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ON POLYNOMIALS IN PRIMES, ERGODIC AVERAGES AND MONOTHETIC GROUPS

JAROSLAV HANČL[†], RADHAKRISHNAN NAIR[‡], AND JEAN-LOUIS VERGER-GAUGRY[◇]

ABSTRACT. Let G denote a compact monothetic group, and let

$$\rho(x) = \alpha_k x^k + \ldots + \alpha_1 x + \alpha_0,$$

where $\alpha_0, \ldots, \alpha_k$ are elements of G one of which is a generator of G. Let $(p_n)_{n\geq 1}$ denote the sequence of rational prime numbers. Suppose $f \in L^p(G)$ for $p > 1$. It is known that if

$$A_N f(x) := \frac{1}{N}\sum_{n=1}^{N} f(x+\rho(p_n)) \qquad (N = 1,2,\ldots),$$

then the limit $\lim_{n\to\infty} A_N f(x)$ exists for almost all x with respect Haar measure. We show that if G is connected then the limit is $\int_G f d\lambda$. In the case where G is the a-adic integers, which is a totally disconnected group, the limit is described in terms of Fourier multipliers which are generalizations of Gauss sums.

Keywords: Monothetic groups, a-adic numbers, polynomial in primes, ergodic averages, Fourier multiplier.

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CONTENTS

1. INTRODUCTION

Topological groups, having a dense cyclic subgroup have been well studied. For the class of locally compact groups, following D. Van Dantzig, who was the first to study them [**vD**], these groups are called monothetic. Such groups are fully classified in [**HeR**]. See also G. Falcone, P. Plaumann and K. Strambach [**FPS**] and D. Dikranjan and A.G. Bruno [**DB**]. Some examples are also given by J.W. Nienhuys [**Ni**].

Following [**HeR**] we say a topological group G is monothetic if it contains an element α such that the closure of $(n\alpha)_{n=1}^{\infty}$ is G. Evidently by approximation, for arbitrary monothetic G, the density and commutativity of $(n\alpha)_{n=1}^{\infty}$ implies the commutativity of G. For a finite set of elements $\alpha_0, \alpha_1, \ldots, \alpha_k \in G$ set

$$\rho(n) = \alpha_k n^k + \alpha_{k-1} n^{k-1} + \ldots + \alpha_1 n + \alpha_0,$$

for each $n \in \mathbb{N}$. Let $(p_n)_{n\geq 1}$ the sequence of rational primes. Let λ denote Haar measure on G and let $L^p(G)$ denote the L^p space of λ-integrable functions on G.

We show the following.

Theorem 1.1. *Suppose G is a compact, connected, monothetic group. Then if one of the elements $\alpha_1, \ldots, \alpha_k \in G$ is a generator and $f \in L^p(G)$ for $p > 1$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x + \rho(p_n)) = \int_G f d\lambda,$$

for almost all x with respect to Haar measure on G.

If we drop the requirement that G is connected the situation is different. We now consider a class of totally disconnected monothetic groups of arithmetic character, of which the p-adic numbers is a special case.

Let $a = (a_i)_{i\in\mathbb{Z}}$ denote a doubly infinite sequence of integers each greater than 1. Set

$$\mathbb{Q}_a := \Pi_{i\in\mathbb{Z}} \{0, 1, \ldots, a_i - 1\},$$

i.e. the space of doubly infinite sequences $a = (a_i)_{i\in\mathbb{Z}}$ with $a_i = 0$ for $i < n_o = n_o(a)$ for some n_o.

Without loss of generality, assume for $x, y \in \mathbb{Q}_a$ that $n_o(x) = n_o(y) = 0$ and that $x = (x_n)_{n=0}^{\infty}$ and $y = (y_n)_{n=0}^{\infty}$ let $z = (z_n)_{n=0}^{\infty}$ be defined as follows. Write $x_0 + y_0 = t_0 a_0 + z_0$, where $z_0 \in \{0, 1, ..., a_0 - 1\}$ and t_0 is a rational integer. Suppose $z_0, \cdots, z_k$ and $t_0, \cdots, t_k$ have been defined. Then write $x_{k+1} + y_{k+1} + t_k = t_{k+1} a_{k+1} + z_{k+1}$, where $z_{k+1} \in \{0, 1, ..., a_{k+1} - 1\}$ and t_{k+1} is a rational integer. We have thus inductively defined the sequence $z = (z_n)_{n=0}^{\infty}$, which we deem to be $x + y$. The binary operation $+$ which we call addition makes $\mathbb{Q}_a$ an abelian group.

For each non-negative integer k let

$$\Lambda_k = \{x \in \mathbb{Q}_a : x_n = 0 \; if \; n < k\}.$$

These sets form a basis at $0 = (0, 0, \cdots)$ for a topology on $\mathbb{Q}_a$. With respect to this topology $\mathbb{Q}_a$ is compact and the group operations are continuous making $\mathbb{Q}_a$ a locally-compact σ-compact abelian topological group. A second binary operation called multiplication, denoted by $\times$ and compatible with addition is defined as follows. Let $u = (1, 0, 0, \cdots)$.

