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EMERGENCE OF WANDERING STABLE COMPONENTS

PIERRE BERGER AND SÉBASTIEN BIEBLER*

To Mikhail Lyubich on his 60th birthday.

Abstract. We prove the existence of a locally dense set of real polynomial automorphisms of $\mathbb{C}^2$ displaying a wandering Fatou component; in particular this solves the problem of their existence, reported by Bedford and Smillie in 1991. These Fatou components have nonempty real trace and their statistical behavior is historic with high emergence. The proof is based on a geometric model for parameter families of surface real mappings. At a dense set of parameters, we show that the dynamics of the model displays a historic, high emergent, stable domain. We show that this model can be embedded into families of Hénon maps of explicit degree and also in an open and dense set of 5-parameter C^r-families of surface diffeomorphisms in the Newhouse domain, for every $2 \leq r \leq \infty$ and $r = \omega$. This implies a complement of the work of Kiriki and Soma (2017), a proof of the last Taken's problem in the C^∞ and C^ω-case. The main difficulty is that here perturbations are done only along finite-dimensional parameter families. The proof is based on the multi-renormalization introduced in [Ber18].

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Introduction: State of the art and main results

The aim of this paper is to reveal two new phenomena in the dynamics of analytic surface diffeomorphisms. In Sections 0.1 and 0.3, we will describe these two phenomena in the iconic Hénon family of diffeomorphisms of the (real and complex) plane. Then in Section 0.2, we will present the general set up of our study. Finally an outline of the proof and of the manuscript is given in Section 0.4. The proof will occupy most of the manuscript and lies to combinatorics, geometry and real or complex analysis.

0.1. **Wandering Fatou components.** Given a holomorphic endomorphism f of a complex manifold X, the *Fatou set* consists of the set of points $x \in X$ which have a neighborhood U such that $(f^n|U)_n$ is normal. By definition, it is open and invariant by the dynamics. In particular the connected components of the Fatou set, called *Fatou components*, are mapped to each other under the dynamics. A main question is:

Question 0.1. *Is the dynamics restricted to the Fatou set always "simple"?*

To answer this "philosophical" question it is fundamental to know whether a dynamics may admit a *wandering Fatou component*, that is a component which is not sent by an iterate of the dynamics to a periodic one.

When X is the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, in a seminal work [Sul85], Sullivan showed that a rational mapping does not have any wandering Fatou component. This result, together with the classification of Fatou components due to Fatou himself, Siegel and Herman, allows for a complete description of the dynamics restricted to the Fatou set: the orbit of every point in the Fatou set eventually lands in an attracting basin, a parabolic basin or a rotation domain (that is, a Siegel disk or a Herman ring). Hence, in this context the answer to Question 0.1 is positive. This was later generalized by Eremenko and Lyubich [EL87] and Goldberg and Keen [GK86] who showed that entire mappings with finitely many singular values have no wandering Fatou component.

The absence of wandering Fatou component for another class of entire maps has been studied in [MBRG13].

On the other hand, when $X = \mathbb{C}$, Baker [Bak76] gave (prior to Sullivan's result) the first example of an entire map displaying a wandering Fatou component, see also[EL87], [Sul85], [Her84]. In all these examples, the singular set is unbounded. Bishop introduced in [Bis15] another example displaying this time a bounded singular set. The rich history of wandering Fatou components for entire maps includes the more recent examples [MPS18, FJL17, FGJ14, KS08, Ber07].

In higher dimension, the fundamental problem of the existence of wandering Fatou components was first studied in 1991 in the work of Bedford and Smillie [Bed91, Theorem 5.6], in the context of polynomial automorphisms of $\mathbb{C}^2$ (see also [FsS99, page 275]). These are the polynomial maps of two complex variables which are invertible with a polynomial inverse. In this setting, a Fatou component is either the open set of points with unbounded forward orbit (which is invariant by the dynamics) or a connected component of the interior of the set K^+ of points with bounded forward orbit. Later the question of the existence of wandering Fatou components was generalized by Fornaess and Sibony in [FS01, Question 2.2] to a holomorphic endomorphism of $\mathbb{P}^k(\mathbb{C})$, for $k \geq 2$. This question was completely open until the recent breakthrough of Astorg, Buff, Dujardin, Peters and Raissy [ABD$^+$16] who proved, following an idea of Lyubich, the existence of a wandering Fatou component for holomorphic endomorphisms of $\mathbb{P}^2(\mathbb{C})$. Their example is a skew product of real polynomial maps and possesses a parabolic point (two eigenvalues equal to 1). Their proof uses parabolic implosion techniques. Developments of this construction allowed recently Hahn and Peters to build examples of polynomial automorphisms of $\mathbb{C}^4$ displaying a wandering Fatou component in [HP18] and then Astorg, Boc Thaler and Peters to replace the parabolic fixed point by an elliptic point for two-dimensional dynamics in [ABTP19b]. However it does not seem possible to adapt these techniques to the long standing problem of the existence of such components for polynomial automorphisms of $\mathbb{C}^2$, for two reasons. First by [FM89], there is no polynomial automorphism of $\mathbb{C}^2$ which is a skew product and has a wandering Fatou component. Secondly, the Jacobian of an automorphism is constant and so the existence of a parabolic fixed point implies that the Jacobian is 1, which is incompatible with their construction (a real disk is shrunk along its orbit). In the same topic, let us mention the example of a transcendental biholomorphic map in $\mathbb{C}^2$ with a wandering Fatou component oscillating to infinity by Fornæss and Sibony in [FS98] and other examples of wandering domains for transcendental mappings in higher dimension in [ABFP17, ABTP19a]. About the non-existence of wandering Fatou components in higher dimension, a few cases study have been done proved [Lil04, Ji18] in some particular cases (skew products with a super-attracting invariant fiber).

The problem of the existence of a wandering Fatou component for polynomial automorphisms of $\mathbb{C}^2$ increased its interest with the recent conjecture of Bedford [Bed15] stating the existence of wandering Fatou components for polynomial automorphisms of $\mathbb{C}^2$ and the spectacular development [LP14] giving a complete classification of the periodic Fatou components for some polynomial automorphisms [1], see also [Wei03].

1. called moderately dissipative, but in a different meaning as we will use.

The (first) main result of the present work is an answer to the original problem raised by Bedford and Smillie [BS91]:

Theorem A. *For any sufficiently small real number $b \neq 0$, there exists a nonempty open subset $\mathcal{P}_b \subset \mathbb{R}^5$ such that for a dense subset of parameter $p = (p_i)_{0 \leq i \leq 4} \in \mathcal{P}_b$, the following polynomial automorphism displays a wandering Fatou component $\mathcal{C}$ in $\mathbb{C}^2$:*

$$f_p : (z, w) \in \mathbb{C}^2 \mapsto (z^6 + \sum_{i=0}^{4} p_i \cdot z^i - w, b \cdot z) \in \mathbb{C}^2 \ .$$

Moreover the Fatou component $\mathcal{C}$ satisfies the following properties:

(1) the real trace $\mathcal{C} \cap \mathbb{R}^2$ of $\mathcal{C}$ is nonempty,

(2) for any compact set $\mathcal{C}' \Subset \mathcal{C}$, it holds $\lim_{n \to \infty} \operatorname{diam} f_p^n(\mathcal{C}') = 0$,

(3) for every $(z, w) \in \mathcal{C}$, the limit set of the orbit of (z, w) contains a (real) horseshoe K,

(4) for every $(z, w) \in \mathcal{C}$, the sequence $(\mathscr{e}_n)_n$ of empirical measures $\mathscr{e}_n := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f_p^i(z,w)}$ diverges. Furthermore, there exists μ in the set of invariant probability measures $\mathcal{M}_p(K)$ of $f_p|K$ such that the limit set of $(\mathscr{e}_n)_n$ contains $t \cdot \mu + (1 - t) \cdot \mathcal{M}_p(K)$ for some $0 < t < 1$.

We recall that a property $\mathcal{P}(a)$ depending on $a \in \mathbb{R}$ is said to be true for any (sufficiently) small a if there exists $a_0 > 0$ such that $\mathcal{P}(a)$ is true when $|a| < a_0$. We recall that a *horseshoe* is both a Cantor set and a hyperbolic basic set. We will see in the sequel (Theorem 0.9) that its set of invariant probability measures $\mathcal{M}_p(K)$ is huge.

Remark 0.2. *We will see in Theorem 1.46 that for every $b \neq 0$, there is a neighborhood of V_b of b such that $\bigcap_{b' \in V_b} \mathcal{P}_{b'}$ has nonempty interior. This implies that the set $\mathcal{P}$ of parameters $((p_i)_{0 \leq i \leq 4}, b) \in \mathbb{R}^5 \times \mathbb{R}$ for which there is a wandering Fatou component is* locally dense *in $\mathbb{R}^6$: its closure has nonempty interior.*

Remark 0.3. *Our techniques give also new examples of endomorphisms of $\mathbb{P}^2(\mathbb{C})$. For instance, the conclusion of Theorem A holds true also for the family of maps:*

$$f_p : [z, w, t] \in \mathbb{P}^2(\mathbb{C}) \mapsto [z^6 + \sum_{i=0}^{4} p_i \cdot z^i \cdot t^{6-i} - w^6, b \cdot z^6, t^6] \ .$$

0.2. **Historical behavior and wandering stable components in the Newhouse domain.** Given a (general) differentiable dynamical system f of a compact manifold M, since the seminal works of Boltzmann and Birkhoff, one is interested in describing the statistical behavior of most (Lebesgue) of the points. The statistical behavior of a point x is described by the sequence $(\mathscr{e}_n(x))_n$ of empirical measures

$$\mathscr{e}_n(x) := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \ .$$

The point x has a *historic behavior* if its sequence of empirical measures $(\mathscr{e}_n(x))_n$ does not converge [Rue01]. The method used to prove Theorem A basically comes from real, smooth dynamical systems and is related to the following well-known:

Takens' Last Problem ([Tak08])**.** *Are there persistent classes of smooth dynamical systems for which the set of initial states which give rise to orbits with historic behavior has positive Lebesgue measure?*

We recall that a *surface diffeomorphism* is a smooth diffeomorphism of a real surface. A strategy to solve this problem is deduced from a scenario introduced by Colli and Vargas [CV01], and takes place in the dissipative Newhouse domain. The *dissipative C^r-Newhouse domain* $\mathcal{N}^r$ is the open set of surface diffeomorphisms leaving invariant a hyperbolic basic set displaying a C^r-robust quadratic homoclinic tangency and an area contracting periodic point (see Definition 2.13 for more details). This strategy was recently implemented by Kiriki and Soma [KS17] to solve this problem in the case of finitely differentiable diffeomorphisms of surface. For every $2 \leq r < \infty$, they proved the existence of a dense set in $\mathcal{N}^r$ of diffeomorphisms f displaying a wandering open set U whose points are asymptotic, but do not lie in the basin of a periodic orbit. Let us formalize this by introducing the following terminology:

Definition 0.4. *A point x is asymptotically* stable *if it has a neighborhood U formed by asymptotic points: for every $y \in U$, it holds $d(f^n(x), f^n(y)) \to 0$ as $n \to +\infty$. A* stable domain *is a connected open subset formed by stable points. A* stable component *is a stable domain which is maximal. In other words, a* stable component *is a component of the set of asymptotically stable points. A stable component is* wandering *if it does not intersect its iterates.*

Note that the first example of a wandering stable component goes back to Denjoy [Den32].

Question 0.5. *Is a stable component either wandering or the basin of a periodic point?*

It can be shown that the construction of Colli, Vargas, Kiriki and Soma yields a wandering stable component [2]. Moreover they showed that the sequence of empirical measures of points in their stable domain does not converge.

We will introduce a geometric model for parameter families of dynamics, and we show that the scenario of [CV01, KS17] occurs at a dense set of parameters of such families, using a new perturbation method. The proof is based on a technique of composed Hénon-like renormalizations [Ber18, Theorem D and Remark 3.5] originally devoted to such pathological behaviors [3]. Also, we prove that this geometric model appears both in the setting of Theorem A and densely inside the Newhouse domain. This implies the second main result of this paper, which gives a positive answer to Takens' last problem in the smooth and even the real analytic category:

Theorem B. *For every $r \in [2, \infty] \cup \{\omega\}$, every $f \in \mathcal{N}^r$ can be C^r-approximated by $\tilde{f} \in \mathcal{N}^r$ such that for each C^r-family $(f_p)_{p \in \mathbb{R}^5}$ in an open dense set of $\tilde{f}$'s unfolding (i.e. $f_0 = \tilde{f}$), there exists a subset of parameters $p \in \mathbb{R}^5$ which is dense in a neighborhood of 0 for which f_p displays a wandering stable component $\mathcal{C}$ satisfying the following:*

(1) *for every $x \in \mathcal{C}$, the limit set of the orbit of x contains a horseshoe K,*

2. The argument is contained in the proof of Corollary 1.49.

3. The renormalizations we will deal with have unbounded combinatorics, in the case of bounded combinatorics, Ou [Ou19] showed recently the nonexistence of wandering stable components.

(2) *for every $x \in \mathcal{C}$, the sequence $(\mathscr{e}_n)_n$ of empirical measures $\mathscr{e}_n := \frac{1}{n}\sum_{i=0}^{n-1}\delta_{f_p^i(x)}$ diverges. Furthermore, there exists μ in the set of invariant probability measures $\mathcal{M}_p(K)$ of $f_p|K$ such that the limit set of $(\mathscr{e}_n)_n$ contains $t\cdot\mu + (1-t)\cdot\mathcal{M}_p(K)$ for some $0<t<1$.*

This theorem extends Kiriki-Soma's work [KS17]. First it solves the C^∞-case and the real analytic case (among analytic mappings displaying a complex extension on a uniform complex strip) among surface diffeomorphisms. Secondly, it does not only show that the set of dynamics displaying a wandering domain with historic behavior is dense in $\mathcal{N}^r$, but also that this phenomenon is of codimension ≤ 5. Thirdly, it implies a stronger description of the statistical behavior of the wandering domain, as we will see in Corollary B via the notion of emergence (see §0.3).

Theorem A, Remark 0.3 and Theorem B are basically consequences of the following result:

Theorem C (Main)**.** *Let $(f_p)_{p\in\mathbb{R}^5}$ be a C^2-family of surface diffeomorphisms. Assume that f_0 has an area contracting periodic saddle point displaying 5 different quadratic homoclinic tangencies.*

If these quadratic tangencies unfold[4] *non-degenerately with $(f_p)_p$, then there is a parameter subset $\mathcal{D}\subset\mathbb{R}^5$ such that $cl(\mathcal{D})\ni 0$, $\mathrm{int}(\mathrm{cl}(\mathcal{D}))\neq\emptyset$ and for every $p\in\mathcal{D}$ the map f_p displays a wandering stable component $\mathcal{C}$ satisfying:*

(1) *for every $x\in\mathcal{C}$, the limit set of the orbit of x contains a horseshoe K,*

(2) *every $x\in\mathcal{C}$ has its sequence $(\mathscr{e}_n(x))_{n\geq 0}$ of empirical measures which diverges. Furthermore, the limit set of $(\mathscr{e}_n(x))_n$ contains $t\cdot\mu+(1-t)\cdot\mathcal{M}_p(K)$ for some $0<t<1$ and $\mu\in\mathcal{M}_p(K)$.*

(3) *If moreover f_p is real analytic, then $\mathcal{C}$ is the real trace of a wandering (complex) Fatou component which is a stable component of the complex extension of f_p.*

All the holomorphic mappings we found with a wandering Fatou component are actually real (even if the Fatou component is an open subset of $\mathbb{C}^2$ or $\mathbb{P}^2(\mathbb{C})$). Nevertheless, it seems to us possible to develop the techniques presented in this work to show the following:

Conjecture 0.6. *There is a locally dense set in the space of polynomial automorphisms $\mathrm{Aut}(\mathbb{C}^2)$ of $\mathbb{C}^2$ formed by dynamics displaying a wandering Fatou component.*

Even if Theorem A is about Hénon maps of degree 6, we know that there exists a family of polynomial automorphisms of degree 5 satisfying the assumptions of Theorem C, hence possessing a wandering Fatou component. We do not know if it is possible for lower degree:

Question 0.7. *Does there exist a polynomial automorphism of $\mathbb{C}^2$ with degree ≤ 4 which displays a wandering Fatou component?*

To answer to this question, one might use a mix of the parabolic renormalization techniques of [ABD+16] together with those of the present paper. A reasonable problem to develop these techniques is:

Problem 0.8. *Give an explicit lower bound on the Hausdorff dimension of the set of parameters having a wandering Fatou component in the space of polynomial automorphisms of $\mathbb{C}^2$ of degree 5.*

4. See Definition 2.9.

The last item in Theorem A indicates that the statistical complexity of the dynamics is high. More specifically this implies that its *emergence is high*, and even that its order is positive and at least the order of the unstable dimension of K. We explain this in the sequel.

0.3. **Statistical complexity of the dynamics: Emergence.** The recent notion of emergence is a natural way to quantify how far from being ergodic a system is. In this subsection we deduce from Theorems A, B and C that the emergence of some dense set of dynamics is more than high.

Roughly speaking, the emergence of a dynamical system is the growth rate as $\epsilon \to 0$ of the number of ϵ-balls necessary to describe the statistical behavior of the system – in mean – up to precision ϵ. The statistical behavior is given by the empirical measures; a natural distance between them is the Wasserstein distance. Recall that by the Kantorovich-Rubinstein's theorem, the (1st) Wasserstein distance between two probability measures μ, ν on M is equal to:

$$d_W(\mu,\nu) = \max_{\phi \in \mathrm{Lip}^1} \int \phi d(\mu - \nu) \ ,$$

where Lip^1 stands for the space of 1-Lipschitz real functions. Let us also recall that on the set of probability measures on any compact metric space, the Wasserstein distance induces the weak $*$ topology. To quantify the complexity of the statistical behavior of typical orbits for general dynamical systems, the notion of emergence has been introduced in [Ber17].

Definition. *The* emergence $\mathcal{E}_f(\epsilon)$ *of* f *at scale* ϵ *is the minimal number* N *of probability measures* $(\mu_i)_{1\leq i\leq N}$ *satisfying:*

$$\limsup_{n\to\infty} \int \min_{1\leq i\leq N} d_W(e_n(x),\mu_i) d\mathrm{Leb} < \epsilon \ .$$

We are interested in understanding the asymptotic behavior of $\mathcal{E}_f(\epsilon)$ when $\epsilon \to 0$. For example, the emergence of a uniformly hyperbolic system is finite (since there are finitely many physical probability measures whose basins cover a full measure subset of M). On the other hand, the emergence of the geodesic flow on the unit tangent bundle of the flat torus $\mathbb{R}^n/\mathbb{Z}^n$ is of the order of ϵ^{-n}, hence polynomial. When the emergence is not polynomial:

$$(\star) \qquad \limsup_{\epsilon\to 0} \frac{\log \mathcal{E}_f(\epsilon)}{-\log \epsilon} = \infty \ ,$$

then the global statistical behavior of the system is deemed very complex. Let us recall:

Conjecture ([Ber17])**.** *Super polynomial emergence is typical in many senses and in many categories of dynamical systems.*

In the recent work [BB21], it has been shown that the order of the emergence $\mathcal{OE}_f$:

$$\mathcal{OE}_f := \limsup_{\epsilon\to 0} \frac{\log\log \mathcal{E}_f(\epsilon)}{-\log \epsilon}$$

of a system f on a compact manifold of dimension d is at most d. Furthermore, it has been shown that this upper-bound is generically attained among conservative, surface C^∞-diffeomorphisms displaying an elliptic periodic point. Note that if a dynamics has emergence of positive order, then its emergence is not polynomial. Thus this confirmed the main Conjecture of [Ber17] in the category of surface conservative diffeomorphisms with local genericity as a version of typicality.

In the situation of Theorem A, we are able to estimate the order of emergence using Theorem A.(4) and the following consequence of [BB21, Theorem A]. Let f be a surface diffeomorphism. Given an invariant set K, we recall that $\mathcal{M}_f(K)$ denotes the set of invariant probability measures of $f|K$.

Theorem 0.9. *If K is a horseshoe of a $C^{1+\alpha}$-surface diffeomorphism f, the covering number $\mathcal{N}(\epsilon)$ at scale ϵ of $\mathcal{M}_f(K)$ for the Wasserstein distance has order at least the unstable dimension*[5] *$d_u > 0$ of K:*

$$\liminf_{\epsilon\to 0} \frac{\log\log \mathcal{N}(\epsilon)}{-\log \epsilon} \geq d_u \ .$$

This together with Theorem A and Remark 0.2 imply:

Corollary A. *There exists a locally dense set in the space of real polynomial automorphisms of $\mathbb{C}^2$ whose emergence has positive order $\mathcal{OE}_f$ (for the canonical normalized, volume form on $\mathbb{P}^2(\mathbb{C})$).*

To this extent, Corollary A gives a negative answer to Question 0.1 in the case of polynomial automorphisms of $\mathbb{C}^2$ and confirms the main conjecture of [Ber17], in the category of real polynomial automorphisms with local density as a (weak) version of typicality.

Note that in contrast to the nowhere dense example of [ABD+16] whose emergence of the wandering domain is linear (as for the well-known Bowen eye), Theorem A and Corollary A exhibit a locally dense set of polynomial automorphisms of $\mathbb{R}^2$ whose emergence has positive order.

Let us communicate the following question raised by Ledrappier:

Question 0.10. *Are there typical examples of transcendental maps of $\mathbb{C}$ with super-polynomial emergence?*

In the same spirit, a positive answer to this question would lead to a negative answer to Question 0.1 in the category of transcendental maps of $\mathbb{C}$. Also the concept of emergence enables to strengthen Conjecture 0.6:

Conjecture 0.11. *There is a locally dense set in the space of polynomial automorphisms* $\mathrm{Aut}(\mathbb{C}^2)$ *of $\mathbb{C}^2$ formed by dynamics displaying a wandering Fatou component with high emergence.*

In the real setting, we notice that Theorems B and 0.9 imply the following contribution to the main conjecture of [Ber17]:

Corollary B. *In the setting of Theorem B, the emergence of $f_p|\bigcup_{n\geq 0} f_p^n(\mathcal{C})$ has order $\geq d_u(K)$.*

This corollary extends the very recent work [KNS19] developing [KS17], where super-polynomial emergence was shown to occur locally densely in the C^r-Newhouse domain for $r < \infty$. We obtained that the emergence of this wandering component $\mathcal{C}$ has positive order (and so super polynomial), and this locally densely (with codimension ≤ 5) in any topology $2 \leq r \leq \infty$ and $r = \omega$.

In this sense, Theorem B and its Corollary B confirm the main conjecture of [Ber17] on typicality of high emergence in the category of C^r-surface diffeomorphisms, $2 \leq r \leq \infty$, $r = \omega$ and even in the category of real polynomial automorphisms of $\mathbb{R}^2$, with density of codimension ≤ 5 as a weak version of typicality.

5. This is the Hausdorff dimension of $K \cap W^u_{loc}(x)$ for any $x \in K$.

0.4. **Outline of the proof and organization of the manuscript.** Theorem C follows from a general result (Theorem 1.44) on a geometric model. We are going to briefly describe the mechanism behind this result. The proofs that Theorem 1.44 implies Theorem C which implies Theorems A and B, are done in Section 2.

Let us first fix a few notations. Given two sequences $(a_j)_j$ and $(b_j)_j$ of non-zero real numbers:

— we say that a_j is small when j is large and write $a_j = o(1)$ if $\lim_{j\to\infty} a_j = 0$,

— we say that a_j is small compared to b_j when j is large and write $a_j = o(b_j)$ if $\lim_{j\to\infty} \frac{a_j}{b_j} = 0$,

— we say that a_j and b_j are equivalent when j is large and write $a_j \sim b_j$ if $\lim_{j\to\infty} \frac{a_j}{b_j} = 1$.

The starting point of the proof is to work with a wild hyperbolic basic set Λ. This is a hyperbolic set such that for any perturbation of the dynamics, there is a local unstable manifold which displays a quadratic tangency with a stable manifold. For such dynamics there are sequences of points $(P_i)_{i\geq 0}$ in $\Lambda^{\mathbb{N}}$ which define chains of heteroclinic quadratic tangencies: the local unstable manifold $W^u_{loc}(P_i)$ of P_i is sent by f tangent to the local stable manifold $W^s_{loc}(P_{i+1})$ and the tangency is quadratic. For the sake of simplicity, let us begin by assuming that each P_i is periodic of some period q_i and the dynamics f^{q_i} near P_i is affine with diagonal linear part:

$$L_i : P_i + (x, y) \mapsto P_i + \left(\frac{x}{\sigma_i}, \lambda_i \cdot y\right) \quad \text{with } 0 < \sigma_i, \lambda_i < 1\,.$$

In particular the local stable manifold of P_i is vertical and the local unstable manifold of P_i is horizontal. Now we consider the tangencies slightly unfolded such that f folds the local unstable manifold $W^u_{loc}(P_i)$ close to a point T_i with $L_{i+1} \circ f(T_i)$ close to T_{i+1} (see Fig. 1). A toy model of the dynamics nearby a homoclinic tangency is given by a Hénon map:

$$f : T_i + (x, y) \mapsto L^{-1}_{i+1}(T_{i+1}) + (x^2 - y/2 + a_i, bx + c_i)\ , \text{ with } \quad (a_i, c_i) = f(T_i) - L^{-1}_{i+1}(T_{i+1})\,.$$

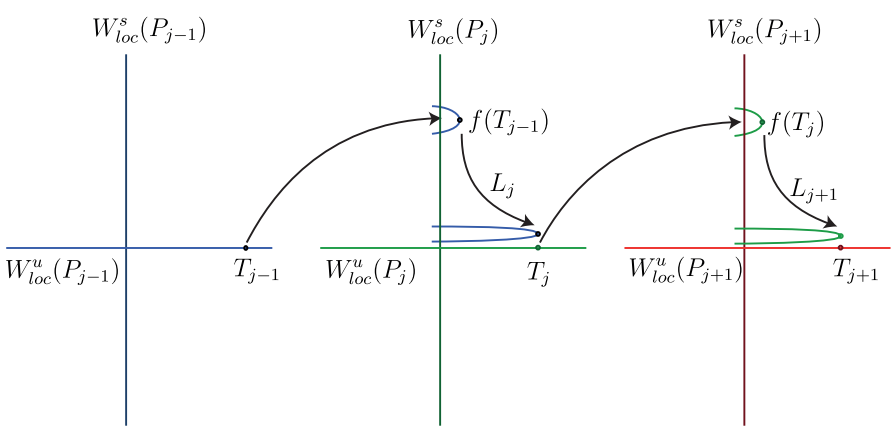

FIGURE 1. Toy model for the heteroclinic tangencies chain leading to a wandering domain.

Then we have:

$$L_{i+1} \circ f : T_i + (x, y) \mapsto T_{i+1} + \left(\frac{x^2 - y/2 + a_i}{\sigma_{i+1}}, \lambda_{i+1}(bx + c_i)\right)\,.$$

In this toy model, if there exists a sequence of neighborhoods $(B_i)_i$ of $(T_i)_i$ such that $\operatorname{diam} B_i \to 0$ and $L_{i+1} \circ f(B_i) \subset B_{i+1}$, then the point T_0 belongs to a stable domain. To find the necessary conditions on $a_i, c_i, \sigma_i, \lambda_i$ to achieve this, we use a variation of the rescaling formula of [Ber18, Theorem D]. We define the rescaling factors and maps by:

$$\gamma_i = \prod_{j=1}^{\infty} \sigma_{i+j}^{2^{-j}} \quad \text{and} \quad \phi_i : (X,Y) \mapsto T_i + (\gamma_i X, \gamma_i^2 Y)\,.$$

We assume that $\gamma_i > 0$. Observe that $\gamma_i^2 = \sigma_{i+1}\gamma_{i+1}$. We now consider the renormalized map $\mathcal{R}_i f(X,Y) := \phi_{i+1}^{-1} \circ L_{i+1} \circ f \circ \phi_i(X,Y)$. A simple computation gives:

$$\mathcal{R}_i f(X,Y) = \left(X^2 - \frac{Y}{2} + \frac{a_i}{\gamma_i^2}, \frac{\lambda_{i+1}}{\gamma_{i+1}^2}(b\gamma_i X + c_i) \right)\,.$$

This is a quadratic Hénon map. Thus the idea is to state conditions implying that its coefficients are small, so that the image by ϕ_i of the ball of radius $1/3$ centered at 0 is contracted and sent into the image by ϕ_{i+1} of the same ball by these iterations (since $1/3^2 + 1/(2\times 3) < 1/3$). Namely our conditions express as:

(C_1) $\prod_{j\geq 0} \sigma_j^{2^{-j}} > 0$ or equivalently $\sum_{j\geq 0} 2^{-j} \log \sigma_j > -\infty$.

(C_2) $a_i = o(\gamma_i^2)$ when i is large; this implies that the first coordinate of $\mathcal{R}_i f$ is close to $X^2 - Y/2$.

(C_3) $\gamma_i \to 0$ and $\lambda_{i+1} = o(\gamma_{i+1}^2)$; this ensures that the second coordinate of $\mathcal{R}_i f$ is small.

To go from this toy model to actual surface diffeomorphisms (including polynomial automorphisms), we first work with a more general model where (in particular) the mapping is not necessarily linearizable at its periodic points. This model of geometrical and combinatorial type is a variation of the one used in [Ber18]. The precise definition is given in Section 1.1 under the name of systems of type $(\mathbf{A}, \mathbf{C})$. A basic and simple example of such system is depicted Fig. 4 Page 21. Then an analogue of the above conditions (C_1)-(C_2)-(C_3) is shown to imply the conclusions of Theorem C in Theorem 1.32 (real case) and Theorem 1.34 (complex case). Their proofs occupy the whole Section 3. In Section 3.1 we recall the notion of the implicit representations for hyperbolic transformations of Palis-Yoccoz [PY01] and introduce a real-analytic generalization. These representations allow us to obtain sharp uniform distortion bounds, without assuming any linearization (i.e. non-resonance) conditions, which is very crucial for our problem. In Section 3.2, we show that the assumptions of Theorems 1.32 and 1.34 imply a normal form for the folding maps which is 'uniformly' close to be Hénon-like (and more precisely close to the assumptions of [Ber18, Theorem D and Remark 3.1]). In Section 3.3, we prove Theorems 1.32 and 1.34. The main point is to show that the domain B_j is well included in the boxes where the implicit representations are defined and where the folding maps are defined. To handle the complex analytic case which is presently not included in [Ber18, Theorem D], we use the Cauchy inequality as a shortcut to obtain the complex distortion bounds. Details are given in Appendix B.

At this point, it is not straightforward at all to see that the assumptions of Theorems 1.32 and 1.34 are satisfied for infinitely smooth surface diffeomorphisms, let alone for polynomial automorphisms of $\mathbb{R}^2$. This context requires to work with a fixed d-parameter family of diffeomorphisms and consider perturbations only along such a family (contrarily to [KS17] where bump functions

are used to perturb the dynamics). Thus in Section 1.2, we consider families $(F_p)_{p\in\mathbb{R}^d}$ of systems of type $(\mathbf{A},\mathbf{C})$. We ask that at every parameter p, there are d pairs of points $(Q_i^u, Q_{i+1}^s)_i$ in Λ such that F_p maps the local unstable manifold $W^u_{loc}(Q_i^u)$ of Q_i^u tangent to the local stable manifold $W^s_{loc}(Q_{i+1}^s)$ and the tangency is quadratic. Then to apply Theorems 1.32 and 1.34 as above, we shadow each $(Q_i^s, Q_i^u)_i$ by a periodic orbit P_i such that if the eigenvalues of P_i are $(1/\sigma_i, \lambda_i)$, the curves $W^s_{loc}(Q_i^s)$ and $W^s_{loc}(P_i)$ are $O(\sigma_i)$-close whereas $W^u_{loc}(Q_i^u)$ and $W^u_{loc}(P_i)$ are $O(\lambda_i)$-close. To satisfy (C_1), we take:

$$\sigma_i \asymp \delta^{\beta^i}\ , \qquad \text{with } \beta\in(1,2) \text{ and } \delta \text{ small.}$$

This implies that

$$\gamma_i \asymp \prod_{j\geq 1} \sigma_i^{(\beta/2)^j} = \sigma_i^{\beta/(2-\beta)}\ .$$

So if we do not perturb the parameter, a priori a_i is of the order of $\sigma_{i+1}+\lambda_i$, which is often large compared to $\gamma_i^2 = \sigma_i^{2\beta/(2-\beta)}$ (since $\gamma_i\to 0$). So we have to perturb the parameter p to obtain a wandering stable component (otherwise we would have found an open set of mappings with a wandering Fatou component!). But then a new difficulty appears: when we perturb the parameter p so that the ith- homoclinic tangency is neatly unfolded, the others are possibly badly unfolded by a length $\sigma_{i+1}+\lambda_i > \sigma_{i+1}$. We recall that all a_j must be small compared to γ_j^2. So if the points $(P_j)_j$ are inductively constructed, for $j<i$ we must have at step i, for every $j<i$:

$$\sigma_{i+1} = o(\gamma_j^2)\ .$$

Note that $\gamma_j^2 = \gamma_{j+1}\cdot\sigma_{j+1}$. So this is equivalent to ask for:

$$\sigma_{i+1} = o(\gamma_{j+1}\cdot\sigma_{j+1}) = o(\sigma_{j+1}^{2/(2-\beta)})\ .$$

And with $k=i-j$, we have $\sigma_{i+1} = \sigma_{j+1}^{\beta^k}$. So this is equivalent to ask:

$$\beta^k > 2/(2-\beta)$$

This inequality does not have any solution $\beta\in(1,2)$ for $k=0,1,2,3$, but $\beta=3/2$ is a solution of this inequality for every $k\geq 4$. One *main new trick of this work* is to work with a several parameters family: while we unfold the ith tangency to plug it at a neat position using one parameter, we use 3 other parameters to maintain the $(i-1)^{th}, (i-2)^{th}, (i-3)^{th}$ folds in their neat position (see Fig. 2). But there is an extra difficulty. Once we have perturbed the parameter in this way, $W^u_{loc}(Q_{i+1})$ might be not anymore tangent to the stable local manifold $W^s_{loc}(Q_{i+2})$. Even worst, $W^u_{loc}(Q_{i+1})$ might be σ_{i+1}-far from being tangent to any stable manifold. So we need to work with a 5-parameter family. The fifth parameter is used to restore this heteroclinic tangency. This explains why all our main results on wandering stable components are stated for 5-parameters families. To handle such an argument we will need five groups of persistent homoclinic tangencies which unfold non-degenerately at every parameter. These are assumptions $(\mathbf{H_1})$ and $(\mathbf{H_2})$ of Definition 1.42 of *unfolding of wild type* $(\mathbf{A},\mathbf{C})$, with $\operatorname{Card}\mathbf{C}=5$.

To satisfy (C_3) we must also have that $\lambda_j = o(\gamma_j^2)$. Using that $\gamma_j \asymp \sigma_j^{\beta/(2-\beta)} = \sigma_j^3$, this condition is equivalent to $(\lambda_j/\sigma_j)\cdot\sigma_j^{-5} = o(1)$. Actually λ_j/σ_j is the Jacobian determinant along the orbit of P_j while σ_j^{-5} is approximately the fifth power of the differential of the dynamics along

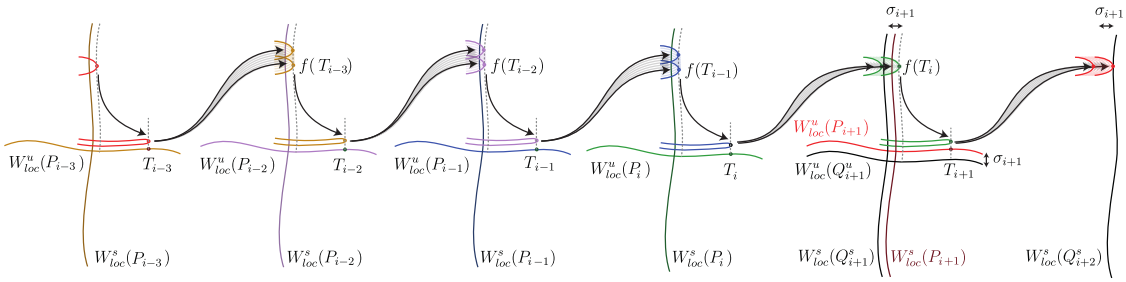

FIGURE 2. This figure depicts the parameter selection; the motion blur reflects how the dynamics varies when the parameter is changed at each inductive step. We assume by induction the existence of a chain of periodic points $(P_j)_{j<i}$ such that $W^u_{loc}(P_j)$ is in nice position w.r.t. $W^s_{loc}(P_{j+1})$ (i.e. satisfying (C_2)), $W^u_{loc}(P_i)$ is tangent to $W^s_{loc}(Q^s_{i+1})$ and $W^u_{loc}(Q^u_{i+1})$ is tangent to the stable lamination of Λ. We consider a periodic point P_{i+1} such that $W^s_{loc}(P_{i+1})$ and $W^u_{loc}(P_{i+1})$ are σ_{i+1}-close to $W^s_{loc}(Q^s_{i+1})$ and $W^u_{loc}(Q^u_{i+1})$. We handle a σ_{i+1}-small perturbation of two parameters such that $W^u_{loc}(P_i)$ is in nice position w.r.t. $W^s_{loc}(P_{i+1})$ (i.e. satisfying (C_2)) and $W^u_{loc}(P_{i+1})$ is tangent to the stable lamination of Λ. Then we use three others parameters in order that $W^u_{loc}(P_j)$ is in nice position w.r.t. $W^s_{loc}(P_{j+1})$ for every $i-3 \leq j \leq i-1$. We need a fifth parameter to make $W^u_{loc}(P_{i+1})$ tangent to a leaf $W^s(Q^s_{i+2})$ of the stable lamination of Λ. By construction, the curve $W^u_{loc}(P_j)$ is in nice position w.r.t. $W^s_{loc}(P_{j+1})$ for every $j \leq i-4$ since $\sigma_{i+1} = o(\gamma_j^2)$.

the orbit of P_j. Hence in order to satisfy this equation, we will require the following moderate dissipativeness hypothesis in Section 1.2:

$$\|Df\| \cdot \|\det Df\|^{\epsilon} < 1 \quad \text{with } \epsilon = \frac{1}{10}.$$

Let us now explain a tricky point of the parameter selection. The number a_j is a function of T_{j+1} which is itself a function of the manifold $W^s(P_{j+2})$. However this manifold is not yet defined by the induction, and depends on the parameter which is moving at the step in progress. So we will need to define the point T_{j+1} independently of $W^s(P_{j+2})$ in a very sharp way. This will be done using *an adapted family* $(\pi_p)_p$ *of projections*, which is basically a C^1-extension of the stable lamination which is equivariant by the dynamics nearby the set Λ. Such a family of projections is actually part of the structure of unfolding of wild type $(\mathbf{A}, \mathbf{C})$. In Appendix C, we show the existence of such a family of projections given any hyperbolic set for a C^2-surface diffeomorphism, following classical techniques.

The main technical theorem in the paper is Theorem 1.44. It asserts that given a moderately dissipative, unfolding (F_p, π_p) of wild type $(\mathbf{A}, \mathbf{C})$ with $\operatorname{Card} \mathbf{C} = 5$, there is a dense set of parameters at which there is a stable component $\mathcal{C}$ satisfying the conclusions of Theorems 1.32 and 1.34 (wandering combinatorics) and moreover the points in $\mathcal{C}$ satisfy conclusions (1) and (2) of Theorem C (ω-limit set of x containing Λ and large emergence). Its proof occupies Section 4. We start in Section 4.1 by showing some uniform bounds on the parameter family for the iterations near a

hyperbolic set. Then in Section 4.2, we give uniform bounds on the parameter family for the normal form nearby the folding map. In Section 4.3, we proceed to the parameter selection using the argument explained above. In Section 4.4, we prove the first part of Theorem 1.44 by showing the existence of a wandering stable component by applying Theorems 1.32 and 1.34. Finally in Section 4.4, we will use some room in our estimates to make the stable component having a large emergence and accumulate onto all the hyperbolic set Λ.

At this stage what remains to do is to show that the families of Theorems A, B and C can be put in the framework of Theorem 1.44. We say that a dynamics satisfies the *geometric model* if an iterate of it leaves invariant a moderately dissipative wild unfolding with Card $\mathbf{C} = 5$. Corollary 1.49 of Theorem 1.44 then asserts that such a system displays a wandering stable component which accumulates onto the embedding of Λ and has stretched exponential emergence. Moreover if the embedding and the dynamics are real analytic this wandering domain is the real trace of a wandering Fatou component.

In Section 2, we show that the families of Theorems A, B and C satisfy the geometric model. We start in Section 2.1 with a proof that the basic and simple example depicted in Fig. 4 defines a moderately dissipative wild unfolding. We remark that this simple example already enables to find a family of polynomial automorphisms which display a wandering Fatou component, of some controlled degree. In Section 2.2 we show that Theorem C implies Theorems A and B. In order to do so we recall some celebrated results of Newhouse [New79], and its extension by [Kra92, HKY93]. These results imply that in the unfolding of a homoclinic tangency there is a wild hyperbolic set with uncountably many tangencies. In Section 2.3 we show that under the assumptions of Theorem C the geometric model occurs. The techniques of the proof are classical but rely on rather sophisticated techniques from real bifurcations theory and uniformly hyperbolic dynamics.

Let us finally emphasize that Section 2 and Appendix C using classical tools from real uniform hyperbolicity theory are completely independent from Sections 3 and 4 and Appendices A and B using tools from both real and complex analysis. All these are independent of Appendix E which is devoted to the proof of Theorem 0.9.

The authors are thankful to Romain Dujardin for his comments on the introduction of this paper. We are grateful to the referees for their deep reading and their valuable suggestions.

1. The geometric model

The geometric model is devoted to offer a nice framework for showing the main theorem on the existence of a wandering stable component and which applies to the following situation: a wild horseshoe which is strongly dissipative and displays five robust homoclinic tangencies which unfold non-degenerately for a 5-parameter family, as we will see in Remark 2.11. Actually, we will show that the geometric model implies the density of the parameters at which there is a wandering stable component, in Corollary 1.49.

Let us define the functional spaces involved in this model. Let $K \subset \mathbb{R}^n$ and $K' \subset \mathbb{R}^m$ be two compact subsets, with $n, m \geq 1$. For $r \in [1, \infty] \cup \{\omega\}$, a map $f : K \to K'$ is of class C^r if it can be extended to a C^r-map from an open neighborhood of K to $\mathbb{R}^m$. It is a C^r-diffeomorphism if f is

bijective and f^{-1} is of class C^r. Then $n = m$. We recall that C^ω is the class of real analytic maps. Likewise, given two compact subsets $K \subset \mathbb{C}^n$ and $K' \subset \mathbb{C}^m$, a map $f : K \to K'$ is holomorphic if it can be extended to a holomorphic map from an open neighborhood of K to $\mathbb{C}^m$. It is a biholomorphism if f is bijective and f^{-1} is holomorphic (and again $n = m$).

1.1. **System of type $(\mathbf{A}, \mathbf{C})$.** In this subsection, we introduce the notion of hyperbolic map of type $\mathbf{A}$ through the definition of hyperbolic transformations. One should think about a hyperbolic map of type $\mathbf{A}$ as a horseshoe Λ and about hyperbolic transformations as the restriction of the dynamics to a neighborhood of a rectangle of a Markov partition (see Propositions 1.13 and 2.25).

Let $I := [-1, 1]$ with topological boundary $\partial I := \{-1, 1\}$. We fix for this subsection $\rho > 0$ and we put $\tilde{I} := I + i \cdot [-\rho, \rho]$ with topological boundary $\partial \tilde{I} \subset \mathbb{C}$. Put $Y^{\mathbf{e}} := I^2$, $\tilde{Y}^{\mathbf{e}} := \tilde{I}^2 \subset \mathbb{C}^2$ and:

$$\partial^s Y^{\mathbf{e}} := (\partial I) \times I \quad , \quad \partial^u Y^{\mathbf{e}} := I \times \partial I \quad , \quad \partial^s \tilde{Y}^{\mathbf{e}} := (\partial \tilde{I}) \times \tilde{I} \quad , \quad \partial^u \tilde{Y}^{\mathbf{e}} := \tilde{I} \times \partial \tilde{I} \ .$$

The sets $Y^{\mathbf{e}}$ and $\tilde{Y}^{\mathbf{e}}$ will be associated to the transformations $F^{\mathbf{e}}$ equal to the identity. The definition of hyperbolic transformations depends on fixed $\theta > 0$ and $\lambda > 1$. For convenience, fix:

$$\theta = \frac{1}{2} \quad \text{and} \quad \lambda = 2 \ .$$

Definition 1.1 (Box)**.** *A* box *is a subset Y of $\mathbb{R}^2$ which is diffeomorphic to $Y^{\mathbf{e}}$ and of the form:*

$$Y = \{(x, y) \in Y^{\mathbf{e}} : \phi^-(y) \leq x \leq \phi^+(y)\} \ ,$$

where ϕ^- and ϕ^+ are C^1-functions on I such that $\phi^- < \phi^+$ with $\|\phi^\pm\|_{C^0} \leq 1$ and $\|D\phi^\pm\|_{C^0} < \theta$, for $\pm \in \{-, +\}$. We put $\partial^u Y = \partial^u Y^{\mathbf{e}} \cap Y$ and $\partial^s Y = cl(\partial Y \setminus \partial^u Y)$.

We notice that the set $Y^{\mathbf{e}}$ is a box. Also in general, a box Y satisfies:

$$\partial^u Y = \{(x, y) \in I \times \partial I : \phi^-(y) \leq x \leq \phi^+(y)\} \quad \text{and} \quad \partial^s Y = \{(x, y) \in I^2 : x \in \{\phi^-(y), \phi^+(y)\}\} \ .$$

For analytic tranformations we will consider the following complex extensions of boxes.

Definition 1.2 (C^ω_ρ-box)**.** *A C^ω_ρ-box $\tilde{Y}$ is a subset of $\tilde{Y}^{\mathbf{e}}$ such that $\tilde{Y} \cap Y^{\mathbf{e}}$ is a box and $\tilde{Y}$ is the range of a biholomorphism of the form $\zeta : (z, w) \in \tilde{I}^2 \mapsto (\mathcal{Z}(z, w), w) \in \tilde{Y}$ satisfying $|\partial_w \mathcal{Z}| < \theta$. Let $\partial^u \tilde{Y} = \partial^u \tilde{Y}^{\mathbf{e}} \cap \tilde{Y}$ and $\partial^s \tilde{Y} = cl(\tilde{Y} \setminus \partial^u \tilde{Y})$.*

Similarly we notice that $\tilde{Y}^{\mathbf{e}}$ is a C^ω_ρ-box. Also in general, a C^ω_ρ-box $\tilde{Y}$ satisfies:

$$\partial^u \tilde{Y} := \zeta(\tilde{I} \times \partial \tilde{I}) \quad \text{and} \quad \partial^s \tilde{Y} := \zeta((\partial \tilde{I}) \times \tilde{I}) \ .$$

The definition of hyperbolic transformation will involve cones.

Definition 1.3 (Cones)**.** *We define the two following cones χ_h and χ_v:*

$$\chi_h := \{(u_x, u_y) \in \mathbb{R}^2 : |u_y| < \theta \cdot |u_x|\} \quad \textit{and} \quad \chi_v := \{(u_x, u_y) \in \mathbb{R}^2 : |u_x| < \theta \cdot |u_y|\} \ .$$

The real cones χ_h and χ_v admit a canonical extension to $\mathbb{C}^2$:

$$\tilde{\chi}_h = \{(u_z, u_w) \in \mathbb{C}^2 : |u_w| < \theta \cdot |u_z|\} \quad \textit{and} \quad \tilde{\chi}_v = \{(u_z, u_w) \in \mathbb{C}^2 : |u_z| < \theta \cdot |u_w|\} \ .$$

Definition 1.4. *A* hyperbolic transformation *(Y, F) is the data of a box Y and a C^2-diffeomorphism from Y onto its image in $Y^{\mathbf{e}}$ such that either $Y = Y^{\mathbf{e}}$ and F is the identity or:*

(1) $F(Y) \subset Y^{\mathbf{e}}$ *and* $F(\partial^s Y) \subset \partial^s Y^{\mathbf{e}}$ *whereas* $Y \cap \partial^s Y^{\mathbf{e}} = \emptyset$ *and* $F(Y) \cap \partial^u Y^{\mathbf{e}} = \emptyset$.

(2) For every $z \in Y$*, every non-zero vector in the complement of* χ_v *is sent into* χ_h *by* $D_z F$ *and has its first coordinate which is more than* λ*-expanded by* $D_z F$.

(3) For every $z \in F(Y)$*, every non-zero vector in the complement of* χ_h *is sent into* χ_v *by* $D_z F^{-1}$ *and has its second coordinate which is more than* λ*-expanded by* $D_z F^{-1}$.

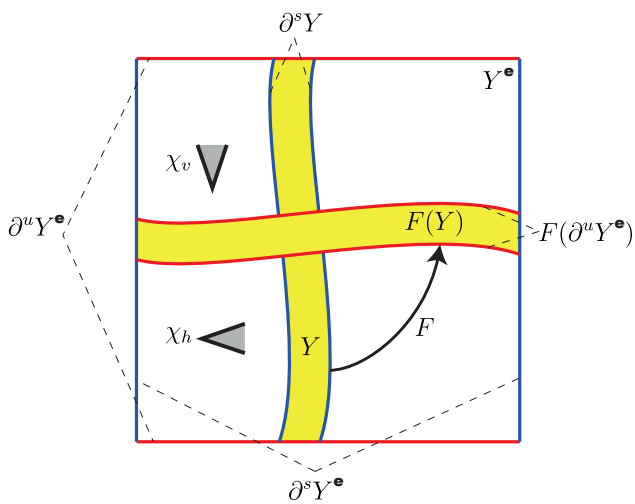

FIGURE 3. A hyperbolic transformation.

Here is the complex counterpart of the latter notion.

