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Predator-Prey Model with Prey Harvesting, Holling Response Function of Type III and SIS Disease

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***Abstract*—The populations of prey and predator interact with prey harvesting. When there is no predator, the logistic equation models the behavior of the preys. For interactions between preys and predators, we use the generalized Holling response function of type III. This function which models the consumption of preys by predators is such that the predation rate of predators increases when the preys are few and decreases when they reach their satiety. Our main goal is to analyze the influence of a SIS infectious disease in the community. The epidemiological SIS model with simple mass incidence is chosen, where only susceptibles and infectious are counted. We assume firstly that the disease spreads only among the prey population and secondly that it spreads only among the predator population. There are many bifurcations as: Hopf bifurcation, transcritical bifurcation and saddle-node bifurcation. The results indicate that either the disease dies out or persists and then, at least one population can disappear because of infection. For some particular choices of the parameters however, there exists endemic equilibria in which both populations survive. Numerical simulations on MATLAB and SCILAB are used to illustrate our results.**

***Keywords*-Predator; Prey; Infectious disease; Response function; Bifurcation; Global Stability**

I. Introduction

There are many epidemiological or ecological models [6], [7], [8], [9], [10], [11], [5] in the literature and also many models which encompass the two fields [3], [4], [8], [9], [10], [11], [12]. Dynamic models for infectious diseases are mostly based on compartment structures that were initially proposed by Kermack and McKendrick (1927,1932) and developed later by many other researchers.

The main questions regarding population dynamics concern the effects of infectious diseases in regulating natural populations, decreasing their population sizes, reducing their natural fluctuations, or causing destabilizations of equilibria into oscillations of the population states. With the Holling function response of type III, it is well known that the predators increase their searching activity when the prey density increases.

Generally, if x denotes the density of prey population, the Holling function of type I is $\phi_1(x) = r\,x$ where r is the intrinsic growth rate of preys. The Holling function of type II is $\phi_2(x) = \dfrac{B\,\omega_0\,x}{1 + B\,\omega_1\,x}$, where ω_0 and ω_1 denote respectively the time taking by a predator to search and

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capture preys, B is the predation rate per unit of time. In the models considered in this work, the Holling function of type III is used for interactions between predators and preys : $\phi_3(x) = \dfrac{m\,x^2}{a\,x^2 + b\,x + 1}$ [2], where m and a are positive constants, b is an arbitrary constant. This function models the consumption of preys by predators. It is well known that with this function, the predation rate of predators increases when the preys are few and decreases when they reach their satiety (a predator increases his searching activity when the prey density increases). The functions ϕ_1, ϕ_2 and ϕ_3 are respectively also referred to as Lotka-Volterra, Michaelis-Menten and sigmoidal response functions. Generally, there are more macroparasitic infections which can affect only preys, only predators or both preys and predators. Our goal in this paper is to analyze the influence of a SIS infectious disease which spreads only in one of the two populations. The models considered and analyzed here are different from all the models in the literature. Moreover, we use numerical simulations on MATLAB and SCILAB to illustrate our results.

II. The Model Formulation

The model (s1) is obtained from the classic Lotka-Volterra model with simple mass action when the disease spreads only inside the prey population. In this model, the infected preys do not reproduce and there is no disease related mortality. The model (s2) is obtained when the disease spreads only inside the predator population. These models are respectively

$$\begin{cases} \dot{x} = \tilde{r}\left(1 - \dfrac{x}{\tilde{k}}\right)x - \dfrac{\tilde{m}x^2y}{\tilde{a}x^2 + \tilde{b}x + 1} - \tilde{\lambda}x\,z \\ \qquad +\tilde{\gamma}z - \tilde{h}_1, \\ \dot{z} = \tilde{\lambda}x\,z - \tilde{\gamma}z - \dfrac{\tilde{m}_1z^2y}{\tilde{a}z^2 + \tilde{b}z + 1}, \qquad (s1) \\ \dot{y} = \dfrac{\tilde{c}\tilde{m}x^2y}{\tilde{a}x^2 + \tilde{b}x + 1} - \dfrac{\tilde{m}_2z^2y}{\tilde{a}z^2 + \tilde{b}z + 1} - \tilde{d}y, \\ x \geq 0,\ z \geq 0,\ y \geq 0, \end{cases}$$

