The Degenerate Scales for Plane Elasticity Problems in Piecewise Homogeneous Media Under General Boundary Conditions

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Abstract The degenerate scale issue for 2D-boundary integral equations and boundary element methods has been already investigated for Laplace equation, antiplane and plane elasticity, bending plate for Dirichlet boundary condition. Recently, the problems of Robin and mixed boundary conditions and of piecewise heterogeneous domains have been considered for the case of Laplace equation. We investigate similar questions for plane elasticity for more general boundary conditions. For interior problems, it is shown that the degenerate scales do not depend on the boundary condition. For exterior problems, the two degenerate scales (homogeneous medium) or two of them (heterogeneous medium) are tightly linked with the behavior at infinity of the solutions. The dependence of this behavior on the boundary conditions is investigated. We give sufficient conditions for the uniqueness of the solution. Numerical applications are provided and validate the set of theoretical results.

Keywords Plane elasticity · Degenerate scale · Heterogeneous media · Boundary integral equation · Boundary conditions

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1 Introduction

When solving some plane problems by boundary integral equations (BIE) (using the single layer indirect method or the direct method) and by the corresponding numerical methods, it appears that a loss of uniqueness of the solution may happen for some specific scales (which are also called degenerate scales). This mathematical issue has been discovered since a long time by [29] and [31] for Laplace equation. A review of the early works can be found in [8]. For Laplace problem, a necessary and sufficient condition for uniqueness is that the logarithmic capacity (or transfinite diameter) of the boundary Γ is different from 1 [11, 26]. As a consequence, a sufficient condition to achieve uniqueness is $\max_{x,y\in\Gamma}|x-y| < 1$. Yan and Sloane extended some results to smooth open contours [41]. A review of mathematical properties of boundary integral operators can be found in [17] as well as results for Lipschitz boundaries. Several methods of regularization have been suggested [2]. For a given problem, we define the factor of homothety applied to the domain that leads to a domain at degenerate scale as the degenerate scale factor, often denoted by ρ.

Concerning plane elasticity, one of the early contributor is Heise [20, 21]. The investigation has been extended to systems of integral equations with a logarithmic term in the kernel [24]. The invertibility of a 3×3 matrix has been suggested as a criterion for invertibility of the single layer elasticity operator [9–11]. Then, using ideas of [18] which were originally applied to biharmonic single layer potential, the invertibility has been linked to a 2×2 matrix $\mathbf{B}$ [38]; one consequence is that there are two degenerate scales or a double degenerate scale. A sufficient condition to ensure invertibility is given: the boundary Γ must be contained within a circular disk of given radius depending of the Poisson ratio [38]. The closed form values of the degenerate scales have been found for some cases as a circle [9], an ellipse [4], a segment [38], an approximate triangle or square [6], a hypocycloid [5]. A more general method is given in [12], if the outside of the boundary is the image of the outside of the unit circle by a conformal mapping which is a rational fraction; another method is also given for some sets of aligned segments and sets of arcs of a circle. Several numerical methods have been suggested to evaluate numerically the degenerate scales: the calculus of the eigenvalues of the matrix $\mathbf{B}$ [39] [3], by inverting the matrix $\mathbf{U}$ in a normal scale or by solving an augmented problem in any scale. A first study has been devoted to the anisotropic case which appears to be far much difficult [40]; we focus here on the isotropic case. An asymptotic property has been found [37], elliptical coordinates have been introduced to solve some specific problems [3].

For plane elasticity, the above cited papers have all considered Dirichlet boundary condition. Recently, the degenerate scale problem for Laplace equation has been extended to Robin condition [14] and to mixed Dirichlet-Neumann boundary condition [15]. The degenerate scales for the interior and the exterior problem are the same for Dirichlet condition and appear to be different for other boundary conditions. The case of a piecewise homogeneous domain for Laplace equation with Dirichlet condition has also been investigated [16]; the number of degenerate scales is equal to the number of homogeneous subdomains. However, even for the Dirichlet condition there are differences between interior and exterior problems. The aim of this paper is to extend these recent results from Laplace to Lamé equation and to give some sufficient conditions for the uniqueness of the solution.

Attention must be paid to the choice of the fundamental solution, since the degenerate scales depend on that choice [38]. In this paper we will consider the following fundamental solution, e.g., [1, 27, 28]:

$$\mathbf{U}(\mathbf{x},\mathbf{y})(\boldsymbol{\xi}) = \Lambda\left(-\kappa\ln|\mathbf{r}|\boldsymbol{\xi} + (\frac{\boldsymbol{\xi}.\mathbf{r}}{\mathbf{r}.\mathbf{r}})\mathbf{r}\right); \tag{1}$$

with $\mathbf{r} = \mathbf{x} - \mathbf{y}$; $\Lambda = \frac{1}{8\pi G(1-\nu)}$; $\kappa = 3 - 4\nu$ and $\boldsymbol{\xi}$ is the force applied at the source point. ν and G are the Poisson's ratio and shear modulus. Using complex potentials leads naturally to a different choice of the fundamental solution [32]. For Laplace equation, we consider the standard fundamental solution $U = -\frac{1}{2\pi}\ln|\mathbf{r}|$ and we denote by ρ_0 the associated degenerate scale factor related to the same boundary.

Among the classes of smoothness of boundaries, the class C^2 has often been considered. For instance in [11], it is proved for a bounded domain with a C^2 boundary, that the Dirichlet, the Neumann and the Robin boundary value problem have only one solution. The uniqueness is proved by applying the Betti's formula to the difference of two solutions. The proof of existence is given by considering the boundary integral equations (complex equation in the case of [32]). In [19], the existence and uniqueness is proved for the mixed Neumann-Dirichlet boundary value problem. The more difficult case of Lyapunov boundaries has been also considered; Lipschitz boundaries appear to be very useful in variational approaches, e.g., [28]. Concerning the piecewise homogeneous problem, we can refer to [27] which deals with the Dirichlet problem with Lipschitz boundaries.

There is in fact a huge mathematical literature on all these various topics and we limit ourselves to the particular issue of the degenerate scales. More specifically, we will assume that the different boundaries are smooth enough to ensure the existence of solutions and the validity of the Betti relations.

We give now an overview of the framework of our paper. In section 2, we provide new upper and lower bounds of the degenerate scales for Dirichlet boundary condition. These bounds depend on the degenerate scale for the Laplace problem and on $\kappa = 3 - 4\nu$. These results complete the ones of [12]. The lower bound allows to give a new sufficient condition for invertibility which is linked to the Laplace problem and the corresponding degenerate scale.

In section 3, a general boundary condition is first defined for a plane elastic problem: a part with Dirichlet boundary condition, other parts of the boundary with Neumann condition or Robin condition or a combined Dirichlet-Neumann condition. Then, it is checked that this very general boundary condition ensures the uniqueness of the solution. Finally, the degenerate scales for any kind of boundary conditions are found to be the same as for the Dirichlet boundary condition.

Section 4 begins the study of the exterior problems. A matrix $\mathbf{B}$ has been defined in [38] for the case of Dirichlet boundary and allows to find the degenerate scales. Combining some ideas of [38] and [25], we build a generalized matrix $\mathbf{B}$ for more general boundary conditions including the case of piecewise homogeneous media. Then, this matrix can be used to obtain the intrinsic degenerate scales for the infinite plane elasticity problem, i.e., the degenerate scales being not the ones of the homogeneous parts and involving the coupling between different subdomains.

Section 5 is devoted to the proof of different inequalities regarding the intrinsic degenerate scale factors. We compare the degenerate scales of problems where the different boundary conditions are changed on some parts of the boundary. These comparisons allow to conclude that it is in the Dirichlet case that the degenerate scale factors are the smallest for a given boundary. Section 6 investigates the influence of the variation of the Poisson's ratio on the intrinsic degenerate scales.

Section 7 proves that the set of degenerate scales of the exterior problem contains the degenerate scales of the bounded subdomains and the intrinsic degenerate scales linked to a specific solution of the boundary value problem (BVP) with a part of the boundary at infinity. These intrinsic degenerate scales can be found by using the matrix $\mathbf{B}$ defined in section 4.

Section 8 displays two sufficient conditions for the uniqueness of the solution of the BIEs system of the exterior problem. One of them is a generalization of a condition given in [38] and the second relies on the conclusion of section 2. An approach to characterize the degenerate scales by a new matrix $\tilde{B}$ is provided in section 9 and numerical applications are given in section 10.

Finally, section 11 summarizes the different results of the paper.

2 Some complements on degenerate scales in elasticity for the homogeneous case with Dirichlet boundary condition

Numerous results can be found in the literature on degenerate scales in elasticity for the case of Dirichlet boundary conditions. One important aspect is the ability to bound the degenerate scale factors by using the degenerate scale factors for Laplace equation with Dirichlet boundary condition, which is largely documented. In this section, we complete the set of bounds by producing a new lower bound for the degenerate scales related to elasticity with boundary conditions. In addition, a new upper bound can be obtained for the smallest of the degenerate scale factors. It allows us to produce a large overview of the possible degenerate scales and their relations with the one related to the case of Laplace equation.

2.1 A new upper bound for the smallest degenerate scale

We denote by ρ_i $i = 1, 2$ the degenerate scale factors that scale the original problem to a degenerate scale in the case of elasticity with Dirichlet boundary condition. We assume that $\rho_2 \geq \rho_1$. It has been proved [12] for the fundamental solution defined in (1) that the following inequality holds:

$$\rho_0 \mathrm{e}^{1/\kappa} \geq \rho_2. \tag{2}$$

where ρ_0 is the degenerate scale factor for the Laplace equation and $\kappa = 3-4\nu$, ν being the Poisson's ratio. This bound is sharp since for a segment $\rho_2 = \rho_0 \mathrm{e}^{1/\kappa}$ [38] . In the same paper [12], it has been proved that $\rho_0^2 \mathrm{e}^{1/\kappa} \geq \rho_1 \rho_2$. Then we have $\rho_0^2 \mathrm{e}^{1/\kappa} \geq \rho_1^2$ since $\rho_2 \geq \rho_1$. We can deduce the following new upper bound for the smallest degenerate scale factor:

$$\rho_0 \mathrm{e}^{1/(2\kappa)} \geq \rho_1. \tag{3}$$

This bound is sharp since for a circle $\rho_1 = \rho_0 \mathrm{e}^{1/(2\kappa)}$ [4].

2.2 A lower bound for the degenerate scale factors

The problem of finding a lower bound was raised in a recent paper [12], the set of known theoretical results allowing to suspect the existence of such a bound. However no proof was presented at that time. We propose here to prove $\rho_1 \geq \rho_0$. Our argument is based on the combination of two preceding works with only marginal and easy extensions [35, 36]. So, we will not reproduce here the rather difficult proofs of these papers but simply indicate the light changes which are necessary for our purpose.

We first refer to [36]. The proof of the coercivity of the single layer operator for Laplace operator (theorem 6.23) is given, assuming that $\mathrm{diam}(\Gamma) < 1$. However, the proof uses only its consequence that the logarithmic capacity of Γ, cap_Γ is < 1. We now turn to [35]. The lemma 3.1 of this paper gives that the ellipticity constant of the plane elasticity single layer operator is > 0 if the ellipticity constant of the Laplace single layer operator is > 0. In other words, the plane elasticity problem is invertible if the Laplace problem is invertible. The paper assumes $1/2 > \nu > 0$ but the proof only needs $1/2 > \nu > -1$. We note $\rho_0 = 1/\mathrm{cap}_\Gamma$ the scale factor such that $\mathrm{cap}_{\rho_0 \Gamma} = 1$. Hence, for any scaling $\alpha < \rho_0$, the Laplace single layer operator is invertible. That means that $\alpha \neq \rho_i, i = 1, 2$, ρ_i being the two degenerate scale factors for the plane elasticity single layer operator. If we assume $\rho_0 > \rho_1$ and choose the scaling $\alpha = \rho_1$ we find that the linear elasticity operator is not invertible and that the Laplace operator is invertible, which is in contradiction with the result of [35] and we conclude:

$$\rho_1 \geq \rho_0\,. \tag{4}$$

This bound is sharp since we have the equality $\rho_1 = \rho_0$ for a segment [38] or a finite set of aligned segments [13].

2.3 Overview of possible values of the couple of degenerate scale factors

Thanks to the preceding results, it is now possible to draw a graph showing the area of possible degenerate scale factors (ρ_1, ρ_2) in relation with ρ_0, studying more precisely the case $(\rho_2 \geq \rho_1)$.

From the previously recalled bounds, the couples $(\kappa \ln(\rho_1/\rho_0), \kappa \ln(\rho_2/\rho_0))$ lie within a triangular area (See Fig. 1) bounded by $x = 0$, $x = y$ and $x+y = 1$. The values for arcs of a circle and a curve with 6 cusps have been inserted [13].