Note that $(nu)_{n=0}^{\infty}$ is dense in $\mathbb{Q}_a$. First on $(nu)_{n=0}^{\infty}$ define $k_1u \times k_2u$ to be k_1k_2u. Deeming multiplication to be continuous on $\mathbb{Q}_a$ defines it off $(nu)_{n=0}^{\infty}$. The binary operations addition and multiplication makes $\mathbb{Q}_a$ a topological ring . We call $\mathbb{Z}_a := \Lambda_0$ the a-adic integers and it is compact sub-ring of $\mathbb{Q}_a$. Let η_{a_i} denoted the counting measure on $\{0,1,\ldots,a_i-1\}$. For each integer let $\mu = \otimes_{i\in\mathbb{Z}}\eta_{a_i}$ denote the corresponding product measure on $\Pi_{n\in\mathbb{Z}}\{0,\ldots,a_n-1\}$, defined first on "cylinder sets". By cylinder sets we mean sets of the form

$$A = \{(b_i)_{i\in\mathbb{Z}} : b_{i_1} = c_{i_1},\ldots,b_{i_r} = c_{i_r}\}$$

for finite r, fixed $\{i_1,\ldots,i_r\} \subset \mathbb{Z}$ and specific $c_{i_1},\ldots c_{i_r}$. We then specify μ by setting

$$\mu(A) = (a_{i_1}\ldots a_{i_r})^{-1}.$$

on cylinder. We then extend and complete to all Haar measurable sets on the topological group $\mathbb{Q}_{\mathbf{a}}$.

The dual group of $\mathbb{Z}_a$, which we denote $\hat{\mathbb{Z}}_a$ consists of all rationals $t = \frac{\ell}{A_r}$ where $A_r = a_0\cdots a_r$ and $0 \le \ell \le A_r$ for some non-negative integer r. To evaluate a character χ_t at x in $\mathbb{Z}_a$ we write

$$\chi_t(x) = e\left(\frac{\ell}{A_r}(x_0 + a_0x_1 + \cdots + a_0\cdots a_{r-1}x_r)\right),$$

where as usual, for a real number x, $e(x)$ denotes $e^{2\pi ix}$. We note that $\hat{\mathbb{Q}}_a = \mathbb{Q}_{a^*}$ where $a^* = (a_{-n})_{n\in\mathbb{Z}}$.

Suppose at least one of the numbers $\alpha_1,\ldots,\alpha_k$ is a generator of $\mathbb{Z}_a$. Let $\delta_j : \mathbb{Z} \to \mathbb{Z}_a$ be the homomorphism defined by $\delta_j(n) = n\alpha_j$ $(j = 1,2,\ldots,k)$ and let $\varepsilon_j : \hat{\mathbb{Z}}_a \to \mathbb{Q}_a$ denote its adjoint. Hence for all $n \in \mathbb{Z}$ we have

$$e^{\varepsilon_j\left(\frac{l}{a_0\ldots a_r}\right)} = \chi_{\frac{l}{a_0\ldots a_r}}\left(\delta_j(n)\right) = \chi_{\frac{l}{a_0\ldots a_r}}(n\alpha_j).$$

We can identify $\hat{\mathbb{Z}}_a$ with the quotient of $\mathbb{Q}_{a^*}$ by $A(\mathbb{Q}_{a^*},\hat{\mathbb{Z}}_a)$ the annihilator of $\hat{\mathbb{Z}}_a$ in $\mathbb{Q}_{a^*}$ ([**HeR**], Lemma 24.5). Let $\Psi : \mathbb{Q}_{a^*} \to \hat{\mathbb{Z}}_a$ be the associated quotient map. Consider the map G, dependent on $\alpha_1,\ldots\alpha_k$, mapping $\hat{\mathbb{Z}}_a$ to $\mathbb{C}$, given by

$$G\left(\chi_{\frac{l}{a_0\ldots a_r}}\right) = \frac{1}{\phi(D_r)}\sum_{\substack{m=1\\(m,D_r)=1}}^{D_r} e\left(\frac{\gamma(m)}{D_r}\right),$$

for all $\frac{l}{a_0\ldots a_r} \in \hat{\mathbb{Z}}_a$. Here, ϕ is the Euler totient function. The integers D_r and polynomial γ are described as follows. Let

$$l_j = l(\alpha_j(0) + \alpha_j(1)a_o + \alpha_j(2)a_oa_1 + \ldots + \alpha_j(r-1)a_0\ldots a_{r-2}). \qquad (j = 1,2,\ldots,k).$$

Here $\alpha_j(r)$ denote the r^{th} term of α_j, viewed as a sequence. In general $\frac{l_j}{a_0\ldots a_r}$ is not in reduced form. Let $\frac{m_j}{B_j}$ with $(m_j,B_j) = 1$ denote $\frac{l_j}{a_0a_1\ldots a_r}$ in reduced terms. Let D_r denote the least common multiple of the numbers $B_1,\ldots,B_k$ and define γ via the identity

$$\frac{\gamma(x)}{D_r} = \frac{m_k}{B_k}x^k + \ldots + \frac{m_1}{B_1}x.$$

Define $m : \mathbb{Q}_{a^*} \to \mathbb{C}$ by $m(\chi) = G(\Psi(\chi))$ for all $\chi \in \mathbb{Q}_{a^*}$. Henceforth $\mathbb{F}(f)$ denotes the Fourier transform of f.