Definition 1.5. *A* C^ω_ρ-hyperbolic transformation $(\tilde{Y}, F)$ *is the data of a* C^ω_ρ*-box* $\tilde{Y}$ *and a biholomorphism* F *from* $\tilde{Y}$ *onto its image in* $\tilde{Y}^{\mathbf{e}}$ *such that either* $(\tilde{Y}, F) = (\tilde{Y}^{\mathbf{e}}, id)$ *or:*

(1) $F(\tilde{Y}) \subset \tilde{Y}^{\mathbf{e}}$ *and* $F(\partial^s \tilde{Y}) \subset \partial^s \tilde{Y}^{\mathbf{e}}$ *whereas* $\tilde{Y} \cap \partial^s \tilde{Y}^{\mathbf{e}} = \emptyset$ *and* $F(\tilde{Y}) \cap \partial^u \tilde{Y}^{\mathbf{e}} = \emptyset$.

(2) For every $z \in \tilde{Y}$*, every non-zero vector in the complement of* $\tilde{\chi}_v$ *is sent into* $\tilde{\chi}_h$ *by* $D_z F$ *and has its first coordinate's modulus which is more than* λ*-expanded by* $D_z F$.

(3) For every $z \in F(\tilde{Y})$*, every non-zero vector in the complement of* $\tilde{\chi}_h$ *is sent into* $\tilde{\chi}_v$ *by* $D_z F^{-1}$ *and has its second coordinate modulus which is more than* λ*-expanded by* $D_z F^{-1}$.

(4) The restriction of F *to the box* $Y := \tilde{Y} \cap \mathbb{R}^2$ *is a real analytic hyperbolic transformation.*

We denote also by F *its restriction* $F|Y$*. The pair* $(\tilde{Y}, F)$ *is called the* complex extension or C^ω_ρ-extension *of* (Y, F).

We define the following operation on hyperbolic transformations.

Definition 1.6 ($\star$-product)**.** *Given two hyperbolic transformations* (Y, F) *and* (Y', F')*, we define:*

$$(Y, F) \star (Y', F') := (Y \cap F^{-1}(Y'), F' \circ F) .$$

Given two C^ω_ρ-hyperbolic transformations $(\tilde{Y}, F)$ and $(\tilde{Y}', F')$, we define:

$$(\tilde{Y}, F) \star (\tilde{Y}', F') := (\tilde{Y} \cap F^{-1}(\tilde{Y}'), F' \circ F) \ .$$

The $\star$-product is a binary operation on the set of hyperbolic transformations:

Proposition 1.7. *The $\star$-product $(Y, F) \star (Y', F')$ of any pair of hyperbolic transformations is a hyperbolic transformation. The $\star$-product $(\tilde{Y}, F) \star (\tilde{Y}', F')$ of any pair of C^ω_ρ-hyperbolic transformations is a C^ω_ρ-hyperbolic transformation, moreover $(\tilde{Y}, F) \star (\tilde{Y}', F')$ is a complex extension of $(Y, F) \star (Y', F')$ if $(\tilde{Y}, F)$ and $(\tilde{Y}', F')$ are respective complex extensions of (Y, F) and (Y', F').*

Proof. In [PY01, §3.2.1] it is noticed that the proof is immediate in the real case. The proof for the complex case is postponed to Appendix A. □

The $\star$-product is associative. In particular, it will not be necessary to use brackets while taking $\star$-products of more than two factors. Also for every hyperbolic transformation (Y, F) it holds:

$$(Y, F) \star (Y^{\mathbf{e}}, F^{\mathbf{e}}) = (Y^{\mathbf{e}}, F^{\mathbf{e}}) \star (Y, F) = (Y, F) \ ,$$

where $F^{\mathbf{e}}$ is the identity of $Y^{\mathbf{e}}$. The same occurs for C^ω_ρ-hyperbolic transformation. This implies:

Proposition 1.8. *The set of (resp. C^ω_ρ-) hyperbolic transformations endowed with the operation $\star$ is a monoid with neutral element $(Y^{\mathbf{e}}, F^{\mathbf{e}})$ (resp. $(\tilde{Y}^{\mathbf{e}}, F^{\mathbf{e}})$).*

Let $(\mathbf{V}, \mathbf{A})$ be an oriented graph where $\mathbf{V}$ is a finite set of vertexes and $\mathbf{A}$ a finite set of arrows. The maps $\mathbf{o} : \mathbf{A} \to \mathbf{V}$ and $\mathbf{t} : \mathbf{A} \to \mathbf{V}$ associate to each arrow $\mathbf{a} \in \mathbf{A}$, its origin vertex $\mathbf{o}(\mathbf{a})$ and its target vertex $\mathbf{t}(\mathbf{a}) \in \mathbf{V}$. Let $\mathbf{B} \subset \mathbf{A}^2$ be the set of pairs $(\mathbf{a}, \mathbf{a}')$ of arrows such that $\mathbf{t}(\mathbf{a}) = \mathbf{o}(\mathbf{a}')$. Then we will always assume that the subshift of finite type defined by $(\mathbf{A}, \mathbf{B})$ is transitive and has positive entropy. This is equivalent to say that the graph is strongly connected and there are two different paths with the same length and the same origin and target vertexes.

The following is inspired from the notion of regular Cantor set [MY10] and of the Markovian structure of [PY09].

Definition 1.9. *A* hyperbolic map of type $\mathbf{A}$ *is a map of the form:*

$$F^{\mathbf{A}} : (z, \mathbf{o}(\mathbf{a})) \in \bigsqcup_{\mathbf{a} \in \mathbf{A}} Y^{\mathbf{a}} \times \{\mathbf{o}(\mathbf{a})\} \mapsto (F^{\mathbf{a}}(z), \mathbf{t}(\mathbf{a})) \in Y^{\mathbf{e}} \times \mathbf{V}$$

where $(Y^{\mathbf{a}}, F^{\mathbf{a}})_{\mathbf{a} \in \mathbf{A}}$ is a finite family of hyperbolic transformations $(Y^{\mathbf{a}}, F^{\mathbf{a}})$ satisfying that $Y^{\mathbf{a}}$ is disjoint from $Y^{\mathbf{a}'}$ for every $\mathbf{a} \neq \mathbf{a}' \in \mathbf{A}$ such that $\mathbf{o}(\mathbf{a}) = \mathbf{o}(\mathbf{a}')$.

*The map $F^{\mathbf{A}}$ is C^ω_ρ-*hyperbolic *if there exists a family $(\tilde{Y}^{\mathbf{a}} \times \{\mathbf{o}(\mathbf{a})\})_{\mathbf{a} \in \mathbf{A}}$ of complex extensions $(\tilde{Y}^{\mathbf{a}}, F^{\mathbf{a}})$ of $(Y^{\mathbf{a}}, F^{\mathbf{a}})$ satisfying that $\tilde{Y}^{\mathbf{a}}$ is disjoint from $\tilde{Y}^{\mathbf{a}'}$ for every $\mathbf{a} \neq \mathbf{a}' \in \mathbf{A}$ such that $\mathbf{o}(\mathbf{a}) = \mathbf{o}(\mathbf{a}')$. We put:*

$$D(\mathbf{A}) := \bigsqcup_{\mathbf{a} \in \mathbf{A}} Y^{\mathbf{a}} \times \{\mathbf{o}(\mathbf{a})\} \quad \textit{and} \quad \textit{in the } C^\omega_\rho\textit{- case } \tilde{D}(\mathbf{A}) := \bigsqcup_{\mathbf{a} \in \mathbf{A}} \tilde{Y}^{\mathbf{a}} \times \{\mathbf{o}(\mathbf{a})\} \ .$$

We notice that a hyperbolic map of type $\mathbf{A}$ is a local C^2-diffeomorphism.

Example 1.10. *Let* $(\mathbf{V},\mathbf{A})$ *be the graph with a single vertex* $\mathbf{V}=\{\diamond\}$ *and* $N\geq 1$ *distinct arrows* $\mathbf{A}=\{\mathbf{a}_1,\cdots,\mathbf{a}_N\}$ *with both origin and target vertexes equal to* $\diamond$*. Let* $\delta>0$ *be small compared to* $1/N$*. For every* $1\leq j\leq N$*, put:*

$$I_j:=\left[\frac{2}{N}(j-1)-1+\delta^2,\frac{2}{N}j-1-\delta^2\right]\ .$$

For $1\leq j\leq N$*, let* C_j *be the affine, orientation preserving map from* I *onto* I_j*, let:*

$$Y^{\mathbf{a}_j}:=I_j\times I\quad\text{and}\quad F^{\mathbf{a}_j}:=(x,y)\in Y^{\mathbf{a}_j}\mapsto(C_j^{-1}(x),\sqrt{\delta}\cdot C_j(y))\ .$$

We notice that $(Y^{\mathbf{a}_j},F^{\mathbf{a}_j})$ *is a hyperbolic transformation for every* $1\leq j\leq N$*. The domains* $(Y^{\mathbf{a}_j})_{1\leq j\leq N}$ *are disjoint since the intervals* $(I_j)_j$ *are disjoint. Also the ranges of* $(F^{\mathbf{a}_j})_j$ *are* $(I\times\sqrt{\delta}\cdot I_j)_j$ *which are disjoint from the one other. Thus* the following diffeomorphism is hyperbolic of type $\mathbf{A}$*:*

$$F^{\mathbf{A}}:(z,\diamond)\in\bigsqcup_{\mathbf{a}\in\mathbf{A}}Y^{\mathbf{a}}\times\{\diamond\}\mapsto(F^{\mathbf{a}}(z),\diamond)\ \text{if}\ z\in Y^{\mathbf{a}}\ .$$

This hyperbolic map with $N=6$ *is depicted on Fig. 4.*

We are going to associate geometric and dynamical objects to the following symbolic objects:

Definition 1.11. *Let* $\mathbf{A}^*\subset\mathbf{A}^{(\mathbb{N})}$ *be the set of* admissible finite words $\mathbf{w}:=\mathbf{a}_1\cdots\mathbf{a}_k\in\mathbf{A}^k$*: for each* $j<k$ *we have* $(\mathbf{a}_j,\mathbf{a}_{j+1})\in\mathbf{B}$*. We denote by* $\mathbf{e}$ *the empty word in* $\mathbf{A}^*$*. The* origin and target vertexes *of* $\mathbf{w}$ *are defined by the maps:*

$$\mathbf{o}:\mathbf{a}_1\cdots\mathbf{a}_k\in\mathbf{A}^*\setminus\{\mathbf{e}\}\mapsto\mathbf{o}(\mathbf{a}_1)\in\mathbf{V}\quad\text{and}\quad\mathbf{t}:\mathbf{a}_1\cdots\mathbf{a}_k\in\mathbf{A}^*\setminus\{\mathbf{e}\}\mapsto\mathbf{t}(\mathbf{a}_k)\in\mathbf{V}$$

If the pair $(\mathbf{w},\mathbf{w}')\in(\mathbf{A}^*)^2$ *is admissible* $(\mathbf{t}(\mathbf{w})=\mathbf{o}(\mathbf{w}'))$*, we denote by* $\mathbf{w}\cdot\mathbf{w}'\in\mathbf{A}^*$ *the concatenation of these two words. The* length $|\mathbf{w}|$ *of* $\mathbf{w}\in\mathbf{A}^*$ *is its number of letter.*

There are three ways of taking limits of these admissible words: by both sides, the right side and the left side. This defines three sets.

Definition 1.12. *Let* $\overleftrightarrow{\mathbf{A}}$ *be the set of sequences* $(\mathbf{a}_i)_{i\in\mathbb{Z}}\in\mathbf{A}^{\mathbb{Z}}$ *which are admissible:* $(\mathbf{a}_j,\mathbf{a}_{j+1})\in\mathbf{B}$ *for every* j*; it is the space of the orbits of the aforementioned transitive subshift of finite type.*

Let $\overrightarrow{\mathbf{A}}$ *be the set of sequences* $\mathbf{s}:=(\mathbf{a}_i)_{i\geq 0}\in\mathbf{A}^{\mathbb{N}}$ *such that* $(\mathbf{a}_j,\mathbf{a}_{j+1})\in\mathbf{B}$ *for every* $j\geq 0$*.*

Let $\overleftarrow{\mathbf{A}}$ *be the set of sequences* $\mathbf{u}:=(\mathbf{a}_i)_{i<0}\in\mathbf{A}^{\mathbb{Z}^*_-}$ *such that* $(\mathbf{a}_{-j-1},\mathbf{a}_{-j})\in\mathbf{B}$ *for every* $j>0$*.*

We endow the sets $\overleftarrow{\mathbf{A}}$, $\overrightarrow{\mathbf{A}}$ and $\overleftrightarrow{\mathbf{A}}$ with the topologies induced by the product topologies of $\mathbf{A}^{\mathbb{Z}^*_-}$, $\mathbf{A}^{\mathbb{N}}$ and $\mathbf{A}^{\mathbb{Z}}$ which are compact by Tychonoff's theorem.

The map $\mathbf{o}$ can be canonically extended from $\mathbf{A}^*$ to $\overrightarrow{\mathbf{A}}$ by $\mathbf{o}((\mathbf{a}_i)_{i\geq 0}):=\mathbf{o}(\mathbf{a}_0)$. We extend similarly $\mathbf{t}$ to $\overleftarrow{\mathbf{A}}$ by $\mathbf{t}((\mathbf{a}_i)_{i<0}):=\mathbf{t}(\mathbf{a}_{-1})$. We extend canonically the concatenation operation to the sets $\mathbf{B}^*$, $\overrightarrow{\mathbf{B}}$ and $\overleftarrow{\mathbf{B}}$ of pairs $(\mathbf{d},\mathbf{d}')$ in respectively $(\mathbf{A}^*)^2$, $\mathbf{A}^*\times\overrightarrow{\mathbf{A}}$ and $\overleftarrow{\mathbf{A}}\times\mathbf{A}^*$ such that $\mathbf{t}(\mathbf{d})=\mathbf{o}(\mathbf{d}')$. For $\mathbf{w}\in\mathbf{A}^*$, we denote:

$$\mathbf{A}^*\cdot\mathbf{w}:=\{\mathbf{d}\cdot\mathbf{w}\in\mathbf{A}^*:(\mathbf{d},\mathbf{w})\in\mathbf{B}^*\}\quad\text{and}\quad\mathbf{w}\cdot\overrightarrow{\mathbf{A}}:=\{\mathbf{w}\cdot\mathbf{s}\in\overrightarrow{\mathbf{A}}:(\mathbf{w},\mathbf{s})\in\overrightarrow{\mathbf{B}}\}$$
$$\text{and}\quad\overleftarrow{\mathbf{A}}\cdot\mathbf{w}:=\{\mathbf{u}\cdot\mathbf{w}\in\overleftarrow{\mathbf{A}}:(\mathbf{u},\mathbf{w})\in\overleftarrow{\mathbf{B}}\}\ .\tag{1.1}$$

Given a hyperbolic map of type $\mathbf{A}$, by associativity of the $\star$-product, we denote for every $\mathbf{w} := \mathbf{a}_1 \cdots \mathbf{a}_k \in \mathbf{A}^*$:

$$(Y^{\mathbf{e}}, F^{\mathbf{e}}) := (Y^{\mathbf{e}}, id) \quad \text{and} \quad (Y^{\mathbf{w}}, F^{\mathbf{w}}) := (Y^{\mathbf{a}_1}, F^{\mathbf{a}_1}) \star \cdots \star (Y^{\mathbf{a}_k}, F^{\mathbf{a}_k}) \tag{1.2}$$

which is a hyperbolic transformation by Proposition 1.7. If F is C^ω_ρ-hyperbolic of type $\mathbf{A}$, we denote similarly $(\tilde{Y}^{\mathbf{e}}, F^{\mathbf{e}}) := (\tilde{Y}^{\mathbf{e}}, id)$ and $(\tilde{Y}^{\mathbf{w}}, F^{\mathbf{w}})$ the $\star$-product of the C^ω_ρ-hyperbolic transformations associated to the letters of $\mathbf{w}$.

We recall that a hyperbolic basic set is a hyperbolic compact set which is transitive and locally maximal.

Proposition 1.13. *The maximal invariant set $\Lambda = \bigcap_{n\in\mathbb{Z}} (F^{\mathbf{A}})^n (Y^{\mathbf{e}} \times \mathbf{V})$ is a hyperbolic basic set.*

Proof. First note that Λ is compact since it is an intersection of compact sets. Furthermore, it is locally maximal since:

$$\Lambda \subset F^{\mathbf{A}}(Y^{\mathbf{e}} \times \mathbf{V}) \cap (F^{\mathbf{A}})^{-1}(Y^{\mathbf{e}} \times \mathbf{V}) = \bigcup_{(\mathbf{a},\mathbf{a}')\in\mathbf{A}^2 : \mathbf{o}(\mathbf{a})=\mathbf{t}(\mathbf{a}')} Y^{\mathbf{a}} \cap F^{\mathbf{a}'}(Y^{\mathbf{a}'}) \times \{\mathbf{o}(\mathbf{a})\}$$

which is strictly included in $Y^{\mathbf{e}} \times \mathbf{V}$ by property (1) of hyperbolic transformation's Definition 1.4. Furthermore it is hyperbolic by the cone properties (2) and (3) of Definition 1.4.

The transitivity of Λ follows from the one of the shift $\overleftrightarrow{\Lambda}$ defined by $(\mathbf{A}, \mathbf{B})$ and the next remark. □

Remark 1.14. *If $F^{\mathbf{A}}$ is a diffeomorphism, the dynamics of Λ is conjugate to the shift on $\overleftrightarrow{\mathbf{A}}$. In general, the dynamics of the inverse limit $\overleftrightarrow{\Lambda} := \{(z_i, \mathbf{v}_i)_{i\in\mathbb{Z}} \in \Lambda^{\mathbb{Z}} : (z_{i+1}, \mathbf{v}_{i+1}) = F^{\mathbf{A}}(z_i, \mathbf{v}_i)\}$ of Λ is conjugate to the shift on $\overleftrightarrow{\mathbf{A}}$ via the map*

$$h : (z_i, \mathbf{v}_i)_{i\in\mathbb{Z}} \in \overleftrightarrow{\Lambda} \mapsto (a_i)_{i\in\mathbb{Z}} \in \overleftrightarrow{\mathbf{A}} \quad \textit{with } \mathbf{a}_i \in \mathbf{A} \textit{ such that } z_i \in Y^{\mathbf{a}_i} \textit{ and } \mathbf{v}_i = \mathbf{o}(\mathbf{a}_i)\ .$$

Conversely, we will see in Proposition 2.25 Page 40 that modulo iterations, any C^2-horseshoe is conjugate to a hyperbolic map $F^{\mathbf{A}}$ of a certain type $\mathbf{A}$.

Definition 1.15. *Given a hyperbolic map of type $\mathbf{A}$, for $\mathbf{s} = (\mathbf{a}_i)_{i\geq 0} \in \overrightarrow{\mathbf{A}}$ and $\mathbf{u} = (\mathbf{a}_i)_{i<0} \in \overleftarrow{\mathbf{A}}$, we denote:*

$$W^{\mathbf{s}} := \bigcap_{i\geq 0} Y^{\mathbf{a}_0\cdots\mathbf{a}_i} \quad \textit{and} \quad W^{\mathbf{u}} := \bigcap_{i>0} F^{\mathbf{a}_{-i}\cdots\mathbf{a}_{-1}}(Y^{\mathbf{a}_{-i}\cdots\mathbf{a}_{-1}})\,.$$

The following is immediate:

Proposition 1.16. *For every $\mathbf{s} \in \overrightarrow{\mathbf{A}}$, the set $W^{\mathbf{s}} \times \{\mathbf{o}(\mathbf{s})\}$ is a local stable manifold of the hyperbolic set Λ with tangent spaces in χ_v and endpoints in the two different components of $\partial^u Y^{\mathbf{e}}$.*

For every $\mathbf{u} \in \overleftarrow{\mathbf{A}}$, the set $W^{\mathbf{u}} \times \{\mathbf{t}(\mathbf{u})\}$ is a local unstable manifold of Λ with tangent spaces in χ_h and with endpoints in the two different components of $\partial^s Y^{\mathbf{e}}$.

By Proposition 1.16, for every $\mathbf{u} \in \overleftarrow{\mathbf{A}}$ and $\mathbf{s} \in \overrightarrow{\mathbf{A}}$, there are θ-Lipschitz functions $w^{\mathbf{u}}$ and $w^{\mathbf{s}}$ on I whose graphs satisfy $\operatorname{Graph} w^{\mathbf{u}} = W^{\mathbf{u}}$ and ${}^t\operatorname{Graph} w^{\mathbf{s}} = W^{\mathbf{s}}$.

Proposition 1.17. *The maps $x \in I \mapsto w^{\mathbf{u}}(x) \in \mathbb{R}$ and $y \in I \mapsto w^{\mathbf{s}}(y) \in \mathbb{R}$ are of class C^2 and depend continuously on $\mathbf{u}$ and $\mathbf{s}$ in the C^2-topology.*

Proof. This is the continuity of the local stable and unstable manifolds of the hyperbolic set Λ. $\square$

The following enables to consider quadratic homoclinic tangencies of the basic set Λ. It is defined using a finite set $\mathbf{C}$ of (new) arrows $\mathbf{c}$ between vertexes of $\mathbf{V}$.

Definition 1.18. *Given $\mathbf{c} \in \mathbf{C}$, a folding transformation $(Y^{\mathbf{c}}, F^{\mathbf{c}})$ is the data of a filled square $Y^{\mathbf{c}} = I^{\mathbf{c}} \times J^{\mathbf{c}}$ included in $Y^{\mathbf{e}} \setminus \partial^s Y^{\mathbf{e}}$ and a C^2-diffeomorphism $F^{\mathbf{c}}$ from $Y^{\mathbf{c}}$ onto its image in $Y^{\mathbf{e}} \setminus \partial^u Y^{\mathbf{e}}$, which satisfies:*

(1) The following subset is nonempty and clopen:

$$\overleftarrow{\mathbf{A}}_{\mathbf{c}} := \{\mathbf{u} \in \overleftarrow{\mathbf{A}} : \mathbf{t}(\mathbf{u}) = \mathbf{o}(\mathbf{c}), W^{\mathbf{u}} \cap Y^{\mathbf{c}} \neq \emptyset \ \& \ W^{\mathbf{u}} \cap \partial^u Y^{\mathbf{c}} = \emptyset\} \quad \text{with } \partial^u Y^{\mathbf{c}} := I^{\mathbf{c}} \times \partial J^{\mathbf{c}}.$$

(2) With $\overrightarrow{\mathbf{A}}_{\mathbf{c}} := \{\mathbf{s} \in \overrightarrow{\mathbf{A}} : \mathbf{o}(\mathbf{s}) = \mathbf{t}(\mathbf{c})\}$, for any $\mathbf{u} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}}$ and $\mathbf{s} \in \overrightarrow{\mathbf{A}}_{\mathbf{c}}$, if the curves $W^{\mathbf{u}}$ and $(F^{\mathbf{c}})^{-1}(W^{\mathbf{s}})$ are tangent, then the tangency is unique, quadratic and lies in $\operatorname{int} Y_{\mathbf{c}}$.

The folding transformation is of class C^ω_ρ if $F^{\mathbf{c}}$ extends to a biholomorphism from $\tilde{Y}^{\mathbf{c}} := Y^{\mathbf{c}} + [-i \cdot \rho, i \cdot \rho]^2$ onto its image in $\mathbb{C}^2$.

Here is the main dynamical object of this subsection.

Definition 1.19. *A system of type $(\mathbf{A}, \mathbf{C})$ is a C^2-local diffeomorphism F of the form:*

$$F : (x, \mathbf{v}) \in D(\mathbf{A}) \sqcup D(\mathbf{C}) \mapsto \begin{cases} F^{\mathbf{A}}(x, \mathbf{v}) & \text{if } (x, \mathbf{v}) \in D(\mathbf{A}), \\ F^{\mathbf{C}}(x, \mathbf{v}) & \text{if } (x, \mathbf{v}) \in D(\mathbf{C}) \end{cases}$$

for a hyperbolic map $F^{\mathbf{A}}$ of type $\mathbf{A}$ and the map $F^{\mathbf{C}}$ is the disjoint union of a finite family of folding transformations $(Y^{\mathbf{c}}, F^{\mathbf{c}})_{\mathbf{c} \in \mathbf{C}}$ for $F^{\mathbf{A}}$ by

$$F^{\mathbf{C}} : (z, \mathbf{o}(\mathbf{c})) \in D(\mathbf{C}) := \bigsqcup_{\mathbf{c} \in \mathbf{C}} Y^{\mathbf{c}} \times \{\mathbf{o}(\mathbf{c})\} \mapsto (F^{\mathbf{c}}(z), \mathbf{t}(\mathbf{c})) \in Y^{\mathbf{e}} \times \mathbf{V}$$

and such that for every $\mathbf{c} \neq \mathbf{c}' \in \mathbf{C}$, the sets $\overleftarrow{\mathbf{A}}_{\mathbf{c}}$ and $\overleftarrow{\mathbf{A}}_{\mathbf{c}'}$ are disjoint.

Summary of symbolic notations

Any object in the category of symbols will be denoted in this **style**. Moreover the following letters will be used canonically:

$\mathbf{v}$ denotes a *vertex*; these form a set $\mathbf{V}$ of vertices.
$\mathbf{a}$ denotes an arrow associated to a *hyperbolic transformation*; these form a set $\mathbf{A}$ of arrows.
$\mathbf{c}$ denotes an arrow associated to a *folding transformation*; these form a set $\mathbf{C}$ of arrows.
$\mathbf{w}$ denotes an admissible *finite* word in an alphabet of arrows.
$\mathbf{s}$ denotes an admissible *forward infinite* word in an alphabet of arrows.
$\mathbf{u}$ denotes an admissible *backward infinite* word in an alphabet of arrows.
$\mathbf{o}(\cdot)$ and $\mathbf{t}(\cdot)$ are maps assigning to an arrow or word its *origin* and *target* vertices, when defined.
Sets with a star, like $\mathbf{A}^*$, are formed by *all* the admissible *finite* words in the given alphabet.
$\mathbf{W}$ denotes a *subset* of admissible words satisfying a certain *extra property*.
Sets with a right arrow, like $\overrightarrow{\mathbf{A}}$, are formed by admissible *forward infinite* words.
Sets with a left arrow, like $\overleftarrow{\mathbf{A}}$, are formed by admissible *backward infinite* words.
Sets with a left-right arrow, like $\overleftrightarrow{\mathbf{A}}$, are formed by admissible *bi-infinite* words.

Example 1.20. *Let $F^{\mathbf{A}}$ be the example of hyperbolic map of type $\mathbf{A}$ of Example 1.10 with N even and $\delta > 0$ small. We recall that $(\mathbf{V}, \mathbf{A})$ is a graph with a unique vertex $\diamond$ and N arrows $\mathbf{a}_j$. Let $\mathbf{C} = \{\mathbf{c}_j : 1 \leq j \leq N-1\}$ be a set of $N-1$ other arrows satisfying $\mathbf{o}(\mathbf{c}_j) = \mathbf{t}(\mathbf{c}_j) = \diamond$. For p in:*

$$\mathcal{P} := \left\{ (p_i)_i \in \mathbb{R}^{N-1} : \forall 1 \leq i \leq N-1, \quad p_i \in I_{i+1} \text{ and } p_i \text{ is } \sqrt[3]{\delta}\text{-distant from } \partial I_{i+1} \right\},$$

we define a folding transformation $F_p^{\mathbf{c}_j}$ for $F^{\mathbf{A}}$ by:

$$F_p^{\mathbf{c}_j} : z = (x, y) \in Y^{\mathbf{c}_j} \mapsto \left(x^2 + y + p_j, -\sqrt{\delta} \cdot x \right)$$

$$\text{with} \quad Y^{\mathbf{c}_j} = I^{\mathbf{c}} \times J^{\mathbf{c}_j}, \quad I^{\mathbf{c}} = \left[-\delta^2/4, \delta^2/4 \right] \quad \text{and} \quad J^{\mathbf{c}_j} := \sqrt{\delta} \cdot I_j .$$

Note that $F_p^{\mathbf{c}_j}$ depends on a parameter $p \in \mathcal{P}$ but its definition domain $Y^{\mathbf{c}_j}$ is independent of p, see Fig. 4.

Fact 1.21. *The map $F_p^{\mathbf{c}_j}$ is a folding transformation at $\diamond$ adapted to $F^{\mathbf{A}}$ with $\overleftarrow{\mathbf{A}}_{\mathbf{c}_j} := \overleftarrow{\mathbf{A}} \cdot \mathbf{a}_j$.*

Proof. First note that the segment $I^{\mathbf{c}}$ is disjoint from $\bigsqcup_k I_k = \bigsqcup_k \left[\frac{2}{N}(k-1) - 1 + \delta^2, \frac{2}{N}k - 1 - \delta^2 \right]$. Thus $Y^{\mathbf{c}_j}$ is disjoint from $\bigcup_{k=1}^N Y^{\mathbf{a}_k}$. Also the image of $F_p^{\mathbf{c}_j}$ is included in $(-1, 1) \times [-\delta^2/4, \delta^2/4]$, which is in the interior of $Y^{\mathbf{e}}$. We notice that for every $\mathbf{u} \in \overleftarrow{\mathbf{A}}$, there exists $y^{\mathbf{u}}$ such that $W^{\mathbf{u}} = I \times \{y^{\mathbf{u}}\}$. Note also that $y^{\mathbf{u}}$ belongs to the interior of $\bigsqcup_k \sqrt{\delta} \cdot I_k$ and so $W^{\mathbf{u}}$ is disjoint from $I \times \bigsqcup_k \sqrt{\delta} \cdot \partial I_k \supset \partial^u Y^{\mathbf{c}_j}$. Also $W^{\mathbf{u}}$ intersects $Y^{\mathbf{c}_j}$ iff $\mathbf{u} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}_j} := \overleftarrow{\mathbf{A}} \cdot \mathbf{a}_j$ which is indeed a clopen set. Also, for $\mathbf{u} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}_j}$, the image by $F_p^{\mathbf{c}_j}$ is:

$$F_p^{\mathbf{c}_j}(W^{\mathbf{u}} \cap Y^{\mathbf{c}_j}) = \left\{ (x^2 + y^{\mathbf{u}} + p_j, -\sqrt{\delta} \cdot x) : -\frac{\delta^2}{4} \leq x \leq \frac{\delta^2}{4} \right\}. \tag{1.3}$$

As every stable manifold $W^{\mathbf{s}}$, for $\mathbf{s} \in \overrightarrow{\mathbf{A}}$, is of the form $\{x_{\mathbf{s}}\} \times I$, if $F_p^{\mathbf{c}_j}(W^{\mathbf{u}} \cap Y^{\mathbf{c}_j})$ is tangent to $W^{\mathbf{s}}$ then the tangency is quadratic. □

As the boxes $(Y^{\mathbf{c}_j})_{1 \leq j \leq N-1}$ are disjoint, we can define:

$$F_p : (x, \diamond) \in \bigsqcup_{j=1}^{N} Y^{\mathbf{a}_j} \sqcup \bigsqcup_{j=1}^{N-1} Y^{\mathbf{c}_j} \times \{\diamond\} \mapsto \begin{cases} (F^{\mathbf{a}}(x), \diamond) & \text{if } x \in Y^{\mathbf{a}},\ \mathbf{a} \in \mathbf{A} \\ (F_p^{\mathbf{c}}(x), \diamond) & \text{if } x \in Y^{\mathbf{c}},\ \mathbf{c} \in \mathbf{C}. \end{cases}$$

Fact 1.22. *The map F_p is a system of type $(\mathbf{A}, \mathbf{C})$ which is a diffeomorphism for every $p \in \mathcal{P}$.*

Proof. We already saw that $F^{\mathbf{A}}$ is a diffeomorphism. Looking at the definition of $\mathcal{P}$ and $F_p^{\mathbf{C}}$, we see that the folding transformations $F_p^{\mathbf{c}}$ have disjoint images from the one other. These images are included in $(-1, 1) \times [-\delta^2/4, \delta^2/4]$ which is disjoint from the range of $F^{\mathbf{A}}$. □

In order to do the parameter selection in Section 4, we will need the following object enabling to follow the tangency point of the homoclinic tangencies in a neat way:

Definition 1.23. *An adapted projection to a map F of type $(\mathbf{A}, \mathbf{C})$ is a C^1-submersion:*

$$\pi : (z, \mathbf{v}) \in \mathbb{R} \times I \times \mathbf{V} \mapsto (\pi^{\mathbf{v}}(z), \mathbf{v}) \in \mathbb{R} \times \mathbf{V}$$

such that with $\mathbb{P}(\chi_v) \subset \mathbb{P}(\mathbb{R})$ the set of lines spanned by vectors in χ_v:

(0) For every $(z, \mathbf{d}) \in \bigcup_{\mathbf{d} \in \mathbf{A} \cup \mathbf{C}} Y^{\mathbf{d}} \times \{\mathbf{o}(\mathbf{d})\}$, the $\pi^{\mathbf{o}(\mathbf{d})}$-fiber of z is included in $Y^{\mathbf{e}}$.

(1) The map $(z, \mathbf{v}) \in Y^{\mathbf{e}} \times \mathbf{V} \mapsto \ker D_z \pi^{\mathbf{v}}$ is of class C^1 and takes its values in $\mathbb{P}(\chi_v)$.

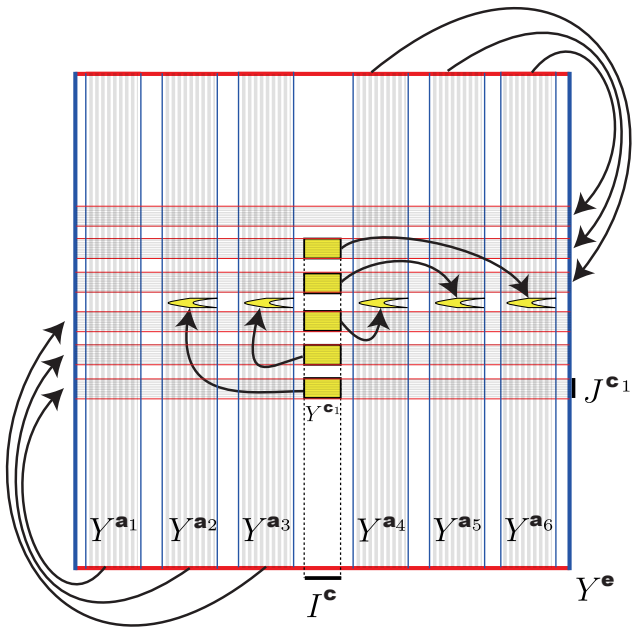

FIGURE 4. An example of a system of type $(\mathbf{A}, \mathbf{C})$.

(2) For any $\mathbf{v} \in \mathbf{V}$, for every $x \in \mathbb{R}$, it holds $\pi^{\mathbf{v}}(x, 0) = x$.

(3) For all $\mathbf{a} \in \mathbf{A}$ and $z, z' \in Y^{\mathbf{a}}$, if $\pi^{\mathbf{o}(\mathbf{a})}(z) = \pi^{\mathbf{o}(\mathbf{a})}(z')$ then $\pi^{\mathbf{t}(\mathbf{a})} \circ F^{\mathbf{a}}(z) = \pi^{\mathbf{t}(\mathbf{a})} \circ F^{\mathbf{a}}(z')$.

(4) For every $\mathbf{u} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}}$ with $\mathbf{c} \in \mathbf{C}$, the curve $W^{\mathbf{u}}$ is tangent to a unique fiber of $\pi^{\mathbf{t}(\mathbf{c})} \circ F^{\mathbf{c}}$ at a unique point $\zeta^{\mathbf{u}} \in \operatorname{int} Y^{\mathbf{c}}$, and the tangency is quadratic.

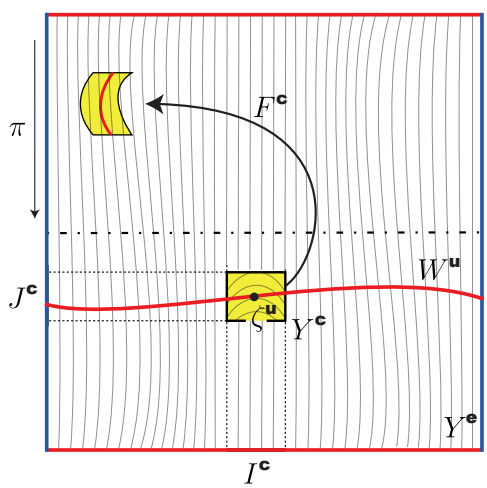

FIGURE 5. Definition of $\zeta^{\mathbf{u}}$.

Let us comment this definition. Item (0) says that for any $\mathbf{a} \in \mathbf{A} \cup \mathbf{C}$, any $\pi^{\mathbf{o}(\mathbf{a})}$-fiber intersecting $Y^{\mathbf{a}}$ is fully included in $Y^{\mathbf{e}}$. Item (1) implies that the fibers of $\pi^{\mathbf{v}}$ are C^2-curves with tangent space in χ_v and with two endpoints, one in each of the components of $\mathbb{R} \times \partial I$. Item (2) implies that π is a retraction. Item (3) implies, as we will prove in Proposition 1.25 that each curve $W^{\mathbf{s}}$ is a fiber of $\pi^{\mathbf{o}(\mathbf{s})}$ for $\mathbf{s}$. Thus π defines an extension of the stable lamination which is invariant by the dynamics $F^{\mathbf{A}}$. The last item is equivalent to say that for every $\mathbf{u} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}}$, the map $x \mapsto \pi^{\mathbf{t}(\mathbf{c})} \circ F^{\mathbf{c}} \circ (x, w^{\mathbf{u}}(x))$ has a unique critical point which is non-degenerate. It enables to describe how far an unstable curve $W^{\mathbf{u}}$, $\mathbf{u} \in \overleftarrow{\mathbf{A}}$ is to be folded tangent to a stable curve. The definition of $\zeta^{\mathbf{u}}$ is depicted in Fig. 5.

Example 1.24. *The projection $\pi : (x, y, \diamond) \mapsto (x, \diamond)$ is adapted to the map F_p of Example 1.20.*

Proof. The three first items of Definition 1.23 are obviously satisfied. Item (3) as well since $F^{\mathbf{A}}$ leaves invariant the vertical foliation. Finally we observe that by Eq. (1.3), for every $1 \leq j \leq N-1$, for every $\mathbf{u} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}_j}$ the curve $F_p^{\mathbf{c}_j}(W^{\mathbf{u}} \cap Y^{\mathbf{c}_j})$ has a unique quadratic tangency with a vertical line. This implies the last item. □

More generally, we will see in Proposition 2.25 that any horseshoe of a surface C^2-diffeomorphism can be modeled by a hyperbolic map of a certain type endowed with a map π satifying items (0-1-2-3-4) of Definition 1.23. As promised let us show:

Proposition 1.25. *If π is adapted to F then for every $\mathbf{s} \in \overrightarrow{\mathbf{A}}$, the curve $W^{\mathbf{s}}$ is a fiber of $\pi^{\mathbf{o}(\mathbf{s})}$.*

Proof. Let $\mathbf{s} = \mathbf{a}_0 \cdots \mathbf{a}_i \cdots \in \overrightarrow{\mathbf{A}}$, $\mathbf{v} = \mathbf{o}(\mathbf{s})$, $z \in W^{\mathbf{s}}$. As $W^{\mathbf{s}} = \bigcap_{n \geq 0} Y^{\mathbf{a}_0 \cdots \mathbf{a}_n}$, it follows that $(F^{\mathbf{A}})^n(z, v)$ is in $Y^{\mathbf{a}_n} \times \{\mathbf{o}(\mathbf{a}_n)\}$. Thus by Definition 1.23-(0), the π-fiber of $(F^{\mathbf{A}})^n(z, v)$ is included in $Y^{\mathbf{e}} \times \mathbf{V}$ for every n. By Definition 1.23 (3), the π-fiber of (z, v) is sent into the π-fiber of $(F^{\mathbf{A}})^n(z, v)$ by $(F^{\mathbf{A}})^n$. The tangent spaces of the fibers of π are in χ_v by Definition 1.23-(1), and every vector in χ_v has its second coordinate which is more than λ-expanded by $D_z(F^{\mathbf{a}})^{-1}$ for any $\mathbf{a} \in \mathbf{A}$ and $z \in F^{\mathbf{a}}(Y^{\mathbf{a}})$ by Definition 1.4 (3). Thus the π-fiber of (z, v) is a local stable manifold of $(z, \mathbf{v})$. By Proposition 1.16 and uniqueness of the local stable manifold of the hyperbolic set Λ, it follows that $W^{\mathbf{s}} \times \{\mathbf{v}\}$ is equal to the π-fiber of $(z, \mathbf{v})$. □

We are going to give sufficient conditions for a system F of type $(\mathbf{A}, \mathbf{C})$ to display a wandering stable component. We will use the following:

Definition 1.26. *For every $\mathbf{w} \in \mathbf{A}^*$, let $H^{\mathbf{w}} := F^{\mathbf{w}}(I \times \{0\} \cap Y^{\mathbf{w}})$.*

Definition 1.27. *For any $\mathbf{w} \in \mathbf{A}^*$, let $|Y^{\mathbf{w}}|$ be the maximum of the lengths of $I \times \{y\} \cap Y^{\mathbf{w}}$ among $y \in I$ and $|F^{\mathbf{w}}(Y^{\mathbf{w}})|$ be the maximum of the lengths of $\{x\} \times I \cap F^{\mathbf{w}}(Y^{\mathbf{w}})$ among $x \in I$.*

The following enables to extend the tangency point $\zeta^{\mathbf{u}}$ for some finite words $\mathbf{w} \in \mathbf{A}^*$.

Proposition 1.28 (definition of $\mathbf{A}^*_{\mathbf{c}}$ and $\zeta^{\mathbf{w}}$)**.** *For every $\mathbf{c} \in \mathbf{C}$, there exists $n_{\mathbf{c}} > 0$ minimal such that for any $\mathbf{w}$ in:*

$$\mathbf{A}^*_{\mathbf{c}} := \{\mathbf{w} \in \mathbf{A}^* : |\mathbf{w}| \geq n_{\mathbf{c}} \text{ and } \mathbf{w} \text{ equals the } |\mathbf{w}| \text{ last letters of a certain } \mathbf{u} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}}\}$$

the curve $H^{\mathbf{w}}$ intersects $Y^{\mathbf{c}}$ but not $\partial^u Y^{\mathbf{c}}$, the curve $H^{\mathbf{w}}$ is tangent to a unique fiber of $\pi^{\mathbf{t}(\mathbf{c})} \circ F^{\mathbf{c}}$ at a unique point $\zeta^{\mathbf{w}} \in \operatorname{int} Y^{\mathbf{c}}$, and the tangency is quadratic.

Proof. It is sufficient to prove that for every sequence $(\mathbf{w}_i)_{\geq 0}$ of words $\mathbf{w}_i \in \mathbf{A}^*$ such that $|\mathbf{w}_i| \to \infty$ and such that there exists $\mathbf{u}_i \in \overleftarrow{\mathbf{A}}_{\mathbf{c}}$ finishing by $\mathbf{w}_i$, the following properties hold true when i is large enough: the curve $H^{\mathbf{w}_i}$ intersects $Y^{\mathbf{c}}$ but not $\partial^u Y^{\mathbf{c}}$, the curve $H^{\mathbf{w}_i}$ is tangent to a unique fiber of $\pi^{\mathbf{t}(\mathbf{c})} \circ F^{\mathbf{c}}$ at a unique point in $\operatorname{int} Y^{\mathbf{c}}$, and the tangency is quadratic. By compactness of $\overleftarrow{\mathbf{A}}_{\mathbf{c}}$, we can assume that $(\mathbf{u}_i)_i$ converges to $\mathbf{u} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}}$. By continuity of $\mathbf{u} \mapsto W^{\mathbf{u}}$, the curve $W^{\mathbf{u}_i}$ is C^2-close to $W^{\mathbf{u}}$ when i is large.

We notice that each of the $H^{\mathbf{w}_i}$ is a component of the image by $(F^{\mathbf{A}})^{|\mathbf{w}_i|}$ of the curve $I \times \{0\} \times \mathbf{V}$ which is uniformly transverse to the stable lamination of Λ. When i is large, by the Inclination Lemma, the curve $H^{\mathbf{w}_i}$ is C^2-close to $W^{\mathbf{u}_i}$ and so to $W^{\mathbf{u}}$.

By definition of $\overleftarrow{\mathbf{A}}_{\mathbf{c}}$, the curve $W^{\mathbf{u}}$ intersects $Y^{\mathbf{c}}$ but not $\partial^u Y^{\mathbf{c}}$. This is equivalent to say that $W^{\mathbf{u}}$ intersects the interior of $Y^{\mathbf{c}}$ but not $\partial^u Y^{\mathbf{c}}$ which is an open condition on curves. Thus for i sufficiently large $H^{\mathbf{w}_i}$ intersects the interior of $Y^{\mathbf{c}}$ but not $\partial^u Y^{\mathbf{c}}$.

Also $W^{\mathbf{u}}$ has a tangency with the foliation defined by $\pi^{\mathbf{t}(\mathbf{c})} \circ F^{\mathbf{c}}$, which is unique, in $\operatorname{int} Y^{\mathbf{c}}$ and quadratic. This is equivalent to say that the curve $TW^{\mathbf{u}} \subset Y^{\mathbf{e}} \times \mathbb{P}(\mathbb{R})$ is transverse to the surface $\{(z, \ker D\pi^{\mathbf{t}(\mathbf{c})} \circ D_z F^{\mathbf{c}}) : z \in Y^{\mathbf{c}}\} \subset Y^{\mathbf{e}} \times \mathbb{P}(\mathbb{R})$ and intersects it at a unique point in $\operatorname{int} Y^{\mathbf{c}} \times \mathbb{P}(\mathbb{R})$. This is an open condition on C^1-curves in $Y^{\mathbf{e}} \times \mathbb{P}(\mathbb{R})$. Thus $H^{\mathbf{w}_i}$ satisfies the same property when i is large. □

Remark 1.29. *We have shown that the set of C^2-curves $\{H^{\mathbf{w}} : \mathbf{w} \in \mathbf{A}^*_{\mathbf{c}}\} \cup \{W^{\mathbf{u}} : \mathbf{u} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}}\}$ is compact. Also the subset $\{\zeta^{\mathbf{w}} : \mathbf{w} \in \mathbf{A}^*_{\mathbf{c}} \cup \overleftarrow{\mathbf{A}}_{\mathbf{c}}\}$ of $\mathbb{R}^2$ is compact.*

When F is real analytic, we define $\rho > 0$ and the sets $(\tilde{Y}^{\mathbf{a}})_{\mathbf{a} \in \mathbf{A}}$ and $\tilde{Y}^{\mathbf{c}}$ by the following:

Proposition 1.30 (and definition of $\rho > 0$)**.** *If F is real analytic, then there exists $\rho > 0$ such that for every $\mathbf{w} \in \mathbf{A}^*$, the hyperbolic transformation $(Y^{\mathbf{w}}, F^{\mathbf{w}})$ extends to a C^ω_ρ-hyperbolic transformation $(\tilde{Y}^{\mathbf{w}}, F^{\mathbf{w}})$ and the map $F^{\mathbf{c}}$ is of class C^ω_ρ on $\tilde{Y}^{\mathbf{c}}$ for every $\mathbf{c} \in \mathbf{C}$.*

Proof. The first condition on $F^{\mathbf{A}}$ is proved in Proposition A.3 and the second condition on $F^{\mathbf{c}}$ is obvious by Definition 1.18. □

Definition 1.31. *If F is analytic, for any $\mathbf{w} \in \mathbf{A}^*$, let $|F^{\mathbf{w}}(\tilde{Y}^{\mathbf{w}})|$ be the maximum of the diameters of $\{z\} \times \tilde{I} \cap F^{\mathbf{w}}(\tilde{Y}^{\mathbf{w}})$ among $z \in \tilde{I}$.*

Here is the proposition which gives sufficient conditions to ensure the existence of a wandering stable component. It is a variation of Theorem D of [Ber18] and its Remark 3.5.

Theorem 1.32. *Let F be a system of type $(\mathbf{A}, \mathbf{C})$ endowed with an adapted projection π. Let $(\mathbf{c}_j)_{j \geq 1} \in \mathbf{C}^{\mathbb{N}}$ and let $(\mathbf{w}_j)_{j \geq 1}$ be a sequence of words $\mathbf{w}_j \in \mathbf{A}^*_{\mathbf{c}_j}$ such that $\mathbf{o}(\mathbf{w}_j) = \mathbf{t}(\mathbf{c}_{j-1})$ and satisfying the following assumptions when j is large:*

(i) the limit exists and is non-zero $\breve{\gamma}_j := |Y^{\mathbf{w}_{j+1}}|^{1/2} \cdot |Y^{\mathbf{w}_{j+2}}|^{1/2^2} \cdots |Y^{\mathbf{w}_{j+k}}|^{1/2^k} \cdots$,

(ii) the distance between $\pi^{\mathbf{t}(\mathbf{w}_{j+1})}$-fibers of $F^{\mathbf{w}_{j+1}} \circ F^{\mathbf{c}_j}(\zeta^{\mathbf{w}_j})$ and $\zeta^{\mathbf{w}_{j+1}}$ is small compared to $\breve{\gamma}_{j+1}$,

(iii) the width $|Y^{\mathbf{w}_j}|$ is small and the height $|F^{\mathbf{w}_j}(Y^{\mathbf{w}_j})|$ is small compared to $\breve{\gamma}_j^2$:

$$|Y^{\mathbf{w}_j}| = o(1) \quad \text{and} \quad |F^{\mathbf{w}_j}(Y^{\mathbf{w}_j})| = o(\breve{\gamma}_j^2).$$

Then there exist $J \geq 0$ and a sequence of nonempty open subsets $(B_j)_{j\geq J}$ of $Y^{\mathbf{e}}$ such that:

$$B_j \subset Y^{\mathbf{c}_j}, \quad F^{\mathbf{c}_j}(B_j) \subset Y^{\mathbf{w}_{j+1}}, \quad F^{\mathbf{w}_{j+1}} \circ F^{\mathbf{c}_j}(B_j) \subset B_{j+1} \quad \text{and} \quad \lim_{k\to\infty} \operatorname{diam} F^k(B_j \times \{\mathbf{o}(\mathbf{c}_j)\}) = 0\,.$$

Remark 1.33. *We can show that (i) is equivalent to ask that $\log|Y^{\mathbf{w}_j}| \asymp |\mathbf{w}_j|$ satisfies that $\sum_j 2^{-j}|\mathbf{w}_j| < \infty$. We will see, using distortion bound, that assumption (ii) is equivalent to say that the distance between the $\pi^{\mathbf{o}(\mathbf{w}_{j+1})}$-fibers of the points $F^{\mathbf{c}_j}(\zeta^{\mathbf{w}_j})$ and $(F^{\mathbf{w}_{j+1}})^{-1}(\zeta^{\mathbf{w}_{j+1}})$ is small compared to $\breve{\gamma}_j^2$. The first part of assumption (iii) implies that $|\mathbf{w}_j|$ is large when j is large.*

Theorem 1.34. *Under the assumptions of Theorem 1.32, if moreover F is analytic and if*

$(\widetilde{iii})$ $\breve{\gamma}_j$ is small compared to $|Y^{\mathbf{w}_{j+1}}|$ and $|F^{\mathbf{w}_j}(\tilde{Y}^{\mathbf{w}_j})|$ is small compared to $\breve{\gamma}_j^2$ when j is large:

$$\breve{\gamma}_j = o(|Y^{\mathbf{w}_{j+1}}|) \quad \text{and} \quad |F^{\mathbf{w}_j}(\tilde{Y}^{\mathbf{w}_j})| = o(\breve{\gamma}_j^2).$$

then each B_j is the real trace of an open subset $\tilde{B}_j \subset \mathbb{C}^2$ such that:

$$\tilde{B}_j \subset \tilde{Y}^{\mathbf{c}_j}, \quad F^{\mathbf{c}_j}(\tilde{B}_j) \subset \tilde{Y}^{\mathbf{w}_{j+1}}, \quad F^{\mathbf{w}_{j+1}} \circ F^{\mathbf{c}_j}(\tilde{B}_j) \subset \tilde{B}_{j+1} \quad \text{and} \quad \lim_{k\to\infty} \operatorname{diam} F^k(\tilde{B}_j \times \{\mathbf{o}(\mathbf{c}_j)\}) = 0\,,$$

where $\tilde{Y}^{\mathbf{c}_j}$ and $\tilde{Y}^{\mathbf{w}_{j+1}}$ are defined by Proposition 1.30.