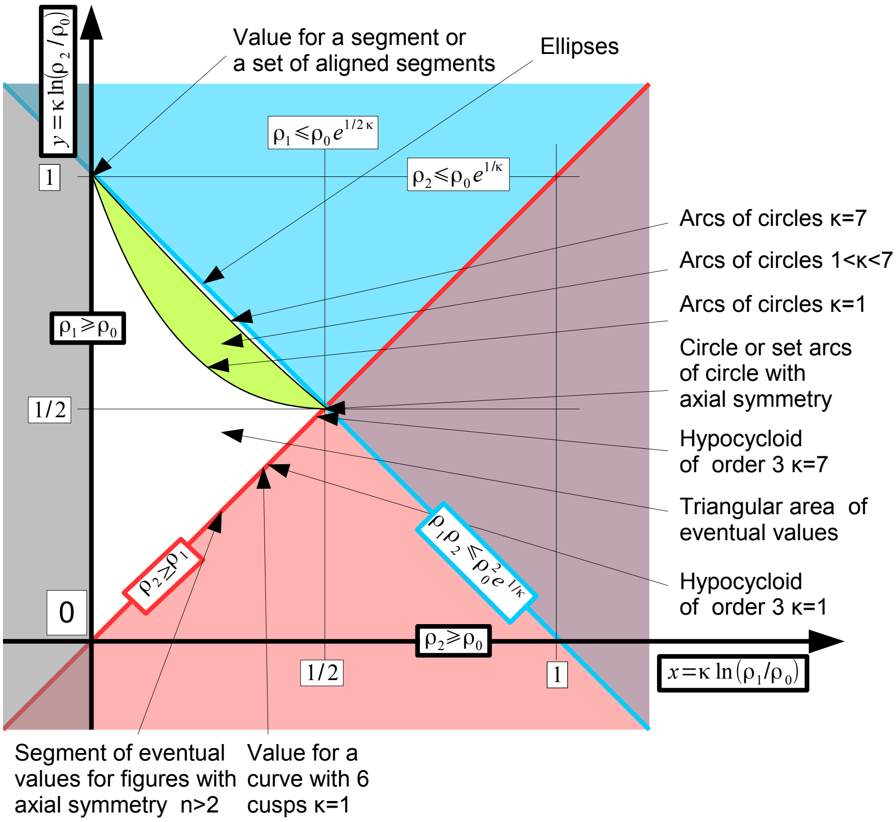

Fig. 1 Triangle containing all the admissible points $(\kappa \ln(\rho_1(\kappa)/\rho_0), \kappa \ln(\rho_2(\kappa)/\rho_0))$

It is still an open question to know more about the couples $(\kappa \ln(\rho_1/\rho_0)$, $\kappa \ln(\rho_2/\rho_0))$ which are possible inside this triangular area. A first answer is that the points between the two curves for arcs of a circle with $\kappa = 1$ and $\kappa = 7$ are possible (green zone of the figure). The existence of points below that zone is a priori also possible, as show the representative points located on the first diagonal $x = y$.

We consider now the influence of κ for a given boundary. It has been shown that if $\kappa_b > \kappa_a$ [12]:

$$\left(\frac{\rho_{i,b}}{\rho_0}\right)^{\kappa_b} \geq \left(\frac{\rho_{i,a}}{\rho_0}\right)^{\kappa_a}, \; i = 1, 2. \tag{5}$$

then we deduce that:

$$x(\kappa_b) = \ln\left(\frac{\rho_{1,b}}{\rho_0}\right)^{\kappa_b} \geq x(\kappa_a); \tag{6}$$

$$y(\kappa_b) = \ln\left(\frac{\rho_{2,b}}{\rho_0}\right)^{\kappa_b} \geq y(\kappa_a). \tag{7}$$

Then if κ increases, the point $(x(\kappa), y(\kappa))$ for a given boundary moves upwards and rightwards in a quarter plane (see Fig. 2). It means that the point related to κ_b lies within the quarter of plane defined by the point related to κ_a as seen on that figure.

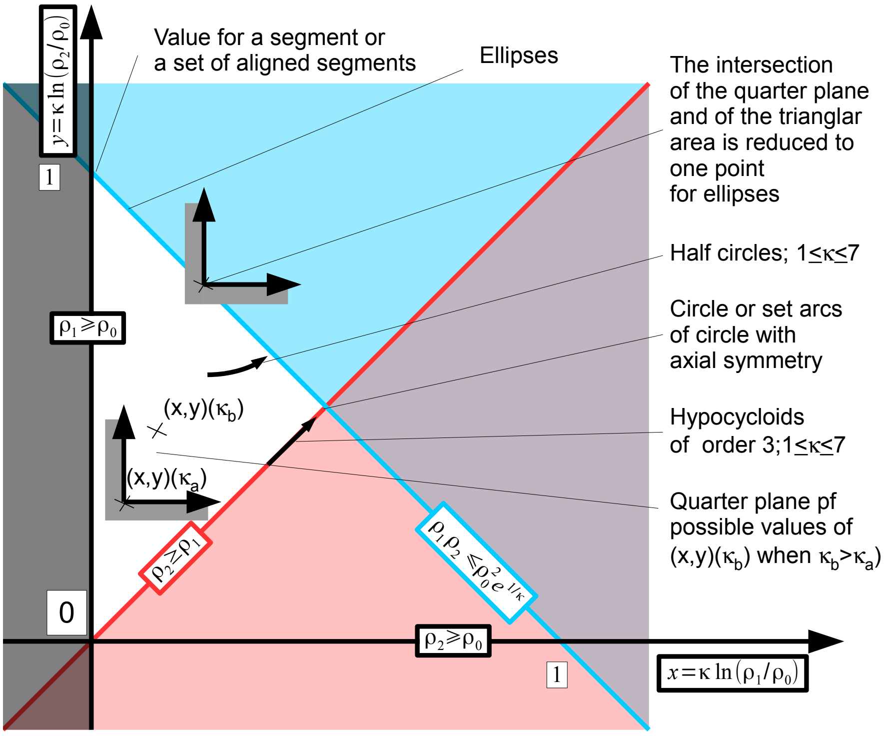

Fig. 2 Effect of the change of κ on the points $(\kappa \ln(\rho_1(\kappa)/\rho_0), \kappa \ln(\rho_2(\kappa)/\rho_0))$

For a boundary with a symmetry axis of order ≥ 3, the points move on the boundary of the triangle like for hypocycloids of order 3. For half circles, the curve $x(\kappa), y(\kappa)$ is no longer a segment. For ellipses with half axis a and b, we have $\kappa \ln(\rho_i/\rho_0) = (1-m)/2, (1+m)/2$, with $m = (a-b)/(a+b)$ [6]. Their representative point is on the upper side of the triangle of eventual values. For such a point, the intersection of the quarter plane and of the triangle is reduced to one point: we check geometrically that for ellipses the point (x,y) depends only on m but does not depend on κ.

3 Degenerate scales for interior problems and various boundary conditions

3.1 Hypotheses and notations

We consider a piecewise homogeneous connected domain $\Omega = \{\cup \Omega_i, i = 1..n\}$ (Fig. 3). The outer boundary is denoted by Γ_0. The inner boundary of an eventual hole is denoted by Γ_{n+1}.

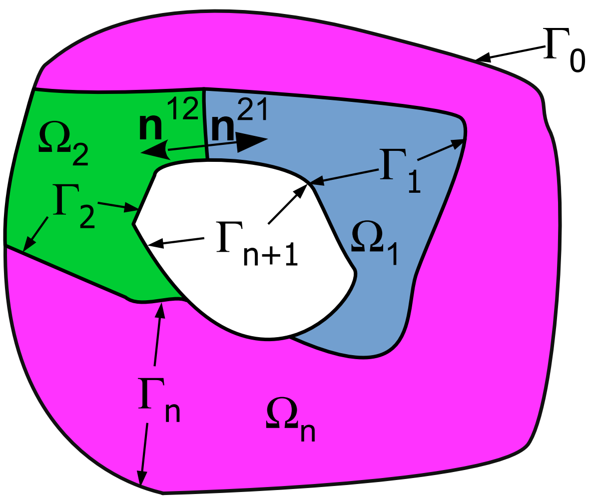

Fig. 3 The interior problem for a piecewise homogeneous medium

We denote by **C** the elasticity tensor and by $\boldsymbol{\Delta}^*$ the elasticity operator for an isotropic medium.

We have:

$$\boldsymbol{\Delta}^* = \begin{pmatrix} G\Delta + (\lambda+G)\partial_1^2 & (\lambda+G)\partial_1\partial_2 \\ (\lambda+G)\partial_1\partial_2 & G\Delta + (\lambda+G)\partial_2^2 \end{pmatrix}; \tag{8}$$

with λ and G the Lamé coefficients.

For the interior problem with no volume forces, the regularized direct BIE writes out, e.g., [1]:

$$\int_{\Gamma} (u_i(\mathbf{y}) - u_i(\mathbf{x}))T_i^k(\mathbf{x},\mathbf{y}) - t_i(\mathbf{y})U_i^k(\mathbf{x},\mathbf{y})\mathrm{d}S_{\mathbf{y}} = 0; \tag{9}$$

where t_i are the components of the traction $t(\mathbf{y})$, where U_i^k are the components of the fundamental solution $\mathbf{U}$ given by (1) and where $T_i^k(\mathbf{x},\mathbf{y})$ are the components of $\mathbf{T}(\mathbf{x},\mathbf{y})$, the traction on the boundary point $\mathbf{y}$ due to $\mathbf{U}(\mathbf{x},\mathbf{y})$.

3.2 The different kinds of conditions on the boundaries

We consider different kinds of boundary conditions which may be different over different parts of the boundary Γ.

Dirichlet condition The displacement is prescribed $\mathbf{u} = \mathbf{u}_D$.

Neumann condition The traction is prescribed $\mathbf{t} = \mathbf{t}_N$.

Robin condition The quantity $\mathbf{u} + \mathbf{kt}$ is prescribed $\mathbf{u} + \mathbf{kt} = \mathbf{s}_R$. It is required that $\mathbf{k}$ is a definitive positive linear matrix function [11]. It can depend on the point of the boundary. A slightly more restrictive condition is named "elastic support" in [30].

Combined condition We consider the combined Dirichlet-Neumann boundary condition. We suppose that there is an orthogonal projector $\mathbf{P}$. The projector can vary along the boundary Γ_C. The combined boundary condition is $\mathbf{Pu} = u_C$; $(\mathbf{I_d} - \mathbf{P})\mathbf{t} = t_C$. This type of problem called "combined Dirichlet-Neumann" problem [28] is also called mixed-mixed problem in [30]. An example in the half-plane is the indentation by a lubricated punch.

Mixed condition We consider that the boundary Γ_0 can be split into a finite number of simple curves submitted to one type of boundary condition. The union of the parts submitted to Dirichlet condition is named Γ_D and we define also Γ_N, Γ_R, Γ_C.

The usual mixed Dirichlet-Neumann condition is a special case where $\Gamma = \Gamma_D \cup \Gamma_N$.

Transmission condition In the case of a piecewise homogeneous problem, a transmission condition between two adjacent subdomains must be introduced. This problem has been considered for example in [19]. At the common boundary $\Gamma_i \cap \Gamma_j$ of two subdomains Ω_i and Ω_j, we write: $\mathbf{u}^i = \mathbf{u}^j$ and $\mathbf{t}^i + \mathbf{t}^j = 0$. We will also consider the condition $\mathbf{t}^i + \mathbf{t}^j = \mathbf{t}_d$, where $\mathbf{t}_d$ is prescribed for some auxiliary problems. When necessary, we write $\mathbf{t}^{i,j}$ for the value of $\mathbf{t}^i$ at a point of $\Gamma_i \cap \Gamma_j$.

3.3 Uniqueness of the BVP solution

We assume that there are two solutions of the Boundary value problem (BVP) $(\mathbf{u}_1, \mathbf{u}_2)$. We write the Betti formula, e.g., [33] for all subdomains denoting by $\mathit{\Gamma}_0$ the exterior boundary of $\mathit{\Omega} = \cup\, \mathit{\Omega}_i$ (see Fig. 3), and we add them:

$$\begin{aligned}
&\sum_{i=1}^{n} \int_{\mathit{\Omega}_i} (\mathbf{u}_1 - \mathbf{u}_2)\boldsymbol{\Delta}_i^*(\mathbf{u}_1 - \mathbf{u}_2) = 0 \\
&= \underbrace{\sum_{i=1}^{n} \int_{\mathit{\Gamma}_i \cap \mathit{\Gamma}_0} (\mathbf{u}_1 - \mathbf{u}_2)(\mathbf{t}_1^{i,0} - \mathbf{t}_2^{i,0}) + \sum_{i=1}^{n} \int_{\mathit{\Gamma}_i \cap \mathit{\Gamma}_{n+1}} (\mathbf{u}_1 - \mathbf{u}_2)(\mathbf{t}_1^{i,n+1} - \mathbf{t}_2^{i,n+1})}_{A_1} \\
&+ \underbrace{\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \int_{\mathit{\Gamma}_i \cap \mathit{\Gamma}_j} (\mathbf{u}_1 - \mathbf{u}_2)(\mathbf{t}_1^{i,j} - \mathbf{t}_2^{i,j})}_{A_2} \underbrace{- \sum_{i=1}^{n} \int_{\mathit{\Omega}_i} (\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2)^\top \mathbf{C}_i (\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2)}_{A_3} .
\end{aligned} \tag{10}$$

The left hand side of the equation is null since $\boldsymbol{\Delta}^*(\mathbf{u}_2 - \mathbf{u}_1) = 0$.