Theorem 1.2. *Suppose that $p \in (1,2]$ and that $f \in L^p \cap L^2(\mathbb{Q}_a)$. Also suppose*

$$\rho(n) = \alpha_k n^k + \alpha_{k-1} n^{k-1} + \ldots + \alpha_1 n + \alpha_0,$$

has degree at least 2, with one of the numbers $\{\alpha_1, \ldots, \alpha_k\} \subset \mathbb{Q}_a$ a generator of $\mathbb{Q}_a$. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x+\rho(p_n)) = \ell(f)(x),$$

for $\ell(f) \in L^p$ almost everywhere with respect to Haar measure on $\mathbb{Q}_a$, where

$$\mathbb{F}(\ell(f))(\chi) = m(\chi)\mathbb{F}(f)(\chi)$$

for almost all x with respect to Haar measure on $\mathbb{Q}_a$.

Theorem 1.3. *Suppose that $p \in (1,2]$ and that $f \in L^p(\mathbb{Z}_a)$. Also suppose that*

$$\rho(n) = \alpha_k n^k + \alpha_{k-1} n^{k-1} + \ldots + \alpha_1 n + \alpha_0,$$

has degree at least 2, with one of the numbers $\{\alpha_1, \ldots, \alpha_k\} \subset \mathbb{Z}_a$ a generator of $\mathbb{Z}_a$. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x+\rho(p_n)) = \ell(f)(x),$$

for $\ell(f) \in L^p(\mathbb{Z}_a)$ almost everywhere with respect to Haar measure on $\mathbb{Z}_a$. Here $\mathbb{F}(\ell(f))(\chi) = G(\chi)(f)(\chi)$ for almost all x with respect to Haar measure on $\mathbb{Z}_a$.

A measurable transformation $T : X \to X$ of a measure space (X, β, μ) is called measure preserving if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \beta$. Here $T^{-1}(A)) := \{x \in X : Tx \in A\}$. For a measure space (X, β, μ), let $T_1, \ldots, T_k$ be a finite case set of commuting measure preserving transformations. Given $f \in L^p(X, \beta, \mu)$ if π_N denotes the number of prime numbers in the interval $[1, N]$, for $(N = 1, 2, \ldots)$ we set

$$A_N f(x) := \frac{1}{\pi_N} \sum_{1 \le p \le N} f(T_1^p T_2^{p^2} \ldots T_k^{p^k} x). \tag{1.0.1}$$

Here the summation parameter p runs over the primes. The pointwise convergence of these averages is proved in [**T**]. This is the new tool, that enables us to prove Theorems 1.1 – 1.3 going beyond the results in [**N3**] and [**N4**].

Some background : A sequence $(x_n)_{n\ge1}$ in a compact topological group is said to be uniformly distibuted on G if for each continuous functions $f : G \to \mathbb{C}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_G f d\lambda.$$

An example is $x_n = \{n\alpha\}$ $(n = 1, 2, \ldots)$, where α is an irrational number and for a real number y we have used $\{y\}$ to denote its fractional part. This early result was due to P. Bohl, W. Sierpinski and H. Weyl independently around the start of the 20th century. See the notes in [**KN**] for the historic background. Subsequently it was shown by H. Weyl [**W**] that of $\rho(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_k x^k$ and one of the numbers $\{\alpha_1, \ldots, \alpha_k\}$ is irrational then the sequences $x_n = \rho(n)$ $(n = 1, 2, \ldots)$ is uniformly distributed modulo 1. Later I. M. Vinogradov

and G. Rhin [**R**] proved that $x_n = \rho(p_n)$ $(n = 1,2,\ldots)$ is uniformly distributed modulo 1. Another result with a form similar in statement is that the result of Bohl, Sierpinski is that if f is Lebesgue integrable on $[0,1)$ then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x+\alpha n\}) = \int_0^1 f(t)dt, \tag{1.0.2}$$

almost everywhere with respect to Lebesgue measure. This is an immediate consequence of Birkhoff's pointwise ergodic theorem and the ergodicty of the Lebesgue measure preserving map $T : x \to x+\alpha$ for irrational α on $[0,1)$. An issue addressed by a number of authors is whether $(\{n\alpha\})_{n\geq 1}$ in (1.0.2) can be replaced by either $(\rho(n))_{n\geq 1}$ or $(\rho(p_n))_{n\geq 1}$ possibly under additional conditions on f. See for instance [**KS**], where the following is shown. Suppose that $\rho(n) = \alpha_0 + \alpha_1 n + \ldots, \alpha_k n^k$ with $(\alpha_1,\ldots,\alpha_k) \notin \mathbb{Q}^k$. Also for $f \in L^2([0,1))$ with Fourier coefficients $(c_n)_{n\in\mathbb{Z}}$ suppose for some $\gamma > 0$ that

$$\sum_{|n|\geq N} |c_n|^2 = O(\log_e N)^{\gamma}). \tag{1.0.3}$$

Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x+\rho(n)\}) = \int_0^1 f(t)dt, \tag{1.0.4}$$

almost everywhere with respect to Lebesgue measure on $[0,1)$.