$$\begin{cases} \dot{x} = \tilde{r}\left(1 - \dfrac{x}{\tilde{k}}\right)x - \dfrac{\tilde{m}x^2y}{\tilde{a}x^2 + \tilde{b}x + 1} - \dfrac{\tilde{\eta}_1x^2\omega}{\tilde{a}x^2 + \tilde{b}x + 1} \\ \qquad -\tilde{h}_1, \\ \dot{y} = \dfrac{\tilde{c}\tilde{m}x^2y}{\tilde{a}x^2 + \tilde{b}x + 1} - \tilde{d}\,y - \tilde{\delta}\,y\,\omega + \tilde{\mu}\,\omega, \qquad (s2) \\ \dot{\omega} = \dfrac{\tilde{e}\tilde{m}x^2\omega}{\tilde{a}x^2 + \tilde{b}x + 1} + \tilde{\delta}y\omega - (\tilde{\mu} + \tilde{d})\omega, \\ x \geq 0,\ y \geq 0,\ \omega \geq 0. \end{cases}$$

where the variables z and ω denotes respectively the infected preys and infected predators, $\tilde{r}$ denotes the intrinsic growth rate of preys, $\tilde{d}$ is the natural death rate of predators, $\tilde{k}$ is the capacity of environment to support the growth of preys, $\tilde{h}_1$ is the rate of preys's harvesting, $\tilde{\gamma}$ and $\tilde{\mu}$ are the recover rates of infected preys and infected predators respectively, $\tilde{\lambda}$ is the adequate contact rate between susceptible preys and infected preys while $\tilde{\delta}$ is the adequate contact rate between susceptible predators and infected predators. We also assume that infected predators still can catch preys at a different rate $\tilde{\eta}_1$ than sound ones. The parameter $\tilde{\eta}_1$ can be thought to be less than $\tilde{m}$, if the disease affects the ability in hunting of the predators or larger than $\tilde{m}$, if we want to emphasize that the interactions with infected predators cause the preys to die for the disease even if they are not caught. $\tilde{a}$ and $\tilde{b}$ are positive constants. $\tilde{m} > 0$ and $\tilde{m}_1 > 0$ denote the adequate predation rate between predators and preys. $\tilde{c}$ and $\tilde{e}$ denote the conversion coefficients. $\tilde{m}_2$ can be negative (conversion of prey's biomass into predator's biomass) or positive (bad effect of the infected preys for the predator population due to disease).

Trough the linear transformation and time scaling $(X, Z, Y, W, T) = \left(\dfrac{x}{\tilde{k}}, \dfrac{z}{\tilde{k}}, \dfrac{y}{\tilde{c}\tilde{k}}, \dfrac{\omega}{\tilde{e}\tilde{k}}, \tilde{c}\tilde{m}\tilde{k}^2t\right)$, the following simplified systems are obtained from (s1) and (s2),

$$\begin{cases} \dot{x} = \rho x(1-x) - p(x)\,y - \lambda\,x\,z + \gamma\,z - h_1, \\ \dot{z} = \lambda\,x\,z - \gamma\,z - m_1\,p(z)\,y, \\ \dot{y} = p(x)\,y - m_2\,p(z)\,y - d\,y, \\ x \geq 0; y \geq 0; z \geq 0, \end{cases} \tag{1}$$

$$\begin{cases} \dot{x} = \rho x(1-x) - p(x)\,y - \eta_1\,p(x)\,\omega - h_1, \\ \dot{y} = p(x)y - dy - \delta\,y\,\omega + \mu\,\omega, \\ \dot{\omega} = e\,p(x)\omega + \delta_1\,y\,\omega - \mu_1\,\omega, \\ x \geq 0; y \geq 0; \omega \geq 0, \end{cases} \tag{2}$$

where the parameters are defined as follow

$$\begin{aligned} &\rho = \frac{\tilde{r}}{\tilde{c}\tilde{m}\tilde{k}^2}, \eta_1 = \frac{\tilde{\eta}_1\tilde{e}}{\tilde{c}\tilde{m}}, \eta_2 = \frac{\tilde{\eta}_2\tilde{e}}{\tilde{c}\tilde{m}}, h_1 = \frac{\tilde{h}_1}{\tilde{c}\tilde{m}\tilde{k}^3}, \lambda = \frac{\tilde{\lambda}}{\tilde{c}\tilde{m}\tilde{k}}, \\ &\gamma = \frac{\tilde{\gamma}}{\tilde{c}\tilde{m}\tilde{k}^2}, m_1 = \frac{\tilde{m}_1}{\tilde{m}}, m_2 = \frac{\tilde{m}_2}{\tilde{c}\tilde{m}}, m_3 = \frac{\tilde{m}_3}{\tilde{c}\tilde{m}}, d = \frac{\tilde{d}}{\tilde{c}\tilde{m}\tilde{k}^2}, \\ &\delta = \frac{\delta\tilde{e}}{\tilde{c}\tilde{m}\tilde{k}}, \mu = \frac{\tilde{\mu}\tilde{e}}{\tilde{c}^2\tilde{m}\tilde{k}^2}, e = \frac{\tilde{e}}{\tilde{c}}, \delta_1 = \frac{\tilde{\delta}}{\tilde{m}\tilde{k}}, \mu_1 = \frac{\tilde{\mu} + d}{\tilde{c}\tilde{m}\tilde{k}^2}, \\ &a = \tilde{a}\tilde{k}^2, b = \tilde{b}\tilde{k}, p(x) = \frac{x^2}{a\,x^2 + b\,x + 1}. \end{aligned} \tag{3}$$