The integral A_1 must be split into integrals on $\mathit{\Gamma}_D$, $\mathit{\Gamma}_N$, $\mathit{\Gamma}_R$, $\mathit{\Gamma}_C$. For instance, $\mathit{\Gamma}_D$ is the part of $\mathit{\Gamma}_0 \cup \mathit{\Gamma}_{n+1}$ submitted to Dirichlet condition.

Integral on $\mathit{\Gamma}_D$ We have $(\mathbf{u}_2 - \mathbf{u}_1) = 0$, hence the integral is null.

Integral on $\mathit{\Gamma}_N$ We have $(\mathbf{t}_2 - \mathbf{t}_1) = 0$, hence the integral is null.

Integral on $\mathit{\Gamma}_R$ We have $(\mathbf{u}_2 - \mathbf{u}_1)(\mathbf{t}_2 - \mathbf{t}_1) = -(\mathbf{u}_2 - \mathbf{u}_1)^\top \mathbf{k}(\mathbf{u}_2 - \mathbf{u}_1) \leq 0$ since $\mathbf{k}$ is definite positive.

Integral on $\mathit{\Gamma}_C$ As $\mathbf{P}$ is an orthogonal projector we can write for any couple of vectors $(\mathbf{a}, \mathbf{b})$, $\mathbf{ab} = \mathbf{P}(\mathbf{a})\mathbf{P}(\mathbf{b}) + (\mathbf{I_d} - \mathbf{P})(\mathbf{a})(\mathbf{I_d} - \mathbf{P})(\mathbf{b})$. Applying this result to $\mathbf{a} = \mathbf{u}_2 - \mathbf{u}_1$, $\mathbf{b} = \mathbf{t}_2 - \mathbf{t}_1$, we find that the integrals on $\mathit{\Gamma}_C$ of the terms $\mathbf{P}(\mathbf{u}_2 - \mathbf{u}_1)$ and $(\mathbf{I_d} - \mathbf{P})(\mathbf{t}_2 - \mathbf{t}_1)$ are null and so is the integral on $\mathit{\Gamma}_C$.

From the examination of these 4 cases of boundary conditions, we conclude that $A_1 \leq 0$. The integral on $\mathit{\Gamma}_i \cap \mathit{\Gamma}_j$ in A_2 is the opposite of the integral on $\mathit{\Gamma}_j \cap \mathit{\Gamma}_i$ due to the transmission conditions, so $A_2 = 0$. As $\mathbf{C}$ is positive definite, we conclude that $\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2 = 0$, then it can be shown, e.g., [33]) that $\mathbf{u}_2 - \mathbf{u}_1$ is a rigid body displacement for a simply connected domain. This constant is null if the part of the boundary submitted to Dirichlet condition or to Robin condition has a non null length. For Robin condition $(\mathbf{u}_2 - \mathbf{u}_1)^\top \mathbf{k}(\mathbf{u}_2 - \mathbf{u}_1) = 0$ and then the constant is also null. The combined Dirichlet-Neumann case needs a little more attention: if the component of $\mathbf{u}$ which is assigned to zero value is the same for the whole boundary, then a translation in the orthogonal direction is also possible. We exclude such a case. In the following, unless otherwise specified, we will assume that this type of generalized mixed boundary condition is met. It ensures the uniqueness of the solution of the interior boundary value problem.

3.4 The system of equations

The degenerate scales are the scales for which the boundary integral equations (with the boundary and transmission conditions) have not a unique solution. We have the following equations written for each subdomain Ω_i $i \in \{1,..,n\}$:

$$\int_{\Gamma_i} (\mathbf{u}^i(\mathbf{y}) - \mathbf{u}^i(\mathbf{x}))\mathbf{T}(\mathbf{x},\mathbf{y}) - \mathbf{t}^i(y)\mathbf{U}(\mathbf{x},\mathbf{y})\mathrm{d}S_y = 0, \mathbf{x} \in \Gamma_i. \tag{11}$$

In the following, we omit $\mathrm{d}S_y$ when it is not necessary for understanding. We have also the following boundary conditions:

$$\mathbf{u}^i(\mathbf{x}) = 0, \mathbf{x} \in \Gamma_D \cap \Gamma_i \cap \Gamma_j \; ; \tag{12a}$$

$$\mathbf{t}^{i,j}(\mathbf{x}) = 0, \mathbf{x} \in \Gamma_N \cap \Gamma_i \cap \Gamma_j; \tag{12b}$$

$$\mathbf{u}^i(\mathbf{x}) + \mathbf{k}\mathbf{t}^{i,j}(\mathbf{x}) = 0, \mathbf{x} \in \Gamma_R \cap \Gamma_i \cap \Gamma_j; \tag{12c}$$

$$\mathbf{P}(\mathbf{u}^i)(\mathbf{x}) = 0; \; (\mathbf{I_d} - \mathbf{P})\mathbf{t}^{i,j}(\mathbf{x}) = 0, \; \mathbf{x} \in \Gamma_C \cap \Gamma_i \cap \Gamma_j. \tag{12d}$$

and the transmission conditions:

$$\mathbf{u}^i(\mathbf{x}) = \mathbf{u}^j(\mathbf{x}), \mathbf{x} \in \Gamma_i \cap \Gamma_j; \tag{13a}$$

$$\mathbf{t}^{i,j}(\mathbf{x}) + \mathbf{t}^{j,i}(\mathbf{x}) = 0, \mathbf{x} \in \Gamma_i \cap \Gamma_j. \tag{13b}$$

3.5 Necessary condition for the loss of uniqueness of the solution

We prove now that the degenerate scale of the global piecewise homogeneous problem is a degenerate scale of one of the subdomains.

We assume that the global problem is at a degenerate scale with $(\mathbf{u}^i, \mathbf{t}^i)$ the non null solution of the system of BIEs. We assume that none of the interior subdomains are at a degenerate scale. Then we can solve the local BVP in each subdomain with the boundary conditions $\mathbf{v}^i = \mathbf{u}^j$ on Γ_i. As the subdomains are not at a degenerate scale, we conclude that $\mathbf{T}(\mathbf{v}^i) = \mathbf{t}^i$. Then $\mathbf{v}$ defined by $\mathbf{v}^i$ on each Ω_i is a solution of a homogeneous problem and is null due to the uniqueness result (section 3.3), but $(\mathbf{u}^i, \mathbf{t}^i)$ is also null. From this contradiction, we conclude that all the degenerate scales of the global problem are a degenerate scale of one of the subdomains.

It means that the number of degenerate scales of this piecewise homogeneous problem is at most $2n$.

3.6 Sufficient condition for the loss of uniqueness of the solution

Let us prove now that a degenerate scale of a homogeneous subdomain is a degenerate scale of the global problem.

Reciprocally, we assume now that the Dirichlet problem is at a degenerate scale for one subdomain, say Ω_1 with boundary Γ_1. So, there is $\mathbf{t}_d \neq 0$ such that: $\int_{\Gamma_i} \mathbf{t}_d(\mathbf{y})\mathbf{U}(\mathbf{x},\mathbf{y}) = 0$. We consider now the solution of the non homogeneous BVP where the conditions are unchanged except those involving Γ_1. For $\Gamma_1 \cap (\Gamma_0 \cup \Gamma_{n+1})$, the boundary conditions are now:

$$\mathbf{v}(\mathbf{x}) = 0, \mathbf{x} \in \Gamma_D \cap \Gamma_1; \quad (14a)$$

$$\mathbf{t}(\mathbf{x}) = \mathbf{t}_d(\mathbf{x}), \mathbf{x} \in \Gamma_N \cap \Gamma_1; \quad (14b)$$

$$\mathbf{v}(\mathbf{x}) + \mathbf{k}\mathbf{t}(\mathbf{x}) = \mathbf{k}\mathbf{t}_d(\mathbf{x}), \mathbf{x} \in \Gamma_R \cap \Gamma_1; \quad (14c)$$

$$\mathbf{P}(\mathbf{v})(\mathbf{x}) = 0;\ (\mathbf{I_d} - \mathbf{P})\mathbf{t}(\mathbf{x}) = (\mathbf{I_d} - \mathbf{P})\mathbf{t}_d(\mathbf{x}),\ \mathbf{x} \in \Gamma_C \cap \Gamma_1; \quad (14d)$$

and the transmission condition:

$$\mathbf{t}^1 + \mathbf{t}^i = \mathbf{t}_d, \mathbf{x} \in \Gamma_1 \cup \Gamma_i. \quad (15)$$

Then $(\mathbf{v}^i, \mathbf{T}\mathbf{v}^i)$ if $i \neq 1$ and $(\mathbf{v}^1, \mathbf{T}\mathbf{v}^1 - \mathbf{t})$ is a solution of the initial system of BIEs and boundary and transmission conditions. If $\mathbf{v}^1$ is non null, we have found a non null solution of the initial systems of BIEs. If $\mathbf{v}^1$ is null, then $\mathbf{T}\mathbf{v}^1$ is null but $\mathbf{T}\mathbf{v}^1 - \mathbf{t}_d \neq 0$. So, we have still found a non null solution of the initial systems of BIEs. We conclude that the global problem is at a degenerate scale.

Finally, the piecewise homogeneous problem with n subdomains has $2n$ degenerate scales and these degenerate scales are the degenerate scales of each subdomain (some of them can be equal).

The holes do not change the degenerate scales as it has been already proved for the homogeneous Dirichlet problem [38] and checked directly for an outer circular boundary [7]. We finally conclude that the degenerate scales for generalized mixed boundary conditions are the same as for the problem with Dirichlet condition. These degenerate scales are those of the constituting homogeneous subdomains.

4 The boundary value problem for exterior domains

We consider a piecewise homogeneous domain where there is only one unbounded subdomain Ω_n (see Fig. 4). Its boundary is constituted of a simple curve or several simple curves. The domain Ω_R is the part of Ω_n included in the circle C_R. We follow the method used in [16] for studying the degenerate scales related to exterior domains with Laplace equation. It was shown in the case of Laplace equation that the degenerate scales comprise the degenerate scales for every bounded subdomain and a specific degenerate scale related to a specific boundary value problem comprising the infinite subdomain, which is called the "intrinsic degenerate scale". In this section, we define a similar boundary value problem for the case of elasticity and show that it defines now two degenerate scales.

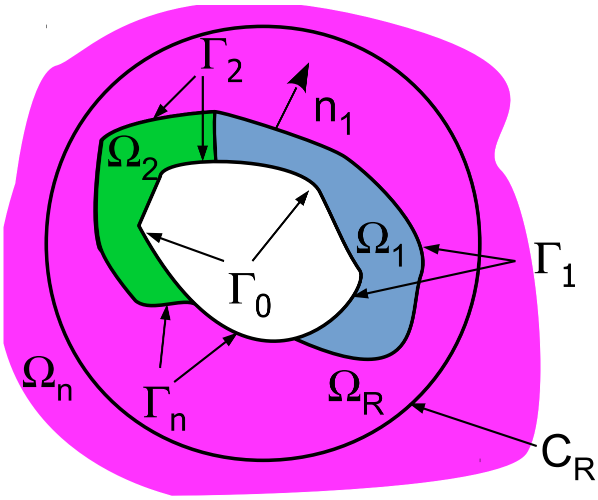

Fig. 4 The exterior problem for a piecewise homogeneous medium

The BIEs, the boundary conditions and the transmission conditions are the same as for the interior problem except for the BIE related to the unbounded subdomain Ω_n, which writes:

$$\mathbf{u}^n(\mathbf{x}) + \int_{\Gamma_n} (\mathbf{u}^n(\mathbf{y}) - \mathbf{u}^n(\mathbf{x}))\mathbf{T}(\mathbf{x},\mathbf{y}) - \mathbf{t}^n(y)\mathbf{U}(\mathbf{x},\mathbf{y})\mathrm{d}S_y = 0, \mathbf{x} \in \Gamma_n. \quad (16)$$

4.1 Solutions of the boundary value problem with a specific behavior at infinity

We intend to give a method for the characterization of a degenerate scale in an infinite elastic domain similar to the one given for Laplace equation by [25]. We recall in the case of Laplace equation that the logarithmic capacity is equal to 1 or equivalently that the problem is at the degenerate scale if there is a non null function defined on the outside of the boundary such that $\Delta u = 0$ in this region, $u = 0$ on the boundary, its behavior at infinity being such that:

$$\begin{aligned} u(\mathbf{r}) &= -\frac{1}{2\pi}\ln|\mathbf{r}|\xi + \mu + \mathrm{O}(\frac{1}{|\mathbf{r}|}); \\ \frac{\partial u}{\partial|\mathbf{r}|} &= -\frac{1}{2\pi}\frac{\xi}{|\mathbf{r}|} + \mathrm{O}(\frac{1}{|\mathbf{r}|^2}); \end{aligned} \quad (17)$$

where ξ is given (usually $\xi = 1$) and where μ is an unknown scalar related to the degenerate scale factor. For the case of plane elasticity we retain the following conditions at infinity (see, e.g., [28, 38]):

$$\mathbf{u}(\mathbf{r}) = \Lambda\left(-\kappa \ln|\mathbf{r}|\boldsymbol{\xi} + (\frac{\boldsymbol{\xi}.\mathbf{r}}{\mathbf{r}.\mathbf{r}})\mathbf{r} - \frac{1}{\Lambda}\boldsymbol{\mu} + \mathrm{O}(\frac{1}{|\mathbf{r}|})\right); \tag{18a}$$

$$\boldsymbol{\sigma}(\mathbf{r}).\frac{\mathbf{r}}{|\mathbf{r}|} = \frac{-1}{4\pi(1-\nu)|\mathbf{r}|}\left((1-2\nu)\boldsymbol{\xi} + 2(\frac{\boldsymbol{\xi}.\mathbf{r}}{\mathbf{r}.\mathbf{r}})\mathbf{r} + \mathrm{O}(\frac{1}{|\mathbf{r}|})\right); \tag{18b}$$

where $\boldsymbol{\xi}$ is a given vector and $\boldsymbol{\mu}$ is an unknown constant vector. The coefficient Λ is given by:

$$\Lambda = \frac{1}{8\pi G(1-\nu)};\ \kappa = 3 - 4\nu. \tag{19}$$

For the Dirichlet problem with prescribed $\boldsymbol{\xi}$, it has been shown that the problem is always uniquely solvable for homogeneous media [28].