The second author was introduced to this topic when asked in private communication independently and at different times by R.C. Baker and M. Weber, whether condition (1.0.3) could be weakened or removed. It turns out the answer is yes under the additional assumption that $f \in L^p([0,1))$ for $p > 1$ and indeed in more general settings than $G = [0,1)$. This is the content of the papers [**N3**], [**N4**], and the current paper. The condition $p \neq 1$ is essential however because of the results in [**BM**]. Notice the x in (1.0.2) cannot in general be chosen to be 0. This is the implication of J. Marstrand's famous result [**M**], that there exists a G_δ set B such that if $f(x) = I_B(x)$, i.e. the indicator set of B, then the limit in (1.0.2) fails to exist with $x = 0$ for almost all α.

2. Some lemmas

The following is a special case of S. Sawyer's Theorem [**S**].

Lemma 2.1. *The pointwise convergence of the averages $(A_N f)_{N\geq 1}$ in* (1.0.1) *for $p > 1$ implies that if*

$$Mf(x) = \sup_{N\geq 1} |A_N f(x)|,$$

then there exists $C > 0$ such that $||Mf||_p \leq C||f||_p$.

Lemma 2.2. *For each $\chi = \chi_{\frac{l}{a_0\ldots a_r}} \in \mathbb{Z}_{a^*}$ we have*

$$G(\chi) = \lim_{N\to\infty} \frac{1}{\pi_N} \sum_{1\leq p\leq N} \chi(\rho(p)),$$

where p runs over the prime numbers in $[1,N]$. Further for $\chi \in \mathbb{Q}_{a^*}$ we have

$$m(\chi) = \lim_{N\to\infty} \frac{1}{\pi_N} \sum_{1\le p\le N} \chi(\rho(p)).$$

Proof. Note that

$$\chi_{\frac{l}{a_0\dots a_r}}(\rho(p_n)) = \chi_{\frac{l}{a_0\dots a_r}}(\alpha_0+\alpha_1 p_n+\ldots+\alpha_k p_n^k)$$

$$= \prod_{j=0}^{k} \chi_{\frac{l}{a_0\dots a_r}}(\alpha_j)^{p_n^j}$$

$$= \prod_{j=0}^{k} \left(e^{2\pi i \frac{l}{a_o\dots a_r} p_n^j(\alpha_j(0)+\alpha_j(1)a_1+\ldots+\alpha_j(r-1)a_0a_1\dots a_{r-2})} \right)$$

$$= \prod_{j=0}^{k} \left(e^{2\pi i \frac{l_j}{a_o\dots a_r}} \right)^{p_n^j} = \prod_{j=0}^{k} \left(e^{\frac{2\pi i m_j}{B_j}} \right)^{p_n^j} = e^{2\pi i \frac{\gamma(p_n)}{D_r}}.$$

By splitting the primes into their reduced residue classes modulo D_r we have

$$\frac{1}{\pi_N} \sum_{1\le p\le N} \chi_{\frac{l}{a_0\dots a_r}}(\rho(p)) = \frac{1}{\pi_N} \sum_{1\le p\le N} e^{2\pi i \frac{\gamma(p)}{D_r}}. \qquad (N=2,3,\ldots)$$

Let $\Lambda : \mathbb{N} \to \mathbb{R}$ denote the Von Mangoldt function defined by $\Lambda(n) = \log_e p$ if $n = p^l$ for some prime p and positive integer l, and zero otherwise. Using partial summation we see that

$$\frac{1}{\pi_N} \sum_{1\le p\le N} e^{2\pi i \frac{\gamma(p)}{D_r}} = \frac{1}{N} \sum_{1\le n\le N} \Lambda(n) e^{\frac{2\pi i \gamma(n)}{D_r}} + O((\log_e N)^{-1}).$$

The Siegel -Walfish prime number theory for arithmetic progressions, says that for a fixed positive u, if $1 \le D_r \le (\log_e N)^u$ and $(m, D_r) = 1$, then for some $C > 0$

$$\sum_{\substack{1\le n\le N \\ n\equiv m \bmod D_r}} \Lambda(n) = \frac{N}{\phi(D_r)} + o(Ne^{-C(\log_e N)^{\frac{1}{2}}}).$$

Now note that

$$\sum_{1\le n\le N} \Lambda(n) e^{\frac{2\pi i \gamma(n)}{D_r}} = \left(\sum_{\substack{m=1 \\ (m,D_r)=1}}^{D_r} e^{2\pi i \frac{\gamma(m)}{D_r}} \right) \left(\sum_{\substack{1\le n\le N \\ n\equiv m \bmod D_r}} \Lambda(n) \right)$$

$$+ O\left(\sum_{p^l\le N; p\mid D_r} \Lambda(p^l) e^{\frac{2\pi i \gamma(p^l)}{D_r}} \right)$$

