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Finite series expressions to evaluate the beam shape coefficients of a Laguerre-Gauss beam focused by a lens in an on-axis configuration.

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Abstract

Pursuing a study concerning the evaluation of Beam Shape Coefficients (BSCs) of Laguerre-Gauss beams, after having dealt with the case of beams freely propagating, the present paper establishes finite series expressions to evaluate the BSCs of Laguerre-Gauss beams focused by a lens, which have the known advantage of speeding up computations with respect to the use of numerical quadrature techniques and which may deal with the limitations of localized approximations in the case of beams exhibiting topological charges and/or axicon angles.

Keywords: Generalized Lorenz-Mie theories; Extended Boundary Condition Method; structured beams; T-matrix; beam shape coefficients; finite series; Laguerre-Gauss beams.

1 Introduction.

When dealing with T-matrix methods for the scattering of light and other electromagnetic radiation by particles, such as analytical Generalized Lorenz-Mie Theories (GLMTs) for particles having a sufficient degree of symmetry

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to allow for the use of a method of separation of variables [1], [2], [3], or semi-analytical Extended Boundary Condition Method (EBCM) for arbitrary shaped particles [4], [5], in particular in the case of structured beams as reviewed in [6], electromagnetic fields may be expanded over a set of vector wave functions [7]. In spherical coordinates, expansions are carried out over a set of Vector Spherical Wave Functions (VSWFs) and expansion coefficients are expressed in terms of coefficients known as Beam Shape Coefficients (BSCs) usually denoted as $g_{n,TM}^{m}$ and $g_{n,TE}^{m}$ (TM: Transverse Magnetic; TE: Transverse Electric).

The original method to evaluate the BSCs of electromagnetic beams is by using quadratures, e.g. [8]. In the case of Laguerre-Gauss beams, either freely propagating or focused by a lens, there is little chance to analytically solve the quadratures to obtain closed form expressions. Hence, quadratures to the evaluation of BSCs of Laguerre-Gauss beams have to be carried out numerically, which is a time-consuming process. To circumvent this difficulty, we may think of using a localized approximation (with several variants) which may speed up the computations by several orders of magnitude as reviewed in [9], see also [10] for an up-dated terminology, and [11], [12] for complements. Unfortunately, it has recently been demonstrated that any existing localized approximation has a limited domain of validity when dealing with helical beams, i.e. having a topological charge [13], as is the case for Laguerre-Gauss beams whether they are freely propagating [14], [15] or focused by a lens [16].

In the case of Laguerre-Gauss beams focused by a lens, there is another limitation due to the fact that field expressions contain axicon angles which are not involved in the case of Laguerre-Gauss beams freely propagating. The fact that the occurrence of axicon angles implies a limited domain of validity of any existing localized approximation has been well documented in a series of papers, namely [17], [18], [19], [20], [21].

The arsenal of methods to evaluate BSCs however contains another technique to speed up the evaluation of BSCs, namely the use of finite series which is a rigorous analytical method [22], [23]. Up to recently, finite series expressions were known only in the case of Gaussian beams, and the method has been given up due to the success encountered by the use of localized approximations. In 2009, one of us [24] wrote that the finite series technique seemingly now possessed only a historical interest. Indeed, after the original papers [22], [23], the use of finite series has been rather scarce and, as far as we know, limited to three papers, two in an acoustical context [25], [26], and one in an electromagnetic context [27].

The time-consuming character of numerical evaluations when using the quadrature method, and the limitations of localized approximations in the case of beams exhibiting axicon angles and/or a topological charge, generated a renewal of interest in the use of finite series to the evaluation of BSCs. The case of Laguerre-Gauss beams freely propagating, exhibiting a topological charge (but without any axicon angle), has been considered in two papers, namely [28] establishing the finite series expressions required to numerically implement the method and [15] dealing with a numerical comparison of three remodeling methods (quadratures, localized approximations, finite series) in the case of

paraxial Laguerre-Gauss beams freely propagating (which do not exactly satisfy Maxwell's equations due to their paraxial character). It was concluded that, notwithstanding a disadvantage of the finite series method, namely its lack of flexibility insofar as it requires a fairly heavy analytical work before numerical implementation, it is to be favoured with respect to the other methods.

This last result implies that, after having examined the case of Laguerre-Gauss beams freely propagating, it is worthwhile to deal with the case of Laguerre-Gauss beams focused by a lens. The present paper is therefore devoted to the building of finite series expressions in the case of Laguerre-Gauss beams focused by a lens which, in contrast with the case of beams freely propagating, exhibit both a topological charge and axicon angles.

Up to now, we have expressed our motivation to the study of the finite series method to evaluate the BSCs of Laguerre-Gauss beams. This motivation is reinforced by the many applications of such beams, in particular due to the fact that they possess a helical wavefront allowing for the transfer of angular momentum to an illuminated object, e.g. Padgett and Allen [29] who discussed orbital angular momentum associated with helical wavefronts and applications, Garbin *et al.* [30] who experimentally and numerically studied the scattering of highly focused Laguerre-Gauss beams by dielectric and metal spheres, O'Holleran *et al.* [31] who provided a 3D view of optical vortices associated with Laguerre-Gauss modes, Friese *et al.* [32] who studied the transfer of optical angular momentum to trapped absorbing particles, a study relevant to the field of optical information. Laguerre-Gauss beams were discussed as well in the framework of a review devoted to optical tweezers by Molloy and Padgett [33]. Other applications concerned quantum information and the entanglement of the polarization of a single photon with its orbital angular momentum by Nagali *et al.* [34], entanglement again but involving many orthogonal quantum states rather than only two states by Mair *et al.* [35], object identification by measuring the joint orbital angular momentum spectrum of two-photon states by Uribe-Patarroyo *et al.* [36], optical traps to study vortices in Bose-Einstein condensates by Tempere *et al.* [37], optical communication systems in atmospheric turbulence by Malik *et al.* [38], use of vortex masks to observe dim exoplanets by Foo *et al.* [39], to provide a few examples without pretending to exhaustiveness.

The paper is organized as follows. Section 2 establishes the field expressions under a formulation which is tailored to the use of the finite series method. Section 3 recalls a background concerning the method to be used to establish finite series expressions. Section 4 deals with TM-BSCs and Section 5 deals with TE–BSCs. Handling finite series expressions requires to deal with recurrence equations which are examined in Section 6. Section 7 is a conclusion. The reader who would like to omit the details could simply use Eqs.96, 102, 131, 140 for TM-BSCs and Eqs.165, 176, 200, 207 for TE-BSCs.

2 Field expressions of Laguerre-Gauss beams focused by a lens.

Expressions for Laguerre-Gauss beams focused by a lens are available from Van de Nes *et al.* [40], see also van de Nes and Török [41] and Török and Munro [42]. In the present paper, we rely on [40] who expressed the field components in terms of cylindrical coordinates (ρ, φ, z) using a time-dependence of the form $\exp(-i\omega t)$ which is opposite to the one usually chosen in GLMT (we shall have later to deal with this issue). The field expressions are claimed to be for nonparaxial beams exactly satisfying Maxwell's equations, e.g. [40], page 682, line 6 after Eq.(16).

2.1 Electric field.

The electric field may be written in terms of cylindrical coordinates as [40], [41]:

$$\mathbf{E} = \frac{1}{2}\mathbf{E}_{pl,0}(\alpha,\beta) + \frac{1}{4}\mathbf{E}_{pl,-2}(\alpha+i\beta, i\alpha-\beta) + \frac{1}{4}\mathbf{E}_{pl,2}(\alpha-i\beta, -i\alpha-\beta) \quad (1)$$

in which:

$$\begin{aligned}\mathbf{E}_{pl,j}(\alpha,\beta) &= \int_0^{kNA} E_{pl,j}(\eta) e^{i(l+j)\varphi + ik_z z}\{(\alpha\mathbf{x}+\beta\mathbf{y})J_{l+j}(\eta\rho) \\ &+\frac{\eta}{2k_z}\mathbf{z}[(i\alpha-\beta)e^{-i\varphi}J_{l+j-1}(\eta\rho) \\ &-(i\alpha+\beta)e^{i\varphi}J_{l+j+1}(\eta\rho)]\}d\eta\end{aligned} \quad (2)$$

with:

$$E_{pl,j}(\eta) = \frac{\eta}{\sqrt{kk_z}}[1+(1-|j|)\frac{k_z}{k}]u_{pl}a(\eta)L_p^l[b(\eta)]\exp[-c(\eta)] \quad (3)$$

and:

$$a(\eta) = (\eta\frac{k_S}{k}R_f)^{|l|} \quad (4)$$

$$b(\eta) = \eta^2 \frac{k_S^2}{k^2} R_f^2 \tag{5}$$

$$c(\eta) = \eta^2 \frac{k_S^2}{2k^2} R_f^2 \tag{6}$$

$$u_{pl} = (-1)^p k_S \sqrt{\frac{p!}{\pi(p+|l|)!}} \tag{7}$$

$$k_z = \sqrt{k^2 - \eta^2} \tag{8}$$

In the quadrature of Eq.2, Van de Nes *et al.* [40] denoted the variable of integration as k_r having the meaning of a radial wavenumber. Insofar as it is actually a dummy variable, we preferred to denote it as η. The latin symbols **x**, **y** and **z** denote unit vectors along the Cartesian axis. The greek symbols α and β denote states of polarization, along **x** and **y** respectively. $J_n(.)$ are Bessel functions of the first kind and $L_p^l(.)$ are generalized Laguerre polynomials defined as follows [43], [44]:

$$L_\mu^\nu(x) = \frac{1}{\mu!} \sum_{i=0}^{\mu} \frac{\mu!}{i!} \begin{pmatrix} \nu+\mu \\ \mu-i \end{pmatrix} (-x)^i \tag{9}$$

in which () denotes a binomial coefficient. The "usual" Laguerre polynomials $L_\mu(x)$ are $L_\mu^0(x)$. The associated Laguerre polynomials can be evaluated recursively from the first two polynomials as follows:

$$L_0^l(x) = 1 \tag{10}$$

$$L_1^l(x) = 1 + l - x \tag{11}$$

$$L_p^l(x) = (A_p - \frac{x}{p}) L_{p-1}^l(x) - B_p L_{p-2}^l(x) \tag{12}$$

in which:

$$A_p = \frac{2p - 1 + l}{p} \tag{13}$$

$$B_p = \frac{p-1+l}{p} \tag{14}$$

Furthermore, according to the authors k_S is given by $\sqrt{k/z_0}$, in which z_0 is the Rayleigh range. This Rayleigh range may be written as $z_0 = kw_0^2/2$, e.g. page 2 of [45] to be compared with Eq.(4.38), page 102, of Gouesbet and Gréhan [3]. Introducing the beam shape factor $s = 1/(kw_0)$, also called the confinement factor, e.g. [3], we obtain $k_S = \sqrt{2}/w_0$. Finally, R_f is the focal length and NA is the numerical aperture of the imaging system.

To proceed further, we now need Cartesian components. Introducing:

$$E_p^l(\eta) = \frac{\eta}{2\sqrt{kk_z}} u_{pl} a(\eta) L_p^l[b(\eta)] \exp[-c(\eta)] e^{il\varphi} e^{ik_z z} \tag{15}$$

the Cartesian components are found to read as:

$$\begin{aligned} E_x &= \int_0^{kNA} E_p^l(\eta)[\alpha(1+\frac{k_z}{k})J_l(\eta\rho) + \frac{\alpha+i\beta}{2}(1-\frac{k_z}{k})e^{-2i\varphi}J_{l-2}(\eta\rho) \\ &\quad + \frac{\alpha-i\beta}{2}(1-\frac{k_z}{k})e^{2i\varphi}J_{l+2}(\eta\rho)]d\eta \end{aligned} \tag{16}$$

$$\begin{aligned} E_y &= \int_0^{kNA} E_p^l(\eta)[\beta(1+\frac{k_z}{k})J_l(\eta\rho) + \frac{i\alpha-\beta}{2}(1-\frac{k_z}{k})e^{-2i\varphi}J_{l-2}(\eta\rho) \\ &\quad + \frac{-i\alpha-\beta}{2}(1-\frac{k_z}{k})e^{2i\varphi}J_{l+2}(\eta\rho)]d\eta \end{aligned} \tag{17}$$

$$E_z = \int_0^{kNA} E_p^l(\eta)\frac{\eta}{k}[(i\alpha-\beta)e^{-i\varphi}J_{l-1}(\eta\rho) - (i\alpha+\beta)e^{i\varphi}J_{l+1}(\eta\rho)]d\eta \tag{18}$$

We now express these results in spherical coordinates (r, θ, φ). For this, we introduce the fact that:

$$\frac{k_S}{k} = \frac{\sqrt{2}}{kw_0} = \sqrt{2}s \tag{19}$$

so that we now have:

$$a(\eta) = (\eta \frac{k_S}{k} R_f)^{|l|} = (\sqrt{2}\eta s R_f)^{|l|} \tag{20}$$

$$b(\eta) = \eta^2 \frac{k_S^2}{k^2} R_f^2 = 2\eta^2 s^2 R_f^2 \tag{21}$$

$$c(\eta) = \eta^2 \frac{k_S^2}{2k^2} R_f^2 = \eta^2 s^2 R_f^2 \tag{22}$$

Let us note that Eq.3 for $E_{pl,j}(\eta)$ used in [40] is different from the one used in Eq.(3) of [41].The ratio of $E_{pl,j}(\eta)$ of Eq.3 over the one of Eq.(3) of [41] is found to be equal to $i^{l-1} R_f / \sqrt{1+\delta_{0l}}$ and can be absorbed in E_0. Then, to get rid of irrelevant constants, we introduce an electric strength E_0 according to:

$$E_0 = \frac{1}{\sqrt{2k}} (-1)^p \sqrt{\frac{p!}{\pi (p+|l|)!}} k s (\sqrt{2} s R_f)^{|l|} \tag{23}$$

and recall, for further use, that $H_0/E_0 = \sqrt{\varepsilon/\mu}$. We then obtain the electric field components in terms of spherical coordinates according to:

$$E_p^l(\eta) = E_0 \frac{\eta^{|l|+1}}{\sqrt{\sqrt{k^2-\eta^2}}} L_p^l[b(\eta)] \exp[-c(\eta)] e^{il\varphi} e^{i\sqrt{k^2-\eta^2} r \cos\theta} \tag{24}$$

$$\begin{aligned} E_x \quad = \quad & \int_0^{kNA} E_p^l(\eta) [\alpha (1 + \frac{\sqrt{k^2-\eta^2}}{k}) J_l(\eta r \sin\theta) \\ & + \frac{\alpha + i\beta}{2} (1 - \frac{\sqrt{k^2-\eta^2}}{k}) e^{-2i\varphi} J_{l-2}(\eta r \sin\theta) \\ & + \frac{\alpha - i\beta}{2} (1 - \frac{\sqrt{k^2-\eta^2}}{k}) e^{2i\varphi} J_{l+2}(\eta r \sin\theta)] d\eta \end{aligned} \tag{25}$$

$$
\begin{aligned}
E_y &= \int_0^{kNA} E_p^l(\eta)[\beta(1+\frac{\sqrt{k^2-\eta^2}}{k})J_l(\eta r\sin\theta) \\
&+\frac{i\alpha-\beta}{2}(1-\frac{\sqrt{k^2-\eta^2}}{k})e^{-2i\varphi}J_{l-2}(\eta r\sin\theta) \\
&+\frac{-i\alpha-\beta}{2}(1-\frac{\sqrt{k^2-\eta^2}}{k})e^{2i\varphi}J_{l+2}(\eta r\sin\theta)]d\eta
\end{aligned} \tag{26}
$$

$$
E_z = \int_0^{kNA} E_p^l(\eta)\frac{\eta}{k}[(i\alpha-\beta)e^{-i\varphi}J_{l-1}(\eta r\sin\theta)-(i\alpha+\beta)e^{i\varphi}J_{l+1}(\eta r\sin\theta)]d\eta \tag{27}
$$

