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CONVERGENCE OF A FINITE ELEMENT METHOD FOR DEGENERATE TWO-PHASE FLOW IN POROUS MEDIA

VIVETTE GIRAULT ^{*}, BEATRICE M. RIVIÈRE[†], AND LOIC CAPPANERA

Abstract. A finite element method with mass-lumping and flux upwinding, is formulated for solving the immiscible two-phase flow problem in porous media. The method approximates directly the wetting phase pressure and saturation, which are the primary unknowns. The discrete saturation satisfies a maximum principle. Theoretical convergence is proved via a compactness argument. The proof is convoluted because of the degeneracy of the phase mobilities and the unboundedness of the capillary pressure.

1. Introduction. This work discretizes on a suitable mesh a degenerate two-phase flow system set in a polyhedral domain by a new finite element scheme that directly approximates the wetting phase pressure and saturation, similar to the formulation proposed in [?]. Mass lumping is used to compute the integrals and a suitable upwinding is used to compute the flux, guaranteeing that the discrete saturation satisfies a maximum principle. The resulting system of discrete equations is a finite element analogue of the finite volume scheme introduced and analyzed by Eymard *et al* in the seminal work [16]. From the point of view of implementation, the advantage of finite elements is that they only use nodal values and a single simplicial mesh. In particular, no orthogonality property is required between the faces and the lines joining the centers of control volumes. From a theoretical point of view, owing that the finite element scheme is based on functions, some steps in its convergence analysis are simpler, but nevertheless the major difficulty in the analysis consists in proving sufficient a priori estimates in spite of the degeneracy. By following closely [16], the degeneracy is remediated by reintroducing in the proofs discrete artificial pressures. From there, convergence of the numerical solutions is shown via a compactness argument.

Incompressible two-phase flow is a popular and important multiphase flow model in reservoirs for the oil and gas industry. Based on conservation laws at the continuum scale, the model assumes the existence of a representative elementary volume. Each wetting phase and non-wetting phase saturation satisfies a mass balance equation and each phase velocity follows the generalized Darcy law [24, 4]. The equations of the mathematical model read

$$\begin{aligned} \partial_t(\varphi s_w) - \nabla \cdot (\eta_w \nabla p_w) &= f_w(s_{\text{in}})\bar{q} - f_w(s_w)\underline{q}, \\ \partial_t(\varphi s_o) - \nabla \cdot (\eta_o \nabla p_o) &= f_o(s_{\text{in}})\bar{q} - f_o(s_w)\underline{q}, \\ p_c(s_w) &= p_o - p_w, \quad s_w + s_o = 1, \end{aligned} \tag{1.1}$$

complemented by initial and boundary conditions. Here p_w, s_w, η_w, f_w , (respectively, p_o, s_o, η_o, f_o), are the pressure, saturation, mobility and fractional flow of the wetting (respectively non-

^{*}Paris VI, visiting professor at Rice

[†]Department of Computational and Applied Mathematics, Rice University, Houston, TX 77005. Supported in part by NSF-DMS

wetting) phase, φ is the porosity, s_{in} is a given input saturation, and \bar{q}, \underline{q} are given flow rates. The capillary pressure, p_c , is a given function that depends nonlinearly on the saturation. Because the phase mobilities are degenerate when they are evaluated at the residual phase saturations and the derivative of the capillary pressure is unbounded, this system of two coupled nonlinear partial differential equations has coefficients that vanish in parts of the domain; this degeneracy makes the numerical analysis challenging.

At the continuous level, this problem has several equivalent formulations, see [7]. They are linked to the choice of primary unknowns selected among wetting phase and non-wetting phase pressure and saturation, or capillary pressure [19, 5]. A good state of the art can be found in the reference [2]. Up to our knowledge, the mathematical analysis of the system of equations was first done in [20, 1]. An equivalent formulation of the model, based on Chavent's global pressure that removes the degeneracy, was analyzed in [8, 9]. Since then, the global pressure formulation has been discretized and analyzed in many references [23, 10, 22], but unfortunately, this formulation is not used in engineering practice because the global pressure is not a physical unknown. Otherwise, with one exception, the numerical analysis of the discrete version of (1.1), has always been done under unrealistic assumptions that cannot be checked at the discrete level [13, 14]. Related to this line of work, the discretization of a degenerate parabolic equation has been studied in the literature [3, 27, 26, 17]. The only paper that performs the complete numerical analysis of the discrete degenerate two-phase flow system written as above (i.e., in the form used by engineers) is the analysis on finite volumes done in reference [16]. This motivates our extension of this work to finite elements.

The remaining part of this introduction makes precise problem (1.1). The numerical scheme is developed in Section 2. Because of the nonlinearity and degeneracy of its equations, existence of a discrete solution requires that the discrete wetting phase saturation satisfies a maximum principle. This is the first object of Section 3, the second one being basic a priori pressure estimates, after which existence is shown in Section 4. The most technical part, done in Section 5, is the derivation of an unconditional bound on an auxiliary pressure, which allows to use a compactness argument. Weak and strong convergences are proved in Section 6 and the equations satisfied by the limit are identified in Section 7, thus confirming existence of a solution of the weak formulation (1.16). Numerical results are presented in Section 8.

1.1. Model problem. Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3 , be a bounded connected Lipschitz domain with boundary $\partial\Omega$ and unit exterior normal \mathbf{n} , and let T be a final time. With the last relation in (1.1), s_w is the only unknown saturation; so we set $s = s_w$, and rewrite (1.1) almost everywhere in $\Omega \times]0, T[$ as

$$\partial_t(\varphi s) - \nabla \cdot (\eta_w \nabla p_w) = f_w(s_{\text{in}})\bar{q} - f_w(s)\underline{q} \quad (1.2)$$

$$-\partial_t(\varphi s) - \nabla \cdot (\eta_o \nabla p_o) = f_o(s_{\text{in}})\bar{q} - f_o(s)\underline{q}, \quad (1.3)$$

complemented by a natural boundary condition almost everywhere on $\partial\Omega \times]0, T[$

$$\eta_w \nabla p_w \cdot \mathbf{n} = 0, \quad \eta_o \nabla p_o \cdot \mathbf{n} = 0, \quad (1.4)$$

and an initial condition almost everywhere in Ω

$$s_w(\cdot, 0) = s_w^0, \quad 0 \leq s_w^0 \leq 1. \quad (1.5)$$

The fractional flows are related to the mobilities by

$$f_w = \frac{\eta_w}{\eta_w + \eta_o}, \quad f_o = 1 - f_w. \quad (1.6)$$

Recall that the phase saturations sum up to 1 and the phase pressures are related by

$$p_c(s_w) = p_o - p_w. \quad (1.7)$$

The first part of this work, up to Section 5.7, is done under the following basic assumptions:

Assumptions

- The porosity φ is piecewise constant in space, independent of time, positive, bounded and uniformly bounded away from zero.
- The mobility of the wetting phase $\eta_w \geq 0$ is continuous and increasing. The mobility of the non-wetting phase $\eta_o \geq 0$ is continuous and decreasing. This implies that the function f_w is increasing and the function f_o is decreasing.
- There is a positive constant η_* such that

$$\eta_w(s) + \eta_o(s) \geq \eta_*, \quad \forall s \in [0, 1]. \quad (1.8)$$

- The capillary pressure p_c is a continuous, strictly decreasing function in $W^{1,1}(0, 1)$.
- The flow rates at the injection and production wells, $\bar{q}, \underline{q} \in L^2(\Omega \times]0, T[)$ satisfy

$$\bar{q} \geq 0, \quad \underline{q} \geq 0, \quad \int_{\Omega} \bar{q} = \int_{\Omega} \underline{q}. \quad (1.9)$$

- The prescribed input saturation s_{in} satisfies almost everywhere in $\Omega \times]0, T[$

$$0 \leq s_{\text{in}} \leq 1. \quad (1.10)$$

Since $p_c, \eta_\alpha, f_\alpha, \alpha = w, o$ are bounded above and below, it is convenient to extend them continuously by constants to \mathbb{R} .

Although the numerical scheme studied below does not discretize the global pressure, following [16], its convergence proof uses a number of auxiliary functions related to the global pressure. First, we introduce the primitive g_c of p_c ,

$$\forall x \in [0, 1], \quad g_c(x) = \int_x^1 p_c(s) ds. \quad (1.11)$$

Since p_c is a continuous function on $[0, 1]$, the function g_c belongs to $\mathcal{C}^1([0, 1])$. Next, we introduce the auxiliary pressures p_{wg} , p_{wo} , and g ,

$$\forall x \in [0, 1], \quad p_{wg}(x) = \int_0^x f_o(s)p'_c(s)ds, \quad p_{og}(x) = \int_0^x f_w(s)p'_c(s)ds, \quad (1.12)$$

$$\forall x \in [0, 1], \quad g(x) = - \int_0^x \frac{\eta_w(s)\eta_o(s)}{\eta_w(s) + \eta_o(s)} p'_c(s)ds. \quad (1.13)$$

Owing to (1.6),

$$\forall x \in [0, 1], \quad p_{wg}(x) + p_{og}(x) = \int_0^x p'_c(s)ds = p_c(x) - p_c(0). \quad (1.14)$$

Moreover, the derivative of g satisfies formally the identities

$$\forall x \in [0, 1], \quad \eta_\alpha(x)p'_{\alpha g}(x) + g'(x) = 0, \quad \alpha = w, o. \quad (1.15)$$

1.2. Weak variational formulation. By multiplying (1.2) and (1.3) with a smooth function v , say $v \in \mathcal{C}^1(\Omega \times [0, T])$ that vanishes at $t = T$, applying Green's formula in time and space, and using the boundary and initial conditions (1.4) and (1.5), we formally derive a weak variational formulation

$$\begin{aligned} - \int_0^T \int_\Omega \varphi s \partial_t v + \int_0^T \int_\Omega \eta_w \nabla p_w \cdot \nabla v &= \int_\Omega \varphi s^0 v(0) + \int_0^T \int_\Omega (f_w(s_{\text{in}})\bar{q} - f_w(s)\underline{q})v, \\ \int_0^T \int_\Omega \varphi s \partial_t v + \int_0^T \int_\Omega \eta_o \nabla p_o \cdot \nabla v &= - \int_\Omega \varphi s^0 v(0) + \int_0^T \int_\Omega (f_o(s_{\text{in}})\bar{q} - f_o(s)\underline{q})v. \end{aligned}$$

But in general, the pressures are not sufficiently smooth to make this formulation meaningful and following [7], by using (1.15), it is rewritten in terms of the artificial pressures,

$$\begin{aligned} - \int_0^T \int_\Omega \varphi s \partial_t v + \int_0^T \int_\Omega (\eta_w \nabla (p_w + p_{wg}(s)) + \nabla g(s)) \cdot \nabla v &= \int_\Omega \varphi s^0 v(0) \\ &+ \int_0^T \int_\Omega (f_w(s_{\text{in}})\bar{q} - f_w(s)\underline{q})v, \\ \int_0^T \int_\Omega \varphi s \partial_t v + \int_0^T \int_\Omega (\eta_o \nabla (p_o - p_{og}(s)) - \nabla g(s)) \cdot \nabla v &= - \int_\Omega \varphi s^0 v(0) \\ &+ \int_0^T \int_\Omega (f_o(s_{\text{in}})\bar{q} - f_o(s)\underline{q})v. \end{aligned} \quad (1.16)$$

The two formulas coincide when the pressures are slightly more regular. With the above assumptions, problem (1.16) has been analyzed in reference [1], where it is shown that it has a solution s in $L^\infty(\Omega \times]0, T[)$ with $g(s)$ in $L^2(0, T; H^1(\Omega))$, p_α , $\alpha = w, o$, in $L^2(\Omega \times]0, T[)$ with both $p_w + p_{wg}(s)$ and $p_o - p_{og}(s)$ in $L^2(0, T; H^1(\Omega))$.

2. Scheme. From now on, we assume that Ω is a polygon ($d = 2$) or Lipschitz polyhedron ($d = 3$) so it can be entirely meshed.

2.1. Meshes and discretization spaces. The mesh \mathcal{T}_h is a regular family of simplices K , with a constraint on the angle that will be used to enforce the maximum principle: each angle is not larger than $\pi/2$, see [6]. This is easily constructed in 2D. In 3D, since we only investigate convergence we can embed the domain in a triangulated box. Moreover, since the porosity φ is a piecewise constant, to simplify we also assume that the mesh is such that φ is a constant per element. The parameter h denotes the mesh size i.e., the maximum diameter of the simplices. On this mesh, we consider the standard finite element space of order one

$$X_h = \{v_h \in \mathcal{C}^0(\bar{\Omega}); \forall K \in \mathcal{T}_h, v_h|_K \in \mathbb{P}_1\}. \quad (2.1)$$

Thus the dimension of X_h is the number of nodes, say M , of \mathcal{T}_h . Let ϕ_i be the Lagrange basis function, that is piecewise linear, and takes the value 1 at node i and the value 0 elsewhere. As usual, the Lagrange interpolation operator $I_h \in \mathcal{L}(\mathcal{C}^0(\bar{\Omega}); X_h)$ is defined by

$$\forall v \in \mathcal{C}^0(\bar{\Omega}), \quad I_h(v) = \sum_{i=1}^M v_i \phi_i, \quad (2.2)$$

where v_i is the value of v at the node of index i . It is easy to see that under the mesh condition, we have

$$\forall K, \quad \int_K \nabla \phi_i \cdot \nabla \phi_j \leq 0, \quad \forall i \neq j. \quad (2.3)$$

For a given node i , we denote by Δ_i the union of elements sharing the node i and by $\mathcal{N}(i)$ the set of indices of all the nodes in Δ_i . In the spirit of [18], we define

$$c_{ij} = \int_{\Delta_i \cap \Delta_j} |\nabla \phi_i \cdot \nabla \phi_j|, \quad \forall i, j. \quad (2.4)$$

Recall that the trapezoidal rule on a triangle or a tetrahedron K is

$$\int_K f \approx \frac{1}{d+1} |K| \sum_{\ell=1}^{d+1} f_{i_\ell},$$

where f_{i_ℓ} is the value of the function f at the ℓ^{th} node (vertex), with global number i_ℓ , of K . For any region \mathcal{O} , the notation $|\mathcal{O}|$ means the measure (volume) of \mathcal{O} .

We define

$$m_i = \frac{1}{d+1} \sum_{K \in \Delta_i} |K| = \frac{1}{d+1} |\Delta_i|,$$

and taking into account the porosity φ , we define more generally

$$\tilde{m}_i(\varphi) = \frac{1}{d+1} \sum_{K \in \Delta_i} \varphi|_K |K|,$$

so that $m_i = \tilde{m}_i(1)$. It is well-known that the trapezoidal rule defines a norm on X_h , $\|\cdot\|_h$, uniformly equivalent to the L^2 norm. Let $U_h \in X_h$ and write

$$U_h = \sum_{i=1}^M U^i \phi_i.$$

The discrete L^2 norm associated with the trapezoidal rule is

$$\|U_h\|_h = \left(\sum_{i=1}^M m_i |U^i|^2 \right)^{\frac{1}{2}}.$$

There exist positive constants \underline{C} and \overline{C} , independent of h and M , such that

$$\forall U_h \in X_h, \quad \underline{C} \|U_h\|_{L^2(\Omega)}^2 \leq \|U_h\|_h^2 \leq \overline{C} \|U_h\|_{L^2(\Omega)}^2. \quad (2.5)$$

This is also true for other piecewise polynomial functions, but with possibly different constants. The scalar product associated with this norm is denoted by $(\cdot, \cdot)_h$,

$$\forall U_h, V_h \in X_h, \quad (U_h, V_h)_h = \sum_{i=1}^M m_i U^i V^i. \quad (2.6)$$

By analogy, we introduce the notation

$$\forall U_h, V_h \in X_h, \quad (U_h, V_h)_h^\varphi = \sum_{i=1}^M \tilde{m}_i(\varphi) U^i V^i. \quad (2.7)$$

The assumptions on the porosity φ imply that (2.7) defines a weighted scalar product associated with the weighted norm $\|\cdot\|_h^\varphi$,

$$\forall U_h \in X_h, \quad \|U_h\|_h^\varphi = ((U_h, U_h)_h^\varphi)^{\frac{1}{2}},$$

that satisfies the analogue of (2.5), with the same constants \underline{C} and \overline{C} ,

$$\forall U_h \in X_h, \quad \underline{C} (\min_{\Omega} \varphi) \|U_h\|_{L^2(\Omega)}^2 \leq (\|U_h\|_h^\varphi)^2 \leq \overline{C} (\max_{\Omega} \varphi) \|U_h\|_{L^2(\Omega)}^2. \quad (2.8)$$

2.2. Motivation of the space discretization. While discretizing the time derivative is fairly straightforward, discretizing the space derivatives is more delicate because we need a scheme that is consistent and satisfies the maximum principle for the saturation. For the moment, we freeze the time variable and focus on consistency in space. First, we recall a standard property of functions of X_h on meshes satisfying (2.3).

PROPOSITION 2.1. *Under condition (2.3), the following identities holds for all U_h and V_h in X_h , with c_{ij} defined in (2.4):*

$$\int_{\Omega} \nabla U_h \cdot \nabla V_h = - \sum_{i=1}^M U^i \sum_{j \neq i, j \in \mathcal{N}(i)} c_{ij} (V^j - V^i) = \frac{1}{2} \sum_{i=1}^M \sum_{j \neq i, j \in \mathcal{N}(i)} c_{ij} (U^j - U^i) (V^j - V^i). \quad (2.9)$$

Proof. The first equality is obtained by using (2.3), (2.4) and the fact that

$$\sum_{j=1}^M \phi_j = 1,$$

as in [15] (Section 12.1). For the second part, we use the symmetry of c_{ij} and the anti-symmetry of $V^j - V^i$ to deduce that

$$-\sum_{i=1}^M U^i \sum_{j \neq i, j \in \mathcal{N}(i)} c_{ij} (V^j - V^i) = \frac{1}{2} \sum_{i=1}^M \sum_{j \neq i, j \in \mathcal{N}(i)} c_{ij} (U^j - U^i) (V^j - V^i),$$

which is the desired result. \square

Note that c_{ij} vanishes when $j \notin \mathcal{N}(i)$. Therefore, when there is no ambiguity it is convenient to write the above double sums on i and j with i and j running from 1 to M .

As an immediate consequence of Proposition 2.1, we have, by taking $V_h = U_h$,

$$\forall U_h \in X_h, \quad \|\nabla U_h\|_{L^2(\Omega)} = \frac{1}{\sqrt{2}} \left(\sum_{i,j=1}^M c_{ij} |U^j - U^i|^2 \right)^{\frac{1}{2}}. \quad (2.10)$$

Now, we consider the case of the product of the gradients by a third function. Beforehand, we introduce the following notation: for indices i and j of two neighboring interior nodes, $\Delta_i \cap \Delta_j$ in two dimensions is the union of two triangles and in three dimensions the union of a number of tetrahedra bounded by a fixed constant, say L , determined by the regularity of the mesh. We shall use the following notation

$$c_{ij,K} = \int_K |\nabla \phi_i \cdot \nabla \phi_j|, \quad w_K = \frac{1}{|K|} \int_K w. \quad (2.11)$$

Note that

$$\sum_{K \subset \Delta_i \cap \Delta_j} c_{ij,K} = c_{ij}. \quad (2.12)$$

Then we have the following proposition:

PROPOSITION 2.2. *Let (2.3) hold. With the notation (2.11), the following identity holds for all w in $L^1(\Omega)$:*

$$\forall U_h, V_h \in X_h, \quad \int_{\Omega} w \nabla U_h \cdot \nabla V_h = - \sum_{i=1}^M U^i \sum_{j=1}^M \left(\sum_{K \subset \Delta_i \cap \Delta_j} c_{ij,K} w_K \right) (V^j - V^i), \quad (2.13)$$

Proof. It is easy to prove that

$$\int_{\Omega} w \nabla U_h \cdot \nabla V_h = \sum_{i,j=1}^M d_{ij} U^i V^j, \quad (2.14)$$

where

$$d_{ij} = \int_{\Delta_i \cap \Delta_j} w (\nabla \phi_i \cdot \nabla \phi_j) = \int_{\Omega} w (\nabla \phi_i \cdot \nabla \phi_j). \quad (2.15)$$

Again, we have for any i ,

$$\sum_{j=1}^M d_{ij} = 0, \quad \text{and} \quad d_{ii} = - \sum_{1 \leq j \leq M, j \neq i} d_{ij},$$

and by substituting this equality into (2.14), we obtain

$$\int_{\Omega} w (\nabla U_h \cdot \nabla V_h) = \sum_{i,j=1}^M U^i d_{ij} (V^j - V^i). \quad (2.16)$$

But, in view of (2.11) and (2.15), and since $\nabla \phi_i \cdot \nabla \phi_j$ is a constant in each element K contained in $\Delta_i \cap \Delta_j$,

$$d_{ij} = - \sum_{K \subset \Delta_i \cap \Delta_j} c_{ij,K} w_K, \quad (2.17)$$

and (2.13) follows by substituting this equation into (2.16). \square

Note that $d_{ij} = d_{ji}$ owing to (2.17). The first consequence of Proposition 2.2 is that the right-hand side of (2.13) is a consistent approximation of $(w, \nabla u \cdot \nabla v)$.

PROPOSITION 2.3. *Let (2.3) hold, let u and v belong to $H^2(\Omega)$ and w to $L^\infty(\Omega)$, and let $U_h = I_h u$, $V_h = I_h v$ be defined by (2.2). Then, there exists a constant C , independent of h , M , u , v , and w , such that*

$$\left| \int_{\Omega} w \nabla u \cdot \nabla v + \sum_{i,j=1}^M U^i \left(\sum_{K \subset \Delta_i \cap \Delta_j} c_{ij,K} w_K \right) (V^j - V^i) \right| \leq C h \|w\|_{L^\infty(\Omega)} \|u\|_{H^2(\Omega)} \|v\|_{H^2(\Omega)}. \quad (2.18)$$

Proof. In view of the identity (2.13), the left-hand side of (2.18) is bounded as follows:

$$\begin{aligned} \left| \int_{\Omega} w (\nabla u \cdot \nabla v - \nabla U_h \cdot \nabla V_h) \right| &\leq \|w\|_{L^\infty(\Omega)} \\ &\times \left(\|\nabla(u - U_h)\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|\nabla(v - V_h)\|_{L^2(\Omega)} \|\nabla U_h\|_{L^2(\Omega)} \right). \end{aligned}$$

From here, (2.18) is a consequence of standard finite element interpolation error. \square

Now, if w is in $W^{1,\infty}(\Omega)$, then again, standard finite element approximation shows that there exists a constant C , independent of h , $K \subset \Delta_i \cap \Delta_j$, and w , such that

$$\|w_K - w\|_{L^\infty(K)} \leq C h |w|_{W^{1,\infty}(K)} \leq C h |w|_{W^{1,\infty}(\Omega)}. \quad (2.19)$$

As a consequence, we will show that in the error formula (2.18), the average w_K can be replaced by any value of w in K . Since all K in $\Delta_i \cap \Delta_j$ share the edge, say e_{ij} , whose end points are the nodes with indices i and j , then we can pick the value of w at any point, say $\tilde{W}^{i,j}$, of e_{ij} . At this stage, we choose this value freely, but we prescribe that it be symmetrical with respect to i and j , i.e.,

$$\tilde{W}^{i,j} = \tilde{W}^{j,i}. \quad (2.20)$$

Then we have the following approximation result.

THEOREM 2.4. *With the assumption and notation of Proposition 2.3, there exists a constant C , independent of h and M , such that for all u , and v in $H^2(\Omega)$ and w in $W^{1,\infty}(\Omega)$,*

$$\int_{\Omega} w \nabla u \cdot \nabla v = - \sum_{i,j=1}^M U^i c_{ij} \tilde{W}^{i,j} (V^j - V^i) + R, \quad (2.21)$$

for any arbitrary value $\tilde{W}^{i,j}$ of w in the common edge e_{ij} satisfying (2.20), and the remainder R satisfies

$$|R| \leq C h |w|_{W^{1,\infty}(\Omega)} \|u\|_{H^2(\Omega)} \|v\|_{H^2(\Omega)}. \quad (2.22)$$

Proof. We infer from (2.12) and (2.13) that

$$\int_{\Omega} w (\nabla U_h \cdot \nabla V_h) = - \sum_{i,j=1}^M U^i (V^j - V^i) \sum_{K \subset \Delta_i \cap \Delta_j} c_{ij,K} (w_K - \tilde{W}^{i,j}) - \sum_{i,j=1}^M U^i c_{ij} (V^j - V^i) \tilde{W}^{i,j}.$$

Let

$$R_{ij} = \sum_{K \subset \Delta_i \cap \Delta_j} c_{ij,K} (w_K - \tilde{W}^{i,j}),$$

which is symmetric in i and j by assumption (2.20). As in Proposition 2.1, the symmetry of R_{ij} and the anti-symmetry of $V^j - V^i$, imply

$$- \sum_{i,j=1}^M U^i R_{ij} (V^j - V^i) \leq \frac{1}{2} \left(\sum_{i,j=1}^M |R_{ij}| (U^j - U^i)^2 \right)^{\frac{1}{2}} \left(\sum_{i,j=1}^M |R_{ij}| (V^j - V^i)^2 \right)^{\frac{1}{2}}. \quad (2.23)$$

From the non negativity of $c_{ij,K}$, (2.12), and (2.19), we infer that

$$|R_{ij}| \leq \left(\sum_{K \subset \Delta_i \cap \Delta_j} c_{ij,K} \right) C h |w|_{W^{1,\infty}(\Omega)} = c_{ij} C h |w|_{W^{1,\infty}(\Omega)}.$$

Hence, with (2.10) and standard finite element approximation,

$$\left| \sum_{i,j=1}^M U^i R_{ij} (V^j - V^i) \right| \leq C h |w|_{W^{1,\infty}(\Omega)} \|\nabla U_h\|_{L^2(\Omega)} \|\nabla V_h\|_{L^2(\Omega)} \leq C h |w|_{W^{1,\infty}(\Omega)} \|u\|_{H^2(\Omega)} \|v\|_{H^2(\Omega)}.$$

The result follows by combining this inequality with (2.18). \square

The above considerations show that

$$- \sum_{i,j=1}^M U^i c_{ij} \tilde{W}^{i,j} (V^j - V^i) \text{ is a consistent approximation of } \int_{\Omega} w \nabla u \cdot \nabla v,$$

for any symmetric choice of $\tilde{W}^{i,j}$ in e_{ij} , the common edge of $\Delta_i \cap \Delta_j$. This will lead to the upwinded space discretization in the next subsection, see also [22]. Furthermore, for all real numbers V^i and $\tilde{W}^{i,j}$ satisfying (2.20), $1 \leq i, j \leq M$, the symmetry of c_{ij} and anti-symmetry of $V^j - V^i$ imply

$$\sum_{i,j=1}^M c_{ij} \tilde{W}^{i,j} (V^j - V^i) = 0. \quad (2.24)$$

2.3. Fully discrete scheme. Let $\tau = \frac{T}{N}$ be the time step, $t_n = n\tau$, the discrete times, $0 \leq n \leq N$. Regarding time, we shall use the standard L^2 projection ρ_τ defined on $]t_{n-1}, t_n]$, for any function f in $L^1(0, T)$, by

$$\rho_\tau(f)^n := \rho_\tau(f)|_{]t_{n-1}, t_n]} := \frac{1}{\tau} \int_{t_{n-1}}^{t_n} f. \quad (2.25)$$

Regarding space, we shall use a standard element-by-element L^2 projection ρ_h as well as a nodal approximation operator r_h defined at each node \mathbf{x}_i for any function $g \in L^1(\Omega)$ by

$$r_h(g)(\mathbf{x}_i) = \frac{1}{|\Delta_i|} \int_{\Delta_i} g, \quad 1 \leq i \leq M, \quad (2.26)$$

and extended to Ω by $r_h(g) \in X_h$. The operator ρ_h is defined for any f in $L^1(\Omega)$ by $\rho_h(f)|_K = \rho_K(f)$ where, in any element K ,

$$\rho_K(f) = \frac{1}{|K|} \int_K f. \quad (2.27)$$

The initial saturation s_w^0 is approximated by the operator r_h ,

$$S_h^0 = r_h(s_w^0). \quad (2.28)$$

The input saturation s_{in} is approximated in space and time by

$$s_{\text{in},h,\tau} = \rho_\tau(r_h(s_{\text{in}})). \quad (2.29)$$

Clearly, (1.10) implies **in space and time**

$$0 \leq s_{\text{in},h,\tau} \leq 1.$$

In order to preserve (1.9), the functions \bar{q} and \underline{q} are approximated by the functions $\bar{q}_{h,\tau}$ and $\underline{q}_{h,\tau}$ defined with r_h and corrected as follows:

$$\bar{q}_{h,\tau} = \rho_\tau \left(r_h(\bar{q}) - \frac{1}{|\Omega|} \int_\Omega (r_h(\bar{q}) - \bar{q}) \right), \quad \underline{q}_{h,\tau} = \rho_\tau \left(r_h(\underline{q}) - \frac{1}{|\Omega|} \int_\Omega (r_h(\underline{q}) - \underline{q}) \right). \quad (2.30)$$

Since $\bar{q}_{h,\tau}$ and $\underline{q}_{h,\tau}$ are piecewise linear in space, they are exactly integrated by the trapezoidal rule and we easily derive from (1.9) and (2.30) that we have for all n ,

$$(\bar{q}_h^n, 1)_h = (\underline{q}_h^n, 1)_h. \quad (2.31)$$

The set of primary unknowns is the discrete wetting phase saturation and the discrete wetting phase pressure, S_h^n and $P_{w,h}^n$, defined pointwise at time t_n by:

$$S_h^n = \sum_{i=1}^M S^{n,i} \phi_i, \quad P_{w,h}^n = \sum_{i=1}^M P_w^{n,i} \phi_i, \quad 1 \leq n \leq N.$$

Then the discrete non-wetting phase pressure $P_{o,h}^n$ defined by

$$P_{o,h}^n = \sum_{i=1}^M P_o^{n,i} \phi_i, \quad 1 \leq n \leq N,$$

is a secondary unknown. The upwind scheme we propose for discretizing (1.2)–(1.3) is inspired by the finite volume scheme introduced and analyzed by Eymard *al* in [16]. For each time step n , $1 \leq n \leq N$, the lines of the discrete equations are

$$\frac{\tilde{m}_i(\varphi)}{\tau} (S^{n,i} - S^{n-1,i}) - \sum_{j=1}^M c_{ij} \eta_w(S_w^{n,ij}) (P_w^{n,j} - P_w^{n,i}) = m_i (f_w(s_{\text{in}}^{n,i}) \bar{q}^{n,i} - f_w(S^{n,i}) \underline{q}^{n,i}), \quad (2.32)$$

$$-\frac{\tilde{m}_i(\varphi)}{\tau} (S^{n,i} - S^{n-1,i}) - \sum_{j=1}^M c_{ij} \eta_o(S_o^{n,ij}) (P_o^{n,j} - P_o^{n,i}) = m_i (f_o(s_{\text{in}}^{n,i}) \bar{q}^{n,i} - f_o(S^{n,i}) \underline{q}^{n,i}), \quad (2.33)$$

$$P_o^{n,i} - P_w^{n,i} = p_c(S^{n,i}), \quad 1 \leq i \leq M, \quad (2.34)$$

$$\sum_{i=1}^M m_i P_w^{n,i} = 0. \quad (2.35)$$

Here i runs from 1 to $M-1$ in (2.32) and from 1 to M in (2.33); the upwind values $S_w^{n,ij}, S_o^{n,ij}$ are defined by

$$S_w^{n,ij} = \begin{cases} S^{n,i} & \text{if } P_w^{n,i} > P_w^{n,j} \\ S^{n,j} & \text{if } P_w^{n,i} < P_w^{n,j} \\ \max(S^{n,i}, S^{n,j}) & \text{if } P_w^{n,i} = P_w^{n,j} \end{cases} \quad (2.36)$$

$$S_o^{n,ij} = \begin{cases} S^{n,i} & \text{if } P_o^{n,i} > P_o^{n,j} \\ S^{n,j} & \text{if } P_o^{n,i} < P_o^{n,j} \\ \min(S^{n,i}, S^{n,j}) & \text{if } P_o^{n,i} = P_o^{n,j} \end{cases} \quad (2.37)$$

We observe that

$$S_w^{n,ij} = S_w^{n,ji}, \quad S_o^{n,ij} = S_o^{n,ji},$$

so that, if we interpret in (2.32) (respectively, (2.33)) $\eta_w(S_w^{n,ij})$ (respectively, $\eta_o(S_o^{n,ij})$) as $\tilde{W}^{i,j}$, then (2.20) and hence (2.24) hold.