Using the fact that the sum inside the O notation is $O((\log_e N)(\log_e \log N))$, and the Siegel-Walfish theorem

$$\frac{1}{N} \sum_{\substack{1\le n\le N \\ n\equiv m \bmod D_r}} \Lambda(n) e^{\frac{2\pi i \gamma(n)}{D_r}} = \left(\frac{1}{\phi(D_r)} \sum_{1\le n\le N} \Lambda(n) e^{\frac{2\pi i \gamma(n)}{D_r}} \right)$$

$$+O\left(\frac{(\log_e N)(\log_e \log_e N)}{N}\right),$$

which proves the first part of Lemma 2.2. The second part follows from the observation for all $x \in \mathbb{Z}_a$ and $\chi \in \mathbb{Q}_{a^*}$ we have $\chi(x) = \Psi(\chi)(x)$. □

3. Proof of Theorem 1.1

For the averages (1.0.1), let X be G, let μ be Haar measure λ on G and let β denote Haar measurable sets in G. Then if we set $T_i = x + \alpha_i$ for $i = 1, \ldots, k$, for $N = 1, 2, \ldots$ and set

$$a_N(f,x) = A_N f(x) := \frac{1}{\pi_N} \sum_{1 \le p \le N} f(x + \rho(p)).$$

We wish to show $a_N(f,x)$ tends to $\int_G f d\lambda$ almost everywhere with respect to Haar measure, as N tends to ∞. First suppose $f : G \to \mathbb{C}$ is continuous. Because G is a compact, connected

and monothetic, for each non-trivial character χ, and each generator $\chi(\alpha) = e^{2\pi i \alpha_*}$ for an irrational number α_* ([**KN**], p. 275). Thus $\phi(\rho(n)) = e^{2\pi i \rho_*(n)}$, for a polynomial ρ_* defined on the real numbers, with at least one irrational coefficient other than the constant term. This means that

$$\frac{1}{N} \sum_{n=1}^{N} \chi(\rho(p_n)) = \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \rho_*(p_n)}. \qquad (N = 1, 2, \ldots)$$

Now, if $\{y\}$ denotes the fractional part of a real number y, then the sequence $(\{\rho_*(p_n)\})_{n \ge 1}$ is uniformly distributed modulo 1. Hence by Weyl's criterion on G, $a_N(f,x)$ tends to $\int_G f d\lambda$ for all continuous $f : G \to \mathbb{C}$. We now deal with the general case of Theorem 1. Suppose (f_n) is a sequence of continuous functions converging to fixed $f \in L^p$. Choose $(n_k)_{k \ge 1}$ such that

$$\sum_{k \ge 1} \int_G |f - f_{n_k}|^p d\lambda < \infty.$$

This means

$$\sum_{k \ge 1} |f - f_{n_k}|^p < \infty,$$

almost everywhere with respect to Haar measure on G. Thus for each $\varepsilon > 0$, there exists a sequence of functions $(f_{\varepsilon,k})_{k \ge 1}$ such that $||f - f_{\varepsilon,k}||_p^p \le \varepsilon^{2k}$ and $f_{\varepsilon,k}$ tends to f as k tends to infinity, almost everywhere with respect to A with respect to Haar measure. Let

$$m(f) = \sup_{N \ge 1} |\alpha_N f|.$$

Notice that m is sub-additive i.e.

$$m(f + g) \le m(f) + m(g).$$

Let

$$E_{\varepsilon,k} = \{x \in G : m(f - f_{\varepsilon,k})(x) > e^{\frac{k}{p}}\}.$$

Notice that,

$$\mu(E_{\varepsilon,k}) \leq \left(\frac{1}{\varepsilon}\right)^k \int_{E_{\varepsilon,k}} [m(f-f_{\varepsilon,k})(x)]^p d\lambda$$

$$\leq \left(\frac{1}{\varepsilon}\right)^k ||m(f-f_{\varepsilon,k})(x)||_P^P,$$

which, using the fact that $||Mf||_p \leq C_p||f||$, is

$$\leq c\left(\frac{1}{\varepsilon}\right)^k ||((f-f_{\varepsilon,k})(x)||_P^P,$$

$$\leq c\left(\frac{1}{\varepsilon}\right)^k .\varepsilon^{2k} = C\varepsilon^k.$$

Now

$$a_N(f,x) = a_n(f-f_{\varepsilon,k},x) + a_N(f_{\varepsilon,k},x).$$

This means that

$$\left|a_N(f,x) - \int_G f d\lambda| \leq |a_N(f-f_{\varepsilon,k},x)| + |a_N(f_{\varepsilon,k},x) - \int_G f d\lambda\right|$$

We have already shown that

$$\lim_{N\to\infty} a_N(f_{\varepsilon,k},x) = \int_G f_{\varepsilon,k} d\lambda,$$

almost everywhere with respect to Haar measure on G. Therefore

$$\limsup_{N\to\infty}\left|a_N(f,x) - \int_G f d\lambda\right| \leq \limsup_{N\to\infty}\left|a_N(f-f_{\varepsilon,k},x)\right| +$$

$$+\left|\int_G f d\lambda - \int_G f_{\varepsilon,k} d\lambda\right|$$

$$\leq m(f-f_{\varepsilon,k}) + \int_G \left|f-f_{\varepsilon,k}\right| d\lambda.$$