4.2 Uniqueness of the solution of the boundary value problem

We consider the problem of a piecewise homogeneous exterior problem Fig. 4. In each Ω_i the operator Δ_i^* is constant. We assume the existence of a solution with the radiation condition (18). This section aims at checking that the solution is also unique for the boundary cases considered in this paper (see section. 3.2).

Assuming that there are two different solution $\mathbf{u}_i$, $i = 1, 2$, satisfying the radiation condition (18) with the same $\boldsymbol{\xi}$ and null on the boundary Γ_0. The Maxwell-Betti reciprocal theorem writes out on the surface limited by Γ_0 and the circle with radius R:

$$\begin{aligned}
&\sum_{i=1}^{(n-1)}\int_{\Omega_i}(\mathbf{u}_2-\mathbf{u}_1)\boldsymbol{\Delta}_i^*(\mathbf{u}_2-\mathbf{u}_1) + \int_{\Omega_R}(\mathbf{u}_2-\mathbf{u}_1)\boldsymbol{\Delta}_n^*(\mathbf{u}_2-\mathbf{u}_1) = 0\\
&= \underbrace{-\sum_{i=1}^{(n-1)}\int_{\Omega_i}(\boldsymbol{\epsilon}_2-\boldsymbol{\epsilon}_1)^\top\mathbf{C}_i(\boldsymbol{\epsilon}_2-\boldsymbol{\epsilon}_1) - \int_{\Omega_R}(\boldsymbol{\epsilon}_2-\boldsymbol{\epsilon}_1)^\top\mathbf{C}_n(\boldsymbol{\epsilon}_2-\boldsymbol{\epsilon}_1)}_{J_1} +\\
&+\underbrace{\sum_{i=1}^{n}\int_{\Gamma_i\cup\Gamma_0}(\mathbf{u}_2-\mathbf{u}_1)(\mathbf{t}_2-\mathbf{t}_1)}_{J_2} + \underbrace{\sum_{i=1}^{n}\sum_{j=1,\neq i}^{n}\int_{\Gamma_i\cap\Gamma_j}(\mathbf{u}_2-\mathbf{u}_1)(\mathbf{t}_2-\mathbf{t}_1)}_{J_3} +\\
&+\underbrace{\int_{C_R}(\mathbf{u}_2-\mathbf{u}_1)(\mathbf{t}_2-\mathbf{t}_1)}_{J_4}.
\end{aligned} \tag{20}$$

The left hand side of the above equation is null since $\mathbf{u}_i$ are solutions of the elastic problem. The term J_2 is similar to the term A_1 of Eq. (10) and we can conclude as in section 3.3 that $J_2 \leq 0$ due to the boundary conditions. The term J_3 is null since the term on $\Gamma_i \cap \Gamma_j$ is the opposite of the term relative to $\Gamma_j \cap \Gamma_i$ as a consequence of the transmission conditions. The integral J_4 tends to zero when $R \to \infty$ due to the radiation condition with the same $\boldsymbol{\xi}$ for $\mathbf{u}_1$ and $\mathbf{u}_1$. Then, we conclude that $J_1 \geq 0$. As $\mathbf{C}$ is positive definite, we have $J_1 \leq 0$. Finally, $J_1 = 0$ and $\mathbf{u}_2 - \mathbf{u}_1$ is a rigid body displacement in each Ω_i. The rigid body displacement is the same everywhere due to the transmission condition. The boundary conditions on Γ finally allow to conclude that $\mathbf{u}_1 = \mathbf{u}_2$ as in section 3.3.

4.3 Definition and properties of a matrix B characterizing the intrinsic degenerate scales

4.3.1 Definition

For any vector $\boldsymbol{\xi}$ the preceding problem defines $\boldsymbol{\mu}$. We can write $\boldsymbol{\mu} = \mathbf{B}\boldsymbol{\xi}$. It is clear that function $\mathbf{B}$ is linear and necessarily continuous as operating on a finite dimension space. This matrix will be useful to find the "intrinsic degenerate scales" these ones being different from the degenerate scales corresponding to the bounded subdomains, as it has been found for Laplace equation [16].

4.3.2 Symmetry

An interesting question is whether $\mathbf{B}$ is symmetric or not. This can be proved by considering the solution $\mathbf{u_i}$ corresponding to the vector $\boldsymbol{\xi_i}$ for the same homogeneous boundary condition on Γ. We write the Betti formula, analogous to the third Green identity (see, e.g., [33, 34]) on the part of the domain bounded by C_R.

$$\sum_{i=1}^{(n-1)} \int_{\Omega_i} (\mathbf{u}_2 \boldsymbol{\Delta}_i^* \mathbf{u}_1 - \mathbf{u}_1 \boldsymbol{\Delta}_i^* \mathbf{u}_2) + \int_{\Omega_R} (\mathbf{u}_2 \boldsymbol{\Delta}_n^* \mathbf{u}_1 - \mathbf{u}_1 \boldsymbol{\Delta}_n^* \mathbf{u}_2) = 0$$
$$= \underbrace{\sum_{i=1}^{n} \int_{\Gamma_i \cup \Gamma_0} (\mathbf{u}_2 \mathbf{t}_1 - \mathbf{u}_1 \mathbf{t}_2)}_{K_1} + \underbrace{\sum_{i=1}^{n} \sum_{j=1,\neq i}^{n} \int_{\Gamma_i \cap \Gamma_j} (\mathbf{u}_2 \mathbf{t}_1 - \mathbf{u}_1 \mathbf{t}_2) +}_{K_2} \tag{21}$$
$$+ \underbrace{\int_{\mathrm{C}_R} (\mathbf{u}_2 \mathbf{t}_1 - \mathbf{u}_1 \mathbf{t}_2)}_{K_3}.$$

The term K_1 is null due to the boundary conditions on Γ_0. The term K_2 is null because the contribution of $\Gamma_i \cap \Gamma_j$ is the opposite of the contribution of $\Gamma_j \cap \Gamma_i$. Then, the term K_3 is null. Using the radiation conditions, we get:

$$K_3 = (\boldsymbol{\xi_2}^\top \mathbf{B} \boldsymbol{\xi_1} - \boldsymbol{\xi_1}^\top \mathbf{B} \boldsymbol{\xi_2}) + \mathrm{O}(1/R). \tag{22}$$

We conclude that $\mathbf{B}$ is symmetric and therefore has two real eigenvalues.

4.3.3 Comparison with a former work in the case of Dirichlet boundary condition

We recall how a symmetric matrix related to the degenerate scale factors has been introduced in [38] in the case of a homogeneous domain for Dirichlet boundary condition. The following problem was considered:

For given $\boldsymbol{\xi}$, *find* $(\mathbf{t}, \boldsymbol{\mu})$ *such that* :

$$\begin{cases} \int_\Gamma \mathbf{t}(\mathbf{y}) \mathbf{U}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\mu}; & \text{(23a)} \\ \int_\Gamma \mathbf{t}(\mathbf{y}) = \boldsymbol{\xi}. & \text{(23b)} \end{cases}$$

This defines a matrix $\mathbf{B}^*$ such that $\mathbf{B}^* \boldsymbol{\xi} = \boldsymbol{\mu}$. Considering the function u defined by

$$\mathbf{u}(\mathbf{x}) = \int_\Gamma \mathbf{t}(\mathbf{y}) \mathbf{U}(\mathbf{x}, \mathbf{y}) - \boldsymbol{\mu}. \tag{24}$$

We see that this function satisfies the radiation condition (18a) and is a solution of the problem for Dirichlet boundary condition and we deduce that $\mathbf{B}^* = \mathbf{B}$. The same matrix has been recovered as the result of an optimization problem in [12], for the same case of Dirichlet boundary condition.

As the matrix $\mathbf{B}^*$ in the case of Dirichlet condition, the eigenvalues of $\mathbf{B}$ allow to evaluate the degenerate scale factors for the elasticity problem.

For an eigenvalue λ, we have an eigenvector $\boldsymbol{\xi}$, and a solution $\mathbf{u}$ and the radiation condition writes out:

$$\mathbf{u}(\mathbf{r}) = \Lambda \left(-\kappa \ln |\mathbf{r}| \boldsymbol{\xi} + (\frac{\boldsymbol{\xi}.\mathbf{r}}{\mathbf{r}.\mathbf{r}}) \mathbf{r} - \frac{\lambda}{\Lambda} \boldsymbol{\xi} + \mathrm{O}(\frac{1}{|\mathbf{r}|}) \right). \tag{25}$$

If we transform Γ in Γ_ρ by using a scaling factor ρ, the solution $\mathbf{u}$ becomes $\mathbf{u}_\rho(x) = \mathbf{u}(x/\rho)$. Choosing $\rho = \mathrm{e}^{\lambda/(\Lambda\kappa)}$ then the radiation condition solution becomes:

$$\mathbf{u}_\rho(\mathbf{r}) = \Lambda \left(-\kappa \ln |\mathbf{r}| \boldsymbol{\xi} + (\frac{\boldsymbol{\xi}.\mathbf{r}}{\mathbf{r}.\mathbf{r}}) \mathbf{r} + \mathrm{O}(\frac{1}{|\mathbf{r}|}) \right). \tag{26}$$

As for the case of Dirichlet boundary condition, this radiation condition is obtained in the case of the boundary problem related to a non null solution of the heterogeneous domain with the various boundary conditions being null. It means that the domain is at a degenerate scale, ρ being the degenerate scale factor. So, we conclude that the eigenvalues of the matrix $\mathbf{B}$ allow to find two degenerate scales (these being possibly identical) linked to the existence of specific solutions of the boundary value problem. As in [16], we will call them the intrinsic degenerate scales.

5 Inequalities concerning the intrinsic degenerate scale factors for different types of boundary conditions

It is of prime importance to understand how the degenerate scale can change when one compares different kinds of boundary conditions. We show a general condition written by using matrices $\mathbf{B}$ related to different boundary conditions. Next, the different conditions allowing to conclude on the comparisons of these matrices are shown.

5.1 A sufficient condition for $\mathbf{B}_2 \geq \mathbf{B}_1$

We consider the solutions $(\mathbf{u}_1, \mathbf{u}_2)$ for the same $\boldsymbol{\xi}$ of two problems with the same boundary but the types of boundary condition are different. The radiation conditions are given by (18).

We write the Betti formula for $(\mathbf{u}_1 - \mathbf{u}_2)$ on Ω_R:

$$\begin{aligned} &\int_{\Omega_R} (\mathbf{u}_1 - \mathbf{u}_2)\boldsymbol{\Delta}^*(\mathbf{u}_1 - \mathbf{u}_2) = 0 \\ &= \underbrace{\int_{\Gamma} (\mathbf{u}_1 - \mathbf{u}_2)(\mathbf{t}_1 - \mathbf{t}_2)}_{I_1} + \underbrace{\int_{C_R} (\mathbf{u}_1 - \mathbf{u}_2)(\mathbf{t}_1 - \mathbf{t}_2)}_{I_2} - \underbrace{\int_{\Omega_R} (\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2)^\top \mathbf{C}(\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2)}_{I_3}. \end{aligned} \tag{27}$$

The integral $I_2 \to 0$ when $R \to \infty$ due to the radiation condition (18). Then as $I_3 \leq 0$ we conclude: $I_1 \geq 0$. We also consider the third Betti formula applied to Ω_R:

$$\begin{aligned} &\int_{\Omega_R} \mathbf{u}_1 \boldsymbol{\Delta}^* \mathbf{u}_2 - \mathbf{u}_2 \boldsymbol{\Delta}^* \mathbf{u}_1 = 0 \\ &= \underbrace{\int_{\Gamma} \mathbf{u}_1 \mathbf{t}_2 - \mathbf{u}_2 \mathbf{t}_1}_{I_4} + \underbrace{\int_{C_R} \mathbf{u}_1 \mathbf{t}_2 - \mathbf{u}_2 \mathbf{t}_1}_{I_5}. \end{aligned} \tag{28}$$

Taking into account the radiation condition (18), the limit of I_5 when $R \to \infty$ is:

$$(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\boldsymbol{\xi} = \boldsymbol{\xi}^\top(\mathbf{B}_1 - \mathbf{B}_2)\boldsymbol{\xi}. \tag{29}$$

Then a necessary and sufficient condition for $\mathbf{B}_2 \geq \mathbf{B}_1$ in the sense of symmetric matrices [23] is that $I_4 \geq 0$ for all $\boldsymbol{\xi}$.