REMARK 2.5. Before setting (2.32)–(2.35) in variational form, observe that:

1. The scheme (2.32)–(2.35) forms a square system in the primary unknowns, S_h^n and P_w^n .
2. Formula (2.32) is also valid for $i = M$. Indeed, we pass to the left-hand side the right-hand side of (2.32) and set A^i the resulting line of index i . Let \tilde{A}^M denote what should be the line of index M , i.e.,

$$\begin{aligned} \tilde{A}^M &= \frac{\tilde{m}_M(\varphi)}{\tau} (S^{n,M} - S^{n-1,M}) - \sum_{j=1}^M c_{Mj} \eta_w(S_w^{n,Mj}) (P_w^{n,j} - P_w^{n,M}) \\ &\quad - m_M (f_w(s_{\text{in}}^{n,M}) \bar{q}^{n,M} - f_w(S^{n,M}) \underline{q}^{n,M}). \end{aligned}$$

Then, in view of (2.24),

$$\tilde{A}^M = \sum_{i=1}^{M-1} A^i + \tilde{A}^M = \sum_{i=1}^M \frac{\tilde{m}_i(\varphi)}{\tau} (S^{n,i} - S^{n-1,i}) - \sum_{i=1}^M m_i (f_w(s_{\text{in}}^{n,i}) \bar{q}^{n,i} - f_w(S^{n,i}) \underline{q}^{n,i}).$$

By summing in the same fashion the lines of (2.33), we obtain

$$\sum_{i=1}^M \frac{\tilde{m}_i(\varphi)}{\tau} (S^{n,i} - S^{n-1,i}) = - \sum_{i=1}^M m_i (f_o(s_{\text{in}}^{n,i}) \bar{q}^{n,i} - f_o(S^{n,i}) \underline{q}^{n,i}).$$

A combination of these two equations yields

$$\tilde{A}^M = - \sum_{i=1}^M m_i \left((f_w(s_{\text{in}}^{n,i}) + f_o(s_{\text{in}}^{n,i})) \bar{q}^{n,i} - (f_w(S^{n,i}) + f_o(S^{n,i})) \underline{q}^{n,i} \right) = - \sum_{i=1}^M m_i (\bar{q}^{n,i} - \underline{q}^{n,i}) = 0,$$

by virtue of (1.6), the definition (2.25), and (1.9).

3. In (2.32) (respectively, (2.33)), any constant can be added to P_w (respectively, P_o), but in view of (2.34), the constant must be the same for both pressures. The last equation (2.35) is added to resolve this constant. ■

As usual, it is convenient to associate time functions $S_{h,\tau}, P_{\alpha,h,\tau}$ with the sequences indexed by n . These are piecewise constant in time in $]0, T[$, for instance

$$P_{\alpha,h,\tau}(t, x) = P_{\alpha,h}^n(x), \quad \alpha = w, o, \quad \forall (t, x) \in \Omega \times]t_{n-1}, t_n]. \quad (2.38)$$

In view of the material of the previous subsection, we introduce the following form:

$$\forall W_h, U_h, V_h, Z_h \in X_h, \quad [Z_h, W_h; V_h, U_h]_h = \sum_{i,j=1}^M U^i c_{ij} \tilde{W}^{ij} (V^j - V^i), \quad (2.39)$$

where the first argument Z_h indicates that the choice of \tilde{W}^{ij} depends on Z_h . Such dependence, used for the upwinding, will be specified further on, but *it is assumed from now on* that \tilde{W}^{ij} satisfies (2.20). Considering (2.24), the form satisfies the following properties,

$$\forall Z_h, W_h, V_h \in X_h, \quad [Z_h, W_h; V_h, \mathbf{1}]_h = 0, \quad (2.40)$$

$$\forall Z_h, W_h, V_h \in X_h, \quad [Z_h, W_h; V_h, V_h]_h = -\frac{1}{2} \sum_{i,j=1}^M c_{ij} \tilde{W}_{ij} (V^i - V^j)^2. \quad (2.41)$$

This last property is derived by the same argument as in proving (2.9).

With the above notation, and taking into account that (2.32) extends to $i = M$, the scheme (2.32)–(2.35) has the equivalent variational form. Starting from S_h^0 , see (2.28),

find S_h^n , $P_{w,h}^n$, and $P_{o,h}^n$ in X_h , for $1 \leq n \leq N$, solution of, for all θ_h in X_h ,

$$\frac{1}{\tau} (S_h^n - S_h^{n-1}, \theta_h)_h^\varphi - [P_{w,h}^n, I_h(\eta_w(S_h^n)); P_{w,h}^n, \theta_h]_h = (I_h(f_w(s_{\text{in},h}^n)) \bar{q}_h^n - I_h(f_w(S_h^n)) \underline{q}_h^n, \theta_h)_h \quad (2.42)$$

$$-\frac{1}{\tau} (S_h^n - S_h^{n-1}, \theta_h)_h^\varphi - [P_{o,h}^n, I_h(\eta_o(S_h^n)); P_{o,h}^n, \theta_h]_h = (I_h(f_o(s_{\text{in},h}^n)) \bar{q}_h^n - I_h(f_o(S_h^n)) \underline{q}_h^n, \theta_h)_h \quad (2.43)$$

$$P_{o,h}^n - P_{w,h}^n = I_h(p_c(S_h^n)), \quad (2.44)$$

$$(P_{w,h}^n, \mathbf{1})_h = 0, \quad (2.45)$$

where the choice of $\eta_w(S_h^n)$ in the left-hand side of (2.42) (respectively, $\eta_o(S_h^n)$ in the left-hand side of (2.43)) is given by (2.36) (respectively (2.37)). Strictly speaking, the interpolation operator I_h is introduced in (2.42) and (2.43) because the forms are defined for functions of X_h , but for the sake of simplicity, since only nodal values are used, it may be dropped further on.

We shall see that under the above basic hypotheses, the discrete problem (2.42)–(2.45) has at least one solution. In the sequel, we shall use the following discrete auxiliary pressures (compare with (1.16)):

$$U_{w,h,\tau} = P_{w,h,\tau} + I_h(p_{wg}(S_{h,\tau})), \quad U_{o,h,\tau} = P_{o,h,\tau} - I_h(p_{og}(S_{h,\tau})). \quad (2.46)$$

The following theorem is the main result of this work:

THEOREM 2.6. *Under the above basic hypotheses and the additional assumptions (5.42)–*

(5.44), the discrete solutions converge up to subsequences as follows:

$$\begin{aligned} \lim_{(h,\tau)\rightarrow(0,0)} S_{h,\tau} &= s \quad \text{strongly in } L^2(\Omega \times]0, T[), \\ \lim_{(h,\tau)\rightarrow(0,0)} U_{w,h,\tau} &= p_w + p_{wg}(s), \quad \lim_{(h,\tau)\rightarrow(0,0)} U_{o,h,\tau} = p_o - p_{og}(s), \quad \text{weakly in } L^2(0, T; H^1(\Omega)), \\ \lim_{(h,\tau)\rightarrow(0,0)} P_{\alpha,h,\tau} &= p_\alpha, \quad \text{weakly in } L^2(\Omega \times]0, T[), \alpha = w, o, \end{aligned}$$

where $p_w + p_{wg}$, $p_o - p_{wg}$, and s solve the weak formulation (1.16).

The proof of the theorem requires several steps that are covered in the remaining of this work.

3. First a priori bounds. This section is devoted to basic a priori bounds used in proving existence of a discrete solution. Existence is fairly technical and will be postponed till Section 4. The first step is a key bound on the discrete saturation. In a second step, this bound will lead to a pressure estimate and in particular to a bound on the discrete analogue of auxiliary pressures.

3.1. Maximum principle. The scheme (2.32)–(2.35) satisfies the maximum principle property. The proof given below uses a standard argument as in [16, ?].

THEOREM 3.1. *The following bounds hold:*

$$0 \leq S_{h,\tau} \leq 1. \quad (3.1)$$

Proof. As $0 \leq s_w^0 \leq 1$ almost everywhere, by construction (2.28), we immediately have

$$0 \leq \min_{\Omega} s_w^0 \leq S_h^0 \leq \max_{\Omega} s_w^0 \leq 1.$$

The proof proceeds by contradiction. Assume that there is an index $n \geq 1$ such that

$$S_h^{n-1} \leq 1, \quad S_h^n > 1.$$

This means there is a node i such that

$$S^{n,i} = \|S_h^n\|_{L^\infty(\Omega)} > 1,$$

and thus

$$S^{n,i} > S^{n-1,i}.$$

Dropping the index n in the rest of the proof, (2.32) and (2.33) imply

$$\sum_{j \neq i, j \in \mathcal{N}(i)} c_{ij} \eta_w(S_w^{ij}) (P_w^j - P_w^i) + m_i (f_w(s_{\text{in}}^i) \bar{q}^i - f_w(S^i) \underline{q}^i) > 0, \quad (3.2)$$

$$- \sum_{j \neq i, j \in \mathcal{N}(i)} c_{ij} \eta_o(S_o^{ij}) (P_o^j - P_o^i) - m_i (f_o(s_{\text{in}}^i) \bar{q}^i - f_o(S^i) \underline{q}^i) > 0. \quad (3.3)$$

We first show that (3.2) holds true with S_w^{ij} replaced by S^i . Indeed if $P_w^i > P_w^j$, then $S_w^{ij} = S^i$. If $P_w^i < P_w^j$, then $S_w^{ij} = S^j$, and as η_w is increasing and by assumption, $S^j \leq S^i$,

$$\eta_w(S_w^{ij})(P_w^j - P_w^i) \leq \eta_w(S^i)(P_w^j - P_w^i).$$

Finally, the term vanishes when $P_w^i = P_w^j$. Therefore we have in all cases

$$\sum_{j \neq i, j \in \mathcal{N}(i)} c_{ij} \eta_w(S^i)(P_w^j - P_w^i) + m_i (f_w(s_{\text{in}}^i) \bar{q}^i - f_w(S^i) \underline{q}^i) > 0. \quad (3.4)$$

A similar argument gives

$$- \sum_{j \neq i, j \in \mathcal{N}(i)} c_{ij} \eta_o(S^i)(P_o^j - P_o^i) - m_i (f_o(s_{\text{in}}^i) \bar{q}^i - f_o(S^i) \underline{q}^i) > 0. \quad (3.5)$$

The substitution of (2.34) into (3.5) yields

$$- \sum_{j \neq i, j \in \mathcal{N}(i)} c_{ij} \eta_o(S^i)((P_w^j - P_w^i) + (p_c(S^j) - p_c(S^i))) - m_i (f_o(s_{\text{in}}^i) \bar{q}^i - f_o(S^i) \underline{q}^i) > 0. \quad (3.6)$$

Since p_c is decreasing and $S^i \geq S^j$, the second term in the above sum is negative. This implies that

$$- \sum_{j \neq i, j \in \mathcal{N}(i)} c_{ij} \eta_o(S^i)(P_w^j - P_w^i) - m_i (f_o(s_{\text{in}}^i) \bar{q}^i - f_o(S^i) \underline{q}^i) > 0. \quad (3.7)$$

The sum on j cancels by multiplying (3.4) by $\eta_o(S^i)$, (3.7) by $\eta_w(S^i)$, and adding the two. The sign is unchanged because either $\eta_o(S^i)$ or $\eta_w(S^i)$ is strictly positive. Hence,

$$m_i \eta_o(S^i) (f_w(s_{\text{in}}^i) \bar{q}^i - f_w(S^i) \underline{q}^i) - m_i \eta_w(S^i) (f_o(s_{\text{in}}^i) \bar{q}^i - f_o(S^i) \underline{q}^i) > 0.$$

By definition of f_w and f_o , this reduces to

$$\eta_o(S^i) f_w(s_{\text{in}}^i) - \eta_w(S^i) f_o(s_{\text{in}}^i) > 0. \quad (3.8)$$

Now consider the function:

$$r(s) = \eta_o(s) f_w(s_{\text{in}}^i) - \eta_w(s) f_o(s_{\text{in}}^i). \quad (3.9)$$

It is decreasing and $r(s_{\text{in}}^i) = 0$. Then, since $S^i > 1 \geq s_{\text{in}}^i$, see (1.10), we have

$$r(S^i) \leq r(s_{\text{in}}^i) = 0,$$

which contradicts (3.8). The proof of the lower bound in (3.1) follows the same lines. \square

3.2. First pressure bounds. The following properties will be used frequently:

LEMMA 3.2. *The fact that p_c is strictly decreasing and (2.34) yield the following:*

$$P_w^i > P_w^j, \quad \text{and} \quad P_o^i \leq P_o^j \quad \text{implies} \quad S^i \geq S^j, \quad (3.10)$$

$$\text{If } P_w^i = P_w^j, \text{ then } P_o^i \geq P_o^j \text{ if and only if } S^i \leq S^j. \quad (3.11)$$

$$\text{If } P_o^i = P_o^j, \text{ then } P_w^i \leq P_w^j, \text{ if and only if } S^i \leq S^j. \quad (3.12)$$

Let us start with a lower bound that removes the degeneracy caused by the mobilities when they multiply the discrete pressures.

LEMMA 3.3. *Let p_{wg} and p_{og} be defined in (1.12). We have for all n and any i and j*

$$\eta_*(U_w^{n,j} - U_w^{n,i})^2 \leq \eta_w(S_w^{n,ij})(P_w^{n,j} - P_w^{n,i})^2 + \eta_o(S_o^{n,ij})(P_o^{n,j} - P_o^{n,i})^2. \quad (3.13)$$

Proof. To simplify the notation, we drop the superscript n . The second mean formula for integrals gives

$$p_{wg}(S^j) - p_{wg}(S^i) = \int_{S^i}^{S^j} f_o(s)p'_c(s)ds = f_o(\xi)(p_c(S^j) - p_c(S^i)), \quad (3.14)$$

for some ξ between S^i and S^j . Using (2.34) we write

$$U_w^j - U_w^i = (1 - f_o(\xi))(P_w^j - P_w^i) + f_o(\xi)(P_o^j - P_o^i) = f_w(\xi)(P_w^j - P_w^i) + f_o(\xi)(P_o^j - P_o^i).$$

Therefore since $f_w + f_o = 1$, we have

$$(U_w^j - U_w^i)^2 \leq \frac{\eta_w(\xi)}{\eta_w(\xi) + \eta_o(\xi)}(P_w^j - P_w^i)^2 + \frac{\eta_o(\xi)}{\eta_w(\xi) + \eta_o(\xi)}(P_o^j - P_o^i)^2. \quad (3.15)$$

We now consider six cases.

1) If $P_w^i > P_w^j$ and $P_o^i \leq P_o^j$, then $\eta_w(S_w^{ij}) = \eta_w(S^i)$ and $\eta_o(S_o^{ij}) = \eta_o(S^j)$ when $P_o^i < P_o^j$; when $P_o^i = P_o^j$, the value of η_o does not matter. From (3.10) we then have $S^i \geq S^j$. Since η_w is increasing, $\eta_w(\xi) \leq \eta_w(S^i)$ and since η_o is decreasing, $\eta_o(\xi) \leq \eta_o(S^j)$. Thus we have

$$(U_w^j - U_w^i)^2 \leq \frac{\eta_w(S_w^{ij})}{\eta_w(\xi) + \eta_o(\xi)}(P_w^j - P_w^i)^2 + \frac{\eta_o(S_o^{ij})}{\eta_w(\xi) + \eta_o(\xi)}(P_o^j - P_o^i)^2,$$

and with (1.8)

$$(U_w^j - U_w^i)^2 \leq \frac{1}{\eta_*} (\eta_w(S_w^{ij})(P_w^j - P_w^i)^2 + \eta_o(S_o^{ij})(P_o^j - P_o^i)^2). \quad (3.16)$$

2) If $\underline{P_w^i > P_w^j}$ and $\underline{P_o^i > P_o^j}$, then $\eta_w(S_w^{ij}) = \eta_w(S^i)$ and $\eta_o(S_o^{ij}) = \eta_o(S^i)$. From

$$\eta_o(S^i)(p_c(S^j) - p_c(S^i)) = (\eta_o(S^i) + \eta_w(S^i)) \int_{S^i}^{S^j} f_o(S^i) p'_c(s) ds,$$

and (3.14), we derive

$$\begin{aligned} & \eta_o(S^i)(p_c(S^j) - p_c(S^i)) - (\eta_o(S^i) + \eta_w(S^i))(p_{wg}(S^j) - p_{wg}(S^i)) \\ &= (\eta_o(S^i) + \eta_w(S^i)) \int_{S^i}^{S^j} (f_o(S^i) - f_o(s)) p'_c(s) ds. \end{aligned}$$

As p_c and f_o are decreasing, the above right-hand side is negative. Hence

$$\eta_o(S^i)(p_c(S^j) - p_c(S^i)) - (\eta_o(S^i) + \eta_w(S^i))(p_{wg}(S^j) - p_{wg}(S^i)) \leq 0. \quad (3.17)$$

We multiply (3.17) by $(P_o^j - P_o^i) + (P_w^j - P_w^i) < 0$ and use (2.34),

$$(\eta_o(S^i)(p_c(S^j) - p_c(S^i)) - (\eta_o(S^i) + \eta_w(S^i))(p_{wg}(S^j) - p_{wg}(S^i))) (2(P_w^j - P_w^i) + p_c(S^j) - p_c(S^i)) \geq 0.$$

By expanding and using the next inequality implied by (3.14), if $f_o(\xi) \neq 0$,

$$(p_{wg}(S^j) - p_{wg}(S^i))(p_c(S^j) - p_c(S^i)) \geq (p_{wg}(S^j) - p_{wg}(S^i))^2,$$

we obtain

$$\begin{aligned} & \eta_o(S^i)(p_c(S^j) - p_c(S^i))^2 + 2\eta_o(S^i)(p_c(S^j) - p_c(S^i))(P_w^j - P_w^i) \geq \\ & (\eta_o(S^i) + \eta_w(S^i))(p_{wg}(S^j) - p_{wg}(S^i)) (2(P_w^j - P_w^i) + p_{wg}(S^j) - p_{wg}(S^i)). \end{aligned}$$

When $(\eta_o(S^i) + \eta_w(S^i))(P_w^j - P_w^i)^2$ is added to both sides, this becomes

$$\eta_w(S^i)(P_w^j - P_w^i)^2 + \eta_o(S^i)(P_o^j - P_o^i)^2 \geq (\eta_o(S^i) + \eta_w(S^i))(U_w^j - U_w^i)^2,$$

and (1.8) implies the desired result. It remains to consider the case $f_o(\xi) = 0$, i.e., $p_{wg}(S^j) = p_{wg}(S^i)$. If $\eta_o(S^i) \neq 0$, then (3.17) yields

$$p_c(S^j) - p_c(S^i) \leq 0 \text{ which implies } P_o^i - P_o^j \geq P_w^i - P_w^j,$$

and we deduce immediately

$$\eta_w(S^i)(P_w^j - P_w^i)^2 + \eta_o(S^i)(P_o^j - P_o^i)^2 \geq (\eta_w(S^i) + \eta_o(S^i))(P_w^j - P_w^i)^2 \geq \eta_*(P_w^j - P_w^i)^2.$$

When $\eta_o(S^i) = 0$, we have trivially

$$\eta_w(S^i)(P_w^j - P_w^i)^2 + \eta_o(S^i)(P_o^j - P_o^i)^2 = \eta_w(S^i)(P_w^j - P_w^i)^2 \geq \eta_*(P_w^j - P_w^i)^2.$$

3) If $\underline{P_w^i \leq P_w^j}$ and $\underline{P_o^i > P_o^j}$, then $\eta_w(S_w^{ij}) = \eta_w(S^j)$ and $\eta_o(S_o^{ij}) = \eta_o(S^i)$ in the case of a strict inequality; also $S^i \leq S^j$. Then (3.15) and the monotonic properties of η_w and η_o yield (3.13). If $P_w^i = P_w^j$, then according to (3.11), $S^i \leq S^j$ and the same conclusion holds.

4) If $\underline{P_w^i \leq P_w^j}$ and $\underline{P_o^i = P_o^j}$, then from (3.12), we have $S^i \leq S^j$ and with (3.15)

$$(U_w^j - U_w^i)^2 \leq \frac{\eta_w(\xi)}{\eta_w(\xi) + \eta_o(\xi)} (P_w^j - P_w^i)^2 \leq \frac{\eta_w(S_w^{ij})}{\eta_w(\xi) + \eta_o(\xi)} (P_w^j - P_w^i)^2,$$

which is the desired result.

5) Similarly, if $\underline{P_w^i = P_w^j}$ and $\underline{P_o^i < P_o^j}$, then from (3.11), we have $S^j \leq S^i$ and with (3.15)

$$(U_w^j - U_w^i)^2 \leq \frac{\eta_o(\xi)}{\eta_w(\xi) + \eta_o(\xi)} (P_o^j - P_o^i)^2 \leq \frac{\eta_o(S_o^{ij})}{\eta_w(\xi) + \eta_o(\xi)} (P_o^j - P_o^i)^2.$$

6) If $\underline{P_w^i < P_w^j}$ and $\underline{P_o^i < P_o^j}$, (3.13) follows from the second case by switching i and j . \square

The pressure bound in the next theorem is the one that arises naturally from the left-hand side of (2.42) and (2.43).

THEOREM 3.4. *There exists a constant C , independent of h and τ , such that*

$$\tau \sum_{n=1}^N \sum_{i,j=1}^M c_{ij} (\eta_w(S_w^{n,ij})(P_w^{n,i} - P_w^{n,j})^2 + \eta_o(S_o^{n,ij})(P_o^{n,i} - P_o^{n,j})^2) \leq C. \quad (3.18)$$

Proof. We test (2.42) by $P_{w,h}^n$, (2.43) by $P_{o,h}^n$, add the two equations, multiply by τ and sum over n from 1 to N . By using (2.44) and (2.41), we obtain

$$\begin{aligned} - \sum_{n=1}^N (S_h^n - S_h^{n-1}, I_h p_c(S_h^n))_h^\varphi + \frac{1}{2} \sum_{n=1}^N \tau \sum_{\alpha=w,o} \sum_{i,j=1}^M c_{ij} \eta_\alpha(S_\alpha^{n,ij}) (P_\alpha^{n,i} - P_\alpha^{n,j})^2 \\ = \sum_{n=1}^N \tau \sum_{\alpha=w,o} (f_\alpha(s_{\text{in},h}^n) \bar{q}_h^n - f_\alpha(S_h^n) \underline{q}_h^n, P_{\alpha,h}^n)_h. \end{aligned} \quad (3.19)$$

Following [16], the first term in (3.19) is treated with the primitive g_c of p_c , see (1.11). Indeed, by the mean-value theorem, there exists ξ between $S^{n,i}$ and $S^{n-1,i}$ such that

$$g_c(S^{n,i}) - g_c(S^{n-1,i}) = -(S^{n,i} - S^{n-1,i}) p_c(\xi).$$

As the function p_c is decreasing, then $p_c(\xi) \geq p_c(S^{n,i})$ when $S^{n,i} \geq S^{n-1,i}$ and $p_c(\xi) \leq p_c(S^{n,i})$ when $S^{n,i} \leq S^{n-1,i}$. In both cases, we have

$$g_c(S^{n,i}) - g_c(S^{n-1,i}) \leq -(S^{n,i} - S^{n-1,i}) p_c(S^{n,i}),$$

and owing that φ is positive and constant in time, (3.19) can be replaced by the inequality

$$\begin{aligned} (g_c(S_h^N) - g_c(S_h^0), 1)_h^\varphi + \frac{1}{2} \sum_{n=1}^N \tau \sum_{\alpha=w,o} \sum_{i,j=1}^M c_{ij} \eta_\alpha(S_\alpha^{n,ij}) (P_\alpha^{n,i} - P_\alpha^{n,j})^2 \\ \leq \sum_{n=1}^N \tau \sum_{\alpha=w,o} (f_\alpha(s_{\text{in},h}^n) \bar{q}_h^n - f_\alpha(S_h^n) \underline{q}_h^n, P_{\alpha,h}^n)_h. \end{aligned} \quad (3.20)$$

As the first term in the above left-hand side is bounded, owing to the continuity of g_c and boundedness of $S_{h,\tau}$, it suffices to handle the right-hand side. Let us drop the superscript n and treat one term in the time sum. Following again [16], in view of Lemma 3.3 we use the auxiliary pressures p_{wg} and p_{wo} , defined in (1.12). Clearly, (1.14) and (2.34) imply

$$P_w^i + p_{wg}(S^i) + p_{og}(S^i) + p_c(0) = P_o^i, \quad \forall i. \quad (3.21)$$

Using this, a generic term, say Y , in the right-hand side of (3.20) can be expressed as

$$\begin{aligned} Y = & (\bar{q}_h - \underline{q}_h, U_{w,h})_h + (f_o(s_{\text{in},h})\bar{q}_h - f_o(S_h)\underline{q}_h, p_c(0))_h \\ & + (f_o(s_{\text{in},h})\bar{q}_h - f_o(S_h)\underline{q}_h, p_{og}(S_h))_h - (f_w(s_{\text{in},h})\bar{q}_h - f_w(S_h)\underline{q}_h, p_{wg}(S_h))_h = T_1 + \dots + T_4. \end{aligned}$$

We now bound each term T_i . For T_1 , (2.31) implies that any constant β can be added to $U_{w,h}$, in particular β can be chosen so that the sum has zero mean value in Ω . Hence, considering the generalized Poincaré inequality

$$\forall v \in H^1(\Omega), \quad \|v\|_{L^2(\Omega)} \leq C(\left| \int_{\Omega} v \right| + \|\nabla v\|_{L^2(\Omega)}), \quad (3.22)$$

with a constant C , depending only on the domain Ω , we have

$$\|U_{w,h} + \beta\|_h \leq C\|U_{w,h} + \beta\|_{L^2(\Omega)} \leq C\|\nabla U_{w,h}\|_{L^2(\Omega)},$$

with another constant C . Then Young's inequality yields

$$|T_1| \leq \frac{C^2}{2\eta_*} \|\bar{q}_h - \underline{q}_h\|_h^2 + \frac{\eta_*}{4} \|\nabla U_{w,h}\|_{L^2(\Omega)}^2,$$

and with Lemma 3.3, this becomes

$$|T_1| \leq \frac{C^2}{2\eta_*} \|\bar{q}_h - \underline{q}_h\|_h^2 + \frac{1}{4} \sum_{i,j=1}^M c_{ij} (\eta_w(S^{ij})(P_w^j - P_w^i)^2 + \eta_o(S^{ij})(P_o^j - P_o^i)^2).$$

The term T_2 is easily bounded since $p_c(0)$ is a number, and so are the terms T_3 and T_4 , in view of the boundedness of the saturation and the continuity of p_{og} and p_{wg} . We thus have

$$|T_2 + T_3 + T_4| \leq C(\|\bar{q}_h\|_{L^1(\Omega)} + \|\underline{q}_h\|_{L^1(\Omega)}).$$

Then substituting these bounds for each n into (3.20), we obtain

$$\begin{aligned} \frac{1}{4}\tau \sum_{n=1}^N \sum_{i,j=1}^M c_{ij} (\eta_w(S_w^{n,ij})(P_w^{n,i} - P_w^{n,j})^2 + \eta_o(S_o^{n,ij})(P_o^{n,i} - P_o^{n,j})^2) \leq C(\|\bar{q}_{h,\tau} - \underline{q}_{h,\tau}\|_{L^2(\Omega \times]0,T])}^2 \\ + \|\bar{q}_{h,\tau}\|_{L^1(\Omega \times]0,T])} + \|\underline{q}_{h,\tau}\|_{L^1(\Omega \times]0,T])}, \end{aligned}$$

thus proving (3.18). \square

By combining Theorem 3.4 with Lemma 3.3, we immediately derive a bound on the discrete auxiliary pressures. The bound (3.23) with $\alpha = o$ follows from the same with $\alpha = w$, (1.14), and (2.34).