We notice that $B_j \times \{\mathbf{o}(\mathbf{c}_j)\}$ and $\tilde{B}_j \times \{\mathbf{o}(\mathbf{c}_j)\}$ are stable domains for the dynamics F which are in wandering stable components (for their combinatorics are wandering). The proof of Theorems 1.32 and 1.34 will occupy all Section 3.3 and will need a development of the techniques of implicit represention of Section 3.1 and a normal form defined in Section 3.2.

1.2. **Unfolding of wild type.** In this section we are going to state natural conditions on families of systems of type $(\mathbf{A}, \mathbf{C})$ so that at a dense set of parameters the map will satisfy the assumptions of Theorems 1.32 and 1.34.

We recall that a closed subset is *regular* if it is nonempty and equal to the closure of its interior. Let $\mathcal{P}$ be a nonempty, regular compact subset of $\mathbb{R}^d$. For the proofs of our main theorems, $\mathcal{P}$ will be a hypercube of $\mathbb{R}^5$. Let $n, m \geq 1$ and let $(K_p)_p$ be a family of compact sets of $\mathbb{R}^n$. For $r \geq 0$, a family $(f_p)_{p\in\mathcal{P}}$ of maps $f_p : K_p \to \mathbb{R}^m$ is *of class C^r* if $p \in \mathcal{P} \mapsto K_p$ is continuous for the Hausdorff topology and the map $(p, z) \in \hat{K} := \bigcup_{p\in\mathcal{P}} \{p\} \times K_p \mapsto f_p(z) \in \mathbb{R}^m$ is of class C^r in the sense we defined Page 14.

We are now going to consider an unfolding of the stable and unstable laminations of a C^2-family of maps $(F_p)_{p\in\mathcal{P}}$ of type $(\mathbf{A}, \mathbf{C})$.

Definition 1.35. *A family $(F_p)_{p\in\mathcal{P}}$ of systems of type $(\mathbf{A}, \mathbf{C})$ is* regular *if each F_p is a system of type $(\mathbf{A}, \mathbf{C})$ with a same set $\overleftarrow{\mathbf{A}}_{\mathbf{c}}$ for every $p \in \mathcal{P}$ at each fixed $\mathbf{c} \in \mathbf{C}$ and so that the family $(F_p)_{p\in\mathcal{P}}$ is of class C^2. Note that for every $p \in \mathcal{P}$, the map F_p is of the form:*

$$F_p : (x, \mathbf{v}) \in \bigsqcup_{\mathbf{a}\in\mathbf{A}} Y_p^{\mathbf{a}} \times \{\mathbf{o}(\mathbf{a})\} \sqcup \bigsqcup_{\mathbf{c}\in\mathbf{C}} Y_p^{\mathbf{c}} \times \{\mathbf{o}(\mathbf{c})\} \mapsto \begin{cases} F_p^{\mathbf{A}}(x, \mathbf{v}) & \text{if } (x, \mathbf{v}) \in \bigsqcup_{\mathbf{a}\in\mathbf{A}} Y_p^{\mathbf{a}} \times \{\mathbf{o}(\mathbf{a})\}, \\ F_p^{\mathbf{C}}(x, \mathbf{v}) & \text{if } (x, \mathbf{v}) \in \bigsqcup_{\mathbf{c}\in\mathbf{C}} Y_p^{\mathbf{c}} \times \{\mathbf{o}(\mathbf{c})\}. \end{cases}$$

A family of projections $(\pi_p)_p$ is adapted to $(F_p)_p$ if each π_p is a projection adapted to F_p, the family $(\pi_p)_p$ is of class C^1 and the family $(\ker D_z\pi_p)_{p\in\mathcal{P}}$ is of class C^1.

Example 1.36. *For instance the family $(F_p)_p$ of systems of type $(\mathbf{A},\mathbf{C})$ of Example 1.20 is regular and the trivial family of projections $(\pi)_{p\in\mathcal{P}}$ of Example 1.24 is adapted to it.*

Let us fix a regular family $(F_p)_p$ of type $(\mathbf{A},\mathbf{C})$ endowed with an adapted family $(\pi_p)_p$ of projections. As in Definition 1.35, for every $p\in\mathcal{P}$, we add an index p to all the geometric objects defined in the last subsection. For instance, we denote also by Λ_p the maximal invariant set of $F_p^{\mathbf{A}}$ defined in Proposition 1.13. For every $\mathbf{w}=\mathbf{a}_1\cdots\mathbf{a}_k\in\mathbf{A}^*$, we denote $(Y_p^{\mathbf{w}},F_p^{\mathbf{w}}):=(Y_p^{\mathbf{a}_1},F_p^{\mathbf{a}_1})\star\cdots(Y_p^{\mathbf{a}_k},F_p^{\mathbf{a}_k})$. For $\mathbf{s}\in\overrightarrow{\mathbf{A}}$ and $\mathbf{u}\in\overleftarrow{\mathbf{A}}$, we denote by $W_p^{\mathbf{s}}$ and $W_p^{\mathbf{u}}$ the corresponding local stable and unstable manifolds of $F_p^{\mathbf{A}}$.

The following is immediate:

Proposition 1.37. *For every $\mathbf{w}\in\mathbf{A}^*$, the family $((Y_p^{\mathbf{w}},F_p^{\mathbf{w}}))_{p\in\mathcal{P}}$ is of class C^2.*

By Proposition 1.16, for every $\mathbf{u}\in\overleftarrow{\mathbf{A}}$ and $\mathbf{s}\in\overrightarrow{\mathbf{A}}$, for every $p\in\mathcal{P}$, there are θ-Lipschitz functions $w_p^{\mathbf{u}}$ and $w_p^{\mathbf{s}}$ on I whose graphs satisfy:

$$\operatorname{Graph} w_p^{\mathbf{u}}=W_p^{\mathbf{u}}\quad\text{and}\quad {}^t\operatorname{Graph} w_p^{\mathbf{s}}=W_p^{\mathbf{s}}\,. \tag{1.4}$$

The following states that the stable and unstable laminations depend C^2 on the parameter:

Proposition 1.38. *The maps $(p,x)\in\mathcal{P}\times I\mapsto w_p^{\mathbf{u}}(x)\in\mathbb{R}$ and $(p,y)\in\mathcal{P}\times I\mapsto w_p^{\mathbf{s}}(y)\in\mathbb{R}$ are of class C^2. Moreover, the two following maps are continuous:*

$$\mathbf{u}\in\overleftarrow{\mathbf{A}}\mapsto(w_p^{\mathbf{u}})_{p\in\mathcal{P}}\in C^2(\mathcal{P}\times I,I)\ \text{ and }\ \mathbf{s}\in\overrightarrow{\mathbf{A}}\mapsto(w_p^{\mathbf{s}})_{p\in\mathcal{P}}\in C^2(\mathcal{P}\times I,I)\,.$$

Proof. This is an immediate consequence of Theorem C.5 of the Appendix applied to the family of hyperbolic sets $(\Lambda_p)_p$. □

Definition 1.39. *For every $\mathbf{c}\in\mathbf{C}$, we define:*

- *The map $p\in\mathcal{P}\mapsto a_p^{\mathbf{u}}:=\pi_p^{\mathbf{t}(\mathbf{c})}\circ F_p^{\mathbf{c}}(\zeta_p^{\mathbf{u}})$, for every $\mathbf{u}\in\overleftarrow{\mathbf{A}}_{\mathbf{c}}$.*
- *The map $p\in\mathcal{P}\mapsto b_p^{\mathbf{s}}:=w_p^s(0)$, for every $\mathbf{s}\in\overrightarrow{\mathbf{A}}$.*
- *The map $p\in\mathcal{P}\mapsto\mathcal{V}(\mathbf{u},\mathbf{s},p)=a_p^{\mathbf{u}}-b_p^{\mathbf{s}}$, for all $\mathbf{u}\in\overleftarrow{\mathbf{A}}_{\mathbf{c}}$ and $\mathbf{s}\in\overrightarrow{\mathbf{A}}_{\mathbf{c}}$.*

As $\pi_p^{\mathbf{v}}(x,0)=x$ for every $(x,\mathbf{v})\in I\times\mathbf{V}$, it holds $\pi_p^{\mathbf{o}(\mathbf{s})}(W_p^{\mathbf{s}})=\{b_p^{\mathbf{s}}\}$. We recall that by Definition 1.18, $\zeta_p^{\mathbf{u}}$ is the unique critical point of $\pi_p^{\mathbf{t}(\mathbf{c})}\circ F_p^{\mathbf{c}}$. Thus $a_p^{\mathbf{u}}$ is the critical value of $\pi_p^{\mathbf{t}(\mathbf{c})}\circ F_p^{\mathbf{c}}\circ w_p^{\mathbf{u}}$ and $\mathcal{V}(\mathbf{u},\mathbf{s},p)$ quantifies how far $F_p^{\mathbf{c}}(W_p^{\mathbf{u}})$ is to be tangent to $W_p^{\mathbf{s}}=(\pi^{\mathbf{t}(\mathbf{c})})^{-1}(\{\mathbf{b}_p^{\mathbf{s}}\})$. In particular we have:

Remark 1.40. *The curve $F_p^{\mathbf{c}}(W_p^{\mathbf{u}}\cap Y_p^{\mathbf{c}})$ is tangent to $W_p^{\mathbf{s}}$ if and only if $\mathcal{V}(\mathbf{u},\mathbf{s},p)=0$.*

Proposition 1.41. *For every $\mathbf{c}\in\mathbf{C}$, the following are C^1-functions depending continuously on $\mathbf{u}\in\overleftarrow{\mathbf{A}}_{\mathbf{c}}$, $\mathbf{s}\in\overrightarrow{\mathbf{A}}_{\mathbf{c}}$:*

$$p\mapsto\zeta_p^{\mathbf{u}}\in\operatorname{int}Y_p^{\mathbf{c}}\,,\quad p\in\mathcal{P}\mapsto a_p^{\mathbf{u}}\in I\,,\quad p\in\mathcal{P}\mapsto b_p^{\mathbf{s}}\in I\quad\text{and}\quad p\mapsto\mathcal{V}(\mathbf{u},\mathbf{s},p)\in\mathbb{R}\,.$$

Proof. We recall that $(p,x)\mapsto w_p^{\mathbf{u}}(x)$ and $(p,y)\mapsto w_p^{\mathbf{s}}(y)$ are C^2-functions depending continuously on $\mathbf{u}\in\overleftarrow{\mathbf{A}}$ and $\mathbf{s}\in\overrightarrow{\mathbf{A}}$ by Proposition 1.38. Thus the map $p\mapsto b_p^{\mathbf{s}}=w_p^{\mathbf{s}}(0)$ is of class C^1 and depends continuously on $\mathbf{s}$. By Definition 1.23 (4), the curve $W_p^{\mathbf{u}}\cap Y_p^{\mathbf{c}}$ displays a unique tangency with a

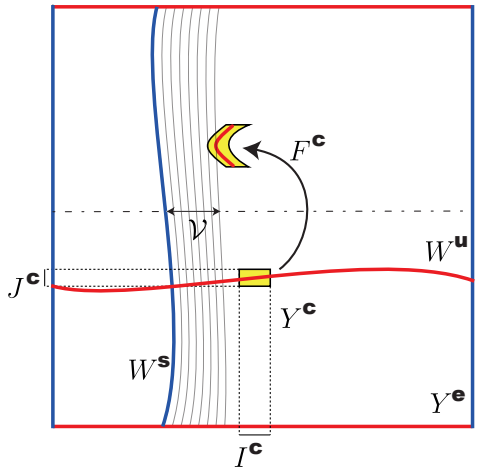

FIGURE 6. Definition of $\mathcal{V}$.

fiber of $\pi_p^{\mathbf{t}(\mathbf{c})} \circ F_p^{\mathbf{c}}$ and this tangency is quadratic. This means that the curve $TW_p^{\mathbf{u}} \cap Y_p^{\mathbf{c}} \times \mathbb{P}(\mathbb{R})$ of $Y^{\mathbf{e}} \times \mathbb{P}(\mathbb{R})$ intersects the surface $\{(z, \ker D\pi^{\mathbf{t}(\mathbf{c})} \circ D_z F^{\mathbf{c}}) : z \in Y^{\mathbf{c}}\}$ of $Y^{\mathbf{e}} \times \mathbb{P}(\mathbb{R})$ at a unique point $(\zeta_p^{\mathbf{u}}, T_{\zeta_p^{\mathbf{u}}} W^{\mathbf{u}})$ and this intersection is transverse. As this curve and this surface vary C^1 with p and continuously with $\mathbf{u}$, by transversality we obtain the sought regularity of $p \in \mathcal{P} \mapsto \zeta_p^{\mathbf{u}}$. Then the sought regularities of $p \in \mathcal{P} \mapsto a_p^{\mathbf{u}} = \pi_p^{\mathbf{t}(\mathbf{c})} \circ F_p^{\mathbf{c}}(\zeta_p^{\mathbf{u}})$ and $p \mapsto \mathcal{V}(\mathbf{u}, \mathbf{s}, p) = a_p^{\mathbf{u}} - b_p^{\mathbf{s}}$ follow. □

The next definition regards a regular family $(F_p)_{p \in \mathcal{P}}$ of systems of type $(\mathbf{A}, \mathbf{C})$ endowed with an adapted family $(\pi_p)_p$ of projections.

Definition 1.42. *The family $(F_p, \pi_p)_{p \in \mathcal{P}}$ is an* unfolding of wild type $(\mathbf{A}, \mathbf{C})$ *if:*

($\mathbf{H}_1$) *For every $p \in \mathcal{P}$, for every $\mathbf{c} \in \mathbf{C}$, there exist $\mathbf{u} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}}$ and $\mathbf{s} \in \overrightarrow{\mathbf{A}}_{\mathbf{c}}$ satisfying that $F_p^{\mathbf{c}}(W_p^{\mathbf{u}} \cap Y_p^{\mathbf{c}})$ has a quadratic tangency with $W_p^{\mathbf{s}}$.*

($\mathbf{H}_2$) *For all $(\mathbf{u}_{\mathbf{c}}, \mathbf{s}_{\mathbf{c}})_{\mathbf{c}} \in \prod_{\mathbf{c} \in \mathbf{C}} \overleftarrow{\mathbf{A}}_{\mathbf{c}} \times \overrightarrow{\mathbf{A}}_{\mathbf{c}}$, the following is a C^1-diffeomorphism onto its image:*

$$p \in \operatorname{int} \mathcal{P} \mapsto (\mathcal{V}(\mathbf{u}_{\mathbf{c}}, \mathbf{s}_{\mathbf{c}}, p))_{\mathbf{c} \in \mathbf{C}} \in \mathbb{R}^{\operatorname{Card} \mathbf{C}}$$

Let us comment this definition. Property ($\mathbf{H}_1$) asks for $\operatorname{Card} \mathbf{C}$ simultaneous homoclinic tangencies, each of which being associated to a different $\mathbf{c} \in \mathbf{C}$. By Proposition 1.41, we already now that the map involved in ($\mathbf{H}_2$) is of class C^1. What is required is its invertibility. It means roughly speaking that the unfoldings of the tangencies are independent and non-degenerated along the parameter space $\operatorname{int} \mathcal{P}$ whose dimension is equal to $\operatorname{Card} \mathbf{C}$.

We introduced the notion of unfolding of wild type $(\mathbf{A}, \mathbf{C})$ to show the existence of a dense set of parameter p for which the the system F_p has a wandering stable domain. To this end, a form of dissipation is necessary:

Definition 1.43. *A hyperbolic map $F^{\mathbf{A}}$ of type $\mathbf{A}$ is* moderately dissipative *if it holds:*

$$\|DF^{\mathbf{A}}\| \cdot \|\det DF^{\mathbf{A}}\|^{\epsilon} < 1 \quad \textit{with } \epsilon = \frac{1}{10}.$$

An unfolding $(F_p, \pi_p)_p$ of wild type is moderately dissipative if $F_p^{\mathbf{A}}$ is moderately dissipative for every $p \in \mathcal{P}$.

Here is the main (abstract) result of this manuscript:

Theorem 1.44 (Main)**.** *Let $(F_p, \pi_p)_{p\in\mathcal{P}}$ be a moderately dissipative unfolding of wild type $(\mathbf{A}, \mathbf{C})$ with $\operatorname{Card}\mathbf{C} = 5$. Then there exists a dense set of parameters $p \in \mathcal{P}$ for which (F_p, π_p) satisfies the conclusions of Theorem 1.32 (and of Theorem 1.34 if F_p is analytic) with subsets $(B_j \times \{\mathbf{o}(\mathbf{c}_j)\})_{j\geq J}$ such that every point $(z, \mathbf{v}) \in B_J \times \{\mathbf{o}(\mathbf{c}_J)\}$ satisfies:*

($\mathcal{E}$) *There exist $0 < t < 1$ and μ in the set of invariant probability measures $\mathcal{M}_p(\Lambda_p)$ of $F_p|\Lambda_p$, such that the limit set of the sequence of empirical measures $e_n := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{F_p^i(z,\mathbf{v})}$ contains $t \cdot \mu + (1-t) \cdot \mathcal{M}_p(\Lambda_p)$.*

The proof of this theorem will occupy the whole Section 4. In this proof we will first show that the assumptions of Theorem 1.32 are satisfied at a dense set of parameter. Finally in Section 4.4, we will show conclusion ($\mathcal{E}$) by selecting the combinatorics of sequences of words $(\mathbf{w}_j)_j$. We recall that Condition ($\mathcal{E}$) is useful to show using Theorem 0.9 that the selected dynamics have emergence of positive order in Corollaries A and B.

Theorem 1.44 being admitted for now, let us show how it implies the main theorems. To this end, we define the following geometric model which applies to surface maps families.

Definition 1.45 (C^r-Geometric model)**.** *A C^r-family $(f_p)_{p\in\mathcal{P}}$ of surface diffeomorphisms $f_p \in \operatorname{Diff}^2(M)$ displays the* geometric model *if there is a moderately dissipative unfolding $(F_p, \pi_p)_{p\in\mathcal{P}}$ of wild type $(\mathbf{A}, \mathbf{C})$ with $\operatorname{Card}\mathbf{C} = 5$ so that $(F_p)_p$ is embedded into M via a C^2-family $(H_p)_p$ of C^r-embeddings $H_p : Y^{\mathbf{e}} \times \mathbf{V} \hookrightarrow M$:*

$$f_p \circ H_p| \bigsqcup_{\mathbf{d}\in\mathbf{A}\sqcup\mathbf{C}} Y_p^{\mathbf{d}} \times \{\mathbf{o}(\mathbf{d})\} = H_p \circ F_p, \quad \forall p \in \mathcal{P}.$$

In Section 2.1 we will prove that the families of maps defined in Example 1.36 define a moderately dissipative unfolding of wild type $(\mathbf{A}, \mathbf{C})$ for any $\delta > 0$ small enough depending on an even $N \geq 2$, and so the conclusions of Theorem 1.44 when moreover $N = 6$.

In Section 2.2, we will state a general Theorem 2.10 implying that the geometric model is displayed in any non-degenerate unfolding of five homoclinic tangencies of a same dissipative saddle periodic point. This will imply Theorem C. Then we will show that this implies the two following theorems. The first one states that the geometric model appears locally densely among generalized Hénon maps:

Theorem 1.46. *For every $b \neq 0$ small, there is a regular compact set $\mathcal{P}_b \subset \mathbb{R}^5$ such that with:*

$$f_p : (x, y) \in \mathbb{R}^2 \mapsto (x^6 + \sum_{i=0}^{4} p_i \cdot x^i - y, b \cdot x) \text{ for every } p = (p_i)_{0\leq i\leq 4} \in \mathcal{P} \ ,$$

the family $(f_p)_{p\in\mathcal{P}}$ displays the C^ω-geometric model 1.45.

Moreover there is a neighborhood V_b of b such that $\bigcap_{b'\in V_b} \mathcal{P}_{b'}$ has nonempty interior.

Remark 1.47. *The same holds true for the family of maps:*

$$f_p : [z, w, t] \in \mathbb{P}^2(\mathbb{C}) \mapsto [z^6 + \sum_{i=0}^{4} p_i \cdot z^i \cdot t^{6-i} - w^6, b \cdot z^6, t^6] \ .$$

The second states that the geometric model appears openly and densely in the space of unfolding of a dissipative homoclinic tangency. We recall that a saddle periodic point displays a *homoclinic tangency* if its unstable manifold is tangent to its stable manifold. The homoclinic tangency is *dissipative* if the determinant of the differential at the periodic cycle has modulus less than 1.

Theorem 1.48. *For every $r \in [2, \infty] \cup \{\omega\}$, if $f \in \mathrm{Diff}^r(M)$ displays a dissipative, quadratic homoclinic tangency, then there exist a C^r-perturbation $\tilde{f}$ of f, a C^r-open-dense set of families $(f_p)_{p\in\mathbb{R}^5}$ satisfying that $f_0 = \tilde{f}$ such that for $n \geq 1$ and a regular compact neighborhood $\mathcal{P}$ of $0 \in \mathbb{R}^5$, the family $(f_p^n)_{p\in\mathcal{P}}$ satisfies the C^r-geometric model 1.45.*

These theorems together with Corollary 1.49 imply the first proof of the existence of a wandering Fatou component of a polynomial automorphism of $\mathbb{C}^2$ and a proof of the last Takens' problem in any regularity:

Proof of Theorems A and B. These theorem are direct consequences of respectively Theorems 1.46 and 1.48 and the following consequence of main Theorem 1.44. □

Corollary 1.49 (Main)**.** *For every $r \in [2, \infty] \cup \{\omega\}$, if a family $(f_p)_{p\in\mathcal{P}}$ satisfies the C^r-geometric model 1.45 with a family of embeddings $(H_p)_{p\in\mathcal{P}}$, then there exists a dense subset of $\mathcal{P}$ of parameters p at which the map f_p has the following properties:*

(1) There exists a wandering stable component $\mathcal{C}$. If $r = \omega$, then the holomorphic extension of f_p has a wandering Fatou component containing $\mathcal{C}$.

(2) For every $x \in \mathcal{C}$, the limit set of the orbit of x contains the horseshoe $K_p := H_p(\Lambda_p)$.

(3) For every $x \in \mathcal{C}$, the sequence $(\mathscr{e}_n^f)_n$ of empirical measures $\mathscr{e}_n^f := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f_p^i(x)}$ diverges. Furthermore, there exist $0 < t < 1$ and μ' in the set of invariant probability measures $\mathcal{M}_p(K_p)$ of $f_p|K_p$ such that the limit set of the sequence $(\mathscr{e}_n^f)_n$ contains $t \cdot \mu' + (1-t) \cdot \mathcal{M}_p(K_p)$.

Proof. Let$(F_p, \pi_p)_{p\in\mathcal{P}}$ be the moderately dissipative unfolding of a certain wild type $(\mathbf{A}, \mathbf{C})$ with $\mathrm{Card}\,\mathbf{C} = 5$ which is embedded into M via $(H_p)_p$ and such that $f_p \circ H_p = H_p \circ F_p$. Let $p \in \mathcal{P}$ be in the dense set of parameters given by Theorem 1.44. Let $(B_j \times \{\mathbf{o}(\mathbf{c}_j)\})_{j\geq J}$ and $(\mathbf{w}_j) \in (\mathbf{A}^*)^{\mathbb{N}}$ be the associated sequences of stable domains and words for the fixed parameter p satisfying both the conclusions of Theorem 1.32 (resp. Theorem 1.34 if F_p is analytic) and condition $(\mathcal{E})$. Put $D_j := H_p(B_j \times \{\mathbf{o}(\mathbf{c}_j)\})$ which is the real trace of $\tilde{D}_j := H_p(\tilde{B}_j \times \{\mathbf{o}(\mathbf{c}_j)\})$ in the analytic case. By the conclusions of Theorem 1.32, it holds for every $j \geq J$:

$$D_j \subset H_p(Y_p^{\mathbf{c}_j} \times \mathbf{o}(\mathbf{c}_j)), \quad f_p(D_j) \subset H_p(Y^{\mathbf{w}_{j+1}} \times \{\mathbf{o}(\mathbf{w}_{j+1})\}) \quad \text{and} \quad f_p^{|\mathbf{w}_{j+1}|+1}(D_j) \subset D_{j+1}$$

and respectively in the analytic case by the conclusions of Theorem 1.34, it holds for every $j \geq J$:

$$\tilde{D}_j \subset H_p(\tilde{Y}_p^{\mathbf{c}_j} \times \mathbf{o}(\mathbf{c}_j)), \quad f_p(\tilde{D}_j) \subset H_p(\tilde{Y}^{\mathbf{w}_{j+1}} \times \{\mathbf{o}(\mathbf{w}_{j+1})\}) \quad \text{and} \quad f_p^{|\mathbf{w}_{j+1}|+1}(\tilde{D}_j) \subset \tilde{D}_{j+1} \ .$$

The last conclusion of Theorem 1.32 (resp. 1.34) implies that D_j (resp. $\tilde{D}_j$) is a stable domain:

$$\lim_{k\to\infty} \operatorname{diam} f_p^k(D_j) = 0 \quad \text{and} \quad \text{resp.} \quad \lim_{k\to\infty} \operatorname{diam} f_p^k(\tilde{D}_j) = 0\ .$$

Thus D_j (resp. $\tilde{D}_j$) is included in a stable component $\mathcal{C}$ (resp. $\tilde{\mathcal{C}}$) . We now prove (1)-(2)-(3) for this fixed parameter p.

Proof of (1). Assume for the sake of contradiction that $\mathcal{C}$ is preperiodic. By taking j larger, we can assume it q periodic for some $q \geq 1$: $f_p^q(\mathcal{C}) = \mathcal{C}$. Let $D'_j \Subset D_j$ be with nonempty interior. We have $D'_j \Subset \mathcal{C}$ and so

$$A := D'_j \cup f_p^q(D'_j) \Subset \mathcal{C}$$

Since $\mathcal{C}$ is a stable component, it holds $\lim_{n\to\infty} \operatorname{diam} f_p^n(A) = 0$.

We recall that $D_p(\mathbf{A}) := \bigsqcup_{\mathbf{a}\in\mathbf{A}} Y_p^{\mathbf{a}} \times \{\mathbf{o}(\mathbf{a})\}$ and $D_p(\mathbf{C}) := \bigsqcup_{\mathbf{c}\in\mathbf{C}} Y_p^{\mathbf{c}} \times \{\mathbf{o}(\mathbf{c})\}$. We observe that for $k \geq j+1$, with $N_k := 1 + |\mathbf{w}_{j+1}| + \cdots + 1 + |\mathbf{w}_k|$, it holds that:

$$\begin{cases} f_p^l(D_j) \subset D_k \subset H_p(D_p(\mathbf{C})) & \text{if } l = N_k \\ \quad f_p^l(D_j) \subset H_p(D_p(\mathbf{A})) & \text{if } l \in \{N_k+1, \ldots, N_{k+1}-1\}. \end{cases}$$

The sets $H_p(D_p(\mathbf{A}))$ and $H_p(D_p(\mathbf{C}))$ are disjoint and compact. As $N_k \to \infty$ and $N_k - N_{k-1} \to \infty$, for k large enough, it holds:

(i) the diameter of $f_p^{N_k-q}(A)$ is smaller than the distance between $H_p(D_p(\mathbf{A}))$ and $H_p(D_p(\mathbf{C}))$.

(ii) the set $f_p^{N_k-q}(A)$ contains $f_p^{N_k-q}(f_p^q(D'_j)) = f_p^{N_k}(D'_j)$ and so intersects $H_p(D_p(\mathbf{C}))$.

(iii) the set $f_p^{N_k-q}(A)$ contains $f_p^{N_k-q}(D'_j)$ and so intersects $H_p(D_p(\mathbf{A}))$ since $N_k - q \in \{N_{k-1} + 1, \ldots, N_k - 1\}$ for k sufficiently large.

Conclusions (i) and (ii) contradict (iii). Thus $\mathcal{C}$ is a wandering stable component.

Literally the same argument (by replacing $\mathcal{C}$ by $\tilde{\mathcal{C}}$) proves that the open set $\tilde{\mathcal{C}} \subset \tilde{M}$ is wandering. We observe that $\tilde{\mathcal{C}}$ is included in a Fatou component $\mathcal{F}$. Let $\mathcal{F}' \Subset \mathcal{F}$ be a precompact open subset of $\mathcal{F}$ which intersects the interior of $\tilde{B}_j$. Then any cluster value for $(f_p^n|\mathcal{F}')_n$ is a holomorphic map constant on $\tilde{B}_j \cap \mathcal{F}'$ and so is constant. Thus any cluster value of $(\operatorname{diam} f_p^n(\mathcal{F}'))_n$ is zero and so $\mathcal{F}'$ is included in $\tilde{\mathcal{C}}$. Since $\mathcal{F}$ can be written as an union of such open sets $\mathcal{F}'$ it follows $\mathcal{F} = \tilde{\mathcal{C}}$ and so $\mathcal{F}$ is a wandering Fatou component.

Proof of (3). Let $j = J$. By (1) the stable component $\mathcal{C}$ containing D_J is wandering. Let $(z, \mathbf{v})$ be a point of $B_J \times \{\mathbf{o}(\mathbf{c}_J)\}$ and $x := H_p(z, \mathbf{v}) \in D_J \subset \mathcal{C}$. By Theorem 1.44 ($\mathcal{E}$), there exist $0 < t < 1$ and $\mu \in \mathcal{M}_p(\Lambda_p)$ such that the limit set of the sequence of empirical measures $e_n := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{F_p^i(z,\mathbf{v})}$ contains $t \cdot \mu + (1-t) \cdot \mathcal{M}_p(\Lambda_p)$. We notice that the pushforward of e_n by H_p is equal to $e_n^f := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f_p^i(x)}$ since $f_p \circ H_p = H_p \circ F_p$. Also the pushforward by H_p of $\mathcal{M}_p(\Lambda_p)$ is equal to $\mathcal{M}_p(K_p)$ since K_p is the image of Λ_p by the embedding H_p. As the pushforward operation on probability measures by an embedding is an embedding in the spaces of probability measures, the limit set of $(e_n^f)_n$ contains $t \cdot \mu' + (1-t) \cdot \mathcal{M}_p(K_p)$ with $\mu' := H_{p\,*}\mu \in \mathcal{M}_p(K_p)$. In particular $(e_n^f)_n$ diverges. Finally, for every x' in $\mathcal{C}$, the limit set of $(\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f_p^i(x')})_n$ is equal to the limit set of $(e_n^f)_{n\geq 0}$ and so contains $t \cdot \mu' + (1-t) \cdot \mathcal{M}_p(K_p)$, since $\mathcal{C}$ is a stable component.

Proof of (2). We now recall that any open set U of K_p has positive mass for a measure $\mu_U \in \mathcal{M}_p(K_p)$, and so for the measure $t \cdot \mu' + (1-t) \cdot \mu_U$. Hence by *(3)* the limit set of any point in $\mathcal{C}$ intersects U and so is dense in K_p. As the limit set is compact, it must contain K_p. □

2. Examples of families displaying the geometric model

In this Section, we give examples satisfying the geometric model (Definition 1.45).

First we show that the family $(F_p, \pi_p)_{p\in\mathcal{P}}$ of Example 1.36 is an unfolding of wild type $(\mathbf{A}, \mathbf{C})$ and we give a sketch of proof of how to deduce simply from this an example of wandering Fatou component for a real polynomial automorphism of $\mathbb{C}^2$.

The main result of this Section is Theorem 2.10 from which we will deduce Theorem 1.46 (and thus examples of wandering Fatou components for real polynomial automorphisms of $\mathbb{C}^2$) and Theorem 1.48. Namely, Theorem 2.10 states that an unfolding of wild type is embedded into some iterate of any d-parameter family which displays a non-degenerate unfolding of d-quadratic homoclinic tangencies. When $d = 5$ and up to a dissipativeness assumption, the geometric model is satisfied. To prove both Theorems 1.48 and 2.10 we will need to recall classical results about Newhouse phenomenon (Theorems 2.15 and 2.16). Let us point out that the proof of Theorem 2.10 is technical and could be omitted in a first reading.

2.1. A simple example of unfolding of wild type $(\mathbf{A}, \mathbf{C})$.

We are going to prove that for any even $N \geq 2$ and $\delta > 0$ small enough, the family $(F_p, \pi_p)_{p\in\mathcal{P}}$ of Example 1.36 is a moderately dissipative unfolding of wild type $(\mathbf{A}, \mathbf{C})$. To this end we must show that it satisfies (in particular) Property $(\mathbf{H_1})$ of Definition 1.42, which by Remark 1.40 is equivalent to say that the Cantor sets $\{a_p^{\mathbf{u}} : \mathbf{u} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}_j}\}$ and $\{b_p^{\mathbf{s}} : \mathbf{s} \in \overrightarrow{\mathbf{A}}_{\mathbf{c}_j}\}$ intersect for all $p \in \mathcal{P}$ and $1 \leq j \leq N-1$. This robust intersection will be obtained using the following well-known notion:

Definition 2.1 (Thickness of a Cantor set)**.** *Given a Cantor set $K \subset \mathbb{R}$, a* gap *of K is a connected component of $\mathbb{R} \setminus K$. Given a bounded gap G of K and x in the boundary of G, the* bridge *B of K at x is the connected component of x in the complement of the union of the gaps larger or equal than $|G|$. The thickness of K at x is $\tau(K, u) = |B|/|G|$. The* thickness *of K, denoted by $\tau(K)$ is the infimum of these $\tau(K, u)$ among all boundary points x of bounded gaps.*

The following is the celebrated Newhouse gap Lemma.

Lemma 2.2 ([New79, Lemma 4])**.** *Let $K_1, K_2 \subset \mathbb{R}$ be Cantor sets with thicknesses τ_1 and τ_2. If $\tau_1 \cdot \tau_2 > 1$, then one of the three following possibilities occurs: K_1 is contained in a component of $\mathbb{R} \setminus K_2$, K_2 is contained in a component of $\mathbb{R} \setminus K_1$ or $K_1 \cap K_2 \neq \emptyset$.*

We are now ready to prove:

Proposition 2.3. *For every even integer $N \geq 2$ and $\delta > 0$ smaller than a positive function of N, the family $(F_p, \pi_p)_{p\in\mathcal{P}}$ of Example 1.36 is an unfolding of wild type $(\mathbf{A}, \mathbf{C})$ which is moderately dissipative.*

Proof. We recall that $\mathbf{A} = \{\mathbf{a}_j : 1 \leq j \leq N\}$ and the hyperbolic transformations are

$$Y^{\mathbf{a}_j} := I_j \times I \quad \text{and} \quad F^{\mathbf{a}_j} := (x, y) \in Y^{\mathbf{a}_j} \mapsto (C_j^{-1}(x), \sqrt{\delta} \cdot C_j(y))$$

with $I_j := \left[\frac{2}{N}(j-1) - 1 + \delta^2, \frac{2}{N}j - 1 - \delta^2\right]$ and C_j the affine, orientation preserving map from I onto I_j. We notice that with $\epsilon = 0.1$, if δ is sufficiently small, then for every $\mathbf{a} \in \mathbf{A}$,

$$\max_{p\in\mathcal{P},\, z\in Y^{\mathbf{a}}} \|D_z F^{\mathbf{a}}\| \cdot |\det D_z F^{\mathbf{a}}|^{\epsilon} = (\frac{1}{N} - \delta^2)^{-1} \delta^{\epsilon/2} < 1 \,.$$

Thus $F^{\mathbf{A}}$ is moderately dissipative. Also the family of systems $(F_p, \pi_p)_{p\in\mathcal{P}}$ of type $(\mathbf{A}, \mathbf{C})$ satisfies property $(\mathbf{H}_2)$. Indeed, for $(\mathbf{u}_j, \mathbf{s}_j)_j \in \prod_{i=1}^{N-1} \overleftarrow{\mathbf{A}}_{\mathbf{c}_j} \times \overrightarrow{\mathbf{A}}_{\mathbf{c}_j}$, by Eq. (1.3) of Example 1.20, we have for every $1 \leq j \leq N-1$:

$$a^{\mathbf{u}_j} = y^{\mathbf{u}_j} + p_j \ , \quad b_p^{\mathbf{s}_j} = x^{\mathbf{s}_j} \quad \text{and} \quad \mathcal{V}(\mathbf{u}_j, \mathbf{s}_{j+1}, p) = y^{\mathbf{u}_j} - x^{\mathbf{s}_{j+1}} + p_j \ . \tag{2.1}$$

Thus the map $p \in \mathcal{P} \mapsto (\mathcal{V}(\mathbf{u}_1, \mathbf{s}_2, p), \cdots, \mathcal{V}(\mathbf{u}_{N-1}, \mathbf{s}_N, p)) \in \mathbb{R}^{N-1}$ is an affine map of linear part Id, and so a diffeomorphism.

Let us show that the regular family $(F_p, \pi_p)_{p\in\mathcal{P}}$ of systems satisfies $(\mathbf{H}_1)$. To this end, we shall study the following Cantor sets:

$$K_s := \{x^{\mathbf{s}} : \mathbf{s} \in \overrightarrow{\mathbf{A}}\} \quad \text{and} \quad K_u := \{y^{\mathbf{u}} : \mathbf{u} \in \overleftarrow{\mathbf{A}}\}$$

The Cantor set K_s is the limit set of the IFS given by the contractions $(C_j)_{j=1}^N$. The Cantor set K_u is the limit set of the IFS given by the contractions $(\sqrt{\delta} \cdot C_j)_{j=1}^N$. It suffices to show that for every $p = (p_j)_j \in \mathcal{P}$ and $1 \leq k \leq N-1$, the Cantor set $\sqrt{\delta} \cdot C_k(K_u) + p_k$ intersects K_s. To show this we compute the thicknesses of these Cantor sets and their diameters. To this end, it is worth to recall:

Fact 2.4. *The intervals $(I_j)_j$ are disjoint and display the same length $2/N - 2\delta^2$. The complement of their union in $[-1+\delta^2, 1-\delta^2]$ is made by intervals of length $2\delta^2$. The left endpoint of I_1 is $-1+\delta^2$ and the right endpoint of I_N is $1-\delta^2$. The component of 0 in $I \setminus \bigcup_{j=1}^N I_j$ is $V^{\mathbf{c}} := (-\delta^2, \delta^2)$.*

We are ready to show:

Lemma 2.5. *The Cantor set K_s is at distance $\frac{N\delta^2}{N-1+N\delta^2} \sim \frac{\delta^2 N}{N-1}$ from the boundary of I and its thickness is $\tau(K_s) := \frac{N-1+N\delta^2}{\delta^2 N^2} - 1 \sim \frac{N-1}{\delta^2 N^2}$ when $\delta \to 0$.*

Proof. Indeed, the Cantor set K_s is the limit set of the IFS given by the contractions $(C_j)_{j=1}^N$ and each C_j is affine, so we only need to compute the ratio $|B|/|G|$ for the $(N-1)$ gaps between $C_j(I) = I_j$ and $C_{j+1}(I) = I_{j+1}$ for $1 \leq j \leq N-1$. By Fact 2.4, each of these gaps has length:

$$|G| := 2\delta^2 \sum_{j\geq 0} (\frac{1}{N} - \delta^2)^j = \frac{2\delta^2}{1 - \frac{1}{N} + \delta^2} = \frac{2\delta^2 N}{N-1+N\delta^2} = 2d(K_s, I^c) \ . \tag{2.2}$$

The bridges between these gaps have length $|B| := \frac{2}{N} - |G|$. Then the proposition follows from the quotient of these two equalities:

$$\tau(K_s) := \frac{2}{N|G|} - 1 = \frac{N-1+N\delta^2}{\delta^2 N^2} - 1 \ .$$

□

Remark 2.6. *By Eq. (2.2), for every j, the lengths of gaps of $K_s^j := K_s \cap I_j$ are $\leq \frac{2\delta^2 N}{N-1} \cdot (\frac{1}{N} - \delta^2) \sim \frac{2\delta^2}{N-1}$. Also $d(I_j^c, K_s^j) \sim (\frac{1}{N} - \delta^2) \cdot \frac{\delta^2 N}{N-1} \sim \frac{\delta^2}{N-1}$.*

Lemma 2.7. *The Cantor set K_u has diameter equivalent to $2\sqrt{\delta}\frac{N-1}{N}$ and its thickness is equivalent to $\sqrt{\delta}\frac{N-1}{N}$ when $\delta \to 0$.*

Proof. The Cantor set K_u is the limit set of the IFS given by the contractions $(\sqrt{\delta}\cdot C_j)_{j=1}^N$ and each $\sqrt{\delta}\cdot C_j$ is affine. Let $[-a,a]$ be the convex hull of K_u. We observe that a is the unique fixed point of $\sqrt{\delta}\cdot C_N$, whereas $-a$ is the fixed point of $\sqrt{\delta}\cdot C_1 = x \mapsto -\sqrt{\delta}\cdot C_N(-x)$. Since by definition C_N is the orientation preserving affine map sending I to I_N, this yields that $\sqrt{\delta}\cdot C_N(x) = \sqrt{\delta}((\frac{1}{N}-\delta^2)(x-1)+1-\delta^2)$ for $x\in I$. Thus the unique fixed point of $\sqrt{\delta}\cdot C_N$ is:

$$a := \sqrt{\delta}(1-\frac{1}{N})\cdot(1-\frac{\sqrt{\delta}}{N}+\delta^{5/2})^{-1} \sim \sqrt{\delta}\frac{N-1}{N} \quad \text{when } \delta\to 0\ .$$

This implies the estimate on the diameter of K_u. To compute the thickness, we only need to compute the ratio $|B|/|G|$ for the $(N-1)$ gaps between $\sqrt{\delta}\cdot C_j([-a,a])$ and $\sqrt{\delta}\cdot C_{j+1}([-a,a])$ for $1\le j\le N-1$. We denote $\tilde{I}_j = \sqrt{\delta}\cdot C_j([-a,a])\subset[-a,a]$. We observe that each $\tilde{I}_j$ is of length $b := 2a\cdot\sqrt{\delta}(\frac{1}{N}-\delta^2)$. There are $N-1$ gaps between the segments $\tilde{I}_j$ which are equal. Thus their size is $\frac{2a-Nb}{N-1}$. Thus the thickness is:

$$b\cdot(\frac{2a-Nb}{N-1})^{-1} = (N-1)\frac{2a\cdot\sqrt{\delta}(\frac{1}{N}-\delta^2)}{2a-N\cdot 2a\cdot\sqrt{\delta}(\frac{1}{N}-\delta^2)} \sim \sqrt{\delta}\frac{N-1}{N}\ .$$

□

For $1\le j\le N$, let us denote:

$$K_s^j := K_s\cap I_j = C_j(K_s) \quad \text{and} \quad K_u^j := \sqrt{\delta}\cdot C_j(K_u)\ . \tag{2.3}$$

The sets K_s^j and K_u^j are Cantor sets of the same thickness as respectively K_s and K_u. Hence by Lemmas 2.5 and 2.7, the product of the thicknesses of K_s^j and K_u^j is greater than 1. By Remark 2.6, all the gaps of K_s^j are smaller than $\sim\frac{2\delta^2}{N-1}$ and $\sim\frac{\delta^2}{N-1}$ distant to the boundary of I_j. By Lemma 2.7, this is small compared to the diameter of K_u^j which is equivalent to $\sim 2\frac{\sqrt{\delta}(N-1)}{N}\cdot\sqrt{\delta}(\frac{1}{N}-\delta^2)$. As $K_u^j+p_j$ is included in I_{j+1}, by the gap Lemma 2.2, it must intersect K_s^{j+1} for every $p\in\mathcal{P}$. This proves Property $(\mathbf{H}_1)$. □

The following remark together with Corollary 1.49 could have been used to prove the existence of a polynomial automorphism of $\mathbb{C}^2$ with a wandering Fatou component using systems of type $(\mathbf{A},\mathbf{C})$ with simple combinatorics (full shift). We just give a sketch of proof of the remark since in the next subsection we will prove the existence of a stronger example of family of automorphisms satisfying the assumption of Corollary 1.49, where we manage to bound the degree by 6. This other example will enable to prove Theorem A. Nevertheless, this will require the full generality of $(\mathbf{A},\mathbf{C})$ systems.

Remark 2.8. *The moderately dissipative unfolding $(F_p)_p$ of wild type $(\mathbf{A},\mathbf{C})$ given by Example 1.36 and Proposition 2.3 with δ small and $N=6$, can be perturbed to be left invariant by a family of polynomial automorphisms. More precisely, from Example 1.36 and Proposition 2.3 one can deduces the existence of a real family of real polynomial automorphisms (of unknown degree) which satisfies the C^ω-geometric model 1.45 and so the assumption of Corollary 1.49.*

Sketch of proof of Remark 2.8. The map F_p has constant Jacobian determinant for every p. Thus by Dacarogna-Moser theorem [DM90], it is possible to extend the family $(F_p)_p$ to a family $(f_p)_p$ of diffeomorphisms f_p of $\mathbb{R}^2$ such that $(f_p)_p$ satisfies the geometric model when $N=6$. Thus the map

f_p has a wandering stable component at a dense set of parameter p. Moreover, it is possible to show the existence of a neighborhood $\mathcal{U}$ of $(f_p)_{p\in\mathcal{P}}$ formed by families displaying the geometric model (the proof is not complicated but rather boring since many items must be checked) via an embedding H_p equal to the canonical inclusion: $Y^{\mathbf{e}} \times \{\diamond\} \hookrightarrow \mathbb{R}^2$. Then by noting that $(\delta^{-1/4} \cdot f_p)_{p\in\mathcal{P}}$ has constant Jacobian equal to 1, by Turaev's Theorem [Tur02, Theorem 2 and Remark 1], this family can be approximated by a smooth family of compositions of generalized Hénon-maps and thus by a smooth family of compositions of polynomial generalized Hénon-maps $(Q_p)_{p\in\mathcal{P}}$. These are polynomial automorphisms of $\mathbb{R}^2[X,Y]$. Remark that $\tilde{f}_p := \delta^{1/4}\cdot Q_p$ is a polynomial automorphism whose restriction to $[-1,1]^2$ is C^2-close to f_p. Moreover $(\tilde{f}_p|[-1,1]^2)_p$ is C^2-close to $(f_p|[-1,1]^2)_p$ and so it displays the geometric model via a smooth family of analytic embeddings. □

2.2. **Natural examples satisfying the geometric model.** In this subsection, we are going to state Theorem 2.10 from which we will deduce Theorems 1.46 and 1.48. Theorem 2.10 states that an unfolding of a wild type is embedded into some iterate of any d-parameter family which displays a non-degenerate unfolding of d-quadratic homoclinic tangencies:

Definition 2.9 (Non-degenerate unfolding)**.** *Let $d \geq 1$, $\hat{\mathcal{P}}$ an open set of $\mathbb{R}^d$, and $(f_p)_{p\in\hat{\mathcal{P}}}$ be a C^2-family of surface diffeomorphisms leaving invariant a continuation*[6] *$(K_p)_{p\in\hat{\mathcal{P}}}$ of hyperbolic basic sets. Given $p_0 \in \hat{\mathcal{P}}$ and d points of quadratic tangencies between the stable and unstable manifolds of K_{p_0}, their* unfolding along $(f_p)_p$ is non-degenerated *if the matrix with the following coefficients $(a_{i,j})_{(i,j)\in\{1,\dots,d\}^2}$ is invertible: $a_{i,j}$ is the derivative following the jth -coordinate of $\mathbb{R}^d = T_{p_0}\mathcal{P}$ of the relative position of the local stable and unstable manifolds associated to the ith-quadratic tangency point.*

We recall that a periodic point P of period n is non-conservative if $|\det D_P f^n| \neq 1$. Here is the general result enabling the aforementioned applications:

Theorem 2.10. *Let $r \in [2,\infty] \cup \{\omega\}$, let $d \geq 1$, let $\hat{\mathcal{P}}$ be a nonempty open subset of $\mathbb{R}^d$ and let $(f_p)_{p\in\hat{\mathcal{P}}}$ be a C^r-family of surface diffeomorphisms. Given $p_0 \in \hat{\mathcal{P}}$, assume that f_{p_0} has a non-conservative*[7] *periodic saddle point P displaying d different quadratic homoclinic tangencies.*

If these quadratic tangencies unfold non-degenerately with $(f_p)_p$, then there exist $k \geq 1$, a regular compact subset $\mathcal{P} \subset \hat{\mathcal{P}}$ arbitrarily close to p_0, an unfolding $(F_p, \pi_p)_{p\in\mathcal{P}}$ of a certain wild type $(\mathbf{A}, \mathbf{C})$ with $\operatorname{Card}\mathbf{C} = d$*, $(F_p)_p$ being of class C^r and a C^r-family of embeddings $(H_p)_p$ such that*

$$f_p^k \circ H_p = H_p \circ F_p \quad \forall p \in \mathcal{P}.$$

Moreover $F_p^{\mathbf{A}}$ is moderately dissipative for every $p \in \mathcal{P}$ if P is dissipative.