Thus as N tends to infinity we know $a_N(f,x)$ tends to $\int_G f d\lambda$ for all x in $A_\varepsilon = \cup_{k\geq 1} E_{\varepsilon,k}$. Let B_ε denote the null set of which $f_{\varepsilon,k}$ tends to f as k tends to infinity. This means that

$$\mu(A_\varepsilon \cup B_\varepsilon) \leq \sum_{k\geq 1} \mu(E_{\varepsilon,k}) \leq C\sum_{k\geq 1} \varepsilon^k = \frac{C\varepsilon}{1-\varepsilon}.$$

Letting $\varepsilon \to 0$ completes the proof of Theorem 1.1, upon noting that $\pi_N \sim \frac{N}{\ln(N)}$ by the prime number theorem.

4. Proof of Theorems 1.2 and 1.3

Fix $p \in (1,\infty)$, let $f \in L^p(\mathbb{Q}_a)$, and suppose the support of f is contained in Λ_k for some non-positive integer k. This means $f = fI_{\Lambda_k}$, where for a set A, we let I_A denotes the indicator function. Applying the Stone-Weierstrass theorem shows that f can be approximated in the L^p norm functions of the form

$$I_{\Lambda_k} \sum_{j=1}^{\nu} b_j \chi_j, \tag{4.0.1}$$

with each $b_j \in \mathbb{C}$ and $\chi \in \mathbb{Q}_{a^*}$. Because compactly supported functions are dense in $L^p(\mathbb{Q}_a)$, it follows functions of the form are also dense in $L^p(\mathbb{Q}_a)$. It is clear for each N that the averaging operator A_N commutes with translations on $\mathbb{Q}_a$. We know by Lemma 2.1 that the function $\ell f(x)$ exists almost everywhere. Also as a consequence of Lemma 2.1 and the Lebesgue dominated covergence theorem, it follows that the functions $(A_N f(x))_{N\geq 1}$ also converges in L^p to the same limit. Evidently the operator ℓ commutes with translations on $\mathbb{Q}_a$. It follows that the linear operator ℓ is a Fourier multiplier on $L^p(\mathbb{Q}_a)$. Hence there is a bounded measurable function m on $\mathbb{Q}_a$ such that for all $f \in L^p \cap L^2(\mathbb{Q}_a)$ we have

$$\mathbb{F}(\ell f)\,(\chi) = m(\chi)\mathbb{F}(f)(\chi),$$

almost everywhere in $\mathbb{Q}_a$. To identify m we evaluate ℓ on functions of the form (4.1) we note that

$$\mathbb{F}(f)(\xi) = \sum_{j=1}^{\nu} b_j \mathbb{F}(\chi_j I_{\Lambda_k})(\xi) = \sum_{j=1}^{\nu} b_j \mathbb{F}(I_{\Lambda_k})(\xi + \chi_j)$$

$$= \lambda(\Lambda_k) \sum_{j=1}^{\nu} b_j I_{\chi_j + A(\mathbb{Q}_{a^*},\Lambda_k)}(\xi), \tag{4.0.2}$$

where $A(\mathbb{Q}_{a^*},\Lambda_k)$ denotes the annihilator in $\mathbb{Q}_{a^*}$ of Λ_k. The penultimate identity follows from the fact that multiplication by a character shifts the Fourier transform. The last identity follows from the identity

$$\mathbb{F}\left(\frac{1}{\lambda(\Lambda_k)} I_{\Lambda_k}\right) = I_{A(\mathbb{Q}_{a^*},\Lambda_k)}. \tag{4.0.3}$$

Indeed if $\chi \in \mathbb{Q}_{a^*}$ the restriction of χ to Λ_k is plainly a continuous character of Λ_k and (4.0.3) follows since the Haar integral on Λ_k of any non-trivial character of Λ_k is zero ([**HeR**], Lemma 23.19, p. 363). For f as in (4.0.1) we have

$$A_N f(x) = \frac{1}{N} \sum_{n=1}^{N} I_{\Lambda_k}(x + \rho(p_n) \sum_{j=1}^{\nu} b_j \chi_j(x + \rho(p_n)).$$

where $(p_n)_{n\geq 1}$ denotes the strictly increasing sequence of all primes. If x is not in Λ_k then nor is $x + \rho(p_n)$ because $\rho(p_n) \in \Lambda_k$. This means that we have

$$A_N f(x) = A_N f(x) I_{\Lambda_k}(x). \tag{4.0.4}$$

For $x \in \Lambda_k$ we have

$$A_N f(x) = \frac{1}{N} \sum_{n=1}^{N} \sum_{j=1}^{\nu} b_j \chi_j(x + \rho(p_n)).$$