Systems (1) and System (2) are new and different from all the models in the literature. These models without disease give us the same system which has been analyzed without disease in [1].

III. Results

A. Results for the Model (1) with Disease only in Prey Population

Let us set $u_1(x) = \dfrac{\rho x(1-x) - h_1}{(1+m)p(x) - md}$, $R_0 = \dfrac{m_2(p(\eta)-d)u_1^2(\eta)}{a(p(\eta)-d)^2u_1^2(\eta) + b(\lambda\eta-\gamma)(p(\eta)-d)u_1(\eta) + (\lambda\eta-\gamma)^2}$ the basic reproduction number, and

$$x_1 = \frac{1-\sqrt{1-4\frac{h_1}{\rho}}}{2},\ x_2 = \frac{1+\sqrt{1-4\frac{h_1}{\rho}}}{2}, \qquad (4)$$
$$x_z = \frac{\gamma}{\lambda}, x_0 = \frac{1}{2}, z_0 \in \mathbb{R}_+^*,$$

the expressions of the positive real values x_0, x_1, x_2, x_z.

Theorem 1: The equilibrium points of System (1), according to the values of the parameters, are given as follow :

- When $h_1 > \dfrac{\rho}{4}$, then there is no equilibrium point.
- When $h_1 = \dfrac{\rho}{4}$, then the unique equilibrium is $B_0(x_0, 0, 0)$ which is a double point if $d \neq \dfrac{1}{a+2b+4}$ and triple point if $d = \dfrac{1}{a+2b+4}$.
- When $h_1 < \dfrac{\rho}{4}$ and $a\,d \geq 1$, then the equilibria are $B_1(x_1, 0, 0)$ and $B_2(x_2; 0; 0)$.
- When $h_1 < \dfrac{\rho}{4}$; $a\,d < 1$ and $x_3 = x_1$, then $B_1(x_1, 0, 0)$ is a double point and $B_2(x_2, 0, 0)$ exists.
- When $h_1 < \dfrac{\rho}{4}$; $a\,d < 1$ and $x_3 = x_2$, then $B_1(x_1, 0, 0)$ is simple and $B_2(x_2, 0, 0)$ is a double point.
- When $h_1 < \dfrac{\rho}{4}$; $a\,d < 1$ and $x_3 \in]x_1; x_2[$, then $B_1(x_1, 0, 0)$; $B_2(x_2, 0, 0)$ and $B_3(x_3, 0, y_3)$ exist, where $y_3 = \dfrac{\rho x_3(1-x_3) - h_1}{d} > 0$.
- When $h_1 < \dfrac{\rho}{4}$; $a\,d < 1$ and $x_3 \in [0; x_1[\cup]x_2; +\infty[$, then $B_1(x_1, 0, 0)$ and $B_2(x_2, 0, 0)$ exist.
- When $h_1 < \dfrac{\rho}{4}$; $ad < 1$; $x_4 \in]\eta; x_2[$, $x_2 > \max\left(x_3; \dfrac{\gamma}{\lambda}\right)$ and $R_0 > 1$, then $B_1(x_1, 0, 0)$; $B_2(x_2, 0, 0)$ and $B_4(x_4, z_4, y_4)$ exist, where $x_4 > 0$, $z_4 > 0$ and $y_4 > 0$.

Proof : These equilibria are obtained by setting the right hand side of (1) equals to zero. For $y = 0$ one has equation $\rho x^2 - \rho x + h_1 = 0$. Then we have B_0, B_1 and B_2. For $z = 0$, one has $p(x) = d \iff (1 - a\,d)x^2 - b\,dx - d = 0$. We deduce x_3 and then B_3. The condition for existence of B_4 is $p(z) = \dfrac{1}{m_2}(p(x) - d) > 0$ ie $p(x) - d > 0 \iff a\,d < 1$ and $x \in]x_3, +\infty[$.