Furthermore, for every C^r-perturbation $(\tilde{f}_p)_{p\in\hat{\mathcal{P}}}$ of $(f_p)_{p\in\hat{\mathcal{P}}}$ the same conclusion holds true for a same regular compact subset $\mathcal{P} \subset \hat{\mathcal{P}}$.

Remark 2.11. *Hence if* $\operatorname{Card}\mathbf{C} = 5$*, the family $(f_p)_{p\in\mathcal{P}}$ satisfies the C^r-geometric model 1.45.*

6. See Theorem C.1 Page 73.
7. This assumption is actually not necessary using Duarte's theorem [Dua08], but sufficient for all our applications needing the moderate dissipativeness assumption.

The proof of this theorem is given in the next subsection 2.3. Let us deduce now rather quickly Theorems 1.46 and 1.48 and Theorem C.

Proof of Theorem C. This is an immediate consequence of Corollary 1.49 and Remark 2.11. (Remind that Corollary 1.49 follows from Theorem 1.44 whose proof will be given in Section 4). □

Proof of Theorem 1.46 and Remark 1.47. Let us consider the Tchebychev polynomial of degree 6:

$$P_0(X) = 32 \cdot X^6 - 48 \cdot X^4 + 18 \cdot X^2 - 1$$

The critical points and values of this polynomial are respectively:

$$(\zeta_{i\,0})_{1\le i\le 5} = (-\sqrt{3}/2, -1/2, 0, 1/2, \sqrt{3}/2) \quad \text{and} \quad (a_{i\,0})_{1\le i\le 5} = (-1, 1, -1, 1, -1).$$

They are all quadratic. All critical values are sent to $\beta_0 = 1$ which is a repelling fixed point of P_0. We notice that $[-\beta_0, \beta_0]$ is the complement of the basin of ∞. For $p = (p_0, \dots, p_4) \in \mathbb{R}^5$, we consider:

$$P_p(X) = P_0(X) + \sum_{j=0}^{4} p_j \cdot X^j \ .$$

Let β_p, $(\zeta_{i\,p})_{1\le i\le 5}$ and $(a_{i\,p})_{1\le i\le 5}$ be the respective continuations of the fixed point β_0, of the critical points and of the critical values for P_p. We have the following:

Lemma 2.12. *The unfolding of the images $P_p(a_{i\,p})$ of the critical values $a_{i\,p}$ w.r.t. the fixed point β_p is non-degenerated:*

$$\det\,[\partial_{p_j}(P_p(a_{i\,p})) - \partial_{p_j}\beta_p]_{1\le i\le 5, 0\le j\le 4} \neq 0 \qquad \textit{at } p = 0\,.$$

Proof. One way to prove this lemma is to use Epstein's transversality (see [BE09], Theorem 1.1 and the subsequent Remark). Let us give a direct proof. We have $DP_0(\beta_0) = 36$ and $\partial_p P_p(X) = \sum_{j=0}^{4} X^j \cdot dp_j$. Thus we have $\big(DP_0(\beta_0)\cdot\partial_p\beta_p + \partial_p P_p(\beta_0)\big)_{|p=0} = (\partial_p\beta_p)_{|p=0}$ and so:

$$(\partial_p\beta_p)_{|p=0} = -\frac{1}{35}\sum_{j=0}^{4} dp_j \ .$$

Also $(\partial_p a_{i\,p})_{|p=0} = (DP_0(\zeta_{i\,0})\cdot\partial_p\zeta_{i\,p} + \partial_p P_p(\zeta_{i\,0}))_{|p=0} = (\partial_p P_p(\zeta_{i\,0}))_{|p=0}$ implies:

$$(\partial_p a_{i\,p})_{|p=0} = \sum_{j=0}^{4}(\zeta_{i\,0})^j \cdot dp_j \quad \text{and} \quad (\partial_p P_p(a_{i\,p}))_{|p=0} = DP_0(a_{i\,0})\cdot(\partial_p a_{i\,p})_{|p=0} + (\partial_p P_p(a_{i\,0}))_{|p=0}$$

and so:

$$(\partial_p P_p(a_{i\,p}))_{|p=0} = (-1)^i \cdot 36 \cdot \sum_{j=0}^{4}(\zeta_{i\,0})^j \cdot dp_j + \sum_{j=0}^{4}(-1)^{i\cdot j}\cdot dp_j\,.$$

Then to show the lemma, it suffices to see the invertibility of the 5×5-matrix whose (i,j) entry is $[\partial_{p_j}(P_p(a_{i\,p})) - \partial_{p_j}\beta_p] = [(-1)^i\cdot 36(\zeta_{i\,0})^j + (-1)^{i\cdot j} + 1/35]$. □

For $b \ge 0$ small and $p \in \mathbb{R}^5$ small, we consider the Hénon map:

$$f_{p\,b} : (x, y) \mapsto (P_p(x) - y, b\cdot x)$$

For $b = 0$, the family $(f_{p\,0})_{p\in\mathbb{R}^5}$ is semi-conjugate to $(P_p)_{p\in\mathbb{R}^5}$ via the maps $(x, y) \mapsto P_p(x) - y$, while for $b > 0$, the maps $f_{p\,b}$ are polynomial automorphisms. For $b = 0$, the point $\beta_{0\,0} := (\beta_0, 0)$ is

fixed and hyperbolic for $f_{0\,0}$. Thus it persists as a hyperbolic fixed point $\beta_{p\,b}$ of $f_{p\,b}$ for $b > 0$. We notice that $W^u_{loc}(\beta_{0\,0}; f_{0\,0}) := (-0.9, 1.1) \times \{0\}$ is a local unstable manifold of $\beta_{0\,0}$: there exists an arbitrarily small neighborhood of $\beta_{0\,0}$ in $\mathbb{R} \times \{0\}$ which is sent diffeomorphically to $W^u_{loc}(\beta_{0\,0}; f_{0\,0})$ by an iterate of $f_{0\,0}$. For p and b small, $W^u_{loc}(\beta_{0\,0}; f_{0\,0})$ persists as a curve $W^u_{loc}(\beta_{p\,b}; f_{p\,b})$ C^2-close to $W^u_{loc}(\beta_{0\,0}; f_{0\,0})$. On the other hand, the set

$$W^s_{loc}(\beta_{0\,0}; f_{0\,0}) := \{(x, y) \in [-2, 2]^2 : P_0(x) - y = \beta_0\}$$

is a local stable manifold of $f_{0\,0}$. For p and b small, it persists as a curve $W^s_{loc}(\beta_{p\,b}; f_{p\,b})$ C^2-close to $W^s_{loc}(\beta_{0\,0}; f_{0\,0})$. As the critical points $(\zeta_{i\,0}, 0)_{1\le i\le 5}$ belong to $W^u_{loc}(\beta_{0\,0}; f_{0\,0})$ and are sent by $f^2_{0\,0}$ to quadratic tangency points between $f^2_{0\,0}(W^u_{loc}(\beta_{0\,0}; f_{0\,0})$ and $W^s_{loc}(\beta_{0\,0}; f_{0\,0})$ which unfold non-degerately by Lemma 2.12, for b small there is a proper compact subset $\hat{\mathcal{P}}$ close to 0 and $p_b \in \hat{\mathcal{P}}$ close to 0 such that $f^2_{p_b\,b}(W^u_{loc}(\beta_{p_b\,b}; f_{p_b\,b})$ and $W^s_{loc}(\beta_{p_b\,b}; f_{p_b\,b})$ have five points of quadratic tangency $(f^2_{p_b\,b}(\tilde{\zeta}_{i\,b}))_{1\le i\le 5}$ which unfold non-degenerately with $(f_{p\,b})_{p\in\hat{\mathcal{P}}}$. Moreover $(\tilde{\zeta}_{i\,b})_{1\le i\le 5}$ are close to $((\zeta_{i\,0}, 0))_{1\le i\le 5}$. Hence we can apply Remark 2.11 of Theorem 2.10 with $P = \beta_{p\,b}$ to deduce Theorem 1.46. The same argument proves Remark 1.47. □

We recall that that a point x of a hyperbolic compact set K displays a homoclinic tangency if its stable manifold is tangent to its unstable manifold. Theorem 2.10 enables also to deduce rather quickly Theorem 1.48 on the Newhouse domain:

Definition 2.13. *Let $r \in [2, \infty] \cup \{\omega\}$. A basic set K for a surface diffeomorphism f is C^r-*wild *if for every C^r-perturbation $\tilde{f}$ of f, there is a point in the hyperbolic continuation of K for $\tilde{f}$ displaying a quadratic homoclinic tangency. The* dissipative C^r-Newhouse domain *$\mathcal{N}^r$ is the open set of surface diffeomorphisms leaving invariant a hyperbolic basic set displaying a C^r-robust quadratic homoclinic tangency and an area contracting periodic point*[8].

The existence of wild hyperbolic basic sets was proved by Newhouse [New74] by extending the concept of thickness to hyperbolic basic sets:

Definition 2.14 (Stable and unstable thicknesses)**.** *The* stable thickness *$\tau_s(K)$ of a horseshoe K for a dynamics f is the infimum of the thicknesses of $K \cap W^u_{loc}(z)$ among $z \in K$. The* unstable thickness *$\tau_u(K)$ of K is is the infimum of the thicknesses of $K \cap W^s_{loc}(z)$ among $z \in K$.*

A point $z \in K$ does not bound an unstable gap *if z does not bound a gap of the Cantor set $W^s_{\rm loc}(z) \cap K$ of $W^s_{\rm loc}(z)$. A point $z \in K$* does not bound a stable gap *if z does not bound a gap of the Cantor set $W^u_{\rm loc}(z) \cap K$ of $W^u_{\rm loc}(z)$.*

Here is Newhouse's theorem implying the existence of wild basic sets:

Theorem 2.15 (Newhouse [New79])**.** *Let f be a surface C^2-dynamics having a hyperbolic basic set Λ containing two basic sets Λ_1 and Λ_2 satisfying the following properties:*

(1) $\tau_u(\Lambda_1) \cdot \tau_s(\Lambda_2) > 1$.

(2) There is a point $P_1 \in \Lambda_1$ which does not bound an unstable gap of Λ_1 and a point $P_2 \in \Lambda_2$ which does not bound a stable gap of Λ_2, such that a local unstable manifold $W^u_{loc}(P_1)$ has a quadratic tangency with a local stable manifold $W^s_{loc}(P_2)$ at a point T,

8. a q-periodic point P satisfying $|\det Df^q(P)| < 1$.

then for any C^2-perturbation $\tilde{f}$ of f, the continuation of Λ has local unstable and stable manifolds $W^u_{loc}(\tilde{P}_1)$ and $W^s_{loc}(\tilde{P}_2)$ close to respectively $W^u_{loc}(P_1)$ and $W^s_{loc}(P_2)$ and displaying a quadratic tangency at a point $\tilde{T}$ close to T.

Proof. Let us briefly recall Newhouse's argument and see how we obtain that $\tilde{T}$ is close to T (this is not explicitly stated in [New79] but it is an easy consequence of his construction). Indeed, there exists a foliation $\mathcal{F}$ whose tangent space is of class C^1 and which extends the stable lamination of Λ (see Theorem C.7) on a neighborhood L of Λ. Up to looking at an iterate, we can assume $T \in L$. We use a local unstable manifold W which is transverse to this foliation and use it to define the holonomy $\hbar o \ell$ from a neighborhood of T in L into W along $\mathcal{F}$. The image of the stable lamination of Λ_2 is a Cantor set K_2 with thickness at least $\tau_s(\Lambda_2)$. Nearby T, the tangency points between the leaves of $\mathcal{F}$ and the unstable lamination of Λ_1 is sent by $\hbar o \ell$ onto a Cantor set K_1 of thickness locally at least close to $\tau_u(\Lambda_1)$. By assumption K_1 intersects K_2 at $\hbar o \ell(T)$ which does not bound a gap of K_1 nor of K_2. Thus any perturbation of K_1 and K_2 cannot be included in the gap of the one other. When we perturb f the tangent space to the foliation $\mathcal{F}$ persists to one which is C^1-close. Thus the compact sets K_1 and K_2 persist. By [New79, Proposition 6] their thicknesses vary also continuously and so by the Gap Lemma 2.2 we get that $\tilde{T}$ is close to T. □

Newhouse showed also that wild hyperbolic sets appear nearby homoclinic tangencies:

Theorem 2.16 (Newhouse [New79, Lemma 7 and Theorem 3])**.** *Let $(f_t)_{t\in(-1,1)}$ be a C^r-family of surface diffeomorphisms such that f_0 has a homoclinic quadratic tangency associated to a n_0-periodic point P_0, which unfolds non-degenerately and such that $|\det Df_0^{n_0}(P_0)| < 1$. Then arbitrarily close to 0, there exists a nonempty open interval I and there is a continuation $(K_t)_{t\in I}$ of hyperbolic basic sets such that for every $t \in I$, the basic set K_t has a homoclinic tangency which is non-degenerately unfolded and K_t contains the continuation P_t of P_0.*

To show this, Newhouse constructed a first horseshoe $\Lambda_1(t) \ni P_t$ with unstable thickness $\tau_u > 0$ (see [New79, Proposition 6]) which persists for t in an open interval $\hat{I}$ nearby 0. Then for any $\tau > 0$ he found for a certain $t \in \hat{I}$ a second horseshoe $\Lambda_2(t)$ such that:

— the stable thickness of $\Lambda_2(t)$ is $> \tau$ (see [New79, ℓ 16 Page 124]).

— A local unstable manifold of $\Lambda_1(t)$ has a quadratic homoclinic tangency with a local stable manifold of $\Lambda_2(t)$ which is non-degenerately unfolded by $(f_t)_t$ (see [New79, Lemma 7]).

— the basic set $\Lambda_1(t)$ and $\Lambda_2(t)$ are included in a basic set K_t (see [New79, Lemmas 7-8]).

Then Theorem 2.16 is a consequence of this construction with $\tau > 1/\tau_u$ and of Theorem 2.15. Taking τ large enough so that both $\tau > \max(\tau_u, (\tau_u^2 + 3\tau_u + 1)/\tau_u^2)$ and $\tau_u > (2\tau + 1)^2/\tau^3$ hold true, the main theorem of [Kra92, HKY93] implies furthermore:

Corollary 2.17. *Under the assumptions of Theorem 2.16, arbitrarily close to 0, there exists a nonempty open interval I and there is a continuation of a hyperbolic basic set $(K_t)_{t\in I}$ containing two basic sets $\Lambda_1(t) \ni P_t$ and $\Lambda_2(t)$ such that*

(i) $\tau_u(\Lambda_1(t)) \cdot \tau_s(\Lambda_2(t)) > 1$, moreover K_t is a Cantor set,

(ii) there is a Cantor set of quadratic homoclinic tangencies between the leaves of an unstable lamination $W^u(\Lambda_1(t))$ and a stable lamination $W^s(\Lambda_2(t))$ which unfold non-degenerately.

Remark 2.18. *Using modern techniques (see* [GST08, Lemma 1], [Ber18, Example 2.3]*, and historically* [TLY86, Eqs. (3.2) and (3.3)]*), the set $\Lambda_2(t)$ constructed by Newhouse is given by a renormalization of a horseshoe of a perturbation of $(x,y) \in [-4,4]^2 \mapsto (x^2-2,0)$. Thus we can assume that f_t is asymptotically moderately dissipative at $\Lambda_2(t)$:*

$$\limsup_{n\to\infty} \|\det Df^n|\Lambda_2(t)\|^{1/\epsilon} \cdot \|Df^n|\Lambda_2(t)\| < 1 \ .$$

We are now ready to show that the geometric model appears typically in the Newhouse domain:

Proof of Theorem 1.48. Let f be a C^r-surface diffeomorphism having a dissipative saddle periodic point P displaying a quadratic homoclinic tangency at a point T. We first apply Proposition 2.19 given below with $d=5$ to perturb f into $\tilde{f}$ so that the map $\tilde{f}$ has 5 different quadratic tangencies between the stable and unstable manifolds of a dissipative periodic saddle point z, which unfold non-degenerately with $(f_p)_{p\in\mathbb{R}^5}$ for any $(f_p)_{p\in\mathbb{R}^5}$ in a C^r-open and dense set of C^r-families containing $\tilde{f}$: $f_0=\tilde{f}$. To conclude, we apply Remark 2.11 of Theorem 2.10 to this open and dense set of 5-dimensional unfoldings of $\tilde{f}$. This gives a regular compact set $\mathcal{P}$ arbitrarily close to 0 such that $(f_p)_{p\in\mathcal{P}}$ satisfies the geometric model. Finally we reparametrize the family so that 0 belongs to the interior of $\mathcal{P}$. □

Proposition 2.19. *Let f be a C^r-surface diffeomorphism having a dissipative saddle periodic point P displaying a quadratic homoclinic tangency at a point T. Then for any $d>1$, there exists $\tilde{f}$ C^r-close to f such that $\tilde{f}$ has d different quadratic homoclinic tangencies associated to some dissipative periodic saddle point z. These tangencies unfold non-degenerately with $(f_p)_{p\in\mathbb{R}^d}$ for any $(f_p)_{p\in\mathbb{R}^d}$ in a C^r-open and dense set of C^r-families containing $\tilde{f}$: $f_0=\tilde{f}$.*

Proof of Proposition 2.19. We begin by a lemma which allows to unfold the initial tangency:

Lemma 2.20. *There exists a C^r-flow $(\phi^t)_t$ such that the quadratic homoclinic tangency of P is non-degenerately unfolded by $(\phi^t\circ f)_t$.*

Proof. For the sake of simplicity we assume that P is a fixed point; the case where P is $n\geq 2$ periodic is left to the reader. First we choose a neighborhood N of T sufficiently small to intersect $W^s_{loc}(P,f)\cup W^u_{loc}(P,f)$ only at the union of two small neighborhoods of T in $W^u_{loc}(P,f)$ and $W^s_{loc}(P,f)$. Now we consider a C^∞-vector field χ supported by N such that $\chi(T)$ is not in the tangent space of $W^u_{loc}(P,f)$ (or equivalently of $W^s_{loc}(P,f)$) at T . Let $(\breve{\phi}^t)_t$ be the flow of χ. We notice that the family $(\breve{\phi}^t\circ f)_t$ unfolds non-degenerately the homoclinic tangency at T. This gives the fact when $r\neq\omega$. Otherwise, let $\tilde{\chi}$ be an analytic vector field which is C^∞-close to χ. Let $(\phi^t)_t$ be its flow. Then the family $(\phi^t\circ f)_t$ unfolds non-degenerately the homoclinic tangency at T. □

Now we use Corollary 2.17 which asserts the existence of $t_0\in\mathbb{R}$ small such that $\tilde{f}:=\phi^{t_0}\circ f$ has a hyperbolic basic set K_0 containing two basic sets Λ_1 and Λ_2 such that

(i) $\tau_u(\Lambda_1)\cdot\tau_s(\Lambda_2)>1$,

(ii) there is a Cantor set of quadratic tangencies between the leaves of $W^u(\Lambda_1)$ and $W^s(\Lambda_2)$

Note that the map $\tilde{f}$ is C^r-close to f since t_0 is small. As a Cantor set is not countable, there exist d points $(P_i)_{1\leq i\leq d}$ with local unstable and stable manifolds $W^u(P_i,\tilde{f})$ and $W^s(P_i,\tilde{f})$ displaying a quadratic tangency at a point T_i, such that the orbits of T_i and T_j are disjoint when $j\neq i$.

Fact 2.21. *The set of C^r-families $(f_p)_{p\in\mathbb{R}^d}$ such that $f_0 = \tilde{f}$ and for which the quadratic tangencies at $(T_i)_{1\le i\le d}$ unfold non-degenerately is C^r-open and dense.*

Proof. Clearly the set of non-degenerate unfoldings is C^r-open. Let us show that it is C^r-dense. Let $(f_p)_{p\in\mathbb{R}^d}$ be a C^r-family containing $\tilde{f}$. Let us show that it can be C^r-approximated by one for which the d homoclinic tangencies unfold non-degenerately.

A proof like the one of Lemma 2.20 shows that there exists d C^r-flows $(\phi_i^t)_{1\le i\le d}$ such that with:

$$t = (t_j)_j \in \mathbb{R}^d \mapsto \phi^t := \phi_d^{t_d} \circ \cdots \circ \phi_1^{t_1}$$

the family $(\phi^t \circ \tilde{f})_{p\in\mathbb{R}^d}$ unfolds non-degenerately the d homoclinic tangencies at $(T_i)_{1\le i\le d}$. This proof assumes first that $1 \le r \le \infty$, and takes the support of each flow ϕ_j disjoint and supported by a small neighborhood of T_j. For the case $r = \omega$, we approximate each flow by an analytic one.

Recall that given a matrix $A \in \mathscr{M}_d(\mathbb{R})$ and a matrix $B \in \mathscr{G}\ell_d(\mathbb{R})$ there is an open and dense set of $\eta \in \mathbb{R}$ such that $A + \eta B$ is invertible. Thus for an open and dense set of small $\eta \in \mathbb{R}$, the C^r-perturbation $(\phi^{\eta\cdot p} \circ f_p)_{p\in\mathbb{R}^d}$ of $(f_p)_{p\in\mathbb{R}^d}$ unfolds non-degenerately the homoclinic tangencies at $(T_i)_{1\le i\le d}$.

□

Using the continuity of the parametrized stable and unstable laminations Theorem C.5 Page 74, and the density of stable and unstable manifolds of a given periodic point $z \in K_0$ in them, up to a small parameter translation, we can assume that $P_i = z$ for every i. Again by Theorem C.5, the unfolding is still non degenerate. □

2.3. **Proof of Theorem 2.10.** Let P be a non-conservative periodic point of f_{p_0} such that $W^u(P; f_{p_0})$ and $W^s(P; f_{p_0})$ have d quadratic tangencies at points $T_1, \dots, T_d$, which unfold non-degenerately with $(f_p)_p$. Up to inversing the dynamics, we can assume that P is dissipative.

The aim of this subsection is to show that there exist $k \ge 1$, a regular compact subset $\mathcal{P} \subset \hat{\mathcal{P}}$ arbitrarily close to p_0, a moderately dissipative unfolding $(F_p, \pi_p)_{p\in\mathcal{P}}$ of a certain wild type $(\mathbf{A}, \mathbf{C})$ with $\operatorname{Card} \mathbf{C} = d$, with $(F_p)_p$ being of class C^r, and a C^r-family of embeddings $(H_p)_p$ such that:

$$f_p^k \circ H_p = H_p \circ F_p \ .$$

In order to do so, we first construct a wild basic set K with d homoclinic tangencies which unfold non-degenerately, then we construct a family of hyperbolic maps $(F_p^{\mathbf{A}})_p$ and finally we will construct a family of folding maps $(F_p^{\mathbf{C}})_p$.

Step 1: Construction of the wild basic set K. The following uses Newhouse's Corollary 2.17.

Lemma 2.22. *There is $p_1 \in \hat{\mathcal{P}}$ close to p_0 such that:*

(1) The map f_{p_1} has a hyperbolic basic set K containing the continuation of P and two basic sets Λ_1 and Λ_2 satisfying $\tau_u(\Lambda_1) \cdot \tau_s(\Lambda_2) > 1$. Moreover K is a Cantor set.

(2) There are d points of quadratic tangencies between local unstable and stable manifolds of points of Λ_1 and Λ_2 which do not bound unstable and stable gaps and which unfold non-degenerately with $(f_p)_p$.

(3) f_{p_1} is asymptotically moderately dissiptative at K:

$$\limsup_{n\to\infty} \|\det Df^n_{p_1}|K\|^{1/\epsilon} \cdot \|Df^n_{p_1}|K\| < 1 \ .$$

Proof. Up to a coordinate change of the parameter set $\hat{\mathcal{P}}$, we can assume that $p_0 = 0$ and that along the one-parameter family $(f_t)_{t\in J}$, where $J := \mathbb{R} \times \{0\} \cap \hat{\mathcal{P}}$, the first homoclinic tangency of P unfolds non degenerately while the $d-1$-other homoclinic tangencies remain tangent. We notice that with $[a_{i,j}]_{1\le i,j\le d}$ the matrix of the unfolding (see Definition 2.9), we have $a_{1,1} \neq 0$ and $a_{i,1} = 0$ for every $2 \le i \le d$. As the matrix $[a_{i,j}]_{1\le i,j\le d}$ is invertible, it holds that $[a_{i,j}]_{2\le i,j\le d}$ is invertible. Now by Corollary 2.17, there exists $t_0 \in J$ small and a hyperbolic basic set K containing two basic sets Λ_1 and Λ_2 such that:

(1) the set Λ_1 contains the hyperbolic continuation of P,

(2) $\tau_u(\Lambda_1) \cdot \tau_s(\Lambda_2) > 1$.

(3) the local unstable and stable manifolds of Λ_1 and Λ_2 have uncountably many points T'_i of quadratic tangency which unfold non-degenerately with $(f_t)_{t\in J}$.

As there are only countably many gaps in a Cantor set, we can assume that T'_1 satisfies the no-ending gap condition (2) of Theorem 2.15.

At $(t_0, 0) \in \hat{\mathcal{P}}$, the matrix of the unfolding of $(T'_1, T_2, T_3, \dots, T_d)$ is of the form $[a'_{i,j}]_{1\le i,j\le d}$ with $[a'_{i,j}]_{2\le i,j\le d} \approx [a_{i,j}]_{2\le i,j\le d}$ and $a'_{i,1} \neq 0$ iff $i = 1$. Thus $[a'_{i,j}]_{1\le i,j\le d}$ is invertible and the unfolding of these d-homoclinic tangencies is non-degenerated. By density of the local unstable and stable manifolds of points in Λ_1 and Λ_2 which do not bound gaps, there is a parameter $p_1 \in \hat{\mathcal{P}}$ nearby $(t_0, 0)$ which displays d homoclinic tangencies at points $(T'_1, T'_2, T'_3, \dots, T'_d)$ between local unstable and stable manifolds of points of Λ_1 and Λ_2 which satisfy the no-ending gap condition (2) of Theorem 2.15. Moreover by continuation, their unfolding is non-degenerate. By [New79, Proposition 6] the unstable and stable thicknesses of these basic sets vary continuously. Thus at the parameter p_1, we still have $\tau_u(\Lambda_1) \cdot \tau_s(\Lambda_2) > 1$. Then the following achieves the proof: □

Fact 2.23. *We can assume that f_{p_1} is asymptotically moderately dissipative at K.*

Proof. Otherwise, as the unstable and stable manifolds of a given periodic point $\tilde{P} \in \Lambda_2$ is dense in $W^u(K)$ and $W^s(K)$, using the non-degeneracy of the unfolding, there is a parameter close to p_0 such that $\tilde{P}$ has d homoclinic tangencies which unfold non-degenerately. By Remark 2.18 , the point $\tilde{P}$ is moderately dissipative. So we can redo the construction starting with this point $\tilde{P}$ instead of P, which is moderately dissipative and so is its dynamics nearby. □

Remark 2.24. *By the proof, we can assume furthermore that the unstable manifolds containing the tangency points have disjoint orbits.*

Step 2: Construction of families of hyperbolic maps $(F^{\mathbf{A}}_p)_p$ and conjugacy maps $(H_p)_p$. The following is a general (but somehow classical) Proposition 2.25 which might be useful in other contexts than the proof of Theorem 2.10. Hence it will be proved in Appendix D.

Let $r \in [2,\infty] \cup \{\omega\}$, let f_0 be a C^r-diffeomorphism of a surface M leaving invariant a horseshoe[9] K_0. Let $d \geq 0$, $\hat{\mathcal{P}}$ be an open neighborhood of $0 \in \mathbb{R}^d$ and let $(f_p)_{p\in\hat{\mathcal{P}}}$ be a C^r-family of diffeomorphisms of M containing f_0. Let $(K_p)_p$ be the hyperbolic continuation of K_0.

Proposition 2.25. *There exist $k \geq 1$ arbitrarily large, a regular compact neighborhood $\mathcal{P}$ of $0 \in \hat{\mathcal{P}}$, an oriented graph $(\mathbf{V},\mathbf{A})$ and a C^r-family $(H_p)_{p\in\mathcal{P}}$ of embeddings $H_p : Y^{\mathbf{e}} \times \mathbf{V} \hookrightarrow M$ such that:*

(1) There exists a C^r-family $(F_p^{\mathbf{A}})_p$ of hyperbolic maps $F_p^{\mathbf{A}} : D_p(\mathbf{A}) \to Y^{\mathbf{e}} \times \mathbf{V}$ of type $\mathbf{A}$ such that $H_p \circ F_p^{\mathbf{A}} := f_p^k \circ H_p|D_p(\mathbf{A})$, and the set $H_p(D_p(\mathbf{A}))$ contains K_p,

(2) if $\limsup_{n\to\infty} \|\det Df_0^n|K_0\|^{1/\epsilon} \cdot \|Df_0^n|K_0\| < 1$, then $(F_p^{\mathbf{A}})_p$ is moderately dissipative,

(3) there is a C^1-family $(\pi_p)_{p\in\mathcal{P}}$ of maps $\pi_p : \mathbb{R} \times I \times \mathbf{V} \to \mathbb{R} \times \mathbf{V}$ which satisfies items (0)-(1)-(2)-(3) of Definition 1.23 and such that $(\ker D\pi_p)_p$ is of class C^1.

A simple consequence of the proof will be:

Corollary 2.26. *If under the assumptions of Proposition 2.25, the family $(f_p)_{p\in\hat{\mathcal{P}}}$ is of class C^{2+}, then the same conclusion holds true and moreover $(\pi_p)_p$ is of class C^{1+}.*

Step 3: Construction of the folding transformations. Let us come back to the proof of Theorem 2.10. Let $p_1 \in \hat{\mathcal{P}}$ and let $K \supset \Lambda_1 \sqcup \Lambda_2$ be the basic set of f_{p_1} given by Lemma 2.22. Let $\mathcal{P}_1$ be a neighborhood of p_1 for which this hyperbolic set persists as $(K_p)_{p\in\mathcal{P}_1}$. Let $(T_i)_{1\leq i\leq d}$ be the d tangencies points between the local unstable and stable manifolds of K satisfying the no-ending gap condition (2) of Theorem 2.15. Also it holds $\tau_u(\Lambda_1) \cdot \tau_s(\Lambda_2) > 1$. Thus by Theorem 2.15, we have:

Fact 2.27. *For every p close to p_1, there are d tangencies points between the local unstable and stable manifolds of K_p which are close to $(T_i)_{1\leq i\leq d}$.*

Let $(F_p^{\mathbf{A}})_{p\in\mathcal{P}_1}$ be the family of hyperbolic maps of type $\mathbf{A}$ given by Proposition 2.25 for the continuation $(K_p)_{p\in\mathcal{P}_1}$. Let $k \geq 1$ and let $(H_p)_p$ be the family of C^r-embeddings such that:

$$f_p^k \circ H_p = H_p \circ F_p^{\mathbf{A}} \ .$$

Let Λ_p be the maximal invariant set of $F_p^{\mathbf{A}}$. Let $(\pi_p)_p$ be the family of projections given by Proposition 2.25 (3). We put also:

$$W^s(\Lambda_p) := \bigcup_{\mathbf{s}\in\overrightarrow{\mathbf{A}}} W_p^{\mathbf{s}} \times \{\mathbf{o}(\mathbf{s})\} \quad \text{and} \quad W^u(\Lambda_p) := \bigcup_{\mathbf{u}\in\overleftarrow{\mathbf{A}}} W_p^{\mathbf{u}} \times \{\mathbf{t}(\mathbf{u})\}$$

Let us construct the folding transformations. As each T_i is in the unstable lamination of K_{p_1}, up to changing T_i by $f_{p_1}^{-m}(T_i)$ for every i, we can assume that each T_i belongs to $H_{p_1}(W_{p_1}^u(\Lambda_{p_1}) \cap \operatorname{int} Y^{\mathbf{e}} \times \mathbf{V})$. Let $\mathbf{u}_i \in \overleftarrow{\mathbf{A}}$ be such that $H_{p_1}^{-1}(T_i)$ belongs to $W_{p_1}^{\mathbf{u}_i} \times \{\mathbf{t}(\mathbf{u}_i)\}$. By Remark 2.24, we have $\mathbf{u}_i \neq \mathbf{u}_j$ for $i \neq j$. Put $W_{loc}^u(T_i, f_p) := H_p(W_p^{\mathbf{u}_i} \times \{\mathbf{t}(\mathbf{u}_i)\})$.

Also each T_i is in the stable lamination of K_{p_1}. Thus for every $n \geq 1$ large enough, each point $f_{p_1}^n(T_i)$ is in $H_{p_1}(W_{p_1}^s(\Lambda_{p_1}) \cap \operatorname{int} Y^{\mathbf{e}} \times \mathbf{V})$. Moreover there exists $\mathbf{s}_i \in \overrightarrow{\mathbf{A}}$ such that $f_{p_1}^n(W_{loc}^u(T_i, f_{p_1}))$ has a quadratic tangency with $H_{p_1}(W_{p_1}^{\mathbf{s}_i} \times \mathbf{o}(\mathbf{s}_i))$ at $f_{p_1}^n(T_i)$. Also the unfolding of these d tangencies

9. A horseshoe is both a hyperbolic basic set and a Cantor set.

induced by $(f_p)_p$ is non degenerated by Lemma 2.22 (2). By regularity of π, there is a small neighborhood of p_1 formed by parameters p for which the curve $W^u_{loc}(T_i, f_p)$ has a quadratic tangency with a fiber of $\pi_p^{\mathbf{o}(\mathbf{s}_i)} \circ (H_p^{\mathbf{o}(\mathbf{s}_i)})^{-1} \circ f_p^n$ at a point $T_i(p)$ nearby T_i. Let $\zeta_p^{\mathbf{u}_i} \in W_p^{\mathbf{u}_i}$ be the preimage by $H_p^{\mathbf{t}(\mathbf{u}_i)}$ of $T_i(p)$. As the unfolding is non degenerate, the following map has invertible differential at $p = p_1$:

$$p \mapsto \left(\pi_p^{\mathbf{o}(\mathbf{s}_i)} \circ (H_p^{\mathbf{o}(\mathbf{s}_i)})^{-1} \circ f_p^n(T_i(p)) - b_p^{\mathbf{s}_i}\right)_{1\le i\le d} \quad \text{with } \{b_p^{\mathbf{s}_i}\} := \pi_p^{\mathbf{o}(\mathbf{s}_i)}(W_p^{\mathbf{s}_i}) \ .$$

By regularity of the stable laminations, there is a clopen neighborhood $\mathbf{W}_i \subset \overrightarrow{\mathbf{A}}$ of each $\mathbf{s}_i$, such that for any $(\tilde{\mathbf{s}}_i)_{1\le i\le d} \in \prod_{1\le i\le d} \mathbf{W}_i$, the following map has invertible differential at $p = p_1$:

$$p \mapsto \left(\pi_p^{\mathbf{o}(\mathbf{s}_i)} \circ (H_p^{\mathbf{o}(\mathbf{s}_i)})^{-1} \circ f_p^n(T_i(p)) - b_p^{\tilde{\mathbf{s}}_i}\right)_{1\le i\le d} \quad \text{with } \{b_p^{\tilde{\mathbf{s}}_i}\} := \pi_p^{\mathbf{o}(\tilde{\mathbf{s}}_i)}(W_p^{\tilde{\mathbf{s}}_i}) \ .$$

Up to reducing each $\mathbf{W}_i$, we can assume it of the form $\mathbf{W}_i = \mathbf{w}_i \cdot \overrightarrow{\mathbf{A}}$ for admissible words $\mathbf{w}_i \in \mathbf{A}^*$ of the same length $|\mathbf{w}|$. Thus up to replacing n by $n + |\mathbf{w}|$, we can assume that $\mathbf{w}_i = \mathbf{e}$ and obtain:

Fact 2.28. *The following maps have invertible differential at $p = p_1$:*

$$p \mapsto \left(\pi_p^{\mathbf{o}(\mathbf{s}_i)} \circ (H_p^{\mathbf{o}(\mathbf{s}_i)})^{-1} \circ f_p^n(T_i(p)) - b_p^{\tilde{\mathbf{s}}_i}\right)_{1\le i\le d} \ , \quad \forall (\tilde{\mathbf{s}}_i)_{1\le i\le d} \in \prod_{i=1}^{d} \{\mathbf{s} \in \overrightarrow{\mathbf{A}} : \mathbf{o}(\mathbf{s}) = \mathbf{o}(\mathbf{s}_i)\}.$$

We recall that k is a large integer such that $f_p^k \circ H_p = H_p \circ F_p^{\mathbf{A}}$ by Proposition 2.25. As n is any large number, we can assume that $n = m \cdot k$ for a certain $m \ge 1$. By replacing $\mathbf{A}$ by $\mathbf{A}^* \cap \mathbf{A}^m$ and $F_p^{\mathbf{A}}$ by $(F_p^{\mathbf{A}})^m$, we can assume that $k = n$.

For every $1 \le i \le d$, let $\mathbf{c}_i$ be a new arrow from $\mathbf{t}(\mathbf{u}_i)$ to $\mathbf{o}(\mathbf{s}_i)$. Let $\mathbf{C} := \{\mathbf{c}_i : 1 \le i \le d\}$. Put:

$$\overrightarrow{\mathbf{A}}_{\mathbf{c}} := \{\mathbf{s} \in \overrightarrow{\mathbf{A}} : \mathbf{o}(\mathbf{s}) = \mathbf{t}(\mathbf{c})\} \quad \text{and} \quad F_p^{\mathbf{c}} := (H_p^{\mathbf{t}(\mathbf{c})})^{-1} \circ f_p^k \circ H_p^{\mathbf{o}(\mathbf{c})} \quad , \qquad \forall \mathbf{c} \in \mathbf{C}.$$

The latter map is defined on a neighborhood of $\zeta_{p_1}^{\mathbf{u}_i}$ when $\mathbf{c} = \mathbf{c}_i$. The following rephrases Fact 2.28:

Fact 2.29. *For any $(\mathbf{s}_{\mathbf{c}})_{\mathbf{c}\in\mathbf{C}} \in \prod_{\mathbf{c}\in\mathbf{C}} \overrightarrow{\mathbf{A}}_{\mathbf{c}}$ and with $(\mathbf{u}_{\mathbf{c}})_{\mathbf{c}\in\mathbf{C}} = (\mathbf{u}_i)_{\mathbf{c}_i\in\mathbf{C}}$, the following map has invertible differential at $p = p_1$:*

$$p \mapsto \left(\pi_p^{\mathbf{t}(\mathbf{c})} \circ F_p^{\mathbf{c}}(\zeta_p^{\mathbf{u}_{\mathbf{c}}}) - b_p^{\mathbf{s}_{\mathbf{c}}}\right)_{\mathbf{c}\in\mathbf{C}} \ .$$

For every i, with $\mathbf{c} = \mathbf{c}_i$, there is a quadratic tangency at the point $\zeta_p^{\mathbf{u}_i}$ between $W_p^{\mathbf{u}_i}$ and a fiber of $\pi_p^{\mathbf{t}(\mathbf{c})} \circ F_p^{\mathbf{c}}$. Thus there is a small clopen neighborhood $\overleftarrow{\mathbf{A}}_{\mathbf{c}}$ of $\mathbf{u}_i$ such that for every $\mathbf{u}_{\mathbf{c}} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}}$ and p close to p_1, there is a quadratic tangency between $W_p^{\mathbf{u}_{\mathbf{c}}}$ and a fiber of $\pi_p^{\mathbf{t}(\mathbf{c})} \circ F_p^{\mathbf{c}}$ at a point $\zeta_p^{\mathbf{u}_{\mathbf{c}}}$ close to $\zeta_p^{\mathbf{u}_i}$. By regularity of the unstable lamination and compactness of $\prod_{\mathbf{c}\in\mathbf{C}} \overrightarrow{\mathbf{A}}_{\mathbf{c}}$, by reducing each $\overleftarrow{\mathbf{A}}_{\mathbf{c}}$, we obtain:

Fact 2.30. *For every $(\mathbf{u}_{\mathbf{c}}, \mathbf{s}_{\mathbf{c}})_{\mathbf{c}\in\mathbf{C}} \in \prod_{\mathbf{c}\in\mathbf{C}} \overleftarrow{\mathbf{A}}_{\mathbf{c}} \times \overrightarrow{\mathbf{A}}_{\mathbf{c}}$, the following map has invertible differential for every p close to p_1:*

$$p \mapsto \left(\pi_p^{\mathbf{t}(\mathbf{c})} \circ F_p^{\mathbf{c}}(\zeta_p^{\mathbf{u}_{\mathbf{c}}}) - b_p^{\mathbf{s}_{\mathbf{c}}}\right)_{\mathbf{c}\in\mathbf{C}} .$$

Up to reducing each $\overleftarrow{\mathbf{A}}_{\mathbf{c}}$, there is a segment $I^{\mathbf{c}}$ (independent of p close to p_1) such that for every $\mathbf{u}_{\mathbf{c}} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}}$ and p close to p_1, the point $\zeta_p^{\mathbf{u}_{\mathbf{c}}}$ is the unique critical point of $\pi_p^{\mathbf{t}(\mathbf{c})} \circ F_p^{\mathbf{c}} | I^{\mathbf{c}} \times I \cap W_p^{\mathbf{u}_{\mathbf{c}}}$ and

belongs to $\mathrm{int} I^{\mathbf{c}} \times I$. Moreover we can assume that the convex hull of $I^{\mathbf{c}} \times I \cap \bigcup_{\mathbf{u_c} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}}} W_p^{\mathbf{u_c}}$ is sent into $\mathrm{int}\, Y^{\mathbf{e}}$ by $F_p^{\mathbf{c}}$.

For every $\mathbf{c} \in \mathbf{C}$, since every local unstable manifold has all its tangent vectors in the cone χ_h, we can reduce $I^{\mathbf{c}}$ if necessary and then find a segment $J^{\mathbf{c}} \Subset I$ independent of p close to p_1 such that, with $Y_p^{\mathbf{c}} = I^{\mathbf{c}} \times J^{\mathbf{c}}$ and $\partial^u Y_p^{\mathbf{c}} = I^{\mathbf{c}} \times \partial J^{\mathbf{c}}$, the following set is a small clopen neighborhood of $\mathbf{u}_i$ included in $\overleftarrow{\mathbf{A}}_{\mathbf{c}}$:

$$\{\mathbf{u} \in \overleftarrow{\mathbf{A}} : \mathbf{t}(\mathbf{u}) = \mathbf{o}(\mathbf{c}), W_p^{\mathbf{u}} \cap Y_p^{\mathbf{c}} \neq \emptyset \ \&\ W_p^{\mathbf{u}} \cap \partial^u Y_p^{\mathbf{c}} = \emptyset\}$$

Thus up to replacing $\overleftarrow{\mathbf{A}}_{\mathbf{c}}$ by the latter set, we can suppose that it holds:

$$\overleftarrow{\mathbf{A}}_{\mathbf{c}} = \{\mathbf{u} \in \overleftarrow{\mathbf{A}} : \mathbf{t}(\mathbf{u}) = \mathbf{o}(\mathbf{c}), W_p^{\mathbf{u}} \cap Y_p^{\mathbf{c}} \neq \emptyset \ \&\ W_p^{\mathbf{u}} \cap \partial^u Y_p^{\mathbf{c}} = \emptyset\}$$

for any p close to p_1. Up to reducing each $\overleftarrow{\mathbf{A}}_{\mathbf{c}}$ a new time and consequently $J^{\mathbf{c}}$ and then reducing $I^{\mathbf{c}}$, we can suppose that $F_p^{\mathbf{c}}$ sends $Y_p^{\mathbf{c}}$ into the interior of $Y^{\mathbf{e}}$.

By Fact 2.27, for every p close to p_1, there are $(\mathbf{u_c}, \mathbf{s_c})_{\mathbf{c} \in \mathbf{C}} \in \prod_{\mathbf{c} \in \mathbf{C}} \overleftarrow{\mathbf{A}}_{\mathbf{c}} \times \overrightarrow{\mathbf{A}}_{\mathbf{c}}$ close to $(\mathbf{u}_i, \mathbf{s}_i)_{\mathbf{c}_i \in \mathbf{C}}$ such that each $W_p^{\mathbf{u_c}}$ has a quadratic tangency with the preimage of $W_p^{\mathbf{s_c}}$ by $F_p^{\mathbf{c}}$ at a point close to $\zeta_p^{\mathbf{u_c}}$. By uniqueness such a point must be $\zeta_p^{\mathbf{u_c}}$. This proves $(\mathbf{H_1})$.

Property $(\mathbf{H_2})$ is given by Fact 2.30 and the local inversion theorem together with the compactness of $\prod_{\mathbf{c} \in \mathbf{C}} \overleftarrow{\mathbf{A}}_{\mathbf{c}} \times \overrightarrow{\mathbf{A}}_{\mathbf{c}}$ to define the regular compact neighborhood $\mathcal{P}$ of p_1. Clearly, the families $(F^{\mathbf{A}})_{p \in \mathcal{P}}$ and $(F^{\mathbf{C}})_{p \in \mathcal{P}}$ define a regular family of systems of type $(\mathbf{A}, \mathbf{C})$.

Now given a C^r-perturbation $(\tilde{f}_p)_p$ of $(f_p)_p$, by structural stability, the wild dissipative basic set K for f_{p_1} persists as a wild dissipative basic set $\tilde{K}$ for $\tilde{f}_{p_1}$ (by [New79, Proposition 6] the unstable and stable thicknesses of these basic sets vary continuously). So Step 1 remains for $(\tilde{f}_p)_p$ at the same parameter $p_1 \in \mathcal{P}$. By structural stability, the Markov partition and so the encoding into a $\mathbf{A}$-system persists as well for $(f_p)_{p \in \mathcal{P}_1}$. So Step 2 remains valid for $(\tilde{f}_p)_p$ for the same regular subset $\mathcal{P}_1 \subset \mathcal{P}$. Likewise Step 3 remains valid since the conditions on the folding maps are open on the elements involved. This shows the last item of Theorem 2.10.

3. Sufficient conditions for a wandering stable domain

In this section, we consider a system of type $(\mathbf{A}, \mathbf{C})$ endowed with an adapted projection π which satisfies the assumptions of Theorems 1.32 and 1.34 and we prove the conclusions of these Theorems. Before performing these proofs, we recall and develop the formalism of implicit representations that will be used throughout this paper.

3.1. Implicit representations and initial bounds.

In the following, we will use the formalism of implicit representations. It was introduced by Palis and Yoccoz in [PY01] and [PY09] in order to get bounds on the iterations of hyperbolic transformations [10]. In fact, this formalism goes back to the generating functions and the Shilnikov cross coordinates, appeared in the work of Shilnikov [Shi67] and his school [GST08].

10. They call such maps affine-like maps.

Proposition 3.1. *Let (Y, F) be a hyperbolic transformation. Then, there are functions $\mathcal{X}_0$ and $\mathcal{Y}_1$ in $C^2(I^2, I)$ satisfying for every $(x_0, y_0) \in Y$ and $(x_1, y_1) \in F(Y)$:*

$$(\mathcal{A}) \qquad F(x_0, y_0) = (x_1, y_1) \Leftrightarrow \begin{cases} x_0 = & \mathcal{X}_0(x_1, y_0) \\ y_1 = & \mathcal{Y}_1(x_1, y_0) \end{cases}$$

Proof. Let $(x_0, y_0) \in Y$ and $(x_1, y_1) \in F(Y)$. By properties *(1)* and *(2)* of Definition 1.4, the image by F of $\{y = y_0\} \cap Y$ is the graph of a function $x \in I \mapsto \mathcal{Y}_1(x, y_0) \in I$. Likewise, by properties *(1)* and *(3)* of Definition 1.4, the image by F^{-1} of $\{x = x_1\} \cap F(Y)$ is the transpose of the graph of a function $y \in I \mapsto \mathcal{X}_0(x_1, y) \in I$. We notice that the functions $(x_1, y_0) \mapsto \mathcal{X}_0(x_1, y_0)$ and $(x_1, y_0) \mapsto \mathcal{Y}_1(x_1, y_0)$ are of class C^2. Observe also that by construction, $F(x_0, y_0) = (x_1, y_1)$ iff the intersection point $\{(x_1, \mathcal{Y}_1(x_1, y_0))\} = \operatorname{Graph} \mathcal{Y}_1(\cdot, y_0) \cap \{x = x_1\}$ is the image by F of $\{(\mathcal{X}_0(x_1, y_0), y_0)\} = {}^t\operatorname{Graph} \mathcal{X}_0(x_1, \cdot) \cap \{y = y_0\}$. □

Definition 3.2. *The pair $(\mathcal{X}_0, \mathcal{Y}_1)$ is the* implicit representation *of (Y, F).*

Let us introduce the natural counterpart of implicit representation in the complex setting:

Proposition 3.3. *Let $(\tilde{Y}, F)$ be a C^ω_ρ-hyperbolic transformation. Then there are holomorphic functions $\mathcal{Z}_0$ and $\mathcal{W}_1$ defined on $\tilde{I}^2$ satisfying for every $(z_0, w_0) \in \tilde{Y}$ and $(z_1, w_1) \in F(\tilde{Y})$:*

$$F(z_0, w_0) = (z_1, w_1) \Leftrightarrow \begin{cases} z_0 = & \mathcal{Z}_0(z_1, w_0) \\ w_1 = & \mathcal{W}_1(z_1, w_0) \end{cases} .$$

Proof. Let $(z_0, w_0) \in \tilde{Y}$ and $(z_1, w_1) \in F(\tilde{Y})$. By properties *(1)* and *(2)* of Definition 1.5, the image by F of $\{w = w_0\} \cap \tilde{Y}$ is the graph of a function $z \in \tilde{I} \mapsto \mathcal{W}_1(z, w_0) \in \tilde{I}$. Likewise, by properties *(1)* and *(3)* of Definition 1.5, the image by F^{-1} of $\{z = z_1\} \cap F(\tilde{Y})$ is the transpose of the graph of a function $w \in \tilde{I} \mapsto \mathcal{Z}_0(z_1, w)$. By transversality, the functions $(z_1, w_0) \mapsto \mathcal{Z}_0(z_1, w_0) \in \tilde{I}$ and $(z_1, w_0) \mapsto \mathcal{W}_1(z_1, w_0)$ are holomorphic. Observe also that by construction, $F(z_0, w_0) = (z_1, w_1)$ iff the intersection point $\{(z_1, \mathcal{W}_1(z_1, w_0))\} = \operatorname{Graph} \mathcal{W}_1(\cdot, w_0) \cap \{z = z_1\}$ is the image by F of $\{(\mathcal{Z}_0(z_1, w_0), w_0)\} = {}^t\operatorname{Graph} \mathcal{Z}_0(z_1, \cdot) \cap \{w = w_0\}$. □

Definition 3.4. *The pair $(\mathcal{Z}_0, \mathcal{W}_1)$ is the* implicit representation *of $(\tilde{Y}, F)$.*

Remark 3.5. *If $(\tilde{Y}, F)$ is a complex extension of a hyperbolic transformation (Y, F), with implicit representations $(\mathcal{Z}_0, \mathcal{W}_1)$ and $(\mathcal{X}_0, \mathcal{Y}_1)$ respectively, it holds: $(\mathcal{Z}_0, \mathcal{W}_1)|I^2 = (\mathcal{X}_0, \mathcal{Y}_1)$.*

Proposition 3.6. *If $(\mathcal{X}_0, \mathcal{Y}_1)$ is the implicit representation of a hyperbolic transformation (Y, F), for every $(x_0, y_0) \in Y$ sent to (x_1, y_1) by F, it holds:*

$$\det DF(x_0, y_0) \cdot \partial_{x_1} \mathcal{X}_0(x_1, y_0) = \partial_{y_0} \mathcal{Y}_1(x_1, y_0) .$$

Proof. By $(\mathcal{A})$, we have $F(\mathcal{X}_0(x_1, y_0), y_0) = (x_1, \mathcal{Y}_1(x_1, y_0))$ and so

$$\det D_{x_0, y_0} F \cdot \det D_{x_1, y_0}(\mathcal{X}_0(x_1, y_0), y_0) = \det D_{x_1, y_0}(x_1, \mathcal{Y}_1(x_1, y_0)).$$

Then the proposition follows from the following equalities:

$$\det D_{x_1, y_0}(\mathcal{X}_0(x_1, y_0), y_0) = \partial_{x_1} \mathcal{X}_0(x_1, y_0) \quad \text{and} \quad \det D_{x_1, y_0}(x_1, \mathcal{Y}_1(x_1, y_0)) = \partial_{y_0} \mathcal{Y}_1(x_1, y_0) .$$

□

An immediate consequence of the latter proof is:

Corollary 3.7. *If $(\tilde{Y}, F)$ is a C^ω_ρ-hyperbolic transformation with implicit representation $(\mathcal{Z}_0, \mathcal{W}_1)$, then for every $(z_0, w_0) \in \tilde{Y}$ sent to (z_1, w_1) by F, it holds:*

$$\det DF(z_0, w_0) \cdot \partial_{z_1} \mathcal{Z}_0(z_1, w_0) = \partial_{w_0} \mathcal{W}_1(z_1, w_0) \ .$$

The following distortion was introduced in [PY09, Page 19].