THEOREM 3.5. *We have for $\alpha = w, o$*

$$\eta_* \|\nabla U_{\alpha,h,\tau}\|_{L^2(\Omega \times]0,T])}^2 \leq C, \quad (3.23)$$

with the constant C of (3.18).

4. Existence of numerical solution. We fix $n \geq 1$ and assume there exists a solution $(S_h^{n-1}, P_{w,h}^{n-1})$ at time t^{n-1} with $0 \leq S_h^{n-1} \leq 1$. We want to show existence of a solution $(S_h^n, P_{w,h}^n)$ by means of the topological degree [11, 12].

Let θ be a constant parameter in $[0, 1]$. For any continuous function $f : [0, 1] \rightarrow \mathbb{R}$, we define the transformed function $\tilde{f} : [0, 1] \rightarrow \mathbb{R}$ by

$$\forall s \in [0, 1], \quad \tilde{f}(s) = f(ts + (1-t)\theta).$$

Since θ is fixed, when $t = 0$, $\tilde{f}(s) = f(\theta)$, a constant independent of s . Now, (2.45) implies that any solution $P_{w,h,\tau}$ of (2.42)–(2.45) belongs to the following subspace $X_{0,h}$ of X_h ,

$$X_{0,h} = \{\Lambda_h \in X_h; \int_{\Omega} \Lambda_h = 0\}. \quad (4.1)$$

This suggests to define the mapping $\mathcal{F} : [0, 1] \times X_h \times X_{0,h} \rightarrow X_h \times X_{0,h}$ by

$$\mathcal{F}(t, \zeta, \Lambda) = (A_h, A_h + B_h),$$

where A_h , respectively B_h , solves for all $\Theta_h \in X_h$,

$$\begin{aligned} (A_h, \Theta_h) &= \frac{1}{\tau} (\zeta_h - S_h^{n-1}, \Theta_h)_h^\varphi - [\Lambda_h, I_h(\tilde{\eta}_w(\zeta_h)); \Lambda_h, \Theta_h]_h \\ &\quad - (I_h(\tilde{f}_w(s_{\text{in},h}^n))t\bar{q}_h^n - I_h(\tilde{f}_w(\zeta_h))t\underline{q}_h^n, \Theta_h)_h, \end{aligned} \quad (4.2)$$

$$\begin{aligned} (B_h, \Theta_h) &= -\frac{1}{\tau} (\zeta_h - S_h^{n-1}, \Theta_h)_h^\varphi - [P_{o,h}, I_h(\tilde{\eta}_o(\zeta_h)); P_{o,h}, \Theta_h]_h \\ &\quad - (I_h(\tilde{f}_o(s_{\text{in},h}^n))t\bar{q}_h^n - I_h(\tilde{f}_o(\zeta_h))t\underline{q}_h^n, \Theta_h)_h, \end{aligned} \quad (4.3)$$

and $P_{o,h}$ is defined by

$$P_{o,h} = \Lambda_h - I_h(\tilde{p}_c(\zeta_h)). \quad (4.4)$$

The choice of $\tilde{\eta}_w(\zeta_h)$ in (4.2) (respectively $\tilde{\eta}_o(\zeta_h)$ in (4.3)) is given by (2.36) (respectively (2.37)) where Λ_h plays the role of $P_{w,h}$ and $P_{o,h}$ is defined in (4.4). As in (2.36) and (2.37), it leads us to introduce the variables ζ_w^{ij} and ζ_o^{ij} for all $1 \leq i, j \leq M$. Clearly, (4.2)–(4.4) determine uniquely A_h and B_h , and it is easy to check that $A_h + B_h$ belongs to $X_{0,h}$.

The mapping $t \mapsto \mathcal{F}(t, \zeta_h, \Lambda_h)$ is continuous. Indeed, since the space has finite dimension, we only need to check continuity of the upwinding. By splitting x into its positive and negative part, $x = x^+ + x^-$, the upwind term, say $\widetilde{\eta}_w(\zeta_w^{ij})(P_w^j - P_w^i)$ reads

$$\widetilde{\eta}_w(\zeta_w^{ij})(P_w^j - P_w^i) = \eta_w(t\zeta^i + (1-t)\theta)((P_w^j - P_w^i)_-) + \eta_w(t\zeta^j + (1-t)\theta)((P_w^j - P_w^i)_+),$$

which is continuous with respect to t .

We remark that $\mathcal{F}(1, \zeta_h, \Lambda_h) = \mathbf{0}$ implies that (ζ_h, Λ_h) solves (2.42)–(2.45). Conversely, if (ζ_h, Λ_h) solves (2.42)–(2.45) then $\mathcal{F}(1, \zeta_h, \Lambda_h) = \mathbf{0}$. Thus showing existence of a solution to the problem (2.42)–(2.45) is equivalent to showing existence of a zero of $\mathcal{F}(1, \zeta_h, \Lambda_h)$. Before proving existence of a zero, we use the estimates established in the previous section to determine an a priori bound of any zero (ζ_h, Λ_h) of $\mathcal{F}(1, \zeta_h, \Lambda_h)$.

4.1. A priori bounds on (ζ_h, Λ_h) . In the following we consider $t \in [0, 1]$ and $(\zeta_h, \Lambda_h) \in X_h \times X_{0,h}$ that satisfy

$$\mathcal{F}(t, \zeta_h, \Lambda_h) = \mathbf{0}. \quad (4.5)$$

We first show that ζ_h satisfies a maximum principle.

PROPOSITION 4.1. *The following bounds hold for all (t, ζ_h, Λ_h) satisfying (4.5):*

$$0 \leq \zeta_h \leq 1. \quad (4.6)$$

Proof. Either $t \in]0, 1]$ or $t = 0$. The proof for $t \in]0, 1]$ follows closely the argument used in proving Theorem 3.1 and is left to the reader. For $t = 0$ we proceed again by contradiction. Assume first that $\|\zeta_h\|_{L^\infty(\Omega)} > 1$, i.e., there is a node i such that

$$\zeta^i = \|\zeta_h\|_{L^\infty(\Omega)} > 1 \geq S^{n-1,i}.$$

As $t = 0$, (4.5) reduces to

$$\sum_{j \neq i} c_{ij} \eta_w(\theta)(\Lambda^i - \Lambda^j) > 0, \quad - \sum_{j \neq i} c_{ij} \eta_o(\theta)(\Lambda^i - \Lambda^j) > 0, \quad \forall 1 \leq i \leq M.$$

Since η_o and η_w are non-negative functions satisfying (1.8), the inequalities above yield a contradiction. A similar argument is used to show that $\zeta_h \geq 0$. \square

Next we show the following bound on Λ_h .

PROPOSITION 4.2. *There is a constant C such that for all $t \in [0, 1]$ we have*

$$\eta_* \sum_{i,j=1}^M c_{ij} (\Lambda^j - \Lambda^i + p_{wg}(t\zeta^j + (1-t)\theta) - p_{wg}(t\zeta^i + (1-t)\theta))^2 \leq C. \quad (4.7)$$

Proof. The proof follows closely that of Theorem 3.5. First we show there exists a constant C_1 independent of t such that

$$\sum_{i,j=1}^M c_{ij} \left(\eta_w (t\zeta_w^{ij} + (1-t)\theta) (\Lambda^j - \Lambda^i)^2 + \eta_o (t\zeta_o^{ij} + (1-t)\theta) (P_{o,h}^j - P_{o,h}^i)^2 \right) \leq C_1,$$

with $P_{o,h}$ defined in (4.4). This bound is obtained via arguments similar to those used in proving Theorem 3.4. The main difference is that the formula is neither summed over n nor multiplied by the time step τ . As a consequence, the constant C_1 includes a term of the form $\tau^{-1} \|g_c\|_{L^\infty(\Omega)}$ arising from the bound of the discrete time derivative. To finish the proof we must show that

$$\eta_* \left(\Lambda^j - \Lambda^i + p_{wg}(t\zeta^j + (1-t)\theta) - p_{wg}(t\zeta^i + (1-t)\theta) \right)^2 \leq \eta_w (t\zeta^{ij} + (1-t)\theta) (\Lambda^j - \Lambda^i)^2 + \eta_o (t\zeta_o^{ij} + (1-t)\theta) (P_o^j - P_o^i)^2.$$

By (1.8), this is trivially satisfied when $t = 0$. When $t \in]0, 1]$, the argument is the same as in the proof of Lemma 3.3. \square

Propositions 4.1 and 4.2 are combined to obtain a bound on $\|\zeta_h\|_h + \|\Lambda_h\|_h$.

PROPOSITION 4.3. *There exists a constant $R_1 > 0$, independent of $t \in [0, 1]$, such that any solution (ζ_h, Λ_h) of (4.5) satisfies*

$$\|\zeta_h\|_h + \|\Lambda_h\|_h \leq R_1. \quad (4.8)$$

Proof. According to Proposition 4.1, there exists a constant C_1 independent of t such that

$$\|\zeta_h\|_h \leq C_1.$$

To establish a bound on $\|\Lambda_h\|_h$, we infer from (1.12) that the function $|p_{wg}|$ is bounded by $p_c(0) - p_c(1)$ because f_o is bounded by one and p_c is a decreasing function. Thus (4.7) implies that there exists a constant C_2 independent of t that satisfies

$$\sum_{i,j=1}^M c_{ij} (\Lambda^j - \Lambda^i)^2 \leq C_2, \quad \text{i.e., } \|\nabla \Lambda_h\|_{L^2(\Omega)} \leq \frac{\sqrt{C_2}}{\sqrt{2}}, \quad (4.9)$$

owing to (2.10). As $\Lambda_h \in X_{0,h}$, the generalized Poincaré inequality (3.22) shows there exists a constant C_3 independent of t such that

$$\|\Lambda_h\|_{L^2(\Omega)} \leq C_3.$$

Then the equivalence of norm (2.5) yields

$$\|\Lambda_h\|_h \leq C_4,$$

and (4.8) follows by setting $R_1 = C_1 + C_4$, a constant independent of t . \square

4.2. Proof of existence. For any $R > 0$, let B_R denote the ball

$$B_R = \{(\zeta_h, \Lambda_h) \in X_h \times X_{0,h} : \|\zeta_h\|_h + \|\Lambda_h\|_h \leq R\}, \quad (4.10)$$

and let $R_0 = R_1 + 1$, where R_1 is the constant of (4.8). Since all solutions (ζ_h, Λ_h) of (4.5) are in the ball B_{R_1} , this function has no zero on the boundary ∂B_{R_0} . Existence of a solution of (2.42)–(2.45) follows from the following result:

THEOREM 4.4. *The equation $\mathcal{F}(1, \zeta_h, \Lambda_h) = \mathbf{0}$ has at least one solution $(\zeta_h, \Lambda_h) \in B_{R_0}$.*

Proof. The proof proceeds in two steps.

First, we show that the system with $t = 0$ has a solution:

$$\mathcal{F}(0, \zeta_h, \Lambda_h) = 0.$$

This is a square linear system in finite dimension, so existence is equivalent to uniqueness. Thus we assume that it has two solutions, and for convenience, we still denote by (ζ_h, Λ_h) the difference between the two solutions. The system reads

$$\frac{\tilde{m}_i}{\tau} \zeta_h^i - \sum_{j \neq i, j \in \mathcal{N}(i)} c_{ij} \eta_w(\theta) (\Lambda^j - \Lambda^i) = 0, \quad 1 \leq i \leq M, \quad (4.11)$$

$$-\frac{\tilde{m}_i}{\tau} \zeta_h^i - \sum_{j \neq i, j \in \mathcal{N}(i)} c_{ij} \eta_o(\theta) (\Lambda^j - \Lambda^i) = 0, \quad 1 \leq i \leq M, \quad (4.12)$$

$$\sum_i m_i \Lambda^i = 0. \quad (4.13)$$

We add the first two equations, multiply by Λ^i , and sum over i . Then (2.10) and (2.41) imply that Λ_h is a constant and finally (4.13) shows that this constant is zero. This yields $\zeta_h = 0$.

Next, we argue on the topological degree. Since the topological degree of a linear map is the sign of its determinant, we have

$$d(\mathcal{F}(0, \zeta_h, \Lambda_h), B_{R_0}, 0) \neq 0.$$

We also know that $d(\mathcal{F}(t, \zeta_h, \Lambda_h), B_{R_0}, 0)$ is independent of t since the mapping $t \mapsto \mathcal{F}(t, \zeta_h, \Lambda_h)$ is continuous and for every $t \in [0, 1]$, if $\mathcal{F}(t, \zeta_h, \Lambda_h) = 0$, then (ζ_h, Λ_h) does not belong to ∂B_{R_0} . Therefore we have

$$d(\mathcal{F}(1, \zeta_h, \Lambda_h), B_{R_0}, 0) = d(\mathcal{F}(0, \zeta_h, \Lambda_h), B_{R_0}, 0) \neq 0.$$

This implies that $\mathcal{F}(1, \zeta_h, \Lambda_h)$ has a zero $(\zeta_h, \Lambda_h) \in B_{R_0}$. \square

5. Additional pressure estimates. The pressure estimates (3.18) and (3.23) are not sufficient to pass to the limit in the scheme (2.42)–(2.45). These are nonlinear equations and we need strong convergences that do not stem directly from (3.18) and (3.23).

Following [16], we propose to derive a bound for the gradient of g , see (1.13), at $s = S_{h,\tau}$. Then, under suitable assumptions on the behavior of η'_w , η'_o , and p'_e , we shall prove the strong convergence of $g(S_{h,\tau})$ in $L^2(\Omega \times]0, T])$ and in turn the strong convergence of $S_{h,\tau}$ in $L^2(\Omega \times]0, T])$.

Estimating the gradient of $g(S_{h,\tau})$ is a long and intricate process; it is based on the fact that

$$|g(S^{n,j}) - g(S^{n,i})|^2 \leq C(f_w(S^{n,i}) - f_w(S^{n,j}))(g(S^{n,i}) - g(S^{n,j})),$$

see (5.59). Therefore, we must derive a bound for the product of the gradients of g and f_w . This is split into several steps.

5.1. A preliminary inequality. Our starting step is the following inequality:

PROPOSITION 5.1. *There exists a constant C_1 independent of h and τ such that*

$$-\sum_{n=1}^N \tau \sum_{\alpha=o,w} [P_{\alpha,h}^n, \eta_\alpha(S_{\alpha,h}^n); f_\alpha(S_h^n), P_{\alpha,h}^n]_h = R_1, \quad (5.1)$$

where the remainder R_1 satisfies $|R_1| \leq C_1$.

Proof. By testing (2.42) with $I_h f_w(S_h^n)$ and (2.43) with $I_h f_o(S_h^n)$, adding the resulting equalities, and multiplying by τ , we obtain

$$\begin{aligned} & \sum_{n=1}^N (S_h^n - S_h^{n-1}, f_w(S_h^n) - f_o(S_h^n))_h^\varphi - \sum_{n=1}^N \tau \sum_{\alpha=o,w} [P_{\alpha,h}^n, \eta_\alpha(S_{\alpha,h}^n); f_\alpha(S_h^n), P_{\alpha,h}^n]_h \\ &= \int_0^T \left((\bar{q}_{h,\tau}, \sum_{\alpha=o,w} f_\alpha(s_{\text{in},h,\tau}) f_\alpha(S_{h,\tau}))_h - (\underline{q}_{h,\tau}, \sum_{\alpha=o,w} (f_\alpha(S_{h,\tau}))^2)_h \right) \leq 4\|\bar{q}\|_{L^1(\Omega \times]0, T])}, \end{aligned} \quad (5.2)$$

in view of (1.6) and (1.9). To control the time difference of $S_{h,\tau}$, we introduce the global flux defined by

$$\forall x \in [0, 1], \quad G(x) = \int_0^x (f_w(s) - f_o(s)) ds, \quad (5.3)$$

and we write

$$(S_h^n - S_h^{n-1})(f_w(S_h^n) - f_o(S_h^n)) = (S_h^n - S_h^{n-1})G'(S_h^n).$$

But by (1.6), $G'(x) = 2f_w(x) - 1$ is increasing. Hence, considering that

$$G(S_h^n) - G(S_h^{n-1}) = (S_h^n - S_h^{n-1})G'(c),$$

for some c between $S_h^n - S_h^{n-1}$, we easily check that

$$G(S_h^n) - G(S_h^{n-1}) \leq (S_h^n - S_h^{n-1})G'(S_h^n).$$

Thus, the properties of φ imply

$$\sum_{n=1}^N (S_h^n - S_h^{n-1}, f_w(S_h^n) - f_o(S_h^n))_h^\varphi \geq (G(S_h^N), 1)_h^\varphi - (G(S_h^0), 1)_h^\varphi.$$

But the boundedness of $S_{h,\tau}$, the continuity of f_α , and the properties of φ imply

$$|(G(S_h^N), 1)_h^\varphi - (G(S_h^0), 1)_h^\varphi| \leq C',$$

with a constant C' independent of h and τ . By substituting these inequalities into (5.2) we derive (5.1) with $C_1 = 4 \|\bar{q}\|_{L^1(\Omega \times]0, T])} + C'$. \square

5.2. Some discrete total flux inequalities. In this section, it is convenient to work directly on the scheme (2.32)–(2.33). For each index i , the sum of the equations (2.32) and (2.33) give, for $1 \leq i \leq M$ and $1 \leq n \leq N$,

$$- \sum_{j \neq i, j \in \mathcal{N}(i)} c_{ij} \left[\eta_w(S_w^{n,ij})(P_w^{n,j} - P_w^{n,i}) + \eta_o(S_o^{n,ij})(P_o^{n,j} - P_o^{n,i}) \right] = m_i(\bar{q}^{n,i} - \underline{q}^{n,i}).$$

Following [16], this suggests to define a discrete anti-symmetric upwinded total flux,

$$F^{n,ij} = -\eta_w(S_w^{n,ij})(P_w^{n,j} - P_w^{n,i}) - \eta_o(S_o^{n,ij})(P_o^{n,j} - P_o^{n,i}); \quad (5.4)$$

it satisfies

$$\sum_{j \neq i, j \in \mathcal{N}(i)} c_{ij} F^{n,ij} = m_i(\bar{q}^{n,i} - \underline{q}^{n,i}). \quad (5.5)$$

This identity yields a first bound for the discrete total flux.

PROPOSITION 5.2. *The discrete total flux $F^{n,ij}$ satisfies the following bounds for $\alpha = w, o$:*

$$\left| \sum_{n=1}^N \tau \sum_{i,j=1}^M f_\alpha^2(S^{n,i}) c_{ij} F^{n,ij} \right| \leq 2 \|\bar{q}\|_{L^1(\Omega \times]0, T])}. \quad (5.6)$$

Proof. The statement follows by multiplying (5.5) with $\tau f_\alpha^2(S^{n,i})$, and summing

$$\sum_{n=1}^N \tau \sum_{i,j=1}^M f_\alpha^2(S^{n,i}) c_{ij} F^{n,ij} = \sum_{n=1}^N \tau \sum_{i=1}^M m_i f_\alpha^2(S^{n,i}) (\bar{q}^{n,i} - \underline{q}^{n,i}) \leq 2 \|\bar{q}\|_{L^1(\Omega \times]0, T])}.$$

\square

To simplify some of the calculations below, it is convenient to drop the time superscript n , when there is no ambiguity, and restore it when needed.

By using the relation (1.7), $F^{i,j}$ can also be written as

$$\begin{aligned} F^{ij} &= -(\eta_w(S_w^{ij}) + \eta_o(S_o^{ij}))(P_w^j - P_w^i) - \eta_o(S_o^{ij})(p_c(S^j) - p_c(S^i)) \\ &= -(\eta_w(S_w^{ij}) + \eta_o(S_o^{ij}))(P_o^j - P_o^i) + \eta_w(S_w^{ij})(p_c(S^j) - p_c(S^i)). \end{aligned} \quad (5.7)$$

In order to insert it into (5.1), we bring forward F^{ij} in the expressions for $\eta_\alpha(S_\alpha^{ij})(P_\alpha^j - P_\alpha^i)$, $\alpha = w, o$. Starting from the identity

$$\begin{aligned} \eta_w(S_w^{ij})(P_w^j - P_w^i) &= f_w(S_w^{ij}) \left[(\eta_w(S_w^{ij}) + \eta_o(S_o^{ij}))(P_w^j - P_w^i) + \eta_o(S_o^{ij})(p_c(S^j) - p_c(S^i)) \right. \\ &\quad \left. - \eta_o(S_o^{ij})(p_c(S^j) - p_c(S^i)) + (\eta_o(S_w^{ij}) - \eta_o(S_o^{ij}))(P_w^j - P_w^i) \right], \end{aligned}$$

the expression (5.7) leads to

$$\eta_w(S_w^{ij})(P_w^j - P_w^i) = f_w(S_w^{ij}) \left[-F^{ij} - \eta_o(S_o^{ij})(p_c(S^j) - p_c(S^i)) + (\eta_o(S_w^{ij}) - \eta_o(S_o^{ij}))(P_w^j - P_w^i) \right]. \quad (5.8)$$

Similarly,

$$\eta_o(S_o^{ij})(P_o^j - P_o^i) = f_o(S_o^{ij}) \left[-F^{ij} + \eta_w(S_w^{ij})(p_c(S^j) - p_c(S^i)) + (\eta_w(S_o^{ij}) - \eta_w(S_w^{ij}))(P_o^j - P_o^i) \right]. \quad (5.9)$$

We also introduce the anti-symmetric quantities that collect the terms other than F^{ij} in (5.8) and (5.9),

$$C_w^{ij} = \eta_o(S_o^{ij})(p_c(S^j) - p_c(S^i)) - (\eta_o(S_w^{ij}) - \eta_o(S_o^{ij}))(P_w^j - P_w^i), \quad (5.10)$$

$$C_o^{ij} = -\eta_w(S_w^{ij})(p_c(S^j) - p_c(S^i)) - (\eta_w(S_o^{ij}) - \eta_w(S_w^{ij}))(P_o^j - P_o^i). \quad (5.11)$$

With this notation, we have

$$\eta_\alpha(S_\alpha^{ij})(P_\alpha^j - P_\alpha^i) = f_\alpha(S_\alpha^{ij}) [-F^{ij} - C_\alpha^{ij}], \quad \alpha = w, o.$$

Thus, the term that is summed over i in (5.1) has the expression

$$- \sum_{\alpha=w,o} f_\alpha(S^i) \sum_{j \neq i, j \in \mathcal{N}(i)} c_{ij} \eta_\alpha(S_\alpha^{ij})(P_\alpha^j - P_\alpha^i) = \sum_{\alpha=w,o} f_\alpha(S^i) \sum_{j \neq i, j \in \mathcal{N}(i)} c_{ij} f_\alpha(S_\alpha^{ij})(F^{ij} + C_\alpha^{ij}). \quad (5.12)$$

Now, we reintroduce the superscript n and to simplify, we set

$$A_{1,i,n} = \sum_{\alpha=w,o} f_\alpha(S^{n,i}) \sum_{j=1}^M c_{ij} f_\alpha(S_\alpha^{n,ij}) F^{n,ij}, \quad (5.13)$$

$$A_{\alpha,i,n} = f_\alpha(S^{n,i}) \sum_{j=1}^M c_{ij} f_\alpha(S_\alpha^{n,ij}) C_\alpha^{n,ij}. \quad (5.14)$$

With this notation, our next proposition is derived by substituting (5.12)–(5.14) into (5.1).

PROPOSITION 5.3. *We have, with the remainder R_1 of (5.1),*

$$\sum_{n=1}^N \tau \sum_{i=1}^M A_{1,i,n} + \sum_{n=1}^N \tau \sum_{i=1}^M \sum_{\alpha=w,o} A_{\alpha,i,n} = R_1. \quad (5.15)$$

We must transform suitably each term in this sum to bring forward g . Let us start with the first term of (5.15), i.e., the combination of the discrete total flux.