Concerning the stability of these equilibria, the following theorem hold.

Theorem 2: Let's consider System (1).

- The equilibria B_0 and B_1 are always unstable.
- The equilibrium B_2 is stable if one of the following conditions is satisfied : $h_1 < \dfrac{\rho}{4}$, $\dfrac{\gamma}{\lambda} \geq x_2$ and $p(x_2) \leq d$, or $h_1 < \dfrac{\rho}{4}$, $\dfrac{\gamma}{\lambda} < x_2$, $p(x_2) = d$ and $p''(x_2) \leq 0$.
- The equilibrium B_3 is stable if one of the following conditions is satisfied. $h_1 < \dfrac{\rho}{4}$, $ad < 1$, $x_3 \in]x_1; x_2[$ and $x_3 = \dfrac{\gamma}{\lambda}$, or $h_1 < \dfrac{\rho}{4}$, $ad < 1$, $x_3 \in]x_1; x_2[$, $x_3 < \dfrac{\gamma}{\lambda}$ and $d > \dfrac{1}{a+2b+4}$, or $h_1 < \dfrac{\rho}{4}$, $ad < 1$, $x_3 \in]x_1; x_2[$, $x_3 < \dfrac{\gamma}{\lambda}$, $d < \dfrac{1}{a+2b+4}$ and $\chi_0(x_3) < 0$, where $\chi_0(x_3)$ is the eigenvalue of x_3.
- The equilibrium point $B_4(x_4, z_4, y_4)$ is asymptotically stable if and only if the following conditions hold : $a_2 < 0$; $a_2a_1 + a_0 > 0$ and $a_1a_0 > 0$, where

$$\begin{cases} a_2 = \rho(1-2x_4) - p'(x_4)y_4 - \lambda z_4 \\ \quad +\lambda x_4 - \gamma - m_1p'(z_4)y_4, \\ a_1 = -\left[\rho(1-2x_4) - p'(x_4)y_4 - \lambda z_4\right] \times \\ \quad \left[\lambda x_4 - \gamma - m_1p'(z_4)y_4\right] \\ \quad -\lambda m_1p(z_4)y_4 - p(x_4)p'(x_4)y_4, \\ a_0 = -\left[\rho(1-2x_4) - p'(x_4)y_4 - \lambda z_4\right] \times \\ \left[\lambda x_4 - \gamma - m_1p'(z_4)y_4\right] + \lambda m_2p(x_4)p'(z_4)y_4z_4 \\ +p'(x_4)y_4m_1p(z_4)(\lambda x_4 - \gamma) \\ +p(x_4)p'(x_4)y_4(\lambda x_4 - \gamma - m_1p'(z_4)y_4). \end{cases} \qquad (5)$$

Proof : The eigenvalues of the jacobian matrix $J(B_0)$ are $\chi_1 = 0$; $\chi_2 = \lambda x_0 - \gamma$ and $\chi_3 = p(x_0) - d$.

a) If $\dfrac{\gamma}{\lambda} < \dfrac{1}{2}$ or $d < \dfrac{1}{a+2b+4} = p(x_0)$, then $\chi_2 > 0$ or $\chi_3 > 0$ and B_0 is unstable.

b) If $\dfrac{\gamma}{\lambda} > \dfrac{1}{2}$ and $d = \dfrac{1}{a+2b+4} = p(x_0)$, then $\chi_2 < 0$ and $\chi_3 = 0$. Hence, the stability of B_0 is given by the center manifold theorem. The translation $(u_1, u_2, u_3) = (x - x_0, z, y)$ brings the singular point B_0 to the origin. In the neighborhood of the origin and, since $h_1 = \dfrac{\rho}{4}$, System (1) has a new form. The Jacobian matrix $J(B_0)$ is not diagonalizable and the passage matrix to the Jordan's basis is