Definition 3.8. *The* distortion $\mathcal{D}(Y, F)$ *of a hyperbolic transformation (Y, F) with implicit representation $(\mathcal{X}, \mathcal{Y})$ is the maximum absolute value attained on I^2 by the six following functions:*

$$\partial_x \log|\partial_x \mathcal{X}|, \quad \partial_y \log|\partial_x \mathcal{X}|, \quad \partial_x \log|\partial_y \mathcal{Y}|, \quad \partial_y \log|\partial_y \mathcal{Y}|, \quad \partial^2_{y^2} \mathcal{X}, \quad \partial^2_{x^2} \mathcal{Y} \ .$$

Let us recall:

Theorem 3.9 ([PY09, (**MP6**), Page 21])**.** *There exists $C_1 > 0$ which depends only on θ and λ such that for every $N \geq 1$, if $((Y_i, F_i))_{1 \leq i \leq N}$ are hyperbolic transformations, then the distortion of $(Y, F) = (Y_1, F_1) \star \cdots \star (Y_N, F_N)$ satisfies:*

$$\mathcal{D}(Y, f) \leq C_1 \cdot \max_{1 \leq i \leq N} \mathcal{D}(Y_i, F_i) \ .$$

For every $\mathbf{w} \in \mathbf{A}^*$, let $(\mathcal{X}^{\mathbf{w}}, \mathcal{Y}^{\mathbf{w}})$ denote the implicit representation of $(Y^{\mathbf{w}}, F^{\mathbf{w}})$.

Corollary 3.10. *If $F^{\mathbf{A}}$ is a hyperbolic map of type $\mathbf{A}$, there exists $\mathcal{D} > 0$ and $\mathcal{B} := 2\exp(2\sqrt{2}\mathcal{D})$ such that for any $\mathbf{w} \in \mathbf{A}^*$ and $(x, y) \in I^2$, it holds both $\mathcal{D}(Y^{\mathbf{w}}, F^{\mathbf{w}}) \leq \mathcal{D}$ and the following:*

$$|Y^{\mathbf{w}}| \cdot \mathcal{B}^{-1} \leq |\partial_x \mathcal{X}^{\mathbf{w}}(x, y)| \leq |Y^{\mathbf{w}}| \cdot \mathcal{B} \quad \textit{and} \quad |F^{\mathbf{w}}(Y^{\mathbf{w}})| \cdot \mathcal{B}^{-1} \leq |\partial_y \mathcal{Y}^{\mathbf{w}}(x, y)| \leq |F^{\mathbf{w}}(Y^{\mathbf{w}})| \cdot \mathcal{B}.$$

Proof. We fix $\mathbf{w} \in \mathbf{A}^*$ and $(x, y) \in I^2$. A consequence of Theorem 3.9 and of the finiteness of $\mathbf{A}$ is:

$$|\partial_x \mathcal{X}^{\mathbf{w}}(x, y)| \cdot (\mathcal{B}/2)^{-1} \leq |\partial_x \mathcal{X}^{\mathbf{w}}(x', y')| \leq |\partial_x \mathcal{X}^{\mathbf{w}}(x, y)| \cdot (\mathcal{B}/2) \quad \forall (x', y') \in I^2 \ , \tag{3.1}$$

$$\text{with } \mathcal{D} := C_1 \cdot \max_{\mathbf{a} \in \mathbf{A}} \mathcal{D}(Y^{\mathbf{a}}, F^{\mathbf{a}}) \quad \text{and} \quad \mathcal{B} := 2\exp(\operatorname{diam}(I^2)\mathcal{D}) = 2\exp(2\sqrt{2}\mathcal{D}) \, .$$

By Definition 1.27, the number $|Y^{\mathbf{w}}|$ is the maximum of the lengths of $I \times \{y'\} \cap Y^{\mathbf{w}}$ among $y' \in I$. Thus there exists $y' \in I$ such that $|Y^{\mathbf{w}}| = |\int_I \partial_x \mathcal{X}^{\mathbf{w}}(x', y') dx'|$. Then Eq. (3.1) together with $\operatorname{diam}(I) = 2$ imply the first bound. The proof of the second one is the same by symmetry. □

3.2. **Normal form and definitions of the sets B_j and $\tilde{B}_j$.** In this subsection, we consider a system F of type $(\mathbf{A}, \mathbf{C})$ endowed with an adapted projection π. Our aim is to prove a normal form useful for the proofs of Theorems 1.32 and 1.34. We start with general initial bounds.

We recall that for $\mathbf{w} \in \mathbf{A}^*$, the curve $H^{\mathbf{w}}$ was defined in Definition 1.26, and for $\mathbf{c} \in \mathbf{C}$, the set $\mathbf{A}^*_{\mathbf{c}}$ and the critical points $(\zeta^{\mathbf{w}})_{\mathbf{w} \in \mathbf{A}^*_{\mathbf{c}}}$ were defined in Proposition 1.28.

Proposition 3.11 (definition of $\epsilon_{\mathbf{C}}$)**.** *There exists $\epsilon_{\mathbf{C}} > 0$ such that for every $\mathbf{c} \in \mathbf{C}$:*

(1) $F^{\mathbf{c}}(Y^{\mathbf{c}}) \subset Y^{\mathbf{e}}$ is $2\epsilon_{\mathbf{C}}$-distant from $\partial^u Y^{\mathbf{e}}$,

(2) $\zeta^{\mathbf{w}} \in Y^{\mathbf{c}}$ is $2\epsilon_{\mathbf{C}}$-distant to $\partial Y^{\mathbf{c}}$, for every $\mathbf{w} \in \mathbf{A}^*_{\mathbf{c}}$,

(3) $H^{\mathbf{w}}$ is $2\epsilon_{\mathbf{C}}$-distant to $\partial^u Y^{\mathbf{c}}$, for every $\mathbf{w} \in \mathbf{A}^*_{\mathbf{c}}$.

Proof. By finiteness of $\mathbf{C}$, it suffices to show that there exists such a constant for every $\mathbf{c} \in \mathbf{C}$.

By Definition 1.18, we have $F^{\mathbf{c}}(Y^{\mathbf{c}}) \subset Y^{\mathbf{e}} \setminus \partial^u Y^{\mathbf{e}}$ and so the first statement is easy.

By Remark 1.29, the set $\{\zeta^{\mathbf{w}} : \mathbf{w} \in \mathbf{A}^*_{\mathbf{c}} \sqcup \overleftarrow{\mathbf{A}}_{\mathbf{c}}\}$ is compact. It is included in the interior of $Y^{\mathbf{c}}$ by Definition 1.23 (4) and Proposition 1.28. This proves the second item.

By Remark 1.29, the set $\bigcup_{\mathbf{u} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}}} W^u \cup \bigcup_{\mathbf{w} \in \mathbf{A}^*_{\mathbf{c}}} H^{\mathbf{w}}$ is compact. By Definition 1.18 (1) and Proposition 1.28, it does not intersect $\partial^u Y^{\mathbf{c}}$. Thus the last item of the proposition follows. □

From now on, we work under the assumptions of Theorem 1.32 (resp. Theorem 1.34), which defined the sequences $(\mathbf{w}_j)$ (and so $(\zeta^{\mathbf{w}_j})_j$) and $(\mathbf{c}_j)_j$. The idea is to define the domains B_j and $\tilde{B}_j$ using a normal form for the folding of the curve $H^{\mathbf{w}_j}$ with respect to the π-fibers. However it is rather complicated to work with $\pi^{\mathbf{o}(\mathbf{w}_{j+1})}$-fibers, for they are a priori not analytic. Hence we will work with:

Definition 3.12. *Let $\breve{V}^{j+1}$ be the pullback of $\{x_{j+1}\} \times I$ by $F^{\mathbf{w}_{j+1}}$, with x_{j+1} the first coordinate of $\zeta^{\mathbf{w}_{j+1}}$.*

We note that:

$$\breve{V}^{j+1} = \{(\breve{\mathfrak{v}}_{j+1}(y), y) : y \in I\} \quad \text{with} \quad \breve{\mathfrak{v}}_{j+1}(\cdot) := \mathcal{X}^{\mathbf{w}_{j+1}}(x_{j+1}, \cdot)\,, \quad \forall j \geq 1. \tag{3.2}$$

We will also use the following notation $\hbar_j$ whose expression is similar to the one of $\breve{\mathfrak{v}}_{j+1}$:

$$H^{\mathbf{w}_j} = \{(x, \hbar_j(x)) : x \in I\} \quad \text{with} \quad \hbar_j(\cdot) := \mathcal{Y}^{\mathbf{w}_j}(\cdot, 0)\,, \quad \forall j \geq 1. \tag{3.3}$$

Let $F_1^{\mathbf{c}_j}$ and $F_2^{\mathbf{c}_j}$ be the x-coordinate and the y-coordinate of $F^{\mathbf{c}_j}$. The x-projection $I^{\mathbf{c}_j} \subset I$ of $Y^{\mathbf{c}_j}$ is the definition domain of $x \mapsto F^{\mathbf{c}_j} \circ (x, \hbar_j(x))$. For every $r > 0$, let $\mathbb{D}(r) := \{z \in \mathbb{C} : |z| < r\}$.

Here is the aforementioned normal form:

Proposition 3.13. *There exists $K > 0$ such that for every j large enough, there exist $\breve{x}_j \in I^{\mathbf{c}_j}$ which is $\epsilon_{\mathbf{C}}$-distant to the boundary of $I^{\mathbf{c}_j}$, real numbers b_j, q_j and a C^2-function r_j such that for every $(x, y) \in [-\epsilon_{\mathbf{C}}, \epsilon_{\mathbf{C}}]^2$ the following is well-defined and holds true:*

$$(F_1^{\mathbf{c}_j} - \breve{\mathfrak{v}}_{j+1} \circ F_2^{\mathbf{c}_j}) \circ \varphi_j(x, y) = q_j \cdot x^2 + b_j \cdot y + r_j(x, y) \tag{3.4}$$

with $\varphi_j(x, y) = (\breve{x}_j + x, \hbar_j(\breve{x}_j + x) + y)$, $1/K \leq |q_j|, |b_j| \leq K$ and r_j satisfying:

$$0 = \partial_x r_j(0) = \partial_y r_j(0) = \partial_x^2 r_j(0) \text{ and } r_j(0) = o(\breve{\gamma}_j^2)\,. \tag{3.5}$$

Also if F is analytic, there exists $\tilde{\epsilon}_{\mathbf{C}} > 0$ independent of j, such that φ_j and r_j extend holomorphically to $\mathbb{D}(\tilde{\epsilon}_{\mathbf{C}})^2$.

We prove this proposition below. This normal form enables us to define rescaling coordinates and the domains B_j and $\tilde{B}_j$. We need just to define the rescaling factors:

Definition 3.14. *Let y_j be the y-coordinate of $F^{\mathbf{c}_j}(\zeta^{\mathbf{w}_j})$. We define:*

$$\sigma_j := \partial_x \mathcal{X}^{\mathbf{w}_j}(x_j, y_j) \quad \text{and} \quad \gamma_j := |\frac{\sigma_{j+1}}{q_j}|^{1/2} \cdot |\frac{\sigma_{j+2}}{q_{j+1}}|^{1/2^2} \cdots |\frac{\sigma_{j+k}}{q_{j+k-1}}|^{1/2^k} \cdots$$

By the latter proposition, $1/K \leq |q_j| \leq K$ and so $\prod_k |1/q_k|^{1/2^k}$ is well-defined and in $[1/K, K]$. Also by Corollary 3.10, it holds $|\sigma_j|/\mathscr{B} \leq |Y^{\mathbf{w}_j}| \leq \mathscr{B}|\sigma_j|$. Thus the limit γ_j exists and satisfies:

$$\frac{\breve{\gamma}_j}{\mathscr{B} \cdot K} \leq \gamma_j \leq \mathscr{B} \cdot K \cdot \breve{\gamma}_j \ . \tag{3.6}$$

Therefore γ_j is well-defined, small and nonzero because $\breve{\gamma}_j$ is so by assumptions (i) and (iii) of Theorem 1.32. In particular, since the domain of φ_j contains $[-\epsilon_{\mathbf{C}}, \epsilon_{\mathbf{C}}]^2$, the following is well-defined:

Definition 3.15. *The rescaling coordinates Φ_j are defined for j large by:*

$$\Phi_j : (X, Y) \in (-0.3, 0.3)^2 \mapsto \varphi_j \left(\gamma_j \cdot X, \frac{q_j \cdot \gamma_j^2}{2b_j} \cdot Y \right) = \left(\breve{x}_j + \gamma_j \cdot X, \hbar_j(\breve{x}_j + \gamma_j \cdot X) + \frac{q_j \cdot \gamma_j^2}{2b_j} \cdot Y \right) .$$

Similarly, if F is analytic, the definition domain of φ_j contains $\mathbb{D}(\tilde{\epsilon}_{\mathbf{C}})^2$. So it holds:

Fact 3.16. *If F is analytic, the rescaling coordinates Φ_j extend holomorphically to $\mathbb{D}(0.3)^2$.*

Definition 3.17. *The real image of Φ_j is denoted by $B_j := \Phi_j((-0.3, 0.3)^2)$. If F is analytic, the complex image of Φ_j is denoted by $\tilde{B}_j := \Phi_j(\mathbb{D}(0.3)^2)$.*

Remark 3.18. *Actually for j large enough, the map φ_j is also well defined on $[-3, 3]^2$. Using* [Ber18, Theorem C and Remark 3.5]*, one can show that when j is large, the following map:*

$$F_j := (X, Y) \in (-0.3, 0.3)^2 \mapsto (\Phi_{j+1})^{-1} \circ F^{\mathbf{w}_{j+1}} \circ F^{\mathbf{c}_j} \circ \Phi_j(X, Y) \ ,$$

is C^2-close to $(X, Y) \in (-0.3, 0.3)^2 \mapsto \operatorname{sgn}(\sigma_{j+1} q_j)(X^2 + \frac{1}{2}Y, 0)$, and obtain Theorem 1.32 (1). However we will give a variation of this proof which works in our complex case.

Proof of Proposition 3.13. To understand the position of the curve $\breve{V}^{j+1} = (F^{\mathbf{w}_{j+1}})^{-1}(\{x_{j+1}\} \times I)$ w.r.t. $F^{\mathbf{c}_j}(H^{\mathbf{w}_j} \cap Y^{\mathbf{c}_j})$, we are going to use the two following curves:

— The $\pi^{\mathbf{o}(\mathbf{w}_{j+1})}$-fiber $\hat{V}^{j+1}$ of $(F^{\mathbf{w}_{j+1}})^{-1}(\zeta^{\mathbf{w}_{j+1}})$.

— The $\pi^{\mathbf{t}(\mathbf{c}_j)}$-fiber V^{j+1} of $F^{\mathbf{c}_j}(\zeta^{\mathbf{w}_j})$.

The two next lemmas states that these curves are close to $\breve{V}^{j+1}$.

Lemma 3.19. *The two curves V^{j+1} and $\hat{V}^{j+1}$ are C^2-close and $o(\breve{\gamma}_j^2)$-C^0-close when j is large.*

Proof. By assumption (ii) of Theorem 1.32, the distance between the $\pi^{\mathbf{t}(\mathbf{w}_{j+1})}$-fibers of the points $F^{\mathbf{w}_{j+1}} \circ F^{\mathbf{c}_j}(\zeta^{\mathbf{w}_j})$ and $\zeta^{\mathbf{w}_{j+1}}$ is small compared to $\breve{\gamma}_{j+1}$. As the horizontal direction is at least $\asymp |Y^{\mathbf{w}_{j+1}}|^{-1}$ expanded by $F^{\mathbf{w}_{j+1}}$ by Corollary 3.10, it follows that the pull back of these fibers by $F^{\mathbf{w}_{j+1}}$ are at C^0-distance small compared to $\breve{\gamma}_{j+1} \cdot |Y^{\mathbf{w}_{j+1}}| = \breve{\gamma}_j^2$. By the equivariance of the fibers given by Definition 1.23.(3), these two pullbacks are equal to respectively V^{j+1} and $\hat{V}^{j+1}$ respectively. Since the π-fibers depend continuously on the based points in the C^2-topology, these two curves are also C^2-close. □

Lemma 3.20. *The two curves V^{j+1} and $\breve{V}^{j+1}$ are C^2-close and $o(\breve{\gamma}_j^2)$-C^0-close when j is large.*

Proof. By Lemma 3.19, it suffices to show that the two curves $\hat{V}^{j+1}$ and $\breve{V}^{j+1}$ are C^2-close and $o(\breve{\gamma}_j^2)$-C^0-close when j is large. The $\pi^{\mathbf{t}(\mathbf{w}_{j+1})}$-fiber going through $\zeta^{\mathbf{w}_{j+1}}$ is a curve included in $Y^{\mathbf{e}}$ with endpoints in $\partial^u Y^{\mathbf{e}}$ and tangent spaces in χ_v by Definition 1.23 (0) and (1). The same occurs

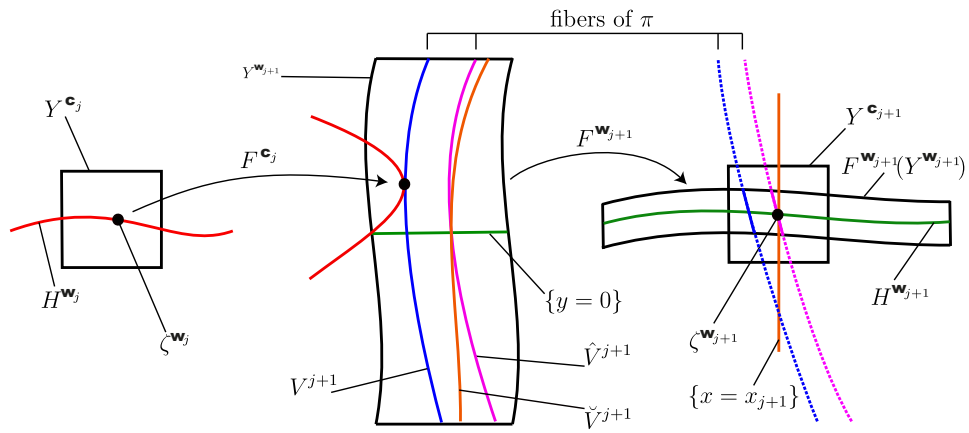

FIGURE 7. Notations for the proof of Proposition 3.13

obviously for the curve $\{x_{j+1}\}\times I$. Hence the curves $\hat{V}^{j+1}$ and $\check{V}^{j+1}$ are the pullbacks by $F^{\mathbf{w}_{j+1}}$ of two curves which are uniformly transverse to the unstable lamination of the maximal invariant set Λ associated to the hyperbolic map $F^{\mathbf{A}}$. Hence by the Inclination Lemma, the curves $\hat{V}^{j+1}$ and $\check{V}^{j+1}$ are C^2-close when $|\mathbf{w}_{j+1}|$ is large and so when j is large.

We now show that the Hausdorff distance between $\hat{V}^{j+1}$ and $\check{V}^{j+1}$ is small compared to $\check{\gamma}_j^2$. Note that the images by $F^{\mathbf{w}_{j+1}}$ of $\hat{V}^{j+1}$ and $\check{V}^{j+1}$ intersects at $\zeta^{\mathbf{w}_{j+1}}$. These curves have tangent spaces in χ_v and are included in $F^{\mathbf{w}_{j+1}}(Y^{\mathbf{w}_{j+1}})$, whose vertical height is smaller than $|F^{\mathbf{w}_{j+1}}(Y^{\mathbf{w}_{j+1}})|$ and thus small compared to $\check{\gamma}_{j+1}^2$ by assumption (iii) of Theorem 1.32. Thus the C^0-distance between the images by $F^{\mathbf{w}_{j+1}}$ of $\hat{V}^{j+1}$ and $\check{V}^{j+1}$ is small compared to $\check{\gamma}_{j+1}^2 \le \check{\gamma}_{j+1}$. As the horizontal direction is at least $\asymp |Y^{\mathbf{w}_{j+1}}|^{-1}$ expanded by $F^{\mathbf{w}_{j+1}}$ by Corollary 3.10, it follows that the C^0-distance between $\hat{V}^{j+1}$ and $\check{V}^{j+1}$ is small compared to $\check{\gamma}_{j+1}\cdot|Y^{\mathbf{w}_{j+1}}| = \check{\gamma}_j^2$. $\square$

The following lemma is the second main step of the proof of Proposition 3.13, it defines the new critical point $\check{x}_j$ defined by $\check{V}^{j+1}$, which is close to the first coordinate x_j of $\zeta^{\mathbf{w}_j}$.

Lemma 3.21. *For every j large enough, the following map:*

$$\check{\Delta}_j : x\in I^{\mathbf{c}_j} \mapsto (F_1^{\mathbf{c}_j} - \check{\mathfrak{v}}_{j+1}\circ F_2^{\mathbf{c}_j})(x,\hbar_j(x))$$

has a unique critical point $\check{x}_j$ in the $\check{\gamma}_j$-neighborhood of x_j in $I^{\mathbf{c}_j}$ which is $\epsilon_{\mathbf{C}}$-distant to $\partial I^{\mathbf{c}_j}$. Moreover $|\check{x}_j - x_j|$ is small compared to $\check{\gamma}_j$ and $|\check{\Delta}_j(\check{x}_j)|$ is small compared to $\check{\gamma}_j^2$.

Proof. By Definition 1.23 (4), for every $\mathbf{c}\in\mathbf{C}$ and $\mathbf{u}\in\overleftarrow{\mathbf{A}}_{\mathbf{c}}$, the curve $W^{\mathbf{u}}$ is sent by $F^{\mathbf{c}}$ tangent to a unique fiber of $\pi^{\mathbf{t}(\mathbf{c})}$ at a unique point $F^{\mathbf{c}}(\zeta^{\mathbf{u}})$, with $\zeta^{\mathbf{u}}\in\operatorname{int} Y^{\mathbf{c}}$, and the tangency is quadratic. By Definition 1.23 (1), there exists a C^2-function $\mathfrak{v}_{\mathbf{u}} : I\to\mathbb{R}$ such that this fiber is equal to $\{(\mathfrak{v}_{\mathbf{u}}(y),y) : y\in I\}$. Thus there exists $r\in(0,\epsilon_{\mathbf{C}}/3)$ such that $[x_{\mathbf{u}}-3r, x_{\mathbf{u}}+3r]\Subset I^{\mathbf{c}}$ and the following map

$$\Delta_{\mathbf{u}} : x\in I^{\mathbf{c}} \mapsto F_1^{\mathbf{c}}(x,w^{\mathbf{u}}(x)) - \mathfrak{v}_{\mathbf{u}}\circ F_2^{\mathbf{c}}(x,w^{\mathbf{u}}(x))$$

has second derivative of absolute value $> 3r$ on $[x_{\mathbf{u}} - 3r, x_{\mathbf{u}} + 3r]$ and a unique critical point in $(x_{\mathbf{u}} - r/3, x_{\mathbf{u}} + r/3)$ equal to $x_{\mathbf{u}}$, with $x_{\mathbf{u}}$ the first coordinate of $\zeta^{\mathbf{u}}$. Note that $\Delta_{\mathbf{u}}$ is indeed well-defined on $I^{\mathbf{c}}$ because $\mathbf{u} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}}$. Since $w^{\mathbf{u}}$ depends continuously on $\mathbf{u}$ in the C^2-topology and $\bigcup_{\mathbf{c}\in\mathbf{C}} \overleftarrow{\mathbf{A}}_{\mathbf{c}}$ is compact, we can take a uniform value of r among $\mathbf{c} \in \mathbf{C}$ and $\mathbf{u} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}}$.

On the other hand, by Proposition 1.28, the curve $H^{\mathbf{w}_j}$ is tangent to a unique fiber V^{j+1} of $\pi^{\mathbf{t}(\mathbf{c}_j)} \circ F^{\mathbf{c}_j}$ at a unique point $\zeta^{\mathbf{w}_j} \in \operatorname{int} Y^{\mathbf{c}_j}$, and the tangency is quadratic. Still by Definition 1.23 (1) there exists a C^2-function $\mathfrak{v}_{j+1} : I \to \mathbb{R}$ such that $V^{j+1} = \{(\mathfrak{v}_{j+1}(y), y) : y \in I\}$.

When j is large we recall that $|\mathbf{w}_j|$ is large. Still by Proposition 1.28, the word $\mathbf{w}_j$ is made by the last letters of some $\mathbf{u}_j \in \overleftarrow{\mathbf{A}}_{\mathbf{c}_j}$. By the Inclination Lemma (as in the proof of Proposition 1.28), the map $\hbar_j$, defined on $I^{\mathbf{c}_j}$, is C^2-close to $w^{\mathbf{u}_j}$ (also defined on $I^{\mathbf{c}_j}$) and $\zeta^{\mathbf{w}_j}$ is close to $\zeta^{\mathbf{u}_j}$ when j is large. Thus x_j is close to $x_{\mathbf{u}_j}$ (in particular $[x_j - 2r, x_j + 2r] \Subset [x_{\mathbf{u}_j} - 3r, x_{\mathbf{u}_j} + 3r]$), and the maps $\mathfrak{v}_{\mathbf{u}_j}$ and $\mathfrak{v}_{j+1}$ are C^2-close. Then the following map is defined on $I^{\mathbf{c}_j}$ and C^2-close to $\Delta_{\mathbf{u}_j}$ on $[x_j - 2r, x_j + 2r]$:

$$\Delta_j : x \in I^{\mathbf{c}_j} \mapsto F_1^{\mathbf{c}_j}(x, \hbar_j(x)) - \mathfrak{v}_{j+1} \circ F_2^{\mathbf{c}_j}(x, \hbar_j(x)) \ .$$

Thus the map Δ_j has second derivative of absolute value $> 2r$ in $[x_j - 2r, x_j + 2r]$ and has a unique critical point in $(x_j - r/2, x_j + r/2)$ which is equal to x_j by Proposition 1.28. Note that $\Delta_j(x_j) = 0$ is a local extremum of Δ_j. By Lemma 3.20, the maps $\mathfrak{v}_{j+1}$ and $\breve{\mathfrak{v}}_{j+1}$ are both C^2-close and $o(\breve{\gamma}_j^2)$-C^0-close when j is large. Then for j large enough the following function:

$$\breve{\Delta}_j : x \in I^{\mathbf{c}_j} \mapsto (F_1^{\mathbf{c}_j} - \breve{\mathfrak{v}}_{j+1} \circ F_2^{\mathbf{c}_j})(x, \hbar_j(x))$$

has a unique critical point $\breve{x}_j$ in $(x_j - r, x_j + r)$ which is close to x_j. Thus by Proposition 3.11 (2), $\breve{x}_j$ is $\epsilon_{\mathbf{C}}$-distant to $\partial I^{\mathbf{c}_j}$ for every j large. Furthermore, the second derivative of $\breve{\Delta}_j$ has absolute value $> r$ on $[-r + \breve{x}_j, r + \breve{x}_j]$. In particular, the C^0-distance between the extremal values $\Delta_j(x_j)$ and $\breve{\Delta}_j(\breve{x}_j)$ of Δ_j and $\breve{\Delta}_j$ is small compared to $\breve{\gamma}_j^2$. Since $\Delta_j(x_j) = 0$, we then have $\breve{\Delta}_j(\breve{x}_j) = o(\breve{\gamma}_j^2)$. Let us estimate the distance between $\breve{x}_j$ and x_j:

$$|\breve{\Delta}_j(x_j) - \breve{\Delta}_j(\breve{x}_j)| = \left| \int_{\breve{x}_j}^{x_j} D\breve{\Delta}_j(t)dt \right| \geq \left| \int_{\breve{x}_j}^{x_j} \int_{\breve{x}_j}^{t} D^2\breve{\Delta}_j(s)dsdt \right| \geq \frac{r}{2}(x_j - \breve{x}_j)^2 \ .$$

As $|\breve{\Delta}_j(\breve{x}_j) - \Delta_j(x_j)| = o(\breve{\gamma}_j^2)$ and by Lemma 3.20 $|\breve{\Delta}_j(x_j) - \Delta_j(x_j)| = o(\breve{\gamma}_j^2)$, it holds $|\breve{x}_j - x_j| = o(\breve{\gamma}_j)$. □

The third step of the proof of Proposition 3.13 will be to obtain the normal form of Eq. (3.4). It is done below in Eq. (3.7). We recall that by Lemma 3.21, $\breve{x}_j$ is $\epsilon_{\mathbf{C}}$-distant from $\partial I^{\mathbf{c}_j}$. On the other hand, the curve $H^{\mathbf{w}_j}$ intersects $Y^{\mathbf{c}_j}$ since $\mathbf{w}_j \in \mathbf{A}^*_{\mathbf{c}_j}$ and is $\epsilon_{\mathbf{C}}$-distant to $\partial^u Y^{\mathbf{c}_j}$ by Proposition 3.11 (3). Thus it follows:

Fact 3.22. *The function $\varphi_j(x, y) = (\breve{x}_j + x, \hbar_j(\breve{x}_j + x) + y)$ is well defined on $(-\epsilon_{\mathbf{C}}, \epsilon_{\mathbf{C}})^2$ and takes its values in $Y^{\mathbf{c}_j}$.*

Similarly it holds:

Fact 3.23. *If F is analytic, there exists $\tilde{\epsilon}_{\mathbf{C}} > 0$ such that for every j large enough, the map φ_j is well-defined on $\mathbb{D}(\tilde{\epsilon}_{\mathbf{C}})^2$ and satisfies both $\varphi_j(\mathbb{D}(\tilde{\epsilon}_{\mathbf{C}})^2) \subset \tilde{Y}^{\mathbf{c}_j}$ and $F^{\mathbf{c}_j} \circ \varphi_j(\mathbb{D}(\tilde{\epsilon}_{\mathbf{C}})^2) \subset \tilde{Y}^{\mathbf{e}}$.*

Proof. We recall that $\breve{x}_j \in I^{\mathbf{c}_j}$ is $\epsilon_{\mathbf{C}}$-distant from $\partial I^{\mathbf{c}_j}$ and so $\min(\rho, \epsilon_{\mathbf{C}})$-distant to $\partial \tilde{I}$. Since the maps $\hbar_j$ extend to $\tilde{I}$ and are uniformly C^1-bounded on $\tilde{I}$ by Definition 1.5 (2), the maps φ_j are well defined on $\mathbb{D}(\tilde{\epsilon}_{\mathbf{C}})^2$ for $\tilde{\epsilon}_{\mathbf{C}} < \min(\rho, \epsilon_{\mathbf{C}})$ and uniformly C^1-bounded. Thus for $\tilde{\epsilon}_{\mathbf{C}}$ small, the set $\varphi_j(\mathbb{D}(\tilde{\epsilon}_{\mathbf{C}})^2)$ has a uniformly small diameter and has real trace $\varphi_j((-\epsilon_{\mathbf{C}}, \epsilon_{\mathbf{C}})^2) \subset \varphi_j(\mathbb{D}(\tilde{\epsilon}_{\mathbf{C}})^2)$. So to get the two last inequalities, it suffices to use Fact 3.22 and remark that the set $\tilde{Y}^{\mathbf{c}_j} \cap (F^{\mathbf{c}_j})^{-1}(\tilde{Y}^{\mathbf{e}})$ contains a uniformly large complex neighborhood of $Y^{\mathbf{c}_j}$. □

Also by Lemma 3.21 there exist real numbers b_j, q_j and a C^2-function r_j such that for every sufficiently large j and $(x, y) \in (-\epsilon_{\mathbf{C}}, \epsilon_{\mathbf{C}})^2$:

$$(F_1^{\mathbf{c}_j} - \breve{\mathfrak{v}}_{j+1} \circ F_2^{\mathbf{c}_j}) \circ \varphi_j(x, y) = q_j \cdot x^2 + b_j \cdot y + r_j(x, y) \ , \tag{3.7}$$

$$\text{with} \quad 0 = \partial_x r_j(0) = \partial_y r_j(0, 0) = \partial_x^2 r_j(0) \text{ and } r_j(0) = o(\breve{\gamma}_j^2) \ .$$

If F is analytic, the map $\breve{\mathfrak{v}}_{j+1}$ extends holomorphically to $\tilde{I}$ and then, by Fact 3.23, the function r_j extends holomorphically to $\mathbb{D}(\tilde{\epsilon}_{\mathbf{C}})^2$. The next lemma achieves the proof of the proposition. □

Lemma 3.24. *The families $(\varphi_j)_j$ and $(r_j)_j$ are normal in the C^2-topology, and if F is analytic, they are also normal on $\mathbb{D}(\tilde{\epsilon}_{\mathbf{C}})^2$. Moreover, there exists $K \geq 1$ such that for j large enough:*

$$K^{-1} < |q_j| < K \quad \textit{and} \quad K^{-1} < |b_j| < K \, .$$

Proof. By finiteness of $\mathbf{C}$, by compactness of $\overrightarrow{\mathbf{A}}$ and $\overleftarrow{\mathbf{A}}_{\mathbf{c}}$ and by the Inclination Lemma, the following are compact subsets of $C^2(I, I)$:

$$\{w^{\mathbf{s}} : \mathbf{s} \in \overrightarrow{\mathbf{A}}\} \cup \{\mathcal{X}^{\mathbf{w}}(x, \cdot) : \mathbf{w} \in \mathbf{A}^*, \, x \in I\} \, ,$$
$$\{w^{\mathbf{u}} : \mathbf{c} \in \mathbf{C}, \mathbf{u} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}}\} \cup \{\mathcal{Y}^{\mathbf{w}}(\cdot, 0) : \mathbf{c} \in \mathbf{C}, \mathbf{w} \in \mathbf{A}_{\mathbf{c}}^*\} \, .$$

We recall that $\breve{\mathfrak{v}}_{j+1}$ belongs to the first compact set and $\hbar_j$ belongs to the second one. Thus the sequence of C^2-diffeomorphisms $(F^{\mathbf{c}_j} \circ \varphi_j)_j$ is normal and $((x, y) \in I^2 \mapsto (x - \breve{\mathfrak{v}}_{j+1}(y), y))_j$ as well, and so also the family of their compositions. Thus $(r_j)_j$ is normal and $(|b_j|)_j$, $(|q_j|)_j$, $(\|r_j\|_{C^2})_j$ are bounded. If F is analytic we do the same proof using this time the Inclination Lemma in $\mathbb{C}^2$. Note also that:

$$(\det DF^{\mathbf{c}_j}) \circ \varphi_j(0) = \det D(F^{\mathbf{c}_j} \circ \varphi_j)(0) = \det D((F_1^{\mathbf{c}_j} - \breve{\mathfrak{v}}_{j+1} \circ F_2^{\mathbf{c}_j}, F_2^{\mathbf{c}_j}) \circ \varphi_j)(0) = -b_j \cdot \partial_x (F_2^{\mathbf{c}_j} \circ \varphi_j)(0) \ ,$$

where the first inequality follows from the fact that φ_j is conservative and the second equality follows from the fact that the determinant of $(x, y) \mapsto (x - \breve{\mathfrak{v}}_{j+1}(y), y)$ is equal to 1. As $|\det DF^{\mathbf{c}_j}|$ is positive on its compact domain and $(F_2^{\mathbf{c}_j} \circ \varphi_j)_j$ is normal, there exists a positive bound from below for $|b_j|$. Finally the existence of the bound from below for $|q_j|$ follows from the uniform bound r at the end of the proof of Lemma 3.21. □

Remark 3.25. *The set B_j contains both $\breve{\zeta}_j := \Phi_j(0)$ and $\zeta^{\mathbf{w}_j}$ when j is large, since $|\breve{x}_j - x_j|$ is small compared to $\breve{\gamma}_j$ by Lemma 3.21 and thus compared to γ_j.*

This Remark with Lemma 3.24 implying $|q_j/b_j| \leq K^2$ and the normality of $(\varphi_j)_j$ give:

Fact 3.26. *For j large enough, the set B_j contains both $\zeta^{\mathbf{w}_j}$ and $\breve{\zeta}_j = \Phi_j(0)$, and has its diameter dominated by γ_j. Likewise the set $\tilde{B}_j \supset B_j$ has its diameter dominated by γ_j.*

3.3. **Proof of Theorems 1.32 and 1.34.** We start by proving one by one each of the six inclusions and then we will prove the two limits of Theorems 1.32 and 1.34.

Claim 3.27. *For j large, the domain B_j is included in $Y^{\mathbf{c}_j}$ and if F is analytic, the domain $\tilde{B}_j$ is included in $\tilde{Y}^{\mathbf{c}_j}$. Moreover they are uniformly distant to respectively $\partial Y^{\mathbf{c}_j}$ and $\partial \tilde{Y}^{\mathbf{c}_j}$.*

Proof. By Fact 3.26 , the sets $B_j \subset \tilde{B}_j$ contain $\breve{\zeta}_j$ and have small diameters. As the point $\breve{\zeta}_j$ is uniformly distant to $\partial Y^{\mathbf{c}_j}$ and $\partial \tilde{Y}^{\mathbf{c}_j}$ by Proposition 3.13 for its x-coordinate $\breve{x}_j$ and by Proposition 3.11 (3) for its y-coordinate, the claim follows. □

Let us show that $F^{\mathbf{c}_j}$ sends B_j into $Y^{\mathbf{w}_{j+1}}$, and if F is analytic, that $F^{\mathbf{c}_j}$ sends $\tilde{B}_j$ into $\tilde{Y}^{\mathbf{w}_{j+1}}$, as claimed in the second inclusions of the two conclusions of Theorems 1.32 and 1.34. We are going to show even more, that they are respectively included in the following sets:

$$Y_{\mathcal{B}}^{\mathbf{w}_{j+1}} := \{(\mathcal{X}_{j+1}(x,y), y) : (x,y) \in \mathcal{B}\} \quad \text{with } \mathcal{B} := \mathrm{pr}_1(B_{j+1}) \times F_2^{\mathbf{c}_j}(B_j)\,,$$

with pr_1 the first coordinate projection and $F_2^{\mathbf{c}_j}$ the second coordinate of $F^{\mathbf{c}_j}$, and respectively:

$$\tilde{Y}_{\mathcal{B}}^{\mathbf{w}_{j+1}} := \{(\mathcal{Z}_{j+1}(z,w), w) : (z,w) \in \tilde{\mathcal{B}}\} \quad \text{with } \tilde{\mathcal{B}} := \mathrm{pr}_1(\tilde{B}_{j+1}) \times F_2^{\mathbf{c}_j}(\tilde{B}_j)\,.$$

Proposition 3.28. *For every $j \geq 1$ sufficiently large, the map $F^{\mathbf{c}_j}$ sends B_j into $Y_{\mathcal{B}}^{\mathbf{w}_{j+1}} \subset Y^{\mathbf{w}_{j+1}}$. If moreover F is analytic, the map $F^{\mathbf{c}_j}$ sends $\tilde{B}_j$ into $\tilde{Y}_{\mathcal{B}}^{\mathbf{w}_{j+1}} \subset \tilde{Y}^{\mathbf{w}_{j+1}}$.*

Proof. We define $\psi_j(x,y) := (x - \breve{\mathfrak{v}}_{j+1}(y), y)$. To prove this proposition, it suffices to show:

$$\psi_j \circ F^{\mathbf{c}_j}(B_j) \subset \psi_j(Y_{\mathcal{B}}^{\mathbf{w}_{j+1}}) \quad \text{and} \quad \text{resp. } \psi_j \circ F^{\mathbf{c}_j}(\tilde{B}_j) \subset \psi_j(\tilde{Y}_{\mathcal{B}}^{\mathbf{w}_{j+1}}).$$

Thus it suffices to prove the two following inclusions:

$$\psi_j \circ F^{\mathbf{c}_j}(B_j) \quad \subset \quad \left(-\frac{1}{4}|\gamma_{j+1}\sigma_{j+1}|, \frac{1}{4}|\gamma_{j+1}\sigma_{j+1}|\right) \times F_2^{\mathbf{c}_j}(B_j) \quad \subset \quad \psi_j(Y_{\mathcal{B}}^{\mathbf{w}_{j+1}})\,, \tag{3.8}$$

$$\text{resp. } \psi_j \circ F^{\mathbf{c}_j}(\tilde{B}_j) \quad \subset \quad \mathbb{D}\left(\frac{1}{4}|\gamma_{j+1}\sigma_{j+1}|\right) \times F_2^{\mathbf{c}_j}(\tilde{B}_j) \quad \subset \quad \psi_j(\tilde{Y}_{\mathcal{B}}^{\mathbf{w}_{j+1}})\,. \tag{3.9}$$

Proof of the first inclusions of Eq. (3.8) and Eq. (3.9). By Definition 3.17, we have $B_j = \Phi_j((-0.3, 0.3)^2)$. Thus it suffices to show that for every $(X,Y) \in (-0.3, 0.3)^2$ the first coordinate x of the following point is in $\left(-\frac{1}{4}|\gamma_{j+1}\sigma_{j+1}|, \frac{1}{4}|\gamma_{j+1}\sigma_{j+1}|\right)$:

$$(x,y) := \psi_j \circ F^{\mathbf{c}_j} \circ \Phi_j(X,Y) = \psi_j \circ F^{\mathbf{c}_j} \circ \varphi_j(\gamma_j \cdot X, \frac{q_j \cdot \gamma_j^2}{2b_j} \cdot Y)\,. \tag{3.10}$$

Note that $|x|$ is the horizontal distance from $F^{\mathbf{c}_j} \circ \Phi_j(X,Y)$ to $\breve{V}^{j+1}$. By Eq. (3.4) it holds:

$$x = q_j \cdot \gamma_j^2 \cdot (X^2 + \frac{Y}{2} + R_j(X,Y)) \quad \text{with } R_j(X,Y) := \frac{1}{q_j\gamma_j^2} r_j(\gamma_j X, \frac{q_j}{2b_j}\gamma_j^2 Y)\,. \tag{3.11}$$

On the other hand, if F is analytic, we have $\tilde{B}_j = \Phi_j(\mathbb{D}(0.3)^2)$ by Definition 3.17. Thus it suffices to show that for every $(Z,W) \in \mathbb{D}(0.3)^2$ the first coordinate z of the following point is in $\mathbb{D}(\frac{1}{4}|\gamma_{j+1}\sigma_{j+1}|)$:

$$(z,w) := \psi_j \circ F^{\mathbf{c}_j} \circ \Phi_j(Z,W) = \psi_j \circ F^{\mathbf{c}_j} \circ \varphi_j(\gamma_j \cdot Z, \frac{q_j \cdot \gamma_j^2}{2b_j} \cdot W)\,. \tag{3.12}$$

Again $|z|$ is the distance from $F^{\mathbf{c}_j} \circ \Phi_j(Z,W)$ to the curve $\check{V}^{j+1}$. By Eq. (3.7) it holds:

$$z = q_j \cdot \gamma_j^2 \cdot (Z^2 + \frac{W}{2} + R_j(Z,W)) \ . \tag{3.13}$$

To conclude, we need the following, whose proof is done below:

Lemma 3.29. *(i) For j large, the map $(X,Y) \in (-0.3,0.3)^2 \mapsto R_j(X,Y)$ is C^0-small,*
(ii) if moreover F is analytic, the map $(Z,W) \in \mathbb{D}(0.3)^2 \mapsto R_j(Z,W)$ is C^0-small.

Since $|X|^2 + \frac{1}{2}|Y| \leq (0.3)^2 + \frac{1}{2} \cdot 0.3 < 1/4$ with $|q_j \cdot \gamma_j^2| = |\gamma_{j+1} \cdot \sigma_{j+1}|$, the latter lemma implies immediately that for j large enough and $(X,Y) \in (-0.3,0.3)^2$, the number x is in $\left(-\frac{1}{4}|\gamma_{j+1}\sigma_{j+1}|, \frac{1}{4}|\gamma_{j+1}\sigma_{j+1}|\right)$, which concludes the proof of the first inclusion of Eq. (3.8).

If F is analytic, since $|Z|^2 + \frac{1}{2}|W| \leq (0.3)^2 + \frac{1}{2} \cdot 0.3 < 1/4$ for every $(Z,W) \in \mathbb{D}(0.3)^2$, the same argument gives that z is in $\mathbb{D}(|\gamma_{j+1}\sigma_{j+1}|/4)$ and so gives the first inclusion of Eq. (3.9).

Proof of Lemma 3.29. Let us first prove (i). We recall that $R_j(X,Y) := \frac{1}{q_j\gamma_j^2} r_j(\gamma_j X, \frac{q_j}{2b_j}\gamma_j^2 Y)$ for every $(X,Y) \in (-0.3,0.3)^2$. We recall that γ_j is small for j large. Moreover by Eq. (3.5) it holds:

$$0 = \partial_x R_j(0) = \partial_y R_j(0) = \partial_x^2 R_j(0) \ .$$

By Eq. (3.5), $r_j(0)$ is small compared to $\check{\gamma}_j^2$ and thus compared to γ_j^2. Thus $R_j(0)$ is small and so the C^0-norm of R_j is dominated by a small constant plus the C^0-norm of its second derivative. We recall that $(r_j)_j$ is a normal family in the C^2-topology by Lemma 3.24. Let μ be a uniform modulus of continuity of their second derivative. We observe that $(\partial_Y^2 R_j)_j$ and $(\partial_X \partial_Y R_j)_j$ converge uniformly to 0 in the C^0 topology. On the other hand $\partial_X^2 R_j(0) = 0$ and so $\partial_X^2 R_j(X,Y) = \frac{1}{q_j}\partial_x^2 r_j(\gamma_j X, \frac{q_j}{2b_j}\gamma_j^2 Y)$ is bounded by $\frac{1}{|q_j|}\mu(\mathrm{diam}(-0.3,0.3)^2 \cdot K^2 \cdot \gamma_j)$ which converges uniformly to 0 since $\gamma_j \to 0$ and $|q_j|^{-1} \leq K$ (by Lemma 3.24). Thus R_j is C^0-small for j large.