5.3. Combination of the discrete total flux. To simplify, let A_1 denote the first term,

$$A_1 = \sum_{n=1}^N \tau \sum_{i,j=1}^M \sum_{\alpha=w,o} \left[f_\alpha(S^{n,i}) c_{ij} f_\alpha(S_\alpha^{n,ij}) F^{n,ij} \right].$$

Inspired by (5.6), we introduce the difference

$$A_2 = A_1 - \sum_{n=1}^N \tau \sum_{i,j=1}^M (f_w^2(S^{n,i}) + f_o^2(S^{n,i})) c_{ij} F^{n,ij}.$$

Clearly, A_2 collects the discrepancies arising from the upwinding,

$$A_2 = \sum_{n=1}^N \tau \sum_{i,j=1}^M \sum_{\alpha=w,o} \left[f_\alpha(S^{n,i}) c_{ij} (f_\alpha(S_\alpha^{n,ij}) - f_\alpha(S^{n,i})) F^{n,ij} \right]. \quad (5.16)$$

As (5.6) yields

$$A_1 = A_2 + R_2, \quad \text{with } |R_2| \leq 4 \|\bar{q}\|_{L^1(\Omega \times]0,T])}, \quad (5.17)$$

a bound for A_1 stems from a bound for A_2 . To this end, in view of (5.16), it is useful to consider the four subsets of indices $j \in \mathcal{N}(i), j \neq i$, union and intersection:

$$\begin{aligned} \mathcal{N}_w(i) &= \{j \in \mathcal{N}(i); P_w^{n,j} > P_w^{n,i}\}, & \mathcal{N}_o(i) &= \{j \in \mathcal{N}(i); P_o^{n,j} > P_o^{n,i}\} \\ \mathcal{N}_{w,S}(i) &= \{j \in \mathcal{N}(i), j \neq i; P_w^{n,j} = P_w^{n,i}, S^{n,j} \geq S^{n,i}\}, \\ \mathcal{N}_{o,S}(i) &= \{j \in \mathcal{N}(i), j \neq i; P_o^{n,j} = P_o^{n,i}, S^{n,j} \leq S^{n,i}\}, \\ \mathcal{UN}(i) &= \mathcal{N}_w(i) \cup \mathcal{N}_o(i) \cup \mathcal{N}_{w,S}(i) \cup \mathcal{N}_{o,S}(i), \\ \mathcal{NF}(i) &= \{j \in \mathcal{N}(i); P_w^{n,i} > P_w^{n,j} \text{ and } P_o^{n,i} > P_o^{n,j}\}. \end{aligned} \quad (5.18)$$

Strictly speaking, these subsets should be written with the superscript n , but we omit it for the sake of simplicity. Then we have the following bound for A_2 :

PROPOSITION 5.4. *There exists a constant C_2 , independent of h and τ , such that*

$$A_2 = -\frac{1}{2} \sum_{n=1}^N \tau \sum_{i=1}^M \sum_{j \in \mathcal{UN}(i)} c_{ij} (f_w(S^{n,j}) - f_w(S^{n,i}))^2 F^{n,ij} + R_3, \quad (5.19)$$

where the remainder R_3 satisfies

$$|R_3| \leq C_2 = 2 \|\bar{q}\|_{L^1(\Omega \times]0,T])}.$$

Proof. Let us drop the superscript n . By definition, $f_w(S_w^{ij}) - f_w(S^i) = 0$ when $P_w^i > P_w^j$ and when $P_w^i = P_w^j$ and $S^i > S^j$. Similarly, $f_o(S_o^{ij}) - f_o(S^i) = 0$ when $P_o^i > P_o^j$ and when $P_o^i = P_o^j$ and $S^{n,i} < S^j$. Therefore, the n th term in A_2 , say a_2 , reduces to

$$a_2 = \sum_{i=1}^M \sum_{\alpha=w,o} f_\alpha(S^i) \sum_{j \in \mathcal{N}_\alpha(i) \cup \mathcal{N}_{\alpha,S}(i)} c_{ij} (f_\alpha(S^j) - f_\alpha(S^i)) F^{ij}.$$

By expanding the products, this can be written

$$a_2 = -\frac{1}{2} \sum_{i=1}^M \sum_{\alpha=w,o} \sum_{j \in \mathcal{N}_\alpha(i) \cup \mathcal{N}_{\alpha,S}(i)} c_{ij} (f_\alpha^2(S^i) - f_\alpha^2(S^j) + (f_\alpha(S^i) - f_\alpha(S^j))^2) F^{ij}. \quad (5.20)$$

Since c_{ij} vanishes when j is not a neighbor of i , we have, by interchanging i and j and using the anti-symmetry of F^{ij} and the symmetry of c_{ij} ,

$$-\sum_{i=1}^M \sum_{j \in \mathcal{N}_w(i)} c_{ij} f_w^2(S^j) F^{ij} = \sum_{i=1, j=1, P_w^j < P_w^i}^M c_{ij} f_w^2(S^i) F^{ij}. \quad (5.21)$$

Similarly,

$$-\sum_{i=1}^M \sum_{j \in \mathcal{N}_{w,S}(i)} c_{ij} f_w^2(S^j) F^{ij} = \sum_{i=1, j=1, P_w^i = P_w^j, S^i \geq S^j}^M c_{ij} f_w^2(S^i) F^{ij}. \quad (5.22)$$

Hence

$$-\frac{1}{2} \sum_{i=1}^M \sum_{j \in \mathcal{N}_w(i)} c_{ij} (f_w^2(S^i) - f_w^2(S^j)) F^{ij} = -\frac{1}{2} \sum_{i=1, j=1, P_w^i \neq P_w^j}^M c_{ij} f_w^2(S^i) F^{ij},$$

and

$$-\frac{1}{2} \sum_{i=1}^M \sum_{j \in \mathcal{N}_{w,S}(i)} c_{ij} (f_w^2(S^i) - f_w^2(S^j)) F^{ij} = -\frac{1}{2} \sum_{i=1, j=1, P_w^i = P_w^j}^M c_{ij} f_w^2(S^i) F^{ij},$$

because there is no contribution from the indices i, j such that $P_w^i = P_w^j, S^i = S^j$ since in this case the factor $F^{ij} = 0$. The same is true for the non-wetting phase. Thus

$$-\frac{1}{2} \sum_{\alpha=w,o} \sum_{i=1}^M \sum_{j \in \mathcal{N}_\alpha(i) \cup \mathcal{N}_{\alpha,S}(i)} c_{ij} (f_\alpha^2(S^i) - f_\alpha^2(S^j)) F^{ij} = -\frac{1}{2} \sum_{\alpha=w,o} \sum_{i=1, j=1}^M c_{ij} f_\alpha^2(S^i) F^{ij}.$$

By comparing with (5.6), we see that

$$\left| \frac{1}{2} \sum_{n=1}^N \tau \sum_{i=1}^M \sum_{\alpha=w,o} \sum_{j \in \mathcal{N}_\alpha(i) \cup \mathcal{N}_{\alpha,S}(i)} c_{ij} (f_\alpha^2(S^{n,i}) - f_\alpha^2(S^{n,j})) F^{n,ij} \right| \leq 2 \|\bar{q}\|_{L^1(\Omega \times]0, T])}. \quad (5.23)$$

This and the equality

$$(f_o(S^{n,j}) - f_o(S^{n,i}))^2 = (f_w(S^{n,j}) - f_w(S^{n,i}))^2,$$

readily imply (5.19). \square

Now, we set

$$A^{ij} = c_{ij} (f_w(S^j) - f_w(S^i))^2 F^{ij}, \quad a_3 = -\frac{1}{2} \sum_{i=1}^M \sum_{j \in \mathcal{UN}(i)} A^{ij}. \quad (5.24)$$

The next proposition simplifies the expression for a_3 .

PROPOSITION 5.5. *We have*

$$a_3 = \sum_{i=1}^M \sum_{j \in \mathcal{N}_{\mathcal{F}}(i)} c_{ij} (f_w(S^j) - f_w(S^i))^2 F^{ij}. \quad (5.25)$$

Proof. By expanding the indices in the set $\mathcal{UN}(i)$, interchanging the indices i and j , and using the anti-symmetry of A^{ij} , we derive

$$a_3 = \frac{1}{2} \left(\left(\sum_{\alpha=w,o} \sum_{P_\alpha^i > P_\alpha^j} A^{ij} \right) + \sum_{P_w^i = P_w^j, S^j \leq S^i} A^{ij} + \sum_{P_o^i = P_o^j, S^i \leq S^j} A^{ij} \right).$$

Now, we split the first two sums above as follows:

$$\sum_{\alpha=w,o} \sum_{P_\alpha^i > P_\alpha^j} A^{ij} = 2 \sum_{P_w^i > P_w^j, P_o^i > P_o^j} A^{ij} + \sum_{P_w^i > P_w^j, P_o^i \leq P_o^j} A^{ij} + \sum_{P_o^i > P_o^j, P_w^i \leq P_w^j} A^{ij}.$$

This leads to

$$a_3 = \sum_{j \in \mathcal{N}_{\mathcal{F}}(i)} A^{ij} + \frac{1}{2} \left(\sum_{P_w^i > P_w^j, P_o^i \leq P_o^j} A^{ij} + \sum_{P_o^i > P_o^j, P_w^i \leq P_w^j} A^{ij} + \sum_{P_w^i = P_w^j, S^j \leq S^i} A^{ij} + \sum_{P_o^i = P_o^j, S^i \leq S^j} A^{ij} \right).$$

The anti-symmetry of A^{ij} gives

$$\sum_{P_w^i > P_w^j, P_o^i \leq P_o^j} A^{ij} = - \sum_{P_w^j > P_w^i, P_o^j < P_o^i} A^{ij} - \sum_{P_w^j > P_w^i, P_o^j = P_o^i} A^{ij}.$$

By substituting and applying twice again the anti-symmetry of A^{ij} , we derive

$$a_3 = \sum_{j \in \mathcal{N}_{\mathcal{F}}(i)} A^{ij} + \frac{1}{2} \left(\sum_{P_o^i > P_o^j, P_w^i = P_w^j} A^{ij} + \sum_{P_w^i > P_w^j, P_o^i = P_o^j} A^{ij} - \sum_{P_w^i = P_w^j, S^i \leq S^j} A^{ij} - \sum_{P_o^i = P_o^j, S^j \leq S^i} A^{ij} \right). \quad (5.26)$$

Note that

$$\sum_{P_o^i > P_o^j, P_w^i = P_w^j} A^{ij} = \sum_{P_o^i \geq P_o^j, P_w^i = P_w^j} A^{ij},$$

since the additional term is zero. Therefore, in view of first (3.11) and next (3.12),

$$\sum_{P_o^i > P_o^j, P_w^i = P_w^j} A^{ij} = \sum_{P_w^i = P_w^j, S^i \leq S^j} A^{ij}, \quad \sum_{P_w^i > P_w^j, P_o^i = P_o^j} A^{ij} = \sum_{P_o^i = P_o^j, S^i \geq S^j} A^{ij}.$$

Thus all terms multiplying $\frac{1}{2}$ in (5.26) are cancelled and we recover (5.25). \square

By applying (5.17) and Propositions 5.4 and 5.5, A_1 has the following expression:

PROPOSITION 5.6. *We have*

$$\begin{aligned} & \sum_{n=1}^N \tau \sum_{i,j=1}^M \sum_{\alpha=w,o} \left[f_\alpha(S^{n,i}) c_{ij} f_\alpha(S^{n,ij}) F^{n,ij} \right] \\ &= \sum_{n=1}^N \tau \sum_{i=1}^M \sum_{j \in \mathcal{N}_{\mathcal{F}}(i)} c_{ij} (f_w(S^{n,j}) - f_w(S^{n,i}))^2 F^{n,ij} + R_4, \end{aligned} \quad (5.27)$$

where

$$|R_4| \leq 6 \|\bar{q}\|_{L^1(\Omega \times]0,T])}. \quad (5.28)$$

This settles the contribution of the first term of (5.15); the second terms are handled in the next subsection.

5.4. Terms involving the capillary pressure and mobility. These are the terms $A_{\alpha,i,n}$ defined in (5.14). By virtue of the anti-symmetry of C^{ij} , we can write for $\alpha = w, o$

$$\sum_{i,j=1}^M f_\alpha(S^i) c_{ij} f_\alpha(S^{ij}) C_\alpha^{ij} = -\frac{1}{2} \sum_{i,j=1}^M (f_\alpha(S^j) - f_\alpha(S^i)) c_{ij} f_\alpha(S^{ij}) C_\alpha^{ij}. \quad (5.29)$$

Owing to (1.6), the term with $\alpha = o$ in the right-hand side is $\frac{1}{2} \sum_{i,j=1}^M (f_w(S^j) - f_w(S^i)) c_{ij} f_o(S^{ij}) C_o^{ij}$. Therefore,

$$\sum_{\alpha=w,o} \sum_{i=1}^M A_{\alpha,i,n} = \frac{1}{2} \sum_{i,j=1}^M c_{ij} (f_w(S^{n,j}) - f_w(S^{n,i})) (-f_w(S_w^{n,ij}) C_w^{n,ij} + f_o(S_o^{n,ij}) C_o^{n,ij}). \quad (5.30)$$

Let K^{ij} denote the symmetric term

$$K^{ij} := c_{ij} (f_w(S^{n,j}) - f_w(S^{n,i})) (-f_w(S_w^{n,ij}) C_w^{n,ij} + f_o(S_o^{n,ij}) C_o^{n,ij});$$

by virtue of this symmetry, we have

$$\sum_{\alpha=w,o} \sum_{i=1}^M A_{\alpha,i,n} = \sum_{P_w^{n,i} > P_w^{n,j}} K^{n,ij} + \frac{1}{2} \sum_{P_w^{n,i} = P_w^{n,j}} K^{n,ij}. \quad (5.31)$$

5.5. Combining all terms. By substituting (5.27) and (5.31) into (5.15), we obtain the next lemma.

LEMMA 5.7. *We have*

$$\begin{aligned} & - \sum_{n=1}^N \tau \sum_{\alpha=o,w} [P_{\alpha,h}^n, \eta_\alpha(S_{\alpha,h}^n); f_\alpha(S_h^n), P_{\alpha,h}^n]_h = \sum_{n=1}^N \tau \left[\sum_{i=1}^M \left(\sum_{j \in \mathcal{N}_{\mathcal{F}}(i)} c_{ij} (f_w(S^{n,j}) - f_w(S^{n,i}))^2 F^{n,ij} \right. \right. \\ & \quad - \sum_{P_w^{n,i} > P_w^{n,j}} c_{ij} (f_w(S^{n,j}) - f_w(S^{n,i})) (f_w(S_w^{n,ij}) C_w^{n,ij} - f_o(S_o^{n,ij}) C_o^{n,ij}) \\ & \quad \left. \left. - \frac{1}{2} \sum_{P_w^{n,i} = P_w^{n,j}} c_{ij} (f_w(S^{n,j}) - f_w(S^{n,i})) (f_w(S_w^{n,ij}) C_w^{n,ij} - f_o(S_o^{n,ij}) C_o^{n,ij}) \right) \right] + R_4. \end{aligned} \quad (5.32)$$

with R_4 bounded by (5.28).

Thus, to bring forward g , we must suitably combine the terms of the above sum over i , and this is done by examining all pairs of indices (i, j) involved in (5.32), i.e., the pairs of indices in the following sets: (i) $P_w^i > P_w^j$ and $P_o^i > P_o^j$, (ii) $P_w^i > P_w^j$ and $P_o^i < P_o^j$, (iii) $P_w^i > P_w^j$ and $P_o^i = P_o^j$, (iv) $P_w^i = P_w^j$ and $P_o^i > P_o^j$, (v) $P_w^i = P_w^j$ and $P_o^i < P_o^j$. Note that the sixth case that would be $P_w^i = P_w^j$ and $P_o^i = P_o^j$ brings no information because it implies that $S^i = S^j$.

For the argument below, we shall use the following intermediate result.

PROPOSITION 5.8. *For each indices i and j , there exist (non unique) points α and α' between S^i and S^j such that*

$$g(S^j) - g(S^i) = -\eta_o(\alpha)f_w(\alpha)(p_c(S^j) - p_c(S^i)) = -\eta_w(\alpha')f_o(\alpha')(p_c(S^j) - p_c(S^i)). \quad (5.33)$$

Proof. By the definition (1.13),

$$g(S^j) - g(S^i) = - \int_{S^i}^{S^j} \eta_o(x)f_w(x)p'_c(x) dx = - \int_{S^i}^{S^j} \eta_w(x)f_o(x)p'_c(x) dx. \quad (5.34)$$

Since the functions $\eta_o f_w$ and $\eta_w f_o$ are continuous and do not change sign between S^i and S^j , (5.33) follows from the second mean formula for integrals. \square

To simplify, the superscript n is dropped.

5.5.1. The case $P_w^i > P_w^j$ and $P_o^i > P_o^j$. The following holds:

PROPOSITION 5.9. *Let $P_w^i > P_w^j$ and $P_o^i > P_o^j$; then the factor of τ in (5.32) satisfies*

$$\begin{aligned} c_{ij}(f_w(S^j) - f_w(S^i)) & \left((f_w(S^j) - f_w(S^i))F^{ij} - (f_w(S^i)C_w^{ij} - f_o(S^i)C_o^{ij}) \right) \\ & \geq c_{ij}(f_w(S^j) - f_w(S^i))(g(S^j) - g(S^i)). \end{aligned} \quad (5.35)$$

Proof. Let E^{ij} denote the left-hand side of (5.35). In the case $P_w^i > P_w^j$ and $P_o^i > P_o^j$, an expansion of F^{ij} , and C_α^{ij} yields

$$\begin{aligned} E^{ij} = & -c_{ij}(f_w(S^j) - f_w(S^i))^2(\eta_w(S^i) + \eta_o(S^i))(P_w^j - P_w^i) \\ & - c_{ij}(f_w(S^j) - f_w(S^i))(p_c(S^j) - p_c(S^i)) \left(\eta_o(S^i)(f_w(S^j) - f_w(S^i)) + 2 \frac{\eta_w(S^i)\eta_o(S^i)}{\eta_w(S^i) + \eta_o(S^i)} \right). \end{aligned}$$

As $P_w^i > P_w^j$, the first line in the above right-hand side is nonnegative and hence

$$E^{ij} \geq -c_{ij}(f_w(S^j) - f_w(S^i))(p_c(S^j) - p_c(S^i))\eta_o(S^i)(f_w(S^j) + f_w(S^i)).$$

Now, either $S^i \leq S^j$ or $S^i > S^j$. If $S^i \leq S^j$, then $f_w(S^j) \geq f_w(S^i)$ and $p_c(S^j) \leq p_c(S^i)$ because f_w is increasing and p_c is decreasing. This implies in particular that

$$\begin{aligned} E^{ij} & \geq -c_{ij}(f_w(S^j) - f_w(S^i))(p_c(S^j) - p_c(S^i))\eta_o(S^i)f_w(S^j) \\ & \geq -c_{ij}(f_w(S^j) - f_w(S^i))(p_c(S^j) - p_c(S^i))\eta_o(x)f_w(x), \end{aligned}$$

for any x between S^i and S^j . Then (5.35) follows from the first part of (5.33).

If $S^i > S^j$, then $f_w(S^j) - f_w(S^i) \leq 0$, $p_c(S^j) - p_c(S^i) \geq 0$, and we infer from (1.7) that E^{ij} reads

$$\begin{aligned} E^{ij} = & c_{ij} (f_w(S^j) - f_w(S^i)) \left[(\eta_w(S^i) + \eta_o(S^i)) (f_w(S^j) - f_w(S^i)) (P_o^i - P_o^j) \right. \\ & + (f_w(S^j) - f_w(S^i)) (\eta_w(S^i) + \eta_o(S^i)) (p_c(S^j) - p_c(S^i)) \\ & \left. + (p_c(S^i) - p_c(S^j)) \left(\eta_o(S^i) (f_w(S^j) - f_w(S^i)) + 2 \frac{\eta_w(S^i) \eta_o(S^i)}{\eta_w(S^i) + \eta_o(S^i)} \right) \right]. \end{aligned}$$

Since $P_o^i - P_o^j > 0$, the first line in the above right-hand side is nonnegative, and thus

$$\begin{aligned} E^{ij} \geq & c_{ij} (f_w(S^i) - f_w(S^j)) (p_c(S^j) - p_c(S^i)) \left[(f_w(S^i) - f_w(S^j)) (\eta_w(S^i) + \eta_o(S^i)) \right. \\ & \left. + (f_w(S^j) - f_w(S^i)) \eta_o(S^i) + 2 \frac{\eta_w(S^i) \eta_o(S^i)}{\eta_w(S^i) + \eta_o(S^i)} \right], \end{aligned}$$

which reduces to

$$E^{ij} \geq c_{ij} (f_w(S^i) - f_w(S^j)) (p_c(S^j) - p_c(S^i)) \eta_w(S^i) (f_o(S^j) + f_o(S^i)).$$

This leads for instance to

$$\begin{aligned} E^{ij} \geq & -c_{ij} (f_w(S^j) - f_w(S^i)) (p_c(S^j) - p_c(S^i)) \eta_w(S^i) f_o(S^j) \\ \geq & -c_{ij} (f_w(S^j) - f_w(S^i)) (p_c(S^j) - p_c(S^i)) \eta_w(x) f_o(x), \end{aligned}$$

for any x between S^i and S^j . Then (5.35) follows from the second part of (5.33). \square

5.5.2. The case $P_w^i > P_w^j$ and $P_o^i < P_o^j$. We have

PROPOSITION 5.10. *Let $P_w^i > P_w^j$ and $P_o^i < P_o^j$; then the factor of τ in (5.32) satisfies*

$$-c_{ij} (f_w(S^j) - f_w(S^i)) (f_w(S^i) C_w^{ij} - f_o(S^j) C_o^{ij}) \geq c_{ij} (f_w(S^j) - f_w(S^i)) (g(S^j) - g(S^i)). \quad (5.36)$$

Proof. Let E^{ij} denote the left-hand side of (5.36). We have $P_w^j - P_w^i < 0$ and $P_o^j - P_o^i > 0$. Then $S_w^{ij} = S^i$ and $S_o^{ij} = S^j$; also $S^j \leq S^i$ which implies that $\eta_w(S^j) \leq \eta_w(S^i)$ and $\eta_o(S^j) \geq \eta_o(S^i)$. The expression for C_α^{ij} becomes (see (5.10) and (5.11))

$$\begin{aligned} C_w^{ij} &= \eta_o(S^j) (p_c(S^j) - p_c(S^i)) - (\eta_o(S^i) - \eta_o(S^j)) (P_w^j - P_w^i), \\ C_o^{ij} &= -\eta_w(S^i) (p_c(S^j) - p_c(S^i)) - (\eta_w(S^j) - \eta_w(S^i)) (P_o^j - P_o^i). \end{aligned}$$

Hence

$$\begin{aligned}
E^{ij} &= -c_{ij}(f_w(S^j) - f_w(S^i)) \left(f_w(S^i)(\eta_o(S^j)(p_c(S^j) - p_c(S^i)) + (\eta_o(S^j) - \eta_o(S^i))(P_w^j - P_w^i)) \right. \\
&\quad \left. + f_o(S^j)(\eta_w(S^i)(p_c(S^j) - p_c(S^i)) + (\eta_w(S^j) - \eta_w(S^i))(P_o^j - P_o^i)) \right) \\
&= -c_{ij}(f_w(S^j) - f_w(S^i)) \left((P_o^j - P_o^i)(f_w(S^i)(\eta_o(S^j) - \eta_o(S^i)) + f_o(S^j)(\eta_w(S^j) - \eta_w(S^i))) \right. \\
&\quad \left. + (p_c(S^j) - p_c(S^i))(f_w(S^i)\eta_o(S^i) + f_o(S^j)\eta_w(S^i)) \right) \\
&= -c_{ij}(f_w(S^j) - f_w(S^i)) \left((P_o^j - P_o^i)(f_w(S^i)(\eta_o(S^j) - \eta_o(S^i)) + f_o(S^j)(\eta_w(S^j) - \eta_w(S^i))) \right. \\
&\quad \left. + (p_c(S^j) - p_c(S^i))\eta_w(S^i)(f_o(S^i) + f_o(S^j)) \right).
\end{aligned}$$

It follows from the above considerations that

$$\begin{aligned}
&-c_{ij}(f_w(S^j) - f_w(S^i))(p_c(S^j) - p_c(S^i))\eta_w(S^i)(f_o(S^i) + f_o(S^j)) \\
&\quad \geq -c_{ij}(f_w(S^j) - f_w(S^i))(p_c(S^j) - p_c(S^i))\eta_w(S^i)f_o(S^j) \\
&\quad \geq -c_{ij}(f_w(S^j) - f_w(S^i))(p_c(S^j) - p_c(S^i))\eta_w(x)f_o(x),
\end{aligned}$$

for any x between S^i and S^j . Now, the sign of the factor

$$f_w(S^i)(\eta_o(S^j) - \eta_o(S^i)) + f_o(S^j)(\eta_w(S^j) - \eta_w(S^i))$$

is not clear. If it is nonnegative, then the whole term

$$-c_{ij}(f_w(S^j) - f_w(S^i))(P_o^j - P_o^i)(f_w(S^i)(\eta_o(S^j) - \eta_o(S^i)) + f_o(S^j)(\eta_w(S^j) - \eta_w(S^i)))$$

is also nonnegative,

$$E^{ij} \geq -c_{ij}(f_w(S^j) - f_w(S^i))(p_c(S^j) - p_c(S^i))\eta_w(x)f_o(x).$$

and (5.36) follows from (5.33). If

$$f_w(S^i)(\eta_o(S^j) - \eta_o(S^i)) + f_o(S^j)(\eta_w(S^j) - \eta_w(S^i)) < 0,$$

then we rewrite E^{ij} in terms of P_w ,

$$\begin{aligned}
E^{ij} &= -c_{ij}(f_w(S^j) - f_w(S^i)) \left[(P_w^j - P_w^i)(f_w(S^i)(\eta_o(S^j) - \eta_o(S^i)) + f_o(S^j)(\eta_w(S^j) - \eta_w(S^i))) \right. \\
&\quad \left. + (p_c(S^j) - p_c(S^i))(f_w(S^j)\eta_o(S^j) + \eta_w(S^j)f_o(S^j)) \right].
\end{aligned}$$

Since the first line is now nonnegative, we infer

$$\begin{aligned}
E^{ij} &\geq -c_{ij}(f_w(S^j) - f_w(S^i))(p_c(S^j) - p_c(S^i))\eta_o(S^j)(f_w(S^j) + f_w(S^i)) \\
&\geq -c_{ij}(f_w(S^j) - f_w(S^i))(p_c(S^j) - p_c(S^i))\eta_o(x)f_w(x),
\end{aligned}$$

again for any x between S^i and S^j , and the result follows from (5.33). \square

5.5.3. The case $P_w^i > P_w^j$ and $P_o^i = P_o^j$. In this case, $p_c(S^j) - p_c(S^i) > 0$, $S^j \leq S^i$, and we have the following result:

PROPOSITION 5.11. *Let $P_w^i > P_w^j$ and $P_o^i = P_o^j$; then the factor of τ in (5.32) satisfies*

$$-c_{ij}(f_w(S^j) - f_w(S^i))(f_w(S^i)C_w^{ij} - f_o(S^j)C_o^{ij}) \geq c_{ij}(f_w(S^j) - f_w(S^i))(g(S^j) - g(S^i)). \quad (5.37)$$

Proof. Let E^{ij} denote the left-hand side of (5.37). We have $C_w^{ij} = \eta_o(S^i)(p_c(S^j) - p_c(S^i))$ and $C_o^{ij} = -\eta_w(S^i)(p_c(S^j) - p_c(S^i))$. Hence

$$\begin{aligned} E^{ij} &= -c_{ij}(f_w(S^j) - f_w(S^i))(f_w(S^i)\eta_o(S^i) + f_o(S^j)\eta_w(S^i))(p_c(S^j) - p_c(S^i)) \\ &= -c_{ij}(f_w(S^j) - f_w(S^i))(p_c(S^j) - p_c(S^i))(\eta_o(S^i)f_w(S^i) + \eta_w(S^i)f_o(S^j)). \end{aligned}$$

From here, (5.37) is derived as in the end of the proof of Proposition 5.10. \square

5.5.4. The case $P_w^i = P_w^j$ and $P_o^i > P_o^j$. In this case, $p_c(S^j) \leq p_c(S^i)$ and $S^j \geq S^i$. We have the following result:

PROPOSITION 5.12. *Let $P_w^i = P_w^j$ and $P_o^i > P_o^j$; then the factor of τ in (5.32) satisfies*

$$-\frac{1}{2}c_{ij}(f_w(S^j) - f_w(S^i))(f_w(S^j)C_w^{ij} - f_o(S^i)C_o^{ij}) \geq \frac{1}{2}c_{ij}(f_w(S^j) - f_w(S^i))(g(S^j) - g(S^i)). \quad (5.38)$$

Proof. In this case, $C_w^{ij} = \eta_o(S^i)(p_c(S^j) - p_c(S^i))$ and $C_o^{ij} = -\eta_w(S^i)(p_c(S^j) - p_c(S^i))$. Then the left-hand side E^{ij} of (5.38) is

$$\begin{aligned} E^{ij} &= -\frac{1}{2}c_{ij}(f_w(S^j) - f_w(S^i))(p_c(S^j) - p_c(S^i))(f_w(S^j)\eta_o(S^i) + f_o(S^i)\eta_w(S^i)) \\ &\geq -\frac{1}{2}c_{ij}(f_w(S^j) - f_w(S^i))(p_c(S^j) - p_c(S^i))f_w(S^j)\eta_o(S^i), \end{aligned}$$

and the proof of (5.38) proceeds as above. \square

5.5.5. The case $P_w^i = P_w^j$ and $P_o^i < P_o^j$. In this case, $p_c(S^j) \geq p_c(S^i)$ and hence $S^i \geq S^j$. We have the following result:

PROPOSITION 5.13. *Let $P_w^i = P_w^j$ and $P_o^i < P_o^j$; then the factor of τ in (5.32) satisfies*

$$-\frac{1}{2}c_{ij}(f_w(S^j) - f_w(S^i))(f_w(S^i)C_w^{ij} - f_o(S^j)C_o^{ij}) \geq \frac{1}{2}c_{ij}(f_w(S^j) - f_w(S^i))(g(S^j) - g(S^i)). \quad (5.39)$$

Proof. In this case, $C_w^{ij} = \eta_o(S^j)(p_c(S^j) - p_c(S^i))$ and $C_o^{ij} = -\eta_w(S^j)(p_c(S^j) - p_c(S^i))$. Then the left-hand side E^{ij} of (5.38) is

$$\begin{aligned} E^{ij} &= -\frac{1}{2}c_{ij}(f_w(S^j) - f_w(S^i))(p_c(S^j) - p_c(S^i))(f_w(S^i)\eta_o(S^j) + f_o(S^j)\eta_w(S^j)) \\ &\geq -\frac{1}{2}c_{ij}(f_w(S^j) - f_w(S^i))(p_c(S^j) - p_c(S^i))\eta_o(S^j)f_w(S^i), \end{aligned}$$

and the proof of (5.39) ends as above. \square

5.6. Auxiliary bound for the gradient of g . The following theorem is the first outcome of this section.

THEOREM 5.14. *There exists a constant C , independent of h and τ , such that*

$$\left| \int_0^T \int_{\Omega} \nabla(I_h(f_{\alpha}(S_{h,\tau}))) \cdot \nabla(I_h(g(S_{h,\tau}))) \right| \leq C, \quad \alpha = w, o. \quad (5.40)$$

Proof. Owing to (1.6), it suffices to prove (5.40) when $\alpha = w$. By applying Propositions 5.9–5.13 to Lemma 5.7 and combining with Proposition 5.1, we readily derive that

$$\sum_{n=1}^N \tau \sum_{i=1}^M \sum_{j \in \mathcal{N}(i), P_w^{n,i} \geq P_w^{n,j}} c_{ij}(f_w(S^{n,j}) - f_w(S^{n,i}))(g(S^{n,j}) - g(S^{n,i})) \leq C, \quad (5.41)$$

with a constant C independent of h and τ . Therefore, (5.40) will follow if we bound the summand for all j such that $P_w^{n,i} < P_w^{n,j}$. But the symmetry of the summand implies that

$$\begin{aligned} \sum_{i=1}^M \sum_{j \in \mathcal{N}(i), P_w^{n,i} < P_w^{n,j}} c_{ij}(f_w(S^{n,j}) - f_w(S^{n,i}))(g(S^{n,j}) - g(S^{n,i})) \\ = \sum_{i=1}^M \sum_{j \in \mathcal{N}(i), P_w^{n,i} > P_w^{n,j}} c_{ij}(f_w(S^{n,j}) - f_w(S^{n,i}))(g(S^{n,j}) - g(S^{n,i})). \end{aligned}$$

Hence

$$\begin{aligned} \int_{\Omega} \nabla(I_h(f_w(S_h^n))) \cdot \nabla(I_h(g(S_h^n))) &= 2 \sum_{i=1}^M \sum_{j \in \mathcal{N}(i), P_w^{n,i} > P_w^{n,j}} c_{ij}(f_w(S^{n,j}) - f_w(S^{n,i}))(g(S^{n,j}) - g(S^{n,i})) \\ &\quad + \sum_{i=1}^M \sum_{j \in \mathcal{N}(i), P_w^{n,i} = P_w^{n,j}} c_{ij}(f_w(S^{n,j}) - f_w(S^{n,i}))(g(S^{n,j}) - g(S^{n,i})), \end{aligned}$$

and (5.40), with another constant C , follows by substituting this equality into (5.41). \square

5.7. Bound for the gradient of g . In order to deduce from (5.40) a direct bound for the gradient of g , we need to sharpen the assumptions on the mobility.

