \xi(t_1^\star,1) \\ q(t_1^\star,1) \\ \tau(t_1^\star,1) \end{bmatrix} = g_1\Big(\begin{bmatrix} \hat{\xi}(t_1^\star,0) \\ q(t_1^\star,0) \\ \tau(t_1^\star,0) \end{bmatrix} \Big) = \begin{bmatrix} \hat{\xi}(t_1^\star,0) \\ 1 \\ 0 \end{bmatrix},$$

where $q(t_1^\star, 1) = 1$ allows repeating the reasoning presented for a flow on $[0, t_1^\star]$. Parallel arguments hold in the case $q(t_1^\star, 0) = 0$ and $\hat{x}(t_1^\star, 0) \in S_{-1}$.

As for $q(t_1^\star, 0) = -1$ and $\hat{x}(t_1^\star, 0) \in S_1$, $\begin{bmatrix} \hat{\xi}(t_1^\star,0) \\ q(t_1^\star,0) \\ \tau(t_1^\star,0) \end{bmatrix} \in \mathcal{D}_0$ in (47f) since for all $t \in [0, t_1^\star)$, $\hat{x}(t, 0) \in S_{-1}$, hence we can deduce $\hat{b}(t,0)\hat{\sigma}(t,0) \leq 0$, $\hat{\sigma}(t,0)\hat{\phi}(t,0) \geq \frac{k_p}{k_i}\hat{\sigma}(t,0)^2$ and $\hat{b}(t,0)\hat{\phi}(t,0) \leq 0$ from (79a). It also holds

$$\begin{bmatrix} \xi(t_1^\star,1) \\ q(t_1^\star,1) \\ \tau(t_1^\star,1) \end{bmatrix} = g_0\Big(\begin{bmatrix} \hat{\xi}(t_1^\star,0) \\ q(t_1^\star,0) \\ \tau(t_1^\star,0) \end{bmatrix} \Big) = \begin{bmatrix} \hat{\xi}(t_1^\star,0) \\ 0 \\ \tau(t_1^\star,0) \end{bmatrix}.$$

Similarly to the previous case with $q(t_1^\star, 0) = 0$ and $\hat{x}(t_1^\star, 0) \in S_1$, we have $\hat{v}(t_1^\star, 0) = 0$ and $\hat{\phi}(t_1^\star, 0) \geq F_s$. $\hat{x}(t_1^\star, 0) \in S_1$ and $\hat{\phi}(t_1^\star, 0) \geq F_s$ imply again $\hat{b}(t_1^\star, 0) = 1$ from the fact that $\hat{\xi}$ flows on $[0, t_1]$ with $t_1 > t_1^\star$, and the condition $\hat{b}\hat{v}\hat{\phi} \geq 0$ in (79a). Hence, $\begin{bmatrix} \xi(t_1^\star,1) \\ q(t_1^\star,1) \\ \tau(t_1^\star,1) \end{bmatrix} \in \bar{\mathcal{D}}_1$ in (49a) and

$$\begin{bmatrix} \xi(t_1^\star,2) \\ q(t_1^\star,2) \\ \tau(t_1^\star,2) \end{bmatrix} = g_1\Big(\begin{bmatrix} \xi(t_1^\star,1) \\ q(t_1^\star,1) \\ \tau(t_1^\star,1) \end{bmatrix} \Big) = \begin{bmatrix} \hat{\xi}(t_1^\star,0) \\ 1 \\ 0 \end{bmatrix},$$

where $q(t_1^\star, 1) = 1$ allows repeating the reasoning presented for a flow on $[0, t_1^\star]$. Parallel arguments hold in the case $q(t_1^\star, 0) = 1$ and $\hat{x}(t_1^\star, 0) \in S_{-1}$. This concludes the examination of all possible cases.

It is then sufficient to repeat the reasoning presented for a flow on $[0, t_1^\star]$ and the reasoning presented for jumps from $\bar{\mathcal{D}}_1$, $\mathcal{D}_0$, $\bar{\mathcal{D}}_{-1}$ to cover the whole interval I^0 by identifying possibly $t_2^\star$, $t_3^\star$, etc.