$P = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. By the transformation $(v_1, v_2, v_3)^T = P^{-1}(u_1, u_2, u_3)^T$, the system becomes:

$$\begin{cases} \dot{v}_1 = v_2 + p'(x_0)(v_1 v_3 - v_3^2) \\ +\dfrac{p''(x_0)}{2}(v_1^2 + v_3^2 - 2v_1 v_3)v_2 \\ +\dfrac{m_1 p''(0)}{2} v_2 v_3^2 + O(|(v_1, v_2, v_3)|^4), \\ \dot{v}_2 = v_3 + p'(x_0)(v_1 v_3 - v_3^2) \\ +\dfrac{p''(x_0)}{2}(v_1^2 + v_3^2 - 2v_1 v_3)v_2 \\ -\dfrac{m_2 p''(0)}{2} v_2 v_3^2 + O(|(v_1, v_2, v_3)|^4), \\ \dot{v}_3 = \chi_2 v_3 - \lambda(v_1 v_3 - v_3^2) + \dfrac{m_1 p''(0)}{2} v_2 v_3^2 \\ +O(|(v_1, v_2, v_3)|^4). \end{cases} \tag{6}$$

We can now find that the center manifold is given by $W^c = \{v_3 = 0\}$. Therefore, the system (6) is topologically equivalent, around the origin, to the following system:

$$\begin{cases} \dot{v}_1 &= v_2 + \dfrac{p''(x_0)}{2} v_1^2 v_2 + O(|(v_1, v_2)|^4), \\ \dot{v}_2 &= O(|(v_1, v_2)|^4), \\ \dot{v}_3 &= O(|(v_1, v_2)|^4). \end{cases}$$

Then, the singular point B_0 is unstable.

c) If $\dfrac{\gamma}{\lambda} = \dfrac{1}{2}$ and $d = \dfrac{1}{a+2b+4} = p(x_0)$, then $\chi_2 = 0$ and $\chi_3 = 0$. Applying the center manifold theory as previously, B_0 is unstable.

d) If $\dfrac{\gamma}{\lambda} = \dfrac{1}{2}$ and $d > \dfrac{1}{a+2b+4} = p(x_0)$, we have $\chi_2 = 0$ and $\chi_3 < 0$. Applying the center manifold theory as previously, B_0 is unstable.

The stability of B_1 is obtained with jacobian matrix. The stability of B_2 is obtained using the center manifold theorem. Taking into account the fact that $p(x_3) = d$, one find that the characteristic polynomial of the linearized system around the singular point B_3 is

$$\begin{aligned} Q(\chi) &= (\chi - \lambda x_3 + \gamma)\left[-\chi^2 + (\rho(1-2x_3) - p'(x_3)y_3)\chi\right] \\ &\quad -d(\chi - \lambda x_3 + \gamma)p'(x_3)y_3. \end{aligned}$$

The discriminant of $Q(\chi)$ is

$$\Delta_3(h_1) = \left(\rho(1-2x_3) - p'(x_3)y_3\right)^2 - 4dp'(x_3)y_3. \tag{7}$$

a) If $x_3 > \frac{\gamma}{\lambda}$, then the eigenvalue $\chi_1 = \lambda x_3 - \gamma > 0$. Hence, B_3 is unstable.

b) If $x_3 < \frac{\gamma}{\lambda}$, then $\chi_1 < 0$.

b1) When $\Delta_3(h_1) = 0$ the Jacobian matrix at B_3 has a double eigenvalue

$$\chi_0(x_3) := \frac{\rho(1-2x_3) - p'(x_3)y_3}{2}. \tag{8}$$

• If $d \geq \dfrac{1}{a+2b+4}$, then $x_3 > \frac{1}{2}$. From where $\chi_0(h_1) < 0$. Therefore, the singular point B_3 is stable.

• If $d < \dfrac{1}{a+2b+4}$, then: When $\chi_0(h_1) < 0$ (resp. $\chi_0(h_1) > 0$) the singular point B_3 is stable (resp. unstable).

b2) When $\Delta_3 > 0$ the eigenvalues of the Jacobian matrix at B_3 are $\chi_1 < 0$, $\chi_2 = \chi_0(h_1) - \dfrac{\sqrt{\Delta_3}}{2}$ and $\chi_3 = \chi_0(h_1) + \dfrac{\sqrt{\Delta_3}}{2}$. We have, $\chi_2\chi_3 = dp'(h_1)y_3 > 0$ and $\chi_2 + \chi_3 = \chi_0(h_1)$, where $\chi_0(h_1)$ is defined by (8).