Let us now prove (ii). Let us notice that $(r_j|\mathbb{D}(\tilde{\epsilon}_{\mathbf{C}})^2)_j$ is normal by Lemma 3.24. Then the proof of (ii) is the same as the proof of (i), replacing $(-0.3,0.3)^2$ by $\mathbb{D}(0.3)^2$. □

Proof of the second inclusions of Eq. (3.8) and Eq. (3.9). We will use the following [11]:

Lemma 3.30. *The following map is C^1-close to the first coordinate projection when j is large:*

$$\Upsilon : (x,y) \in \mathcal{B} \mapsto \frac{\mathcal{X}_{j+1}(x,y) - \mathcal{X}_{j+1}(x_{j+1},y)}{\sigma_{j+1}} + x_{j+1} \ .$$

If F is analytic, the following map is C^1-close to the first coordinate projection when j is large:

$$\tilde{\Upsilon} : (z,w) \in \tilde{\mathcal{B}} \mapsto \frac{\mathcal{Z}_{j+1}(z,w) - \mathcal{Z}_{j+1}(x_{j+1},w)}{\sigma_{j+1}} + x_{j+1} \ .$$

Proof. To show this lemma it suffices to use Proposition B.3, whose assumptions must be fulfilled. The first is that when j is large, the diameter of $\mathcal{B}$ must be small and when F is analytic, that the diameter of $\tilde{\mathcal{B}}$ must be small compared to σ_{j+1}. Indeed by Remark 1.33, it holds $\check{\gamma}_j = o(|Y^{\mathbf{w}_{j+1}}|)$ and so $\check{\gamma}_j = o(|\sigma_{j+1}|)$ by Corollary 3.10. Thus by Eq. (3.6), we have $\gamma_j = o(|\sigma_{j+1}|)$. The second assumption in the real case is that the distortion of $(\mathcal{X}_j)_j$ is bounded; this is the case by Theorem 3.9. The second assumption in the complex case is that $\tilde{\mathcal{B}}$ is uniformly distant to the boundary of $\tilde{I}^2$. This is indeed the case since $\tilde{B}_j \ni \zeta^{\mathbf{w}_j}$ has a small diameter by Fact 3.26, while

11. This lemma is a consequence of [Ber18, Lemma 2.30] in the real case.

the first coordinate of $\zeta^{\mathbf{w}_j} \in Y^{\mathbf{c}_j}$ and the second coordinate of $F^{\mathbf{c}_j}(\zeta^{\mathbf{w}_j}) \in F^{\mathbf{c}_j}(Y^{\mathbf{c}_j})$ are uniformly distant from $\partial\tilde{I}$, by Proposition 3.11 (1) and (2). □

Let us now use this lemma to prove the second inclusions of Eq. (3.8) and Eq. (3.9). We notice that it suffices to show:

$$\{\mathcal{X}_{j+1}(x,y) - \breve{\mathfrak{v}}_{j+1}(y) : x \in \mathrm{pr}_1(B_{j+1})\} \supset (-|\gamma_{j+1}\sigma_{j+1}|/4, |\gamma_{j+1}\sigma_{j+1}|/4) \quad \forall y \in \ F_2^{\mathbf{c}_j}(B_j)\,.$$

$$\text{resp. } \{\mathcal{Z}_{j+1}(z,w) - \breve{\mathfrak{v}}_{j+1}(w) : z \in \mathrm{pr}_1(\tilde{B}_{j+1})\} \supset \mathbb{D}(|\gamma_{j+1}\sigma_{j+1}|/4) \quad \forall w \in \ F_2^{\mathbf{c}_j}(\tilde{B}_j)\,.$$

Now we recall that by Eq. (3.2) we have $\breve{\mathfrak{v}}_{j+1}(y) = \mathcal{X}_{j+1}(x_{j+1}, y)$ (resp. by the discussion following Eq. (3.7) we have $\breve{\mathfrak{v}}_{j+1}(w) = \mathcal{Z}_{j+1}(x_{j+1}, w)$). So it suffices to show that:

$$\{\Upsilon(x,y) - x_{j+1} : x \in \mathrm{pr}_1(B_{j+1})\} \supset (-|\gamma_{j+1}|/4, |\gamma_{j+1}|/4) \quad \forall y \in \ F_2^{\mathbf{c}_j}(B_j)\,.$$

$$\text{resp. } \{\tilde{\Upsilon}(z,w) - x_{j+1} : z \in \mathrm{pr}_1(\tilde{B}_{j+1})\} \supset \mathbb{D}(|\gamma_{j+1}|/4) \quad \forall w \in \ F_2^{\mathbf{c}_j}(\tilde{B}_j)\,.$$

Now we notice that $|\breve{x}_{j+1} - x_{j+1}|$ is small compared to $\breve{\gamma}_{j+1}$ by Lemma 3.21 and so compared to γ_{j+1} by Eq. (3.6). As $\mathrm{pr}_1(B_{j+1}) = \breve{x}_{j+1} + (-0.3\cdot\gamma_{j+1}, 0.3\cdot\gamma_{j+1})$ (resp. $\mathrm{pr}_1(\tilde{B}_{j+1}) = \breve{x}_{j+1} + \mathbb{D}(0.3\cdot\gamma_{j+1})$), the result follows from Lemma 3.30 and $1/4 < 0.3$. □

The following (with Proposition 3.28) shows the two remaining inclusions of Theorems 1.32 and 1.34:

Proposition 3.31. *For every $j \geq 1$ large enough, the map $F^{\mathbf{w}_{j+1}}$ sends $Y_{\mathcal{B}}^{\mathbf{w}_{j+1}}$ into B_{j+1}. If moreover F is analytic, the map $F^{\mathbf{w}_{j+1}}$ sends $\tilde{Y}_{\mathcal{B}}^{\mathbf{w}_{j+1}}$ into $\tilde{B}_{j+1}$.*

Proof. Since $Y_{\mathcal{B}}^{\mathbf{w}_{j+1}} = \{(\mathcal{X}_{j+1}(x,y), y) : (x,y) \in \mathcal{B}\}$ and $\tilde{Y}_{\mathcal{B}}^{\mathbf{w}_{j+1}} = \{(\mathcal{Z}_{j+1}(z,w), w) : (z,w) \in \tilde{\mathcal{B}}\}$, by definition of the implicit representation, it holds:

Fact 3.32. *The map $F^{\mathbf{w}_{j+1}}$ sends $Y_{\mathcal{B}}^{\mathbf{w}_{j+1}}$ to $\{(x, \mathcal{Y}_{j+1}(x,y)) : (x,y) \in \mathcal{B}\}$. If moreover F is analytic, the map $F^{\mathbf{w}_{j+1}}$ sends $\tilde{Y}_{\mathcal{B}}^{\mathbf{w}_{j+1}}$ to $\{(z, \mathcal{W}_{j+1}(z,w)) : (z,w) \in \tilde{\mathcal{B}}\}$.*

Thus we shall prove that $\{(x, \mathcal{Y}_{j+1}(x,y)) : (x,y) \in \mathcal{B}\}$ is included in B_{j+1} and respectively that $\{(z, \mathcal{W}_{j+1}(z,w)) : (z,w) \in \tilde{\mathcal{B}}\})$ is included in $\tilde{B}_{j+1}$. By the skew product form of the sets B_{j+1} and $\tilde{B}_{j+1}$ (see Definitions 3.15 and 3.17), it suffices to show that:

$$|\mathcal{Y}_{j+1}(x,y) - \hbar_{j+1}(x)| = |\mathcal{Y}_{j+1}(x,y) - \mathcal{Y}_{j+1}(x,0)|$$
$$\text{and} \quad \text{resp.} \quad |\mathcal{W}_{j+1}(z,w) - \hbar_{j+1}(z)| = |\mathcal{W}_{j+1}(z,w) - \mathcal{W}_{j+1}(z,0)|$$

are small compared to γ_{j+1}^2 for any $(x,y) \in \mathcal{B}$ and resp. $(z,w) \in \tilde{\mathcal{B}}$. They are respectively smaller than $|F^{\mathbf{w}_{j+1}}(Y^{\mathbf{w}_{j+1}})|$ and $|F^{\mathbf{w}_{j+1}}(\tilde{Y}^{\mathbf{w}_{j+1}})|$. By Theorem 1.32 (iii) and Theorem 1.34 $(\widetilde{iii})$, we have $|F^{\mathbf{w}_{j+1}}(Y^{\mathbf{w}_{j+1}})| = o(\breve{\gamma}_{j+1}^2)$, and moreover $|F^{\mathbf{w}_{j+1}}(\tilde{Y}^{\mathbf{w}_{j+1}})| = o(\breve{\gamma}_{j+1}^2)$ if F is analytic. Since $\breve{\gamma}_{j+1}^2 = O(\gamma_{j+1}^2)$ by Eq. (3.6), the result follows. □

We are now in position to prove:

Proof of Theorems 1.32 and 1.34. The six inclusions of Theorems 1.32 and 1.34 are given by Claim 3.27, Proposition 3.28 and then Proposition 3.31. Hence it remains only to show:

(3.14)
$$\lim_{k\to\infty} \mathrm{diam}\, F^k(B_j \times \{\mathbf{o}(\mathbf{c}_j)\}) = 0 \quad \text{and} \quad \lim_{k\to\infty} \mathrm{diam}\, F^k(\tilde{B}_j \times \{\mathbf{o}(\mathbf{c}_j)\}) = 0 \quad \text{if } F \text{ is analytic,}$$

when j is large. To this end, it suffices to show that for every $\eta > 0$, for every j sufficiently large, the $|\mathbf{w}_{j+1}|$ first iterates of $F^{\mathbf{c}_j}(B_j)$ (resp. $F^{\mathbf{c}_j}(\tilde{B}_j)$) have diameters smaller than η. To this end we use:

Lemma 3.33. *When N is large,* $\max_{(\mathbf{p},\mathbf{q})\in\mathbf{B}^*:|\mathbf{p}|,|\mathbf{q}|\geq N} \operatorname{diam} Y^{\mathbf{p}} \cap F^{\mathbf{q}}(Y^{\mathbf{q}})$ *is small.*

Proof. For every $s, t \in I$, let $\mathcal{H}_s$ be the image of $\{y = s\} \cap Y^{\mathbf{q}}$ under $F^{\mathbf{q}}$ and $\mathcal{V}_t$ be the image of $\{x = t\} \cap F^{\mathbf{p}}(Y^{\mathbf{p}})$ under $(F^{\mathbf{p}})^{-1}$. We notice that $\mathcal{H}_s$ and $\mathcal{V}_t$ intersects at a unique point $\phi(s,t) \in Y^{\mathbf{p}} \cap F^{\mathbf{q}}(Y^{\mathbf{q}})$, since by the cone properties of Definition 1.4, $\mathcal{H}_s$ is a graph over $x \in I$ with tangent spaces in χ_h and $\mathcal{V}_t$ is a graph over $y \in I$ with tangent spaces in χ_v. It is straightforward that ϕ sends I^2 onto $Y^{\mathbf{p}} \cap F^{\mathbf{q}}(Y^{\mathbf{q}})$. Also by transversality, ϕ is differentiable. For every $(s,t) \in I^2$, the vector $\partial_t\phi(s,t)$ is tangent to $\mathcal{H}_s$ and so in χ_h and the norm of its first coordinate projection is at most $\lambda^{-|\mathbf{p}|}$ by property (2) of Definition 1.4. Thus the norm of $\partial_t\phi(s,t)$ is at most $\sqrt{1+\theta^2}\cdot\lambda^{-|\mathbf{p}|}$. Likewise the norm of $\partial_s\phi(s,t)$ is at most $\sqrt{1+\theta^2}\cdot\lambda^{-|\mathbf{q}|}$. Thus the norm of $D\phi$ is small when $|\mathbf{p}|$ and $|\mathbf{q}|$ are large and so the diameter of $\phi(I^2) = Y^{\mathbf{p}} \cap F^{\mathbf{q}}(Y^{\mathbf{q}})$ is small. □

By Lemma 3.33, there exists N independent of j such that the kth iterate of $F^{\mathbf{c}_j}(B_j)$ has diameter smaller than η for every $N \leq k \leq |\mathbf{w}_{j+1}| - N$. By Fact 3.26, the diameter of B_j is dominated by γ_j which is small when j is large and likewise for $F^{\mathbf{w}_{j+1}} \circ F^{\mathbf{c}_j}(B_j) \subset B_{j+1}$. Thus their N first iterates and preimages have also small diameter when j is large. This implies the first limit in Eq. (3.14).

If F is analytic, we prove the second limit in Eq. (3.14) exactly the same way, replacing Lemma 3.33 by Lemma 3.34 below. □

Lemma 3.34. *When N is large,* $\max_{(\mathbf{p},\mathbf{q})\in\mathbf{B}^*:|\mathbf{p}|,|\mathbf{q}|\geq N} \operatorname{diam} \tilde{Y}^{\mathbf{p}} \cap F^{\mathbf{q}}(\tilde{Y}^{\mathbf{q}})$ *is small.*

Proof. The proof is exactly the same as for Lemma 3.33, up to changing I to $\tilde{I}$ and Y to $\tilde{Y}$. □

4. Parameter selection

In this Section, we will work with an unfolding $(F_p, \pi_p)_{p\in\mathcal{P}}$ of wild type $(\mathbf{A}, \mathbf{C})$. Under extra assumptions ($\#\mathbf{C} = 5$ and moderate dissipativeness), given any $p_0 \in \operatorname{int}\mathcal{P}$, we will define in Section 4.3 a sequence of parameters $(p_i)_{i\geq 0}$ which converges to a certain p_∞ close to p_0 such that at each p_i, the map F_{p_i} will satisfy tangency conditions. Then we prove Theorem 1.44 in Section 4.4. To this aim, we first show that F_{p_∞} satisfies the assumptions of Theorem 1.32, and those of Theorem 1.34 if F_{p_∞} is analytic, and finally we will show property $(\mathcal{E})$.

4.1. Estimates for parameter families.

We set up here some analytic constants useful for the proof. For every $\mathbf{w} \in \mathbf{A}^*$ and $p \in \mathcal{P}$, let $(\mathcal{X}_p^{\mathbf{w}}, \mathcal{Y}_p^{\mathbf{w}})$ denote the implicit representation of $(Y_p^{\mathbf{w}}, F_p^{\mathbf{w}})$. We recall that by Proposition 1.37, the family $(\mathcal{X}_p^{\mathbf{w}}, \mathcal{Y}_p^{\mathbf{w}})_{p\in\mathcal{P}}$ is of class C^2. Here is a consequence of Theorem 3.9:

Corollary 4.1. *There exists $\overline{\mathfrak{D}} > 0$ such that for all $\mathbf{w} \in \mathbf{A}^*$ and $p \in \mathcal{P}$: $\mathfrak{D}(Y_p^{\mathbf{w}}, F_p^{\mathbf{w}}) \leq \overline{\mathfrak{D}}$.*

Proof. As the domain of each F_p is compact, and the set $\mathcal{P}$ is compact, the C^2-norm of F_p and its inverse are bounded independently of $p \in \mathcal{P}$. Thus for every $\mathbf{a} \in \mathbf{A}$, the distortion of each

hyperbolic transformation $(Y_p^{\mathbf{a}}, F_p^{\mathbf{a}})$ is bounded independently of $p \in \mathcal{P}$. By finiteness of $\mathbf{A}$, these distortions are bounded independently of $p \in \mathcal{P}$ and $\mathbf{a} \in \mathbf{A}$. We conclude using Theorem 3.9. $\square$

An immediate consequence of this bound is:

Corollary 4.2. *For every $\mathbf{w} \in \mathbf{A}^*$, for every $(x,y),(x',y') \in I^2$, it holds for every $p \in \mathcal{P}$:*

$$\left|\frac{\partial_x \mathcal{X}_p^{\mathbf{w}}(x,y)}{\partial_x \mathcal{X}_p^{\mathbf{w}}(x',y')}\right| \leq \exp(\bar{\mathcal{D}} \cdot \|(x-x',y-y')\|) \leq \bar{\mathcal{B}}, \quad \textit{with } \bar{\mathcal{B}} := \exp(2\sqrt{2} \cdot \bar{\mathcal{D}}) < \infty \,.$$

Definition 4.3. *Given a nonempty compact subset $\mathcal{P}_0 \subset \mathcal{P}$, we define the* maximal and minimal widths of $\mathbf{w} \in \mathbf{A}^*$ [among $p \in \mathcal{P}_0$] *by respectively:*

$$\underline{w}(\mathbf{w},\mathcal{P}_0) := \min_{p \in \mathcal{P}_0,\, I^2} |\partial_{x_1} \mathcal{X}_p^{\mathbf{w}}| \quad \textit{and} \quad \overline{w}(\mathbf{w},\mathcal{P}_0) := \max_{p \in \mathcal{P}_0,\, I^2} |\partial_{x_1} \mathcal{X}_p^{\mathbf{w}}| \,.$$

The maximal and minimal vertical heights *of $\mathbf{w}$ are respectively:*

$$\underline{\hbar}(\mathbf{w},\mathcal{P}_0) := \min_{p \in \mathcal{P}_0,\, I^2} |\partial_{y_0} \mathcal{Y}_p^{\mathbf{w}}| \quad \textit{and} \quad \overline{\hbar}(\mathbf{w},\mathcal{P}_0) := \max_{p \in \mathcal{P}_0,\, I^2} |\partial_{y_0} \mathcal{Y}_p^{\mathbf{w}}| \,.$$

Remark 4.4. *For any $p \in \mathcal{P}_0$ and $\mathbf{w} \in \mathbf{A}^*$, we notice that any line $\mathbb{R} \times \{y\}$, $y \in I$, intersects $Y_p^{\mathbf{w}}$ at a segment of length in $[2\underline{w}(\mathbf{w},\mathcal{P}_0), 2\overline{w}(\mathbf{w},\mathcal{P}_0)]$; whereas any line $\{x\} \times \mathbb{R}$, $x \in I$, intersects $F_p^{\mathbf{w}}(Y_p^{\mathbf{w}})$ at a segment of length in $[2\underline{\hbar}(\mathbf{w},\mathcal{P}_0), 2\overline{\hbar}(\mathbf{w},\mathcal{P}_0)]$. In particular, with the notations of Definition 1.27, is holds:*

$$2\underline{w}(\mathbf{w},\mathcal{P}_0) \leq |Y_p^{\mathbf{w}}| \leq 2\overline{w}(\mathbf{w},\mathcal{P}_0) \quad \textit{and} \quad 2\underline{\hbar}(\mathbf{w},\mathcal{P}_0) \leq |F_p^{\mathbf{w}}(Y_p^{\mathbf{w}})| \leq 2\overline{\hbar}(\mathbf{w},\mathcal{P}_0) \,.$$

From now on, we will suppose that $(F_p,\pi_p)_p$ is moderately dissipative and we fix a parameter $p_0 \in \operatorname{int}\mathcal{P}$.

The following will be used many times and proved in Appendix B:

Proposition 4.5 (and definition of η_0 and $\mathcal{P}_0$)**.** *There exists $\eta_0 > 0$ and a compact neighborhood $\mathcal{P}_0 \Subset \mathcal{P}$ of $p_0 \in \mathcal{P}$ satisfying the following properties for every $\mathbf{w} \in \mathbf{A}^*$ s.t. $\overline{w}(\mathbf{w},\mathcal{P}_0) < \eta_0$:*

(1) $\underline{w}(\mathbf{w}) \leq \overline{w}(\mathbf{w}) < \underline{w}(\mathbf{w})^{1-\epsilon^2}$, *with $\underline{w}(\mathbf{w}) := \underline{w}(\mathbf{w},\mathcal{P}_0)$ and $\overline{w}(\mathbf{w}) := \overline{w}(\mathbf{w},\mathcal{P}_0)$,*

(2) $\overline{\hbar}(\mathbf{w})^{\epsilon} \leq \underline{w}(\mathbf{w})$, *with $\overline{\hbar}(\mathbf{w}) := \overline{\hbar}(\mathbf{w},\mathcal{P}_0)$.*

If at some $\tilde{p} \in \mathcal{P}_0$ the map $F_{\tilde{p}}$ is analytic, then for ρ sufficiently small, for every $\mathbf{w} \in \mathbf{A}^*$, the hyperbolic transformation $(Y_{\tilde{p}}^{\mathbf{w}}, F_{\tilde{p}}^{\mathbf{w}})$ extends to a C_ρ^ω-hyperbolic transformation whose implicit representation $(\mathcal{Z}_{\tilde{p}}^{\mathbf{w}}, \mathcal{W}_{\tilde{p}}^{\mathbf{w}})$ satisfies:

(2') $\max_{\tilde{I}^2} |\partial_{w_0} \mathcal{W}_{\tilde{p}}^{\mathbf{w}}|^{\epsilon} \leq \min_{\tilde{I}^2} |\partial_{z_1} \mathcal{Z}_{\tilde{p}}^{\mathbf{w}}|$ *if* $\overline{w}(\mathbf{w},\mathcal{P}_0) < \eta_0$.

We first notice the following monotonicity property for widths and heights:

Lemma 4.6. *For every finite words $\mathbf{w} \in \mathbf{A}^*$ and $\mathbf{w}' \in \mathbf{A}^*$ such that $\mathbf{w} \cdot \mathbf{w}'$ is admissible, it holds:*

$$\overline{w}(\mathbf{w} \cdot \mathbf{w}') \leq \overline{w}(\mathbf{w}) \quad \textit{and} \quad \overline{\hbar}(\mathbf{w} \cdot \mathbf{w}') \leq \overline{\hbar}(\mathbf{w}')$$

Proof. Let us prove that $\overline{w}(\mathbf{w}\cdot\mathbf{w}') \leq \overline{w}(\mathbf{w})$, the proof of $\overline{\hbar}(\mathbf{w}\cdot\mathbf{w}') \leq \overline{\hbar}(\mathbf{w}')$ is similar. The case where $\mathbf{w}$ or $\mathbf{w}'$ is the empty word being trivial, let us suppose that $\min(|\mathbf{w}|,|\mathbf{w}'|) \geq 1$. By Lemma B.1 of the appendix, we have:

$$\overline{w}(\mathbf{w}\cdot\mathbf{w}') \leq \frac{1}{1-\theta^2}\overline{w}(\mathbf{w})\cdot\overline{w}(\mathbf{w}').$$

According to the cone condition (2) of Definition 1.4, we have $|\partial_x \mathcal{X}_p^{\mathbf{w}'}| \leq \lambda^{-|\mathbf{w}'|} \leq \lambda^{-1}$ and so:

$$\overline{w}(\mathbf{w}\cdot\mathbf{w}') \leq \frac{1}{1-\theta^2}\cdot\overline{w}(\mathbf{w})\cdot\lambda^{-|\mathbf{w}'|} < \overline{w}(\mathbf{w}).$$

□

Definition 4.7 ($\mathbf{p}_m$)**.** *For every $m \in \mathbb{N}^*$, for every $\mathbf{s} = (\mathbf{a}_j)_{j\geq 0} \in \overrightarrow{\mathbf{A}}$ and $\mathbf{u} = (\mathbf{a}_j)_{j<0} \in \overleftarrow{\mathbf{A}}$, let $\mathbf{p}_m(\mathbf{s}) := \mathbf{a}_0\cdots\mathbf{a}_{m-1}$ be the concatenation of the m first terms of $\mathbf{s}$ and let $\mathbf{p}_m(\mathbf{u}) = \mathbf{a}_{-m}\cdots\mathbf{a}_{-1}$ be the concatenation of the m last terms of $\mathbf{u}$.*

The following will be used many times.

Lemma 4.8 (and definition of η_1)**.** *There exists $\eta_1 \in (0,\eta_0)$ such that for every $\mathbf{s} \in \overrightarrow{\mathbf{A}}$, $\mathbf{u} \in \overleftarrow{\mathbf{A}}$, for every $\delta \in (0,\eta_1)$, if $m \geq 1$ and $m' \geq 1$ are maximal such that:*

$$\overline{w}(\mathbf{p}_m(\mathbf{s})) > \delta \quad \text{and} \quad \overline{\hbar}(\mathbf{p}_{m'}(\mathbf{u})) > \delta\ ,$$

then for every $\mathbf{d} \in \mathbf{A}^*$ with $\underline{w}(\mathbf{d}) \geq \delta^{\epsilon^4}$ and such that $\mathbf{w} := \mathbf{p}_{m+1}(\mathbf{s})\cdot\mathbf{d}\cdot\mathbf{p}_{m'+1}(\mathbf{u}) \in \mathbf{A}^*$, it holds:

$$\delta \geq \overline{w}(\mathbf{w}) \geq \underline{w}(\mathbf{w}) \geq \delta^{1+\epsilon+2\epsilon^2} \quad \text{and} \quad \delta \geq \overline{\hbar}(\mathbf{w}) \geq \underline{\hbar}(\mathbf{w})\ .$$

Moreover such $\mathbf{d} \in \mathbf{A}^*$ always exist.

Proof. By maximality of m and m', the maximal widths of $\mathbf{p}_m(\mathbf{s})$ and $\mathbf{p}_{m'}(\mathbf{u})$ are small when η_1 is small, and sufficiently to apply Proposition 4.5 to these words. Also by maximality, it holds $\delta \geq \overline{w}(\mathbf{p}_{m+1}(\mathbf{s}))$ and $\delta \geq \overline{\hbar}(\mathbf{p}_{m'+1}(\mathbf{u}))$. Thus by Lemma 4.6 it holds:

$$\delta \geq \overline{w}(\mathbf{w}) \geq \underline{w}(\mathbf{w}) \quad \text{and} \quad \delta \geq \overline{\hbar}(\mathbf{w}) \geq \underline{\hbar}(\mathbf{w})\ .$$

It remains to prove that $\underline{w}(\mathbf{w}) \geq \delta^{1+\epsilon+2\epsilon^2}$. By Lemma B.1 we have:

$$\underline{w}(\mathbf{w}) \geq \underline{w}(\mathbf{p}_m(\mathbf{s}))\cdot\min_{\mathbf{a}}\underline{w}(\mathbf{a})\cdot\underline{w}(\mathbf{d})\cdot\min_{\mathbf{a}}\underline{w}(\mathbf{a})\cdot\underline{w}(\mathbf{p}_{m'}(\mathbf{u}))\cdot\frac{1}{(1+\theta^2)^4}\ .$$

We now use Proposition 4.5 (1) with $\underline{w}(\mathbf{p}_m(\mathbf{s}))^{1-\epsilon^2} > \overline{w}(\mathbf{p}_m(\mathbf{s})) > \delta$ to obtain:

$$\underline{w}(\mathbf{w}) > \delta^{(1-\epsilon^2)^{-1}}\cdot(\min_{\mathbf{a}}\underline{w}(\mathbf{a}))^2\cdot\underline{w}(\mathbf{p}_{m'}(\mathbf{u}))\cdot\underline{w}(\mathbf{d})\cdot\frac{1}{(1+\theta^2)^4}\ .$$

By Proposition 4.5 (2), $\underline{w}(\mathbf{p}_{m'}(\mathbf{u})) \geq \overline{\hbar}(\mathbf{p}_{m'}(\mathbf{u}))^{\epsilon}$. Also $\underline{w}(\mathbf{d}) \geq \delta^{\epsilon^4}$. Thus:

$$\underline{w}(\mathbf{w}) > \delta^{(1-\epsilon^2)^{-1}}\cdot(\min_{\mathbf{a}}\underline{w}(\mathbf{a}))^2\cdot\delta^{\epsilon}(1+\theta^2)^{-4}\cdot\delta^{\epsilon^4}$$

We now use the logarithm to compare this with the sought inequality:

(4.1)

$$\begin{aligned}\log(\underline{w}(\mathbf{w})\delta^{-1-\epsilon-2\epsilon^2}) &> ((1-\epsilon^2)^{-1}+\epsilon+\epsilon^4-1-\epsilon-2\epsilon^2)\log\delta + 2\log\min_{\mathbf{a}}\underline{w}(\mathbf{a}) - 4\log(1+\theta^2)\\ &= (\epsilon^4(1-\epsilon^2)^{-1}+\epsilon^4-\epsilon^2)\log\delta + 2\log\min_{\mathbf{a}}\underline{w}(\mathbf{a}) - 4\log(1+\theta^2)\end{aligned}$$

Note that $(\epsilon^4(1-\epsilon^2)^{-1}+\epsilon^4-\epsilon^2)<0$, hence for $\eta_1>\delta$ sufficiently small, the right most sought inequality is obtained. Also, by connectedness of the graph, when δ is small, given any two vertexes in $\mathbf{V}$, there exists a word $\mathbf{d}$ which joints them and so that $\underline{w}(\mathbf{d})>\delta^{\epsilon^4}$. □

Remark 4.9. *Let $0<\mathscr{W}\leq 1$ and $\mathcal{L}$ be numbers such that every pair of vertexes in $\mathbf{V}$ can be connected by a path $\mathbf{d}^r$ such that $\underline{w}(\mathbf{d}^r)>\mathscr{W}$ and $|\mathbf{d}^r|\leq\mathcal{L}$. Thus, under the setting of Lemma 4.8 fixing m,m', $\mathbf{s}$ and $\mathbf{u}$, for every word $\mathbf{d}^o\in\mathbf{A}^*$ satisfying*

$$\underline{w}(\mathbf{d}^o)>\delta^{\epsilon^4}\cdot\frac{1+\theta^2}{\mathscr{W}}\quad\text{and}\quad\mathbf{o}(\mathbf{d}^o)=\mathbf{t}(\mathbf{p}_{m+1}(\mathbf{s}))\ ,$$

there exists $\mathbf{d}^r\in\mathbf{A}^*$ starting at $\mathbf{t}(\mathbf{d}^o)$, ending at $\mathbf{o}(\mathbf{p}_{m'+1}(\mathbf{u}))$ and such that $\underline{w}(\mathbf{d}^r)>\mathscr{W}$, $|\mathbf{d}^r|\leq\mathcal{L}$. Thus in Lemma 4.8, we can take $\mathbf{d}=\mathbf{d}^o\cdot\mathbf{d}^r$.

The following provides some uniform bounds on the unfolding.

Proposition 4.10 (and definition of ϑ and L_0)**.** *There exist $\vartheta>0$ and $L_0>0$ such that for every $p'\in\mathcal{P}_0$, $\mathbf{m}:=(\mathbf{u_c},\mathbf{s_c})_{\mathbf{c}\in\mathbf{C}}\in\prod_{\mathbf{c}\in\mathbf{C}}\overleftarrow{\mathbf{A}}_{\mathbf{c}}\times\overrightarrow{\mathbf{A}}_{\mathbf{c}}$, the map $\Psi_{\mathbf{m}}:p\in\mathcal{P}\mapsto(\mathcal{V}(\mathbf{u_c},\mathbf{s_c},p))_{\mathbf{c}\in\mathbf{C}}$ is $\frac{L_0}{2}$-biLipschitz from a neighborhood of p' in $\mathcal{P}$ onto the 2ϑ-neighborhood of $\Psi_{\mathbf{m}}(p')$.*

Proof. Let $\alpha>0$ be such that for every $p'\in\mathcal{P}_0$, the euclidean closed ball $B(p',\alpha)$ is included in $\operatorname{int}\mathcal{P}$. Let $\mathbf{m}=(\mathbf{u_c},\mathbf{s_c})_{\mathbf{c}\in\mathbf{C}}\in\prod_{\mathbf{c}\in\mathbf{C}}\overleftarrow{\mathbf{A}}_{\mathbf{c}}\times\overrightarrow{\mathbf{A}}_{\mathbf{c}}$. By $(\mathbf{H}_2)$, the restriction $\Psi_{\mathbf{m}}|B(p',\alpha)$ is a diffeomorphism onto its image. By compactness of $B(p',\alpha)$, the restriction $\Psi_{\mathbf{m}}|B(p',\alpha)$ is biLipschitz; let $L(p',\mathbf{m})/2>0$ be its biLipschitz constant. Also let $\rho(p',\mathbf{m})>0$ be maximal such that $\Psi_{\mathbf{m}}(B(p',\alpha))$ contains the $2\rho(p',\mathbf{m})$-neighborhood of $\Psi_{\mathbf{m}}(p')$. By Proposition 1.41, the functions ρ and L are continuous on the compact set $\mathcal{P}\times\prod_{\mathbf{c}\in\mathbf{C}}\overleftarrow{\mathbf{A}}_{\mathbf{c}}\times\overrightarrow{\mathbf{A}}_{\mathbf{c}}$ and so bounded from below by positive constants ϑ and L_0. □

4.2. **Dynamics at the folding map.** In this subsection, we continue to consider a moderately dissipative unfolding $(F_p,\pi_p)_{p\in\mathcal{P}}$ of wild type $(\mathbf{A},\mathbf{C})$ and a fixed parameter $p_0\in\operatorname{int}\mathcal{P}$. The compact neighborhood $\mathcal{P}_0\Subset\mathcal{P}$ of p_0 was defined by Proposition 4.5, whereas ϑ, L_0 and η_1 were defined in Proposition 4.10 and Lemma 4.8.

To study the folding map $F_p^{\mathbf{c}}|Y_p^{\mathbf{c}}$ for any $\mathbf{c}\in\mathbf{C}$, we will use analogous functions to $\mathcal{V}(\mathbf{u},\mathbf{s},p)=a_p^{\mathbf{u}}-b_p^{\mathbf{s}}$ defined in Definition 1.39, when $\mathbf{u}$ and/or $\mathbf{s}$ are replaced by finite words $\mathbf{w}\in\mathbf{A}^*$. The definitions of the counterparts of $a_p^{\mathbf{u}}$ and $b_p^{\mathbf{s}}$ use the following counterparts of $W_p^{\mathbf{u}}$ or $W_p^{\mathbf{s}}$:

Definition 4.11 ($H_p^{\mathbf{w}}$ and $V_p^{\mathbf{w}}(\zeta)$)**.** *For every $p\in\mathcal{P}_0$, let $H_p^{\mathbf{w}}$ be the push forward by $F_p^{\mathbf{w}}$ of $I\times\{0\}$ and given a point $\zeta\in Y^{\mathbf{e}}$, let $V_p^{\mathbf{w}}(\zeta)$ be the pullback by $F_p^{\mathbf{w}}$ of the $\pi_p^{\mathbf{t}(\mathbf{w})}$-fiber containing ζ:*

$$H_p^{\mathbf{w}}=F_p^{\mathbf{w}}(Y^{\mathbf{w}}\cap I\times\{0\})\quad\text{and}\quad V_p^{\mathbf{w}}(\zeta)=(\pi_p^{\mathbf{t}(\mathbf{w})}\circ F_p^{\mathbf{w}})^{-1}(\{\pi_p^{\mathbf{t}(\mathbf{w})}(\zeta)\})\ .$$

The following is an immediate consequence of Definition 1.23*(3)*:

Fact 4.12. *The curve $V_p^{\mathbf{w}}(\zeta)$ is the fiber of $\pi_p^{\mathbf{o}(\mathbf{w})}$ containing $(F_p^{\mathbf{w}})^{-1}(\zeta)$.*

We now introduce the analogous of Proposition 1.28 for a parameter family.

Proposition 4.13 (and definition of $\mathbf{W_c}$, $\zeta_p^{\mathbf{w}}$ and η_2)**.** *There exists $\eta_2\in(0,\eta_1)$ such that for every $\mathbf{c}\in\mathbf{C}$, $p\in\mathcal{P}_0$ and $\mathbf{w}\in\mathbf{A}^*$ in the set:*

$$\mathbf{W_c}:=\{\mathbf{w}\in\mathbf{A}^*:\mathbf{w}\text{ coincides with the }|\mathbf{w}|\text{ last letters of a certain }\mathbf{u}\in\overleftarrow{\mathbf{A}}_{\mathbf{c}}\quad\text{and}\quad\overline{\hbar}(\mathbf{w})<\eta_2\},$$

the curve $H_p^{\mathbf{w}}$ intersects $Y_p^{\mathbf{c}}$ but not $\partial^u Y_p^{\mathbf{c}}$, the curve $H_p^{\mathbf{w}}$ is tangent to a unique fiber of $\pi_p^{\mathbf{t}(\mathbf{c})} \circ F_p^{\mathbf{c}}$ at a unique point $\zeta_p^{\mathbf{w}} \in \operatorname{int} Y_p^{\mathbf{c}}$, and the tangency is quadratic.

Proof. First observe that when η_2 is small, the minimal length of the words in $\mathbf{W_c}$ is large. Thus, the proof of this Proposition is the same as the one of Proposition 1.28 but for a parameter family this time. Indeed, note that $\overleftarrow{\mathbf{A}}_{\mathbf{c}}$ does not depend on p for every $\mathbf{c} \in \mathbf{C}$ (by Definition 1.35), so it suffices to replace the Inclination Lemma and the condition on $|\mathbf{w}|$ by respectively the parametric Inclination Lemma C.6 Page 74 and the condition on $\overline{\hbar}(\mathbf{w})$. By finiteness of $\mathbf{C}$, one can take the same value η_2 for every $\mathbf{c} \in \mathbf{C}$. □

Remark 4.14. *Up to reducing η_2, we can assume that $\mathbf{A}_{\mathbf{c}}^*(F_p) \supset \mathbf{W_c}$ for any $p \in \mathcal{P}_0$, where $\mathbf{A}_{\mathbf{c}}^*(F_p)$ was defined in Proposition 1.28 as $\mathbf{A}_{\mathbf{c}}^*$.*

Definition 4.15. *For every $\mathbf{c} \in \mathbf{C}$ and $\mathbf{w} \in \mathbf{W_c}$, we denote:*

$$a_p^{\mathbf{w}} := \pi_p^{\mathbf{t}(\mathbf{c})} \circ F_p^{\mathbf{c}}(\zeta_p^{\mathbf{w}}) \in I \quad \text{and} \quad b_p^{\mathbf{w}} := \pi_p^{\mathbf{o}(\mathbf{w})} \circ (F_p^{\mathbf{w}})^{-1}(\zeta_p^{\mathbf{w}}) \ .$$

The following compares $a_p^{\mathbf{w}}$ and $b_p^{\mathbf{w}}$ to the limit case $a_p^{\mathbf{u}}$ and $b_p^{\mathbf{s}}$:

Proposition 4.16. *For every $\mathbf{c} \in \mathbf{C}$, $\mathbf{w} \in \mathbf{W_c}$ and $p \in \mathcal{P}_0$, it holds:*

(1) For every $\mathbf{s} \in \overrightarrow{\mathbf{A}}$ starting with $\mathbf{w}$, it holds $|b_p^{\mathbf{w}} - b_p^{\mathbf{s}}| \leq 2 \cdot \overline{\varpi}(\mathbf{w})$.

(2) For every $\mathbf{u} \in \overleftarrow{\mathbf{A}}$ ending with $\mathbf{w}$, it holds $|a_p^{\mathbf{w}} - a_p^{\mathbf{u}}| \leq \|D(\pi_p^{\mathbf{t}(\mathbf{c})} \circ F_p^{\mathbf{c}})\|_{C^0} \cdot 2 \cdot \overline{\hbar}(\mathbf{w})$.

Proof. We first prove (1). Let $\mathbf{s}' \in \overrightarrow{\mathbf{A}}$ be such that $\mathbf{s} = \mathbf{w} \cdot \mathbf{s}'$. We recall that $\zeta_p^{\mathbf{w}} \in Y_p^{\mathbf{c}}$ for every p. Thus by Definition 1.23 (0) and (1), the $\pi_p^{\mathbf{t}(\mathbf{w})}$-fiber of $\zeta_p^{\mathbf{w}}$ is a curve with tangent space in χ_v, included in $Y^{\mathbf{e}}$ and with endpoint in and endpoints in $I \times \partial I$. These properties also hold true for $W_p^{\mathbf{s}'}$ (by Proposition 1.25 for $W_p^{\mathbf{s}'}$). By Definition 1.4 (3), their respective pullbacks $V_p^{\mathbf{w}}(\zeta_p^{\mathbf{w}})$ and $W_p^{\mathbf{s}}$ by $F_p^{\mathbf{w}}$ are then also graphs over $y \in I$ with direction in χ_v and endpoints in $I \times \partial I$, which are moreover included in $Y_p^{\mathbf{w}}$. They are then $|Y_p^{\mathbf{w}}|$-C^0-close and so $2\overline{\varpi}(\mathbf{w})$-$C^0$-close by Remark 4.4. By Definition 1.23 (3), they are also fibers of $\pi_p^{\mathbf{o}(\mathbf{w})}$. By Definition 1.23 (2), their images by $\pi_p^{\mathbf{o}(\mathbf{w})}$ (equal to $b_p^{\mathbf{w}}$ and $b_p^{\mathbf{s}}$) are the x-coordinates of the intersection points of these fibers with $I \times \{0\}$. This implies $|b_p^{\mathbf{w}} - b_p^{\mathbf{s}}| \leq 2 \cdot \overline{\varpi}(\mathbf{w})$.

Let us prove the second item. Similarly to the case of vertical curves, we can show that $H_p^{\mathbf{w}} \cap Y_p^{\mathbf{c}} \subset H_p^{\mathbf{w}}$ and $W_p^{\mathbf{u}} \cap Y_p^{\mathbf{c}} \subset W_p^{\mathbf{u}}$ are $2\overline{\hbar}(\mathbf{w})$-$C^0$-close. Their respective images by $\pi_p^{\mathbf{t}(\mathbf{c})} \circ F_p^{\mathbf{c}}$ are consequently $\|D(\pi_p^{\mathbf{t}(\mathbf{c})} \circ F_p^{\mathbf{c}})\|_{C^0} \cdot 2\overline{\hbar}(\mathbf{w})$-close. Thus the endpoints of $\pi_p^{\mathbf{t}(\mathbf{c})} \circ F_p^{\mathbf{c}}(H_p^{\mathbf{w}} \cap Y_p^{\mathbf{c}})$ and $\pi_p^{\mathbf{t}(\mathbf{c})} \circ F_p^{\mathbf{c}}(W_p^{\mathbf{u}} \cap Y_p^{\mathbf{c}})$ are $2 \cdot \|D(\pi_p^{\mathbf{t}(\mathbf{c})} \circ F_p^{\mathbf{c}})\|_{C^0} \cdot \overline{\hbar}(\mathbf{w})$-close. We conclude by noting that $a_p^{\mathbf{w}}$ and $a_p^{\mathbf{u}}$ are endpoints of these segments. □

Definition 4.17. *For all $\mathbf{c}, \mathbf{c}' \in \mathbf{C}$, $\mathbf{s} \in \overrightarrow{\mathbf{A}}_{\mathbf{c}}$, $\mathbf{w} \in \mathbf{W_c}$ and $\mathbf{w}' \in \mathbf{W}_{\mathbf{c}'}$ such that $\mathbf{t}(\mathbf{c}) = \mathbf{o}(\mathbf{w}')$, for every $p \in \mathcal{P}_0$, we put:*

$$\mathcal{V}(\mathbf{w}, \mathbf{w}', p) := a_p^{\mathbf{w}} - b_p^{\mathbf{w}'} \quad \text{and} \quad \mathcal{V}(\mathbf{w}, \mathbf{s}, p) := a_p^{\mathbf{w}} - b_p^{\mathbf{s}} \ .$$

Remark 4.18. *We notice that $\mathcal{V}(\mathbf{w}, \mathbf{s}, p) = 0$ if and only if $F_p^{\mathbf{c}}(H_p^{\mathbf{w}} \cap Y_p^{\mathbf{c}})$ is tangent to $W_p^{\mathbf{s}}$. Similarly, $\mathcal{V}(\mathbf{w}, \mathbf{w}', p) = 0$ if and only if $F_p^{\mathbf{c}}(H_p^{\mathbf{w}} \cap Y_p^{\mathbf{c}})$ is tangent to $V_p^{\mathbf{w}'}(\zeta_p^{\mathbf{w}'})$. We notice that the tangency holds at $F_p^{\mathbf{c}}(\zeta_p^{\mathbf{w}})$.*

From now on we work under the assumptions of Theorem 1.44. In other words, the unfolding $(F_p, \pi_p)_p$ of wild type $(\mathbf{A}, \mathbf{C})$ is moderately dissipative and $\#\mathbf{C} = 5$.

Now we are ready to define the counterpart of the function involved in $(\mathbf{H}_2)$. We set:

$$\mathbf{C} =: \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5\}\,.$$

Given $i \in \mathbb{Z}$, let $\mathbf{c}_i := \mathbf{c}_{[i]}$ where $[i]$ is the equivalence class of i in $\mathbb{Z}/5\mathbb{Z}$. In particular $\mathbf{c}_{i+5} = \mathbf{c}_i$ for every $i \in \mathbb{Z}$.

Definition 4.19. *For all $i_0 \geq 0$ and $(\mathbf{w}_i)_{i_0 \leq i \leq i_0+4} \in \prod_{i=i_0}^{i_0+4} \mathbf{W}_{\mathbf{c}_i}$ and $\mathbf{s} \in \overrightarrow{\mathbf{A}}_{\mathbf{c}_{i_0+4}}$ such that $\mathbf{t}(\mathbf{c}_i) = \mathbf{o}(\mathbf{w}_{i+1})$ for every $i_0 \leq i \leq i_0+3$, we put:*

$$\Psi_{(\mathbf{w}_i)_i, \mathbf{s}} : p \in \mathcal{P}_0 \mapsto \Big(\big(\mathcal{V}(\mathbf{w}_i, \mathbf{w}_{i+1}, p)\big)_{i_0 \leq i \leq i_0+3}, \mathcal{V}(\mathbf{w}_{i_0+4}, \mathbf{s}, p) \Big)\,.$$

Here is the counterpart of Proposition 4.10:

Proposition 4.20 (and definition of η_3)**.** *There exists $\eta_3 \in (0, \eta_2)$ such that for every $(\mathbf{w}_i)_{i_0 \leq i \leq i_0+4} \in \prod_{i=i_0}^{i_0+4} \mathbf{W}_{\mathbf{c}_i}$ such that $\mathbf{t}(\mathbf{c}_i) = \mathbf{o}(\mathbf{w}_{i+1})$ for every $i_0 \leq i \leq i_0+3$ and $\mathbf{w}_i$ satisfies $\overline{\varpi}(\mathbf{w}_i) < \eta_3$ for every $i_0 \leq i \leq i_0+4$, the following property holds true.*

For every $p_1 \in \mathcal{P}_0$ and $\mathbf{s}_{i_0+5} \in \overrightarrow{\mathbf{A}}_{\mathbf{c}_{i_0+4}}$, the restriction of $\Psi_{(\mathbf{w}_i)_i, \mathbf{s}_{i_0+5}}$ to a neighborhood of p_1 is a L_0-biLipschitz map onto the ϑ-neighborhood of $\Psi_{(\mathbf{w}_i)_i, \mathbf{s}_{i_0+5}}(p_1)$.

Proof. Let us take for each $i_0 \leq i \leq i_0+4$ a sequence $\mathbf{u}_i \in \overleftarrow{\mathbf{A}}_{\mathbf{c}_i}$ such that the last letters of $\mathbf{u}_i$ are equal to $\mathbf{w}_i$. By Proposition 4.5, we have $\overline{\hbar}(\mathbf{w}_i) < \overline{\hbar}(\mathbf{w}_i)^\epsilon \leq \underline{\varpi}(\mathbf{w}_i) < \eta_3 < \eta_2 < \eta_0$. Let $K \geq 1$ be greater than the supremum of the C^1-norms of $p \in \mathcal{P}_0 \mapsto F_p^{\mathbf{c}}(\zeta_p^{\mathbf{u}})$ among $\mathbf{c} \in \mathbf{C}$ and $\mathbf{u} \in \overleftarrow{\mathbf{A}}_{\mathbf{c}}$. By finiteness of $\mathbf{C}$ and compactness of $\mathcal{P}_0$ and $\overleftarrow{\mathbf{A}}_{\mathbf{c}}$, the constant K is finite. By the parametric Inclination Lemma C.6 Page 74 when η_3 is small, the following pairs of maps are C^1-close:

$$p \in \mathcal{P}_0 \mapsto \zeta_p^{\mathbf{w}_i} \quad \text{and} \quad p \in \mathcal{P}_0 \mapsto \zeta_p^{\mathbf{u}_i} \quad \forall i_0 \leq i \leq i_0+4\,.$$

By Proposition 1.41 and compactness of $\bigcup_{\mathbf{c} \in \mathbf{C}} \overleftarrow{\mathbf{A}}_{\mathbf{c}}$, the maps $p \in \mathcal{P}_0 \mapsto \zeta_p^{\mathbf{u}_i}$ are C^1-uniformly bounded among all tuples $(\mathbf{u}_i)_{i_0 \leq i \leq i_0+4}$ of sequences $\mathbf{u}_i \in \overleftarrow{\mathbf{A}}_{\mathbf{c}_i}$ and so also among all $(\mathbf{u}_i)_{i_0 \leq i \leq i_0+4}$ defined as before. Then the maps $p \in \mathcal{P}_0 \mapsto \zeta_p^{\mathbf{w}_i}$ are C^1-uniformly bounded among all tuples $(\mathbf{w}_i)_i$ satisfying the hypotheses of the proposition. Also by looking at their images by $\pi_p^{\mathbf{t}(\mathbf{c}_i)} \circ F_p^{\mathbf{c}_i}$, the following pairs of maps are C^1-close, when $\eta_3 > 0$ is small:

$$p \in \mathcal{P}_0 \mapsto a_p^{\mathbf{w}_i} \quad \text{and} \quad p \in \mathcal{P}_0 \mapsto a_p^{\mathbf{u}_i} \quad \forall i_0 \leq i \leq i_0+4\,.$$

Let us take for each $i_0 \leq i \leq i_0+4$ a sequence $\mathbf{s}_i \in \overrightarrow{\mathbf{A}}$ such that the first letters of $\mathbf{s}_i$ are equal to $\mathbf{w}_i$. As each map $p \mapsto \zeta_p^{\mathbf{w}_i}$ is uniformly C^1-bounded, the family of the $\pi_p^{\mathbf{t}(\mathbf{w}_i)}$-fibers going through $\zeta_p^{\mathbf{w}_i}$ indexed by $p \in \mathcal{P}_0$ is a C^1-bounded family, uniformly transverse to the unstable lamination of the horseshoe Λ_p. Applying a last time the parametric Inclination Lemma C.6 Page 74, we obtain that its pull back by $(F_p^{\mathbf{w}_i})_p$ is C^1-close to $(W_p^{\mathbf{s}_i})_{p \in \mathcal{P}_0}$, and is equal by definition to $(V_p^{\mathbf{w}_i}(\zeta_p^{\mathbf{w}_i}))_p$. By Fact 4.12, the image by $(\pi_p^{\mathbf{o}(\mathbf{w}_i)})_p$ of this family of curves is $(\pi_p^{\mathbf{o}(\mathbf{w}_i)} \circ (F^{\mathbf{w}_i})^{-1}(\zeta_p^{\mathbf{w}_i}))_p =: (b_p^{\mathbf{w}_i})_p$ which is therefore uniformly C^1-close to $(b_p^{\mathbf{s}_i})_p$. In other words, the following pairs of maps are C^1-close, when $\eta_3 > 0$ is small:

$$p \in \mathcal{P}_0 \mapsto b_p^{\mathbf{w}_i} \quad \text{and} \quad p \in \mathcal{P}_0 \mapsto b_p^{\mathbf{s}_i} \quad \forall i_0 \leq i \leq i_0+4\,.$$

Consequently when $\eta_3 > 0$ is small, the map $\Psi_{(\mathbf{w}_i)_i,\mathbf{s}_{i_0+5}}$ is uniformly C^1-close to $p \in \mathcal{P}_0 \mapsto (\mathcal{V}(\mathbf{u}_i, \mathbf{s}_{i+1}, p))_i$ among all the $(\mathbf{w}_i)_i$. Thus Proposition 4.10 implies the proposition by the inverse local theorem and compactness of the spaces involved. □

4.3. **Infinite chain of nearly heteroclinic tangencies.** In this subsection, we work under the assumptions of Theorem 1.44. We consider a moderately dissipative unfolding $(F_p, \pi_p)_p$ of wild type $(\mathbf{A}, \mathbf{C})$ such that $\mathbf{C}$ is formed by five elements $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5\}$. We fix $p_0 \in \operatorname{int} \mathcal{P}$ as in the previous subsection.