2.2 Magnetic field.

Similarly, the magnetic field is expressed as:

$$
\mathbf{H} = \frac{1}{2}\mathbf{H}_{pl,0}(\alpha,\beta)+\frac{1}{4}\mathbf{H}_{pl,-2}(\alpha+i\beta,i\alpha-\beta)+\frac{1}{4}\mathbf{H}_{pl,2}(\alpha-i\beta,-i\alpha-\beta) \tag{28}
$$

in which:

$$
\begin{aligned}
\mathbf{H}_{pl,j}(\alpha,\beta) &= \sqrt{\frac{\varepsilon}{\mu}}\int_0^{kNA}\frac{E_{pl,j}(\eta)}{2kk_z}e^{i(l+j)\varphi+ik_z z} \\
&\{(-\beta\mathbf{x}+\alpha\mathbf{y})(2k^2-\eta^2)J_{l+j}(\eta\rho) \\
&+\frac{\eta^2}{2}[(\mathbf{x}+i\mathbf{y})(i\alpha-\beta)e^{-2i\varphi}J_{l+j-2}(\eta\rho) \\
&-(\mathbf{x}-\mathbf{i}\mathbf{y})(i\alpha+\beta)e^{2i\varphi}J_{l+j+2}(\eta\rho)] \\
&-\eta k_z\mathbf{z}[(\alpha+i\beta)J_{l+j-1}e^{-i\varphi}(\eta\rho)+(\alpha-i\beta)e^{i\varphi}J_{l+j+1}(\eta\rho)]\}d\eta
\end{aligned} \tag{29}
$$

in which we used the same notations than for the electric field, including Eq.3 for $E_{pl,j}(\eta)$. Let us also introduce :

$$
H_p^l(\eta) = \frac{\eta^{|l|+1}}{k\sqrt{2}(k^2-\eta^2)^{3/4}}L_p^l[b(\eta)]\exp[-c(\eta)] \tag{30}
$$

From Eqs.28 and 29 we then establish :

$$
\begin{aligned}
[\mathbf{H}_{pl,0}(\alpha,\beta)]_x &= \int_0^{kNA} H_0 H_p^l(\eta) e^{il\varphi} e^{i\sqrt{k^2-\eta^2}z} (1+\frac{\sqrt{k^2-\eta^2}}{k}) \\
&\{-\beta(2k^2-\eta^2) J_l(\eta\rho) \\
&+\frac{\eta^2}{2}[(i\alpha-\beta)e^{-2i\varphi} J_{l-2}(\eta\rho) - (i\alpha+\beta)e^{2i\varphi} J_{l+2}(\eta\rho)]\}d\eta
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
[\mathbf{H}_{pl,-2}(\alpha,\beta)]_x &= \int_0^{kNA} H_0 H_p^l(\eta) e^{il\varphi} e^{i\sqrt{k^2-\eta^2}z} (1-\frac{\sqrt{k^2-\eta^2}}{k}) \\
&\{-\beta(2k^2-\eta^2) e^{-2i\varphi} J_{l-2}(\eta\rho) \\
&+\frac{\eta^2}{2}[(i\alpha-\beta)e^{-4i\varphi} J_{l-4}(\eta\rho) - (i\alpha+\beta) J_l(\eta\rho)]\}d\eta
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
[\mathbf{H}_{pl,2}(\alpha,\beta)]_x &= \int_0^{kNA} H_0 H_p^l(\eta) e^{il\varphi} e^{i\sqrt{k^2-\eta^2}z} (1-\frac{\sqrt{k^2-\eta^2}}{k}) \\
&\{-\beta(2k^2-\eta^2) e^{2i\varphi} J_{l+2}(\eta\rho) \\
&+\frac{\eta^2}{2}[(i\alpha-\beta) J_l(\eta\rho) - (i\alpha+\beta)e^{4i\varphi} J_{l+4}(\eta\rho)]\}d\eta
\end{aligned}
\tag{33}
$$

from which we obtain $[\mathbf{H}_{pl,-2}(\alpha+i\beta, i\alpha-\beta)]_x$, $[\mathbf{H}_{pl,2}(\alpha-i\beta,-i\alpha-\beta)]_x$ and, eventually:

$$
\begin{aligned}
H_x &= \int_0^{kNA} H_0 H_p^l(\eta) e^{il\varphi} e^{i\sqrt{k^2-\eta^2}z} \\
&\{\frac{1}{2} J_l(\eta\rho)\beta[-(1+\frac{\sqrt{k^2-\eta^2}}{k})(2k^2-\eta^2) + (1-\frac{\sqrt{k^2-\eta^2}}{k})\eta^2] \\
&+\frac{1}{4} e^{-2i\varphi} J_{l-2}(\eta\rho)(i\alpha-\beta)[\eta^2(1+\frac{\sqrt{k^2-\eta^2}}{k}) - (1-\frac{\sqrt{k^2-\eta^2}}{k})(2k^2-\eta^2)] \\
&-\frac{1}{4} e^{2i\varphi} J_{l+2}(\eta\rho)(i\alpha+\beta)[\eta^2(1+\frac{\sqrt{k^2-\eta^2}}{k}) - (1-\frac{\sqrt{k^2-\eta^2}}{k})(2k^2-\eta^2)]\}d\eta
\end{aligned}
\tag{34}
$$

For the y-component, we first write:

$$
\begin{aligned}
[\mathbf{H}_{pl,0}(\alpha,\beta)]_y &= \int_0^{kNA} H_0 H_p^l(\eta) e^{il\varphi} e^{i\sqrt{k^2-\eta^2}z} (1+\frac{\sqrt{k^2-\eta^2}}{k}) \qquad (35)\\
&\{\alpha(2k^2-\eta^2) J_l(\eta\rho)\\
&+\frac{i\eta^2}{2}[(i\alpha-\beta)e^{-2i\varphi}J_{l-2}(\eta\rho)+(i\alpha+\beta)e^{2i\varphi}J_{l+2}(\eta\rho)]\}d\eta
\end{aligned}
$$

$$
\begin{aligned}
[\mathbf{H}_{pl,-2}(\alpha,\beta)]_y &= \int_0^{kNA} H_0 H_p^l(\eta) e^{il\varphi} e^{i\sqrt{k^2-\eta^2}z} (1-\frac{\sqrt{k^2-\eta^2}}{k}) \qquad (36)\\
&\{\alpha(2k^2-\eta^2) e^{-2i\varphi} J_{l-2}(\eta\rho)\\
&+\frac{i\eta^2}{2}[(i\alpha-\beta)e^{-4i\varphi}J_{l-4}(\eta\rho)+(i\alpha+\beta)J_l(\eta\rho)]\}d\eta
\end{aligned}
$$

$$
\begin{aligned}
[\mathbf{H}_{pl,2}(\alpha,\beta)]_y &= \int_0^{kNA} H_0 H_p^l(\eta) e^{il\varphi} e^{i\sqrt{k^2-\eta^2}z} (1-\frac{\sqrt{k^2-\eta^2}}{k}) \qquad (37)\\
&\{\alpha(2k^2-\eta^2) e^{2i\varphi} J_{l+2}(\eta\rho)\\
&+\frac{i\eta^2}{2}[(i\alpha-\beta)J_l(\eta\rho)+(i\alpha+\beta)e^{4i\varphi}J_{l+4}(\eta\rho)]\}d\eta
\end{aligned}
$$

from which we obtain $[\mathbf{H}_{pl,-2}(\alpha+i\beta, i\alpha-\beta)]_y$, $[\mathbf{H}_{pl,2}(\alpha-i\beta,-i\alpha-\beta)]_y$ and, eventually:

$$
\begin{aligned}
H_y &= \int_0^{kNA} H_0 H_p^l(\eta) e^{il\varphi} e^{i\sqrt{k^2-\eta^2}z} \qquad (38)\\
&\{\frac{1}{2}J_l(\eta\rho)\alpha[(1+\frac{\sqrt{k^2-\eta^2}}{k})(2k^2-\eta^2)-(1-\frac{\sqrt{k^2-\eta^2}}{k})\eta^2]\\
&+\frac{i}{4}e^{-2i\varphi}J_{l-2}(\eta\rho)(i\alpha-\beta)[\eta^2(1+\frac{\sqrt{k^2-\eta^2}}{k})-(1-\frac{\sqrt{k^2-\eta^2}}{k})(2k^2-\eta^2)]\\
&+\frac{i}{4}e^{2i\varphi}J_{l+2}(\eta\rho)(i\alpha+\beta)[\eta^2(1+\frac{\sqrt{k^2-\eta^2}}{k})-(1-\frac{\sqrt{k^2-\eta^2}}{k})(2k^2-\eta^2)]\}d\eta
\end{aligned}
$$

Finally, for the z-component:

$$[\mathbf{H}_{pl,0}(\alpha,\beta)]_z = \int_0^{kNA} H_0 \frac{\eta^{|l|+1}}{k\sqrt{2}} L_p^l[b(\eta)] \exp[-c(\eta)] e^{il\varphi} e^{i\sqrt{k^2-\eta^2}z} \frac{-1}{\sqrt{\sqrt{k^2-z^2}}} \tag{39}$$
$$(1+\frac{\sqrt{k^2-\eta^2}}{k})\eta[(\alpha+i\beta)e^{-i\varphi}J_{l-1}(\eta\rho)+(\alpha-i\beta)e^{i\varphi}J_{l+1}(\eta\rho)]\}d\eta$$

$$[\mathbf{H}_{pl,-2}(\alpha,\beta)]_z = \int_0^{kNA} H_0 \frac{\eta^{|l|+1}}{k\sqrt{2}} L_p^l[b(\eta)] \exp[-c(\eta)] e^{il\varphi} e^{i\sqrt{k^2-\eta^2}z} \frac{-1}{\sqrt{\sqrt{k^2-z^2}}} \tag{40}$$
$$(1-\frac{\sqrt{k^2-\eta^2}}{k})\eta[(\alpha+i\beta)e^{-3i\varphi}J_{l-3}(\eta\rho)+(\alpha-i\beta)e^{-i\varphi}J_{l-1}(\eta\rho)]\}d\eta$$

$$[\mathbf{H}_{pl,2}(\alpha,\beta)]_z = \int_0^{kNA} H_0 \frac{\eta^{|l|+1}}{k\sqrt{2}} L_p^l[b(\eta)] \exp[-c(\eta)] e^{il\varphi} e^{i\sqrt{k^2-\eta^2}z} \frac{-1}{\sqrt{\sqrt{k^2-z^2}}} \tag{41}$$
$$(1-\frac{\sqrt{k^2-\eta^2}}{k})\eta[(\alpha+i\beta)e^{i\varphi}J_{l+1}(\eta\rho)+(\alpha-i\beta)e^{3i\varphi}J_{l+3}(\eta\rho)]\}d\eta$$

from which we obtain $[\mathbf{H}_{pl,-2}(\alpha+i\beta, i\alpha-\beta)]_z$, $[\mathbf{H}_{pl,2}(\alpha-i\beta,-i\alpha-\beta)]_z$ and:

$$H_z = \int_0^{kNA} H_0 \frac{\eta^{|l|+2}}{k\sqrt{2}} L_p^l[b(\eta)] \exp[-c(\eta)] e^{il\varphi} e^{i\sqrt{k^2-\eta^2}z} \frac{-1}{\sqrt{\sqrt{k^2-z^2}}} \tag{42}$$
$$\frac{1}{2}\{(1+\frac{\sqrt{k^2-\eta^2}}{k})[(\alpha+i\beta)e^{-i\varphi}J_{l-1}(\eta\rho)+(\alpha-i\beta)e^{i\varphi}J_{l+1}(\eta\rho)]$$
$$+(1-\frac{\sqrt{k^2-\eta^2}}{k})[(\alpha+i\beta)e^{-i\varphi}J_{l-1}(\eta\rho)+(\alpha-i\beta)e^{i\varphi}J_{l+1}(\eta\rho)]\}d\eta$$

which simplifies to:

$$H_z = \int_0^{kNA} H_0 \frac{\eta^{|l|+2}}{k\sqrt{2}} L_p^l[b(\eta)] \exp[-c(\eta)] e^{il\varphi} e^{i\sqrt{k^2-\eta^2}z} \frac{-1}{\sqrt{\sqrt{k^2-z^2}}} \tag{43}$$
$$[(\alpha+i\beta)e^{-i\varphi}J_{l-1}(\eta\rho)+(\alpha-i\beta)e^{i\varphi}J_{l+1}(\eta\rho)]d\eta$$

3 Neumann expansion theorem (NET) and NET-procedure.

In this section, we express the Neumann expansion theorem (NET-theorem) and the NET-procedure to be used to derive the finite series expressions to the evaluations of BSCs.

3.1 Neumann Expansion Theorem (NET).

Although the way to use the NET, relying on Neumann or Bessel function expansions, has already been published in the literature, e.g. [3], pp. 121-124, [28], and references therein, it is concisely reproduced here for the convenience of the reader. It starts with a result published by Watson [46], pp. 524-525. Let us consider an equation of the form:

$$x^{1/2}g(x) = \sum_{n=0}^{\infty} c_n J_{n+1/2}(x) \tag{44}$$

in which $J_{n+1/2}(.)$ are classical half-order Bessel functions (with x not to be confused with the Cartesian coordinate) and $g(x)$ is a function (not to be confused with a BSC, which also uses the symbol g). The Maclaurin expansion of the function $g(x)$ reads as:

$$g(x) = \sum_{n=0}^{\infty} b_n x^n \tag{45}$$

Then the NET states that the coefficients c_n are given by:

$$c_n = (n + \frac{1}{2}) \sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m} \frac{\Gamma(\frac{1}{2} + n - m)}{m!} b_{n-2m} \tag{46}$$

in which Γ is the celebrated Gamma function satisfying $\Gamma(n) = (n-1)!$ and $\Gamma(z+1) = z\Gamma(z)$. We then have two ways of expressing the coefficients c_n, either from the Bessel function expansion of Eq.44 or from the Maclaurin expansion of Eq.45, using Eq.46. Equating the two resulting expressions will allow one to establish finite series expressions for the BSCs. This is done using what may be called the NET-procedure.