- If $d \geq \dfrac{1}{a+2b+4}$, then the singular point B_3 is stable.
- If $d < \dfrac{1}{a+2b+4}$, then: When $\chi_0(h_1) < 0$ (resp. $\chi_0(h_1) > 0$) the singular point B_3 is stable (resp. unstable).

b3) If $\Delta_3 < 0$, then the eigenvalues of the Jacobian matrix at B_3 are $\chi_1 < 0$, $\chi_2 = \chi_0(h_1) - i\dfrac{\sqrt{-\Delta_3}}{2}$ and $\chi_3 = \chi_0(h_1) + i\dfrac{\sqrt{-\Delta_3}}{2}$, where $\chi_0(h_1)$ is defined by (8). If $d \geq \dfrac{1}{a+2b+4}$, then the singular point B_3 is stable. If $d < \dfrac{1}{a+2b+4}$ and $\chi_0(h_1) < 0$ then, the singular point B_3 is stable. If $d < \dfrac{1}{a+2b+4}$ and $\chi_0(h_1) > 0$ then, the singular point is unstable. If $d < \dfrac{1}{a+2b+4}$ and $\chi_0(h_1) = 0$ then, the real central and stable spaces are respectively defined by $E^c = \langle(1,0,0);(0,0,1)\rangle$ and $E^s = \left\langle(1, -1 - \dfrac{dp'(x_3)y_3}{\chi_1^2}, \dfrac{p'(x_3)y_3}{\chi_1})\right\rangle$. Then applying the center manifold theorem it comes that the singular point B_3 is unstable.

The stability of B_4 is obtained using the Routh-Hurwitz conditions.

B. Results for the Model (2) with Disease only inside Predator Population

Let us set $u_2(x) = \frac{e}{\delta}\left[\frac{\mu_1}{e} - p(x)\right]$ and $v_2(x) = \frac{(p(x)-d)u_2(x)}{e\left[\frac{d}{e} - p(x)\right]}$. Let x_5 the eventual positive root of equation $p(x_5) = \frac{d}{e}$ and the function $g_2(x) = \rho x(1-x) - h_1 - p(x)u_2(x) - \eta_1 p(x)v_2(x)$.

Hypothesis 1 : The attack of non-infected predators is more important than the one of the infected predators i.e. $e = \frac{\tilde{e}}{\tilde{c}} \leq 1$.

Theorem 3: The equilibria of System (2), where x_0; x_1 and x_2 are given by (4), according to the values of the parameters, are given as follow.

- When $h_1 > \frac{\rho}{4}$, then there is no equilibrium point.
- When $h_1 = \frac{\rho}{4}$, then $C_0(x_0; 0; 0)$ is a double point if $d \neq \frac{1}{a+2b+4}$ and triple point if $d = \frac{1}{a+2b+4}$.
- When $h_1 < \frac{\rho}{4}$ and $a\,d \geq 1$, then $C_1(x_1; 0; 0)$ and $C_2(x_2; 0; 0)$ exist.
- When $h_1 < \frac{\rho}{4}$; $a\,d < 1$ and $x_3 = x_1$, then $C_1(x_1; 0; 0)$ is a double point and $C_2(x_2; 0; 0)$ exists.
- When $h_1 < \frac{\rho}{4}$; $a\,d < 1$ and $x_3 = x_2$, then $C_1(x_1; 0; 0)$ exists and $C_2(x_2; 0; 0)$ is a double point.
- When $h_1 < \frac{\rho}{4}$; $a\,d < 1$ and $x_3 \in]x_1; x_2[$, then the equilibria are $C_1(x_1; 0; 0)$; $C_2(x_2; 0; 0)$ and $C_3(x_3; y_3; 0)$, where $y_3 = \frac{\rho x_3(1-x_3) - h_1}{d} > 0$.
- When $h_1 < \frac{\rho}{4}$; $a\,d < 1$ and $x_3 \in [0; x_1[\cup]x_2; +\infty[$, then the equilibria are $C_1(x_1; 0; 0)$ and $C_2(x_2; 0; 0)$.
- When $h_1 < \frac{\rho}{4}$; $a\,d < 1$; $\frac{a\,d}{e} \geq 1$, $x_6 \in]x_1; x_2[\cap]x_3; +\infty[$; $x_2 > x_3$ or $h_1 < \frac{\rho}{4}$; $\frac{a\,d}{e} < 1$, $x_6 \in]x_1; x_2[\cap]x_3; x_5[$; $x_2 > x_3$; $x_1 < x_5$, then the equilibria are $C_1(x_1; 0; 0)$; $C_2(x_2; 0; 0)$ and $C_4(x_6; y_6; \omega_6)$, $y_6 = u_2(x_6)$ and $\omega_6 = v_2(x_6)$.

Proof : The equilibria C_0, C_1, C_2 and C_3 are obtained in the same way as in theorem 1, setting the right hand side of the system equals to zero. Equilibrium C_4 exists when the previous conditions are satisfied.