5.7.1. Assumptions on the mobility. Here we assume that

$$\eta_w(0) = \eta_o(1) = 0,$$

and $\eta_\alpha \in W^{1,\infty}(0,1)$, $\alpha = w, o$. Furthermore, we assume that, for all $x \in]0, 1[$,

$$\alpha_w x^{\theta_w-1} \leq \eta'_w(x) \leq \frac{1}{\alpha_w} x^{\theta_w-1}, \quad \theta_w \geq 1, 0 < \alpha_w \leq 1, \quad (5.42)$$

$$\alpha_o(1-x)^{\theta_o-1} \leq -\eta'_o(x) \leq \frac{1}{\alpha_o}(1-x)^{\theta_o-1}, \quad \theta_o \geq 1, 0 < \alpha_o \leq 1, \quad (5.43)$$

$$\frac{1}{\alpha_3} x^{\beta_3-1}(1-x)^{\beta_4-1} \geq -p'_c(x) \geq \alpha_3 x^{\beta_3-1}(1-x)^{\beta_4-1}, \quad \beta_3, \beta_4 > 0, 0 < \alpha_3 \leq 1. \quad (5.44)$$

From (5.42) and (5.43), we deduce respectively, for all $x \in]0, 1[$,

$$\frac{\alpha_w}{\theta_w} x^{\theta_w} \leq \eta_w(x) \leq \frac{1}{\alpha_w \theta_w} x^{\theta_w}, \quad (5.45)$$

$$\frac{\alpha_o}{\theta_o} (1-x)^{\theta_o} \leq \eta_o(x) \leq \frac{1}{\alpha_o \theta_o} (1-x)^{\theta_o}. \quad (5.46)$$

The sum of these two inequalities reads for all $x \in]0, 1[$,

$$\frac{\alpha_w}{\theta_w} x^{\theta_w} + \frac{\alpha_o}{\theta_o} (1-x)^{\theta_o} \leq \eta_w(x) + \eta_o(x) \leq \frac{1}{\alpha_w \theta_w} x^{\theta_w} + \frac{1}{\alpha_o \theta_o} (1-x)^{\theta_o}.$$

Let ℓ denote the lower bound in this inequality. It is easy to check that ℓ is a nonnegative continuous function of x on $[0, 1]$, hence uniformly continuous. Therefore, it is bounded and as it does not vanish in this interval, it is bounded away from zero. Thus there exists a positive constant C_{\min} such that

$$\forall x \in [0, 1], \quad C_{\min} \leq \ell(x) := \frac{\alpha_w}{\theta_w} x^{\theta_w} + \frac{\alpha_o}{\theta_o} (1-x)^{\theta_o} \leq C_{\max}, \quad (5.47)$$

where

$$C_{\max} = \max_{x \in [0,1]} \left(\frac{1}{\alpha_w \theta_w} x^{\theta_w} + \frac{1}{\alpha_o \theta_o} (1-x)^{\theta_o} \right). \quad (5.48)$$

5.7.2. Properties of the derivatives of f_w and g . By definition, we have

$$g'(x) = -\frac{\eta_w(x)\eta_o(x)}{\eta_w(x) + \eta_o(x)} p'_c(x),$$

which is positive in $]0, 1[$. Considering (1.8), (5.45), and (5.46), we infer

$$g'(x) \leq \frac{1}{\eta_* \alpha_3} \frac{1}{\alpha_w \theta_w} \frac{1}{\alpha_o \theta_o} x^{\theta_w-1+\beta_3} (1-x)^{\theta_o-1+\beta_4}, \quad (5.49)$$

thus implying that g' is a bounded function, i.e., g is Lipschitz continuous. Note that the Lipschitz constant L of g is bounded by

$$L \leq \frac{1}{\alpha\alpha_3} \frac{1}{\alpha_w\theta_w} \frac{1}{\alpha_o\theta_o} \max_{x \in [0,1]} (x^{\theta_w-1+\beta_3} (1-x)^{\theta_o-1+\beta_4}). \quad (5.50)$$

On the other hand, (5.44)–(5.47) yield for all $x \in]0, 1[$,

$$g'(x) \geq \frac{\alpha_3}{C_{\max}} \frac{\alpha_w}{\theta_w} \frac{\alpha_o}{\theta_o} x^{\theta_w-1+\beta_3} (1-x)^{\theta_o-1+\beta_4} > 0. \quad (5.51)$$

Thus $g \in W^{1,\infty}(0, 1)$ is a strictly monotonic increasing function on $[0, 1]$ with range $[0, \beta]$ for some $\beta > 0$, hence invertible with inverse $g^{-1} \in W^{1,\infty}(0, \beta)$.

Now, we turn to f_w . By definition, we have

$$f'_w(x) = \frac{1}{(\eta_w(x) + \eta_o(x))^2} (\eta_o(x)\eta'_w(x) - \eta_w(x)\eta'_o(x)). \quad (5.52)$$

The inequalities (5.42)–(5.47) imply that

$$f'_w(x) \geq \frac{1}{C_{\max}^2} \alpha_o \alpha_w \left[\frac{1}{\theta_o} x^{\theta_w-1} (1-x)^{\theta_o} + \frac{1}{\theta_w} x^{\theta_w} (1-x)^{\theta_o-1} \right].$$

Thus,

$$\forall x \in [0, \frac{3}{4}], \quad f'_w(x) \geq \frac{\alpha_o \alpha_w}{C_{\max}^2 \theta_o} \left(\frac{1}{4}\right)^{\theta_o} x^{\theta_w-1}, \quad (5.53)$$

and

$$\forall x \in [\frac{1}{4}, 1], \quad f'_w(x) \geq \frac{\alpha_o \alpha_w}{C_{\max}^2 \theta_w} \left(\frac{1}{4}\right)^{\theta_w} (1-x)^{\theta_o-1}. \quad (5.54)$$

Let us use these results to compare g' and f'_w . It follows from (5.49) that

$$\forall x \in [0, \frac{3}{4}], \quad g'(x) \leq \left(\frac{1}{\alpha\alpha_3} \frac{1}{\alpha_w\theta_w} \frac{1}{\alpha_o\theta_o} \frac{\theta_o C_{\max}^2}{\alpha_o \alpha_w} \right) \frac{\alpha_o \alpha_w}{C_{\max}^2 \theta_o} x^{\theta_w-1},$$

and by setting

$$C_1 = \left(\frac{1}{\alpha\alpha_3} \frac{1}{\alpha_w\theta_w} \frac{1}{\alpha_o\theta_o} 4^{\theta_o} \frac{\theta_o C_{\max}^2}{\alpha_o \alpha_w} \right)$$

and comparing with (5.53), we obtain

$$\forall x \in [0, \frac{3}{4}], \quad g'(x) \leq C_1 f'_w(x). \quad (5.55)$$

Similarly,

$$\forall x \in [\frac{1}{4}, 1], \quad g'(x) \leq \left(\frac{1}{\alpha\alpha_3} \frac{1}{\alpha_w\theta_w} \frac{1}{\alpha_o\theta_o} \frac{\theta_w C_{\max}^2}{\alpha_o \alpha_w} \right) \frac{\alpha_o \alpha_w}{C_{\max}^2 \theta_w} (1-x)^{\theta_o-1},$$

so that, by setting

$$C_2 = \left(\frac{1}{\alpha\alpha_3} \frac{1}{\alpha_w\theta_w} \frac{1}{\alpha_o\theta_o} 4^{\theta_w} \frac{\theta_w C_{\max}^2}{\alpha_o\alpha_w} \right)$$

and comparing with (5.54), we deduce

$$\forall x \in \left[\frac{1}{4}, 1\right], \quad g'(x) \leq C_2 f'_w(x). \quad (5.56)$$

This leads to the desired relation between the derivative of f_w and g :

$$\forall x \in [0, 1], \quad g'(x) \leq C f'_w(x), \quad (5.57)$$

where $C = \max(C_1, C_2)$.

The main result of this section follows by combining (5.57) with (5.40).

THEOREM 5.15. *Under the assumptions (5.42)–(5.44) on the derivatives of the mobilities and capillary pressure, there exists a constant C , independent of h and τ , such that*

$$\|\nabla(I_h(g(S_{h,\tau})))\|_{L^2(\Omega \times]0,T])} \leq C. \quad (5.58)$$

Proof. Let (i, j) be any pair of indices. If $S^{n,i} \leq S^{n,j}$, then by (5.57),

$$f_w(S^{n,j}) - f_w(S^{n,i}) = \int_{S^{n,i}}^{S^{n,j}} f'_w(x) dx \geq C \int_{S^{n,i}}^{S^{n,j}} g'(x) dx = C(g(S^{n,j}) - g(S^{n,i})).$$

As g is increasing, we have $g(S^{n,j}) - g(S^{n,i}) \geq 0$. Therefore

$$(f_w(S^{n,j}) - f_w(S^{n,i}))(g(S^{n,j}) - g(S^{n,i})) \geq C|g(S^{n,j}) - g(S^{n,i})|^2. \quad (5.59)$$

By changing both signs, the same result holds when $S^{n,j} < S^{n,i}$. Then (5.58) follows from (5.40). \square

6. Convergence. The interpolants of $p_{\alpha g}(S_{h,\tau})$, $g(S_{h,\tau})$, and $p_c(S_{h,\tau})$ play an important part in this work, see Theorems 3.5 and 5.15, and (2.44). Therefore, we begin by studying convergence properties first of $I_h(g(S_{h,\tau}))$ and $I_h(p_{\alpha g}(S_{h,\tau}))$, $\alpha = w, o$, and next $I_h(p_c(S_{h,\tau}))$. Some results will stem from an interesting relation between differences in values of $S_{h,\tau}$ and $g(S_{h,\tau})$.

6.1. Properties of $I_h(g(S_{h,\tau}))$ and $I_h(p_{\alpha g}(S_{h,\tau}))$, $\alpha = w, o$.

6.1.1. Convergence properties of $I_h(g(S_{h,\tau}))$. Let K be an element of \mathcal{T}_h with vertices \mathbf{a}_i , $1 \leq i \leq d+1$ (local numbers); then

$$\int_K |g(S_{h,\tau})(t_n)|^2 = \int_K |g(\sum_{i=1}^{d+1} S^{n,i} \phi_i)|^2.$$

As $0 \leq S^{n,i}, \phi_i \leq 1$ and g is increasing, we have

$$0 \leq \sum_{i=1}^{d+1} S^{n,i} \phi_i \leq \sum_{i=1}^{d+1} S^{n,i}, \quad g\left(\sum_{i=1}^{d+1} S^{n,i} \phi_i\right) \leq \sum_{i=1}^{d+1} g(S^{n,i}).$$

Hence

$$\int_K |g(S_{h,\tau})(t_n)|^2 \leq (d+1)|K| \sum_{i=1}^{d+1} (g(S^{n,i}))^2.$$

As a consequence, there exist constants C, D, E , independent of n, h and τ , such that

$$\|g(S_{h,\tau})(t_n)\|_{L^2(\Omega)} \leq C \|I_h(g(S_{h,\tau})(t_n))\|_h \leq D \|I_h(g(S_{h,\tau})(t_n))\|_h^\varphi \leq E \|I_h(g(S_{h,\tau})(t_n))\|_{L^2(\Omega)}, \quad (6.1)$$

owing to (2.5) and (2.8). These inequalities carry over to the norm in $L^2(\Omega \times]0, T[)$.

Now, let us prove the following convergence property of $I_h(g(S_{h,\tau}))$.

LEMMA 6.1. *Under the assumptions of Theorem 5.15, we have*

$$\lim_{(h,\tau) \rightarrow (0,0)} \|g(S_{h,\tau}) - I_h(g(S_{h,\tau}))\|_{L^2(\Omega \times]0, T[)} = 0. \quad (6.2)$$

Proof. For any \mathbf{x} in any element K of \mathcal{T}_h , we have

$$I_h(g(S_{h,\tau}))(\mathbf{x}, t_n) - g(S_{h,\tau})(\mathbf{x}, t_n) = \sum_{i=1}^{d+1} g(S^{n,i}) \phi_i(\mathbf{x}) - g\left(\sum_{i=1}^{d+1} S^{n,i} \phi_i(\mathbf{x})\right).$$

As $S_{h,\tau}$ is a polynomial of degree one in K , it attains its maximum and its minimum in space at vertices of K , say $g(S^{n,\ell})$ and $g(S^{n,r})$ are its maximum and minimum respectively. Thus, recalling that g is a nonnegative monotonically increasing function,

$$\sum_{i=1}^{d+1} g(S^{n,i}) \phi_i(\mathbf{x}) \leq g(S^{n,\ell}), \quad g\left(\sum_{i=1}^{d+1} S^{n,i} \phi_i(\mathbf{x})\right) \geq g(S^{n,r}).$$

Hence

$$\|I_h(g(S_{h,\tau})) - g(S_{h,\tau})\|_{L^2(\Omega \times]0, T[)}^2 \leq \sum_{n=1}^N \tau \sum_{K \in \mathcal{T}_h} |K| |g(S^{n,\ell}) - g(S^{n,r})|^2. \quad (6.3)$$

For any node i , let

$$\kappa_i = \text{Max } |K|,$$

where the maximum is taken over all elements K in Δ_i . Then we can readily check that

$$\sum_{n=1}^N \tau \sum_{K \in \mathcal{T}_h} |K| |g(S^{n,\ell}) - g(S^{n,r})|^2 \leq C \sum_{n=1}^N \tau \sum_{i=1}^M \kappa_i \sum_{j \in \mathcal{N}(i)} |g(S^{n,j}) - g(S^{n,i})|^2,$$

where C is a bound for the maximum number of elements that share a common edge, bound independent of h and τ by virtue of the regularity of the mesh. Now, recall the classical formula in each d -simplex K ,

$$\int_K |\nabla \phi_i \cdot \nabla \phi_j| = \frac{1}{d^2} \frac{|F_i||F_j|}{|K|} |\mathbf{n}_i \cdot \mathbf{n}_j|, \quad (6.4)$$

where F_i is the face opposite to the vertex \mathbf{a}_i and \mathbf{n}_i is the exterior (to K) unit normal to the face F_i . The regularity of the mesh implies that there exists a constant c_0 , independent of h and τ , such that

$$|\mathbf{n}_i \cdot \mathbf{n}_j| \geq c_0.$$

Hence, using again the regularity of the mesh, we obtain

$$\int_K |\nabla \phi_i \cdot \nabla \phi_j| \geq C h_K^{d-2},$$

and denoting by ρ_{ij} the minimum of h_K for all K in $\Delta_i \cap \Delta_j$, we deduce

$$c_{ij} \geq C \rho_{ij}^{d-2}, \quad (6.5)$$

with another constant C independent of h and τ . By collecting these results, we derive

$$\|I_h(g(S_{h,\tau})) - g(S_{h,\tau})\|_{L^2(\Omega \times]0,T])}^2 \leq C \sum_{n=1}^N \tau \sum_{i=1}^M \kappa_i \sum_{j \in \mathcal{N}(i)} \left(\frac{1}{\rho_{ij}^{d-2}} \right) c_{ij} |g(S^{n,j}) - g(S^{n,i})|^2. \quad (6.6)$$

With another application of the regularity of the mesh, this becomes

$$\|I_h(g(S_{h,\tau})) - g(S_{h,\tau})\|_{L^2(\Omega \times]0,T])}^2 \leq C h^2 \|\nabla(I_h(g(S_{h,\tau})))\|_{L^2(\Omega \times]0,T])}^2, \quad (6.7)$$

(note that the power of h is independent of the dimension) and the limit (6.2) follows from Theorem 5.15. \square

6.1.2. Relation between $g(S^{n,j}) - g(S^{n,i})$ and $S^{n,j} - S^{n,i}$. Here, we derive an upper bound for $S^{n,j} - S^{n,i}$ in terms of $g(S^{n,j}) - g(S^{n,i})$.

LEMMA 6.2. *Under the assumptions (5.42)–(5.44) on the derivatives of the mobilities and capillary pressure, there exists a constant C , independent of h and τ , such that for all i, j , and n*

$$|S^{n,j} - S^{n,i}| \leq C |g(S^{n,j}) - g(S^{n,i})|^{\frac{1}{r}}, \quad (6.8)$$

where $r = \max(\theta_o + \beta_4, \theta_w + \beta_3) > 1$.

Proof. To simplify, we set $c = S^{n,i}$, $d = S^{n,j}$ and assume $c < d$. From (5.51), it follows that

$$g(d) - g(c) \geq \frac{\alpha_3}{C_{\max}} \frac{\alpha_w}{\theta_w} \frac{\alpha_o}{\theta_o} \int_c^d x^{\theta_w + \beta_3 - 1} (1 - x)^{\theta_o + \beta_4 - 1}. \quad (6.9)$$

For the sake of brevity, we do not specify the constant factor in (6.9) and write

$$g(d) - g(c) \geq C_1 \int_c^d x^{\theta_w + \beta_3 - 1} (1 - x)^{\theta_o + \beta_4 - 1}.$$

Now, we argue according to the positions of c and d . There are four cases.

1) If $\frac{1}{8} \leq c \leq \frac{7}{8}$, then (6.9) gives

$$\begin{aligned} g(d) - g(c) &\geq C_1 \left(\frac{1}{8}\right)^{\theta_w + \beta_3 - 1} \int_c^d (1 - x)^{\theta_o + \beta_4 - 1} \\ &= C_1 \left(\frac{1}{8}\right)^{\theta_w + \beta_3 - 1} \frac{1}{\theta_o + \beta_4} \left((1 - c)^{\theta_o + \beta_4} - (1 - d)^{\theta_o + \beta_4} \right). \end{aligned}$$

But

$$\begin{aligned} (1 - c)^{\theta_o + \beta_4} - (1 - d)^{\theta_o + \beta_4} &= (1 - c)^{\theta_o + \beta_4 - 1} (d - c) + (1 - d) \left((1 - c)^{\theta_o + \beta_4 - 1} - (1 - d)^{\theta_o + \beta_4 - 1} \right) \\ &\geq (d - c) \left(\frac{1}{8}\right)^{\theta_o + \beta_4 - 1}. \end{aligned}$$

Hence

$$g(d) - g(c) \geq \frac{C_1}{\theta_o + \beta_4} \left(\frac{1}{8}\right)^{\theta_w + \beta_3 + \theta_o + \beta_4 - 2} (d - c). \quad (6.10)$$

2) If $c > \frac{7}{8}$, then $d > \frac{7}{8}$ and (6.9) gives

$$g(d) - g(c) \geq C_1 \left(\frac{7}{8}\right)^{\theta_w + \beta_3 - 1} \frac{1}{\theta_o + \beta_4} \left((1 - c)^{\theta_o + \beta_4} - (1 - d)^{\theta_o + \beta_4} \right).$$

Let us set $a = 1 - d$, $b = d - c$, $\gamma = \theta_o + \beta_4 - 1 > 0$. We can also write

$$(1 - c)^{\theta_o + \beta_4} - (1 - d)^{\theta_o + \beta_4} = a^{\gamma + 1} \left(\left(1 + \frac{b}{a}\right)^{\gamma + 1} - 1 \right).$$

It is easy to check that the function

$$x \mapsto (1 + x)^{\gamma + 1} - 1 - x^{\gamma + 1}$$

vanishes at $x = 0$ and is strictly monotonic increasing, hence is strictly positive for $x > 0$.

Hence

$$a^{\gamma + 1} \left(\left(1 + \frac{b}{a}\right)^{\gamma + 1} - 1 \right) > a^{\gamma + 1} \left(\frac{b}{a}\right)^{\gamma + 1} = b^{\gamma + 1}.$$

Thus

$$(1 - c)^{\theta_o + \beta_4} - (1 - d)^{\theta_o + \beta_4} \geq (d - c)^{\theta_o + \beta_4},$$

and

$$g(d) - g(c) \geq C_1 \left(\frac{7}{8}\right)^{\theta_w + \beta_3 - 1} \frac{1}{\theta_o + \beta_4} (d - c)^{\theta_o + \beta_4}. \quad (6.11)$$

3) If $c < \frac{1}{8}$ and $d < \frac{7}{8}$, then the integrand $1 - x \geq 1 - d > \frac{1}{8}$ and by the above argument,

$$\begin{aligned} g(d) - g(c) &\geq C_1 \left(\frac{1}{8}\right)^{\theta_o + \beta_4 - 1} \frac{1}{\theta_w + \beta_3} \left(d^{\theta_w + \beta_3} - c^{\theta_w + \beta_3} \right) \\ &\geq C_1 \left(\frac{1}{8}\right)^{\theta_o + \beta_4 - 1} \frac{1}{\theta_w + \beta_3} (d - c)^{\theta_w + \beta_3}. \end{aligned} \quad (6.12)$$

4) If $c < \frac{1}{8}$ and $d > \frac{7}{8}$, then $c < \frac{1}{6}(d - c) < \frac{1}{2}(d - c) < d$. Therefore, we can write

$$\begin{aligned} g(d) - g(c) &\geq C_1 \int_{\frac{1}{6}(d-c)}^{\frac{1}{2}(d-c)} x^{\theta_w + \beta_3 - 1} (1 - x)^{\theta_o + \beta_4 - 1} \\ &\geq C_1 \left(\frac{1}{2}\right)^{\theta_o + \beta_4 - 1} \frac{1}{\theta_w + \beta_3} \left(\left(\frac{1}{2}(d - c)\right)^{\theta_w + \beta_3} - \left(\frac{1}{6}(d - c)\right)^{\theta_w + \beta_3} \right) \\ &\geq C_1 \left(\frac{1}{2}\right)^{\theta_o + \theta_w + \beta_3 + \beta_4 - 1} \frac{1}{\theta_w + \beta_3} \left(1 - \left(\frac{1}{3}\right)^{\theta_w + \beta_3} \right) (d - c)^{\theta_w + \beta_3}. \end{aligned} \quad (6.13)$$

Since $d - c \leq 1$, $\theta_o + \beta_4 > 1$, and $\theta_w + \beta_3 > 1$, we have in all cases

$$g(d) - g(c) \geq C_2 (d - c)^{\max(\theta_o + \beta_4, \theta_w + \beta_3)},$$

where C_2 is the minimum of the constant factors in (6.10)–(6.13). \square

The convergence to zero of the differences $I_h(p_{\alpha g}(S_{h,\tau})) - p_{\alpha g}(S_{h,\tau})$, $\alpha = w, o$, follows from this lemma and Theorem 5.15.

LEMMA 6.3. *Under the assumptions (5.42)–(5.44) on the derivatives of the mobilities and capillary pressure, there exists a constant C , independent of h and τ , such that*

$$\|I_h(p_{\alpha g}(S_{h,\tau})) - p_{\alpha g}(S_{h,\tau})\|_{L^2(\Omega \times]0, T])} \leq C h^{\gamma_\alpha}, \quad \alpha = w, o, \quad (6.14)$$

where $\gamma_w = \frac{\beta_3}{r}$, $\gamma_o = \frac{\beta_4}{r}$ and in both cases, r is the exponent of Lemma 6.2.

Proof. Let us start with $\alpha = w$. Arguing as in the proof of Lemma 6.1, with $-p_{wg}$ (monotonic increasing) instead of g , the analogue of (6.6) holds for $-p_{wg}(S_{h,\tau})$, with the same notation

$$\|I_h(p_{wg}(S_{h,\tau})) - p_{wg}(S_{h,\tau})\|_{L^2(\Omega \times]0, T])}^2 \leq C \sum_{n=1}^N \tau \sum_{i=1}^M \sum_{j \in \mathcal{N}(i)} \left(\frac{\kappa_i}{C_{ij}} \right) c_{ij} |p_{wg}(S^{n,j}) - p_{wg}(S^{n,i})|^2, \quad (6.15)$$

and the result will stem from an adequate upper bound for $p_{wg}(S^{n,j}) - p_{wg}(S^{n,i})$, for all neighbors j of i . To this end, we proceed as in Lemma 6.2. Let $c = S^{n,i}$, $d = S^{n,j}$ and suppose again that $c < d$; then by (1.12), (5.44), (5.46), and (5.47),

$$|p_{wg}(S^{n,j}) - p_{wg}(S^{n,i})| \leq \frac{1}{C_{\min} \alpha_3 \alpha_o \theta_o} \int_c^d x^{\beta_3 - 1} (1 - x)^{\theta_o + \beta_4 - 1}, \quad (6.16)$$

that we write as

$$|p_{wg}(S^{n,j}) - p_{wg}(S^{n,i})| \leq C'_1 \int_c^d x^{\beta_3-1} (1-x)^{\theta_o+\beta_4-1}.$$

Here, the discussion reduces to three cases.

1) If $\frac{1}{8} \leq c \leq \frac{7}{8}$, since $\theta_o + \beta_4 - 1 > 0$,

$$\int_c^d x^{\beta_3-1} (1-x)^{\theta_o+\beta_4-1} \leq 8^{1-\beta_3} \int_c^d (1-x)^{\theta_o+\beta_4-1} \leq 8^{1-\beta_3} (d-c). \quad (6.17)$$

2) Likewise, if $c > \frac{7}{8}$,

$$\int_c^d x^{\beta_3-1} (1-x)^{\theta_o+\beta_4-1} \leq \left(\frac{8}{7}\right)^{1-\beta_3} (d-c). \quad (6.18)$$

3) If $c < \frac{1}{8}$,

$$\int_c^d x^{\beta_3-1} (1-x)^{\theta_o+\beta_4-1} \leq \int_c^d x^{\beta_3-1} = \frac{1}{\beta_3} (d^{\beta_3} - c^{\beta_3}) \leq \frac{1}{\beta_3} (d-c)^{\beta_3}. \quad (6.19)$$

Indeed, by Jensen's inequality, valid for $0 < \beta_3 \leq 1$,

$$d = c + (d-c) \leq (c^{\beta_3} + (d-c)^{\beta_3})^{\frac{1}{\beta_3}}, \quad \text{i.e., } d^{\beta_3} \leq c^{\beta_3} + (d-c)^{\beta_3}.$$

Consequently, in all cases,

$$|p_{wg}(S^{n,j}) - p_{wg}(S^{n,i})| \leq C'_2 |S^{n,j} - S^{n,i}|^{\beta_3}. \quad (6.20)$$

Thus, by substituting into (6.15), applying Lemma 6.2, and setting $\gamma_w = \frac{\beta_3}{r}$, we infer

$$\begin{aligned} \|I_h(p_{wg}(S_{h,\tau})) - p_{wg}(S_{h,\tau})\|_{L^2(\Omega \times]0,T])}^2 &\leq C \sum_{n=1}^N \sum_{i=1}^M \sum_{j \in \mathcal{N}(i)} \frac{\kappa_i}{c_{ij}} \tau c_{ij} |S^{n,j} - S^{n,i}|^{2\beta_3} \\ &\leq C \sum_{n=1}^N \sum_{i=1}^M \sum_{j \in \mathcal{N}(i)} \frac{\kappa_i}{c_{ij}} \tau c_{ij} A_{ij}^{2\gamma_w}, \end{aligned}$$

where $A_{ij} = |g(S^{n,j}) - g(S^{n,i})|$. Note that $r > \beta_3$, hence $\gamma_w < 1$. Then

$$\begin{aligned} \|I_h(p_{wg}(S_{h,\tau})) - p_{wg}(S_{h,\tau})\|_{L^2(\Omega \times]0,T])}^2 &\leq C \sum_{n=1}^N \sum_{i=1}^M \sum_{j \in \mathcal{N}(i)} \frac{\kappa_i}{c_{ij}} (\tau c_{ij})^{1-\gamma_w} (\tau c_{ij})^{\gamma_w} A_{ij}^{2\gamma_w} \\ &\leq C \left(\sum_{n=1}^N \sum_{i=1}^M \sum_{j \in \mathcal{N}(i)} \tau c_{ij} A_{ij}^2 \right)^{\gamma_w} \left(\sum_{n=1}^N \sum_{i=1}^M \sum_{j \in \mathcal{N}(i)} \left(\frac{\kappa_i}{c_{ij}} \right)^{\frac{1}{1-\gamma_w}} \tau c_{ij} \right)^{1-\gamma_w}. \end{aligned}$$

But

$$\left(\sum_{n=1}^N \sum_{i=1}^M \sum_{j \in \mathcal{N}(i)} \left(\frac{\kappa_i}{c_{ij}} \right)^{\frac{1}{1-\gamma_w}} \tau c_{ij} \right)^{1-\gamma_w} \leq C(T|\Omega|)^{1-\gamma_w} \sup_{i,j} \left(\frac{\kappa_i}{c_{ij}} \right)^{\gamma_w},$$

and (6.14) with $\alpha = w$ follows from (6.5), the regularity of the mesh, and Theorem 5.15.