The parameter selection is given by Proposition 4.23 below and proved along this subsection. Its statement involves the following constants which are positive and finite:

$$C_1 := 2\sqrt{5} \cdot L_0 \cdot \sup_{p \in \mathcal{P}_0} (\|D(\pi_p \circ F_p^{\mathbf{C}})\| + 1) \quad \text{and} \quad \eta_4 := \min(L_0 \vartheta / C_1, \eta_3) \ , \tag{4.2}$$

where the compact subset $\mathcal{P}_0 \subset \mathcal{P}$ was defined by Proposition 4.5, the constant η_3 was defined in Proposition 4.20, whereas the constants ϑ and L_0 were defined in Proposition 4.10.

We are going to define a sequence of parameters $(p_i)_{i \geq 1}$ which converges to a certain p_∞ such that each p_i is close to p_{i-1} and at p_i the map F_{p_i} satisfy some tangency's conditions. To this end we will ask that the distance between p_i and p_{i+1} is at most of the order δ^{β^i} for a certain δ small and a fixed $\beta > 1$. Hence the following immediate fact will be useful to ensure that all the p_i remain in $\mathcal{P}_0$ (at which all our bounds were done):

Fact 4.21. *For every $\beta > 1$, there exists $M_\beta > 0$ independent of $1/2 > \delta > 0$ such that $\sum_{j \geq 0} \delta^{\beta^j} \leq M_\beta \cdot \delta$ for every $1/2 > \delta > 0$.*

This fact enables us to fix the following constant:

Definition 4.22 (β and $(\delta_j)_j$)**.** *Let $\beta = 3/2$ and let $1/2 > \delta_0 > 0$ be any number smaller than η_4 such that the ball centered at p_0 with radius $C_1 \cdot M_\beta \cdot \delta_0$ is included in $\mathcal{P}_0$. We put*

$$\delta_j := \delta_0^{\beta^j} \quad \text{with } \beta = \frac{3}{2}. \tag{4.3}$$

We notice that the sequence $(\delta_j)_j$ decreases super exponentially fast to 0. Also it satisfies the following for every $i, m \geq 0$:

$$\delta_{i+m} = (\delta_i)^{\beta^m} \quad \text{and} \quad \sum_{j \geq i} \delta_j \leq M_\beta \cdot \delta_i \ . \tag{4.4}$$

We recall that for every $i \in \mathbb{Z}$, we put $\mathbf{c}_i = \mathbf{c}_{[i]}$, where $[i]$ is the equivalent class of i in $\mathbb{Z}/5\mathbb{Z}$. The following involves the curves $H_p^{\mathbf{w}}$ and $V_p^{\mathbf{w}}(\zeta)$ introduced in Definition 4.11.

Proposition 4.23 (Main)**.** *There exist a sequence of words $(\mathbf{w}_i)_{i \geq 1} \in \prod_{i \geq 1} \mathbf{W}_{\mathbf{c}_i}$ and a sequence of parameters $(p_i)_{i \geq 1} \in \mathcal{P}_0^{\mathbb{N}}$, such that it holds $\mathbf{t}(\mathbf{c}_i) = \mathbf{o}(\mathbf{w}_{i+1})$ and the following for any $i \geq 1$:*

($\mathbf{C}_1$) $\delta_i^{1+\epsilon+2\epsilon^2} \leq \underline{w}(\mathbf{w}_i) \leq \overline{w}(\mathbf{w}_i) \leq \delta_i$,

($\mathbf{C}_2$) *the map $F_{p_i}^{\mathbf{c}_i}$ sends $H_{p_i}^{\mathbf{w}_i} \cap Y_{p_i}^{\mathbf{c}_i}$ tangent to $V_{p_i}^{\mathbf{w}_{i+1}}(\zeta_{p_i}^{\mathbf{w}_{i+1}})$ at the point $F_{p_i}^{\mathbf{c}_i}(\zeta_{p_i}^{\mathbf{w}_i})$,*

($\mathbf{C}_3$) $\|p_0 - p_1\| < C_1 \delta_1$ *and* $\|p_i - p_{i-1}\| < C_1 \delta_{i+4}$ *whenever* $i \geq 2$.

We will prove this proposition below. From this we deduce the following result involving the constants L_0, M_β and C_1 defined respectively in Proposition 4.10, Fact 4.21 and Eq. (4.2) Page 59:

Corollary 4.24. *There exist $p_\infty \in \mathcal{P}_0$ which is $C_1 M_\beta \delta_0$-close to p_0 and a sequence of words $(\mathbf{w}_i)_{i\geq 1} \in \prod_{i\geq 1} \mathbf{W}_{\mathbf{c}_i}$ such that for any $i \geq 1$ it holds $\mathbf{t}(\mathbf{c}_i) = \mathbf{o}(\mathbf{w}_{i+1})$ and:*

($\mathbf{C}_1$) $\quad \delta_i^{1+\epsilon+2\epsilon^2} \leq \underline{w}(\mathbf{w}_i) \leq \overline{w}(\mathbf{w}_i) \leq \delta_i,$

($\mathbf{C}_2'$) $\quad |\mathcal{V}(\mathbf{w}_i, \mathbf{w}_{i+1}, p_\infty)| \leq C_1 L_0 M_\beta \delta_{i+5}.$

Proof. We take the sequence of words $(\mathbf{w}_i)_{i\geq 1}$ and the sequence of parameters $(p_i)_{i\geq 1}$ given by Proposition 4.23. In particular, they satisfy ($\mathbf{C}_1$). By Conclusion ($\mathbf{C}_3$) of Proposition 4.23, we have $\|p_{i+1} - p_i\| < C_1 \delta_{i+5}$ whenever $i \geq 1$ with $(\delta_i)_{i\geq 1}$ decreasing super exponentially fast to 0. Thus the sequence $(p_i)_{i\geq 1}$ converges to a parameter $p_\infty \in \mathbb{R}^5$. By Eq. (4.4), it holds:

$$\|p_0 - p_\infty\| \leq \|p_1 - p_0\| + \sum_{i\geq 1} \|p_{i+1} - p_i\| < C_1\delta_1 + C_1 \sum_{i\geq 1} \delta_{i+5} \leq C_1 M_\beta \delta_0 \ .$$

By Definition 4.22, this implies that p_∞ is in $\mathcal{P}_0$. Still using Conclusion ($\mathbf{C}_3$) of Proposition 4.23 and then Eq. (4.4), we have $\|p_i - p_\infty\| \leq C_1 M_\beta \delta_{i+5}$ for $i \geq 1$. We notice that $\mathcal{V}(\mathbf{w}_i, \mathbf{w}_{i+1}, p_i) = 0$ by Conclusion ($\mathbf{C}_2$) of Proposition 4.23. Then ($\mathbf{C}_2'$) follows from Proposition 4.20. □

In Section 4.4, we will show that F_{p_∞} displays a stable wandering domain by showing that the conclusions of Corollary 4.24 imply the assumptions of Theorem 1.32. Proposition 4.23 will be proved by induction. The first step will be implied by the following:

Lemma 4.25. *There exist $(\mathbf{w}_i)_{1\leq i\leq 5} \in \prod_{i=1}^5 \mathbf{W}_{\mathbf{c}_i}$ with $\mathbf{t}(\mathbf{c}_i) = \mathbf{o}(\mathbf{w}_{i+1})$ for any $1 \leq i \leq 4$, $\mathbf{s}_6 \in \overrightarrow{\mathbf{A}}_{\mathbf{c}_5}$ and a parameter $p_1 \in \mathcal{P}_0$ which is $C_1\delta_1$-close to p_0 such that:*

($\mathbf{C}_1$) *For every $1 \leq i \leq 5$, it holds $\delta_i^{1+\epsilon+2\epsilon^2} \leq \underline{w}(\mathbf{w}_i) \leq \overline{w}(\mathbf{w}_i) \leq \delta_i$.*

($\mathbf{C}_2''$) *For any $1 \leq i \leq 4$, it holds $\mathcal{V}(\mathbf{w}_i, \mathbf{w}_{i+1}, p_1) = 0$ and $\mathcal{V}(\mathbf{w}_5, \mathbf{s}_6, p_1) = 0$.*

Proof. By ($\mathbf{H_1}$) of Definition 1.42, there exist

$$\mathbf{u}_1 \in \overleftarrow{\mathbf{A}}_{\mathbf{c}_1},\ \mathbf{u}_2 \in \overleftarrow{\mathbf{A}}_{\mathbf{c}_2},\ \mathbf{u}_3 \in \overleftarrow{\mathbf{A}}_{\mathbf{c}_3},\ \mathbf{u}_4 \in \overleftarrow{\mathbf{A}}_{\mathbf{c}_4},\ \mathbf{u}_5 \in \overleftarrow{\mathbf{A}}_{\mathbf{c}_5},$$
$$\text{and} \quad \mathbf{s}_2 \in \overrightarrow{\mathbf{A}}_{\mathbf{c}_1}, \mathbf{s}_3 \in \overrightarrow{\mathbf{A}}_{\mathbf{c}_2}, \mathbf{s}_4 \in \overrightarrow{\mathbf{A}}_{\mathbf{c}_3}, \mathbf{s}_5 \in \overrightarrow{\mathbf{A}}_{\mathbf{c}_4}, \mathbf{s}_6 \in \overrightarrow{\mathbf{A}}_{\mathbf{c}_5}$$

such that:

(T') for every $1 \leq j \leq 5$, the curve $F_{p_0}^{\mathbf{c}_j}(W_{p_0}^{\mathbf{u}_j} \cap Y_{p_0}^{\mathbf{c}_j})$ is tangent to $W_{p_0}^{\mathbf{s}_{j+1}}$.

We also pick any $\mathbf{s}_1 \in \overrightarrow{\mathbf{A}}_{\mathbf{c}_0}$. For every $1 \leq i \leq 5$, let $m_i \geq 1$ and $m_i' \geq 1$ be maximal such that:

$$\overline{w}(\mathbf{p}_{m_i}(\mathbf{s}_i)) > \delta_i \quad \text{and} \quad \overline{\hbar}(\mathbf{p}_{m_i'}(\mathbf{u}_i)) > \delta_i.$$

As $\delta_i < \eta_4 < \eta_1$, by Lemma 4.8, for $1 \leq i \leq 5$, there exists $\mathbf{d}_i \in \mathbf{A}^*$ such that $\mathbf{w}_i := \mathbf{p}_{1+m_i}(\mathbf{s}_i) \cdot \mathbf{d}_i \cdot \mathbf{p}_{1+m_i'}(\mathbf{u}_i) \in \mathbf{A}^*$ is well-defined and satisfies:

$$\delta_i \geq \overline{\hbar}(\mathbf{w}_i) \geq \underline{\hbar}(\mathbf{w}_i) \quad \text{and} \quad \delta_i \geq \overline{w}(\mathbf{w}_i) \geq \underline{w}(\mathbf{w}_i) \geq \delta_i^{1+\epsilon+2\epsilon^2} \ . \tag{4.5}$$

Moreover $\mathbf{w}_i \in \mathbf{W}_{\mathbf{c}_i}$ since $\mathbf{u}_i \in \overleftarrow{\mathbf{A}}_{\mathbf{c}_i}$ and since $\overline{\hbar}(\mathbf{w}_i) \leq \delta_i < \eta_4 < \eta_2$. We remark that $\mathbf{t}(\mathbf{c}_i) = \mathbf{o}(\mathbf{s}_{i+1}) = \mathbf{o}(\mathbf{w}_{i+1})$. This proves ($\mathbf{C}_1$).

Let us prove ($\mathbf{C}_2''$). In Definition 4.19, we introduced:

$$\Psi_{(\mathbf{w}_i)_i, \mathbf{s}_6} : p \in \mathcal{P}_0 \mapsto \Big(\big(\mathcal{V}(\mathbf{w}_i, \mathbf{w}_{i+1}, p)\big)_{1\leq i\leq 4}, \mathcal{V}(\mathbf{w}_5, \mathbf{s}_6, p) \Big) \ .$$

By Proposition 4.16, the latter function is $2\sqrt{5}\cdot\sup_{p\in\mathcal{P}_0}(\|D(\pi_p\circ F_p^{\mathbf{c}})\|+1)\cdot\delta_1 = C_1\delta_1/L_0$-$C^0$-close to the following:

$$\Psi_{(\mathbf{u}_i,\mathbf{s}_{i+1})_i} : p\in\mathcal{P}_0 \mapsto (\mathcal{V}(\mathbf{u}_i,\mathbf{s}_{i+1},p))_{1\le i\le 5}$$

where $\mathcal{V}(\mathbf{u}_i,\mathbf{s}_j,p)$ was defined in Definition 1.39. By definition of $(\mathbf{u}_i,\mathbf{s}_{i+1})_i$, this function vanishes at $p=p_0$. Thus it holds:

$$\|\Psi_{(\mathbf{w}_i)_i,\mathbf{s}_6}(p_0)\| \le C_1\delta_1/L_0\ .$$

By Proposition 4.20, a restriction of $\Psi_{(\mathbf{w}_i)_i,\mathbf{s}_6}$ to a neighborhood of p_0 is L_0-biLipschitz onto the ϑ-ball about $\Psi_{(\mathbf{w}_i)_i,\mathbf{s}_6}(p_0)$. As $C_1\delta_1/L_0 \le C_1\eta_4/L_0 \le \vartheta$, this ball contains 0. Its preimage p_1 is $C_1\delta_1$-small. Thus Conclusion $(\mathbf{C}_2'')$ holds true. □

Here is the proposition which will give the step $i_0\to i_0+1$ of the aforementioned induction.

Lemma 4.26. *If $p_{i_0}\in\mathcal{P}_0$, $(\mathbf{w}_i)_{i_0\le i\le i_0+4}\in\prod_{i=i_0}^{i_0+4}\mathbf{W}_{\mathbf{c}_i}$ with $\mathbf{t}(\mathbf{c}_i)=\mathbf{o}(\mathbf{w}_{i+1})$ when $i\le i_0+3$ and $\mathbf{s}_{i_0+5}\in\overrightarrow{\mathbf{A}}_{\mathbf{c}_{i_0+4}}$ satisfy:*

(h_1) $\overline{\varpi}(\mathbf{w}_i)$ is smaller than η_4 for any $i_0\le i\le i_0+4$.

(h_2) For every $i_0\le i\le i_0+3$, it holds $\mathcal{V}(\mathbf{w}_i,\mathbf{w}_{i+1},p_{i_0})=0$ and $\mathcal{V}(\mathbf{w}_{i_0+4},\mathbf{s}_{i_0+5},p_{i_0})=0$.

Then there exist $\mathbf{w}_{i_0+5}\in\mathbf{W}_{\mathbf{c}_{i_0+5}}$, $\mathbf{s}_{i_0+6}\in\overrightarrow{\mathbf{A}}_{\mathbf{c}_{i_0+5}}$ with $\mathbf{t}(\mathbf{c}_{i_0+4})=\mathbf{o}(\mathbf{w}_{i_0+5})$ and a parameter $p_{i_0+1}\in\mathcal{P}_0$ which is $C_1\delta_{i_0+5}$-close to p_{i_0} such that:

$(\mathbf{C}_1)$ $\delta_{i_0+5}^{1+\epsilon+2\epsilon^2}\le\underline{\varpi}(\mathbf{w}_{i_0+5})\le\overline{\varpi}(\mathbf{w}_{i_0+5})\le\delta_{i_0+5}$.

$(\mathbf{C}_2'')$ It holds $\mathcal{V}(\mathbf{w}_i,\mathbf{w}_{i+1},p_{i_0+1})=0$ for every $i_0+1\le i\le i_0+4$ and $\mathcal{V}(\mathbf{w}_{i_0+5},\mathbf{s}_{i_0+6},p_{i_0+1})=0$.

Proof. The proof is similar to the one of Lemma 4.25. By $(\mathbf{H_1})$ of Definition 1.42, there exist $\mathbf{u}_{i_0+5}\in\overleftarrow{\mathbf{A}}_{\mathbf{c}_{i_0+5}}$ and $\mathbf{s}_{i_0+6}\in\overrightarrow{\mathbf{A}}_{\mathbf{c}_{i_0+5}}$ such that the curve $W_{p_{i_0}}^{\mathbf{u}_{i_0+5}}\cap Y_{p_{i_0}}^{\mathbf{c}_{i_0+5}}$ is sent tangent to $W_{p_{i_0}}^{\mathbf{s}_{i_0+6}}$ by $F_{p_{i_0}}^{\mathbf{c}_{i_0+5}}$. Let $m_{i_0+5}\ge 1$ and $m'_{i_0+5}\ge 1$ be maximal such that:

$$\overline{\varpi}(\mathbf{p}_{m_{i_0+5}}(\mathbf{s}_{i_0+5}))>\delta_{i_0+5}\quad\text{and}\quad\overline{\hbar}(\mathbf{p}_{m'_{i_0+5}}(\mathbf{u}_{i_0+5}))>\delta_{i_0+5}.$$

As $\delta_{i_0+5}<\eta_4<\eta_1$, by Lemma 4.8 there exists $\mathbf{d}_{i_0+5}\in\mathbf{A}^*$ such that the word $\mathbf{w}_{i_0+5}:=\mathbf{p}_{1+m_{i_0+5}}(\mathbf{s}_{i_0+5})\cdot\mathbf{d}_{i_0+5}\cdot\mathbf{p}_{1+m'_{i_0+5}}(\mathbf{u}_{i_0+5})\in\mathbf{A}^*$ is well-defined and satisfies $\mathbf{t}(\mathbf{c}_{i_0+4})=\mathbf{o}(\mathbf{s}_{i_0+5})=\mathbf{o}(\mathbf{w}_{i_0+5})$ with:

$$\delta_{i_0+5}\ge\overline{\hbar}(\mathbf{w}_{i_0+5})\ge\underline{\hbar}(\mathbf{w}_{i_0+5})\quad\text{and}\quad\delta_{i_0+5}\ge\overline{\varpi}(\mathbf{w}_{i_0+5})\ge\underline{\varpi}(\mathbf{w}_{i_0+5})\ge\delta_{i_0+5}^{1+\epsilon+2\epsilon^2}\ . \tag{4.6}$$

Moreover $\mathbf{w}_{i_0+5}\in\mathbf{W}_{\mathbf{c}_{i_0+5}}$ since $\mathbf{u}_{i_0+5}\in\overleftarrow{\mathbf{A}}_{\mathbf{c}_{i_0+5}}$ and since $\overline{\hbar}(\mathbf{w}_{i_0+5})\le\delta_{i_0+5}<\eta_4<\eta_2$. This proves $(\mathbf{C}_1)$. Let us prove $(\mathbf{C}_2'')$. We recall:

$$\Psi_{(\mathbf{w}_i)_i,\mathbf{s}_{i_0+6}} : p\in\mathcal{P}_0\mapsto\Big(\big(\mathcal{V}(\mathbf{w}_i,\mathbf{w}_{i+1},p)\big)_{i_0+1\le i\le i_0+4},\mathcal{V}(\mathbf{w}_{i_0+5},\mathbf{s}_{i_0+6},p)\Big)\ .$$

By Proposition 4.16, the latter function is $C_1\delta_{i_0+5}/L_0$-C^0-close to the following:

$$\mathring{\Psi} : p\in\mathcal{P}_0\mapsto\Big(\big(\mathcal{V}(\mathbf{w}_i,\mathbf{w}_{i+1},p)\big)_{i_0+1\le i\le i_0+3},\mathcal{V}(\mathbf{w}_{i_0+4},\mathbf{s}_{i_0+5},p),\mathcal{V}(\mathbf{u}_{i_0+5},\mathbf{s}_{i_0+6},p)\Big)\ .$$

By (h_2) and definition of $(u_i,\mathbf{s}_{i+1})_i$, this function vanishes at $p=p_{i_0}$. Thus it holds:

$$\|\Psi_{(\mathbf{w}_i)_i,\mathbf{s}_{i_0+6}}(p_{i_0})\|\le C_1\delta_{i_0+5}/L_0\ .$$

By Proposition 4.20, a restriction of $\Psi_{(\mathbf{w}_i)_i,\mathbf{s}_{i_0+6}}$ to a neighborhood of p_{i_0} is L_0-biLipschitz onto the ϑ-ball about $\Psi_{(\mathbf{w}_i)_i,\mathbf{s}_{i_0+6}}(p_1)$. As $C_1\delta_{i_0+5}/L_0 \leq C_1\eta_4/L_0 \leq \vartheta$, this ball contains 0. Its preimage p_{i_0+1} is $C_1\delta_{i_0+5}$-close to p_{i_0}. Thus conclusion $(\mathbf{C}_2'')$ holds true. □

We are now ready to:

Proof of Proposition 4.23. We prove by induction on $i_0 \geq 1$ the existence of parameters $(p_i)_{1\leq i\leq i_0}$, words $(\mathbf{w}_i)_{i\leq i_0+4} \in \prod_{i=1}^{i_0+4} \mathbf{W}_{\mathbf{c}_i}$ with $\mathbf{t}(\mathbf{c}_i) = \mathbf{o}(\mathbf{w}_{i+1})$ for $i \leq i_0+3$ and of $\mathbf{s}_{i_0+5} \in \overrightarrow{\mathbf{A}}_{\mathbf{c}_{i_0+4}}$ satisfying:

$(\mathbf{C}_1)$ $\delta_i^{1+\epsilon+2\epsilon^2} \leq \underline{w}(\mathbf{w}_i) \leq \overline{w}(\mathbf{w}_i) \leq \delta_i$ for every $i \leq i_0+4$.

$(\mathbf{C}_2'')$ $\mathcal{V}(\mathbf{w}_i, \mathbf{w}_{i+1}, p_{i_0}) = 0$, for every $i_0 \leq i \leq i_0+3$ and $\mathcal{V}(\mathbf{w}_{i_0+4}, \mathbf{s}_{i_0+5}, p_{i_0}) = 0$.

$(\mathbf{C}_3)$ The parameters p_{i_0-1} and p_{i_0} are $< C_1\delta_{i_0+4}$ distant whenever $i_0 \geq 2$. The parameters p_0 and p_1 are $C_1\delta_1$ distant.

Observe that by Remark 4.18, this statement implies Proposition 4.23.

The proof of this statement is done by induction on $i_0 \geq 1$ using Lemmas 4.25 and 4.26. We notice that step $i_0 = 1$ is an immediate consequence of Lemma 4.25. Let $i_0 \geq 1$ and assume $(\mathbf{w}_i)_{i\leq i_0+4} \in (\mathbf{A}^*)^{i_0+4}$ constructed such that $(\mathbf{C}_1 - \mathbf{C}_2'' - \mathbf{C}_3)$ are satisfied at every $i \leq i_0$. By Fact 4.21, the distance between the parameter p_{i_0} and p_0 is at most $C_1 \sum_{i=1}^{i_0+4} \delta_i < C_1 \cdot M_\beta \cdot \delta_0$, and so by Definition 4.22 the parameter p_{i_0} belongs to $\mathcal{P}_0$. Thus we can use Lemma 4.26 which implies $(\mathbf{C}_1 - \mathbf{C}_2'' - \mathbf{C}_3)$ at step i_0+1. □

Here is a consequence of the above proof which will be useful to obtain Conclusions $(\mathcal{E})$ of Theorem 1.44. We recall that $(\mathbf{A} \sqcup \mathbf{C})^*$ denote the set of words $\mathbf{g} = \mathbf{a}_1 \cdots \mathbf{a}_m$ formed by letters $\mathbf{a}_i \in \mathbf{A} \sqcup \mathbf{C}$ such that $\mathbf{t}(\mathbf{a}_i) = \mathbf{o}(\mathbf{a}_{i+1})$ for every $i < m$.

Corollary 4.27. *Under the assumptions of Corollary 4.24, there exists $\nu > 0$ such that given any function $\mathsf{Cod} : (\mathbf{A} \sqcup \mathbf{C})^* \mapsto \mathbf{A}^*$ satisfying*

$$|\mathsf{Cod}(\mathbf{w})| \leq \nu|\mathbf{w}| \quad \textit{and} \quad \mathbf{t}(\mathbf{w}) = \mathbf{o}(\mathsf{Cod}(\mathbf{w})) \quad \forall \mathbf{w} \in (\mathbf{A} \sqcup \mathbf{C})^*,$$

then Conclusions $(\mathbf{C}_1)$ and $(\mathbf{C}_2')$ of Corollary 4.24 hold true with words $(\mathbf{w}_i)_i$ satisfying:

$(\mathbf{C}_0)$ *the sequence $\mathbf{f} := \mathbf{w}_1 \cdot \mathbf{c}_1 \cdot \mathbf{w}_2 \cdot \mathbf{c}_2 \cdots \mathbf{w}_i \cdot \mathbf{c}_i \cdots$ has infinitely many $m \geq m'$ such that:*

$$\mathbf{p}_m(\mathbf{f}) = \mathbf{p}_{m'}(\mathbf{f}) \cdot \mathsf{Cod}(\mathbf{p}_{m'}(\mathbf{f})) \ .$$

Proof. In the proof of Lemma 4.26, the word $\mathbf{w}_{i_0+5}$ is chosen in function of $(\mathbf{u}_{i_0+5}, \mathbf{s}_{i_0+5})$ and of the form:

$$\mathbf{w}_{i_0+5} = \mathbf{p}_{1+m_{i_0+5}}(\mathbf{s}_{i_0+5}) \cdot \mathbf{d}_{i_0+5} \cdot \mathbf{p}_{1+m'_{i_0+5}}(\mathbf{u}_{i_0+5}).$$

By using Remark 4.9 instead of Lemma 4.8, the word $\mathbf{d}_{i_0+5}$ can be chosen of the form

$$\mathbf{d}_{i_0+5} = \mathbf{d}^o_{i_0+5} \cdot \mathbf{d}^r_{i_0+5},$$

for any word $\mathbf{d}^o_{i_0+5}$ in $\mathbf{A}^*$ such that:

$$\underline{w}(\mathbf{d}^o_{i_0+5}) > \delta_{i_0+5}^{\epsilon^4} \cdot \frac{1+\theta^2}{\mathscr{W}} \quad \text{and} \quad \mathbf{o}(\mathbf{d}^o_{i_0+5}) = \mathbf{t}(\mathbf{p}_{1+m_{i_0+5}}(\mathbf{s}_{i_0+5})) \ , \tag{4.7}$$

and where $\mathbf{d}^r_{i_0+5}$ is a word in $\mathbf{A}^*$ such that $\underline{w}(\mathbf{d}^r_{i_0+5}) > \mathscr{W}$ and such that $\mathbf{w}_{i_0+5}$ is in $\mathbf{A}^*$. By definition of $\mathscr{W}$, the word $\mathbf{d}^r_{i_0+5}$ always exists whatever are $\mathbf{d}^o_{i_0+5}$ and $\mathbf{p}_{1+m'_{i_0+5}}(\mathbf{u}_{i_0+5})$. Now we

recall that $\mathbf{p}_{1+m_{i_0+5}}(\mathbf{s}_{i_0+5})$ depends only on p_{i_0} and i_0 which do not depend on $\mathbf{d}^o_{i_0+5}$. So really $\mathbf{d}^o_{i_0+5}$ can be any word satisfying Eq. (4.7). For $j \geq 1$, let

$$\mathbf{g}_j := \mathbf{w}_1 \cdot \mathbf{c}_1 \cdot \mathbf{w}_2 \cdots \mathbf{w}_{j-1} \cdot \mathbf{c}_{j-1} \cdot \mathbf{p}_{1+m_j}(\mathbf{s}_j)$$

With $j = i_0 + 5$, Eq. (4.7), is equivalent to:

$$\log \underline{w}(\mathbf{d}^o_j) > \epsilon^4 \log \delta_j + \log((1+\theta^2)/\mathscr{W}) \quad \text{and} \quad \mathbf{o}(\mathbf{d}^o_j) = \mathbf{t}(\mathbf{p}_{1+m_j}(\mathbf{s}_j)) \ , \tag{4.8}$$

Put:

$$\mu := -\inf_{\mathbf{d} \in \mathbf{A}^*} \frac{\log \underline{w}(\mathbf{d})}{|\mathbf{d}|}$$

We observe that $0 < \mu < \infty$ since $\mathbf{A}$ is finite. Then Eq. (4.8) is implied by:

$$|\mathbf{d}^o_j| < -\frac{\epsilon^4}{\mu} \log \delta_j - \frac{\log((1+\theta^2)/\mathscr{W})}{\mu} \quad \text{and} \quad \mathbf{o}(\mathbf{d}^o_j) = \mathbf{t}(\mathbf{p}_{1+m_j}(\mathbf{s}_j)) \ , \tag{4.9}$$

Now let us replace the latter inequality by a stronger one involving $|\mathbf{g}_j|$ instead of δ_j. It holds:

$$|\mathbf{g}_j| = j + |\mathbf{w}_1| + \cdots + |\mathbf{w}_{j-1}| + m_j \leq 2(|\mathbf{w}_1| + \cdots + |\mathbf{w}_j|) \ .$$

For every i we have by $(\mathbf{C}_1)$:

$$\delta_i^{1+\epsilon+2\epsilon^2} \leq \overline{w}(\mathbf{w}_i) \leq \lambda^{-|\mathbf{w}_i|} \Rightarrow |\mathbf{w}_i| \log \lambda \leq -(1+\epsilon+2\epsilon^2) \log \delta_i = -(1+\epsilon+2\epsilon^2)\beta^{i-j} \log \delta_j \ .$$

Thus:

$$|\mathbf{g}_j| \leq -2(1+\epsilon+2\epsilon^2) \sum_{i=1}^{j} \beta^{i-j} \log_\lambda \delta_j \leq -2(1+\epsilon+2\epsilon^2) \frac{\beta}{\beta-1} \log_\lambda \delta_j \leq -10 \cdot \log \delta_j \ .$$

Thus Eq. (4.9) and so Eq. (4.7) is implied by

$$|\mathbf{d}^o_j| < \frac{\epsilon^4}{10 \cdot \mu} |\mathbf{g}_j| - \frac{\log((1+\theta^2)/\mathscr{W})}{\mu} \quad \text{and} \quad \mathbf{o}(\mathbf{d}^o_j) = \mathbf{t}(\mathbf{p}_{1+m_j}(\mathbf{s}_j)) \ , \tag{4.10}$$

We fix $\nu := \frac{\epsilon^4}{20 \cdot \mu}$. Since $(|\mathbf{g}_j|)_j$ tends to $+\infty$, if $|\mathbf{d}^o_j| \leq \nu \cdot |\mathbf{g}_j|$ for any $j \geq 1$, then the left inequality of Eq. (4.10) will be satisfied when j is large.

In particular, given any map $\mathsf{Cod} : (\mathbf{A} \sqcup \mathbf{C})^* \mapsto \mathbf{A}^*$ satisfying $|\mathsf{Cod}(\mathbf{w})| \leq \nu |\mathbf{w}|$ and $\mathbf{t}(\mathbf{w}) = \mathbf{o}(\mathsf{Cod}(\mathbf{w}))$ for any $\mathbf{w} \in (\mathbf{A} \sqcup \mathbf{C})^*$, it is therefore possible to pick a sequence of words $\mathbf{w}_j$ satisfying both the conclusions $(\mathbf{C}_1)$ and $(\mathbf{C}'_2)$ of Corollary 4.24 and $\mathbf{d}^o_j = \mathsf{Cod}(\mathbf{g}_j)$ for any $j \geq 1$. The proof is complete. □

4.4. **Proof of Theorem 1.44.** To show the first part of Theorem 1.44 regarding the existence of a family of sets $B_j \times \{o(\mathbf{c}_j)\}$ satisfying the conclusions of Theorem 1.32 (resp. Theorem 1.34), it suffices to show that the assumptions of these theorems are verified at a dense set of parameters.

Proof of the first part of Theorem 1.44. Let $(F_p, \pi_p)_{p \in \mathcal{P}}$ be a moderately dissipative unfolding of wild type $(\mathbf{A}, \mathbf{C})$ with $\#\mathbf{C} = 5$. In Section 4.3, we showed that for every $p_0 \in \mathcal{P}$ and every sufficiently small δ_0, we can find $p_\infty \in \mathcal{P}$ which is $C_1 M_\beta \delta_0$-close to p_0, a sequence $(\mathbf{c}_j)_{j \geq 1} \in \mathbf{C}^{\mathbb{N}^*}$ and a sequence of words $(\mathbf{w}_j)_{j \geq 1} \in \prod_{j \geq 1} \mathbf{W}_{\mathbf{c}_j}$ such that $\mathbf{t}(\mathbf{c}_j) = \mathbf{o}(\mathbf{w}_{j+1})$ for every $j \geq 1$ and satisfying the conclusions $(\mathbf{C_1})$ and $(\mathbf{C'_2})$ of Corollary 4.24. We are going to apply Theorem 1.32, or respectively Theorem 1.34 if F_{p_∞} is analytic.

Proposition 4.28. *The system F_{p_∞} of type $(\mathbf{A},\mathbf{C})$ endowed with the adapted projection π_{p_∞} and the sequences $(\mathbf{c}_j)_{j\geq 1}$, $(\mathbf{w}_j)_{j\geq 1}$ satisfies the assumptions of Theorem 1.32, and moreover those of Theorem 1.34 if F_{p_∞} is analytic.*

The proof of Proposition 4.28 is given below. Together with Theorem 1.32 (resp. Theorem 1.34), it implies the existence of families of nonempty subsets B_j of $Y^{\mathbf{e}}$ satisfying the conclusions of Theorem 1.32 (resp. Theorem 1.34). □

The rest of this subsection is devoted to the proof of Proposition 4.28. In the following, we fix a parameter p_∞ as in Corollary 4.24. We are going to show that F_{p_∞} endowed with π_{p_∞}, $(\mathbf{c}_j)_{j\geq 1}$ and $(\mathbf{w}_j)_{j\geq 1}$ satisfies the assumptions of Theorem 1.32, by checking these assumptions one by one.

For simplicity and since we work at fixed parameter, we will omit to note p_∞ in index in the following. We begin by proving the following crucial estimates between the coefficients δ_j (defined in Definition 4.22), $|Y^{\mathbf{w}_j}|$ (defined in Definition 1.27) and $\breve{\gamma}_j$ (defined in Theorem 1.32):

Lemma 4.29. *For every $j\geq 1$, the coefficient $\breve{\gamma}_j$ is non zero and it holds:*

$$2\delta_j^{1+\epsilon+2\epsilon^2}\leq |Y^{\mathbf{w}_j}|\leq 2\delta_j \quad \textit{and} \quad 2\delta_j^{3(1+\epsilon+2\epsilon^2)}\leq \breve{\gamma}_j\leq 2\delta_j^3\,.$$

Proof. To prove the first inequalities, we recall that by Remark 4.4 we have $2\underline{w}(\mathbf{w}_j)\leq |Y^{\mathbf{w}_j}|\leq 2\overline{w}(\mathbf{w}_j)$ for every $j\geq 1$. Thus the inequalities follow immediately from Corollary 4.24 ($\mathbf{C}_1$).

We now prove the second ones. We first recall that:

$$\breve{\gamma}_j=|Y^{\mathbf{w}_{j+1}}|^{1/2}\cdot|Y^{\mathbf{w}_{j+2}}|^{1/2^2}\cdots|Y^{\mathbf{w}_{j+k}}|^{1/2^k}\cdots$$

if it is well-defined. To show that $\breve{\gamma}_j$ is indeed well-defined and positive, we compute the logarithm of the latter infinite product (with $\log 0=-\infty$ by convention) and we show that it is finite:

$$\log\breve{\gamma}_j=\sum_{k>j}\log|Y^{\mathbf{w}_k}|^{1/2^{k-j}}=\sum_{k>j}\log|Y^{\mathbf{w}_k}|^{2^{j-k}}=\sum_{k>j}2^{j-k}\log|Y^{\mathbf{w}_k}|=\sum_{k>0}2^{-k}\log|Y^{\mathbf{w}_{j+k}}|\,.$$

We just proved that $2\delta_k^{1+\epsilon+2\epsilon^2}\leq|Y^{\mathbf{w}_k}|\leq 2\delta_k$ for every $k>0$. Thus we have:

$$\log(2)+(1+\epsilon+2\epsilon^2)\sum_{k>0}2^{-k}\log\delta_{k+j}\leq\log\breve{\gamma}_j\leq\log(2)+\sum_{k>0}2^{-k}\log\delta_{k+j}\,.$$

We recall that $\delta_{k+j}=\delta_j^{\beta^k}$ by Eq. (4.4). Thus it holds:

$$\sum_{k>0}2^{-k}\log\delta_{k+j}=\sum_{k>0}\left(\frac{\beta}{2}\right)^k\log\delta_j=\left(\frac{\beta}{2-\beta}\right)\log\delta_j\,.$$

Then $\log(2)+\left(\frac{\beta}{2-\beta}\right)(1+\epsilon+2\epsilon^2)\log\delta_j\leq\log\breve{\gamma}_j\leq\log(2)+\left(\frac{\beta}{2-\beta}\right)\log\delta_j$. In particular, $\breve{\gamma}_j$ is positive and:

$$2\delta_j^{\frac{\beta}{2-\beta}(1+\epsilon+2\epsilon^2)}\leq\breve{\gamma}_j\leq 2\delta_j^{\frac{\beta}{2-\beta}} \tag{4.11}$$

which concludes the proof of the second inequalities since $\frac{\beta}{2-\beta}=3$. □

Here is an important consequence of the latter lemma:

Fact 4.30. *When j is large, δ_{j+5} is small compared to $\breve{\gamma}_j^2$.*

Proof. By Eq. (4.4), we have $\delta_{j+5} = \delta_j^{\beta^5} = \delta_j^{7.59375}$. By Eq. (4.11), we have $\breve{\gamma}_j \geq 2\delta_j^{\frac{\beta}{2-\beta}(1+\epsilon+2\epsilon^2)}$. Since $\frac{2\beta}{2-\beta}(1+\epsilon+2\epsilon^2) = 6(1+\epsilon+2\epsilon^2) < 7.59375$, the fact follows. □

Remark 4.31. *The proof of Fact 4.30 enables to see why we needed 5 parameters in the proof of the main theorems of this article. Indeed, for a k-parameter family, we would have to find a $2 > \beta > 1$ which statisfies:*

$$\frac{\beta}{2-\beta}(1+\epsilon+2\epsilon^2) < \beta^k$$

By taking ϵ small this gives the inequality:

$$\frac{\beta}{2-\beta} < \beta^k$$

which does not have any solution for $2 > \beta > 1$ and $k < 5$.

Proof of Proposition 4.28. Let us show that F endowed with π, $(\mathbf{c}_j)_j$, $(\mathbf{w}_j)_j$ satisfies the assumptions of Theorem 1.32. First, by Lemma 4.29, the coefficient $\breve{\gamma}_j$ is positive. This shows (i).

We now show (ii). We remark that $\mathbf{t}(\mathbf{c}_j) = \mathbf{o}(\mathbf{w}_{j+1})$ by Corollary 4.24. We want to prove that the $\pi^{\mathbf{t}(\mathbf{c}_j)}$-fiber of $F^{\mathbf{c}_j}(\zeta^{\mathbf{w}_j})$ is $o(\breve{\gamma}_j^2)$-C^0-close to the $\pi^{\mathbf{o}(\mathbf{w}_{j+1})}$-fiber of $(F^{\mathbf{w}_{j+1}})^{-1}(\zeta^{\mathbf{w}_{j+1}})$. By Corollary 4.24 $(\mathbf{C}_2')$, we have $|\mathcal{V}(\mathbf{w}_j, \mathbf{w}_{j+1})| = O(\delta_{j+5})$, where, by Definitions 4.15 and 4.17, it holds:

$$\mathcal{V}(\mathbf{w}_j, \mathbf{w}_{j+1}) = a^{\mathbf{w}_j} - b^{\mathbf{w}_{j+1}} = \pi^{\mathbf{t}(\mathbf{c}_j)} \circ F^{\mathbf{c}_j}(\zeta^{\mathbf{w}_j}) - \pi^{\mathbf{o}(\mathbf{w}_{j+1})} \circ (F^{\mathbf{w}_{j+1}})^{-1}(\zeta^{\mathbf{w}_{j+1}}).$$

By Fact 4.30, we have $\delta_{j+5} = o(\breve{\gamma}_j^2)$. Since $F^{\mathbf{w}_{j+1}}$ expands the horizontal distance by a factor dominated by $|Y^{\mathbf{w}_{j+1}}|^{-1}$ and since $\breve{\gamma}_j^2 = |Y^{\mathbf{w}_{j+1}}| \cdot \breve{\gamma}_{j+1}$ we obtain (ii).

By Lemma 4.29, $|Y^{\mathbf{w}_j}| \leq 2\delta_j$ and so $|Y^{\mathbf{w}_j}|$ is small when j is large.

This shows the first statement of (iii). Finally, by Remark 4.4 and Proposition 4.5 (2), we have:

$$|F^{\mathbf{w}_j}(Y^{\mathbf{w}_j})| \leq 2\overline{\hbar}(\mathbf{w}_j) \leq 2\underline{w}(\mathbf{w}_j)^{1/\epsilon} \leq 2|Y^{\mathbf{w}_j}|^{10}.$$

By Lemma 4.29, we have $|Y^{\mathbf{w}_j}|^{6(1+\epsilon+2\epsilon^2)} = O(\breve{\gamma}_j^2)$. As $6(1+\epsilon+2\epsilon^2) = 6 \times 1.12 < 10$, we obtain $|F^{\mathbf{w}_j}(Y^{\mathbf{w}_j})| = o(\breve{\gamma}_j^2)$ and the proof of (iii) is complete.

If moreover F is analytic, by Proposition 4.5 $(2')$, we have $\max_{\tilde{I}^2} |\partial_{w_0}\mathcal{W}^{\mathbf{w}_j}|^\epsilon \leq \min_{\tilde{I}^2} |\partial_{z_1}\mathcal{Z}^{\mathbf{w}_j}|$ when j is large, after reducing ρ if necessary, and we conclude the same. This shows assumption $(\widetilde{iii})$ of Theorem 1.34. □

We are now going to prove the second part of Theorem 1.44 regarding Property $(\mathcal{E})$. This will be done using a new symbolic aforemeithm. We recall that $\overrightarrow{\mathbf{A}} \subset \overrightarrow{\mathbf{A} \sqcup \mathbf{C}}$ denote the sets of infinite words $\mathbf{s} = \mathbf{s}_1 \cdots \mathbf{s}_m \cdots$ in respectively the alphabets $\mathbf{A}$ and $\mathbf{A} \sqcup \mathbf{C}$ which are admissible: $\mathbf{t}(\mathbf{a}_i) = \mathbf{o}(\mathbf{a}_{i+1})$, for every i. The space $\overrightarrow{\mathbf{A} \sqcup \mathbf{C}}$ is compact endowed with the following distance:

$$d(\mathbf{s}, \mathbf{s}') = 2^{-\min(i : \mathbf{s}_i \neq \mathbf{s}'_i)} \qquad \text{for } \mathbf{s} = \mathbf{s}_1\mathbf{s}_2 \cdots, \mathbf{s}' = \mathbf{s}'_1\mathbf{s}'_2 \cdots \in \overrightarrow{\mathbf{A} \sqcup \mathbf{C}}.$$

Let g and $g^{\mathbf{A}}$ be the shift dynamics on $\overrightarrow{\mathbf{A} \sqcup \mathbf{C}}$ and $\overrightarrow{\mathbf{A}}$:

$$g : \mathbf{s}_1 \cdots \mathbf{s}_m \cdots \in \overrightarrow{\mathbf{A} \sqcup \mathbf{C}} \mapsto \mathbf{s}_2 \cdots \mathbf{s}_{m+1} \cdots \in \overrightarrow{\mathbf{A} \sqcup \mathbf{C}} \quad \text{and} \quad g^{\mathbf{A}} := g|\overrightarrow{\mathbf{A}}.$$

Let $\mathcal{M}(g)$ be the space of g-invariant measure on $\overrightarrow{\mathbf{A} \sqcup \mathbf{C}}$ and let $\mathcal{M}(g^{\mathbf{A}})$ be the space of g-invariant measures supported by $\overrightarrow{\mathbf{A}}$. We endow $\mathcal{M}(g) \supset \mathcal{M}(g^{\mathbf{A}})$ with the Wasserstein distance d_W introduced in the introduction on Page 7. Given $\mathbf{f} \in \overrightarrow{\mathbf{A} \sqcup \mathbf{C}}$, let $\mathsf{e}_n(\mathbf{f})$ be the probability measure on $\overrightarrow{\mathbf{A} \sqcup \mathbf{C}}$

which is equidistributed on the n first g-iterates of $\mathbf{f}$. For every $\mathbf{w} \in \mathbf{A} \sqcup \mathbf{C}^*$, let us chose $\tilde{\mathbf{w}} \in \overrightarrow{\mathbf{A} \sqcup \mathbf{C}}$ such that $\mathbf{p}_{|\mathbf{w}|}(\tilde{\mathbf{w}}) = \mathbf{w}$ and put $\mathsf{e}_n(\mathbf{w}) := \mathsf{e}_n(\tilde{\mathbf{w}})$ for every $n \leq |\mathbf{w}|$. Here is the algorithm, it uses $\nu > 0$ defined in Corollary 4.27.

Definition 4.32. *Let* $\mathsf{Cod} : (\mathbf{A} \sqcup \mathbf{C})^* \to \mathbf{A}^*$ *be a map sending* $\mathbf{w} \in (\mathbf{A} \sqcup \mathbf{C})^*$ *to a word* $\mathbf{d}$ *such that:*

(1) $\mathbf{t}(\mathbf{w}) = \mathbf{o}(\mathbf{d})$ *and* $|\mathbf{d}| \leq \nu|\mathbf{w}|$.

(2) for every word $\mathbf{d}' \in \mathbf{A}^*$ *such that* $\mathbf{t}(\mathbf{w}) = \mathbf{o}(\mathbf{d}')$ *and* $|\mathbf{d}'| \leq \nu|\mathbf{w}|$*, it holds:*

$$d_W(e_{|\mathbf{w}\cdot\mathbf{d}'|}(\mathbf{w}\cdot\mathbf{d}'), \{e_n(\mathbf{w}) : 1 \leq n \leq |\mathbf{w}|\}) \leq d_W(e_{|\mathbf{w}\cdot\mathbf{d}|}(\mathbf{w}\cdot\mathbf{d}), \{e_n(\mathbf{w}) : 1 \leq n \leq |\mathbf{w}|\}) \ .$$

Corollary 4.33. *Under the assumptions of Corollary 4.24, there is a parameter* $p_\infty \in \mathcal{P}_0$ *which is* $C_1 M_\beta \delta_0$*-close to* p_0 *and a sequence of words* $(\mathbf{w}_i)_i$ *satisfying Conclusion* $(\mathbf{C}_1)$ *and* $(\mathbf{C}_2')$ *of Corollary 4.24 and moreover the point* $\mathbf{f} := \mathbf{w}_1 \cdot \mathbf{c}_1 \cdot \mathbf{w}_2 \cdot \mathbf{c}_2 \cdots \mathbf{w}_i \cdot \mathbf{c}_i \cdots \in \overrightarrow{\mathbf{A} \sqcup \mathbf{C}}$ *satisfies:*

$(\mathbf{C}_0)$ *the limit set* L *of* $(e_n(\mathbf{f}))_{n\geq 0}$ *contains* $\frac{1}{1+\nu} \cdot \check{\mu} + \frac{\nu}{1+\nu} \cdot \mathcal{M}(g^{\mathbf{A}})$ *for a certain* $\check{\mu} \in \mathcal{M}(g^{\mathbf{A}})$.