3.2 The NET-procedure.

Before explaining the NET-procedure, we have to introduce what we call the P- and N-conventions, "P" standing for "Positive" and "N" standing for "Negative". The P-convention and the N-convention assume that we deal with a time-dependence of the form $\exp(i\omega t)$ and $\exp(-i\omega t)$, respectively. The letters P and N may be used between parentheses to specify which convention is to be used. For instance the electromagnetic fields derived in Section 2 may then be more explicitly denoted as $E(N)$ and $H(N)$. Conversely, the NET-procedure described in the present subsection relies on the use of the P-convention which is the usual convention used in GLMT.

The procedure then starts with the expressions of the radial electric E_r and magnetic H_r field components in spherical coordinates $(r,\, \theta,\, \varphi)$ which may be expressed as, e.g. Eqs.(3.10) and (3.19) of [3]:

$$\begin{pmatrix} E_r(P) \\ H_r(P) \end{pmatrix} = \begin{pmatrix} E_0(P) \\ H_0(P) \end{pmatrix} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} c_n^{pw} \begin{pmatrix} g_{n,TM}^m(P) \\ g_{n,TE}^m(P) \end{pmatrix} \frac{n(n+1)}{r} \Psi_n^{(1)}(kr) P_n^{|m|}(\cos\theta) \exp(im\varphi) \tag{47}$$

in which E_0, H_0 are field strengths, $g_{n,TM}^m(P)$ and $g_{n,TE}^m(P)$ are the BSCs to be evaluated within the framework of the NET-procedure, k is the wavenumber, $\Psi_n^{(1)}(.)$ denote the first-order spherical Bessel functions also denoted as $j_n(.)$, and $P_n^{|m|}(.)$ are associated Legendre functions defined by using Hobson's notation [47]. Furthermore c_n^{pw} (with pw standing for "plane wave") denotes expansion coefficients which appear in the Bromwich formulation of the classical Lorenz-Mie theory [48] according to:

$$c_n^{pw} = \frac{(-i)^n}{ik} \frac{2n+1}{n(n+1)} \tag{48}$$

Afterward, we discard the φ-dependency by using the orthogonality relation:

$$\int_0^{2\pi} \exp[i(m-m')\varphi] d\varphi = 2\pi\delta_{mm'} \tag{49}$$

to obtain:

$$\int_0^{2\pi} \left(\begin{array}{c} E_r(P) \\ H_r(P) \end{array} \right) \exp(-im\varphi) d\varphi \tag{50}$$

$$= \quad 2\pi \left(\begin{array}{c} E_0(P) \\ H_0(P) \end{array} \right) \sum_{n=|m|}^{\infty} c_n^{pw} \left(\begin{array}{c} g_{n,TM}^m(P) \\ g_{n,TE}^m(P) \end{array} \right) \frac{n(n+1)}{r} \Psi_n^{(1)}(kr) P_n^{|m|}(\cos\theta)$$

in which we took account of the fact that $P_n^{|m|} = 0$ if $n < |m|$. The next step is to discard the θ-dependency as well. This may be done in two ways. The first way is to specify $\theta = \pi/2$ in Eq.50 and to invoke the following relations for associated Legendre functions [47]:

$$P_n^m(0) = (-1)^{\frac{n+m}{2}} \frac{(n+m-1)!!}{2^{\frac{n-m}{2}} (\frac{n-m}{2})!}, \; (n-m) \text{ even} \tag{51}$$

$$P_n^m(0) = 0, \; (n-m) \text{ odd} \tag{52}$$

in which:

$$\left. \begin{array}{c} n!! = 1.3.5...n \\ (-1)!! = 1 \end{array} \right\} \tag{53}$$

to be completed with, for a better-looking expression:

$$n!! = \frac{2^{\frac{n+1}{2}}}{\sqrt{\pi}} \Gamma(\frac{n}{2} + 1) \tag{54}$$

Furthermore, we have [46], [49]:

$$\Psi_n^{(1)}(kr) = \sqrt{\frac{\pi}{2kr}} J_{n+\frac{1}{2}}(kr) \tag{55}$$

Eq.50 may then be rewritten as:

$$(kr)^{1/2} \int_0^{2\pi} r \left(\begin{array}{c} E_r(\theta = \pi/2, P) \\ H_r(\theta = \pi/2, P) \end{array} \right) \exp(-im\varphi) d\varphi \tag{56}$$

$$= \quad \pi\sqrt{2\pi} \left(\begin{array}{c} E_0(P) \\ H_0(P) \end{array} \right) \sum_{n=|m|,(n-m) \text{ even}}^{\infty} c_n^{pw} n(n+1) \left(\begin{array}{c} g_{n,TM}^m(P) \\ g_{n,TE}^m(P) \end{array} \right) P_n^{|m|}(0) J_{n+1/2}(kr)$$

Later on, this equation, specified for the Laguerre-Gauss beams under study, will be given the form of Eq.44 and will allow one to express the BSCs using finite series expressions, with however the restriction that we must have $(n-m)$ even. For $(n-m)$ odd, Eq.50 is differentiated with respect to $\cos\theta$, in order to take advantage of the relations:

$$[\frac{dP_n^m(\cos\theta)}{d\cos\theta}]_{\cos\theta=0} = 0,\ (n-m)\ \text{even} \tag{57}$$

$$[\frac{dP_n^m(\cos\theta)}{d\cos\theta}]_{\cos\theta=0} = (-1)^{\frac{n+m-1}{2}} \frac{(n+m)!!}{2^{\frac{n-m-1}{2}}(\frac{n-m-1}{2})!},\ (n-m)\ \text{odd} \tag{58}$$

Then, instead of Eq.56, we obtain:

$$\begin{aligned} & (kr)^{1/2} \int_0^{2\pi} r[\frac{\partial}{\partial\cos\theta} \begin{pmatrix} E_r(P) \\ H_r(P) \end{pmatrix}]_{\theta=\pi/2} \exp(-im\varphi)d\varphi \qquad (59) \\ = \ & \pi\sqrt{2\pi} \begin{pmatrix} E_0(P) \\ H_0(P) \end{pmatrix} \sum_{n=|m|,(n-m)\ \text{odd}}^{\infty} c_n^{pw} n(n+1) \begin{pmatrix} g_{n,TM}^m(P) \\ g_{n,TE}^m(P) \end{pmatrix} [\frac{dP_n^{|m|}(\cos\theta)}{d\cos\theta}]_{\cos\theta=0} J_{n+1/2}(kr) \end{aligned}$$

which will later be given the form of Eq.44 and will allow one to obtain finite series expressions for the BSCs of Laguerre-Gauss beams for $(n-m)$ odd.

3.3 Relationships between P- and N-conventions.

In the sequel, instead of using Eq.47, we found convenient to use:

$$\begin{aligned} \begin{pmatrix} E_r(N) \\ H_r(N) \end{pmatrix} \ = \ & \begin{pmatrix} E_0(N) \\ H_0(N) \end{pmatrix} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} \qquad (60) \\ & c_n^{pw} \begin{pmatrix} g_{n,TM}^m(N) \\ g_{n,TE}^m(N) \end{pmatrix} \frac{n(n+1)}{r} \Psi_n^{(1)}(kr) P_n^{|m|}(\cos\theta) \exp(im\varphi) \end{aligned}$$

But we have [11]:

$$X_r(N) = X_r(P)^*,\ X = E \text{ or } H \tag{61}$$

Inserting Eq.61 into Eq.60, taking the complex conjugate from the result, changing m to $(-m)$, substracting from Eq.47, and using Eq.48, we obtain:

$$\left(\begin{array}{c} g^m_{n,TM}(P) \\ g^m_{n,TE}(P) \end{array} \right) = (-1)^{n+1} \left(\begin{array}{c} g^{-m*}_{n,TM}(N) \\ g^{-m*}_{n,TE}(N) \end{array} \right) \tag{62}$$

As a remark, let us immediately note that we shall find, in the present paper, that BSCs with the N-convention will be different from 0 only for $m = l+1$ and $m = l-1$. Eq.62 then implies that BSCs with the P-convention will be different from 0 only for $m = -l-1$ and $m = -l+1$. This is indeed what we observed using quadratures and localized approximations with the P-convention, see Eqs.(47)-(48) and Table 2 in [50].

4 Transverse Magnetic Beam Shape Coefficients.

4.1 *(n-m)* even.

We start from Eq.56 specified for the electric field and rewritten using the N-convention:

$$\begin{aligned} & (kr)^{1/2} \int_0^{2\pi} r E_r(\theta = \pi/2, N) \exp(-im\varphi) d\varphi \qquad (63) \\ = \quad & \pi\sqrt{2\pi} E_0(N) \sum_{n=|m|,(n-m) \text{ even}}^{\infty} c_n^{pw} n(n+1) g^m_{n,TM}(N) P_n^{|m|}(0) J_{n+1/2}(kr) \end{aligned}$$

From Eqs.16 and 17, we readily establish expressions for $E_x(\theta = \pi/2, N)$ and $E_y(\theta = \pi/2, N)$ and, converting cos. and sin. to exponentials, we find:

$$E_r(\theta = \pi/2, N) = E_0(N)[(F_p^l)_1 e^{i(l+1)\varphi} + (F_p^l)_{-1} e^{i(l-1)\varphi}] \tag{64}$$

in which:

$$(F_p^l)_1 = \frac{\alpha - i\beta}{2} \int_0^{kNA} F_p^l(\eta)[(1 + \frac{\sqrt{k^2-\eta^2}}{k}) J_l(\eta r) + (1 - \frac{\sqrt{k^2-\eta^2}}{k}) J_{l+2}(\eta r)] d\eta \tag{65}$$

$$(F_p^l)_{-1} = \frac{\alpha + i\beta}{2} \int_0^{kNA} F_p^l(\eta)[(1 + \frac{\sqrt{k^2 - \eta^2}}{k})J_l(\eta r) + (1 - \frac{\sqrt{k^2 - \eta^2}}{k})J_{l-2}(\eta r)]d\eta \tag{66}$$

with:

$$F_p^l(\eta) = \frac{\eta^{|l|+1}}{\sqrt{\sqrt{k^2 - \eta^2}}} L_p^l[b(\eta)] \exp[-c(\eta)] \tag{67}$$

We then insert Eq.64 into Eq.63, use Eq.48 and rearrange to obtain:

$$\begin{aligned} & \frac{i(kr)^{1/2}kr}{\pi\sqrt{2\pi}} \int_0^{2\pi} [(F_p^l)_1 e^{i(l+1)\varphi} + (F_p^l)_{-1} e^{i(l-1)\varphi}] e^{-im\varphi} d\varphi \qquad (68) \\ = & \sum_{n=|m|,(n-m) \text{ even}}^{\infty} (-i)^n (2n+1) g_{n,TM}^m(N) P_n^{|m|}(0) J_{n+1/2}(kr) \end{aligned}$$

To get rid of the integral over φ in Eq.68, we then use the orthogonality relation of Eq.49, and introduce $x = kr$ to obtain:

$$\begin{aligned} & i\sqrt{\frac{2}{\pi}} x^{1/2} x[(F_p^l)_1 \delta_{m,l+1} + (F_p^l)_{-1} \delta_{m,l-1}] \qquad (69) \\ = & \sum_{n=|m|,(n-m) \text{ even}}^{\infty} (-i)^n (2n+1) g_{n,TM}^m(N) P_n^{|m|}(0) J_{n+1/2}(x) \end{aligned}$$

We now have to distinguish three cases, as follows.

First case : $m = l + 1$, $l \geq 0$

Eq.69 becomes (introducing an extra-subscript "p" for further convenience) :

$$x^{1/2}[g(x)]_p = \sum_{n=l+1,(n-l) \text{ odd}}^{\infty} (-i)^n (2n+1) [g_{n,TM}^{l+1}(N)]_p P_n^{l+1}(0) J_{n+1/2}(x) \tag{70}$$

in which we recall that l is an index for the Laguerre-Gauss function in Eq.3 and:

$$[g(x)]_p = i\sqrt{\frac{2}{\pi}}x(F_p^l)_1 \tag{71}$$

Hence :

$$c_n = (-i)^n(2n+1)[g_{n,TM}^{l+1}(N)]_p P_n^{l+1}(0),\ n \geq l+1,\ n-l \text{ odd} \tag{72}$$

$$c_n = 0 \text{ otherwise} \tag{73}$$

For the sequel, we shall use Eqs.10-12 and [43]:

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\frac{x^{n+2s}}{2^{n+2s}} \tag{74}$$

We are now going to evaluate the coefficients b_n, starting with $p = 0, 1$ which shall initialize recurrence relations. For $p = 0$, we have from Eqs.71, 67, 65 and 10:

$$[g(x)]_0 = i\sqrt{\frac{2}{\pi}}x(F_0^l)_1 \tag{75}$$

in which :

$$(F_0^l)_1 = \frac{\alpha - i\beta}{2}\int_0^{kNA} \frac{\eta^{|l|+1}}{\sqrt{\sqrt{k^2-\eta^2}}}\exp[-c(\eta)][(1+\frac{\sqrt{k^2-\eta^2}}{k})J_l(\frac{\eta x}{k})+(1-\frac{\sqrt{k^2-\eta^2}}{k})J_{l+2}(\frac{\eta x}{k})]d\eta \tag{76}$$

Eq.75 contains the quantities $xJ_l(\eta x/k)$ and $xJ_{l+2}(\eta x/k)$ that we are now considering. From 74, we have :

$$
\begin{aligned}
xJ_l(\eta x/k) &= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(l+s)!}(\frac{\eta}{k})^{l+2s}\frac{1}{2^{l+2s}}x^{l+2s+1} \qquad (77)\\
&= \sum_{n=l+1} \frac{(-1)^{(n-l-1)/2}}{(\frac{n-l-1}{2})!(l+\frac{n-l-1}{2})!}(\frac{\eta}{k})^{n-1}\frac{1}{2^{n-1}}x^n\\
&= \sum_{n=0} \varepsilon(n;0,1,...,l)\frac{(-1)^{(n-l-1)/2}}{(\frac{n-l-1}{2})!(l+\frac{n-l-1}{2})!}(\frac{\eta}{k})^{n-1}\frac{1}{2^{n-1}}x^n
\end{aligned}
$$

We conveniently introduce the notation :

$$F_n^u = \frac{(-1)^{\frac{n-u}{2}}}{(\frac{n-u}{2})!(u+\frac{n-u}{2})!} = \frac{(-1)^{\frac{n-u}{2}}}{(\frac{n-u}{2})!(\frac{n+u}{2})!} \qquad (78)$$

Hence, we have :

$$xJ_l(\eta x/k) = \sum_{n=0}^{\infty} \varepsilon(n;0,1,...,l)F_{n-1}^l(\frac{\eta}{k})^{n-1}\frac{1}{2^{n-1}}x^n \qquad (79)$$