Combining (27) and (28), we get:

$$I_4 = I_1 + \int_\Gamma (2\mathbf{u}_1\mathbf{t}_2 - \mathbf{u}_1\mathbf{t}_1 - \mathbf{u}_2\mathbf{t}_2). \tag{30}$$

So a sufficient condition for $I_4 \geq 0$ is:

$$I_6 = \int_\Gamma (2\mathbf{u}_1\mathbf{t}_2 - \mathbf{u}_1\mathbf{t}_1 - \mathbf{u}_2\mathbf{t}_2) \geq 0. \tag{31}$$

Finally, a sufficient condition for $\mathbf{B}_2 \geq \mathbf{B}_1$ is:

$$a_{12}(\mathbf{x}) = (2\mathbf{u}_1\mathbf{t}_2 - \mathbf{u}_1\mathbf{t}_1 - \mathbf{u}_2\mathbf{t}_2)(\mathbf{x}) \geq 0,\ \mathbf{x} \in \Gamma. \tag{32}$$

From (27) we get $I_1 \geq 0$ due to $I_2 = 0; I_3 \leq 0$. From (28) and (30), we get $I_5 = -I_4 = -I_1 - I_6 \leq -I_1 \leq 0$. And we finally conclude:

$$\boldsymbol{\xi}^\top\mathbf{B}_1\boldsymbol{\xi} \leq \boldsymbol{\xi}^\top\mathbf{B}_2\boldsymbol{\xi} \tag{33}$$

for all values of $\boldsymbol{\xi}$. We can write $\mathbf{B}_2 \geq \mathbf{B}_1$ in the sense of symmetric matrices [23]. Then the ordered eigenvalues (ρ_1, ρ_2) of the symmetric matrix $\mathbf{B}_2$ are larger than the ones of (the symmetric matrix) $\mathbf{B}_1$. It is worthwhile mentioning that the local condition must be met for every point of the boundary. Now, we will study how this local condition can be met when the boundary condition changes at the local point $\mathbf{x}$.

5.2 The sufficient condition on a_{12} is satisfied locally if the boundary conditions are locally of the same kind over the boundary

If the boundary conditions are the same for the two problems on one part of the boundary, the quantity a_{12} defined by (32) is positive or null at any point of the considered part of the boundary.

Case of Dirichlet condition Substituting $\mathbf{u}_1 = \mathbf{u}_2 = 0$ in (33), we get $a_{12} = 0$.

Case of Neumann condition Substituting $\mathbf{t}_1 = \mathbf{t}_2 = 0$ in (33), we get $a_{12} = 0$.

Case of combined condition We assume that the projector $\mathbf{P}$ is the same for the two problems. Then all the terms $\mathbf{u}_i\mathbf{t}_j$ in a_{12} can be expanded in the following way:

$$\mathbf{u}_i\mathbf{t}_j = \mathbf{P}(\mathbf{u}_i)\mathbf{P}(\mathbf{t}_j) + (\mathbf{I_d} - \mathbf{P})(\mathbf{u}_i)(\mathbf{I_d} - \mathbf{P})(\mathbf{t}_j). \tag{34}$$

As $\mathbf{P}(\mathbf{u}_i) = 0$ and $(\mathbf{I_d} - \mathbf{P})(\mathbf{t}_j) = 0$, we conclude $a_{12} = 0$.

Case of Robin condition with $\mathbf{k}_1 = \mathbf{k}_2$ We assume that the matrix $\mathbf{k}$ is the same for the two problems. Then we can substitute $\mathbf{u}_i = -\mathbf{k}\mathbf{t}_i$ in a_{12}:

$$a_{12} = -2\mathbf{u}_1^\top \mathbf{k}\mathbf{u}_2 + \mathbf{u}_1^\top \mathbf{k}\mathbf{u}_1 + \mathbf{u}_2^\top \mathbf{k}\mathbf{u}_2 = (\mathbf{u}_1 - \mathbf{u}_2)^\top \mathbf{k}(\mathbf{u}_1 - \mathbf{u}_2) \geq 0, \qquad (35)$$

since we have assumed that $\mathbf{k}$ is positive definite.

Remark If the boundary conditions are the same for the two problems on the whole boundary, we will conclude that $\mathbf{B}_2 \geq \mathbf{B}_1$, but, in that case, we can exchange the two problems and finally conclude $\mathbf{B}_2 = \mathbf{B}_1$.

5.3 Study of different cases which ensure the sufficient condition (32) locally

5.3.1 Comparison between Dirichlet and Neumann boundary condition (BC)

We assume that boundary value problem 1 (BVP1) has Dirichlet boundary condition on part Γ_{DN} of the boundary when problem 2 (BVP2) has Neumann condition. Then for $\mathbf{x} \in \Gamma_{DN}$ we have $\mathbf{u_1} = 0$; $\mathbf{t}_2 = 0$ and $a_{12} = 0$ on Γ_{DN} .

5.3.2 Comparison between Dirichlet and combined BC

We assume that BVP1 has Dirichlet boundary condition on the part Γ_{DC} when BVP2 has combined boundary condition.

For $\mathbf{x} \in \Gamma_{DC}$, we have $\mathbf{u_1} = 0$. Since $\mathbf{P}(\mathbf{u}_2) = 0$ and $(\mathbf{I_d} - \mathbf{P})(\mathbf{t}_2) = 0$, we have also $\mathbf{u_2}\mathbf{t_2} = \mathbf{P}(\mathbf{u}_2)\mathbf{P}(\mathbf{t}_2) + (\mathbf{I_d} - \mathbf{P})(\mathbf{u}_2)(\mathbf{I_d} - \mathbf{P})(\mathbf{t}_2) = 0$. Then we conclude $a_{12} = 0$ on Γ_{DC}.

5.3.3 Comparison between combined BC and Neumann BC

We assume that BVP1 has a combined boundary condition on the part Γ_{CN} when BVP2 has Neumann boundary condition. Then on Γ_{CN}, $\mathbf{t}_2 = 0$ and $\mathbf{u}_1\mathbf{t}_1 = 0$, and we conclude $a_{12} = 0$ on Γ_{CN}.

5.3.4 Comparison between Dirichlet BC and Robin BC

We assume that BVP1 has a Dirichlet boundary condition on the part Γ_{DR} when BVP2 has Robin boundary condition. Then on Γ_{DR}, $\mathbf{u}_1 = 0$ and $\mathbf{u}_2 = -\mathbf{k}\mathbf{t}_2$. We conclude:

$$a_{12}(\mathbf{x}) = \mathbf{u}_2^\top \mathbf{k}_2 \mathbf{u}_2 \geq 0; \; \mathbf{x} \in \Gamma_{DR}. \qquad (36)$$

5.3.5 Comparison between Robin BC and Neumann BC

We assume that BVP1 has a Robin boundary condition on the part Γ_{RN} when BVP2 has Neumann boundary condition. Then on Γ_{DR}, we have $\mathbf{u}_1 = -\mathbf{k}_1\mathbf{t}_1$ and $\mathbf{t}_2 = 0$. We conclude:

$$a_{12}(\mathbf{x}) = \mathbf{u}_1^\top \mathbf{k}_1 \mathbf{u}_1 \geq 0; \; \mathbf{x} \in \Gamma_{RN}. \qquad (37)$$

5.3.6 Comparison between Robin BC with $\mathbf{k}_1 \geq \mathbf{k}_2$

We assume that the two problems have the Robin boundary conditions Γ_{RR} with $\mathbf{k}_1 \geq \mathbf{k}_2$. We write a_{12}, substituting $\mathbf{t}_i = -\mathbf{k}_i \mathbf{u}_i$ and using the symmetry of $\mathbf{k}_2$:

$$\begin{aligned} a_{12} =& \mathbf{u}_1^\top \mathbf{k}_1 \mathbf{u}_1 + \mathbf{u}_2^\top \mathbf{k}_2 \mathbf{u}_2 - 2\mathbf{u}_1^\top \mathbf{k}_2 \mathbf{u}_2 \\ =& (\mathbf{u}_1 - \mathbf{u}_2)^\top \mathbf{k}_2 (\mathbf{u}_1 - \mathbf{u}_2) + \mathbf{u}_1^\top (\mathbf{k}_1 - \mathbf{k}_2) \mathbf{u}_1. \end{aligned} \tag{38}$$

As $\mathbf{k}_1 \geq \mathbf{k}_2 \geq 0$, we conclude $a_{12} \geq 0$ on Γ_{RR}.

5.4 Remarks and conclusion

Using the results of the two preceding subsections allows the comparison of all cases of piecewise boundary conditions given in Table 1.

BVP$_2$ \ BVP$_1$	Dirichlet	Combined	Robin	Neumann
Dirichlet	0	(a)	(c)	(a)
Combined	0	0 if $\mathbf{p}_1$=$\mathbf{p}_2$	(b)	(a)
Robin	$\geq$0	(b)	$\geq$0 if $\mathbf{k}_1 \geq \mathbf{k}_2$	(c)
Neumann	0	0	$\geq$0	0

Table 1 The value of a_{12} when comparing the local boundary conditions. (a) in these cases $a_{21} = 0$; (b) the sign is a priori unknown; (c) in these cases $a_{21} \geq 0$.

Finally, if the two problems BVP$_1$, BVP$_2$ are such that $a_{12} \geq 0$ holds on the whole boundary, then we conclude that $\mathbf{B}_2 \geq \mathbf{B}_1$ and that the degenerate scales of BVP$_2$ are larger than those of BVP$_1$.

An especially interesting case is the comparison of a problem with Dirichlet boundary condition and any problem with parts of the boundary submitted to Neumann, Robin or combined BC: we have always the intrinsic degenerate scale factors which are larger than for the case with Dirichlet condition on the whole boundary. These intrinsic degenerate scale factors are also larger than the inverse of the logarithmic capacity of the boundary. Table 1 allows also to conclude that replacing any boundary condition by a Neumann boundary condition leads to higher values of degenerate scales.

6 Influence of the ratio of the elasticity coefficients on matrix B in the case of a 2-subdomains problem

We consider a 2-subdomains problem (Fig. 5) and we study the influence on $\mathbf{B}$ of a change of the elasticity constants in one of the subdomains, ν being unchanged.

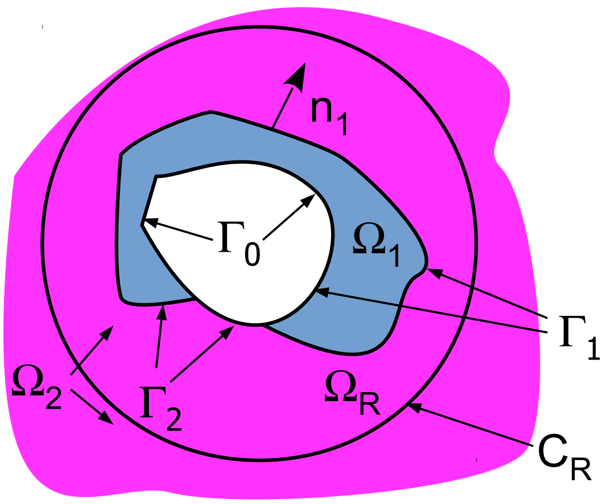

Fig. 5 Cases of two subdomains

As the degenerate scales do not vary if we multiply all the coefficients λ_i, G_i by a same constant, we will assume without loss of generality that the elasticity constants are the same in the unbounded subdomain Ω_2. So we compare the problems a and b which differ only by the elasticity tensor in Ω_1: $\mathbf{C}_{1a} = k\mathbf{C}_{2a}$. We can also write: $\Delta^*_{1a} = k\Delta^*_{1b}$. We consider the solutions $(\mathbf{v}_a, \mathbf{v}_b)$ of these two problems for the same $\boldsymbol{\xi}$ in the radiation condition.