Concerning the Stability analysis of these equilibria, the following theorem holds.

Theorem 4: Let's consider the System (2) and suppose that Hypothesis 1 holds.

- The equilibria C_0 and C_1 are always unstable.
- The equilibrium C_2 is stable if $h_1 < \frac{\rho}{4}$ and $p(x_2) < d$.
- The equilibrium C_3 is stable if and only if one of these conditions is satisfied : $h_1 < \frac{\rho}{4}$, $ad < 1$, $x_3 \in]x_1; x_2[$ and $y_3 = \frac{e}{\delta_1}\left(\frac{\mu_1}{e} - d\right)$ or $h_1 < \frac{\rho}{4}$, $ad < 1$, $x_3 \in]x_1; x_2[$, $y_3 < \frac{e}{\delta_1}\left(\frac{\mu_1}{e} - d\right)$, $d > p(x_0)$, or $h_1 < \frac{\rho}{4}$, $ad < 1$, $x_3 \in]x_1; x_2[$, $y_3 < \frac{e}{\delta_1}\left(\frac{\mu_1}{e} - d\right)$, $d < p(x_0)$, $\xi_0(x_3) < 0$.
- The singular point $C_4(x_6, y_6, \omega_6)$ is asymptotically stable if and only if the following conditions are satisfied : $b_2 < 0$; $b_2 b_1 + b_0 > 0$ and $b_1 b_0 > 0$, where

$$\begin{cases} b_2 = \rho(1-2x_6) - p'(x_6)(y_6 + \eta_1\omega_6) \\ \quad +p(x_6) - d - \delta\omega_6; \\ b_1 = -\left(\rho(1-2x_6) - p'(x_6)(y_6+\eta_1\omega_6)\right) \times \\ \quad (p(x_6) - d - \delta\omega_6) + \delta_1\omega_6(\mu - \delta y_6) \\ \quad -p(x_6)p'(x_6)y_6 - e\eta_1 p(x_6)p'(x_6)\omega_6; \\ \\ b_0 = ep(x_6)p'(x_6)\omega_6\left[\delta y_6 - \mu + \eta_1(p(x_6) - d - \delta\omega_6)\right] \\ -\delta_1\eta_1 p(x_6)p'(x_6)y_6\omega_6 \\ -\delta_1\omega_6(\mu - \delta y_6)\left(\rho(1-2x_6) - p'(x_6)(y_6+\eta_1\omega_6)\right). \end{cases} \tag{9}$$

Proof : The stability of C_0 is deduce as for B_0 in theorem 2. The jacobian matrix always has a positive eigenvalue. Then, C_1 is unstable. We obtain the stability of C_2 and C_3 applying the same arguments as for B_2 and B_3 in theorem 2. The stability of C_4 is obtained using the Routh-Hurwitz conditions.

IV. Hopf Bifurcation

Let us introduce the following parameters

$$h_{10} = \frac{\rho x_3}{bx_3 + 2}\left[2ax_3^3 + (b-a)x_3^2 + 1\right], \tag{10}$$

and

$$\Pi = \frac{1}{16}\left[p^{(2)}(x_3) + p^{(3)}(x_3)\right] - \frac{(p'(x_3))^2}{4\sqrt{-\Delta_3(h_{10})}}. \tag{11}$$

Recalling (4), the flow of System (1) and System (2) respectively undergo a supercritical Hopf bifurcation around h_{10} given by the following result

Theorem 5: (Hopf bifurcation) Let $h_1 < \frac{\rho}{4}$; $ad < 1$; $x_3 \in]x_1, \min\left(\frac{1}{2}, \frac{\gamma}{\lambda}\right)[$. Thanks to Hypothesis 1. Then, a unique stable curve of periodic solution bifurcates from the singular points B_3 and C_3 into the regions $h_1 > h_{10}$ if $\Pi < 0$ or $h_1 < h_{10}$ if $\Pi > 0$. The singular points B_3 and C_3 are stable for $h_1 < h_{10}$ and unstable for

$h_1 \geq h_{10}$. This correspond to supercritical stable Hopf bifurcation.

Proof : The proof can be obtained as in [13].