Similarly, we establish :

$$xJ_{l+2}(\eta x/k) = \sum_{n=0}^{\infty} \varepsilon(n;0,1,...,l+2)F_{n-1}^{l+2}(\frac{\eta}{k})^{n-1}\frac{1}{2^{n-1}}x^n \qquad (80)$$

and, for further use :

$$xJ_{l-2}(\eta x/k) = \sum_{n=0}^{\infty} \varepsilon(n;0,1,...,l-2)F_{n-1}^{l-2}(\frac{\eta}{k})^{n-1}\frac{1}{2^{n-1}}x^n \qquad (81)$$

Inserting Eqs.79 and 80 into Eqs.75-76, we obtain :

$$
\begin{aligned}
[g(x)]_0 &= i\sqrt{\frac{2}{\pi}}\frac{\alpha-i\beta}{2}\sum_{n=0}^{\infty}\frac{1}{2^{n-1}}[\varepsilon(n;0,1,...,l)F_{n-1}^l A_{l0}^n \qquad (82)\\
&\quad +\varepsilon(n;0,1,...,l+2)F_{n-1}^{l+2}B_{l0}^n]x^n
\end{aligned}
$$

in which :

$$A_{lp}^{n} = \int_{0}^{kNA} \frac{\eta^{|l|+1}}{\sqrt{\sqrt{k^2-\eta^2}}} \exp[-c(\eta)][(1+\frac{\sqrt{k^2-\eta^2}}{k})(\frac{\eta}{k})^{n-1}d\eta,\ p=0 \qquad (83)$$

$$B_{lp}^{n} = \int_{0}^{kNA} \frac{\eta^{|l|+1}}{\sqrt{\sqrt{k^2-\eta^2}}} \exp[-c(\eta)][(1-\frac{\sqrt{k^2-\eta^2}}{k})(\frac{\eta}{k})^{n-1}d\eta,\ p=0 \qquad (84)$$

Hence :

$$\begin{aligned}[b_n]_0 &= i\sqrt{\frac{2}{\pi}}\frac{\alpha-i\beta}{2^n}[\varepsilon(n;0,1,...,l)F_{n-1}^{l}A_{l0}^{n} \\ &\quad +\varepsilon(n;0,1,...,l+2)F_{n-1}^{l+2}B_{l0}^{n}]\end{aligned} \qquad (85)$$

For $p=1$, the procedure is similar. To begin with, we have :

$$[g(x)]_1 = i\sqrt{\frac{2}{\pi}}x(F_1^l)_1 \qquad (86)$$

in which :

$$(F_1^l)_1 = \frac{\alpha-i\beta}{2}\int_{0}^{kNA} \frac{\eta^{|l|+1}}{\sqrt{\sqrt{k^2-\eta^2}}} L_1^l[b(\eta)]\exp[-c(\eta)][(1+\frac{\sqrt{k^2-\eta^2}}{k})J_l(\frac{\eta x}{k})+(1-\frac{\sqrt{k^2-\eta^2}}{k})J_{l+2}(\frac{\eta x}{k})]d\eta \qquad (87)$$

Then, using Eq.11 instead of Eq.10 :

$$[g(x)]_1 = (l+1)[g(x)]_0 - i\sqrt{\frac{2}{\pi}}\frac{\alpha-i\beta}{2}\sum_{n=0}^{\infty}\frac{1}{2^{n-1}}[\varepsilon(n;0,1,...,l)F_{n-1}^{l}A_{l1}^{n}+\varepsilon(n;0,1,...,l+2)F_{n-1}^{l+2}B_{l1}^{n}]x^n \qquad (88)$$

in which :

$$A_{lp}^{n}=\int_{0}^{kNA}\frac{\eta^{|l|+1}}{\sqrt{\sqrt{k^2-\eta^2}}}b(\eta)L_{p-1}^{l}[b(\eta)]\exp[-c(\eta)][(1+\frac{\sqrt{k^2-\eta^2}}{k})(\frac{\eta}{k})^{n-1}d\eta,\ p\geq 1 \tag{89}$$

$$B_{lp}^{n}=\int_{0}^{kNA}\frac{\eta^{|l|+1}}{\sqrt{\sqrt{k^2-\eta^2}}}b(\eta)L_{p-1}^{l}[b(\eta)]\exp[-c(\eta)][(1-\frac{\sqrt{k^2-\eta^2}}{k})(\frac{\eta}{k})^{n-1}d\eta,\ p\geq 1 \tag{90}$$

Hence, recalling Eq.82, we obtain :

$$\begin{aligned}[b_n]_1 &= i\sqrt{\frac{2}{\pi}}\frac{\alpha-i\beta}{2^n}\{\varepsilon(n;0,1,...,l)F_{n-1}^{l}[(l+1)A_{l0}^{n}-A_{l1}^{n}] \\ &\quad +\varepsilon(n;0,1,...,l+2)F_{n-1}^{l+2}[(l+1)B_{l0}^{n}-B_{l1}^{n}]\}\end{aligned} \tag{91}$$

For the general case, starting from :

$$[g(x)]_p=i\sqrt{\frac{2}{\pi}}x(F_p^l)_1 \tag{92}$$

and using Eq.12, we arrive to a fairly obvious generalization of Eq.88 reading as, for $p>1$:

$$\begin{aligned}[g(x)]_p &= \frac{2p-1+l}{p}[g(x)]_{p-1}-\frac{p-1+l}{p}[g(x)]_{p-2} \\ &\quad -\frac{i}{p}\sqrt{\frac{2}{\pi}}\frac{\alpha-i\beta}{2}\sum_{n=0}^{\infty}\frac{1}{2^{n-1}}[\varepsilon(n;0,1,...,l)F_{n-1}^{l}A_{lp}^{n}+\varepsilon(n;0,1,...,l+2)F_{n-1}^{l+2}B_{lp}^{n}]x^n\end{aligned} \tag{93}$$

Hence :

$$\begin{aligned}[b_n]_p &= \frac{2p-1+l}{p}[b_n]_{p-1}-\frac{p-1+l}{p}[b_n]_{p-2} \\ &\quad -\frac{i}{p}\sqrt{\frac{2}{\pi}}\frac{\alpha-i\beta}{2^n}[\varepsilon(n;0,1,...,l)F_{n-1}^{l}A_{lp}^{n}+\varepsilon(n;0,1,...,l+2)F_{n-1}^{l+2}B_{lp}^{n}]\end{aligned} \tag{94}$$

Knowing the b_n's, we know the b_{n-2m}'s and have a new expression for the c_n's, namely :

$$c_n = (n+\frac{1}{2}) \sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m} \frac{\Gamma(\frac{1}{2}+n-m)}{m!} [b_{n-2m}]_p \tag{95}$$

Then, using Eqs.72, 95, 51, and rearranging, we have:

$$[g^{l+1}_{n,TM}(N)]_p = i^n \frac{\sqrt{\pi}}{2^{l+2}} (-1)^{\frac{n+l+1}{2}} \frac{(\frac{n-l-1}{2})!}{\Gamma(\frac{n+l}{2}+1)} \sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m} \frac{\Gamma(\frac{1}{2}+n-m)}{m!} [b_{n-2m}]_p \tag{96}$$

Second case : $m = l-1$, $l > 0$

Instead of Eqs.70 and 71, we now have :

$$x^{1/2}[g(x)]_p = \sum_{n=l-1,(n-l)\ \text{odd}}^{\infty} (-i)^n (2n+1) [g^{l-1}_{n,TM}]_p P^{l-1}_n(0) J_{n+1/2}(x) \tag{97}$$

in which :

$$[g(x)]_p = i\sqrt{\frac{2}{\pi}} x (F^l_p)_{-1} \tag{98}$$

Comparing $(F^l_p)_1$ and $(F^l_p)_{-1}$, see Eqs.65 and 66, we see that, from the previous case, we have to change $(\alpha - i\beta)$ to $(\alpha + i\beta)$, and $xJ_{l+2}(x)$ to $xJ_{l-2}(x)$, i.e. we have to use Eq.81 instead of Eq.80. We also have to change $(l+1)$ to $(l-1)$. Hence, Eqs.85, 91 and 94 are changed to :

$$\begin{aligned} [b_n]_0 &= i\sqrt{\frac{2}{\pi}} \frac{\alpha+i\beta}{2^n} [\varepsilon(n;0,1,...,l) F^l_{n-1} A^n_{l0} \\ &\quad + \varepsilon(n;0,1,...,l-2) F^{l-2}_{n-1} B^n_{l0}] \end{aligned} \tag{99}$$

$$\begin{aligned} [b_n]_1 &= i\sqrt{\frac{2}{\pi}} \frac{\alpha+i\beta}{2^n} \{\varepsilon(n;0,1,...,l) F^l_{n-1} [(l+1)A^n_{l0} - A^n_{l1}] \\ &\quad + \varepsilon(n;0,1,...,l-2) F^{l-2}_{n-1} [(l+1)B^n_{l0} - B^n_{l1}]\} \end{aligned} \tag{100}$$

$$[b_n]_p = \frac{2p-1+l}{p}[b_n]_{p-1} - \frac{p-1+l}{p}[b_n]_{p-2} \qquad (101)$$
$$-\frac{i}{p}\sqrt{\frac{2}{\pi}}\frac{\alpha+i\beta}{2^n}[\varepsilon(n;0,1,...,l)F^l_{n-1}A^n_{lp} + \varepsilon(n;0,1,...,l-2)F^{l-2}_{n-1}B^n_{lp}]$$

Then, proceeding as in the first case, we obtain:

$$[g^{l-1}_{n,TM}(N)]_p = i^n\frac{\sqrt{\pi}}{2^l}(-1)^{\frac{n+l-1}{2}}\frac{(\frac{n-l+1}{2})!}{\Gamma(\frac{n+l}{2})}\sum_{m=0}^{\leq n/2}2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_p \qquad (102)$$

Third case : $m = l-1$, $l = 0$.

This case corresponds to $m = -1$. From the fact that the formulation involves $P_n^{|m|}$ rather than P_n^m, this case identifies with the first case for $m = 1$, that is to say with the first case for $l = 0$. The result for the present third case is then obtained by forcing $l = 0$ in the r.h.s. of Eq.96 leading to:

$$[g^{-1}_{n,TM}(N)]_p = i^n\frac{\sqrt{\pi}}{4}(-1)^{\frac{n+1}{2}}\frac{(\frac{n-1}{2})!}{\Gamma(\frac{n}{2}+1)}\sum_{m=0}^{\leq n/2}2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_p \qquad (103)$$

Note however that the corresponding recurrence equation Eq.101 is not valid for $l = 0$ (and incidently for $l = 1$) as well due to the inefficient definition of the expression $\varepsilon(n;0,1,...,l-2)F^{l-2}_{n-1}$ involved in it. This difficulty may be overcome by rederiving the corresponding results using however the relation $J_{-n}(x) = (-1)_n J_n(x)$. The cases $l = 0$ and 1 are therefore to be viewed as special cases which would deserve a particular study which is postponed to future work. These remarks are valid as well for three other "third cases" described below.

4.2 *(n-m)* odd.

For the case $(n-m)$ odd, instead of Eq.63, we have to use, from Eq.59:

$$(kr)^{1/2}\int_0^{2\pi} r[\frac{\partial E_r(\cos\theta,N)}{\partial\cos\theta}]_{\theta=\pi/2}\exp(-im\varphi)d\varphi \tag{104}$$

$$= \pi\sqrt{2\pi}E_0(N)\sum_{n=|m|,(n-m)\ \mathrm{odd}}^{\infty} c_n^{pw}n(n+1)g_{n,TM}^m(N)[\frac{dP_n^{|m|}(\cos\theta)}{d\cos\theta}]_{\cos\theta=0}J_{n+1/2}(kr)$$

From results in Section 2, we derive after a bit of standard algebra:

$$[\frac{\partial E_r(\cos\theta)}{\partial\cos\theta}]_{\theta=\pi/2} = E_0[(G_p^l)_1 e^{i(l+1)\varphi} + (G_p^l)_{-1}e^{i(l-1)\varphi}] \tag{105}$$

in which :

$$\begin{aligned}(G_p^l)_1 &= \int_0^{kNA} G_p^l(\eta)\frac{i\alpha+\beta}{2}\{r\sqrt{k^2-\eta^2}[(1+\frac{\sqrt{k^2-\eta^2}}{k})J_l(\eta r) \\ &\quad +(1-\frac{\sqrt{k^2-\eta^2}}{k})J_{l+2}(\eta r)] - 2\frac{\eta}{k}J_{l+1}(\eta r)\}d\eta\end{aligned} \tag{106}$$

$$\begin{aligned}(G_p^l)_{-1} &= \int_0^{kNA} G_p^l(\eta)\frac{i\alpha-\beta}{2}\{r\sqrt{k^2-\eta^2}[(1+\frac{\sqrt{k^2-\eta^2}}{k})J_l(\eta r) \\ &\quad +(1-\frac{\sqrt{k^2-\eta^2}}{k})J_{l-2}(\eta r)] + 2\frac{\eta}{k}J_{l-1}(\eta r)\}d\eta\end{aligned} \tag{107}$$

and :

$$G_p^l(\eta) = F_p^l(\eta) = \frac{\eta^{|l|+1}}{\sqrt{\sqrt{k^2-\eta^2}}}L_p^l[b(\eta)]\exp[-c(\eta)] \tag{108}$$

Proceeding as in the case $(n-m)$, we then arrive at:

$$i\sqrt{\frac{2}{\pi}}x^{1/2}x[(G_p^l)_1\delta_{m,l+1} + (G_p^l)_{-1}\delta_{m,l-1}] \tag{109}$$

$$= \sum_{n=|m|,(n-m)\ \mathrm{odd}}^{\infty}(-i)^n(2n+1)g_{n,TM}^m[\frac{dP_n^{|m|}(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2}J_{n+1/2}(x)$$

As before, we now have to distinguish three cases, as follows:

First case : $m = l + 1$, $l \geq 0$

Eq.109 becomes (introducing an extra-subscript "p") :

$$x^{1/2}[g(x)]_p = \sum_{n=l+1,(n-l)\ \text{even}}^{\infty} (-i)^n (2n+1)[g_{n,TM}^{l+1}]_p [\frac{dP_n^{l+1}(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2} J_{n+1/2}(x) \tag{110}$$

in which :

$$[g(x)]_p = i\sqrt{\frac{2}{\pi}} x (G_p^l)_1 \tag{111}$$

Hence :

$$c_n = (-i)^n (2n+1)[g_{n,TM}^{l+1}]_p [\frac{dP_n^{l+1}(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2},\ n \geq l+1,\ n-l \text{ even} \tag{112}$$

$$c_n = 0 \text{ otherwise} \tag{113}$$

To pursue the calculations for the case $(n - m)$ odd, we shall need:

$$\begin{aligned} x^2 J_l(\eta x/k) &= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(l+s)!} (\frac{\eta}{k})^{l+2s} \frac{1}{2^{l+2s}} x^{l+2s+2} \\ &= \sum_{n=l+2}^{\infty} \frac{(-1)^{(n-l-2)/2}}{(\frac{n-l-2}{2})!(l+\frac{n-l-2}{2})!} (\frac{\eta}{k})^{n-2} \frac{1}{2^{n-2}} x^n \\ &= \sum_{n=0}^{\infty} \varepsilon(n; 0, 1, ..., l+1) F_{n-2}^l (\frac{\eta}{k})^{n-2} \frac{1}{2^{n-2}} x^n \end{aligned} \tag{114}$$

$$x^2 J_{l+2}(\eta x/k) = \sum_{n=0}^{\infty} \varepsilon(n; 0, 1, ..., l+3) F_{n-2}^{l+2} (\frac{\eta}{k})^{n-2} \frac{1}{2^{n-2}} x^n \tag{115}$$

$$xJ_{l+1}(\eta x/k) = \sum_{n=0}^{\infty} \varepsilon(n;0,1,...,l+1) F_{n-1}^{l+1} (\frac{\eta}{k})^{n-1} \frac{1}{2^{n-1}} x^n \tag{116}$$

$$x^2 J_{l-2}(\eta x/k) = \sum_{n=0}^{\infty} \varepsilon(n;0,1,...,l-1) F_{n-2}^{l-2} (\frac{\eta}{k})^{n-2} \frac{1}{2^{n-2}} x^n \tag{117}$$

$$xJ_{l-1}(\eta x/k) = \sum_{n=0}^{\infty} \varepsilon(n;0,1,...,l-1) F_{n-1}^{l-1} (\frac{\eta}{k})^{n-1} \frac{1}{2^{n-1}} x^n \tag{118}$$

Proceeding as in the case $(n-m)$ even, we then obtain:

$$\begin{aligned} [g(x)]_0 &= i\sqrt{\frac{2}{\pi}} \frac{i\alpha+\beta}{2} [\sum_{n=0}^{\infty} \varepsilon(n;0,1,...,l+1) \frac{F_{n-2}^{l}}{2^{n-2}} C_{l0}^{n} \\ &+ \sum_{n=0}^{\infty} \varepsilon(n;0,1,...,l+3) \frac{F_{n-2}^{l+2}}{2^{n-2}} D_{l0}^{n} \\ &- \sum_{n=0}^{\infty} \varepsilon(n;0,1,...,l+1) \frac{F_{n-1}^{l+1}}{2^{n-2}} E_{l0}^{n}] x^n \end{aligned} \tag{119}$$

in which:

$$C_{l0}^{n} = \int_0^{kNA} \eta^{|l|+1} \exp[-c(\eta)] \sqrt{\sqrt{k^2-\eta^2}} \frac{1}{k} (1 + \frac{\sqrt{k^2-\eta^2}}{k}) (\frac{\eta}{k})^{n-2} d\eta \tag{120}$$

$$D_{l0}^{n} = \int_0^{kNA} \eta^{|l|+1} \exp[-c(\eta)] \sqrt{\sqrt{k^2-\eta^2}} \frac{1}{k} (1 - \frac{\sqrt{k^2-\eta^2}}{k}) (\frac{\eta}{k})^{n-2} d\eta \tag{121}$$

$$E_{l0}^{n} = \int_0^{kNA} \frac{\eta^{|l|+1}}{\sqrt{\sqrt{k^2-\eta^2}}} \exp[-c(\eta)] (\frac{\eta}{k})^{n} d\eta \tag{122}$$

leading to:

$$
\begin{aligned}
[b_n]_0 &= i\sqrt{\frac{2}{\pi}}\frac{i\alpha+\beta}{2^{n-1}}[\varepsilon(n;0,1,...,l+1)F^l_{n-2}C^n_{l0} \\
&+\varepsilon(n;0,1,...,l+3)F^{l+2}_{n-2}D^n_{l0} \\
&-\varepsilon(n;0,1,...,l+1)F^{l+1}_{n-1}E^n_{l0}]x^n
\end{aligned} \tag{123}
$$

We also obtain:

$$
\begin{aligned}
[g(x)]_1 &= (l+1)[g(x)]_0 \\
&-i\sqrt{\frac{2}{\pi}}\frac{i\alpha+\beta}{2}[\sum_{n=0}^{\infty}\varepsilon(n;0,1,...,l+1)\frac{F^l_{n-2}}{2^{n-2}}C^n_{l1} \\
&+\sum_{n=0}^{\infty}\varepsilon(n;0,1,...,l+3)\frac{F^{l+2}_{n-2}}{2^{n-2}}D^n_{l1} \\
&-\sum_{n=0}^{\infty}\varepsilon(n;0,1,...,l+1)\frac{F^{l+1}_{n-1}}{2^{n-2}}E^n_{l1}]x^n
\end{aligned} \tag{124}
$$

in which we introduced, for $p \geq 1$:

$$
C^n_{lp} = \int_0^{kNA} \eta^{|l|+1}\exp[-c(\eta)][b(\eta)L^l_{p-1}[b(\eta)]\sqrt{\sqrt{k^2-\eta^2}}\frac{1}{k}(1+\frac{\sqrt{k^2-\eta^2}}{k})(\frac{\eta}{k})^{n-2}d\eta \tag{125}
$$

$$
D^n_{lp} = \int_0^{kNA} \eta^{|l|+1}\exp[-c(\eta)][b(\eta)L^l_{p-1}[b(\eta)]\sqrt{\sqrt{k^2-\eta^2}}\frac{1}{k}(1-\frac{\sqrt{k^2-\eta^2}}{k})(\frac{\eta}{k})^{n-2}d\eta \tag{126}
$$

$$
E^n_{lp} = \int_0^{kNA} \frac{\eta^{|l|+1}}{\sqrt{\sqrt{k^2-\eta^2}}}[b(\eta)L^l_{p-1}[b(\eta)]\exp[-c(\eta)](\frac{\eta}{k})^n d\eta \tag{127}
$$

and:

$$
\begin{aligned}
[b_n]_1 &= i\sqrt{\frac{2}{\pi}}\frac{i\alpha+\beta}{2^{n-1}}\{\varepsilon(n;0,1,...,l+1)F^l_{n-2}[(l+1)C^n_{l0}-C^n_{l1}] \\
&+\varepsilon(n;0,1,...,l+3)F^{l+2}_{n-2}[(l+1)D^n_{l0}-D^n_{l1}] \\
&-\varepsilon(n;0,1,...,l+1)F^{l+1}_{n-1}[(l+1)E^n_{l0}-E^n_{l1}]\}
\end{aligned} \tag{128}
$$

For the general case, we obtain:

$$
\begin{aligned}
[g(x)]_p &= \frac{2p-1+l}{p}[g(x)]_{p-1}-\frac{p-1+l}{p}[g(x)]_{p-2} \\
&-\frac{i}{p}\sqrt{\frac{2}{\pi}}\frac{i\alpha+\beta}{2}[\sum_{n=0}^{\infty}\varepsilon(n;0,1,...,l+1)\frac{F^l_{n-2}}{2^{n-2}}C^n_{lp} \\
&+\sum_{n=0}^{\infty}\varepsilon(n;0,1,...,l+3)\frac{F^{l+2}_{n-2}}{2^{n-2}}D^n_{lp} \\
&-\sum_{n=0}^{\infty}\varepsilon(n;0,1,...,l+1)\frac{F^{l+1}_{n-1}}{2^{n-2}}E^n_{lp}]x^n
\end{aligned} \tag{129}
$$

and:

$$
\begin{aligned}
[b_n]_p &= \frac{2p-1+l}{p}[b_n]_{p-1}-\frac{p-1+l}{p}[b_n]_{p-2} \\
&-\frac{i}{p}\sqrt{\frac{2}{\pi}}\frac{i\alpha+\beta}{2^{n-1}}[\varepsilon(n;0,1,...,l+1)F^l_{n-2}C^n_{lp} \\
&+\varepsilon(n;0,1,...,l+3)F^{l+2}_{n-2}D^n_{lp} \\
&-\varepsilon(n;0,1,...,l+1)F^{l+1}_{n-1}E^n_{lp}]
\end{aligned} \tag{130}
$$

Proceeding again similarly as for the case $(n-m)$ even, and recalling Eq.58, we then obtain:

$$
\begin{aligned}
[g^{l+1}_{n,TM}]_p &= \frac{i^n\sqrt{\pi}}{2^{l+3}}\frac{(-1)^{\frac{n+l}{2}}}{\Gamma(\frac{n+l}{2}+\frac{3}{2})}(\frac{n-l}{2}-1)! \\
&\sum_{m=0}^{\leq n/2}2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_p, \\
\text{for } n &\geqslant l+1,(n-l) \text{ even}, l\geqslant 0
\end{aligned} \tag{131}
$$

Second case : $m = l - 1$, $l > 0$

Instead of Eqs.110 and 111, we now have :

$$x^{1/2}[g(x)]_p = \sum_{n=l-1,(n-l)\ \text{even}}^{\infty} (-i)^n (2n+1)[g_{n,TM}^{l-1}]_p [\frac{dP_n^{l-1}(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2} J_{n+1/2}(x) \tag{132}$$

in which :

$$[g(x)]_p = i\sqrt{\frac{2}{\pi}} x (G_p^l)_{-1} \tag{133}$$

i.e. we now use $(G_p^l)_{-1}$instead of $(G_p^l)_1$. Comparing Eqs.106 and 107, we see that, with respect to the previous case, we have to change $(i\alpha + \beta)$ to $(i\alpha - \beta)$, J_{l+2} to J_{l-2} (using Eq.117 instead of Eq.115) and J_{l+1} to $(-J_{l-1})$ (using Eq.118 instead of Eq.116). Without having to redo the computations, we then obtain:

$$\begin{aligned} [g(x)]_0 \quad = \quad & i\sqrt{\frac{2}{\pi}} \frac{i\alpha - \beta}{2} [\sum_{n=0}^{\infty} \varepsilon(n; 0, 1, ..., l+1) \frac{F_{n-2}^l}{2^{n-2}} C_{l0}^n \\ & + \sum_{n=0}^{\infty} \varepsilon(n; 0, 1, ..., l-1) \frac{F_{n-2}^{l-2}}{2^{n-2}} D_{l0}^n \\ & + \sum_{n=0}^{\infty} \varepsilon(n; 0, 1, ..., l-1) \frac{F_{n-1}^{l-1}}{2^{n-2}} E_{l0}^n] x^n \end{aligned} \tag{134}$$

$$\begin{aligned} [b_n]_0 \quad = \quad & i\sqrt{\frac{2}{\pi}} \frac{i\alpha - \beta}{2^{n-1}} [\varepsilon(n; 0, 1, ..., l+1) F_{n-2}^l C_{l0}^n \\ & + \varepsilon(n; 0, 1, ..., l-1) F_{n-2}^{l-2} D_{l0}^n \\ & + \varepsilon(n; 0, 1, ..., l-1) F_{n-1}^{l-1} E_{l0}^n] \end{aligned} \tag{135}$$

$$
\begin{aligned}
[g(x)]_1 &= (l+1)[g(x)]_0 \\
&\quad -i\sqrt{\frac{2}{\pi}}\frac{i\alpha-\beta}{2}[\sum_{n=0}^{\infty}\varepsilon(n;0,1,...,l+1)\frac{F_{n-2}^{l}}{2^{n-2}}C_{l1}^{n} \\
&\quad +\sum_{n=0}^{\infty}\varepsilon(n;0,1,...,l-1)\frac{F_{n-2}^{l-2}}{2^{n-2}}D_{l1}^{n} \\
&\quad +\sum_{n=0}^{\infty}\varepsilon(n;0,1,...,l-1)\frac{F_{n-1}^{l-1}}{2^{n-2}}E_{l1}^{n}]x^n
\end{aligned}
\tag{136}
$$

$$
\begin{aligned}
[b_n]_1 &= i\sqrt{\frac{2}{\pi}}\frac{i\alpha-\beta}{2^{n-1}}\{\varepsilon(n;0,1,...,l+1)F_{n-2}^{l}[(l+1)C_{l0}^{n}-C_{l1}^{n}] \\
&\quad +\varepsilon(n;0,1,...,l-1)F_{n-2}^{l-2}[(l+1)D_{l0}^{n}-D_{l1}^{n}] \\
&\quad +\varepsilon(n;0,1,...,l-1)F_{n-1}^{l-1}[(l+1)E_{l0}^{n}-E_{l1}^{n}]\}
\end{aligned}
\tag{137}
$$

$$
\begin{aligned}
[g(x)]_p &= \frac{2p-1+l}{p}[g(x)]_{p-1}-\frac{p-1+l}{p}[g(x)]_{p-2} \\
&\quad -\frac{i}{p}\sqrt{\frac{2}{\pi}}\frac{i\alpha-\beta}{2}[\sum_{n=0}^{\infty}\varepsilon(n;0,1,...,l+1)\frac{F_{n-2}^{l}}{2^{n-2}}C_{lp}^{n} \\
&\quad +\sum_{n=0}^{\infty}\varepsilon(n;0,1,...,l-1)\frac{F_{n-2}^{l-2}}{2^{n-2}}D_{lp}^{n} \\
&\quad +\sum_{n=0}^{\infty}\varepsilon(n;0,1,...,l-1)\frac{F_{n-1}^{l-1}}{2^{n-2}}E_{lp}^{n}]x^n
\end{aligned}
\tag{138}
$$

$$
\begin{aligned}
[b_n]_p &= \frac{2p-1+l}{p}[b_n]_{p-1}-\frac{p-1+l}{p}[b_n]_{p-2} \\
&\quad -\frac{i}{p}\sqrt{\frac{2}{\pi}}\frac{i\alpha-\beta}{2^{n-1}}[\varepsilon(n;0,1,...,l+1)F_{n-2}^{l}C_{lp}^{n} \\
&\quad +\varepsilon(n;0,1,...,l-1)F_{n-2}^{l-2}D_{lp}^{n} \\
&\quad +\varepsilon(n;0,1,...,l-1)F_{n-1}^{l-1}E_{lp}^{n}]
\end{aligned}
\tag{139}
$$

and, eventually:

$$
\begin{aligned}
[g^{l-1}_{n,TM}]_p &= \frac{i^n\sqrt{\pi}}{2^{l+1}}\frac{(-1)^{\frac{n+l}{2}-1}}{\Gamma(\frac{n+l}{2}+\frac{1}{2})}(\frac{n-l}{2})! \\
&\quad \sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_p, \\
\text{for } n &\geqslant l-1, (n-l) \text{ even}, l>0
\end{aligned}
\tag{140}
$$

Third case : $m = l-1$, $l = 0$

Again, similarly as for the third case of (n-m) even, we simply have to force $l = 0$ in the r.h.s of Eq.131 leading to:

$$
[g^{-1}_{n,TM}]_p = i^n\frac{\sqrt{\pi}}{8}(-1)^{\frac{n}{2}}\frac{(\frac{n}{2}-1)!}{\Gamma(\frac{n}{2}+\frac{3}{2})}\sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_p \tag{141}
$$

5 Transverse Electric Beam Shape Coefficients.

Expressions for TE-BSCs are derived similarly as for the TM-BSCs, using H_r instead of E_r, and shall therefore be reported more concisely.

