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► **To cite this version:**

| Alexandru Iosif. The scalar orbit of the virtual Kähler-Ricci flow. 2020. hal-02453356

HAL Id: hal-02453356

<https://hal.science/hal-02453356v1>

Preprint submitted on 23 Jan 2020

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THE SCALAR ORBIT OF THE VIRTUAL KÄHLER-RICCI FLOW

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ABSTRACT. We introduce the virtual Kähler-Ricci flow and propose it as a conjectural framework for representing meromorphic functions in one complex variable.

Let (M, g_0) be a Kähler manifold of finite dimension and let Ric_0 be its Ricci curvature tensor. Let P be a polynomial in the variables $g(s)$ and $\text{Ric}(s)$ and coefficients in the field \mathbb{M} of meromorphic functions in one complex variable s . We would like to generalize the notion of Kähler-Ricci flow as follows.

Definition 1. The *virtual Kähler-Ricci flow* $\mathfrak{R}_{P;(M,g_0)}$ of (M, g_0) is

$$\{(M, g(t)) : t \in \mathbb{C}, \frac{dg}{ds} = P(g(s), \text{Ric}(s)), g(0) = g_0\}.$$

By setting $P = -2\text{Ric}(s)$, one obtains the following result due to Bando (e.g. [1, Section 1.1.3]).

Theorem 2. *Every element of $\mathfrak{R}_{-2\text{Ric}(s);(M,g_0)}$ is a Kähler manifold.*

Proof. If $P = -2\text{Ric}(s)$, then the virtual Kähler-Ricci Flow is a Kähler-Ricci flow. But the Kähler condition is preserved under the Kähler-Ricci flow. \square

Let $(M, g(t)) \in \mathfrak{R}_{P;(M,g_0)}$. If $g(t)$ is a positive definite Hermitian matrix at each point of M , then $\text{Ric}(t)$ is the Ricci curvature of $g(t)$. This motivates the following generalization.

Definition 3. The *virtual Ricci curvature* is $\text{Ric}(t)$ and the *mean virtual scalar curvature* $\sigma(t)$ is the mean of $\text{tr}_{g(t)}\text{Ric}(t)$. The *Ricci orbit* of (M, g_0) is $\text{Ric}(s)$ and its *scalar orbit* is $\sigma(s)$.

The following problems are the motivation for the above generalizations.

Problem 4. *Given a meromorphic function $\mu(s)$, does there exist a Kähler manifold (M, g_0) and a polynomial P such that $\mu(s)$ is the scalar orbit of $\mathfrak{R}_{P;(M,g_0)}$ and, for every zero t of $\mu(t)$, $g(t)$ is a real, positive definite, symmetric matrix?*

If the answer to the previous problem were yes, then we could associate to each zero of $\mu(s)$ a (flat) Kähler manifold.

Problem 5. *Given a Kähler manifold (M, g_0) , find a P , or prove that there is none, such that $(M, g(s))$ is Kähler if and only if $\sigma(s) = 0$.*

It would be interesting to restrict this study to polynomials P such that the virtual Kähler-Ricci Flow has a unique solution. This would make the notion of orbit in the definition above a nicer object. A natural question that arises is the following.

Question 6. *What is the algebraic structure of the set of polynomials P which lead to a virtual Kähler-Ricci Flow with a unique solution?*

Note that a generalization of the Chern-Gauss-Bonnet theorem (or, more generally, of the Atiyah-Singer Theorem) is needed. It should be more general in the sense that it should cover the case of degenerated manifolds (that is, which fail to have a real curvature, or, in particular, a Hermitian metric).

REFERENCES

1. Jian Song and Ben Weinkove, *An introduction to the Kähler-Ricci flow*, Springer, 2013, pp. 89–188.

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