$$= \sum_{j=1}^{\nu} \chi(x) \frac{1}{N} \sum_{n=1}^{N} \chi_j(\rho(p_n)).$$

By Lemma 2.2 and (4.0.4) for all $x \in \mathbb{Q}_a$ we get

$$\ell(f)(x) = \lim_{N\to\infty} I_{\Lambda_k}(x) A_N f(x) = I_{\Lambda_k}(x) \sum_{j=1}^{\nu} b_j G(\Psi(\chi_j))\chi_j(x) \tag{4.0.5}$$

$$= I_{\Lambda_k}(x) \sum_{j=1}^{\nu} b_j G(\Psi m(\chi_j))\chi_j(x).$$

Computations similar to (4.0.2) and (4.0.5), show for all $\chi \in \mathbb{Q}_{a^*}$

$$\mathbb{F}(\ell f)(\chi) = \lambda(\Lambda_k) \sum_{j=1}^{\nu} b_j G(\Psi(\chi_j)) I_{\lambda_j + A(\mathbb{Z}_{a^*}, \Lambda_k)}(\chi). \tag{4.0.6}$$

If $\chi \in \chi_j + A(\mathbb{Q}_{a^*}, \Lambda_k)$ then $\chi = \chi_j + \chi'$ with $\chi' \in A(\mathbb{Q}_{a^*}, \Lambda_k)$. Consequently $\Psi(\chi) = \Psi(\chi_j) + \Psi(\chi')$. Recall $\Psi : \mathbb{Q}_{a^*} \to \mathbb{Q}_{a^*} \backslash A(\mathbb{Q}_{a^*}, \Lambda_k)$ and χ' is in $A(\mathbb{Q}_{a^*}, \Lambda_k)$, which is a subset of $A(\mathbb{Q}_{a^*}, \mathbb{Z}_a)$ because $\mathbb{Z}_a$ is a subset of Λ_k. Hence if $\chi \in \chi_j + A(\mathbb{Q}_{a^*}, \mathbb{Z}_a)$, then $\Psi(\chi) = \Psi(\chi_j)$. Using (4.0.2) and (4.0.4) we have that for all $x \in \mathbb{Q}_{a^*}$

$$\mathbb{F}(\ell(f))(x)(\chi) = m(\chi)\mathbb{F}(f)(x)(\chi),$$

establishing the theorem for all functions of the form (4.0.1). To prove it in general, given $f \in L^p \cap L^2(\mathbb{Q}_a)$, let $(f_n)_{n\geq 1}$ be a sequence of functions of the form (4.0.1) which converge to $f \in L^2$. Because ℓ is continuous $(\ell(f_n))_{n\geq 1}$ converges to $\ell(f)$ in $L^2(\mathbb{Q}_a)$. Hence a subsequence of $(\mathbb{F}(\ell(f_n))_{n\geq 1}$ converges almost everywhere in $\mathbb{Q}_{a^*}$ to $\mathbb{F}(\ell(f))$. The desired conclusion follows from the fact that $\mathbb{F}(\ell(f_n)) = m(\chi)\mathbb{F}(f_n)$ and a subsequence of $(\mathbb{F}(f_n))_{n\geq 1}$ converges to $\mathbb{F}(f)$ almost everywhere.

To prove Theorem 1.3, we need only observe that $f \in L^p(\mathbb{Z}_a)$ then it may be approximated by functions in $f \in L^p(\mathbb{Z}_a)$ of the form (4.0.1) with $k = 0$ as required.

5. More on the Limit Function $\ell(\mathbf{f})$

Define $H : \hat{\mathbb{Z}}_a \to \mathbb{C}$, dependent on $\alpha_1, \ldots \alpha_k$ by

$$H\left(\frac{l}{a_0 \ldots a_k}\right) := \frac{1}{D_k} \sum_{m=1}^{D_k} e\left(\frac{\gamma(m)}{D_k}\right).$$

Also define $n : \mathbb{Q}_{a^*} \to \mathbb{C}$ by $n(\chi) := H(\Psi(\chi))$. In [**AN**] we show the following lemma and the two theorems

Lemma 5.1. *For each $\chi = \chi_{\frac{l}{a_0 \ldots a_r}} \in \mathbb{Z}_{a^*}$ we have*

$$H(\chi) = \lim_{q\to\infty} \frac{1}{N} \sum_{1\leq q\leq N} \chi(\rho(q)).$$

Further for $\chi \in \mathbb{Q}_{a^*}$ we have

$$n(\chi) = \lim_{N\to\infty} \frac{1}{N} \sum_{1\leq q\leq N} \chi(\rho(q)).$$

We also showed the following theorems.