When $\alpha = o$, the proof is based on fact that $-p_{og}$ is nonnegative, monotonically increasing, and satisfies the inequality

$$-p_{og}(x) \leq \frac{1}{C_{\min}} \frac{1}{\alpha_3 \alpha_w \theta_w} \int_0^x x^{\theta_w + \beta_3 - 1} (1-x)^{\beta_4 - 1}.$$

By comparing with (6.16), we see that the above argument carries over to p_{og} with β_3 replaced by β_4 . \square

Finally, with the notation of Lemma 6.3, the following bound regarding $p_c(S_{h,\tau})$ follows from (6.14) and (1.14), and the fact that $p_c(0)$ is a constant:

$$\|I_h(p_c(S_{h,\tau})) - p_c(S_{h,\tau})\|_{L^2(\Omega \times]0, T])} \leq C h^\gamma, \quad (6.21)$$

where $\gamma = \frac{1}{r} \min(\beta_3, \beta_4)$.

6.2. Weak convergence. All constants below are independent of h and τ .

The bound (3.1) on the discrete saturation $S_{h,\tau}$ implies that there exists a function $\bar{s} \in L^\infty(\Omega \times]0, T])$ and a subsequence of (h, τ) not indicated, such that

$$\lim_{(h,\tau) \rightarrow (0,0)} S_{h,\tau} = \bar{s} \quad \text{weakly* in } L^\infty(\Omega \times]0, T]). \quad (6.22)$$

PROPOSITION 6.4. *The limit function \bar{s} satisfies*

$$\forall (x, t) \text{ a.e. in } \Omega \times]0, T[, \quad 0 \leq \bar{s}(x, t) \leq 1. \quad (6.23)$$

Proof. The convergence (6.22) means that for all $\psi \in L^1(\Omega \times]0, T])$,

$$\int_{\Omega \times]0, T[} S_{h,\tau} \psi \rightarrow \int_{\Omega \times]0, T[} \bar{s} \psi \text{ and } \int_{\Omega \times]0, T[} (1 - S_{h,\tau}) \psi \rightarrow \int_{\Omega \times]0, T[} (1 - \bar{s}) \psi.$$

We argue by contradiction. Suppose that $\bar{s} > 1$ on a set of positive measure, say D , and take $\psi = (\bar{s} - 1)_+$, the positive part of $\bar{s} - 1$. Then

$$0 \leq \int_{\Omega \times]0, T[} (1 - S_{h,\tau}) \psi \rightarrow \int_{\Omega \times]0, T[} (1 - \bar{s})(\bar{s} - 1)_+ = \int_D (1 - \bar{s})(\bar{s} - 1)_+,$$

thus contradicting the fact that $(1 - \bar{s}) < 0$ on D . This proves that $\bar{s} \leq 1$. The proof that $\bar{s} \geq 0$ is similar. \square

Regarding the pressure, the bound (3.23) yields weak convergence, up to a subsequence, of the gradient of $U_{\alpha,h,\tau}$. We can deduce weak convergence of the sequences **themselves** by applying (3.22). Indeed,

$$\int_{\Omega} U_{w,h,\tau} = (U_{w,h,\tau}, 1)_h = (I_h(p_{wg}(S_{h,\tau})), 1)_h,$$

owing to (2.45). Then the properties of p_{wg} and the boundedness of $S_{h,\tau}$ imply that

$$|(I_h(p_{wg}(S_{h,\tau})), 1)_h| \leq C.$$

Similarly,

$$\int_{\Omega} U_{o,h,\tau} = (I_h(p_{wg}(S_{h,\tau})) + p_c(0), 1)_h,$$

a bounded quantity. Then we infer from (3.22) that

$$\|U_{\alpha,h,\tau}\|_{L^2(\Omega \times]0, T])} \leq C, \quad \alpha = w, o. \quad (6.24)$$

With this, (3.23) implies that there exist functions $\bar{W}_{\alpha} \in L^2(0, T; H^1(\Omega))$, $\alpha = w, o$, and a subsequence of h and τ (not indicated) such that,

$$\lim_{(h,\tau) \rightarrow (0,0)} U_{\alpha,h,\tau} = \bar{W}_{\alpha}, \quad \text{weakly in } L^2(0, T; H^1(\Omega)). \quad (6.25)$$

Likewise, the function $I_h(g(S_{h,\tau}))$ is bounded in $L^2(\Omega \times]0, T])$ and it follows from this and (5.58) that there exists a function $\bar{K} \in L^2(0, T, H^1(\Omega))$ such that, up to a subsequence,

$$\lim_{(h,\tau) \rightarrow (0,0)} I_h(g(S_{h,\tau})) = \bar{K} \quad \text{weakly in } L^2(0, T, H^1(\Omega)). \quad (6.26)$$

This implies in particular that for almost every time t , $I_h(g(S_{h,\tau}))$ converges strongly in $L^2(\Omega)$. But as is well-known, these convergences are not sufficient to pass to the limit in the nonlinear terms: they must be supplemented by a bound for a fractional derivative in time of $S_{h,\tau}$ that yields compactness in time. This will stem via a bound for a fractional derivative in time of $g(S_{h,\tau})$.

6.3. Compactness in time. Following the argument introduced by Kazhikhov, see [21], and recalling that $\|\cdot\|_h^{\varphi}$ is equivalent to the L^2 norm in finite dimension, we want to derive first a fractional estimate in time for $I_h(g(S_{h,\tau}))$ and next for $g(S_{h,\tau})$. The following lemma is a preliminary bound written in terms of sums of the pointwise values in time.

LEMMA 6.5. *Under the assumptions of Theorem 5.15, there exist constants C , independent of h and τ , such that for all integers $1 \leq \ell \leq N - 1$,*

$$\sum_{m=1}^{N-\ell} \tau (\|g(S_h^{m+\ell}) - g(S_h^m)\|_h^{\varphi})^2 \leq C(\ell\tau), \quad \sum_{m=1}^{N-\ell} \tau \|g(S_h^{m+\ell}) - g(S_h^m)\|_{L^2(\Omega)}^2 \leq C(\ell\tau). \quad (6.27)$$

Proof. The starting point is the inequality

$$\sum_{m=1}^{N-\ell} \tau(\|g(S_h^{m+\ell}) - g(S_h^m)\|_h^\varphi)^2 \leq L \sum_{m=1}^{N-\ell} \tau\left(g(S_h^{m+\ell}) - g(S_h^m), S_h^{m+\ell} - S_h^m\right)_h^\varphi, \quad (6.28)$$

owing that g is Lipschitz continuous and increasing. Thus, by writing

$$S_h^{m+\ell} - S_h^m = \sum_{k=1}^{\ell} (S_h^{m+k} - S_h^{m+k-1}),$$

testing each line of (2.42) taken at level $m+k$ with $I_h(g(S_h^{m+\ell}) - g(S_h^m))$, and applying (6.28), we obtain

$$\begin{aligned} \sum_{m=1}^{N-\ell} \tau(\|g(S_h^{m+\ell}) - g(S_h^m)\|_h^\varphi)^2 &\leq L \sum_{m=1}^{N-\ell} \tau \sum_{k=1}^{\ell} \tau \left| (f_w(s_{\text{in},h}^{m+k}) \bar{q}_h^{m+k} - f_w(S_h^{m+k}) \underline{q}_h^{m+k}, g(S_h^{m+\ell}) - g(S_h^m))_h \right. \\ &\quad \left. + [P_{w,h}^{m+k}, I_h(\eta_w(S_h^{m+k}))]; P_{w,h}^{m+k}, I_h(g(S_h^{m+\ell}) - g(S_h^m))]_h \right|. \end{aligned} \quad (6.29)$$

It is easy to check that, on one hand, with $r = \ell$ or $r = 0$,

$$\begin{aligned} &\left| [P_{w,h}^{m+k}, I_h(\eta_w(S_h^{m+k}))]; P_{w,h}^{m+k}, I_h(g(S_h^{m+r}))]_h \right| \\ &= \frac{1}{2} \left| \sum_{i,j=1}^M (g(S^{m+r,j}) - g(S^{m+r,i})) c_{ij} \eta_w(S_w^{m+k,ij}) (P_w^{m+k,j} - P_w^{m+k,i}) \right| \\ &\leq \frac{1}{4} \sum_{i,j=1}^M c_{ij} \eta_w(S_w^{m+k,ij}) \left(|g(S^{m+r,j}) - g(S^{m+r,i})|^2 + |P_w^{m+k,j} - P_w^{m+k,i}|^2 \right) \\ &\leq \frac{1}{4} \sum_{i,j=1}^M c_{ij} \left(\eta_w(1) |g(S^{m+r,j}) - g(S^{m+r,i})|^2 + \eta_w(S_w^{m+k,ij}) |P_w^{m+k,j} - P_w^{m+k,i}|^2 \right), \end{aligned}$$

since η_w is increasing and $S_{h,\tau}$ is bounded by one. On the other hand,

$$\left| (f_w(s_{\text{in},h}^{m+k}) \bar{q}_h^{m+k} - f_w(S_h^{m+k}) \underline{q}_h^{m+k}, g(S_h^{m+\ell}) - g(S_h^m))_h \right| \leq C(\|\bar{q}^{m+k}\|_{L^1(\Omega)} + \|\underline{q}^{m+k}\|_{L^1(\Omega)}),$$

where here and below, C denotes constants that are independent of ℓ , h , and τ . Therefore, in view of (2.10)

$$\begin{aligned}
& \sum_{m=1}^{N-\ell} \tau (\|g(S_h^{m+\ell}) - g(S_h^m)\|_h^\varphi)^2 \leq L \sum_{m=1}^{N-\ell} \tau \left(\left[\frac{1}{8} \eta_w(1) (\ell\tau) \sum_{r=\ell,0} \|\nabla I_h(g(S_h^{m+r}))\|_{L^2(\Omega)}^2 \right. \right. \\
& \left. \left. + \frac{1}{2} \sum_{k=1}^{\ell} \tau \sum_{i,j=1}^M c_{ij} \eta_w(S_w^{m+k,ij}) |P_w^{m+k,j} - P_w^{m+k,i}|^2 \right] + C \sum_{k=1}^{\ell} \tau (\|\bar{q}^{m+k}\|_{L^1(\Omega)} + \|\underline{q}^{m+k}\|_{L^1(\Omega)}) \right) \\
& \leq \frac{1}{8} \eta_w(1) L (\ell\tau) \left[\sum_{m=1+\ell}^N \tau \|\nabla I_h(g(S_h^m))\|_{L^2(\Omega)}^2 + \sum_{m=1}^{N-\ell} \tau \|\nabla I_h(g(S_h^m))\|_{L^2(\Omega)}^2 \right] \\
& \left. + \frac{1}{2} L \sum_{m=1}^{N-\ell} \tau \sum_{k=1}^{\ell} \tau \left(\sum_{i,j=1}^M c_{ij} \eta_w(S_w^{m+k,ij}) |P_w^{m+k,j} - P_w^{m+k,i}|^2 + C (\|\bar{q}^{m+k}\|_{L^1(\Omega)} + \|\underline{q}^{m+k}\|_{L^1(\Omega)}) \right). \right. \\
& \hspace{15em} (6.30)
\end{aligned}$$

By (5.58), it suffices to bound the terms in the last line above. This is achieved by interchanging the sums over m and k . Let $n = m + k$; n runs from 2 to N and m runs from $\max(1, n - \ell)$ to $\min(n - 1, N - \ell)$. Thus

$$\begin{aligned}
& \sum_{m=1}^{N-\ell} \tau \sum_{k=1}^{\ell} \tau \sum_{i,j=1}^M c_{ij} \eta_w(S_w^{m+k,ij}) |P_w^{m+k,j} - P_w^{m+k,i}|^2 \\
& = \sum_{n=2}^N \tau \left(\sum_{m=\max(1, n-\ell)}^{\min(n-1, N-\ell)} \tau \right) \sum_{i,j=1}^M c_{ij} \eta_w(S_w^{n,ij}) |P_w^{n,j} - P_w^{n,i}|^2.
\end{aligned}$$

But $\min(n - 1, N - \ell) - \max(1, n - \ell) \leq \ell - 1$. Hence

$$\sum_{m=1}^{N-\ell} \tau \sum_{k=1}^{\ell} \tau \sum_{i,j=1}^M c_{ij} \eta_w(S_w^{m+k,ij}) |P_w^{m+k,j} - P_w^{m+k,i}|^2 \leq (\ell\tau) \sum_{n=2}^N \tau \sum_{i,j=1}^M c_{ij} \eta_w(S_w^{n,ij}) |P_w^{n,j} - P_w^{n,i}|^2, \quad (6.31)$$

and we know from (3.18) that this last sum over n is bounded. In the same fashion,

$$\sum_{m=1}^{N-\ell} \tau \sum_{k=1}^{\ell} \tau (\|\bar{q}^{m+k}\|_{L^1(\Omega)} + \|\underline{q}^{m+k}\|_{L^1(\Omega)}) \leq (\ell\tau) (\|\bar{q}\|_{L^1(\Omega \times]0, T]) + \|\underline{q}\|_{L^1(\Omega \times]0, T]}). \quad (6.32)$$

Then, under the assumptions of Theorem 5.15, (6.27) follows by substituting (5.58), (6.31), and (6.32) into (6.30). The second inequality stems from the first and (6.1). \square

The next theorem transforms (6.27) into integrals.

THEOREM 6.6. *Under the assumptions of Theorem 5.15, there exists a constant C , independent of h , and τ , such that for all real numbers δ , $0 < \delta < T$,*

$$\int_0^{T-\delta} \|g(S_{h,\tau}(t + \delta)) - g(S_{h,\tau}(t))\|_{L^2(\Omega)}^2 dt \leq C\delta. \quad (6.33)$$

Similarly,

$$\int_0^{T-\delta} \|I_h(g(S_{h,\tau}(t+\delta)) - g(S_{h,\tau}(t)))\|_{L^2(\Omega)}^2 dt \leq C\delta, \quad (6.34)$$

with another constant C , independent of h , and τ .

Proof. The argument is not new, see for instance [25], but we recall it for the reader's convenience. The discussion depends on the value of δ ; there are three cases: (i) $0 < \delta < \tau$, (ii) $\delta = \ell\tau$, $1 \leq \ell \leq N-1$, (iii) $\delta = \tau(\ell + \eta)$, $1 \leq \ell \leq N-1$, $0 < \eta < 1$.

(i) If $0 < \delta < \tau$, we have for all f in $L^1(0, T)$

$$\int_0^{T-\delta} f(t) dt = \sum_{m=0}^{N-2} \left(\int_{t_m}^{t_{m+1}-\delta} f(t) dt + \int_{t_{m+1}-\delta}^{t_{m+1}} f(t) dt \right) + \int_{t_{N-1}}^{t_{N-1}-\delta} f(t) dt.$$

On the interval $(t_m, t_{m+1} - \delta)$, by convention, see (2.38) applied to $S_{h,\tau}$, $S_{h,\tau}(t + \delta) = S_h^{m+1} = S_{h,\tau}(t)$ and on the interval $(t_{m+1} - \delta, t_{m+1})$, $S_{h,\tau}(t + \delta) = S_h^{m+2}$. Therefore

$$\begin{aligned} \int_0^{T-\delta} \|g(S_{h,\tau}(t+\delta)) - g(S_{h,\tau}(t))\|_{L^2(\Omega)}^2 dt &= \sum_{m=0}^{N-2} \delta \|g(S_h^{m+2}) - g(S_h^{m+1})\|_{L^2(\Omega)}^2 \\ &= \sum_{m=1}^{N-1} \delta \|g(S_h^{m+1}) - g(S_h^m)\|_{L^2(\Omega)}^2, \end{aligned}$$

and (6.27) with $\ell = 1$ yields (6.33) for this value of δ .

(ii) Let $\delta = \ell\tau$, for instance consider $\ell = 2$. We have

$$\int_{t_m}^{t_{m+1}} \|g(S_{h,\tau}(t+2\tau)) - g(S_{h,\tau}(t))\|_{L^2(\Omega)}^2 dt = \tau \|g(S_h^{m+3}) - g(S_h^{m+1})\|_{L^2(\Omega)}^2.$$

Thus

$$\int_0^{T-2\tau} \|g(S_{h,\tau}(t+\delta)) - g(S_{h,\tau}(t))\|_{L^2(\Omega)}^2 dt = \tau \sum_{m=1}^{N-2} \|g(S_h^{m+2}) - g(S_h^m)\|_{L^2(\Omega)}^2,$$

and (6.27) with $\ell = 2$ yields (6.33). This argument applies to any ℓ with $1 \leq \ell \leq N-1$.

(iii) Let $\delta = \tau(\ell + \eta)$ for some $0 < \eta < 1$, and for instance consider again $\ell = 2$. Then for any integer m , $0 \leq m \leq N-3$,

$$\int_{t_m}^{t_{m+1}-\tau\eta} \|g(S_{h,\tau}(t+\tau(2+\eta))) - g(S_{h,\tau}(t))\|_{L^2(\Omega)}^2 dt = \tau(1-\eta) \|g(S_h^{m+3}) - g(S_h^{m+1})\|_{L^2(\Omega)}^2,$$

and

$$\int_{t_{m+1}-\tau\eta}^{t_{m+1}} \|g(S_{h,\tau}(t+\tau(2+\eta))) - g(S_{h,\tau}(t))\|_{L^2(\Omega)}^2 dt = \tau\eta \|g(S_h^{m+4}) - g(S_h^{m+1})\|_{L^2(\Omega)}^2.$$

Hence

$$\begin{aligned} & \int_0^{T-\tau(2+\eta)} \|g(S_{h,\tau}(t + \tau(2 + \eta))) - g(S_{h,\tau}(t))\|_{L^2(\Omega)}^2 dt \\ &= \sum_{m=1}^{N-2} \tau(1 - \eta) \|g(S_h^{m+2}) - g(S_h^m)\|_{L^2(\Omega)}^2 + \sum_{m=1}^{N-3} \tau\eta \|g(S_h^{m+3}) - g(S_h^m)\|_{L^2(\Omega)}^2. \end{aligned}$$

Then (6.27) with $\ell = 2$ and $\ell = 3$ implies that

$$\int_0^{T-\tau(2+\eta)} \|g(S_{h,\tau}(t + \tau(2 + \eta))) - g(S_{h,\tau}(t))\|_{L^2(\Omega)}^2 dt \leq C(2\tau(1 - \eta) + 3\tau\eta) = C\tau(2 + \eta).$$

More generally, we have

$$\begin{aligned} & \int_0^{T-\tau(\ell+\eta)} \|g(S_{h,\tau}(t + \tau(\ell + \eta))) - g(S_{h,\tau}(t))\|_{L^2(\Omega)}^2 dt \\ &= \tau(1 - \eta) \sum_{m=1}^{N-\ell} \|g(S_h^{m+\ell}) - g(S_h^m)\|_{L^2(\Omega)}^2 + \tau\eta \sum_{m=1}^{N-(\ell+1)} \|g(S_h^{m+\ell+1}) - g(S_h^m)\|_{L^2(\Omega)}^2, \end{aligned}$$

and we apply (6.27) with ℓ to the first sum and $\ell + 1$ to the second sum. This proves the first part of (6.33). The proof of (6.34) follows likewise from (6.27) with $\varphi = 1$. \square

6.4. Strong convergence. With Theorem 6.6, it follows from Kolmogorov's theorem that the sequence $I_h(g(S_{h,\tau}))$ is compact in $L^2(\Omega \times]0, T[)$, see [21]. Thus, again up to a subsequence, $I_h(g(S_{h,\tau}))$ converges strongly in $L^2(\Omega \times]0, T[)$. Since it converges weakly to \bar{K} in $L^2(0, T; H^1(\Omega))$ (\bar{K} belongs also to $L^\infty(\Omega \times]0, T[)$), uniqueness of the limit implies

$$\lim_{(h,\tau) \rightarrow (0,0)} I_h(g(S_{h,\tau})) = \bar{K} \quad \text{strongly in } L^2(\Omega \times]0, T[). \quad (6.35)$$

By Lemma 6.1, this also implies

$$\lim_{(h,\tau) \rightarrow (0,0)} g(S_{h,\tau}) = \bar{K} \quad \text{strongly in } L^2(\Omega \times]0, T[). \quad (6.36)$$

From here, let us prove the strong convergence of $S_{h,\tau}$. Recall that g is invertible with range $]0, \beta[$ and inverse $g^{-1} \in W^{1,\infty}(]0, \beta[)$. Let $F_{h,\tau} = g(S_{h,\tau})$; then

$$S_{h,\tau} = g^{-1}(F_{h,\tau}).$$

The strong convergence of $F_{h,\tau}$ and the continuity of g^{-1} imply the strong convergence of $S_{h,\tau}$ to $g^{-1}(\bar{K})$ in $L^2(\Omega \times]0, T[)$, and since $S_{h,\tau}$ converges weakly to \bar{s} , uniqueness of the limit implies that $\bar{s} = g^{-1}(\bar{K})$, i.e.,

$$\lim_{(h,\tau) \rightarrow (0,0)} S_{h,\tau} = \bar{s} = g^{-1}(\bar{K}) \quad \text{strongly in } L^2(\Omega \times]0, T[). \quad (6.37)$$

This strong convergence and the continuity of g , $p_{\alpha g}$, $\alpha = w, o$, and p_c , also imply that

$$\lim_{(h,\tau)\rightarrow(0,0)} g(S_{h,\tau}) = g(\bar{s}), \quad \lim_{(h,\tau)\rightarrow(0,0)} p_{\alpha g}(S_{h,\tau}) = p_{\alpha g}(\bar{s}), \quad \alpha = w, o, \quad \lim_{(h,\tau)\rightarrow(0,0)} p_c(S_{h,\tau}) = p_c(\bar{s}), \quad (6.38)$$

all strongly in $L^2(\Omega \times]0, T[)$. Furthermore Lemma 6.3 and (6.21) yield

$$\lim_{(h,\tau)\rightarrow(0,0)} I_h(p_{\alpha g}(S_{h,\tau})) = p_{\alpha g}(\bar{s}), \quad \lim_{(h,\tau)\rightarrow(0,0)} I_h(p_c(S_{h,\tau})) = p_c(\bar{s}), \quad \text{strongly in } L^2(\Omega \times]0, T[). \quad (6.39)$$

In view of (6.25), this convergence implies that $P_{\alpha,h,\tau}$ converges weakly in $L^2(\Omega \times]0, T[)$ to some function $\bar{p}_\alpha \in L^2(\Omega \times]0, T[)$, $\alpha = w, o$. Furthermore, uniqueness of the limit implies that \bar{W}_α , the limit function of $U_{\alpha,h,\tau}$ has the form

$$\bar{W}_w = \bar{p}_w + p_{wg}(\bar{s}), \quad \bar{W}_o = \bar{p}_o - p_{og}(\bar{s}). \quad (6.40)$$

7. Identification of the limit. Let us pass to the limit in the equations of the scheme. This is done in several steps because we do not have convergence of the pressure gradient.

7.1. The upwind diffusions. Since the discrete auxiliary pressures $U_{\alpha,h,\tau}$ converge weakly to \bar{W}_α in $L^2(0, T; H^1(\Omega))$, instead of treating directly the upwind diffusion terms $[P_{\alpha,h,\tau}, I_h(\eta_\alpha(S_{h,\tau})); P_{\alpha,h,\tau}, \theta_h]_h$, we begin with $[P_{\alpha,h,\tau}, I_h(\eta_\alpha(S_{h,\tau})); U_{\alpha,h,\tau}, \theta_h]_h$.

7.1.1. Discrete auxiliary pressure. Let us start with the wetting phase, the treatment of the non-wetting phase being much the same.

Let v be a smooth function, say $v \in \mathcal{C}^1(\bar{\Omega} \times [0, T])$ and let $V_{h,\tau} = \rho_\tau(I_h(v))$. Assume for the moment that \bar{s} , the limit of $S_{h,\tau}$, is sufficiently smooth, say $\bar{s} \in W^{1,\infty}(\Omega \times]0, T[)$ and let $\bar{s}_\tau = \bar{s}(t_n)$ in $]t_{n-1}, t_n]$. Then assumption (5.42) implies

$$\left\| \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \eta_w(\bar{s}) dt - \eta_w(\bar{s}_\tau^n) \right\|_{L^\infty(\Omega)} \leq C\tau \|\eta'_w\|_{L^\infty(0,1)} \|\partial_t \bar{s}\|_{L^\infty(\Omega \times]0, T])}. \quad (7.1)$$

Proceeding as in Section 2.2, we treat the upwinding in several steps and consider first

$$\int_0^T \int_\Omega \eta_w(\bar{s}) \nabla U_{w,h,\tau} \cdot \nabla V_{h,\tau} = \int_0^T \int_\Omega \nabla U_{w,h,\tau} \cdot \nabla V_{h,\tau} (\rho_\tau(\eta_w(\bar{s})) - \eta_w(\bar{s}_\tau) + \eta_w(\bar{s}_\tau)). \quad (7.2)$$

But in view of (7.1),

$$\left| \int_0^T \int_\Omega \nabla U_{w,h,\tau} \cdot \nabla V_{h,\tau} (\rho_\tau(\eta_w(\bar{s})) - \eta_w(\bar{s}_\tau)) \right| \leq C\tau \|\eta'_w\|_{L^\infty(0,1)} \|\partial_t \bar{s}\|_{L^\infty(\Omega \times]0, T])} \\ \times \|U_{w,h,\tau}\|_{L^2(0,T;H^1(\Omega))} \|V_{h,\tau}\|_{L^2(0,T;H^1(\Omega))},$$

and the boundedness of all factors of τ , owing to (3.23) and the regularity of v , implies

$$\lim_{(h,\tau)\rightarrow(0,0)} \int_0^T \int_\Omega \nabla U_{w,h,\tau} \cdot \nabla V_{h,\tau} (\rho_\tau(\eta_w(\bar{s})) - \eta_w(\bar{s}_\tau)) = 0. \quad (7.3)$$

Next the weak convergence of $U_{w,h,\tau}$ to \bar{W}_w in $L^2(0, T; H^1(\Omega))$, the strong convergence of $V_{h,\tau}$ to v in $L^\infty(0, T; W^{1,\infty}(\Omega))$, the continuity of η_w , the regularity of \bar{s} , and (7.3) imply

$$\lim_{(h,\tau) \rightarrow (0,0)} \int_0^T \int_\Omega \eta_w(\bar{s}_\tau) \nabla U_{w,h,\tau} \cdot \nabla V_{h,\tau} = \int_0^T \int_\Omega \eta_w(\bar{s}) \nabla \bar{W}_w \cdot \nabla v.$$

Let us expand the expression in the above left-hand side. With the notation (2.11), in view of Proposition 2.2 we have

$$\int_0^T \int_\Omega \eta_w(\bar{s}_\tau) \nabla U_{w,h,\tau} \cdot \nabla V_{h,\tau} = - \sum_{n=1}^N \tau \sum_{i,j=1}^M U_w^{n,i} \left(\sum_{K \subset \Delta_i \cap \Delta_j} c_{ij,K}(\eta_w(\bar{s}_\tau))_K \right) (V^{n,j} - V^{n,i}).$$

By symmetry, this becomes

$$\int_0^T \int_\Omega \eta_w(\bar{s}_\tau^n) \nabla U_{w,h,\tau} \cdot \nabla V_{h,\tau} = \frac{1}{2} \sum_{n=1}^N \tau \sum_{i,j=1}^M \left(\sum_{K \subset \Delta_i \cap \Delta_j} c_{ij,K}(\eta_w(\bar{s}_\tau^n))_K \right) (U_w^{n,j} - U_w^{n,i}) (V^{n,j} - V^{n,i}). \quad (7.4)$$

Hence

$$\lim_{(h,\tau) \rightarrow (0,0)} \frac{1}{2} \sum_{n=1}^N \tau \sum_{i,j=1}^M \left(\sum_{K \subset \Delta_i \cap \Delta_j} c_{ij,K}(\eta_w(\bar{s}_\tau^n))_K \right) (U_w^{n,j} - U_w^{n,i}) (V^{n,j} - V^{n,i}) = \int_0^T \int_\Omega \eta_w(\bar{s}) \nabla \bar{W}_w \cdot \nabla v. \quad (7.5)$$

According to (5.42) and the regularity of \bar{s} , $\eta_w(\bar{s})$ belongs to $L^\infty(0, T; W^{1,\infty}(\Omega))$, and (2.19) gives

$$\|(\eta_w(\bar{s}_\tau^n))_K - \eta_w(\bar{s}_\tau)\|_{L^\infty(K)} \leq C h \|\eta'_w\|_{L^\infty(0,1)} \|\nabla \bar{s}\|_{L^\infty(\Omega \times]0,T])},$$

that allows to replace $(\eta_w(\bar{s}_\tau^n))_K$ by any value of $\eta_w(\bar{s}_\tau^n)$ in K . Let us choose the upwind value of \bar{s}_τ^n as in (2.36), i.e.,

$$\bar{s}_{w,\tau}^{n,ij} = \begin{cases} \bar{s}_\tau^n(\mathbf{x}_i) & \text{if } P_w^{n,i} > P_w^{n,j} \\ \bar{s}_\tau^n(\mathbf{x}_j) & \text{if } P_w^{n,i} < P_w^{n,j} \\ \max(\bar{s}_\tau^n(\mathbf{x}_i), \bar{s}_\tau^n(\mathbf{x}_j)) & \text{if } P_w^{n,i} = P_w^{n,j}, \end{cases} \quad (7.6)$$

and set

$$R_{ij} = \sum_{K \subset \Delta_i \cap \Delta_j} c_{ij,K} \left((\eta_w(\bar{s}_\tau^n))_K - \eta_w(\bar{s}_{w,\tau}^{n,ij}) \right).$$