5.1 *(n-m)* even.

Instead of Eq.63, we now start from:

$$
\begin{aligned}
&(kr)^{1/2}\int_0^{2\pi} rH_r(\theta=\pi/2,N)\exp(-im\varphi)d\varphi \\
= \quad &\pi\sqrt{2\pi}H_0(N)\sum_{n=|m|,(n-m)\text{ even}}^{\infty} c_n^{pw}n(n+1)g^m_{n,TE}(N)P_n^{|m|}(0)J_{n+1/2}(kr)
\end{aligned}
\tag{142}
$$

in which, from results in Section 2 and proceeding similarly as for the TM-BSCs, we derive:

$$H_r(\theta=\pi/2,N)=H_0(N)[(H_p^l)_1 e^{i(l+1)\varphi}+(H_p^l)_{-1}e^{i(l-1)\varphi}] \tag{143}$$

in which :

$$(H_p^l)_1=\frac{i\alpha+\beta}{4}\int_0^{kNA} H_p^l(\eta)[J_l(\eta r)A(\eta)-J_{l+2}(\eta r)B(\eta)]d\eta \tag{144}$$

$$(H_p^l)_{-1}=\frac{\beta-i\alpha}{4}\int_0^{kNA} H_p^l(\eta)[J_l(\eta r)A(\eta)-J_{l-2}(\eta r)B(\eta)]d\eta \tag{145}$$

in which $H_p^l(\eta)$ is given in Eq.30 and:

$$A(\eta)=(1-\frac{\sqrt{k^2-\eta^2}}{k})\eta^2-(1+\frac{\sqrt{k^2-\eta^2}}{k})(2k^2-\eta^2) \tag{146}$$

$$B(\eta)=(1+\frac{\sqrt{k^2-\eta^2}}{k})\eta^2-(1-\frac{\sqrt{k^2-\eta^2}}{k})(2k^2-\eta^2) \tag{147}$$

Next, inserting Eq.143 into Eq.142, and dealing with the integration over φ, we obtain:

$$\begin{aligned} & i\sqrt{\frac{2}{\pi}}x^{1/2}x[(H_p^l)_1\delta_{m,l+1}+(H_p^l)_{-1}\delta_{m,l-1}] \\ = & \sum_{n=|m|,(n-m)\text{ even}}^{\infty}(-i)^n(2n+1)g_{n,TE}^m(N)P_n^{|m|}(0)J_{n+1/2}(x) \end{aligned} \tag{148}$$

We now again consider different cases, using an extra-subscript p for convenience.

First case : $m=l+1$, $l\geq 0$.

We have :

$$x^{1/2}[g(x)]_p = \sum_{n=l+1,(n-l)\ \mathrm{odd}}^{\infty} (-i)^n(2n+1)[g_{n,TE}^{l+1}(N)]_p P_n^{l+1}(0) J_{n+1/2}(x) \quad (149)$$

in which :

$$[g(x)]_p = i\sqrt{\frac{2}{\pi}} x(H_p^l)_1 \quad (150)$$

Then we have :

$$c_n = (-i)^n(2n+1)[g_{n,TE}^{l+1}]_p P_n^{l+1}(0),\ n \geq l+1,\ (n-l)\ \mathrm{odd} \quad (151)$$

$$c_n = 0\ \mathrm{otherwise} \quad (152)$$

We afterward express the b_n-coefficients, similarly as for the TM-case, and establish:

$$\begin{aligned} [g(x)]_0 &= \frac{i\beta-\alpha}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} [\varepsilon(n;0,1,...,l) F_{n-1}^l P_{l0}^n \\ &\quad -\varepsilon(n;0,1,...,l+2) F_{n-1}^{l+2} Q_{l0}^n] \end{aligned} \quad (153)$$

in which :

$$P_{l0}^n = \int_0^{kNA} (\frac{\eta}{k})^n \frac{\eta^{|l|}}{(k^2-\eta^2)^{3/4}} \exp[-c(\eta)] A(\eta) d\eta \quad (154)$$

$$Q_{l0}^n = \int_0^{kNA} (\frac{\eta}{k})^n \frac{\eta^{|l|}}{(k^2-\eta^2)^{3/4}} \exp[-c(\eta)] B(\eta) d\eta \quad (155)$$

and :

$$\begin{aligned} [b_n]_0 &= \frac{i\beta-\alpha}{\sqrt{\pi}} \frac{1}{2^{n+1}} [\varepsilon(n;0,1,...,l) F_{n-1}^l P_{l0}^n \\ &\quad -\varepsilon(n;0,1,...,l+2) F_{n-1}^{l+2} Q_{l0}^n] \end{aligned} \quad (156)$$

For $p = 1$, we obtain:

$$\begin{aligned}[g(x)]_1 &= (l+1)[g(x)]_0 \\ &-\frac{i\beta-\alpha}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}[\varepsilon(n;0,1,...,l)F_{n-1}^{l}P_{l1}^{n} \\ &-\varepsilon(n;0,1,...,l+2)F_{n-1}^{l+2}Q_{l1}^{n}]x^n\end{aligned} \tag{157}$$

in which :

$$P_{l1}^{n} = \int_0^{kNA}(\frac{\eta}{k})^n\frac{\eta^{|l|}}{(k^2-\eta^2)^{3/4}}b(\eta)\exp[-c(\eta)]A(\eta)d\eta \tag{158}$$

$$Q_{l1}^{n} = \int_0^{kNA}(\frac{\eta}{k})^n\frac{\eta^{|l|}}{(k^2-\eta^2)^{3/4}}b(\eta)\exp[-c(\eta)]B(\eta)d\eta \tag{159}$$

Hence :

$$\begin{aligned}[b_n]_1 &= \frac{i\beta-\alpha}{\sqrt{\pi}}\frac{1}{2^{n+1}}\{\varepsilon(n;0,1,...,l)F_{n-1}^{l}[(l+1)P_{l0}^{n}-P_{l1}^{n}] \\ &-\varepsilon(n;0,1,...,l+2)F_{n-1}^{l+2}[(l+1)Q_{l0}^{n}-Q_{l1}^{n}]\}\end{aligned} \tag{160}$$

For the general case, we find:

$$\begin{aligned}[g(x)]_p &= \frac{2p-1+l}{p}[g(x)]_{p-1}-\frac{p-1+l}{p}[g(x)]_{p-2} \\ &-\frac{i\beta-\alpha}{p\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}[\varepsilon(n;0,1,...,l)F_{n-1}^{l}P_{lp}^{n} \\ &-\varepsilon(n;0,1,...,l+2)F_{n-1}^{l+2}Q_{lp}^{n}]\end{aligned} \tag{161}$$

and :

$$\begin{aligned}[b_n]_p &= \frac{2p-1+l}{p}[b_n]_{p-1}-\frac{p-1+l}{p}[b_n]_{p-2} \\ &-\frac{i\beta-\alpha}{p\sqrt{\pi}}\frac{1}{2^{n+1}}[\varepsilon(n;0,1,...,l)F_{n-1}^{l}P_{lp}^{n} \\ &-\varepsilon(n;0,1,...,l+2)F_{n-1}^{l+2}Q_{lp}^{n}]\end{aligned} \tag{162}$$

in which :

$$P_{lp}^{n} = \int_{0}^{kNA} (\frac{\eta}{k})^{n} \frac{\eta^{|l|}}{(k^2-\eta^2)^{3/4}} b(\eta) L_{p-1}^{l}[b(\eta)] \exp[-c(\eta)] A(\eta) d\eta \tag{163}$$

$$Q_{lp}^{n} = \int_{0}^{kNA} (\frac{\eta}{k})^{n} \frac{\eta^{|l|}}{(k^2-\eta^2)^{3/4}} b(\eta) L_{p-1}^{l}[b(\eta)] \exp[-c(\eta)] B(\eta) d\eta \tag{164}$$

These relations are valid for $p = 1$ as well by virtue of $\{L_{p-1}^{l}[b(\eta)]\}_{p=1} = 1$. These coefficients being established, we then establish:

$$[g_{n,TE}^{l+1}(N)]_p = \frac{i^n \sqrt{\pi} (-1)^{\frac{n+l+1}{2}} (\frac{n-l-1}{2})!}{2^{l+2} \Gamma(\frac{n+l}{2}+1)} \sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m} \frac{\Gamma(\frac{1}{2}+n-m)}{m!} [b_{n-2m}]_p \tag{165}$$

Second case : $m = l-1$, $l > 0$.

We start from:

$$x^{1/2}[g(x)]_p = \sum_{n=l-1,(n-l)\text{ odd}}^{\infty} (-i)^n (2n+1) [g_{n,TE}^{l-1}]_p P_n^{l-1}(0) J_{n+1/2}(x) \tag{166}$$

in which :

$$[g(x)]_p = i\sqrt{\frac{2}{\pi}} x (H_p^l)_{-1} \tag{167}$$

Then we have :

$$c_n = (-i)^n (2n+1) [g_{n,TE}^{l-1}]_p P_n^{l-1}(0),\ n \geq l-1,\ (n-l) \text{ odd} \tag{168}$$

$$c_n = 0 \text{ otherwise} \tag{169}$$

Next the b_n-coefficients are found to be:

$$[g(x)]_0 = \frac{\alpha+i\beta}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}[\varepsilon(n;0,1,...,l)F_{n-1}^{l}P_{l0}^{n} - \varepsilon(n;0,1,...,l-2)F_{n-1}^{l-2}Q_{l0}^{n}]x^n \tag{170}$$

$$[b_n]_0 = \frac{\alpha+i\beta}{\sqrt{\pi}}\frac{1}{2^{n+1}}[\varepsilon(n;0,1,...,l)F_{n-1}^{l}P_{l0}^{n} - \varepsilon(n;0,1,...,l-2)F_{n-1}^{l-2}Q_{l0}^{n}] \tag{171}$$

$$[g(x)]_1 = (l+1)[g(x)]_0 - \frac{\alpha+i\beta}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}[\varepsilon(n;0,1,...,l)F_{n-1}^{l}P_{l1}^{n} - \varepsilon(n;0,1,...,l-2)F_{n-1}^{l-2}Q_{l1}^{n}]x^n \tag{172}$$

$$[b_n]_1 = \frac{\alpha+i\beta}{\sqrt{\pi}}\frac{1}{2^{n+1}}\{\varepsilon(n;0,1,...,l)F_{n-1}^{l}[(l+1)P_{l0}^{n} - P_{l1}^{n}] - \varepsilon(n;0,1,...,l-2)F_{n-1}^{l-2}[(l+1)Q_{l0}^{n} - Q_{l1}^{n}]\} \tag{173}$$

$$[g(x)]_p = \frac{2p-1+l}{p}[g(x)]_{p-1} - \frac{p-1+l}{p}[g(x)]_{p-2} - \frac{\alpha+i\beta}{p\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}[\varepsilon(n;0,1,...,l)F_{n-1}^{l}P_{lp}^{n} - \varepsilon(n;0,1,...,l-2)F_{n-1}^{l-2}Q_{lp}^{n}]x^n \tag{174}$$

$$[b_n]_p = \frac{2p-1+l}{p}[b_n]_{p-1} - \frac{p-1+l}{p}[b_n]_{p-2} - \frac{\alpha+i\beta}{p\sqrt{\pi}}\frac{1}{2^{n+1}}[\varepsilon(n;0,1,...,l)F_{n-1}^{l}P_{lp}^{n} - \varepsilon(n;0,1,...,l-2)F_{n-1}^{l-2}Q_{lp}^{n}] \tag{175}$$

And the corresponding BSCs are found to be given by:

$$[g^{l-1}_{n,TE}(N)]_p = \frac{i^n\sqrt{\pi}(-1)^{\frac{n+l-1}{2}}(\frac{n-l+1}{2})!}{2^l\Gamma(\frac{n+l}{2})}\sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_p \tag{176}$$

Third case : $m = l-1$, $l = 0$.

Similarly as for previously encountered "third cases", the result for $m = l-1$, $l = 0$ can be obtained by forcing $l = 0$ in the r.h.s. of Eq.165, leading to:

$$[g^{-1}_{n,TE}(N)]_p = \frac{i^n\sqrt{\pi}(-1)^{\frac{n+1}{2}}(\frac{n-1}{2})!}{4\Gamma(\frac{n}{2}+1)}\sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_p \tag{177}$$

5.2 *(n-m)* odd.

We now start from:

$$\begin{aligned} & (kr)^{1/2}\int_0^{2\pi} r[\frac{\partial H_r(\cos\theta,N)}{\partial\cos\theta}]_{\theta=\pi/2}\exp(-im\varphi)d\varphi \qquad (178)\\ = \quad & \pi\sqrt{2\pi}H_0(N)\sum_{n=|m|,(n-m)\text{ odd}}^{\infty} c_n^{pw}n(n+1)g^m_{n,TE}(N)[\frac{dP_n^{|m|}(\cos\theta)}{d\cos\theta}]_{\cos\theta=0}J_{n+1/2}(kr)\end{aligned}$$

in which we evaluate, after a bit of algebra:

$$\begin{aligned}[\frac{\partial H_r(\cos\theta,N)}{\partial\cos\theta}]_{\theta=\pi/2} \quad = \quad & \int_0^{kNA} H_0(N)\frac{\eta^{|l|+1}}{k\sqrt{2}}L_p^l[b(\eta)]\exp[-c(\eta)]e^{il\varphi} \qquad (179)\\ & [e^{i\varphi}A_1 + e^{-i\varphi}A_{-1}]d\eta\end{aligned}$$

in which :

$$A_1 = \frac{ix}{4k}\frac{i\alpha+\beta}{(k^2-\eta^2)^{1/4}}[J_l(\eta x/k)A(\eta) - J_{l+2}(\eta x/k)B(\eta)] - \frac{\eta}{\sqrt{k}}(\alpha - i\beta)J_{l+1}(\eta x/k) \tag{180}$$

$$A_{-1} = \frac{ix}{4k}\frac{\beta-i\alpha}{(k^2-\eta^2)^{1/4}}[J_l(\eta x/k)A(\eta) - J_{l-2}(\eta x/k)B(\eta)] - \frac{\eta}{\sqrt{k}}(\alpha + i\beta)J_{l-1}(\eta x/k) \tag{181}$$

Eq.179 may conveniently be rewritten as :

$$[\frac{\partial H_r(\cos\theta)}{\partial\cos\theta}]_{\theta=\pi/2} = H_0[\widetilde{(H_p^l)_1}e^{i(l+1)\varphi} + \widetilde{(H_p^l)_{-1}}e^{i(l-1)\varphi}] \tag{182}$$

in which :

$$\widetilde{(H_p^l)_i} = \int_0^{kNA} H_0\frac{\eta^{|l|+1}}{k\sqrt{2}}L_p^l[b(\eta)]\exp[-c(\eta)]A_i(\eta)d\eta \tag{183}$$

Afterward, we deal with the integration over φ and obtain:

$$\begin{aligned} & i\sqrt{\frac{2}{\pi}}x^{1/2}x[\widetilde{(H_p^l)_1}\delta_{m,l+1} + \widetilde{(H_p^l)_{-1}}\delta_{m,l-1}] \qquad (184)\\ = & \sum_{n=|m|,(n-m)\ \text{odd}}^{\infty}(-i)^n(2n+1)g_{n,TE}^m[\frac{\partial P_n^{|m|}(\cos\theta)}{\partial\cos\theta}]_{\theta=\pi/2}J_{n+1/2}(x) \end{aligned}$$

The reader being now used with the details of the procedure, we shall be content to provide the results.