Theorem 5.2. *Suppose that $p \in (1,2]$ and that $f \in L^p \cap L^2(\mathbb{Q}_a)$. Also suppose that*

$$\rho(n) = \alpha_k n^k + \alpha_{k-1} n^{k-1} + \ldots + \alpha_1 n + \alpha_0,$$

has degree at least 2 with one of the numbers $\{\alpha_1, \ldots, \alpha_k\} \subset \mathbb{Q}_a$ a generator of $\mathbb{Q}_a$. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x+\rho(n)) = k(f)(x),$$

for $k(f) \in L^p$ almost everywhere with respect to Haar measure on $\mathbb{Q}_a$, where

$$\mathbb{F}(k(f))(\chi) = n(\chi)\mathbb{F}(f)(\chi).$$

Theorem 5.3. *Suppose that $p \in (1,2]$ and that $f \in L^p(\mathbb{Z}_a)$. Also suppose*

$$\rho(n) = \alpha_k n^k + \alpha_{k-1} n^{k-1} + \ldots + \alpha_1 n + \alpha_0,$$

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$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x+\rho(n)) = k(f)(x),$$

for $k(f) \in L^p(\mathbb{Z}_a)$ almost everywhere with respect to Haar measure on $\mathbb{Z}_a$, where

$$\mathbb{F}(k(f))(\chi) = n(\chi)(f)(\chi).$$

The Riesz representation theorem tells us that there exist measures μ_ℓ and μ_k on $\mathbb{Z}_a$ such that at least for $f \in C(\mathbb{Z}_a)$ we have $\ell(f) = \int f d\mu_\ell$ and $k(f) = \int f d\mu_k$. The same is true for with $\mathbb{Z}_a$ replaced by $\mathbb{Q}_a$. A natural question is whether the measures μ_ℓ or μ_k are in fact Haar measure. One might conjecture that if ρ has degree at least two the answer is no. A special case we can deal with is $\rho(n) = n^2$. To see this, choose $l \neq 0$ such that $t = l/a_0 \ldots a_s$, is of the form $\frac{a}{q}$ for a prime q, with $a \neq 0$. In this case $H(\chi_t)$ is of the form $C\sum_{r=1}^{q} e^{\frac{2\pi i a r^2}{q}}$ for a non-zero constant C. As is well known $\left|\sum_{r=1}^{q} e^{\frac{2\pi i a r^2}{q}}\right| = q^{\frac{1}{2}}$ ([**A**], p. 166). This means that the Fourier coefficient of the measure μ_ℓ of the squares on $\mathbb{Z}_a$ is non-zero. For μ_ℓ to be Haar measure you would need $H(t) = 0$ for every non-zero t which we have discounted.

Unfortunately, dealing with polynomials more general than $\rho(n) = n^2$ or $s > 1$, is a good deal more complex and is as yet unresolved. This is because it relies on lower bounds for exponential sums of the form $\left|\sum_{r=1}^{q} e^{\frac{2\pi i a \rho(r)}{q}}\right|$, which seems a serious undertaking and yet to appear in the literature. We can however show that μ_ℓ and μ_k are always continuous which we now prove.

For a measure μ defined on the group $\mathbb{Z}_a$ set

$$\mathbb{F}(\mu)(\chi_t) = \int_{\mathbb{Z}_a} \overline{\chi}_t d\mu, \qquad (t \in \hat{\mathbb{Z}}_a)$$

i.e. the Fourier transform of μ. The analogue of Wiener's lemma on the group $\mathbb{Z}_a$ ([**GM**], p. 236) requires us to show that

$$\lim_{r\to\infty} \frac{1}{A_r} \sum_{\{\frac{l}{A_s}:0\le s\le r\}} |\mathbb{F}(\mu)(\chi_t)|^2 = 0.$$

We need the following Lemma [**N5**].

Lemma 5.4. *Suppose that*

$$\psi(x) \;=\; a_d x^d \;+\; \cdots \;+\; a_1 x$$

for integers a_i $(i \;=\; 1,2,\cdots,d)$ *and let*

$$S(\psi|q) \;=\; \sum_{r=0}^{q-1} e^{2\pi i \psi(r) q^{-1}}.$$

Then there exist $\delta_0 \;>\; 0$, *and* $C_{\delta_0} \;>\; 0$ *such that*

$$|S(\psi|q)| \;\le\; \frac{C_{\delta_0}}{q^{\delta_0}}.$$

This means that there exist $\delta_H > 0$ such that $H(\frac{l}{A_r}) \ll A_r^{-\delta_H}$. Also

$$\sum_{\substack{1\le m\le q\\(m,q)=1}} e(\rho(m)) = \sum_{1\le m\le q} e(\rho(m)) - \sum_{\frac{p}{q}} \sum_{\substack{1\le m\le q\\ p|m}} e(\rho(m)),$$

so there exist $\delta_\ell > 0$ such that $G(\frac{l}{A_r}) \ll A_r^{-\delta_\ell}$. In either case there is $\delta > 0$ such that

$$\frac{1}{A_r} \sum_{\{t=\frac{l}{A_s}:0\le s\le r\}} |\mathbb{F}(\mu)(\chi_t)|^2 \ll A_r^{-\delta},$$

as required, where μ is either μ_ℓ or μ_k. Hence both measures μ_k and μ_ℓ are continuous on $\mathbb{Z}_a$.

If we consider $\mathbb{Q}_a$, we notice that $m(\chi) = G(\Psi(\chi))$ and $n(\chi) = H(\Psi(\chi))$. The argument for $\mathbb{Z}_a$ with $G(\chi)$ replaced by $G(\Psi(\chi))$ and with $H(\chi)$ replaced by $H(\Psi(\chi))$ now implies that both μ_ℓ and μ_k are continuous.

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