We apply the first Green identity to $(\alpha\mathbf{v}_a - \mathbf{v}_b)$ on Ω_1, α being constant. We get

$$\begin{aligned}
&\int_{\Omega_1} (\alpha\mathbf{v}_a - \mathbf{v}_b)\Delta^*_{1a}(\alpha\mathbf{v}_a - \mathbf{v}_b) = 0 \\
&= \underbrace{\int_{\Gamma_1\cap\Gamma_0} (\alpha\mathbf{v}_a - \mathbf{v}_b)(\alpha\mathbf{t}_a^{1,0} - k\mathbf{t}_b^{1,0})}_{I_0} + \underbrace{\int_{\Gamma_1\cap\Gamma_2} (\alpha\mathbf{v}_a - \mathbf{v}_b)(\alpha\mathbf{t}_a^{1,2} - k\mathbf{t}_b^{1,2})}_{I_1} \\
&\underbrace{-\int_{\Omega_1} ((\alpha\boldsymbol{\epsilon}_a - \boldsymbol{\epsilon}_b)^\top \mathbf{C}_{1a}(\alpha\boldsymbol{\epsilon}_a - \boldsymbol{\epsilon}_b))}_{I_2 \leq 0}.
\end{aligned} \tag{39}$$

We restrict ourselves to the case of a boundary with parts submitted to Dirichlet, Neumann and combined Dirichlet-Neumann conditions. We can then conclude that $I_0 = 0$ and finally $I_1 \geq 0$. We define $A_{aa} = \int_{\Gamma_1\cap\Gamma_2} \mathbf{v}_a \mathbf{t}_a^{1,2}$; $A_{ab} = \int_{\Gamma_1\cap\Gamma_2} \mathbf{v}_a \mathbf{t}_b^{1,2}$; $A_{ba} = \int_{\Gamma_1\cap\Gamma_2} \mathbf{v}_b \mathbf{t}_a^{1,2}$; $A_{bb} = \int_{\Gamma_1\cap\Gamma_2} \mathbf{v}_b \mathbf{t}_b^{1,2}$. Hence, the inequality on I_1 can be written:

$$\alpha^2 A_{aa} - \alpha k A_{ab} - \alpha A_{ba} + k A_{bb} \geq O. \tag{40}$$

We apply the second Betti formula on Ω_1:

$$\begin{aligned}&\int_{\Omega_1} \mathbf{v}_a \Delta_{1a}^*(\mathbf{v}_b) - \mathbf{v}_b \Delta_{1a}^*(\mathbf{v}_a) = 0\\ &= \underbrace{\int_{\Gamma_1\cap\Gamma_0} \mathbf{v}_a k \mathbf{t}_b^{1,0} - \mathbf{v}_b \mathbf{t}_a^{1,0}}_{I_3=0} + \underbrace{\int_{\Gamma_1\cap\Gamma_2} k \mathbf{v}_a \mathbf{t}_b^{1,2} - \mathbf{v}_b \mathbf{t}_a^{1,2}}_{I_4}.\end{aligned} \tag{41}$$

Under the same assumption as before, $I_3 = 0$ and we conclude:

$$k A_{ab} - A_{ba} = 0. \tag{42}$$

Writing the first Betti formula on Ω_R, we get:

$$\begin{aligned}&\int_{\Omega_R} (\mathbf{v}_a - \mathbf{v}_b) \Delta_2^*(\mathbf{v}_a - \mathbf{v}_b) = 0\\ &= \underbrace{\int_{\Gamma_2\cap\Gamma_0} (\mathbf{v}_a - \mathbf{v}_b)(\mathbf{t}_a^{2,0} - k\mathbf{t}_b^{2,0})}_{I_5} + \underbrace{\int_{\Gamma_2\cap\Gamma_1} (\mathbf{v}_a - \mathbf{v}_b)(\mathbf{t}_a^{2,1} - k\mathbf{t}_b^{2,1})}_{I_6}\\ &\underbrace{+\int_{C_R} (\mathbf{v}_a - \mathbf{v}_b)(\mathbf{t}_a^{2,R} - k\mathbf{t}_b^{2,R})}_{I_7} \underbrace{-\int_{\Omega_R} ((\boldsymbol{\epsilon}_a - \boldsymbol{\epsilon}_b)^\top \mathbf{C}_2 (\boldsymbol{\epsilon}_a - \boldsymbol{\epsilon}_b))}_{I_8 \leq 0}.\end{aligned} \tag{43}$$

We have $I_5 = 0$ as there is no part of the boundary submitted to Robin condition, we have $I_7 \to 0$ when $R \to \infty$, and we finally conclude:

$$-A_{aa} + A_{ab} + A_{ba} - A_{bb} \geq 0. \tag{44}$$

Finally, we use the second Betti formula on Ω_R:

$$\begin{aligned}&\int_{\Omega_R} \mathbf{v}_a \Delta_2^*(\mathbf{v}_b) - \mathbf{v}_b \Delta_2^*(\mathbf{v}_a) = 0\\ &= \underbrace{\int_{\Gamma_2\cap\Gamma_0} \mathbf{v}_a \mathbf{t}_b^{2,0} - \mathbf{v}_b \mathbf{t}_a^{2,0}}_{I_9=0} + \underbrace{\int_{\Gamma_2\cap\Gamma_1} \mathbf{v}_a \mathbf{t}_b^{2,1} - \mathbf{v}_b \mathbf{t}_a^{2,1}}_{I_{10}} + \underbrace{\int_{C_R} \mathbf{v}_a \mathbf{t}_b^{2,R} - \mathbf{v}_b \mathbf{t}_a^{2,R}}_{I_{11}}.\end{aligned} \tag{45}$$

The asymptotic value of I_{11} for $r \to \infty$ is $\boldsymbol{\mu}_a.\boldsymbol{\xi} - \boldsymbol{\mu}_b.\boldsymbol{\xi} = \boldsymbol{\xi}.(\mathbf{B}_a - \mathbf{B}_b).\boldsymbol{\xi}$. Finally we have:

$$-A_{ab} + A_{ba} + \boldsymbol{\xi}^\top (\mathbf{B}_a - \mathbf{B}_b) \boldsymbol{\xi} = 0. \tag{46}$$

Now we choose $\alpha = \sqrt{k}$ and we divide (40) by k and we add it to (44). We get:

$$(1 - \sqrt{k})A_{ab} + (1 - \frac{1}{\sqrt{k}})A_{ba} \geq 0. \tag{47}$$

Substituting (42) in the above equation we get $(1-\sqrt{k})^2 A_{ab} \geq 0$. Substituting (42) in (46) we get:

$$\boldsymbol{\xi}^\top(\mathbf{B}_a - \mathbf{B}_b)\boldsymbol{\xi} = (1-k)A_{ab}. \tag{48}$$

As $A_{ab} \geq 0$ and the above equality is true for all $\boldsymbol{\xi}$ we conclude:

$$\mathbf{B}_a \geq \mathbf{B}_b \text{ if } k < 1;\ \mathbf{B}_a \leq \mathbf{B}_b \text{ if } k > 1. \tag{49}$$

A similar result has been shown in [16] for Laplace equation, when one compares the same infinite subdomain with different conductivities.

7 Characterization of the degenerate scales of the systems of BIEs

Up to now, only the case of the intrinsic degenerate scales has been considered. We turn now to the consideration of the set of all degenerate scales of the system of BIEs.

7.1 Necessary condition for the loss of uniqueness

As for the interior problem, we assume that the system of BIEs is at a degenerate scale, i.e., there is a non null solution with homogeneous boundary conditions on Γ_0. If we assume that none of the bounded subdomains Ω_i $1 \leq i \leq (n-1)$ is at a degenerate scale, then it is possible to build a solution of the BVP in the union of the bounded subdomains.

We consider now the solution $\mathbf{v}^n$ in Ω_n with Dirichlet condition on Γ_n $\mathbf{v}^n = \mathbf{u}^n$ and a radiation condition defined by $\boldsymbol{\xi} = \int_{\Gamma_n} \mathbf{t}^n$. Then $\mathbf{v}$ satisfies the following BIE:

$$\mathbf{v}^n(\mathbf{x}) + \int_{\Gamma_n} ([\mathbf{v}^n(\mathbf{y}) - \mathbf{v}^n(\mathbf{x})]\mathbf{T}(\mathbf{x},\mathbf{y}) - \mathbf{t}^n(\mathbf{y})\mathbf{U}(\mathbf{x},\mathbf{y}))\, \mathrm{d}S_y = \boldsymbol{\mu}. \tag{50}$$

The comparison with the BIE satisfied by $(\mathbf{u}^n, \mathbf{t}^n)$ gives $\int_{\Gamma_n}(\mathbf{t}^n - \mathbf{t})\mathbf{U} = \boldsymbol{\mu}$ with $\int_{\Gamma_n}(\mathbf{t}^n - \mathbf{t}) = 0$, then it can be deduced that $\mathbf{t}_n = \mathbf{t}$ (see, e.g., the work of reference [38]). Therefore, we conclude that there is a solution of the BVP with the radiation condition such that $\boldsymbol{\mu} = 0$. This solution is $\neq 0$, otherwise all $\mathbf{u}^i, \mathbf{t}^i$ would be null.

So we can conclude that a necessary condition for the loss of uniqueness of the BIEs is that one of the bounded subdomains is at a degenerate scale (for Dirichlet condition) or that there is a non null solution of the BVP with $\boldsymbol{\mu} = 0$.

7.2 Sufficient condition for the loss of uniqueness

If there is a non-null solution of the BVP with $\boldsymbol{\mu} = 0$ then we find a solution of the systems of BIEs.

If a bounded subdomain is at a degenerate scale, say Ω_1, we can, as for the interior problem, change the boundary and transmission condition of this subdomain and find a solution of a BVP with this modified conditions and a radiation condition written with $\boldsymbol{\xi} = 0$. If the solution is such that $\boldsymbol{\mu} = 0$, then it satisfies the corresponding BIE in Ω_n. Then, as for the interior case, we can find a non null solution of the BIEs relative to the bounded subdomains. If $\boldsymbol{\mu} \neq 0$, and if the scale is not an intrinsic degenerate scale, we can find $\boldsymbol{\xi}'$ such that there is a solution $\mathbf{u}'$ with a radiation condition $\boldsymbol{\xi}'$ and $\boldsymbol{\mu}' = -\boldsymbol{\mu}$. (See section 4). Then, adding $\mathbf{u} + \mathbf{u}'$ we find a solution of the BVP which satisfies the BIEs relative to all subdomains except Ω_1 and all the boundary and transmission conditions except those relative to Γ_1. For this boundary we replace $\mathbf{t}_1 + \mathbf{t}'_1$ by $\mathbf{t}_1 + \mathbf{t}'_1 - \mathbf{t}_d$, and we have found a non null solution (since the corresponding radiation condition is with $\boldsymbol{\xi}' \neq 0$).

We finally conclude that for the exterior problems with n subdomains, the set of degenerate scales comprises the $2(n-1)$ degenerate scales of the bounded subdomains and the two intrinsic degenerate scales linked to the non null solution of a BVP with a specific radiation condition and determined by the matrix $\mathbf{B}$.

8 Two sufficient conditions for uniqueness of the solution of the system of BIEs

In a first step, the global problem with its various boundary conditions is compared with the problem with Dirichlet boundary conditions. In a second step, a condition of uniqueness can be deduced.

8.1 Comparison of the intrinsic degenerate scale factors of the global problem with those of a Dirichlet problem on Γ_n

We compare the problem of the piecewise homogeneous problem with generalized mixed boundary condition with the Dirichlet problem on the unbounded homogeneous subdomain of the first problem. We consider the solution $\mathbf{u}_1$ of the BVP on the whole domain and $\mathbf{u}_2$ the solution of the boundary problem on Ω_n with null Dirichlet boundary condition on Γ_n. The radiation condition is the same for the two problems with the same $\boldsymbol{\xi}$.

We write first the following quantity:

$$
\begin{aligned}
&\sum_{i=1}^{n-1}\int\limits_{\Omega_i}\mathbf{u}_1\boldsymbol{\Delta}_i^*\mathbf{u}_1+\int\limits_{\Omega_R}(\mathbf{u}_1-\mathbf{u}_2)\boldsymbol{\Delta}_n^*(\mathbf{u}_1-\mathbf{u}_2)=0\\
&=\underbrace{\sum_{i=1}^{n-1}\int\limits_{\Gamma_i\cap\Gamma_0}\mathbf{u}_1\mathbf{t}_1^{i,0}}_{L_1}+\underbrace{\sum_{i=1}^{n-1}\sum_{j=1,j\neq i}^{n-1}\int\limits_{\Gamma_i\cap\Gamma_j}\mathbf{u}_1\mathbf{t}_1^{i,j}}_{L_2}+\underbrace{\sum_{i=1}^{n-1}\int\limits_{\Gamma_i\cap\Gamma_n}\mathbf{u}_1\mathbf{t}_1^{i,n}}_{L_3}\\
&\quad+\underbrace{\sum_{i=1}^{n}\int\limits_{\Gamma_i\cap\Gamma_n}(\mathbf{u}_1-\mathbf{u}_2)(\mathbf{t}_1^{n,i}-\mathbf{t}_2^{n,i})}_{L_4}+\underbrace{\int\limits_{\mathbf{C_R}}(\mathbf{u}_1-\mathbf{u}_2)(\mathbf{t}_1^{n,R}-\mathbf{t}_2^{n,R})}_{L_5}\\
&\quad\underbrace{-\sum_{i=1}^{n}\int\limits_{\Omega_i}\boldsymbol{\epsilon}_1^\top\mathbf{C}_i\boldsymbol{\epsilon}_1}_{L_6}\underbrace{-\int\limits_{\Omega_R}(\boldsymbol{\epsilon}_1-\boldsymbol{\epsilon}_2)^\top\mathbf{C}_n(\boldsymbol{\epsilon}_1-\boldsymbol{\epsilon}_2)}_{L_7}\,.
\end{aligned}
\tag{51}
$$

Due to the boundary conditions applied to $\mathbf{u}_1$ on Γ_0, we have $L_1 \leq 0$; the quantity $L_6 + L_7$ is also ≤ 0. Due to the transmission condition, we have $L_2 = 0$. We must consider now the quantity $L_3 + L_4$.

We have: $u_2 = 0$ on Γ_n, and $t^{i,n} = -t^{n,i}$. Finally, $L_3 + L_4$ reduces to $-\sum_{i=1}^{n-1}\int_{\Gamma_i\cap\Gamma_n}\mathbf{u}_1\mathbf{t}_2^{n,i}$ which is ≥ 0 as $L_5 \to 0$ when $R \to \infty$.