First case : $m = l+1$, $l \geq 0$.

We obtain the functions $[g(x)]_p$ and the b_n-coefficients reading as:

$$\begin{aligned} [g(x)]_0 \quad = \quad & \frac{-(i\alpha+\beta)}{4k^2\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{x^n}{2^{n-2}}[\varepsilon(n;0,1,...,l+1)F_{n-2}^l W_{l0}^n \qquad (185)\\ & -\varepsilon(n;0,1,...,l+3)F_{n-2}^{l+2}X_{l0}^n]\\ & -\frac{(i\alpha+\beta)}{k^{3/2}\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{x^n}{2^{n-1}}\varepsilon(n;0,1,...,l+1)F_{n-1}^{l+1}Y_{l0}^n \end{aligned}$$

in which :

$$W_{l0}^n = \int_0^{kNA} \frac{\eta^{|l|+1}}{(k^2-\eta^2)^{1/4}} \exp[-c(\eta)](\frac{\eta}{k})^{n-2} A(\eta) d\eta \tag{186}$$

$$X_{l0}^n = \int_0^{kNA} \frac{\eta^{|l|+1}}{(k^2-\eta^2)^{1/4}} \exp[-c(\eta)](\frac{\eta}{k})^{n-2} B(\eta) d\eta \tag{187}$$

$$Y_{l0}^n = \int_0^{kNA} \eta^{|l|+2} \exp[-c(\eta)](\frac{\eta}{k})^{n-1} d\eta \tag{188}$$

and:

$$\begin{aligned} [b_n]_0 &= \frac{-(i\alpha+\beta)}{k^2\sqrt{\pi}} \frac{1}{2^n} [\varepsilon(n;0,1,...,l+1) F_{n-2}^l W_{l0}^n \\ &\quad -\varepsilon(n;0,1,...,l+3) F_{n-2}^{l+2} X_{l0}^n] \\ &\quad -\frac{(i\alpha+\beta)}{k^{3/2}\sqrt{\pi}} \frac{1}{2^{n-1}} \varepsilon(n;0,1,...,l+1) F_{n-1}^{l+1} Y_{l0}^n \end{aligned} \tag{189}$$

For $p = 1$, we obtain:

$$\begin{aligned} [g(x)]_1 &= (l+1)[g(x)]_0 \\ &\quad \frac{+(i\alpha+\beta)}{4k^2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{x^n}{2^{n-2}} [\varepsilon(n;0,1,...,l+1) F_{n-2}^l W_{l1}^n - \varepsilon(n;0,1,...,l+3) F_{n-2}^{l+2} X_{l1}^n] \\ &\quad + \frac{(i\alpha+\beta)}{k^{3/2}\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{x^n}{2^{n-1}} \varepsilon(n;0,1,...,l+1) F_{n-1}^{l+1} Y_{l1}^n \end{aligned} \tag{190}$$

in which :

$$W_{l1}^n = \int_0^{kNA} \frac{\eta^{|l|+1}}{(k^2-\eta^2)^{1/4}} b(\eta) \exp[-c(\eta)](\frac{\eta}{k})^{n-2} A(\eta) d\eta \tag{191}$$

$$X_{l1}^n = \int_0^{kNA} \frac{\eta^{|l|+1}}{(k^2-\eta^2)^{1/4}} b(\eta) \exp[-c(\eta)](\frac{\eta}{k})^{n-2} B(\eta) d\eta \tag{192}$$

$$Y_{l1}^n = \int_0^{kNA} \eta^{|l|+2} b(\eta) \exp[-c(\eta)] (\frac{\eta}{k})^{n-1} d\eta \tag{193}$$

and:

$$\begin{aligned} [b_n]_1 &= \frac{-(i\alpha+\beta)}{k^2\sqrt{\pi}} \frac{1}{2^n} \{\varepsilon(n;0,1,...,l+1) F_{n-2}^l [(l+1) W_{l0}^n - W_{l1}^n] \\ &\quad -\varepsilon(n;0,1,...,l+3) F_{n-2}^{l+2} [(l+1) X_{l0}^n - X_{l1}^n]\} \\ &\quad -\frac{(i\alpha+\beta)}{k^{3/2}\sqrt{\pi}} \frac{1}{2^{n-1}} \varepsilon(n;0,1,...,l+1) F_{n-1}^{l+1} [(l+1) Y_{l0}^n - Y_{l1}^n] \end{aligned} \tag{194}$$

For the general case, we find:

$$\begin{aligned} [g(x)]_p &= \frac{2p-1+l}{p} [g(x)]_{p-1} - \frac{p-1+l}{p} [g(x)]_{p-2} \\ &\quad \frac{+(i\alpha+\beta)}{4k^2 p\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{x^n}{2^{n-2}} [\varepsilon(n;0,1,...,l+1) F_{n-2}^l W_{lp}^n - \varepsilon(n;0,1,...,l+3) F_{n-2}^{l+2} X_{lp}^n] \\ &\quad + \frac{(i\alpha+\beta)}{k^{3/2} p\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{x^n}{2^{n-1}} \varepsilon(n;0,1,...,l+1) F_{n-1}^{l+1} Y_{lp}^n \end{aligned} \tag{195}$$

and :

$$\begin{aligned} [b_n]_p &= \frac{2p-1+l}{p} [b_n]_{p-1} - \frac{p-1+l}{p} [b_n]_{p-2} \\ &\quad \frac{+(i\alpha+\beta)}{4k^2 p\sqrt{\pi}} \frac{1}{2^{n-2}} [\varepsilon(n;0,1,...,l+1) F_{n-2}^l W_{lp}^n - \varepsilon(n;0,1,...,l+3) F_{n-2}^{l+2} X_{lp}^n] \\ &\quad + \frac{(i\alpha+\beta)}{k^{3/2} p\sqrt{\pi}} \frac{1}{2^{n-1}} \varepsilon(n;0,1,...,l+1) F_{n-1}^{l+1} Y_{lp}^n \end{aligned} \tag{196}$$

in which :

$$W_{lp}^n = \int_0^{kNA} \frac{\eta^{|l|+1}}{(k^2-\eta^2)^{1/4}} b(\eta) L_{p-1}^l [b(\eta)] \exp[-c(\eta)] (\frac{\eta}{k})^{n-2} A(\eta) d\eta \tag{197}$$

$$X_{lp}^{n}=\int_{0}^{kNA}\frac{\eta^{|l|+1}}{(k^{2}-\eta^{2})^{1/4}}b(\eta)L_{p-1}^{l}[b(\eta)]\exp[-c(\eta)](\frac{\eta}{k})^{n-2}B(\eta)d\eta \tag{198}$$

$$Y_{lp}^{n}=\int_{0}^{kNA}\eta^{|l|+2}b(\eta)L_{p-1}^{l}[b(\eta)]\exp[-c(\eta)](\frac{\eta}{k})^{n-1}d\eta \tag{199}$$

These relations are valid for $p=1$ as well by virtue of $\{L_{p-1}^{l}[b(\eta)]\}_{p=1}=1$. The BSCs then read as:

$$[g_{n,TE}^{l+1}(N)]_{p}=\frac{i^{n}\sqrt{\pi}(-1)^{\frac{n+l}{2}}(\frac{n-l}{2}-1)!}{2^{l+3}\Gamma(\frac{n+l}{2}+\frac{3}{2})}\sum_{m=0}^{\leq n/2}2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_{p} \tag{200}$$

Second case : $m=l-1$, $l>0$.

We obtain:

$$\begin{aligned}[g(x)]_{0} &= \frac{i\alpha-\beta}{4k^{2}\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{x^{n}}{2^{n-2}}[\varepsilon(n;0,1,...,l+1)F_{n-2}^{l}W_{l0}^{n} \\ &\quad -\varepsilon(n;0,1,...,l-1)F_{n-2}^{l-2}X_{l0}^{n}] \\ &\quad -\frac{(i\alpha-\beta)}{k^{3/2}\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{x^{n}}{2^{n-1}}\varepsilon(n;0,1,...,l-1)F_{n-1}^{l-1}Y_{l0}^{n}\end{aligned} \tag{201}$$

in which the integrals W,X,Y are the ones previously given, and:

$$\begin{aligned}[b_{n}]_{0} &= \frac{i\alpha-\beta}{4k^{2}\sqrt{\pi}}\frac{1^{n}}{2^{n-2}}[\varepsilon(n;0,1,...,l+1)F_{n-2}^{l}W_{l0}^{n} \\ &\quad -\varepsilon(n;0,1,...,l-1)F_{n-2}^{l-2}X_{l0}^{n}] \\ &\quad -\frac{(i\alpha-\beta)}{k^{3/2}\sqrt{\pi}}\frac{1^{n}}{2^{n-1}}\varepsilon(n;0,1,...,l-1)F_{n-1}^{l-1}Y_{l0}^{n}\end{aligned} \tag{202}$$

For $p=1$, we obtain:

$$
\begin{aligned}
[g(x)]_1 &= (l+1)[g(x)]_0 \qquad (203)\\
&-\frac{i\alpha-\beta)}{4k^2\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{x^n}{2^{n-2}}[\varepsilon(n;0,1,...,l+1)F_{n-2}^{l}W_{l1}^{n}-\varepsilon(n;0,1,...,l-1)F_{n-2}^{l-2}X_{l1}^{n}]\\
&+\frac{(i\alpha-\beta)}{k^{3/2}\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{x^n}{2^{n-1}}\varepsilon(n;0,1,...,l-1)F_{n-1}^{l-1}Y_{l1}^{n}
\end{aligned}
$$

in which the integrals W,X,Y have again already been previously given, and:

$$
\begin{aligned}
[b_n]_1 &= \frac{i\alpha-\beta}{4k^2\sqrt{\pi}}\frac{1^n}{2^{n-2}}\{\varepsilon(n;0,1,...,l+1)F_{n-2}^{l}[(l+1)W_{l0}^{n}-W_{l1}^{n}] \qquad (204)\\
&-\varepsilon(n;0,1,...,l-1)F_{n-2}^{l-2}[(l+1)X_{l0}^{n}-X_{l1}^{n}]\}\\
&-\frac{(i\alpha-\beta)}{k^{3/2}\sqrt{\pi}}\frac{1^n}{2^{n-1}}\varepsilon(n;0,1,...,l-1)F_{n-1}^{l-1}[(l+1)Y_{l0}^{n}-Y_{l1}^{n}]
\end{aligned}
$$

For the general case, we find:

$$
\begin{aligned}
[g(x)]_p &= \frac{2p-1+l}{p}[g(x)]_{p-1}-\frac{p-1+l}{p}[g(x)]_{p-2} \qquad (205)\\
&-\frac{i\alpha-\beta)}{4k^2p\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{x^n}{2^{n-2}}[\varepsilon(n;0,1,...,l+1)F_{n-2}^{l}W_{lp}^{n}-\varepsilon(n;0,1,...,l-1)F_{n-2}^{l-2}X_{lp}^{n}]\\
&+\frac{(i\alpha-\beta)}{k^{3/2}p\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{x^n}{2^{n-1}}\varepsilon(n;0,1,...,l-1)F_{n-1}^{l-1}Y_{lp}^{n}
\end{aligned}
$$

and :

$$
\begin{aligned}
[b_n]_p &= \frac{2p-1+l}{p}[b_n]_{p-1}-\frac{p-1+l}{p}[b_n]_{p-2} \qquad (206)\\
&-\frac{i\alpha-\beta)}{4k^2p\sqrt{\pi}}\frac{1^n}{2^{n-2}}[\varepsilon(n;0,1,...,l+1)F_{n-2}^{l}W_{lp}^{n}-\varepsilon(n;0,1,...,l-1)F_{n-2}^{l-2}X_{lp}^{n}\\
&+\frac{(i\alpha-\beta)}{k^{3/2}p\sqrt{\pi}}\frac{1^n}{2^{n-1}}\varepsilon(n;0,1,...,l-1)F_{n-1}^{l-1}Y_{lp}^{n}
\end{aligned}
$$

in which the integrals W, X, Y have again already been given previously. These relations are valid for $p=1$ as well by virtue of $\{L_{p-1}^{l}[b(\eta)]\}_{p=1}=1$.

The BSCs then are found to read as:

$$[g^{l-1}_{n,TE}(N)]_p = \frac{i^n\sqrt{\pi}(-1)^{\frac{n+l}{2}-1}(\frac{n-l}{2})!}{2^{l+1}\Gamma(\frac{n+l}{2}+\frac{1}{2})}\sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_p \tag{207}$$

Third case, $m = l-1, l = 0$.

Again, for this third case, the expressions for the BSCs are obtained by forcing $l = 0$ in the r.h.s. of Eq.200:

$$[g^{-1}_{n,TE}(N)]_p = \frac{i^n\sqrt{\pi}(-1)^{\frac{n}{2}}(\frac{n}{2}-1)!}{8\Gamma(\frac{n}{2}+\frac{3}{2})}\sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_p \tag{208}$$

6 Recurrence equations.

In this section, we examine how to deal with the recurrence equations in which $[b_n]_p$ is expressed in terms of $[b_n]_{p-1}$ and $[b_n]_{p-2}$. We begin with the case of TE-BSCs for $(n-m)$ even, $m = l+1, l \geq 0$ for which we shall provide details.