Suppose $\hat{\xi}$ jumps from $\hat{\mathcal{D}}_\sigma$ at $(0,0)$. We make the following observation. If $\hat{\xi}$ jumps from $\hat{\mathcal{D}}_\sigma$ at $(0,0)$, $\hat{b}(0,1) = -1$ so $\hat{\xi}$ cannot jump from $\hat{\mathcal{D}}_\sigma$ at $(0,1)$. If $\hat{\xi}$ jumps from $\hat{\mathcal{D}}_v$ at $(0,1)$, then $\hat{v}(0,2) = \hat{v}(0,1) = 0$ (otherwise a jump from $\hat{\mathcal{D}}_v$ cannot occur), $\hat{\sigma}(0,2) = \hat{\sigma}(0,1) = \hat{\sigma}(0,0) = 0$ (otherwise a jump from $\hat{\mathcal{D}}_\sigma$ could not have occurred) and $\hat{\phi}(0,2) = \frac{k_p}{k_i}\hat{\sigma}(0,1) = 0$ due to $\hat{g}_v$ in (21e). Then, two consecutive jumps from $\hat{\mathcal{D}}_\sigma$ and $\hat{\mathcal{D}}_v$ are such that $\hat{\xi}(0,2) \in \hat{\mathcal{A}}$ and we do not need to prove anything in this case.

Based on this observation, the only case to consider is when $\hat{\xi}$ flows in $\hat{\mathcal{C}}$ after the jump from $\hat{\mathcal{D}}_\sigma$. If $\hat{x}(0,1) \in S_1$ and $\hat{\xi}$ flows in $\hat{\mathcal{C}}$, $\hat{v}(0,1) = \hat{v}(0,0) > 0$ (indeed, the case with $\hat{v} = 0$, $\hat{\phi} = F_s$, $\hat{\sigma} > 0$ in S_1 is excluded, and the case with $\hat{v} = 0$, $\hat{\phi} > F_s$ in S_1 is excluded as well because $\hat{v}$ would become positive by Lemma 1(ii), $\hat{b}(0,1) = -1$, and the constraint $\hat{b}\hat{v}\hat{\phi} \geq 0$ in (79a) would be violated). If $\hat{x}(0,1) \in S_1$, we need $q(0,1)$ to be 1 and this is achieved by selecting $q(0,0) = 1$. Since $\hat{v}(0,0) > 0$ (as just motivated), $\hat{\sigma}(0,0) = 0$ (by (80a)), $\hat{b}(0,0) = 1$ (by (80a)), $q(0,0)\hat{v}(0,0) \geq 0$ and $q(0,0)\hat{\phi}(0,0) \geq 0$ (by $\hat{v}(0,0) > 0$ and $\hat{v}\hat{\phi} \geq 0$ in (80a). $\begin{bmatrix} \hat{\xi}(0,0) \\ q(0,0) \\ \tau(0,0) \end{bmatrix} \in \mathcal{D}_\sigma$ and

$$\begin{bmatrix} \hat{\xi}(0,1) \\ q(0,1) \\ \tau(0,1) \end{bmatrix} = g_\sigma\Big(\begin{bmatrix} \hat{\xi}(0,0) \\ q(0,0) \\ \tau(0,0) \end{bmatrix} \Big)$$

because the first four components of g_σ in (47e) coincide with $\hat{g}_\sigma$ in (21e), and $q(0,1) = q(0,0) = 1$ as needed. If $\hat{x}(0,1) \in S_{-1}$, parallel arguments yield the same conclusion by selecting $q(0,0) = -1$. If $\hat{x}(0,1) \in S_0$, $\hat{v}(0,1) = \hat{v}(0,0) = 0$ and $|\hat{\phi}(0,1)| \leq F_s$. Since $\hat{\sigma}(0,1) = 0$, $\hat{\xi}(0,1) \in \hat{\mathcal{A}}$ and we do not need to prove anything in this case[8].

Suppose $\hat{\xi}$ jumps from $\hat{\mathcal{D}}_v$ at $(0,0)$. As noted for the case of $\hat{\xi}$ jumping from $\hat{\mathcal{D}}_\sigma$, $\hat{\xi}(0,1)$ cannot jump from $\hat{\mathcal{D}}_v$ again and if it jumps from $\hat{\mathcal{D}}_\sigma$, $\hat{\xi}(0,2) \in \hat{\mathcal{A}}$. Then, the only case to consider is when $\hat{\xi}$ flows in $\hat{\mathcal{C}}$ after the jump from $\hat{\mathcal{D}}_v$. Then, $\hat{\xi}$ flows in either S_0, S_1, or S_{-1}, depending on $\hat{\sigma}(0,1)$ (recall $\hat{\phi}(0,1) = \frac{k_p}{k_i}\hat{\sigma}(0,1)$), and in all cases we select $q(0,0) =$

[8]The lemma requires that (48) is satisfied for all $t \geq 0$ such that $\hat{\xi}(t, j(t)) \notin \hat{\mathcal{A}}$.