We consider now:

$$
\int\limits_{\Omega_R}\mathbf{u}_1\boldsymbol{\Delta}_i^*\mathbf{u}_2-\mathbf{u}_2\boldsymbol{\Delta}_i^*\mathbf{u}_2=0=\underbrace{\sum_{i=1}^{n}\int\limits_{\Gamma_i\cap\Gamma_n}\mathbf{u}_1\mathbf{t}_2^{n,i}-\mathbf{u}_2\mathbf{t}_1^{n,i}}_{L_8}+\underbrace{\int\limits_{\mathbf{C}_R}\mathbf{u}_1\mathbf{t}_2^{n,R}-\mathbf{u}_2\mathbf{t}_1^{n,R}}_{L_9}\,.
\tag{52}
$$

Then we notice that L_8 reduces to $\sum_{i=1}^{n}\int_{\Gamma_i\cap\Gamma_n}\mathbf{u}_1\mathbf{t}_2^{n,i} \leq 0$. As the limit of L_9 when $R \to \infty$ is $(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{B}_1 - \mathbf{B}_2)\boldsymbol{\xi}$ (from Eq. (29)), we conclude that $\mathbf{B}_1 \geq \mathbf{B}_2$. The couple of intrinsic degenerate scale factors for the problem in the exterior domain bounded by Γ_0 with generalized mixed condition is equal or larger than those for the exterior domain bounded by Γ_n with Dirichlet condition.

This result is similar to the partial result for the 2 concentric circles in the Laplace problem: the intrinsic degenerate scale factor of the heterogeneous problem is always larger that the degenerate scale factor for the largest circle [16]. The general result that we get for elasticity can be straightforwardly extended to the case of Laplace equation by using the same approach.

8.2 Sufficient conditions

We can deduce two sufficient conditions for the uniqueness of the solution. A sufficient condition for the intrinsic degenerate scale factors to be > 1 is that the two eigenvalues of $\mathbf{B}_2$ are > 0.

This is the case if the boundary Γ_n is included in a circle of radius $< \mathrm{e}^{1/2\kappa_m}$ as seen in [38] with $\kappa_m \geq \kappa_i, i \in \{1, .., n\}$. Then all the boundaries Γ_i, $1 \leq i \leq n$ are also included in the circle of radius $< \mathrm{e}^{1/2\kappa_i}$ if $\kappa_i \geq \kappa$ and the condition is sufficient to ensure the uniqueness of the solution.

Another sufficient condition is that the logarithmic capacity of Γ_n is < 1. According to section 2, we then deduce that the degenerate scale factors associated with the Dirichlet problem on Γ_n are > 1. But that is also true for all the degenerate scales associated to Γ_i, $1 \geq i \geq n-1$ since the logarithmic capacity of Γ_i is smaller or equal to the logarithmic capacity of Γ_n (see for example [22]). These two sufficient conditions are also valid for interior problems if we consider Γ_0 instead of Γ_n.

9 Definition of a matrix $\tilde{\mathbf{B}}$

We presented in section 4 a matrix $\mathbf{B}$ allowing to recover the intrinsic degenerate scale factors. In this section, we introduce a matrix related to the whole set of degenerate scales for the heterogeneous domain.

We consider the following problem to find $\mathbf{u}^i, \mathbf{t}^i$ defined on Γ_i satisfying all the boundary and transmission conditions. We modify the initial BIEs by adding a right member $\boldsymbol{\mu}_i$ and conditions $\int_{\Gamma_i} \mathbf{t}^i = \boldsymbol{\xi}^i$. We define $\tilde{\boldsymbol{\mu}} = (\boldsymbol{\mu}^1, .., \boldsymbol{\mu}^n)^\top$ and $\tilde{\boldsymbol{\xi}} = (\boldsymbol{\xi}^1, .., \boldsymbol{\xi}^n)^\top)$. This system of augmented BIEs provides a matrix $\tilde{\mathbf{B}}$ defined by $\tilde{\boldsymbol{\mu}} = \tilde{\mathbf{B}}\tilde{\boldsymbol{\xi}}$. It is a generalization of the matrix defined in [38] for finding the degenerate scales of the homogeneous Dirichlet problem.

This matrix is well defined: for given $\tilde{\boldsymbol{\xi}}$, there is at most one solution $\tilde{\boldsymbol{\mu}}$. If there were two solutions their difference $\mathbf{u}^{*i}, \mathbf{t}^{*i}$ would satisfy all the initial BIEs with $\int_{\Gamma_i} \mathbf{t}^{*i} = 0$. For the exterior problem we add $\boldsymbol{\xi} = 0$ in the radiation condition. Then solving the Dirichlet BVP in each Γ_i, we find $\hat{\mathbf{v}}^i, \hat{t}^i$ which satisfy the BIEs with $\int_{\Gamma_i} \hat{\mathbf{t}}_i = 0$. Then we have $\int_{\Gamma_i} \mathbf{U}(\hat{\mathbf{t}}^i - \mathbf{t}^{*i}) = 0$ with $\int_{\Gamma_i} (\hat{\mathbf{t}}^i - \mathbf{t}^{*i}) = 0$. This ensures that $\hat{\mathbf{t}}^i - \mathbf{t}^{*i} = 0$ [38]. And then we have found a global solution of the BVP with the radiation condition $\boldsymbol{\xi} = 0$ and this solution is null (section 4.2). So the solution $\tilde{\boldsymbol{\mu}}$ is unique. It is easy to see that the function $\tilde{\boldsymbol{\mu}} \to \tilde{\boldsymbol{\xi}}$ is linear. So, in order to check that there is a solution for all $\tilde{\boldsymbol{\xi}}$, it is enough to check that it is true for a basis of the vectors $\tilde{\boldsymbol{\xi}}$.

We consider an orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$ in the plane and we assume that all the BIEs are written using this basis. We define the basis for the vectors $\tilde{\boldsymbol{\xi}}$, the following set of vectors $\tilde{\boldsymbol{\xi}}^{i,1}, \tilde{\boldsymbol{\xi}}^{i,2}$ defined as having all its components null except $\tilde{\xi}^{2i-1} = 1$ for $\tilde{\boldsymbol{\xi}}^{i,1}$ and $\tilde{\xi}^{2i} = 1$ for $\tilde{\boldsymbol{\xi}}^{i,2}$.

We focus on the exterior problem. The solutions for $\tilde{\boldsymbol{\xi}}^i = 0$ if $i \neq n$ and $\tilde{\boldsymbol{\xi}}^n \neq 0$ are defined by a solution of the boundary value problem of section 4.1 and we have $\boldsymbol{\mu}^i = 0$ if $i \neq n$.

We consider now the ξ^i, $i \neq n$, linked to the bounded subdomains, for example $\tilde{\boldsymbol{\xi}}^{1,1} = 1$ and all other components of vectors $\tilde{\xi}^i = 0$. Then, we can find $\mathbf{t}_d^1$ on Ω_1 such that $\int_{\Gamma_1} \mathbf{U}\mathbf{t}_d^1 = \boldsymbol{\mu}^1$ and $\int_{\Gamma_1} \mathbf{t}_d^1 = \binom{1}{0}$. Next, we consider the global auxiliary boundary $\mathbf{u}$ value problem with the conditions (14), (15) (case of the interior problem). As a consequence, the solution of this problem satisfies the initial BIEs (without any $\boldsymbol{\mu}^i$) for $i \neq n$. If we consider $\tilde{\mathbf{t}}^1 = \mathbf{t}^1 - \mathbf{t}_d^1$, and $\tilde{\mathbf{u}}^i = \mathbf{u}^i$ for all i and $\tilde{\mathbf{t}}^i = \mathbf{t}^i$ if $i \neq 1$. Then $(\tilde{\mathbf{u}}^i, \tilde{\mathbf{t}}^i)$ satisfies all the boundary conditions and transmission conditions of the initial problem and the initial BIEs except the one relative to Γ_1 which has the right hand term $\boldsymbol{\mu}^1$ and possibly $\boldsymbol{\xi}^n \neq 0$ and also the augmented BIEs with $\tilde{\boldsymbol{\xi}}^i = 0$, except $i = 1$ and $i = n$. Combining with the solutions with $\boldsymbol{\xi}^n \neq 0$ we build a solution with $\boldsymbol{\xi} = 0$ if $i \neq 1$ and $\boldsymbol{\mu}^i = 0$ for $i \neq 1, \neq n$. The case of the interior problem is the same except for the unbounded domain Ω_n for which $\tilde{\mathbf{u}}$ is found as the solution of a global BVP with $\boldsymbol{\xi}^n$ in the radiation condition.

The way used to find $\tilde{\boldsymbol{\mu}}$ shows that the matrix $\left[\tilde{\mathbf{B}}\right]$ with the chosen basis is as follows:

$$
[\tilde{\mathbf{B}}] = \begin{bmatrix}
\mathbf{B}_{11} & 0 & \ldots & & \ldots & 0 \\
0 & \mathbf{B}_{22} & 0 & & & \vdots \\
\vdots & \ddots & \ddots & & \ddots & \vdots \\
0 & \ldots & 0 & \mathbf{B}_{(n-1)(n-1)} & & 0 \\
\mathbf{B}_{n1} & \ldots & \ldots & \mathbf{B}_{n(n-1)} & & \mathbf{B}_{nn}
\end{bmatrix}; \tag{53}
$$

where $\left[\mathbf{B}_{ij}\right]$ is a 2×2 matrix.

The case of the interior problem is the same except that there are no longer the two rows corresponding to the unbounded domain Ω_n.

It can been checked directly that the $2n$ eigenvalues of $\left[\tilde{\mathbf{B}}\right]$ are linked to the degenerate scale factors of the problem. We assume that α is an eigenvalue of $\left[\tilde{\mathbf{B}}\right]$ with eigenvector $\tilde{\boldsymbol{\xi}}$, then there is a solution $\mathbf{u}, \mathbf{t}, \tilde{\boldsymbol{\mu}}$) such that:

$$
-\frac{1}{2}\mathbf{u}^i + \int_{\Gamma_i} \mathbf{U}\mathbf{t}^i - \int_{\Gamma_i} \mathbf{T}^i\mathbf{u}^i = \tilde{\boldsymbol{\mu}}^i = \alpha\tilde{\boldsymbol{\xi}}^i \text{ with } \int_{\Gamma_i} \mathbf{t}^i = \tilde{\boldsymbol{\xi}}^i. \tag{54}
$$

If we change the scale of the problem by a factor ρ and we use $\int_{\rho\Gamma_i} \mathbf{U}_\rho \mathbf{t}^i(y/\rho) = \rho(\int_{\Gamma_i} \mathbf{U}_\rho \mathbf{t}^i(y) - \Lambda\kappa \ln\rho \int_{\Gamma_i} \mathbf{t}^i)$ [38]the BIEs becomes in this new scale:

$$
-\frac{1}{2}\mathbf{u}^i(\frac{\mathbf{x}}{\rho}) + \int_{\rho\Gamma_i} \mathbf{U}(\mathbf{x},\mathbf{y})\frac{\mathbf{t}_\rho^i(\mathbf{y}/\rho)}{\rho} - \int_{\rho\Gamma_i} \mathbf{T}^i(\mathbf{x},\mathbf{y})\mathbf{u}^i(\frac{\mathbf{y}}{\rho}) = \tilde{\boldsymbol{\mu}}^i - \Lambda\kappa \ln\rho \int_{\Gamma_i} \mathbf{t}^i. \tag{55}
$$

If we choose $\rho = \exp\frac{\alpha}{\Lambda\kappa}$, then $\tilde{\boldsymbol{\mu}}_i - \Lambda\kappa\ln\rho\int_{\Gamma_i}\mathbf{t}^i = \tilde{\boldsymbol{\mu}}^i - \alpha\tilde{\boldsymbol{\xi}}^i = 0$, and ρ is a degenerate scale factor. The boundary conditions are satisfied by $\mathbf{u}^i(\frac{\mathbf{x}}{\rho})$, $\frac{\mathbf{t}^i}{\rho}$ for the scaled problem with Dirichlet, Neumann or combined Dirichlet-Neumann boundary conditions. For the Robin condition it is also necessary to scale $\mathbf{k}$: if $\mathbf{u} = \mathbf{k}\mathbf{t}$, then, $\mathbf{u} = (\rho\mathbf{k})(\frac{1}{\rho}\mathbf{t})$.

10 Numerical applications

10.1 BEM formulation for the mixed problem

The practical applications rest on the discretization of equation (54). The BEM formulation of this equation can be written for a mixed problem with homogeneous boundary conditions as:

$$\begin{cases} [U_D][t_D] - [T_N][u_N] = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_1 \\ \mu_2 \\ \dots \end{bmatrix} ; & (56a) \\ [S][t_D] = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} . & (56b) \end{cases}$$

where $[t_D]$ contains the unknown components of nodal tractions on the boundary and $[U_D]$ the unknown components of nodal displacements, the interaction matrices $[U_D]$ and $[T_N]$ contain the columns consistent with the unknown nodal values. $[S]$ is the matrix computed from interpolation functions N_i related to nodes of Γ_D by:

$$[S] = \begin{bmatrix} \int_\Gamma N_1 ds & 0 & \int_\Gamma N_2 ds \dots \\ 0 & \int_\Gamma N_1 ds & 0 \quad \dots \end{bmatrix} . \tag{57}$$

This leads to the linear system written in a matrix form as:

$$\begin{bmatrix} U_D & -T_D & V_1 & V_2 \\ S & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t_D \\ u_N \\ \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \xi_1 \\ \xi_2 \end{bmatrix} ; \tag{58}$$

with the notation

$$V_1 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ \dots \end{bmatrix} ; V_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ \dots \end{bmatrix} . \tag{59}$$

This matrix has the dimension $2.n_D \times 2$, where n_D is the number of nodes at a Dirichlet boundary condition.