By proceeding as in Theorem 2.4 and applying (3.23), the regularity of v , and the approximation properties of I_h , we obtain

$$\begin{aligned} & \frac{1}{2} \sum_{n=1}^N \tau \sum_{i,j=1}^M R_{ij} (U_w^{n,j} - U_w^{n,i}) (V^{n,j} - V^{n,i}) \\ & \leq \frac{1}{2} \sum_{n=1}^N \tau \left(\sum_{i,j=1}^M |R_{ij}| (U_w^{n,j} - U_w^{n,i})^2 \right)^{\frac{1}{2}} \left(\sum_{i,j=1}^M |R_{ij}| (V^{n,j} - V^{n,i})^2 \right)^{\frac{1}{2}} \\ & \leq C h \|\eta'_w\|_{L^\infty(0,1)} \|\nabla \bar{s}\|_{L^\infty(\Omega \times]0,T])} \|\nabla U_{w,h,\tau}\|_{L^2(\Omega \times]0,T])} \|\nabla V_{h,\tau}\|_{L^2(\Omega \times]0,T])} \\ & \leq C h \|\eta'_w\|_{L^\infty(0,1)} \|\nabla \bar{s}\|_{L^\infty(\Omega \times]0,T])} \|v\|_{H^1(0,T;H^2(\Omega))}. \end{aligned}$$

With (7.5), this implies

$$\lim_{(h,\tau)\rightarrow(0,0)} \frac{1}{2} \sum_{n=1}^N \tau \sum_{i,j=1}^M c_{ij} \eta_w(\bar{s}_{w,\tau}^{n,ij}) (U_w^{n,j} - U_w^{n,i}) (V^{n,j} - V^{n,i}) = \int_0^T \int_{\Omega} \eta_w(\bar{s}) \nabla \bar{W}_w \cdot \nabla v. \quad (7.7)$$

To recover $\int_0^T [P_{w,h,\tau}, I_h(\eta_w(S_{h,\tau})); U_{w,h,\tau}, V_{h,\tau}]_h$, we write

$$\eta_w(\bar{s}_{w,\tau}^{n,ij}) = \eta_w(\bar{s}_{w,\tau}^{n,ij}) - \eta_w(S_w^{n,ij}) + \eta_w(S_w^{n,ij}),$$

and we must examine the convergence of

$$X := \frac{1}{2} \sum_{n=1}^N \tau \sum_{i,j=1}^M c_{ij} (\eta_w(\bar{s}_{w,\tau}^{n,ij}) - \eta_w(S_w^{n,ij})) (U_w^{n,j} - U_w^{n,i}) (V^{n,j} - V^{n,i}).$$

On the one hand, owing to the smoothness of v , we have

$$|V^{n,j} - V^{n,i}| \leq Ch_i \|\nabla v\|_{L^\infty(\Omega \times]0, T])}, \quad (7.8)$$

where h_i is the length of the edge whose endpoints are the vertices i and j . On the other hand,

$$|\eta_w(\bar{s}_{w,\tau}^{n,ij}) - \eta_w(S_w^{n,ij})| \leq C \|\eta'_w\|_{L^\infty(0,1)} |\bar{s}_{w,\tau}^{n,ij} - S_w^{n,ij}|.$$

Hence

$$|X| \leq C \|\nabla v\|_{L^\infty(\Omega \times]0, T])} \|\nabla U_{w,h,\tau}\|_{L^2(\Omega \times]0, T])} \left(\sum_{n=1}^N \tau \sum_{i,j=1}^M c_{ij} h_i^2 |\bar{s}_{w,\tau}^{n,ij} - S_w^{n,ij}|^2 \right)^{\frac{1}{2}}.$$

It is easy to check that

$$\sum_{i,j=1}^M c_{ij} h_i^2 |\bar{s}_{w,\tau}^{n,ij} - S_w^{n,ij}|^2 \leq C \sum_{i=1}^M m_i |\bar{s}_\tau^{n,i} - S^{n,i}|^2.$$

Therefore

$$\begin{aligned} |X| &\leq C \|\nabla v\|_{L^\infty(\Omega \times]0, T])} \|\nabla U_{w,h,\tau}\|_{L^2(\Omega \times]0, T])} \left(\sum_{n=1}^N \tau \|I_h(\bar{s}_\tau^n) - S_{h,\tau}^n\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \\ &= C \|\nabla v\|_{L^\infty(\Omega \times]0, T])} \|\nabla U_{w,h,\tau}\|_{L^2(\Omega \times]0, T])} \|I_h(\bar{s}_\tau) - S_{h,\tau}\|_{L^2(\Omega \times]0, T])}, \end{aligned}$$

where we have used the equivalence (2.5). Then, we write

$$\|I_h(\bar{s}_\tau) - S_{h,\tau}\|_{L^2(\Omega \times]0, T])} \leq \|I_h(\bar{s}_\tau) - \bar{s}_\tau\|_{L^2(\Omega \times]0, T])} + \|\bar{s}_\tau - \bar{s}\|_{L^2(\Omega \times]0, T])} + \|\bar{s} - S_{h,\tau}\|_{L^2(\Omega \times]0, T])},$$

and the approximation properties of I_h , the strong convergence of \bar{s}_τ to \bar{s} and of $S_{h,\tau}$ to \bar{s} , all in $L^2(\Omega \times]0, T])$ imply that

$$\lim_{(h,\tau)\rightarrow(0,0)} \sum_{n=1}^N \tau \sum_{i,j=1}^M c_{ij} (\eta_w(\bar{s}_{w,\tau}^{n,ij}) - \eta_w(S_w^{n,ij})) (U_w^{n,j} - U_w^{n,i}) (V^{n,j} - V^{n,i}) = 0. \quad (7.9)$$

A combination of (7.9) and (7.7) yields the intermediate convergence result when the limit function \bar{s} is smooth,

$$\lim_{(h,\tau)\rightarrow(0,0)} - \sum_{n=1}^N \tau [P_{w,h,\tau}, I_h(\eta_w(S_{h,\tau})); U_{w,h,\tau}, V_{h,\tau}]_h = \int_0^T \int_{\Omega} \eta_w(\bar{s}) \nabla \bar{W}_w \cdot \nabla v. \quad (7.10)$$

It remains to lift the regularity restriction on \bar{s} . Let $(S_m)_{m \geq 1}$ be a sequence of smooth functions that tend to \bar{s} in $L^2(\Omega \times]0, T[)$. Then for each $\varepsilon > 0$, there exists an integer M_0 such that

$$\|S_{M_0} - \bar{s}\|_{L^2(\Omega \times]0, T[)} \leq \varepsilon. \quad (7.11)$$

From (7.11), the projection properties, and the fact that M_0 is fixed, we infer

$$\begin{aligned} \|\rho_{\tau}(\eta_w(\bar{s})) - \eta_w(\bar{s})\|_{L^2(\Omega \times]0, T[)} &\leq \|\rho_{\tau}(\eta_w(\bar{s}) - \eta_w(S_{M_0}))\|_{L^2(\Omega \times]0, T[)} \\ &\quad + \|\rho_{\tau}(\eta_w(S_{M_0})) - \eta_w(S_{M_0})\|_{L^2(\Omega \times]0, T[)} + \|\eta_w(S_{M_0}) - \eta_w(\bar{s})\|_{L^2(\Omega \times]0, T[)} \\ &\leq (2\varepsilon + C\tau) \|\eta'_w\|_{L^\infty(0,1)}. \end{aligned} \quad (7.12)$$

Now, we replace (7.2) by

$$\begin{aligned} \int_0^T \int_{\Omega} \eta_w(\bar{s}) \nabla U_{w,h,\tau} \cdot \nabla V_{h,\tau} &= \int_0^T \int_{\Omega} \rho_{\tau}(\eta_w(\bar{s}) - \eta_w(S_{M_0})) \nabla U_{w,h,\tau} \cdot \nabla V_{h,\tau} \\ &\quad + \int_0^T \int_{\Omega} \rho_{\tau}(\eta_w(S_{M_0})) \nabla U_{w,h,\tau} \cdot \nabla V_{h,\tau} \\ &= \int_0^T \int_{\Omega} \rho_{\tau}(\eta_w(\bar{s}) - \eta_w(S_{M_0})) \nabla U_{w,h,\tau} \cdot \nabla V_{h,\tau} + \int_0^T \int_{\Omega} \eta_w(S_{M_0}) \nabla U_{w,h,\tau} \cdot \nabla V_{h,\tau}. \end{aligned} \quad (7.13)$$

For the first term, owing to (7.11), the projection properties, and (5.42), we have

$$\begin{aligned} \left| \int_0^T \int_{\Omega} \rho_{\tau}(\eta_w(\bar{s}) - \eta_w(S_{M_0})) \nabla U_{w,h,\tau} \cdot \nabla V_{h,\tau} \right| &\leq \|\nabla V_{h,\tau}\|_{L^\infty(\Omega \times]0, T[)} \\ &\quad \times \|\nabla U_{w,h,\tau}\|_{L^2(\Omega \times]0, T[)} \|\eta_w(\bar{s}) - \eta_w(S_{M_0})\|_{L^2(\Omega \times]0, T[)} \\ &\leq \varepsilon \|\eta'_w\|_{L^\infty(0,1)} \|\nabla V_{h,\tau}\|_{L^\infty(\Omega \times]0, T[)} \|\nabla U_{w,h,\tau}\|_{L^2(\Omega \times]0, T[)}. \end{aligned}$$

Then the uniform boundedness of $U_{w,h,\tau}$ and $V_{h,\tau}$ yield

$$\left| \int_0^T \int_{\Omega} \rho_{\tau}(\eta_w(S) - \eta_w(S_{M_0})) \nabla U_{w,h,\tau} \cdot \nabla V_{h,\tau} \right| \leq C\varepsilon, \quad (7.14)$$

with a constant C independent of h and τ . Thus, we must examine the limit of the second term. Since M_0 is fixed and S_{M_0} is smooth, by reproducing the previous steps, we derive

the analogue of (7.7) for the function S_{M_0} ,

$$\begin{aligned} \lim_{(h,\tau) \rightarrow (0,0)} \frac{1}{2} \sum_{n=1}^N \tau \sum_{i,j=1}^M c_{ij} \eta_w((S_{M_0})_{w,\tau}^{n,ij}) (U_w^{n,j} - U_w^{n,i}) (V^{n,j} - V^{n,i}) \\ = \int_0^T \int_{\Omega} \eta_w(S_{M_0}) \nabla \bar{W}_w \cdot \nabla v = \int_0^T \int_{\Omega} \eta_w(\bar{s}) \nabla \bar{W}_w \cdot \nabla v + R, \end{aligned} \quad (7.15)$$

where

$$\begin{aligned} |R| &= \left| \int_0^T \int_{\Omega} (\eta_w(S_{M_0}) - \eta_w(\bar{s})) \nabla \bar{W}_w \cdot \nabla v \right| \\ &\leq \|\eta'_w\|_{L^\infty(0,1)} \|S_{M_0} - \bar{s}\|_{L^2(\Omega \times]0,T])} \|\nabla \bar{W}_w\|_{L^2(\Omega \times]0,T])} \|\nabla v\|_{L^\infty(\Omega \times]0,T])} \leq C \varepsilon. \end{aligned} \quad (7.16)$$

To relate the left-hand side of (7.15) to $[P_{w,h,\tau}, I_h(\eta_w(S_{h,\tau})); U_{w,h,\tau}, V_h]_h$, we split

$$\eta_w((S_{M_0})_{w,\tau}^{n,ij}) = \eta_w(S_w^{n,ij}) + \eta_w((S_{M_0})_{w,\tau}^{n,ij}) - \eta_w(S_w^{n,ij}),$$

and examine the convergence of

$$Y := \frac{1}{2} \sum_{n=1}^N \tau \sum_{i,j=1}^M c_{ij} (\eta_w((S_{M_0})_{w,\tau}^{n,ij}) - \eta_w(S_w^{n,ij})) (U_w^{n,j} - U_w^{n,i}) (V^{n,j} - V^{n,i}).$$

By arguing as above and using the interpolant I_h , we derive

$$|Y| \leq C \|\eta'_w\|_{L^\infty(0,1)} \|\nabla v\|_{L^\infty(\Omega \times]0,T])} \|\nabla U_{w,h,\tau}\|_{L^2(\Omega \times]0,T])} \|I_h((S_{M_0})_\tau) - S_{h,\tau}\|_{L^2(\Omega \times]0,T])}.$$

Finally, we write

$$\begin{aligned} \|I_h((S_{M_0})_\tau) - S_{h,\tau}\|_{L^2(\Omega \times]0,T])} &\leq \|I_h((S_{M_0})_\tau) - (S_{M_0})_\tau\|_{L^2(\Omega \times]0,T])} \\ &\quad + \|(S_{M_0})_\tau - S_{M_0}\|_{L^2(\Omega \times]0,T])} + \|S_{M_0} - \bar{s}\|_{L^2(\Omega \times]0,T])} + \|\bar{s} - S_{h,\tau}\|_{L^2(\Omega \times]0,T])} \\ &\leq Ch \|S_{M_0}\|_{L^\infty(0,T;H^2(\Omega))} + C\tau \|S_{M_0}\|_{H^1(0,T;L^2(\Omega))} + \varepsilon + \|\bar{s} - S_{h,\tau}\|_{L^2(\Omega \times]0,T])}, \end{aligned}$$

so that

$$|Y| \leq C(h + \tau + \varepsilon) + \|\bar{s} - S_{h,\tau}\|_{L^2(\Omega \times]0,T])}. \quad (7.17)$$

In the next theorem, the limit (7.10) when \bar{s} is only in $L^2(\Omega \times]0,T])$ follows by combining (7.13)–(7.17). The same argument holds when w is replaced by o .

THEOREM 7.1. *Let $v \in \mathcal{C}^1(\bar{\Omega} \times [0, T])$ be a smooth function and let $V_{h,\tau} = I_h(v)(t_n)$ in $[t_{n-1}, t_n]$. Under the assumptions and notation on the mobility (5.42)–(5.46),*

$$\lim_{(h,\tau) \rightarrow (0,0)} - \int_0^T [P_{\alpha,h,\tau}, I_h(\eta_\alpha(S_{h,\tau})); U_{\alpha,h,\tau}, V_{h,\tau}]_h = \int_0^T \int_{\Omega} \eta_\alpha(\bar{s}) \nabla \bar{W}_\alpha \cdot \nabla v \quad (7.18)$$

where \bar{s} is the strong limit of $S_{h,\tau}$ and \bar{W}_α the weak limit of $U_{\alpha,h,\tau}$, $\alpha = w, o$.

7.1.2. The additional term . This paragraph is dedicated to the limit of

$$\int_0^T [P_{\alpha,h,\tau}, I_h(\eta_\alpha(S_{h,\tau})); I_h(p_{\alpha g}(S_{h,\tau})), V_{h,\tau}]_h, \quad \alpha = w, o.$$

It shall be split below, as suggested by the following observation, derived from (1.12) and (1.13):

$$\begin{aligned} \eta_w(S_w^{ij})p_{wg}(S^j) + g(S^j) &= \int_0^{S^j} f_o(x)(\eta_w(S_w^{ij}) - \eta_w(x))p'_c(x) dx, \\ \eta_o(S_o^{ij})p_{og}(S^j) + g(S^j) &= \int_0^{S^j} f_w(x)(\eta_o(S_o^{ij}) - \eta_o(x))p'_c(x) dx. \end{aligned}$$

Thus, we add and subtract g and write by applying (2.9),

$$\begin{aligned} &\int_0^T [P_{\alpha,h,\tau}, I_h(\eta_\alpha(S_{h,\tau})); I_h(p_{\alpha g}(S_{h,\tau})), V_{h,\tau}]_h \\ &= \sum_{n=1}^N \tau \sum_{i,j=1}^M V^{n,i} c_{ij} \left[\eta_\alpha(S_\alpha^{n,ij})(p_{\alpha g}(S^{n,j}) - p_{\alpha g}(S^{n,i})) + g(S^{n,j}) - g(S^{n,i}) \right] \\ &+ \int_0^T \int_\Omega \nabla g(S_{h,\tau}) \cdot \nabla V_{h,\tau} = T_1 + T_2. \end{aligned}$$

Since

$$\lim_{(h,\tau) \rightarrow (0,0)} T_2 = \int_0^T \int_\Omega \nabla g(\bar{s}) \cdot \nabla v, \quad (7.19)$$

we must prove that the first term tends to zero. When $\alpha = w$, it has the form

$$T_1 = -\frac{1}{2} \sum_{n=1}^N \tau \sum_{i,j=1}^M c_{ij} \left(\int_{S^{n,i}}^{S^{n,j}} f_o(x)(\eta_w(S_w^{n,ij}) - \eta_w(x))p'_c(x) dx \right) (V^{n,j} - V^{n,i}), \quad (7.20)$$

with an analogous expression in the non-wetting phase. Then (7.8) yields,

$$|T_1| \leq \frac{C}{2} \|\nabla v\|_{L^\infty(\Omega \times]0,T])} \sum_{n=1}^N \tau \sum_{i,j=1}^M h_i c_{ij} \left| \int_{S^{n,i}}^{S^{n,j}} f_o(x)(\eta_w(S_w^{n,ij}) - \eta_w(x))p'_c(x) dx \right|. \quad (7.21)$$

Showing that T_1 is small requires a technical argument that we split into several steps.

PROPOSITION 7.2. *For the wetting phase, we have*

$$\left| \int_{S^i}^{S^j} f_o(x)(\eta_w(S_w^{ij}) - \eta_w(x))p'_c(x) dx \right| \leq -(\eta_w(S^j) - \eta_w(S^i))(p_{wg}(S^j) - p_{wg}(S^i)). \quad (7.22)$$

For the non-wetting phase, the corresponding expression is bounded by

$$\left| \int_{S^i}^{S^j} f_w(x)(\eta_o(S_o^{ij}) - \eta_o(x))p'_c(x) dx \right| \leq (\eta_o(S^j) - \eta_o(S^i))(p_{og}(S^j) - p_{og}(S^i)). \quad (7.23)$$

Proof. Let us prove (7.22), the proof of (7.23) being similar. The discussion depends on the respective values of S^j and S^i . There are two cases: $S^i < S^j$ or $S^i > S^j$. Of course $S^i = S^j$ brings nothing.

1) If $S^i < S^j$ and $S_w^{ij} = S^i$, then $\eta_w(S_w^{ij}) - \eta_w(x) = \eta_w(S^i) - \eta_w(x)$, and, as p_{wg} is decreasing,

$$0 \leq \int_{S^i}^{S^j} f_o(x)(-p'_c(x))(\eta_w(x) - \eta_w(S^i)) dx \leq -(\eta_w(S^j) - \eta_w(S^i))(p_{wg}(S^j) - p_{wg}(S^i)).$$

If $S_w^{ij} = S^j$, then $\eta_w(S_w^{ij}) - \eta_w(x) = \eta_w(S^j) - \eta_w(x)$, and

$$0 \leq \int_{S^i}^{S^j} f_o(x)(-p'_c(x))(\eta_w(S^j) - \eta_w(x)) dx \leq -(\eta_w(S^j) - \eta_w(S^i))(p_{wg}(S^j) - p_{wg}(S^i)).$$

2) If $S^i > S^j$ and $S_w^{ij} = S^i$, then

$$0 \leq \int_{S^j}^{S^i} f_o(x)(-p'_c(x))(\eta_w(S^i) - \eta_w(x)) dx \leq -(\eta_w(S^i) - \eta_w(S^j))(p_{wg}(S^i) - p_{wg}(S^j)).$$

Finally, suppose that $S_w^{ij} = S^j$. Then

$$0 \leq \int_{S^j}^{S^i} f_o(x)p'_c(x)(\eta_w(S^j) - \eta_w(x)) dx \leq -(\eta_w(S^i) - \eta_w(S^j))(p_{wg}(S^i) - p_{wg}(S^j)).$$

This proves (7.22). \square

By substituting (7.22) into (7.21), we arrive at

$$|T_1| \leq \frac{C}{2} \|\nabla v\|_{L^\infty(\Omega \times]0, T])} \sum_{n=1}^N \tau \sum_{i,j=1}^M h_i c_{ij} \left(-(\eta_w(S^{n,j}) - \eta_w(S^{n,i})) (p_{wg}(S^{n,j}) - p_{wg}(S^{n,i})) \right), \quad (7.24)$$

with an analogous bound in the non-wetting phase. Up to the factor h_i , they behave like $\int_0^T \int_\Omega \nabla(I_h(\eta_\alpha(S_{h,\tau}))) \cdot \nabla(I_h(p_{\alpha g}(S_{h,\tau})))$, $\alpha = w, o$. Thus T_1 tends to zero if this quantity is bounded or is of the order of a negative power of h that is larger than -1 . We have no direct bound for it, but as we do have a bound for $\int_0^T \int_\Omega \nabla(I_h(f_\alpha(S_{h,\tau}))) \cdot \nabla(I_h(g(S_{h,\tau})))$, see (5.40), we can gain some insight by relating the two integrands. Again, we examine the wetting phase, the treatment of the non-wetting phase being the same. The proposition below will be applied to $x_1 = S^{n,i}$ and $x_2 = S^{n,j}$. The condition $x_1 < x_2$ is not a restriction because if it does not hold, the indices i and j can be interchanged without changing the value of the two integrands.

PROPOSITION 7.3. *Under the assumptions and notation on the mobility (5.42)–(5.46), we have for all pairs x_1, x_2 with $0 \leq x_1 < x_2 \leq \frac{3}{4}$,*

$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leq C(x_2^{\theta_w} - x_1^{\theta_w})(x_2^{\beta_3} - x_1^{\beta_3}), \quad (7.25)$$

$$(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)) \geq C(x_2^{\theta_w} - x_1^{\theta_w})(x_2^{\theta_w+\beta_3} - x_1^{\theta_w+\beta_3}). \quad (7.26)$$

Similarly, we have for all pairs x_1, x_2 with $\frac{1}{4} \leq x_1 < x_2 \leq 1$,

$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leq C(x_2 - x_1)((1 - x_1)^{\theta_o+\beta_4} - (1 - x_2)^{\theta_o+\beta_4}), \quad (7.27)$$

$$(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)) \geq C((1 - x_1)^{\theta_o} - (1 - x_2)^{\theta_o})((1 - x_1)^{\theta_o+\beta_4} - (1 - x_2)^{\theta_o+\beta_4}). \quad (7.28)$$

Finally, we have for all pairs x_1, x_2 with $0 \leq x_1 \leq \frac{1}{4}$ and $\frac{3}{4} \leq x_2 \leq 1$,

$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leq C(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)). \quad (7.29)$$

All constants C above are independent of x_1 and x_2 .

Proof. According to (5.42),

$$\eta_w(x_2) - \eta_w(x_1) \leq \frac{1}{\alpha_w \theta_w} (x_2^{\theta_w} - x_1^{\theta_w}).$$

Next, recalling that $p'_{wg}(x) = f_o(x)p'_c(x)$, we have, owing to (5.44), (5.46), and (1.8),

$$\begin{aligned} p_{wg}(x_1) - p_{wg}(x_2) &= \int_{x_1}^{x_2} f_o(x)(-p'_c(x)) dx \leq \frac{1}{\eta_*} \frac{1}{\alpha_3} \frac{1}{\alpha_o \theta_o} \int_{x_1}^{x_2} x^{\beta_3-1} (1-x)^{\theta_o+\beta_4-1} dx \\ &\leq \frac{1}{\eta_*} \frac{1}{\alpha_3} \frac{1}{\alpha_o \theta_o} \int_{x_1}^{x_2} x^{\beta_3-1} dx \leq \frac{1}{\eta_*} \frac{1}{\alpha_3 \beta_3} \frac{1}{\alpha_o \theta_o} (x_2^{\beta_3} - x_1^{\beta_3}), \end{aligned} \quad (7.30)$$

and (7.25), valid on $[0, 1]$, follows from these two inequalities.

For (7.26), we use (5.53) that gives

$$f_w(x_2) - f_w(x_1) \geq \frac{\alpha_o \alpha_w}{C_{\max}^2} \frac{1}{\theta_o \theta_w} \left(\frac{1}{4}\right)^{\theta_o} (x_2^{\theta_w} - x_1^{\theta_w}), \quad (7.31)$$

and we use (5.51) that gives

$$g(x_2) - g(x_1) \geq \frac{\alpha_3}{C_{\max}} \frac{\alpha_w}{\theta_w} \frac{\alpha_o}{\theta_o} \frac{1}{\theta_w + \beta_3} \left(\frac{1}{4}\right)^{\theta_o+\beta_4-1} (x_2^{\theta_w+\beta_3} - x_1^{\theta_w+\beta_3}).$$

The product of the two leads to (7.26).

Regarding (7.27), (7.25), albeit valid for all $x \in [0, 1]$, is not adequate for the comparison we have in mind, and instead we use that

$$\eta'_w(x) \leq \frac{1}{\alpha_w},$$

which implies that

$$\eta_w(x_2) - \eta_w(x_1) \leq \frac{1}{\alpha_w} (x_2 - x_1).$$

Similarly, we use

$$-p'_{wg}(x) \leq \frac{1}{\eta_*} \frac{1}{\alpha_3} \frac{1}{\alpha_o \theta_o} 4^{1-\beta_3} (1-x)^{\theta_o+\beta_4-1},$$

so that

$$p_{wg}(x_1) - p_{wg}(x_2) \leq \frac{1}{\eta_*} \frac{1}{\alpha_3} \frac{1}{\alpha_o \theta_o} \frac{1}{\theta_o + \beta_4} 4^{1-\beta_3} ((1-x_1)^{\theta_o+\beta_4} - (1-x_2)^{\theta_o+\beta_4}),$$

thus proving (7.27). Next, by applying (5.54), we have

$$f_w(x_2) - f_w(x_1) \geq \frac{1}{C_{\max}^2} \frac{\alpha_o \alpha_w}{\theta_w \theta_o} \left(\frac{1}{4}\right)^{\theta_w} ((1-x_1)^{\theta_o} - (1-x_2)^{\theta_o}).$$

Likewise, by applying (5.51), we obtain

$$g(x_2) - g(x_1) \geq \frac{\alpha_3}{C_{\max}} \frac{\alpha_w \alpha_o}{\theta_w \theta_o} \frac{1}{\theta_o + \beta_4} \left(\frac{1}{4}\right)^{\theta_w-1+\beta_3} ((1-x_1)^{\theta_o+\beta_4} - (1-x_2)^{\theta_o+\beta_4}).$$

The product of the two yields (7.28).

Finally, when $0 \leq x_1 \leq \frac{1}{4}$ and $\frac{3}{4} \leq x_2 \leq 1$, since both η_w and $-p_{wg}$ are both increasing, they satisfy

$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leq \eta_w(1)(-p_{wg}(1)) > 0.$$

Likewise, as both f_w and g are increasing, they satisfy

$$(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)) \geq (f_w(\frac{3}{4}) - f_w(\frac{1}{4}))(g(\frac{3}{4}) - g(\frac{1}{4})) =: D > 0.$$

Hence

$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leq -\frac{1}{D} (\eta_w p_{wg})(1) (f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)),$$

whence (7.29). Clearly all constants involved are independent of x_1 and x_2 . \square

It stems from (7.25) and (7.26), that the two left-hand sides cannot be compared when x_1 and x_2 are too small. The same observation applies to (7.27) and (7.28) when $1-x_1$ and $1-x_2$ are too small. But in this case, there is no need for comparison because the expression we want to bound is sufficiently small, as is shown in the next proposition where again, $x_1 = S^{n,i}$ and $x_2 = S^{n,j}$.

PROPOSITION 7.4. *Suppose that $x_1 < x_2 \leq h_i^{\gamma_1}$ for some exponent $\gamma_1 > 0$ such that*

$$\gamma_1 > \frac{1}{\theta_w + \beta_3}. \tag{7.32}$$

Then

$$h_i(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leq C h_i^2 h_i^{\delta_1}, \tag{7.33}$$

where

$$0 < \delta_1 \leq \gamma_1(\theta_w + \beta_3) - 1. \quad (7.34)$$

Similarly, suppose that $1 - x_2 < 1 - x_1 \leq h_i^{\gamma_2}$ for some exponent $\gamma_2 > 0$ such that

$$\gamma_2 > \frac{1}{1 + \theta_o + \beta_4}. \quad (7.35)$$

Then

$$h_i(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leq C h_i^2 h_i^{\delta_2}, \quad (7.36)$$

where

$$0 < \delta_2 \leq \gamma_2(1 + \theta_o + \beta_4) - 1. \quad (7.37)$$

In both cases, the constants C are independent of x_1 , x_2 , and h_i .

Proof. In the first case, according to (7.25), the choice (7.34) and (7.32) on γ_1 , we have

$$h_i(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leq C h_i^{1+\gamma_1(\theta_w+\beta_3)} = h_i^2 h_i^{\gamma_1(\theta_w+\beta_3)-1},$$

with the constant C of (7.25), which gives (7.33). In the second case, the same argument leads to

$$h_i(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leq h_i^{1+\gamma_2(1+\theta_o+\beta_4)} = h_i^2 h_i^{\gamma_2(1+\theta_o+\beta_4)-1},$$

with the constant C of (7.27), thus implying (7.36) with the choice (7.37) for δ_2 and the condition (7.35) on γ_2 . \square

Now, we turn to the case when x_2 is not too small.