0 in order to jump from the corresponding $\mathcal{D}_v$ in (47f). If $\hat{x}(0,1) \in S_0$, we need $q(0,1)$ to be 0. Since $q(0,0) = 0$, $\hat{v}(0,0) = 0$, $\hat{\sigma}(0,0)\hat{\phi}(0,0) \geq \frac{k_p}{k_i}\hat{\sigma}(0,0)^2$, $\hat{b}(0,0) = -1$ (the last three by (81a)), $\begin{bmatrix} \hat{\xi}(0,0) \\ q(0,0) \\ \tau(0,0) \end{bmatrix} \in \mathcal{D}_v$ and

$$\begin{bmatrix} \hat{\xi}(0,1) \\ q(0,1) \\ \tau(0,1) \end{bmatrix} = g_v\left(\begin{bmatrix} \hat{\xi}(0,0) \\ q(0,0) \\ \tau(0,0) \end{bmatrix} \right)$$

because the first four components of g_v coincide with $\hat{g}_v$ in (21e) and $q(0,1) = q(0,0) = 0$ as needed. If $\hat{x}(0,1) \in S_1$, we need $q(0,2)$ to be 1, which is achieved by jumping additionally from $\bar{\mathcal{D}}_1$ in (49a). Indeed, we have $q(0,1) = 0$, $\hat{v}(0,1) = 0$, $\hat{\sigma}(0,1)\hat{\phi}(0,1) = \frac{k_p}{k_i}\hat{\sigma}(0,1)^2$ (because $\hat{\phi}(0,1) = \frac{k_p}{k_i}\hat{\sigma}(0,0)$ and $\hat{\sigma}(0,1) = \hat{\sigma}(0,0)$), $\hat{\phi}(0,1) \geq F_s$ (because $\hat{x}(0,1) \in \hat{S}_1$), and $\hat{b}(0,1) = -\hat{b}(0,0) = 1$ so that $\begin{bmatrix} \hat{\xi}(0,1) \\ q(0,1) \\ \tau(0,1) \end{bmatrix} \in \bar{\mathcal{D}}_1$ and

$$\begin{bmatrix} \xi(0,2) \\ q(0,2) \\ \tau(0,2) \end{bmatrix} = g_1\left(\begin{bmatrix} \hat{\xi}(0,1) \\ 0 \\ \tau(0,1) \end{bmatrix} \right) = \begin{bmatrix} \hat{\xi}(0,1) \\ 1 \\ 0 \end{bmatrix}$$

with $q(0,2) = 1$ as needed. The case $\hat{x}(0,1) \in S_{-1}$ follows from parallel arguments.

Up to now, we have shown that if $\hat{\xi}$ flows in $\hat{\mathcal{C}}$, jumps from $\hat{\mathcal{D}}_\sigma$ or jumps from $\hat{\mathcal{D}}_v$ at $(0,0)$, then the hybrid signal q can be selected suitably. As we mentioned earlier, we can discard in the proof without loss of generality the cases of two consecutive jumps from $\hat{\mathcal{D}}_\sigma$ and $\hat{\mathcal{D}}_v$, or from $\hat{\mathcal{D}}_v$ and $\hat{\mathcal{D}}_\sigma$, since after these two jumps, $\hat{\xi}$ would belong to $\hat{\mathcal{A}}$. For the proof, this implies that each jump from $\hat{\mathcal{D}}_\sigma$ or from $\hat{\mathcal{D}}_v$ is preceded (except at $(0,0)$, which we have already addressed) and followed by a flow in $\hat{\mathcal{C}}$. In the latter scenario, we have already shown how to select q so that the appropriate flow for $(\hat{\xi}, q, \tau)$ occurs in $\mathcal{H}$. Hence, if we show that, regardless of the selection of q dictated by the preceding flow (the former scenario), a jump from $\mathcal{D}_\sigma$ or from $\mathcal{D}_v$ for $(\hat{\xi}, q, \tau)$ can be achieved, then the procedure outlined for $\hat{\xi}$ flowing in $\hat{\mathcal{C}}$, jumping from $\hat{\mathcal{D}}_\sigma$ or from $\hat{\mathcal{D}}_v$ at $(0,0)$, can be easily extended for all $t \geq 0$ such that $\hat{\xi}(t, j(t)) \notin \hat{\mathcal{A}}$ and the proof of the lemma is complete. We show then this last point, i.e., that regardless of the selection of q dictated by the preceding flow, a jump from $\mathcal{D}_\sigma$ or from $\mathcal{D}_v$ for $(\hat{\xi}, q, \tau)$ can be achieved.