V. Numerical Simulations

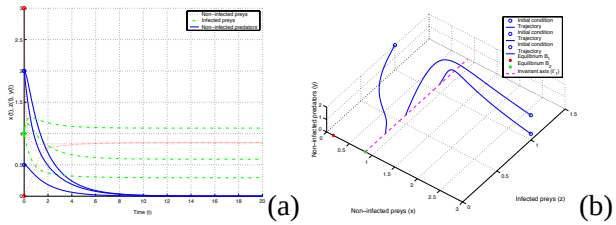

Fig. 1. Phase portraits of System (1) for $h_1 < \frac{\rho}{4}$; $\frac{\gamma}{\lambda} = x_2$ and $p(x_2) < d$. B_1 and B_2 are unstable. The axis $x = \frac{\gamma}{\lambda}$ is stable.

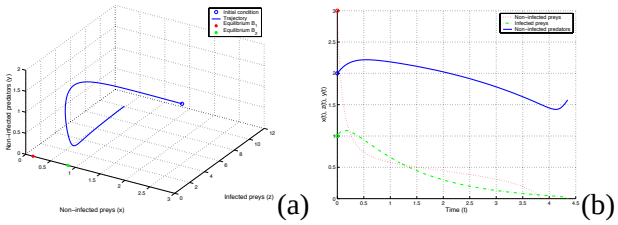

Fig. 2. Phase portraits of System (1). The case (a) corresponds to $h_1 < \frac{\rho}{4}$; $\frac{\gamma}{\lambda} < x_2$ and $p(x_2) < d$. The case (b) corresponds to $h_1 = \frac{\rho}{4}$; $\frac{\gamma}{\lambda} > 1/2$ and $d = \frac{1}{a + 2b + 4}$. Unstability of B_1 and B_2.

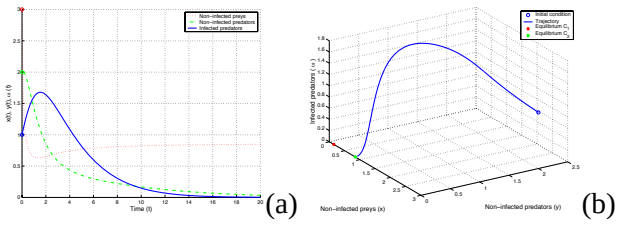

Fig. 3. Phase portraits of System (2) for $h_1 < \frac{\rho}{4}$ and $d > p(x_2)$. Stability of C_2.

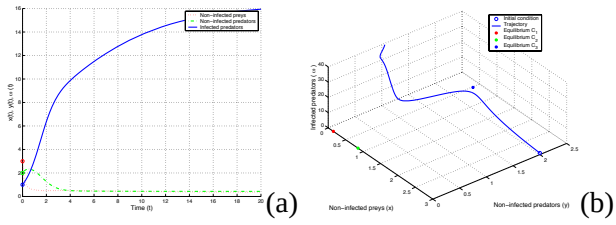

Fig. 4. Phase portraits of System (2) for $h_1 < \frac{\rho}{4}$ and $d < p(x_2)$. Unstability of C_1, C_2 and C_3.

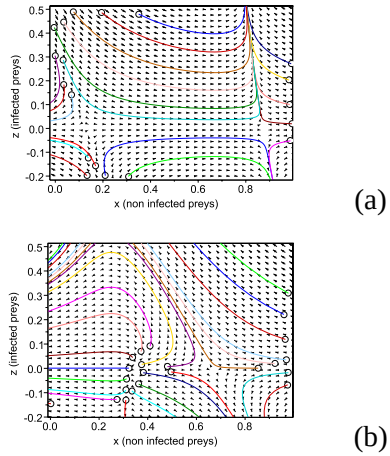

Fig. 5. Phase portraits of System (1). The case (a) corresponds to $h_1 < \rho\frac{\gamma}{\lambda}\left(1 - \frac{\gamma}{\lambda}\right)$. The case (b) corresponds to $h_1 > \rho\frac{\gamma}{\lambda}\left(1 - \frac{\gamma}{\lambda}\right)$ and $\frac{\gamma}{\lambda} < \frac{1}{2}$. Illustration of saddle-node bifurcation phenomenon.

VI. Conclusion

Our goal was to analyze the modifications on a predator prey model (generalized Gause model) with prey harvesting and Holling response type III : $\frac{m\,x^2}{a\,x^2 + b\,x + 1}$, to account for a disease spreading among one of the two species. The simple epidemiological model SIS has been chosen, where only susceptibles and infectives are counted. The results indicate that either the disease dies out, leaving only neutral cycles of generalized Gause model, or one species disappears and all individuals in the other one eventually become infected. For some particular choices of the parameters however, endemic equilibria in which both populations survive seem to arise.

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