The full system has the dimension $(2N + 2) \times (2N + 2)$ for a number N of nodes. Solving this system with unknown μ_1, μ_2 for $\xi_1 = 1, \xi_2 = 0$ (resp. $\xi_1 = 0, \xi_2 = 1$) gives $\mu_1 = B_{11}, \mu_2 = B_{21}$ (resp. $\mu_1 = B_{12}, \mu_2 = B_{22}$), where B_{ij} are the components of matrix $\tilde{B}$. Then, the degenerate scales are obtained from the eigenvalues of $\tilde{B}$, as explained previously.

10.2 BEM formulation of the heterogeneous problem

We consider now the case of a heterogeneous domain, made of an exterior domain Ω_A containing an interior domain Ω_B, the two domains being in contact along Γ_I. We study Dirichlet boundary conditions applied on the interior boundary of Ω_B as in Fig. 6.

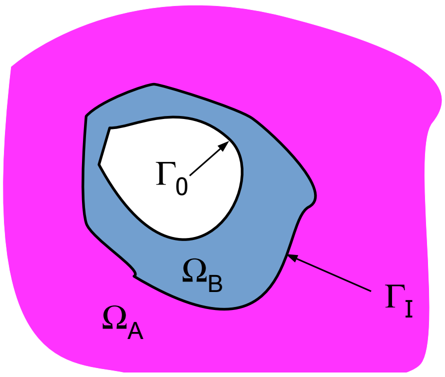

Fig. 6 Notations for BEM formulation of the heterogeneous problem

The BEM system of equations can be written:

$$\begin{cases} [U_{AI}][t_I] - [T_{AI}][u_I] = [\mu_A]; & \text{(60a)} \\ [U_B|U_{BI}] \begin{bmatrix} t_B \\ -t_I \end{bmatrix} - [T_{BI}][u_I] = [\mu_B]; & \text{(60b)} \\ [S_I][t_I] = [\xi_A]; & \text{(60c)} \\ [S_B][t_B] - [S_I][t_I] = [\xi_B]. & \text{(60d)} \end{cases}$$

As shown previously, the degenerate scales contain the one of Ω_B and the intrinsic degenerate scales. From the previous section, the intrinsic degenerate scales can be obtained by looking for μ_A and μ_B related to the infinite domain with $\xi_B = 0$. In addition, it has been seen that changing the scale involves the multiplication by the term $\Lambda\kappa$ of the logarithmic term. Finally, the system of linear equations can be synthetized into the following matrix form:

$$\begin{bmatrix} 0 & U_{AI} & -T_{AI} & V_{A1} & V_{A2} & 0 & 0 \\ U_B & -U_{BI} & -T_{BI} & 0 & 0 & V_{B1} & V_{B2} \\ 0 & S_I & 0 & 0 & 0 & 0 & 0 \\ 0 & -S_I & S_B & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t_B \\ t_I \\ u_I \\ \mu_{A1} \\ \mu_{A2} \\ \mu_{B1} \\ \mu_{B2} \end{bmatrix} = \begin{bmatrix} 0 \\ \dots \\ 0 \\ \xi_{A1} \\ \xi_{A2} \\ 0 \\ 0 \end{bmatrix}; \qquad (61)$$

where the components of vectors V_{A1}, V_{A2} are given by:

$$[V_{A1}] = \Lambda_A \kappa_A [V_1]\,; [V_{A2}] = \Lambda_A \kappa_A [V_2]\,; \tag{62}$$

and similarly for V_{B1}, V_{B2}.

10.3 Numerical results for the mixed boundary condition

10.3.1 Comparison with an analytical result

The exact values of degenerate scale factors in the case of mixed boundary conditions are not found in the literature.

We present here a case related to a segment of length $2b$ along axis Ox_1. On the central part of the segment of length $2a$, the boundary condition is mixed, corresponding to the values of u_2, t_1. On the remaining part, the boundary condition is of Dirichlet type. For such a configuration, one of the degenerate scale factors is given exactly by $\rho = 2e^{1/\kappa}/\sqrt{b^2 - a^2}$. This is found by considering an auxiliary problem: the degenerate scales of the set $[-b, -a] \cup [a, b]$ included in the axis Ox_1 for Dirichlet condition. The degenerate scale with the resultant of the applied forces parallel to Ox_1 is given by $\rho_0 e^{(1/\kappa)}$ [13] with $\rho_0 = 2/\sqrt{b^2 - a^2}$. The corresponding displacement field satisfies the Dirichlet condition on $[-b, -a] \cup [a, b]$; but, due to the symmetry of the problem, Ox_1 is a symmetry axis and we conclude also that $u_2 = 0$ and $t_1 = 0$ on $[-a, a]$. So we have found one of the degenerate scales of the initial problem.

The numerical test has been effected with $a = 1, b = 3, \nu = 0.25$ and 600 constant elements for the segment $[-b, b]$. The corresponding theoretical degenerate scale factor is $\rho = 1.1658$. The numerical test produces two values of ρ: $\rho_1 = 0.6671$ and $\rho_2 = 1.1668$. The second value corresponds to the theoretical value with a relative difference inferior to 10^{-3}. This is a first validation of the numerical computation.

10.3.2 Case of an exterior problem on a circle with an increasing part at Neumann boundary condition

One considers the case of an exterior problem for a circle. The boundary condition is of Dirichlet type on a part and of Neumann type on the remaining part. The boundary of the circle has been discretized with the Boundary Element Method by using 500 constant elements.

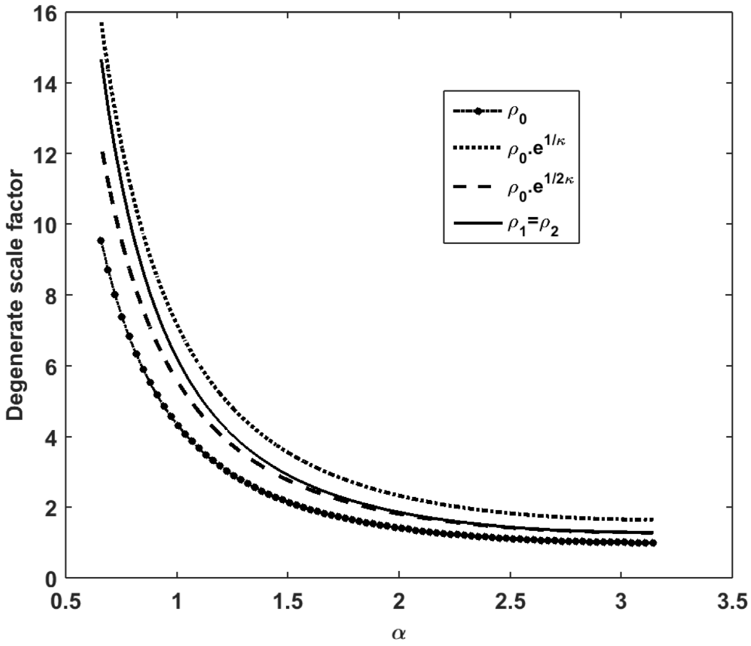

Fig. 7 Degenerate scale factors as a function of α, for increasing part at Dirichlet condition characterized by angle 2α

A first result is that both degenerate scale factors are identical. Figure 7 displays the values of $\rho_1 = \rho_2$ as a function of α, the angle characterizing the part at Dirichlet condition being given by 2α.

Three curves obtained from the degenerate scale factor for Laplace equation in a similar configuration (obtained from [15]) and from the value of κ are also given. Indeed, for the case of Dirichlet boundary condition, the following inequalities hold, as recalled in section 2:

$$\begin{cases} \rho_i < \rho_0 e^{1/\kappa}; \\ \min(\rho_1, \rho_2) < \rho_0 e^{1/2\kappa}; \\ \rho_i > \rho_0. \end{cases} \tag{63}$$

It can be seen that the common degenerate scale factor for elasticity is still comprised between ρ_0 and $\rho_0 e^{1/\kappa}$. However, the second inequality is no more valid. When the part of the domain at Neumann boundary condition increases, the degenerate scale factor increases. It tends to infinity for the case of the full boundary at Neumann condition($\alpha = 0$).

10.3.3 Case of an exterior and heterogeneous problem

We consider the case of a heterogeneous domain made of a first subdomain Ω_B being a ring between radii 2 and 3 and of the infinite domain Ω_B outside the circle of radius $R = 3$. The discretization of the boundaries has been effected by using 300 constant elements on each circular boundary.

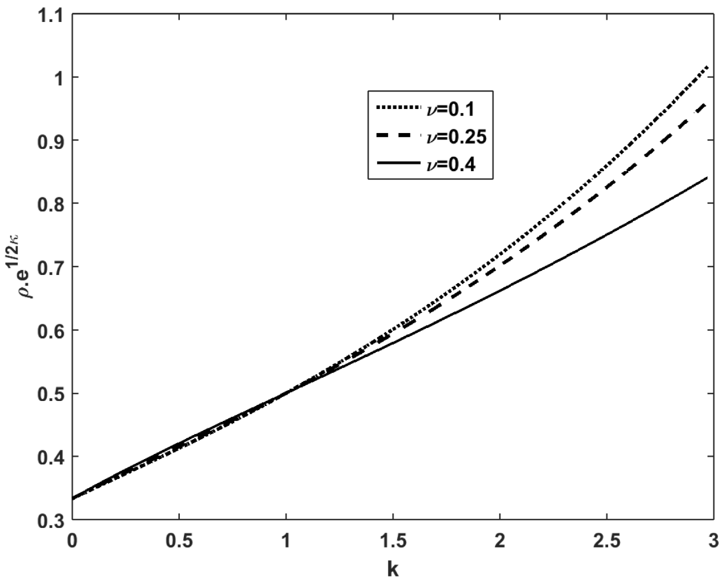

Fig. 8 Variation of $\rho e^{1/2\kappa}$ as a function of the ratio of elastic moduli, for different values of the Poisson's ratio

The degenerate scale factors have been computed for an increasing value of the ratio $k = \Lambda_B/\Lambda_A$ and for three values of Poisson's ratio, $\nu = 0.1, 0.25$, 0.4 that are the same for both domains. Both degenerate scale factors are again identical, $\rho_1 = \rho_2 = \rho$.

The results can be shown on Fig. 8 where the value of $\rho\, e^{1/2\kappa}$ can be found as a function of the contrast k. This presentation of the results is due to the fact that for the homogeneous case with Dirichlet boundary condition the degenerate scale factors are both given by $\rho_0\, e^{-1/2\kappa}$. It can be seen that the value of $\rho\, e^{1/2\kappa}$ for all curves is equal to 1/3 when the contrast is null, which corresponds to the degenerate scale factor ρ_0 of the homogeneous ring for Laplace equation. When there is no contrast ratio, i.e., $k = 1$, the problem becomes the one of a homogeneous domain with a hole of radius $R = 2$ and all curves recover the degenerate scale factor $\rho_0 = 1/2$ for Laplace equation related to this homogeneous domain.

For k between 0 and 1, the three curves are nearly the same, but differ significantly when k increases.

For very large values of k, the computation shows that the degenerate scale factor becomes very large, the problem evolving toward the case of Neumann boundary condition on the outside circle for which the degenerate scale factor is infinite.

11 Conclusion

The degenerate scales for the homogeneous Dirichlet plane elasticity problem appear to be less than or equal to the degenerate scale factor for the Laplace problem with the same boundary (section 2). The degenerate scales for homogeneous or piecewise homogeneous interior problems with mixed boundary conditions have been found to be equal to the degenerate scales for the Dirichlet problem in the case of a homogeneous domain or several homogeneous subdomains. In section 3, a matrix **B** has been introduced which gives information on the solutions of the global exterior BVP. The degenerate scales for the exterior problem are the degenerate scales of the bounded subdomains and the intrinsic degenerate scales linked to the existence of specific solutions of the global exterior BVP (section 7). The intrinsic degenerate factors ρ_1, ρ_2 for mixed boundary conditions are larger or equal to the couple of intrinsic degenerate scale factors for the Dirichlet boundary condition (section 5). A sufficient condition for uniqueness of the solution (section 8) is that all the boundaries are included in a disk with a radius smaller than $\mathrm{e}^{1/(2\kappa_m)}$ or that the boundary Γ_n has a logarithmic capacity strictly smaller than 1.

The numerical BEM application for obtaining degenerate scale factors in the case of mixed boundary conditions or heterogeneous domains has been presented. It allows the obtaining of the degenerate scale factors through the computation of the **B** matrix. Several numerical applications have been presented and are consistent with the theoretical developments.

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