PROPOSITION 7.5. *In addition to (7.32), suppose that the exponent γ_1 of Proposition 7.4 satisfies*

$$\gamma_1 < \frac{1}{\theta_w}. \quad (7.38)$$

Suppose that $x_1 < x_2$ and $\frac{3}{4} \geq x_2 > h_i^{\gamma_1}$. Then

$$h_i(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leq C h_i^{\delta'_1} (f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)), \quad (7.39)$$

where

$$0 < \delta'_1 = \min(1 - \gamma_1 \theta_w, \delta_1). \quad (7.40)$$

Again, the constant C is independent of x_1 , x_2 , and h_i .

Proof. Either $x_1 \leq \frac{1}{2}x_2$ or $x_1 > \frac{1}{2}x_2$, and we examine each case.

1) When $x_1 \leq \frac{1}{2}x_2$, formula (7.26) leads to

$$(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)) \geq \left(1 - \left(\frac{1}{2}\right)^{\theta_w}\right) \left(1 - \left(\frac{1}{2}\right)^{\theta_w+\beta_3}\right) C x_2^{2\theta_w+\beta_3},$$

with the constant C of (7.26), whereas

$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leq Cx_2^{\theta_w + \beta_3},$$

with the constant C of (7.25). Hence

$$h_i(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leq C \frac{h_i}{x_2^{\theta_w}} (f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)),$$

with another constant C independent of x_1 , x_2 , and h_i . Now, we use the assumption that $x_2 > h_i^{\gamma_1}$. Then, owing to (7.40),

$$\frac{h_i}{x_2^{\theta_w}} \leq h_i^{1 - \gamma_1 \theta_w} \leq h_i^{\delta'_1},$$

and we recover (7.39).

2) When $x_1 > \frac{1}{2}x_2$, we infer from the next to last inequality in (7.30) that

$$p_{wg}(x_1) - p_{wg}(x_2) \leq \frac{1}{\eta_*} \frac{1}{\alpha_3} \frac{1}{\alpha_o \theta_o} (x_2 - x_1) x_1^{\beta_3 - 1} \leq \frac{1}{\eta_*} \frac{1}{\alpha_3} \frac{1}{\alpha_o \theta_o} 2^{1 - \beta_3} \frac{1}{x_2^{1 - \beta_3}} (x_2 - x_1).$$

Thus, on the one hand,

$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leq C \frac{1}{x_2^{1 - \beta_3}} (x_2 - x_1) (x_2^{\theta_w} - x_1^{\theta_w}), \quad (7.41)$$

where C is the above constant divided by α_w . On the other hand, we use the lower bound (7.31) for the difference in f_w and we need a lower bound for the difference in g . It is derived from (5.51),

$$\begin{aligned} g(x_2) - g(x_1) &\geq \frac{\alpha_3}{C_{\max}} \frac{\alpha_w}{\theta_w} \frac{\alpha_o}{\theta_o} \left(\frac{1}{4}\right)^{\theta_o + \beta_4 - 1} x_1^{\theta_w + \beta_3 - 1} (x_2 - x_1) \\ &\geq \frac{\alpha_3}{C_{\max}} \frac{\alpha_w}{\theta_w} \frac{\alpha_o}{\theta_o} \left(\frac{1}{4}\right)^{\theta_o + \beta_4 - 1} \left(\frac{1}{2}\right)^{\theta_w + \beta_3 - 1} x_2^{\theta_w + \beta_3 - 1} (x_2 - x_1). \end{aligned} \quad (7.42)$$

Hence (7.31) and (7.42) yield

$$(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)) \geq Cx_2^{\theta_w + \beta_3 - 1} (x_2^{\theta_w} - x_1^{\theta_w})(x_2 - x_1), \quad (7.43)$$

with the product of the constants of (7.31) and (7.42). Then by combining (7.41) and (7.43), we deduce that

$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leq \frac{C}{h_i^{\gamma_1 \theta_w}} (f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1));$$

which is (7.39) when δ'_1 satisfies (7.40). \square

The case when $1 - x_1$ is not too small is handled by the next proposition.

PROPOSITION 7.6. *In addition to (7.35), suppose that the exponent γ_2 of Proposition 7.4 satisfies*

$$\gamma_2 < \frac{1}{\theta_o - 1}. \quad (7.44)$$

Suppose that $\frac{1}{4} < x_1 < x_2 \leq 1$ and $1 - x_1 > h_i^{\gamma_2}$. Then

$$h_i(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leq Ch_i^{\delta'_2}(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)), \quad (7.45)$$

where

$$0 < \delta'_2 = \min(\delta_2, 1 - \gamma_2(\theta_o - 1)). \quad (7.46)$$

Again, the constant C is independent of x_1 , x_2 , and h_i .

Proof. The proof is analogous to that of Proposition 7.5, but we sketch the steps for the reader's convenience. We skip the constants' details, but stress that they are independent of x_1 , x_2 , and h_i . Again, there are two possibilities, either $1 - x_2 \leq \frac{1}{2}(1 - x_1)$ or $1 - x_2 > \frac{1}{2}(1 - x_1)$, and we examine each case.

1) In the first case,

$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leq C(1 - x_1)^{1+\theta_0+\beta_4},$$

and

$$(f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)) \geq C(1 - x_1)^{2\theta_0+\beta_4}.$$

Hence

$$\begin{aligned} (\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) &\leq C \frac{1}{(1 - x_1)^{\theta_o - 1}} (f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)) \\ &\leq C \frac{1}{h_i^{\gamma_2(\theta_o - 1)}} (f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)). \end{aligned}$$

With (7.44) and (7.46), this implies (7.45).

2) In the second case, we have on the one hand,

$$p_{wg}(x_1) - p_{wg}(x_2) \leq C(x_2 - x_1)(1 - x_1)^{\theta_o+\beta_4-1},$$

so that

$$(\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) \leq C(x_2 - x_1)^2(1 - x_1)^{\theta_o+\beta_4-1}.$$

On the other hand,

$$f_w(x_2) - f_w(x_1) \geq C(x_2 - x_1)(1 - x_1)^{\theta_o-1},$$

and

$$g(x_2) - g(x_1) \geq C(x_2 - x_1)(1 - x_1)^{\theta_o+\beta_4-1},$$

and thus

$$\begin{aligned} (\eta_w(x_2) - \eta_w(x_1))(p_{wg}(x_1) - p_{wg}(x_2)) &\leq C \frac{1}{(1-x_1)^{\theta_o-1}} (f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)) \\ &\leq C \frac{1}{h_i^{\gamma_2(\theta_o-1)}} (f_w(x_2) - f_w(x_1))(g(x_2) - g(x_1)), \end{aligned}$$

whence (7.45). \square

In view of (7.32), (7.34), (7.38), and (7.40), let us choose

$$\delta_1 = \delta'_1 = \frac{\beta_3}{2\theta_w + \beta_3}, \quad \gamma_1 = \frac{2}{2\theta_w + \beta_3}. \quad (7.47)$$

Then (7.32) and (7.34) are satisfied, as well as (7.38) and (7.40). Likewise, in view of (7.35), (7.37), (7.44), and (7.46), the choice

$$\delta_2 = \delta'_2 = \frac{2 + \beta_4}{2\theta_o + \beta_4}, \quad \gamma_2 = \frac{2}{2\theta_o + \beta_4}, \quad (7.48)$$

satisfies (7.35), (7.37), (7.44), (7.46). Then the desired limit follows by collecting these results.

LEMMA 7.7. *Under the assumptions and notation on the mobility (5.42)–(5.46), the term T_1 defined in (7.20) tends to zero, with a similar limit in the non-wetting phase, i.e.,*

$$\begin{aligned} \lim_{(h,\tau) \rightarrow (0,0)} \sum_{n=1}^N \tau \sum_{i,j=1}^M c_{ij} \left(\int_{S^{n,i}}^{S^{n,j}} f_o(x) (\eta_w(S_w^{n,ij}) - \eta_w(x)) p'_c(x) dx \right) (V^{n,j} - V^{n,i}) &= 0, \\ \lim_{(h,\tau) \rightarrow (0,0)} \sum_{n=1}^N \tau \sum_{i,j=1}^M c_{ij} \left(\int_{S^{n,i}}^{S^{n,j}} f_w(x) (\eta_o(S_o^{n,ij}) - \eta_o(x)) p'_c(x) dx \right) (V^{n,j} - V^{n,i}) &= 0. \end{aligned} \quad (7.49)$$

Proof. We prove the first limit. Here the parameters of Propositions 7.4 and 7.5 are chosen by (7.47) and (7.48). It stems from the above considerations that, for each index n , the set of all indices (i, j) from 1 to M can be partitioned into three subsets,

$$\mathcal{O}_1 = \{(i, j); 0 \leq S^{n,i} < S^{n,j} \leq \frac{3}{4}\}, \quad \mathcal{O}_2 = \{(i, j); \frac{1}{4} \leq S^{n,i} < S^{n,j} \leq 1\},$$

$$\mathcal{O}_3 = \{(i, j); 0 \leq S^{n,i} \leq \frac{1}{4} \text{ and } \frac{3}{4} \leq S^{n,j} \leq 1\}.$$

In turn, \mathcal{O}_1 and \mathcal{O}_2 can each be partitioned into two subsets

$$\begin{aligned} \mathcal{O}_{1,1} &= \{(i, j) \in \mathcal{O}_1; S^{n,j} \leq h_T^{\gamma_1}\}, & \mathcal{O}_{1,2} &= \{(i, j) \in \mathcal{O}_1; S^{n,j} > h_T^{\gamma_1}\}, \\ \mathcal{O}_{2,1} &= \{(i, j) \in \mathcal{O}_2; 1 - S^{n,i} \leq h_T^{\gamma_2}\}, & \mathcal{O}_{2,2} &= \{(i, j) \in \mathcal{O}_2; 1 - S^{n,i} > h_T^{\gamma_2}\}. \end{aligned}$$

To simplify, let

$$A_{i,j} = c_{ij} \left(\int_{S^{n,i}}^{S^{n,j}} f_o(x) (\eta_w(S_w^{n,ij}) - \eta_w(x)) p'_c(x) dx \right) (V^{n,j} - V^{n,i}).$$

In view of (7.33) and (7.36), for all pairs (i, j) in $\mathcal{O}_{\ell,1}$, $\ell = 1, 2$, $A_{i,j}$ satisfies

$$|A_{i,j}| \leq C \|\nabla v\|_{L^\infty(\Omega \times]0, T])} h_i^{2+\delta_\ell} c_{ij}.$$

Owing to (7.39) and (7.45), for all pairs (i, j) in $\mathcal{O}_{\ell,2}$, $\ell = 1, 2$, we have

$$|A_{i,j}| \leq C \|\nabla v\|_{L^\infty(\Omega \times]0, T])} h_i^{\delta_\ell} c_{ij} (f_w(S^{n,j}) - f_w(S^{n,i})) (g(S^{n,j}) - g(S^{n,i})).$$

Finally, for all pairs (i, j) in \mathcal{O}_3 ,

$$|A_{i,j}| \leq C \|\nabla v\|_{L^\infty(\Omega \times]0, T])} h_i c_{ij} (f_w(S^{n,j}) - f_w(S^{n,i})) (g(S^{n,j}) - g(S^{n,i})).$$

According to (5.40), the sum of the terms over all (i, j) in $\mathcal{O}_{\ell,2}$ and \mathcal{O}_3 tends to zero. For the remaining terms, observe that by definition,

$$h_i^2 c_{ij} \leq C |\Delta_i \cap \Delta_j|,$$

so that the sum over all (i, j) in $\mathcal{O}_{\ell,1}$ is bounded by $Ch_i^{\delta_\ell}$ that also tends to zero, whence the first part of the limit (7.49). The same limit to zero holds for the non-wetting phase. \square

With (7.19), this lemma leads to the desired limit of the term with the auxiliary pressures.

THEOREM 7.8. *Let $v \in \mathcal{C}^1(\bar{\Omega} \times [0, T])$ be a smooth function and let $V_{h,\tau}(t) = I_h(v)(t_n)$ in $]t_{n-1}, t_n]$. Under the assumptions and notation on the mobility (5.42)–(5.46),*

$$\lim_{(h,\tau) \rightarrow (0,0)} \int_0^T [P_{\alpha,h,\tau}, I_h(\eta_\alpha(S_{h,\tau})); I_h(p_{\alpha g}(S_{h,\tau})), V_{h,\tau}]_h = \int_0^T \int_\Omega \nabla g(\bar{s}) \cdot \nabla v, \quad \alpha = w, o, \quad (7.50)$$

where \bar{s} is the limit of $S_{h,\tau}$.

Finally, Theorems 7.1 and 7.8, together with (1.15) and (6.40), give the desired convergence of the upwind diffusion terms.

THEOREM 7.9. *With the notation and assumptions of Theorem 7.1, we have for all functions $v \in \mathcal{C}^1(\bar{\Omega} \times [0, T])$,*

$$\begin{aligned} & \lim_{(h,\tau) \rightarrow (0,0)} - \int_0^T [P_{\alpha,h,\tau}, I_h(\eta_\alpha(S_{h,\tau})); P_{\alpha,h,\tau}, V_{h,\tau}]_h \\ &= \int_0^T \int_\Omega (\eta_w(\bar{s}) \nabla(\bar{p}_w + p_{wg}(\bar{s})) + \nabla g(\bar{s})) \cdot \nabla v \quad \text{if } \alpha = w, \\ &= \int_0^T \int_\Omega (\eta_o(\bar{s}) \nabla(\bar{p}_o - p_{og}(\bar{s})) - \nabla g(\bar{s})) \cdot \nabla v \quad \text{if } \alpha = o. \end{aligned} \quad (7.51)$$

7.2. Convergence of the right-hand sides. In order to pass to the limit in the right-hand sides of (2.42)–(2.43) it is convenient to replace the quadrature formulas by integrals. Since the quadrature formulas are exact for polynomials of degree one, this is achieved by approximating some functions with the operator ρ_h , see (2.27). As s_{in} belongs to $L^\infty(\Omega \times]0, T[)$, standard approximation properties of ρ_τ and r_h and a density argument imply

$$\lim_{(h,\tau) \rightarrow (0,0)} \rho_\tau(\rho_K(s_{\text{in}})) = s_{\text{in}} \text{ in } L^\infty(K \times]0, T[). \quad (7.52)$$

Then the continuity of f_α , for $\alpha = w, o$, yields

$$\lim_{(h,\tau) \rightarrow (0,0)} f_\alpha(\rho_\tau(\rho_K(s_{\text{in}}))) = f_\alpha(s_{\text{in}}) \text{ in } L^\infty(K \times]0, T[). \quad (7.53)$$

Similarly, since \bar{q} belongs to $L^2(\Omega \times]0, T[)$,

$$\lim_{(h,\tau) \rightarrow (0,0)} \rho_\tau(\rho_K(\bar{q})) = \bar{q} \text{ in } L^2(K \times]0, T[).$$

Also the (constant in space) correction added to $\rho_\tau(r_h(\bar{q}))$ satisfies

$$\lim_{(h,\tau) \rightarrow (0,0)} \rho_\tau\left(\frac{1}{|\Omega|} \int_\Omega (r_h(\bar{q}) - \bar{q})\right) = 0 \text{ in } L^2(\Omega \times]0, T[).$$

Therefore

$$\lim_{(h,\tau) \rightarrow (0,0)} \bar{q}_{h,\tau} = \bar{q} \text{ in } L^2(\Omega \times]0, T[). \quad (7.54)$$

With the same function $V_{h,\tau}$, consider the first term in the right-hand sides of (2.42)–(2.43)

$$X := \sum_{n=1}^N \tau (I_h(f_\alpha(s_{\text{in},h}^n)) \bar{q}_h^n, V_h^n)_h = \int_0^T (I_h(f_\alpha(s_{\text{in},h,\tau})) \bar{q}_{h,\tau}, V_{h,\tau})_h.$$

By definition of the quadrature formula, X has the following expression:

$$X = \sum_{n=1}^N \tau \sum_{K \in \bar{\Omega}} \frac{|K|}{d+1} \sum_{\ell=1}^{d+1} f_\alpha(s_{\text{in},h,\tau}^{n,\ell_i}) \bar{q}_{h,\tau}^{n,\ell_i} V_{h,\tau}^{n,\ell_i}.$$

By inserting $f_\alpha(\rho_\tau(\rho_K(s_{\text{in}})))$ and $\rho_\tau(\rho_K(\bar{q}))$, this becomes

$$\begin{aligned} X &= \sum_{n=1}^N \tau \sum_{K \in \bar{\Omega}} \frac{|K|}{d+1} \sum_{\ell=1}^{d+1} (f_\alpha(s_{\text{in},h,\tau}^{n,\ell_i}) - f_\alpha(\rho_\tau(\rho_K(s_{\text{in}})))) \bar{q}_{h,\tau}^{n,\ell_i} V_{h,\tau}^{n,\ell_i} \\ &+ \sum_{n=1}^N \tau \sum_{K \in \bar{\Omega}} \frac{|K|}{d+1} \sum_{\ell=1}^{d+1} f_\alpha(\rho_\tau(\rho_K(s_{\text{in}}))) (\bar{q}_{h,\tau}^{n,\ell_i} - \rho_\tau(\rho_K(\bar{q}))) V_{h,\tau}^{n,\ell_i} \\ &+ \int_0^T \int_\Omega f_\alpha(\rho_\tau(\rho_K(s_{\text{in}}))) \rho_\tau(\rho_K(\bar{q})) V_{h,\tau} = X_1 + X_2 + X_3, \end{aligned}$$

since the last summand is a polynomial of degree one. We have

$$\lim_{(h,\tau)\rightarrow(0,0)} X_3 = \int_0^T \int_{\Omega} f_o(s_{\text{in}}) \bar{q} v.$$

It remains to show that X_1 and X_2 tend to zero. For X_1 , since f_o and f_w have the same derivative (up to the sign), we deduce from (5.52), (5.42), (5.43), (5.45), (5.46), and (5.47) that f'_α is bounded in $[0, 1]$; hence

$$|f_\alpha(s_{\text{in},h,\tau}^{n,\ell_i}) - f_\alpha(\rho_\tau(\rho_K(s_{\text{in}})))| \leq C |s_{\text{in},h,\tau}^{n,\ell_i} - \rho_\tau(\rho_K(s_{\text{in}}))|.$$

Thus, the summand is bounded by polynomials and the equivalence of norms yields

$$|X_1| \leq C \|v\|_{L^\infty(\Omega \times]0, T])} \|s_{\text{in},h,\tau} - \rho_\tau(\rho_K(s_{\text{in}}))\|_{L^2(\Omega \times]0, T])} \|\bar{q}_{h,\tau}\|_{L^2(\Omega \times]0, T])},$$

that tends to zero with (h, τ) . It is easy to check that the same holds for X_2 . Hence

$$\lim_{(h,\tau)\rightarrow(0,0)} \int_0^T (I_h(f_\alpha(s_{\text{in},h,\tau})) \bar{q}_{h,\tau}, V_{h,\tau})_h = \int_0^T \int_{\Omega} f_\alpha(s_{\text{in}}) \bar{q} v. \quad (7.55)$$

The argument for the second term in the right-hand side of (2.42) is much the same; we insert $\rho_\tau(\rho_K(\bar{s}))$ and we use the fact that

$$\lim_{(h,\tau)\rightarrow(0,0)} \|S_{h,\tau} - \rho_\tau(\rho_K(\bar{s}))\|_{L^2(\Omega \times]0, T])} = 0.$$

Then the argument used for the first term readily gives

$$\lim_{(h,\tau)\rightarrow(0,0)} \int_0^T (I_h(f_\alpha(S_{h,\tau})) \underline{q}_{h,\tau}, V_{h,\tau})_h = \int_0^T \int_{\Omega} f_\alpha(\bar{s}) \bar{q} v. \quad (7.56)$$

By combining (7.55) and (7.56), we obtain convergence of the right-hand sides,

$$\lim_{(h,\tau)\rightarrow(0,0)} \left(\int_0^T (I_h(f_\alpha(s_{\text{in},h,\tau})) \bar{q}_{h,\tau} - I_h(f_\alpha(S_{h,\tau})) \underline{q}_{h,\tau}, V_{h,\tau})_h = \int_0^T \int_{\Omega} (f_\alpha(s_{\text{in}}) \bar{q} - f_\alpha(\bar{s}) \bar{q}) v. \quad (7.57)$$

7.3. The full scheme. It remains to pass to the limit in the time derivative, say in (2.42), summed over n , and tested with the same $V_{h,\tau}$ as previously, except that here we take $v(T) = 0$. After summation by parts, this term reads

$$\sum_{n=1}^N (S_h^n - S_h^{n-1}, V_h^n)_h^\varphi = - \sum_{n=1}^{N-1} (V_h^{n+1} - V_h^n, S_h^n)_h^\varphi - (V_h^1, S_h^0)_h^\varphi. \quad (7.58)$$

By definition,

$$(V_h^{n+1} - V_h^n, S_h^n)_h^\varphi = \sum_{K \in \bar{\Omega}} \frac{|K|}{d+1} \varphi|_K \sum_{\ell=1}^{d+1} (V^{n+1, i_\ell} - V^{n, i_\ell}) S^{n, i_\ell}.$$

By inserting $\rho_K(V^{n+1,i\ell} - V^{n,i\ell})$ in each element, this becomes

$$(V_h^{n+1} - V_h^n, S_h^n)_h^\varphi = (V_h^{n+1} - V_h^n - \rho_h(V^{n+1} - V^n), S_h^n)_h^\varphi + \int_{\Omega} \varphi \rho_h(V^{n+1} - V^n) S_h^n.$$

The first term has the bound

$$|(V_h^{n+1} - V_h^n - \rho_h(V^{n+1} - V^n), S_h^n)_h^\varphi| \leq \|\varphi\|_{L^\infty(\Omega)} \|V_h^{n+1} - V_h^n - \rho_h(V^{n+1} - V^n)\|_h \|S_h^n\|_h.$$

Since the functions are piecewise polynomials, the equivalence of norms yields

$$\begin{aligned} & \left| \sum_{n=1}^{N-1} (V_h^{n+1} - V_h^n - \rho_h(V^{n+1} - V^n), S_h^n)_h^\varphi \right| \leq C \|\varphi\|_{L^\infty(\Omega)} \\ & \quad \times \left(\sum_{n=1}^{N-1} \tau \left\| \frac{1}{\tau} (I_h(v^{n+1} - v^n) - \rho_h(V^{n+1} - V^n)) \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{N-1} \tau \|S_h^n\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Then the regularity of v , the approximation properties of I_h and ρ_h and the boundedness of $S_{h,\tau}$ imply that

$$\lim_{(h,\tau) \rightarrow (0,0)} \left| \sum_{n=1}^{N-1} (V_h^{n+1} - V_h^n - \rho_h(V^{n+1} - V^n), S_h^n)_h^\varphi \right| = 0.$$

Similarly, it is easy to check from the convergence of $S_{h,\tau}$ that

$$- \lim_{(h,\tau) \rightarrow (0,0)} \sum_{n=1}^{N-1} \int_{\Omega} \varphi \rho_h(V^{n+1} - V^n) S_h^n = - \int_0^T \int_{\Omega} \varphi(\partial_t v) \bar{s}.$$

The treatment of the initial term is the same. Hence

$$\lim_{(h,\tau) \rightarrow (0,0)} \sum_{n=1}^N (S_h^n - S_h^{n-1}, V_h^n)_h^\varphi = - \int_0^T \int_{\Omega} \varphi(\partial_t v) \bar{s} - \int_{\Omega} \varphi s^0 v. \quad (7.59)$$

By combining (7.59), with (7.9) and (7.57), we readily see that the limit functions \bar{s} , \bar{p}_α and $p_{\alpha g}(\bar{s})$ satisfy the weak formulation (1.16). This proves Theorem 2.6.

8. Numerical validation. This section proposes a numerical validation of our algorithm with a two dimensional finite difference code. Details on the algorithm implemented are given. A problem with manufactured solutions is then considered to study the convergence properties of our algorithm.

8.1. Implementation of the model. To avoid dealing with nonlinear terms, we implement a modified version of the algorithm proposed in section 2.3. The main difference

consists of approximating the terms $S_w^{n+1,ij}$, $S_o^{n+1,ij}$ and P_o^{n+1} with first time order extrapolation. For each node $1 \leq i \leq M$, the unknowns $(S^{n+1,i}, P_w^{n+1,i})$ are computed as the solution of the following problem:

$$\begin{aligned} \frac{\tilde{m}_i}{\Delta t}(S^{n+1,i} - S^{n,i}) - \sum_{j \neq i, j \in N(i)} c_{ij} \eta_w(S_w^{*,n+1,ij})(P_w^{n+1,j} - P_w^{n+1,i}) \\ = m_i f_1^{n+1,i}, \quad 1 \leq i \leq M, \end{aligned} \quad (8.1)$$

$$\begin{aligned} -\frac{\tilde{m}_i}{\Delta t}(S^{n+1,i} - S^{n,i}) - \sum_{j \neq i, j \in N(i)} c_{ij} \eta_o(S_o^{*,n+1,ij})(P_w^{n+1,j} - P_w^{n+1,i}) \\ - \sum_{j \neq i, j \in N(i)} c_{ij} \eta_o(S_o^{*,n+1,ij})(p_c^{*,n+1,j} - p_c^{*,n+1,i}) = m_i f_2^{n+1,i}, \quad 1 \leq i \leq M, \end{aligned} \quad (8.2)$$

where the pressure P_o has been substituted with $P_w + p_c$ with respect to (2.34). The solution P_w is enforced to satisfy (2.35) a posteriori by subtracting its integral $\sum_{i=1}^M m_i P_w^i$ after solving the above problem. The terms $S_w^{*,n+1,ij}$ and $S_o^{*,n+1,ij}$ are approximated at time iteration n by setting them to $S_w^{n,ij}$ and $S_o^{n,ij}$. Eventually, the capillary pressure $p_c^{*,n+1}$ is approximated with a first order Taylor expansion with respect to the saturation S , it reads:

$$p_c^{*,n+1} = p_c^n + \left(\frac{\partial p_c}{\partial S} \right)^n (S^{n+1} - S^n). \quad (8.3)$$

We note that to facilitate the implementation of this algorithm in a two dimensional finite difference code, the source terms of the equations (2.32)-(2.33) have been replaced by functions denoted by f_1 and f_2 .

8.2. Numerical test with a manufactured solution. The numerical validation of the algorithm is done by approximating the analytical solutions defined by

$$P_w(t, x, y) = 2 + x^2 y - y^2 + x^2 \sin(t + y), \quad (8.4)$$

$$S(t, x, y) = 0.2(2 + 2xy + \cos(t + x)), \quad (8.5)$$

on the computational domain $\Omega = [0, 1]^2$. Dirichlet boundary conditions are applied on $\partial\Omega$ on both unknowns P_w and S . The initial conditions of the problem satisfy (8.4)-(8.5). The porosity of the domain is set to:

$$\phi(t, x, y) = 0.2(1 + xy). \quad (8.6)$$

The mobilities η_w and η_o , introduced in section 1.1, are defined as follows:

$$\eta_w = 4S^2, \quad \eta_o = 0.4(1 - S)^2. \quad (8.7)$$

The capillary pressure is based on the Brooks-Corey model, it reads:

$$P_c = \begin{cases} AS^{-0.5} & \text{if } S > 0.05, \\ A(1.5 - 10S) \times 0.05^{-0.5} & \text{otherwise.} \end{cases} \quad (8.8)$$

where A is a constant set to 50. The term sources f_1 and f_2 are computed accordingly. The convergence tests are performed on a set of six uniform grids with respective mesh size $h \in \{0.2, 0.1, 0.05, 0.025, 0.0125, 0.00625\}$. The convergence properties are evaluated by using a time step τ set to the mesh size h with a final time $T = 1$. As the time derivatives and the saturations $S_w^{n+1,ij}, S_o^{n+1,ij}$ are computed with first order time approximation, we expect the convergence rate in the L^2 norm to be of order one. The results of the convergence

L^2 -norm of error		Water pressure P_w		Water saturation S	
h	n_{df}	Error	Rate	Error	Rate
0.2	25	8.50E-3	-	4.21E-3	-
0.1	100	4.15E-3	1.03	2.30E-3	0.87
0.05	400	2.08E-3	1.00	1.14E-4	1.01
0.025	1600	1.04E-3	1.00	5.57E-4	1.03
0.0125	6400	5.23E-4	0.99	2.75E-4	1.02

TABLE 8.1

Results of convergence tests where the mesh size is denoted by h and the number of degrees of freedom per unknown by n_{df} . The time step τ is set to h and errors are computed at final time $T = 1$.

tests are presented in Table 8.1. The theoretical order of convergence, equal to one, is recovered for both unknowns which confirms the correct behavior of the algorithm.

9. Conclusions. This paper formulates a \mathbb{P}_1 finite element method to solve the immiscible two-phase flow problem in porous media. The unknowns are the phase pressure and saturation, which are the preferred unknowns in industrial reservoir simulators. The numerical method employs mass lumping for integration and an upwind flux technique. As a consequence, the saturation is shown to be bounded between zero and one. The discrete approximations of pressure and saturation converge to the weak solution as the time step and mesh sizes tend to zero.

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