6.1 TE-BSCs, $(n-m)$ even, $m = l+1, l \geq 0$.

Eqs.154, 155, 158, 159, 163, 164 lead us to define:

$$\begin{pmatrix} I^{TE}_e(X(\eta) \\ J^{TE}_e(X(\eta \end{pmatrix} = \int_0^{kNA} (\frac{\eta}{k})^n \frac{\eta^{|l|}}{(k^2-\eta^2)^{3/4}}\exp[-c(\eta)]\begin{pmatrix} A(\eta) \\ B(\eta) \end{pmatrix} X(\eta)d\eta \tag{209}$$

in which $A(\eta)$ and $B(\eta)$ are given by Eqs.146-147. From Eqs.154-155, we have:

$$P^n_{l0} = I^{TE}_e(1),\, Q^n_{l0} = J^{TE}_e(1) \tag{210}$$

and, from Eqs.158-159, 163-164:

$$P_{lp}^{n} = I_{e}^{TE}[b(\eta)L_{p-1}^{l}(b(\eta))],\ Q_{lp}^{n} = J_{e}^{TE}[b(\eta)L_{p-1}^{l}(b(\eta))],\ p \geq 1 \tag{211}$$

in which we used $L_{0}^{l}(b(\eta)) = 1$. With these notations, Eqs.156, 160 are used to initiate the recurrence according to:

$$\begin{aligned}[b_n]_0 &= \frac{i\beta-\alpha}{\sqrt{\pi}}\frac{1}{2^{n+1}}[\varepsilon(n;0,1,...,l)F_{n-1}^{l}I_{e}^{TE}(1) \\ &\quad -\varepsilon(n;0,1,...,l+2)F_{n-1}^{l+2}J_{e}^{TE}(1)]\end{aligned} \tag{212}$$

$$\begin{aligned}[b_n]_1 &= \frac{i\beta-\alpha}{\sqrt{\pi}}\frac{1}{2^{n+1}}[\varepsilon(n;0,1,...,l)F_{n-1}^{l}I_{e}^{TE}(l+1-b) \\ &\quad -\varepsilon(n;0,1,...,l+2)F_{n-1}^{l+2}J_{e}^{TE}(l+1-b)]\end{aligned} \tag{213}$$

We now introduce an operator T such as:

$$[b_n]_p = T^{TE}(X_p(\eta)) \tag{214}$$

in which:

$$\begin{aligned}T^{TE}(X_p(\eta)) &= \frac{i\beta-\alpha}{\sqrt{\pi}}\frac{1}{2^{n+1}}[\varepsilon(n;0,1,...,l)F_{n-1}^{l}I_{e}^{TE}(X_p(\eta)) \\ &\quad -\varepsilon(n;0,1,...,l+2)F_{n-1}^{l+2}J_{e}^{TE}(X_p(\eta))]\end{aligned} \tag{215}$$

which is valid for $p = 0,\ 1$ with:

$$X_0(\eta) = 1 = L_{0}^{l}(b(\eta)) \tag{216}$$

$$X_1(\eta) = l + 1 - b(\eta) = L_{1}^{l}(b(\eta)) \tag{217}$$

in which we must recall $L_0^l(b(\eta)) = 1$ and $L_1^l(b(\eta)) = l+1-b(\eta))$. Assuming that Eq.214 is valid for p changed to $(p-1)$ and $(p-2)$, we afterward readily show that it is valid $\forall p$. The recurrence relation Eq.162 then simply becomes:

$$[b_n]_p = T^{TE}(X_p(\eta)) = T^{TE}(A_p X_{p-1}(\eta)) - B_p X_{p-2}(\eta)) - \frac{1}{p} b(\eta) L_{p-1}^l(\eta)) \quad (218)$$

so that, instead of having to solve the recurrence relation Eq.162, we have to solve the simpler reduced recurrence relation:

$$X_p(\eta) = A_p X_{p-1}(\eta) - B_p X_{p-2}(\eta) - \frac{1}{p} b(\eta) L_{p-1}^l(\eta) \quad (219)$$

with initial values given by Eqs.216-217. We shall later return to this equation.

6.2 Other TE-BSCs.

In all cases that we have to investigate, the same reduced recurrence relation Eq.219 is obtained. The differences between the different cases are only involved in modifications of Eqs.214-215. For the TE-BSCs, $(n-m)$ even, $m = l-1$, $l > 0$, we obtain:

$$[b_n]_p = U^{TE}(X_p(\eta)) \quad (220)$$

in which:

$$\begin{aligned} U^{TE}(X_p(\eta)) \quad = \quad & \frac{\alpha + i\beta}{\sqrt{\pi}} \frac{1}{2^{n+1}} [\varepsilon(n; 0, 1, ..., l) F_{n-1}^l I_e^{TE}(X_p(\eta)) \\ & -\varepsilon(n; 0, 1, ..., l-2) F_{n-1}^{l-2} J_e^{TE}(X_p(\eta))] \end{aligned} \quad (221)$$

For the TE-BSCs, $(n-m)$ odd, $m = l+1$, $l \geq 0$, we introduce:

$$\begin{pmatrix} I_o^{TE}(X(\eta) \\ J_o^{TE}(X(\eta) \end{pmatrix} = \int_0^{kNA} (\frac{\eta}{k})^{n-2} \frac{\eta^{|l|+1}}{(k^2-\eta^2)^{1/4}} \exp[-c(\eta)] \begin{pmatrix} A(\eta) \\ B(\eta) \end{pmatrix} X(\eta) d\eta \quad (222)$$

$$K_o^{TE} = \int_0^{kNA} (\frac{\eta}{k})^{n-1} \eta^{|l|+2} \exp[-c(\eta)] X(\eta) d\eta \tag{223}$$

and we have:

$$[b_n]_p = V^{TE}(X_p(\eta)) \tag{224}$$

in which:

$$\begin{aligned} V^{TE}(X_p(\eta)) &= \frac{-(i\alpha+\beta)}{k^2\sqrt{\pi}} \frac{1}{2^n} [\varepsilon(n;0,1,...,l+1) F_{n-2}^{l} I_o^{TE}(X_p(\eta)) \\ &\quad -\varepsilon(n;0,1,...,l+3) F_{n-2}^{l+2} J_o^{TE}(X_p(\eta))] \\ &\quad -\frac{i\alpha+\beta}{k^{3/2}\sqrt{\pi}} \frac{1}{2^{n-1}} \varepsilon(n;0,1,...,l+1) F_{n-1}^{l+1} K_o^{TE}(X_p(\eta)) \end{aligned} \tag{225}$$

For the TE-BSCs, $(n-m)$ odd, $m = l-1$, $l > 0$, we have:

$$[b_n]_p = W^{TE}(X_p(\eta)) \tag{226}$$

in which:

$$\begin{aligned} W^{TE}(X_p(\eta)) &= \frac{i\alpha-\beta}{k^2\sqrt{\pi}} \frac{1}{2^n} [\varepsilon(n;0,1,...,l+1) F_{n-2}^{l} I_o^{TE}(X_p(\eta)) \\ &\quad -\varepsilon(n;0,1,...,l-1) F_{n-2}^{l-2} J_o^{TE}(X_p(\eta))] \\ &\quad -\frac{i\alpha-\beta}{k^{3/2}\sqrt{\pi}} \frac{1}{2^{n-1}} \varepsilon(n;0,1,...,l-1) F_{n-1}^{l-1} K_o^{TE}(X_p(\eta)) \end{aligned} \tag{227}$$

For the TM-BSCs, $(n-m)$ even, $m = l+1$, $l \geq 0$, we introduce:

$$\begin{pmatrix} I_e^{TM}(X(\eta) \\ J_e^{TM}(X(\eta \end{pmatrix} = \int_0^{kNA} (\frac{\eta}{k})^{n-1} \frac{\eta^{|l|+1}}{(k^2-\eta^2)^{1/4}} \exp[-c(\eta)] \begin{pmatrix} C(\eta) \\ D(\eta) \end{pmatrix} X(\eta) d\eta \tag{228}$$

in which:

$$C(\eta) = 1 + \frac{\sqrt{k^2 - \eta^2}}{k} \tag{229}$$

$$D(\eta) = 1 - \frac{\sqrt{k^2 - \eta^2}}{k} \tag{230}$$

and we obtain:

$$[b_n]_p = T^{TM}(X_p(\eta)) \tag{231}$$

in which:

$$\begin{aligned} T^{TM}(X_p(\eta)) &= i\sqrt{\frac{2}{\pi}}\frac{\alpha - i\beta}{2^n}[\varepsilon(n;0,1,...,l)F_{n-1}^{l} I_e^{TM}(X_p(\eta)) \\ &\quad +\varepsilon(n;0,1,...,l+2)F_{n-1}^{l+2} J_e^{TM}(X_p(\eta))] \end{aligned} \tag{232}$$

For the TM-BSCs, $(n-m)$ even, $m = l-1$, $l > 0$, we have:

$$[b_n]_p = U^{TM}(X_p(\eta)) \tag{233}$$

in which:

$$\begin{aligned} U^{TM}(X_p(\eta)) &= i\sqrt{\frac{2}{\pi}}\frac{\alpha + i\beta}{2^n}[\varepsilon(n;0,1,...,l)F_{n-1}^{l} I_e^{TM}(X_p(\eta)) \\ &\quad +\varepsilon(n;0,1,...,l+2)F_{n-1}^{l+2} J_e^{TM}(X_p(\eta))] \end{aligned} \tag{234}$$

For the TM-BSCs, $(n-m)$ odd, $m = l+1$, $l \geq 0$, we introduce:

$$\begin{pmatrix} I_o^{TM}(X(\eta)) \\ J_o^{TM}(X(\eta)) \end{pmatrix} = \int_0^{kNA} \frac{1}{k}\eta^{|l|+1}(\frac{\eta}{k})^{n-2}(k^2-\eta^2)^{1/4}\exp[-c(\eta)]\begin{pmatrix} C(\eta) \\ D(\eta) \end{pmatrix} X(\eta)d\eta \tag{235}$$

$$K_o^{TM}(X(\eta)) = \int_0^{kNA} \frac{\eta^{|l|+1}}{(k^2-\eta^2)^{1/4}} (\frac{\eta}{k})^n \exp[-c(\eta)] X(\eta) d\eta \tag{236}$$

and we obtain:

$$[b_n]_p = V^{TM}(X_p(\eta)) \tag{237}$$

in which:

$$\begin{aligned} V^{TM}(X_p(\eta)) &= i\sqrt{\frac{2}{\pi}} \frac{i\alpha+\beta}{2^{n-1}} [\varepsilon(n;0,1,...,l+1) F_{n-2}^l I_o^{TM}(X_p(\eta)) \\ &+\varepsilon(n;0,1,...,l+3) F_{n-2}^{l+2} J_o^{TM}(X_p(\eta)) \\ &-\varepsilon(n;0,1,...,l+1) F_{n-1}^{l+1} K_o^{TM}(X_p(\eta))] \end{aligned} \tag{238}$$

and, finally, for TM-BSCs, $(n-m)$ odd, $m = l-1$, $l > 0$, we have:

$$[b_n]_p = W^{TM}(X_p(\eta)) \tag{239}$$

in which:

$$\begin{aligned} W^{TM}(X_p(\eta)) &= i\sqrt{\frac{2}{\pi}} \frac{i\alpha-\beta}{2^{n-1}} [\varepsilon(n;0,1,...,l+1) F_{n-2}^l I_o^{TM}(X_p(\eta)) \\ &+\varepsilon(n;0,1,...,l-1) F_{n-2}^{l-2} J_o^{TM}(X_p(\eta)) \\ &+\varepsilon(n;0,1,...,l-1) F_{n-1}^{l-1} K_o^{TM}(X_p(\eta))] \end{aligned} \tag{240}$$

6.3 Solving the reduced recurrence relation.

The reduced recurrence relation involves Laguerre-Gauss polynomials. To get a still simpler recurrence relation, i.e. not involving any Laguerre-Gauss polynomials, we set:

$$X_p(\eta) = \sum_{\mu=0}^{p} k_p^\mu L_\mu^l(b(\eta)), \forall p \tag{241}$$

This is readily shown to be valid for $p = 0,\ 1$ with initial values given by Eqs.216-217, reading as:

$$k_0^0 = 1,\ k_1^0 = 0,\ k_1^1 = 1 \tag{242}$$

and afterward demonstrated to be valid $\forall p$ by recurrence. Inserting Eq.241 in the reduced recurrence relation Eq.219, we obtain recurrence relations of the coefficients k_p^μ reading as, for $p \geq 2$:

$$\begin{aligned} k_p^p &= 0 \\ k_p^{p-1} &= A_p k_{p-1}^{p-1} - \frac{b}{p} \\ k_p^\mu &= A_p k_{p-1}^\mu - B_p k_{p-2}^\mu,\ \mu = 0...(p-2) \end{aligned} \tag{243}$$

with initial values for $p = 0, 1$ given by Eq.242. The first two lines of Eq.243 may then be rewritten as:

$$\begin{aligned} k_0^0 &= k_1^1 = 1, k_p^p = 0 \text{ for } p \geq 2 \\ k_1^0 &= 0, k_2^1 = (A_2 - \frac{b}{p}), k_p^{p-1} = -\frac{b}{p} \text{ for } p \geq 3 \end{aligned} \tag{244}$$

We examined whether it would be possible to solve these last equations for obtain closed form solutions. From this last investigation, it has been found that the answer is likely to be positive but that the closed form solutions obtained would be very clumsy and inefficient. We therefore believe that Eqs.243-244 are the best which can be done.

7 Conclusion.

After the success encountered when designing finite solutions in the case of (paraxial) Laguerre-Gauss beams freely propagating [15], [28], it has been decided to investigate the use of finite series to the evaluations of BSCs in the case of Laguerre-Gauss beams focused by a lens, in order to overcome the limitations inherent to the use of quadratures (too time-consuming) and of localized approximations (lack of accuracy in the case of helical beams). The present paper therefore establishes the finite series expressions required to evaluate BSCs of Laguerre-Gauss beams focused by a lens.

Let us however note that the present work is devoted to Laguerre-Gauss beams focused by a lens, with the restriction that we have considered BSCs in spherical coordinates for parallel illumination, in on-axis situations. For the sake of completeness, we now serve again comments already introduced in [28]. and in [15]. BSCs for other kinds of curvilinear systems of coordinates can afterward be obtained from the BSCs in spherical coordinates by using an extrinsic method, see [51] for the definition of intrinsic and extrinsic methods, and the introduction of [52] for a review of extrinsic methods in spheroidal and cylindrical coordinates. BSCs for oblique illumination may be deduced from the BSCs for parallel illumination by using rules of transformations of BSCs under rotation of coordinate systems [53], [54], [55], [56], [57]. BSCs for off-axis situations can be obtained from the BSCs for on-axis situations by using translation theorems [58] or by introducing the off-axis expressions of the fields right at the beginning of the algebraic treatment, as actually done in the case of Gaussian beams, see [23]. The last procedure however requires an extra-algebraic work. It is indeed a defect of the finite series technique that each modification of the field expressions requires an extra-algebraic work before implementation in computer programs, although it has been stated that the "whole process may be in principle carried out in an automatic way by using a formal computation procedure which would furthermore generate FORTRAN sources" ([3], p. 121).

A next paper shall present numerical results, including the comparison between the three basic methods used to evaluate BSCs of structured beams (quadratures, localized approximations, finite series). Another prospect must be mentioned. In the case of Laguerre-Gauss beams freely propagating, the expressions for the electric field and for the magnetic field presented a high degree of symmetry, so that the TE-BSCs and the TM-BSCs were related by very simple relations. This is not the case in the present work devoted to Laguerre-Gauss beams focused by a lens due to a lack of symmetry between the expressions of the electric and magnetic fields. This suggests that it should be possible to introduce improved expressions of Laguerre-Gauss beams by a symmetrization of the fields. Such a symmetrization of fields has already been carried out in the case of paraxial Gaussian beams described in the Davis framework [59], [60]. As a final remark related to forthcoming numerical computations, and relying on previous works concerning the calculations of BSCs by using finite series, particularly concerning Laguerre-Gauss beam freely propagating [15], no concergence problem is expected concerning the numerical implementation of the expressions derived in the present paper.

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