Suppose $\hat{\xi}$ jumps from $\hat{\mathcal{D}}_\sigma$ at (t,j) after a flow in $\hat{\mathcal{C}}$. Note that because of the extra jumps from $\bar{\mathcal{D}}_1$, $\mathcal{D}_0$, $\bar{\mathcal{D}}_{-1}$ that may have appeared so far, we may need to reparametrize the jump counter as follows. For each $(t,j) \in \operatorname{dom} \hat{\xi}$, there exist $j^\star \geq 0$ such that $(t, j+j^\star) \in \operatorname{dom} q$. If $|q(t,j+j^\star)| = 1$ from the preceding flow, a jump from $\mathcal{D}_\sigma$ is achieved since $|q(t,j+j^\star)| = 1$, $\hat{\sigma}(t,j) = 0$ and $\hat{b}(t,j) = 1$ (both by (80a)), $q(t,j+j^\star)\hat{v}(t,j) \geq 0$ and $q(t,j+j^\star)\hat{\phi}(t,j) \geq 0$ (both since $(\hat{\xi}, q, \tau)$ flowed in $\mathcal{C}_{\text{slip}}$), . If $q(t,j+j^\star) = 0$ from the preceding flow, $(\hat{\xi}, q, \tau)$ flowed in $\mathcal{C}_{\text{stick}}$ so $\hat{v}(t,j) = 0$ and $|\hat{\phi}(t,j)| \leq F_s$. These two conditions together with $\hat{\sigma}(t,j) = 0$ (by (80a)), imply that $\hat{\xi}(t,j) \in \hat{\mathcal{A}}$ so there is nothing to check.

Suppose $\hat{\xi}$ jumps from $\hat{\mathcal{D}}_v$ at (t,j) after a flow in $\hat{\mathcal{C}}$. Adopt the same jump reparametrization through $j^\star$ described for a jump from $\hat{\mathcal{D}}_\sigma$ at (t,j). If $q(t,j+j^\star) = 0$ from the preceding flow, a jump from $\mathcal{D}_v$ is achieved thanks to (81a). If $|q(t,j+j^\star)| = 1$ from the preceding flow, $(\hat{\xi}, q, \tau)$ flowed in $\mathcal{C}_{\text{slip}}$ so that

$$\begin{aligned} &\hat{b}(t,j)q(t,j+j^\star)\hat{\sigma}(t,j) \geq 0 \\ &\hat{\sigma}(t,j)\hat{\phi}(t,j) \geq \tfrac{k_p}{k_i}\hat{\sigma}(t,j)^2 \\ &\hat{b}(t,j)q(t,j+j^\star)\hat{\phi}(t,j) \geq 0. \end{aligned} \tag{87}$$

Then, a jump from $\mathcal{D}_0$ is possible since $|q(t,j+j^\star)| = 1$, $\hat{v}(t,j) = 0$ (by (81a)) and (87) holds since $(\hat{\xi}, q, \tau)$ flowed in $\mathcal{C}_{\text{slip}}$. By jumping from $\mathcal{D}_0$, $\hat{\xi}$ does not change and $q(t,j+j^\star+1) = 0$ so that we fall back to the case $q(t,j+j^\star) = 0$ just analyzed.

In summary, for each solution $\hat{\xi}$ to $\hat{\mathcal{H}}$ with $\hat{\xi}(0,0) = \hat{\xi}_0 \in \mathcal{K}$ that flows in $\hat{\mathcal{C}}$, jumps from $\hat{\mathcal{D}}_\sigma$ or jumps from $\hat{\mathcal{D}}_v$, we have shown how to construct a suitable hybrid arc q so that $(\hat{\xi}, q, \tau)$ is a solution to $\mathcal{H}_{\delta(\mathcal{K})}$ (modulo a reparametrization of the jump counter of $\hat{\xi}$) for all $t \geq 0$ such that $\hat{\xi}(t, j(t)) \notin \hat{\mathcal{A}}$, hence (48) holds by construction and the proof is complete.

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