Proof. We recall that for every $m \geq 0$ and $\mathbf{s} = \mathbf{s}_0 \cdots \mathbf{s}_j \cdots \in \overrightarrow{\mathbf{A} \sqcup \mathbf{C}}$, we defined $\mathbf{p}_m(\mathbf{s}) = \mathbf{s}_0 \cdots \mathbf{s}_{m-1}$. We apply Corollary 4.27 with the function Cod of Definition 4.32. It gives a parameter $p_\infty \in \mathcal{P}_0$ which is $C_1 M_\beta \delta_0$-close to p_0 and a sequence of words $(\mathbf{w}_i)_i$ satisfying Conclusions $(\mathbf{C}_1)$ and $(\mathbf{C}_2')$ of Corollary 4.24 and moreover $\mathbf{f} := \mathbf{w}_1 \cdot \mathbf{c}_1 \cdot \mathbf{w}_2 \cdot \mathbf{c}_2 \cdots \mathbf{w}_i \cdot \mathbf{c}_i \cdots$ satisfies:

$(\mathbf{C}_0')$ there are infinitely many $m \geq 0$ such that with $K_m := \{e_n(\mathbf{p}_n(\mathbf{f})) : 1 \leq n \leq m\}$ and $m' := m + |\mathsf{Cod}(\mathbf{p}_m(\mathbf{f}))|$, it holds:

$$d_W\left(e_{m'}(\mathbf{p}_{m'}(\mathbf{f}), K_m\right) \geq \max_{\{\mathbf{d}\in\mathbf{A}^* : \mathbf{t}(\mathbf{p}_m(\mathbf{f}))=\mathbf{o}(\mathbf{d}) \text{ and } |\mathbf{d}|\leq\nu m\}} d_W\left(e_{m+|\mathbf{d}|}(\mathbf{p}_m(\mathbf{f})\cdot\mathbf{d}), K_m\right) \ .$$

Now we use:

Lemma 4.34. *For every* $\mathbf{s} \in \overrightarrow{\mathbf{A} \sqcup \mathbf{C}}$*, the measures* $e_n(\mathbf{s})$ *and* $e_n(\mathbf{p}_n(\mathbf{s}))$ *are* $2/n$*-close.*

Proof. The measures $e_n(\mathbf{s})$ and $e_n(\mathbf{p}_n(\mathbf{s}))$ are equidistributed on n atoms which pairwise 2^{-n+k}-close for $0 \leq k \leq n$. Thus the transport cost is at most $\frac{1}{n}\sum 2^{-n+k} = 2/n$. □

Thus $e_{m'}(\mathbf{p}_{m'}(\mathbf{f}))$ is close to $e_{m'}(\mathbf{f})$ when m is large. Also $e_{m+|\mathbf{d}|}(\mathbf{p}_m(\mathbf{f})\cdot\mathbf{d})$ is close to $\frac{m}{m+|\mathbf{d}|}e_m(\mathbf{f}) + \frac{|\mathbf{d}|}{m+|\mathbf{d}|}e_{|\mathbf{d}|}(\mathbf{d})$ when m is large. Consequently when m is large:

$$d_W\left(e_{m'}(\mathbf{f}), K_m\right) \geq \max_{\{\mathbf{d}\in\mathbf{A}^* : \mathbf{t}(\mathbf{p}_m(\mathbf{f}))=\mathbf{o}(\mathbf{d}) \text{ and } |\mathbf{d}|\leq\nu m\}} d_W\left(\frac{m}{m+|\mathbf{d}|}e_m(\mathbf{f}) + \frac{|\mathbf{d}|}{m+|\mathbf{d}|}e_{|\mathbf{d}|}(\mathbf{d}), K_m\right) + o(1) \ .$$

As $(\mathbf{V}, \mathbf{A})$ is a finite and connected graph, we do not need to assume that $\mathbf{p}_m(\mathbf{f})\cdot\mathbf{d}$ is admissible in the above inequality, also we can restrict this inequality to the case $|\mathbf{d}| = [\nu \cdot m]$:

$$d_W\left(e_{m'}(\mathbf{f}), K_m\right) \geq \max_{\{\mathbf{d}\in\mathbf{A}^* : |\mathbf{d}|=[\nu\cdot m]\}} d_W\left(\frac{m}{m+|\mathbf{d}|}e_m(\mathbf{f}) + \frac{|\mathbf{d}|}{m+|\mathbf{d}|}e_{|\mathbf{d}|}(\mathbf{d}), K_m\right) + o(1) \ .$$

Thus with $K_\infty = cl(\bigcup_{m\geq 0} K_m)$ the limit of $(K_m)_m$ in the Hausdorff topology, we have:

$$0 = d_W\left(e_{m'}(\mathbf{f}), K_\infty\right) + o(1) \geq \max_{\{\mathbf{d}\in\mathbf{A}^* : |\mathbf{d}|=[\nu\cdot m]\}} d_W\left(\frac{1}{1+\nu}e_m(\mathbf{f}) + \frac{\nu}{1+\nu}e_{|\mathbf{d}|}(\mathbf{d}), K_\infty\right) + o(1) \ .$$

Hence for any accumulation point $\check{\mu}$ of such $\mathscr{e}_m(\mathbf{f})$, as $\mathscr{e}_{|\mathbf{d}|}(\mathbf{d})$ can approximate any measure μ' of $\mathcal{M}(g^{\mathbf{A}})$, it holds:

$$\max_{\mu'\in\mathcal{M}(g^{\mathbf{A}})} d_W\left(\frac{1}{1+\nu}\check{\mu}+\frac{\nu}{1+\nu}\mu', K_\infty\right)=0.$$

Hence we obtained that the connected set $\frac{1}{1+\nu}\cdot\check{\mu}+\frac{\nu}{1+\nu}\cdot\mathcal{M}(g^{\mathbf{A}})$ is included in K_∞. Finally we notice that the proportion of letters in $\mathbf{C}$ is asymptotically small by $(\mathbf{C}_1)$. Thus $\check{\mu}$ belongs to $\mathcal{M}(g^{\mathbf{A}})$. As K_∞ is equal to the union of the limit set L of $(\mathscr{e}_m(\mathbf{f}))$ with the discrete set $\bigcup_m K_m$, the limit set L contains $\frac{1}{1+\nu}\cdot\check{\mu}+\frac{\nu}{1+\nu}\cdot\mathcal{M}(g^{\mathbf{A}})$. □

Proof of property $(\mathcal{E})$ of Theorem 1.44. We still consider a moderately dissipative unfolding $(F_p,\pi_p)_{p\in\mathcal{P}}$ of wild type $(\mathbf{A},\mathbf{C})$, such that $\mathbf{C}$ is formed by five elements $\{\mathbf{c}_1,\mathbf{c}_2,\mathbf{c}_3,\mathbf{c}_4,\mathbf{c}_5\}$, and a fixed parameter $p_0\in\operatorname{int}\mathcal{P}$. We define $\mathcal{P}_0$ as in Proposition 4.5. We apply Corollary 4.33 which gives the existence of a parameter $p_\infty\in\mathcal{P}_0$ which is $C_1M_\beta\delta_0$-close to p_0 and a sequence of words $(\mathbf{w}_i)_i$ satisfying Conclusions $(\mathbf{C}_1)$ and $(\mathbf{C}_2')$ of Corollary 4.24 and moreover:

$(\mathbf{C}_0)$ the point $\mathbf{f}:=\mathbf{w}_1\cdot\mathbf{c}_1\cdot\mathbf{w}_2\cdot\mathbf{c}_2\cdots\mathbf{w}_i\cdot\mathbf{c}_i\cdots\in\overrightarrow{\mathbf{A}\sqcup\mathbf{C}}$ satisfies that the limit set L of $(\mathscr{e}_n(\mathbf{f}))_{n\geq 0}$ contains $\frac{1}{1+\nu}\cdot\check{\mu}+\frac{\nu}{1+\nu}\cdot\mathcal{M}(g^{\mathbf{A}})$ for a certain $\check{\mu}\in\mathcal{M}(g^{\mathbf{A}})$.

Let $B:=\bigcap_{n\geq 0}F_{p_\infty}^{-n}(Y^{\mathbf{e}}\times\mathbf{V})$. The map F_{p_∞} sends B into itself, and the restriction $F_{p_\infty}|B$ is semi-conjugate to g via the following map:

$$\mathbf{h}:(z,\mathbf{v})\in B\mapsto(\mathbf{d}_i)_{i\geq 0}\quad\text{with } \mathbf{d}_i\in\mathbf{A}\sqcup\mathbf{C} \text{ such that } F_{p_\infty}^i(z,\mathbf{v})\in Y_{p_\infty}^{\mathbf{d}_i}\times\{\mathbf{o}(\mathbf{d}_i)\}\ .$$

We are going to use the following proved below:

Lemma 4.35. *The set of invariant probability measures $\mathcal{M}(F_{p^\infty}^{\mathbf{A}})$ of $F_{p^\infty}^{\mathbf{A}}$ is homeomorphic to $\mathcal{M}(g^{\mathbf{A}})$ via the pushforward $\mathbf{h}_*$ induced by $\mathbf{h}$.*

By Proposition 4.28 and Theorem 1.32, there exists $J\geq 0$ and a stable domain $B_J\subset B$ which is sent by $\mathbf{h}$ to $\{g^J(\mathbf{f})\}$. By $(\mathbf{C}_0)$ and Lemma 4.35, the limit set of the empirical measures of points in B_J contains:

$$\mathbf{h}_*^{-1}(\frac{1}{1+\nu}\cdot\check{\mu}+\frac{\nu}{1+\nu}\cdot\mathcal{M}(g^{\mathbf{A}}))=\frac{1}{1+\nu}\cdot\mathbf{h}_*^{-1}\check{\mu}+\frac{\nu}{1+\nu}\cdot\mathbf{h}_*^{-1}(\mathcal{M}(g^{\mathbf{A}}))=\frac{1}{1+\nu}\cdot\mu+\frac{\nu}{1+\nu}\cdot\mathcal{M}(F_{p^\infty}^{\mathbf{A}})\ ,$$

with $\mu=\mathbf{h}_*^{-1}\check{\mu}\in\mathcal{M}(F_{p^\infty}^{\mathbf{A}})$. □

Proof of Lemma 4.35. The map $\mu'\in\mathcal{M}(F_{p^\infty}^{\mathbf{A}})\mapsto\mathbf{h}_*\mu'\in\mathcal{M}(g^{\mathbf{A}})$ is continuous on its domain which is compact. Thus it suffices to show that it is a bijection. In order to show this, it suffices to recall that the space of invariant measures of a continuous map are homeomorphic to those of its inverse limit, and that the dynamics on the inverse limit $\overleftrightarrow{\mathbf{A}}$ of $\overrightarrow{\mathbf{A}}$ is conjugate to the dynamics on the inverse limit $\overleftrightarrow{\Lambda}$ of Λ by Remark 1.14. □

Appendix A. Some results on C_ρ^ω-hyperbolic transformations

Proof of Proposition 1.7. Let us just do the proof for C_ρ^ω-hyperbolic transformations, the one for (real) hyperbolic transformations is similar. We assume that $(\tilde{Y},F)\neq(\tilde{Y}^{\mathbf{e}},id)\neq(\tilde{Y}',F')$, otherwise the proof is obvious. Put:

$$(\tilde{Y}'',F'':=(\tilde{Y},F)\star(\tilde{Y}',F')\ .$$

The proposition is the consequence of the two following Lemmas A.1 and A.2. □

Lemma A.1. *The set $\tilde{Y}''$ is a C^ω_ρ-box.*

Proof. Note first that $\tilde{Y}'' \subset \tilde{Y} \subset \tilde{Y}^{\mathbf{e}}$. Let $\zeta : (z,w) \in \tilde{I}^2 \to (\mathcal{Z}_1(z,w),w) \in \tilde{Y}$ and $\zeta' : (z,w) \in \tilde{I}^2 \to (\mathcal{Z}_2(z,w),w) \in \tilde{Y}'$ be two biholomorphisms given by Definition 1.2 for the C^ω_ρ-boxes $\tilde{Y}$ and $\tilde{Y}'$. By the cone property (2) of Definition 1.5, at $w \in \tilde{I}$ fixed, the (complex) curve $\mathbb{D}_{1\,w} := \{(\mathcal{Z}_1(z,w),w) : z \in \tilde{I}\}$ is sent by F to a curve with tangent spaces in $\tilde{\chi}_h$. By property (1) of Definition 1.5, the curve $F(\mathbb{D}_{1\,w})$ is disjoint from $\partial^u\tilde{Y}^{\mathbf{e}}$ and with boundary in $\partial^s\tilde{Y}^{\mathbf{e}}$. Still by property (1) of Definition 1.5, for every $z_2 \in \tilde{I}$, the curve $\{(\mathcal{Z}_2(z_2,w),w) : w \in \tilde{I}\}$ is disjoint from $\partial^s\tilde{Y}^{\mathbf{e}}$. By Definition 1.2, it has its tangent spaces in $\tilde{\chi}_v$ and its boundary is in $\partial^u\tilde{Y}^{\mathbf{e}}$. Thus $\{(\mathcal{Z}_2(z_2,w),w) : w \in \tilde{I}\}$ intersects $F(\mathbb{D}_{1\,w})$ transversally at a unique point. We denote by $\zeta''(z_2,w) = (\mathcal{Z}_{12}(z_2,w),w)$ its preimage by F. By transversality, ζ'' is a holomorphic map from $\tilde{I}^2$ onto $\tilde{Y}''$. Since $|\partial_w\mathcal{Z}_2| < \theta$ and by the cone property (3) of Definition 1.5, it holds $|\partial_w\mathcal{Z}_{12}| < \theta$. Also $(\{(\mathcal{Z}_2(z_2,w),w) : w \in \tilde{I}\})_{z_2\in\tilde{I}}$ are the leaves of a foliation. Each leaf intersects transversally the curve $F(\mathbb{D}_{1\,w})$ at the unique point $F\circ\zeta''(z_2,w)$, so the derivative of $F\circ\zeta''$ w.r.t. z_2 is non-zero. Thus, the derivative of $\zeta''(z_2,w)$ w.r.t. z_2 is non-zero. By definition, $\zeta''(z_2,w)$ belongs to $\tilde{I}\times\{w\}$ and so $\partial_{z_2}\zeta''$ is horizontal and non zero. Thus $D\zeta''$ is invertible and so ζ'' is a biholomorphism.

Let us denote by Y, Y', Y'' the real traces $\tilde{Y}\cap Y^{\mathbf{e}}$, $\tilde{Y}'\cap Y^{\mathbf{e}}$, $\tilde{Y}''\cap Y^{\mathbf{e}}$ of $\tilde{Y}$, $\tilde{Y}'$, $\tilde{Y}''$. It remains to show that Y'' is a (real) box. It is enough to show that $Y'' = \breve{Y}''$ where we set $(\breve{Y}'',F'') := (Y,F)\star(Y',F')$. Indeed both Y and Y' are boxes by Definition 1.2 and so by [PY01, §3.2.1] it is also the case for $\breve{Y}''$. Then one just has to remark that:

$$Y'' = \tilde{Y}\cap F^{-1}(\tilde{Y}')\cap Y^{\mathbf{e}} = Y\cap F^{-1}(\tilde{Y}') = Y\cap F^{-1}(Y') = \breve{Y}''\,.$$

where the first equality is true by definition of $\tilde{Y}''$, the second equality is true by definition of Y, the fourth one is given by definition of the real $\star$-product and the third one uses that $F|Y$ is a bijection onto $F(Y)$ whose inverse is conjugate to a hyperbolic transformation via the involution $(z,w)\mapsto(w,z)$. This shows that $\tilde{Y}''$ is a C^ω_ρ-box. □

Lemma A.2. *The pair $(\tilde{Y}'',F'')$ is a C^ω_ρ-hyperbolic transformation.*

Proof. The map $F'' = F'\circ F$ is a biholomorphism on $\tilde{Y}''$ as a composition of two biholomorphisms. We now continue with the same notations as in the previous lemma. Let us prove (1) of Definition 1.5. Note that when $z_2 \in \partial\tilde{I}$, then $\{(\mathcal{Z}_{12}(z_2,w),w) : w \in \tilde{I}\}$ is sent by F into $\{(\mathcal{Z}_2(z_2,w),w) : w \in \tilde{I}\} \subset \partial^s\tilde{Y}'$. Thus by this property (1) for $(\tilde{Y}',F)$, the set $\{(\mathcal{Z}_{12}(z_2,w),w) : w \in \tilde{I}\}$ is sent by F'' into $\partial^s\tilde{Y}^{\mathbf{e}}$: $F''(\partial^s\tilde{Y}'') \subset \partial^s\tilde{Y}^{\mathbf{e}}$. By definition, the set $\tilde{Y}''$ is included in the definition domain of F'' which sends it into $F'(\tilde{Y}') \subset \tilde{Y}^{\mathbf{e}}$. As $(\tilde{Y},F) \neq (\tilde{Y}^{\mathbf{e}},id) \neq (\tilde{Y}',F')$, it holds $\tilde{Y}''\cap\partial^s\tilde{Y}^{\mathbf{e}} \subset \tilde{Y}\cap\partial^s\tilde{Y}^{\mathbf{e}} = \emptyset$ and $F''(\tilde{Y}'')\cap\partial^u\tilde{Y}^{\mathbf{e}} \subset F'(\tilde{Y}')\cap\partial^u\tilde{Y}^{\mathbf{e}} = \emptyset$. Properties (2), (3) and (4) of Definition 1.5 for $(\tilde{Y}'',F'')$ come from the same properties for $(\tilde{Y},F)$ and $(\tilde{Y}',F')$. □

The remaining of this section is devoted to show the following given a finite graph $(\mathbf{V},\mathbf{A})$:

Proposition A.3. *If $F^{\mathbf{A}}$ is a hyperbolic map of type $\mathbf{A}$ and if $F^{\mathbf{A}}$ is real analytic, then there exists $\rho > 0$ such that for every $\mathbf{w} \in \mathbf{A}^*$, there exists a C^ω_ρ-box $\tilde{Y}^{\mathbf{w}}$ such that $Y^{\mathbf{w}}$ is the real trace $\tilde{Y}^{\mathbf{w}}\cap\mathbb{R}^2$ of $\tilde{Y}^{\mathbf{w}}$, $F^{\mathbf{w}}$ extends to a biholomorphism from $\tilde{Y}^{\mathbf{w}}$ and $(\tilde{Y}^{\mathbf{w}},F^{\mathbf{w}})$ is a C^ω_ρ-hyperbolic transformation.*

Proof. We prove below the following:

Lemma A.4. *If (Y, F) is a hyperbolic transformation and F is real analytic, then for every $\rho > 0$ sufficiently small, there exists a C^ω_ρ-box $\tilde{Y}$ such that Y is the real trace $\tilde{Y} \cap \mathbb{R}^2$ of $\tilde{Y}$, F extends to a biholomorphism from $\tilde{Y}$ and $(\tilde{Y}, F)$ is a C^ω_ρ-hyperbolic transformation.*

Thus by finiteness of $\mathbf{A}$ and by Lemma A.4, there exist $\rho > 0$ such that each hyperbolic transformation $(Y^{\mathbf{a}}, F^{\mathbf{a}})$ with $\mathbf{a} \in \mathbf{A}$ admits an extension as a C^ω_ρ-hyperbolic transformation $(\tilde{Y}^{\mathbf{a}}, F^{\mathbf{a}})$. Then by Proposition 1.7, for every $\mathbf{w} \in \mathbf{A}^*$, we can define the C^ω_ρ-hyperbolic transformation:

$$(\tilde{Y}^{\mathbf{w}}, F^{\mathbf{w}}) := (\tilde{Y}^{\mathbf{a}_1}, F^{\mathbf{a}_1}) \star \cdots \star (\tilde{Y}^{\mathbf{a}_k}, F^{\mathbf{a}_k}) \,, \quad \forall \mathbf{w} = \mathbf{a}_1 \cdots \mathbf{a}_k \in \mathbf{A}^* \,.$$

By Proposition 1.7, the pair $(\tilde{Y}^{\mathbf{w}}, F^{\mathbf{w}})$ is a C^ω_ρ-hyperbolic transformation extending $(Y^{\mathbf{w}}, F^{\mathbf{w}})$. □

Proof of Lemma A.4. Let us assume that F is not the identity (otherwise we take the holomorphic extension $(\tilde{Y}^{\mathbf{e}}, id)$).

As $F|Y$ is a real analytic diffeomorphism onto its image, it extends to a biholomorphism from a neighborhood $\hat{Y}$ of Y in $\mathbb{C}^2$ onto its image $F(\hat{Y})$. If $\hat{Y}$ is small enough, from cone properties (2) and (3) of Definition 1.4, at every point in $\hat{Y}$, every non-zero vector in the complement of $\tilde{\chi}_v$ is sent into $\tilde{\chi}_h$ by DF and has its first coordinate's modulus which is more than λ-expanded by DF. Also at every point in $F(\hat{Y})$, every non-zero vector in the complement of $\tilde{\chi}_h$ is sent into $\tilde{\chi}_v$ by DF^{-1} and has its second coordinate modulus which is more than λ-expanded by DF^{-1}.

Let $\mathcal{X} : I^2 \to \operatorname{int} I$ and $\mathcal{Y} : I^2 \to \operatorname{int} I$ be the implicit representation of the hyperbolic transformation (F, Y) as defined in Proposition 3.1. By the transversality used in the proof of this proposition, the maps $\mathcal{X}$ and $\mathcal{Y}$ are real analytic. Thus for ρ-small enough they extend to holomorphic functions $\mathcal{Z}$ and $\mathcal{W}$ on $\tilde{I}^2$, with $\tilde{I} = I + i[-\rho, \rho]$. By [PY01, Lemma 3.2], it holds:

$$|\partial_x \mathcal{X}| + |\partial_y \mathcal{X}| < \theta \quad \text{and} \quad |\partial_x \mathcal{Y}| + |\partial_y \mathcal{Y}| < \theta \,, \quad \text{on } I^2 \,.$$

Thus for ρ sufficiently small:

$$|\partial_z \mathcal{Z}| + |\partial_w \mathcal{Z}| < \theta \quad \text{and} \quad |\partial_z \mathcal{W}| + |\partial_w \mathcal{W}| < \theta \,, \quad \text{on } \tilde{I}^2 \,.$$

In particular $\mathcal{Z}$ and $\mathcal{W}$ are contracting. As they map I^2 into the interior of I, for ρ-sufficiently small $\mathcal{Z}$ and $\mathcal{W}$ map $\tilde{I}^2$ into the interior of $\tilde{I}$. Hence the following set is included in $\tilde{Y}^{\mathbf{e}} \setminus \partial^s \tilde{Y}^{\mathbf{e}}$:

$$\tilde{Y} := \{(\mathcal{Z}(z_1, w_0), w_0) : (z_1, w_0) \in \tilde{I}^2\}$$

Also $|\partial_w \mathcal{Z}| < \theta$. Since $\partial_x \mathcal{X} \neq 0$, reducing ρ if necessary, we also have $\partial_z \mathcal{Z} \neq 0$ and so the map $(z, w) \in \tilde{I}^2 \mapsto (\mathcal{Z}(z, w), w) \in \tilde{Y}$ is a biholomorphism. Thus $\tilde{Y}$ is C^ω_ρ-box. Note that for ρ small, $\tilde{Y}$ is close to Y and so included in $\hat{Y}$. Thus cone properties (2) and (3) of Definition 1.5 are satisfied. Put:

$$\partial^u \tilde{Y} := \zeta(\tilde{I} \times \partial \tilde{I}) \quad \text{and} \quad \partial^s \tilde{Y} := \zeta((\partial \tilde{I}) \times \tilde{I}) \,.$$

As for every $(x, y) \in I^2$, it holds $F(\mathcal{X}(x, y), y) = (x, \mathcal{Y}(x, y))$ by Proposition 3.1 and by analyticity, we have:

$$F(\mathcal{Z}(z, w), w) = (z, \mathcal{W}(z, w)) \quad \forall (z, w) \in \tilde{I}^2 \,,$$

from which property (1) of Definition 1.5 is immediate. Finally note that if for $w_0 \in I$ and $z_1 \in \tilde{I}$, the number $\mathcal{Z}(z_1, w_0)$ is real, then its image $(z_1, \mathcal{W}(z_1, w_0))$ by F is real. Thus z_1 is real and so in I. Thus

$$\tilde{Y} \cap \mathbb{R}^2 = \{(\mathcal{Z}(z_1, w_0), w_0) : (z_1, w_0) \in I^2\} = Y\,.$$

This achieves the proof of Definition 1.5 (4). □

Appendix B. Bounds on implicit representations and distortion results

Let us recall:

Lemma B.1 ([PY09, Ineq. (A.8) p.195])**.** *Let (Y, F) and (Y', F') be hyperbolic transformations with implicit representations $(\mathcal{X}_0, \mathcal{Y}_1)$ and $(\mathcal{X}_1, \mathcal{Y}_2)$. Then the implicit representation $(\tilde{\mathcal{X}}_0, \tilde{\mathcal{Y}}_2)$ of $(Y, F) \star (Y', F')$ satisfies:*

$$\frac{1}{1+\theta^2} \leq \frac{|\partial_{x_2}\tilde{\mathcal{X}}_0(x_2, y_0)|}{|\partial_{x_1}\mathcal{X}_0\big(x_1, y_0\big)| \cdot |\partial_{x_2}\mathcal{X}_1\big(x_2, y_1\big)|} \leq \frac{1}{1-\theta^2}\,,$$

where F sends $(x_0, y_0) \in Y$ to $(x_1, y_1) \in Y'$ and (x_1, y_1) is sent to (x_2, y_2) by F'.

We will also need the following complex analogous of the latter result:

Lemma B.2. *Let $(\tilde{Y}, F)$ and $(\tilde{Y}', F')$ be C^ω_ρ-hyperbolic transformations with implicit representations $(\mathcal{Z}_0, \mathcal{W}_1)$ and $(\mathcal{Z}_1, \mathcal{W}_2)$. Then the implicit representation $(\tilde{\mathcal{Z}}_0, \tilde{\mathcal{W}}_2)$ of $(\tilde{Y}, F) \star (\tilde{Y}', F')$ satisfies:*

$$\frac{1}{1+\theta^2} \leq \frac{|\partial_{z_2}\tilde{\mathcal{Z}}_0(z_2, w_0)|}{|\partial_{z_1}\mathcal{Z}_0\big(z_1, w_0\big)| \cdot |\partial_{z_2}\mathcal{Z}_1\big(z_2, w_1\big)|} \leq \frac{1}{1-\theta^2}\,,$$

$$\frac{1}{1+\theta^2} \leq \frac{|\partial_{w_0}\tilde{\mathcal{W}}_2(z_2, w_0)|}{|\partial_{w_0}\mathcal{W}_1\big(z_1, w_0\big)| \cdot |\partial_{w_1}\mathcal{W}_2\big(z_2, w_1\big)|} \leq \frac{1}{1-\theta^2}\,,$$

where F sends $(z_0, w_0) \in \tilde{Y}$ to $(z_1, w_1) \in \tilde{Y}'$ and (z_1, w_1) is sent to (z_2, w_2) by F'.

Proof. We just prove the inequalities on $\tilde{\mathcal{Z}}_0$, those on regarding $\tilde{\mathcal{W}}_2$ are obtained similarly. We consider the map $\Psi : (z_1, w_1, z_2, w_0) \in \tilde{I}^4 \to (z_1 - \mathcal{Z}_1(z_2, w_1), w_1 - \mathcal{W}_1(z_1, w_0))$, which is holomorphic. For all $(z_0, w_0) \in \tilde{Y}$, $(z_1, w_1) \in \tilde{Y}'$ and $(z_2, w_2) \in \tilde{Y}^{\mathsf{e}}$ such that F sends (z_0, w_0) to (z_1, w_1) and F' sends (z_1, w_1) to (z_2, w_2), it holds $\Psi(z_1, w_1, z_2, w_0) = (0, 0)$. By the cone properties of C^ω_ρ-hyperbolic transformations, we have $|\partial_{w_1}\mathcal{Z}_1| < \theta$ and $|\partial_{z_1}\mathcal{W}_1| < \theta$. Then $\Delta := |\det(\partial_{(z_1,w_1)}\Psi)| = |1 - \partial_{w_1}\mathcal{Z}_1 \cdot \partial_{z_1}\mathcal{W}_1|$ satisfies:

$$1 + \theta^2 > \Delta > 1 - \theta^2 > 0 \tag{B.1}$$

and so $\partial_{(z_1,w_1)}\Psi$ is invertible. By the implicit function Theorem, there exist implicitly defined holomorphic maps $\tilde{\mathcal{Z}}_1$ and $\tilde{\mathcal{W}}_1$ such that:

$$\Psi\big(z_1, w_1, z_2, w_0\big) = 0 \Longleftrightarrow \Big(z_1 = \tilde{\mathcal{Z}}_1(z_2, w_0) \text{ and } w_1 = \tilde{\mathcal{W}}_1(z_2, w_0)\Big)\,. \tag{B.2}$$

Furthermore, this theorem gives:

$$\begin{pmatrix} \partial_{z_2}\tilde{\mathcal{Z}}_1 & \partial_{w_0}\tilde{\mathcal{Z}}_1 \\ \partial_{z_2}\tilde{\mathcal{W}}_1 & \partial_{w_0}\tilde{\mathcal{W}}_1 \end{pmatrix} = -\partial_{(z_1,w_1)}\Psi^{-1} \cdot \partial_{z_2,w_0}\Psi = -\begin{pmatrix} 1 & -\partial_{w_1}\mathcal{Z}_1 \\ -\partial_{z_1}\mathcal{W}_1 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -\partial_{z_2}\mathcal{Z}_1 & 0 \\ 0 & -\partial_{w_0}\mathcal{W}_1 \end{pmatrix}\,.$$

Thus, it holds $\partial_{z_2}\tilde{\mathcal{Z}}_1 = \Delta^{-1}\cdot\partial_{z_2}\mathcal{Z}_1$. As $\tilde{\mathcal{Z}}_0(z_2,w_0) = \mathcal{Z}_0(\tilde{\mathcal{Z}}_1(z_2,w_0),w_0)$, we then obtain $\partial_{z_2}\tilde{\mathcal{Z}}_0 = \partial_{z_1}\mathcal{Z}_0\cdot\partial_{z_2}\tilde{\mathcal{Z}}_1 = \Delta^{-1}\cdot\partial_{z_1}\mathcal{Z}_0\cdot\partial_{z_2}\mathcal{Z}_1$ and the sought inequalities follow from Eq. (B.1). □

We are now in position to prove:

Proof of Proposition 4.5. Proof of (1). The left inequality being obvious, let us show the right inequality. Let $\mathbf{w}\in\mathbf{A}^*$ and $(\mathcal{X}_p^{\mathbf{w}},\mathcal{Y}_p^{\mathbf{w}})$ be the implicit representation of $(Y_p^{\mathbf{w}},F_p^{\mathbf{w}})$ for $p\in\mathcal{P}$. According to the cone condition (2) of Definition 1.4, we have $|\partial_x\mathcal{X}_p^{\mathbf{w}}|\le\lambda^{-|\mathbf{w}|}$ and so:

$$(1+\epsilon^2)\max_{I^2}\log|\partial_x\mathcal{X}_p^{\mathbf{w}}|\le\max_{I^2}\log|\partial_x\mathcal{X}_p^{\mathbf{w}}|-\epsilon^2\cdot|\mathbf{w}|\cdot\log\lambda$$

Then using Corollary 4.2, we obtain:

$$(1+\epsilon^2)\max_{I^2}\log|\partial_x\mathcal{X}_p^{\mathbf{w}}|\le\min_{I^2}\log|\partial_x\mathcal{X}_p^{\mathbf{w}}|+\log\overline{\mathcal{B}}-\epsilon^2\cdot|\mathbf{w}|\cdot\log\lambda$$

Let $N_0>0$ be such that

$$\log\overline{\mathcal{B}}-\epsilon^2\cdot N_0\cdot\log\lambda<-\log(1+\theta^2)+(1+\epsilon^2)\log(1-\theta^2)\ . \tag{B.3}$$

Then:

$$(1+\epsilon^2)\max_{I^2}\log|(1-\theta^2)^{-1}\cdot\partial_x\mathcal{X}_p^{\mathbf{w}}|<\min_{I^2}\log|(1+\theta^2)^{-1}\cdot\partial_x\mathcal{X}_p^{\mathbf{w}}|\quad\forall\mathbf{w}\in\mathbf{A}^*\text{ with }|\mathbf{w}|\ge N_0.$$

As the latter inequality is strict, given any $p_0\in\mathcal{P}$, there exists a compact neighborhood $\mathcal{P}_0\subset\mathcal{P}$ of p_0 such that for every word $\mathbf{w}\in\mathbf{A}^*$ with at least N_0 and at most $2N_0$ letters, for every $p,p'\in\mathcal{P}_0$, we obtain:

$$(1-\theta^2)^{-1-\epsilon^2}\cdot\max_{I^2}|\partial_x\mathcal{X}_p^{\mathbf{w}}|^{1+\epsilon^2}<(1+\theta^2)^{-1}\cdot\min_{I^2}|\partial_x\mathcal{X}_{p'}^{\mathbf{w}}|. \tag{B.4}$$

Consequently for every word $\mathbf{w}\in\mathbf{A}^*$ with at least N_0 and at most $2N_0$ letters,

$$(1-\theta^2)^{-1-\epsilon^2}\cdot\overline{w}(\mathbf{w})^{1+\epsilon^2}<(1+\theta^2)^{-1}\cdot\underline{w}(\mathbf{w})\ . \tag{B.5}$$

Let $\mathbf{w}\in\mathbf{A}^*$ be such that $|\mathbf{w}|\ge N_0$. It can be written as a concatenation $\mathbf{w}=\mathbf{w}_1\cdots\mathbf{w}_m$ where each $\mathbf{w}_i$ has at least N_0 and at most $2N_0$ letters. According to Lemma B.1, we have:

$$\overline{w}(\mathbf{w})\le(1-\theta^2)^{-m}\cdot\overline{w}(\mathbf{w}_1)\cdots\overline{w}(\mathbf{w}_m)\quad\text{and}\quad(1+\theta^2)^{-m}\cdot\underline{w}(\mathbf{w}_1)\cdots\underline{w}(\mathbf{w}_m)\le\underline{w}(\mathbf{w})\ . \tag{B.6}$$

Together with Eq. (B.5), this yields:

$$\overline{w}(\mathbf{w})^{1+\epsilon^2}\le\underline{w}(\mathbf{w})\ . \tag{B.7}$$

Since $1-\epsilon^2<(1+\epsilon^2)^{-1}$, this proves (1) for every word with more than N_0 letters. Thus it suffices to assume $\eta_0>0$ small enough such that every $\mathbf{w}\in\mathbf{A}^*$ such that $\overline{w}(\mathbf{w})<\eta_0$ has at least N_0 letters.

Proof of (2). Let $\mathbf{w}\in\mathbf{A}^*$. By Proposition 3.6 and since $F_p^{\mathbf{A}}$ is moderately dissipative for every $p\in\mathcal{P}$, for every $z=(x_0,y_0)\in Y_p^{\mathbf{w}}$ sent to (x_1,y_1) by $F_p^{\mathbf{w}}$, it holds:

$$|\partial_{y_0}\mathcal{Y}_p^{\mathbf{w}}(x_1,y_0)|^\epsilon=|\det D_{(x_0,y_0)}F_p^{\mathbf{w}}|^\epsilon\cdot|\partial_{x_1}\mathcal{X}_p^{\mathbf{w}}(x_1,y_0)|^\epsilon\le\min_{Y_p^{\mathbf{w}}}\|DF_p^{\mathbf{w}}\|^{-1}\cdot|\partial_{x_1}\mathcal{X}_p^{\mathbf{w}}(x_1,y_0)|^\epsilon\ .$$

As $\|DF_p^{\mathbf{w}}\|^{-1}\le|\partial_{x_1}\mathcal{X}_p^{\mathbf{w}}|$, it follows:

$$\max_{Y_p^{\mathbf{w}}}|\partial_{y_0}\mathcal{Y}_p^{\mathbf{w}}|^\epsilon\le\min_{Y_p^{\mathbf{w}}}|\partial_{x_1}\mathcal{X}_p^{\mathbf{w}}|\cdot\max_{Y_p^{\mathbf{w}}}|\partial_{x_1}\mathcal{X}_p^{\mathbf{w}}|^\epsilon\le\max_{Y_p^{\mathbf{w}}}|\partial_{x_1}\mathcal{X}_p^{\mathbf{w}}|^{1+\epsilon}\ . \tag{B.8}$$

Thus $\overline{\mathsf{h}}(\mathbf{w})^\epsilon \leq \overline{w}(\mathbf{w})^{1+\epsilon}$. Also by Eq. (B.7) and since $0<\epsilon<1$, if $\bar{w}(\mathbf{w}) \leq \eta_0$ then $\overline{\mathsf{h}}(\mathbf{w})^\epsilon < \underline{w}(\mathbf{w})$.

Proof of (2') (and definition of ρ). Let F_p be real analytic for a certain $p \in \mathcal{P}_0$. By Proposition A.3, let $\rho>0$ be such that for every $\mathbf{w} \in \mathbf{A}^*$, the hyperbolic transformation $(Y_p^{\mathbf{w}}, F_p^{\mathbf{w}})$ admits a C^ω_ρ-extension $(\tilde{Y}_p^{\mathbf{w}}, F_p^{\mathbf{w}})$. Then by Eq. (B.8), reducing ρ if necessary, we have:

$$\max_{\tilde{Y}_p^{\mathbf{w}}} |\partial_{w_0} \mathcal{W}_p^{\mathbf{w}}|^\epsilon \leq \min_{\tilde{Y}_p^{\mathbf{w}}} |\partial_{z_1} \mathcal{Z}_p^{\mathbf{w}}| \cdot \max_{\tilde{Y}_p^{\mathbf{w}}} |\partial_{z_1} \mathcal{Z}_p^{\mathbf{w}}|^\epsilon \leq \lambda^{-N_0 \cdot \epsilon} \cdot \min_{\tilde{Y}_p^{\mathbf{w}}} |\partial_{z_1} \mathcal{Z}_p^{\mathbf{w}}| \quad \text{if } N_0 \leq |\mathbf{w}| \leq 2N_0 \,. \tag{B.9}$$

By Eq. (B.3), $-\epsilon \cdot N_0 \cdot \log \lambda < \epsilon \log(1-\theta^2) - \log(1+\theta^2)$ and so it holds:

$$(1-\theta^2)^{-\epsilon} \max_{\tilde{I}^2} |\partial_{w_0} \mathcal{W}_p^{\mathbf{w}}|^\epsilon < (1+\theta^2)^{-1} \cdot \min_{\tilde{I}^2} |\partial_{z_1} \mathcal{Z}_p^{\mathbf{w}}| \quad \forall \mathbf{w} \in \mathbf{A}^* \text{ with } N_0 \leq |\mathbf{w}| \leq 2N_0 \,. \tag{B.10}$$

Then, given any $\mathbf{w} \in \mathbf{A}^*$ with $|\mathbf{w}| \geq N_0$, we split $\mathbf{w}$ into words of lengths in $[N_0, 2N_0]$ and proceed as in the proof of Eq. (B.6) and Eq. (B.7), using Lemma B.2 instead of Lemma B.1, to obtain:

$$\max_{\tilde{I}^2} |\partial_{z_1} \mathcal{W}_p^{\mathbf{w}}|^\epsilon \leq \min_{\tilde{I}^2} |\partial_{z_1} \mathcal{Z}_p^{\mathbf{w}}| \,, \tag{B.11}$$

for every $\mathbf{w} \in \mathbf{A}^*$ such that $|\mathbf{w}| \geq N_0$ and so every $\mathbf{w} \in \mathbf{A}^*$ such that $\bar{w}(\mathbf{w}) \leq \eta_0$. □

In Section 3.3, we needed the following distortion result:

Proposition B.3. *(i) Let $(\mathcal{X}_j)_j$ be a sequence of C^2-functions $\mathcal{X}_j : I^2 \to \mathbb{R}$ such that $\partial_x \mathcal{X}_j$ does not vanish. Let $(\mathcal{B}_j)_j$ be a sequence of subsets of I^2 whose diameters are small when j is large and let $(x_j^0, y_j^0)_j$ be a sequence of points in $\mathcal{B}_j$. If the distortion $(\|\partial_x \log|\partial_x \mathcal{X}_j|\|_{C^0}, \|\partial_y \log|\partial_x \mathcal{X}_j|\|_{C^0})_j$ is bounded, then the following map is C^1-close to the first coordinate projection when j is large:*

$$\Upsilon_j : (x,y) \in \mathcal{B}_j \mapsto \frac{\mathcal{X}_j(x,y) - \mathcal{X}_j(x_j^0, y)}{\partial_x \mathcal{X}_j(x_j^0, y_j^0)} + x_j^0 \,.$$

(ii) Let $(\mathcal{Z}_j)_j$ be a sequence of holomorphic functions $\mathcal{Z}_j : \tilde{I}^2 \to \tilde{I}$ such that $0 < |\partial_z \mathcal{Z}_j| < 1$. Let $(\mathcal{B}_j)_j$ be a sequence of subsets of $\tilde{I}^2$ and $(z_j, w_j)_j$ be a sequence of points such that $(z_j, w_j) \in \mathcal{B}_j$ for every j. We suppose that the diameter of $\mathcal{B}_j$ is small compared to $|\partial_z \mathcal{Z}_j(z_j, w_j)|$ when j is large and that the points (z_j, w_j) are uniformly distant to $\partial \tilde{I}^2$. Then the following map is C^1-close to the first coordinate projection when j is large:

$$\tilde{\Upsilon}_j : (z,w) \in \mathcal{B}_j \mapsto \frac{\mathcal{Z}_j(z,w) - \mathcal{Z}_j(z_j, w)}{\partial_z \mathcal{Z}_j(z_j, w_j)} + z_j \,.$$

Proof of Proposition B.3. Let us first prove (i). Up to replacing $\mathcal{B}_j$ by $\mathrm{pr}_1(\mathcal{B}_j) \times \mathrm{pr}_2(\mathcal{B}_j)$ (whose diameter is still small when j is large), we can suppose that $\mathcal{B}_j$ is a product. Note that $\Upsilon_j(x_j^0, y_j^0) = x_0^j$, $\partial_x \Upsilon_j(x_j^0, y_j^0) = 1$ and $\partial_y \Upsilon_j(x_j^0, y) = 0$ for every $y \in \mathrm{pr}_2(\mathcal{B}_j)$. Thus by the mean value inequality, it suffices to show that the map $\partial_x^2 \Upsilon_j = \partial_x^2 \mathcal{X} / \partial_x \mathcal{X}(x_j^0, y_j^0)$ and $\partial_x \partial_y \Upsilon_j = \partial_y \partial_x \mathcal{X} / \partial_x \mathcal{X}(x_j^0, y_j^0)$ are bounded since the diameter of $\mathcal{B}_j$ is small. This is a direct consequence of the distortion bound:

$$\left| \frac{\partial_x^2 \mathcal{X}(x,y)}{\partial_x \mathcal{X}(x_j^0, y_j^0)} \right| \leq |\partial_x \log|\partial_x \mathcal{X}_j(x,y)|| \cdot \left| \frac{\partial_x \mathcal{X}(x,y)}{\partial_x \mathcal{X}(x_j^0, y_j^0)} \right| \leq \|D \log|\partial_x \mathcal{X}_j|\| \cdot e^{|\partial_x \log|\partial_x \mathcal{X}_j(x,y)|| \cdot \mathrm{diam}\, \mathcal{B}_j} \,,$$

$$\left| \frac{\partial_y \partial_x \mathcal{X}(x,y)}{\partial_x \mathcal{X}(x_j^0, y_j^0)} \right| \leq |\partial_y \log|\partial_x \mathcal{X}_j(x,y)|| \cdot \left| \frac{\partial_x \mathcal{X}(x,y)}{\partial_x \mathcal{X}(x_j^0, y_j^0)} \right| \leq \|D \log|\partial_x \mathcal{X}_j|\| \cdot e^{|\partial_y \log|\partial_x \mathcal{X}_j(x,y)|| \cdot \mathrm{diam}\, \mathcal{B}_j} \,.$$

We now prove (ii). Let us denote $\sigma_j := \partial_z \mathcal{Z}_j(z_j, w_j)$ for every $j \in \mathbb{N}$ and recall that $0 < |\sigma_j| < 1$. We notice that for any $(z, w) \in \mathcal{B}_j$, it holds:

$$\partial_z \tilde{\Upsilon}_j(z, w) = \frac{1}{\sigma_j} \partial_z \mathcal{Z}_j(z, w) \quad \text{and} \quad \partial_w \tilde{\Upsilon}_j(z, w) = \frac{1}{\sigma_j} (\partial_w \mathcal{Z}_j(z, w) - \partial_w \mathcal{Z}_j(z_j, w)) .$$

In particular, we have $\partial_z \tilde{\Upsilon}_j(z_j, w_j) = 1$ and $\partial_w \tilde{\Upsilon}_j(z_j, w_j) = 0$. Since $\tilde{\Upsilon}_j(z_j, w_j) = z_j$, the tangent map of $\tilde{\Upsilon}_j$ at (z_j, w_j) is simply the first coordinate projection.

On the other hand, the maps $\mathcal{Z}_j : \tilde{I}^2 \to \tilde{I}$ are holomorphic on the interior of $\tilde{I}^2$ and the subsets $\mathcal{B}_j$ of $\tilde{I}^2$ are uniformly distant to $\partial \tilde{I}^2$ (since by assumption the points $(z_j, w_j) \in \mathcal{B}_j$ are uniformly distant to the boundary and $\operatorname{diam}(\mathcal{B}_j)$ is small). By the Cauchy integral formula, the second differential of $\mathcal{Z}_j$ is then uniformly C^0-bounded on $\mathcal{B}_j$ among $j \in \mathbb{N}$ by a constant C. Thus the second differential of $\tilde{\Upsilon}_j$ is C^0-bounded by $2C \cdot |\sigma_j|^{-1}$ on $\mathcal{B}_j$. Since $\operatorname{diam}(\mathcal{B}_j) \cdot 2C \cdot |\sigma_j|^{-1}$ is small when j is large (by assumption) and $(z_j, w_j) \in \mathcal{B}_j$, the mean value inequality implies that $\tilde{\Upsilon}_j$ is C^1-close on $\mathcal{B}_j$ to its tangent map at (z_j, w_j). □

Appendix C. Standard results from hyperbolic theory

Let M be a manifold and $1 \leq r \leq \infty$. We recall that a compact subset $K \subset U$ is *hyperbolic* for a C^r-endomorphism f of M if it is invariant ($K = f(K)$) and there exists $n \geq 1$ and open cone fields χ_s and χ_u such that for every $z \in K$, it holds:

(1) the union of $\chi_s(z)$ and $\chi_u(z)$ is $T_z M \setminus \{0\}$,

(2) $D_z f$ sends the closure of $\chi_u(z)$ into $\chi_u(f(z)) \cup \{0\}$ and the preimage by $D_z f$ of the closure of $\chi_s(f(z))$ is in $\chi_s(z) \cup \{0\}$,

(3) every vector in $\chi_u(z)$ is expanded by $D_z f^n$, every vector in $D_{f^n(z)} f^{-n}(\chi_s(f^n(z)))$ is contracted by $D_z f^n$.

We recall that the inverse limit $\overleftrightarrow{K} := \{(x_i)_{i \in \mathbb{Z}} \in K^{\mathbb{Z}} : f(x_{i-1}) = x_i\}$ is endowed with the topology induced by the product one. The map $\overleftrightarrow{f} := (x_i)_{i \in \mathbb{Z}} \in \overleftrightarrow{K} \mapsto (f(x_i))_{i \in \mathbb{Z}} \in \overleftrightarrow{K}$ is semi-conjugate to $f|K$ via $h_0 : (x_i)_{i \in \mathbb{Z}} \mapsto x_0$. Note that if $f|K$ is a homeomorphism then h_0 is a homeomorphism.

Theorem C.1 (Anosov, Quandt [Qua88, Proposition 1])**.** *For every f' C^1-close to f, there exists a unique continuous map $h_{f'} : \overleftrightarrow{K} \to M$ which is C^0-close to h_0 and such that:*

— $h_{f'} \circ \overleftrightarrow{f} = f' \circ h_{f'}$,

— $K_{f'} := h_{f'}(\overleftrightarrow{K})$ *is hyperbolic for f', and called the hyperbolic continuation of K,*

— *the map $f' \mapsto h_{f'}(\overleftrightarrow{k})$ is of class C^∞ and depends continuously on $\overleftrightarrow{k} \in \overleftrightarrow{K}$ for the C^1-topology,*

— *if $f'|K_{f'}$ is a homeomorphism, then $h_{f'}$ is a homeomorphism.*

Remark C.2. *Quandt did not state that $f' \mapsto h_{f'}(\overleftrightarrow{k})$ is of class C^∞ but he uses the fact that this map is the fixed point of a hyperbolic operator, and so the implicit function theorem implies that $f' \mapsto h_{f'}(\overleftrightarrow{k})$ is of class C^∞.*

Remark C.3. *If f is a diffeomorphism, then $h_{f'}$ is bi-Hölder with exponent close to 1 when f' is close to f.*

For a proof see for instance [Yoc95, Theorem §2.6]. For every $k \in K$ and $\overleftrightarrow{k} \in \overleftrightarrow{K}$, we define their respective stable and unstable manifolds by:

$$\text{(C.1)} \quad W^s(k,f) = \{k' \in M : \lim_{i\to\infty} d(f^i(k), f^i(k')) = 0\} \quad \text{and}$$
$$W^u(\overleftrightarrow{k}, f) = \{k'_0 \in M : \exists (k'_i)_{i<0}, f(k'_{i-1}) = k'_i, \lim_{i\to-\infty} d(k_i, k'_i) = 0\}$$

and for $\eta > 0$ let:

$$\text{(C.2)} \quad W^s_\eta(k,f) = \{k' \in M : \ \eta > d(f^i(k), f^i(k')) \overset{i\to\infty}{\longrightarrow} 0\} \quad \text{and}$$
$$W^u_\eta(\overleftrightarrow{k}, f) = \{k'_0 \in M : \exists (k'_i)_{i<0}, f(k'_{i-1}) = k'_i, \eta > d(k_i, k'_i) \overset{i\to-\infty}{\longrightarrow} 0\}$$

These sets are properly embedded C^r manifolds which depend continuously on $k \in K$ and $\overleftrightarrow{k} \in \overleftrightarrow{K}$ for the C^r-topology (see for instance [Yoc95, Theorem §3.6] or [Ber10, Proposition 9.1]). For the sake of simplicity let us assume that all the unstable manifolds have the same dimension d_u and all the stable manifolds have the same dimension d_s. By continuity and invariance, this is the case when K is transitive. Let us recall the following that we are going to generalize and prove in Lemma C.6:

Lemma C.4 (Inclination Lemma for endomorphism)**.** *Let Γ be a d_u-submanifold which is transverse to $W^s_\eta(k,f)$ for $k \in K$ at a point z. Then for every n sufficiently large, there exists a neighborhood Γ' of z in Γ such that $f^n(\Gamma')$ is a submanifold which is uniformly C^r-close to $W^u_\eta(\overleftrightarrow{k^n}, f)$, among any $\overleftrightarrow{k^n} \in \overleftrightarrow{K}$ such that $h_0(\overleftrightarrow{k^n}) = f^n(k)$.*

For $d \geq 0$ and let $\mathcal{P}$ be a regular compact subset of $\mathbb{R}^d$: it is nonempty and equal to the closure of its interior. We recall that a family $(f_p)_{p\in\mathcal{P}}$ of maps $f_p \in C^r(M,M)$ is of class C^r if the map $(p,x) \in \mathcal{P} \times M \mapsto f_p(x) \in M$ extends to a C^r map from a neighborhood $\hat{\mathcal{P}} \times M$ of $\mathcal{P} \times M$. By regularity of $\mathcal{P} \times M$, the C^r-jet of this map is uniquely defined at every point in $\mathcal{P} \times M$. We say that two families are C^r-close if their C^r-jet functions are close for the C^0-compact-open topology.

Assume that $0 \in \mathcal{P}$ and a compact hyperbolic set K_0 for f_0 persists by Theorem C.1 for every $p \in \mathcal{P}$. We denote $h_p := h_{f_p}$ the semi-conjugacy and $K_p := h_p(K_0)$ its hyperbolic continuation at the parameter $p \in \mathcal{P}$. For $p \in \mathcal{P}$ and $\overleftrightarrow{k_0} \in \overleftrightarrow{K}_0$, we put $k_p := h_p(\overleftrightarrow{k_0})$ and $\overleftrightarrow{k}_p := (h_p \circ \overleftrightarrow{f_0}^n(\overleftrightarrow{k_0}))_{n\in\mathbb{Z}}$.

Theorem C.5. *The families $(W^s_\eta(k_p, f_p))_{p\in\mathcal{P}}$ and $(W^u_\eta(\overleftrightarrow{k_p}, f_p))_{p\in\mathcal{P}}$ of C^r-submanifolds are of class C^r and depend continuously on respectively $k_0 \in K_0$ and $\overleftrightarrow{k_0} \in \overleftrightarrow{K}_0$.*

We will prove this result below. The following is a generalization of Lemma C.4 for families.

Lemma C.6 (Parametric Inclination Lemma [Ber16, Ber19, Lemma 1.7])**.** *Let $(\Gamma_p)_{p\in\mathcal{P}}$ be a C^r-family of d_u-submanifolds such that for some $k_0 \in K_0$, the manifold Γ_p intersects uniformly transversally $W^s_\eta(k_p, f_p)$ at a point z_p depending continuously on $p \in \mathcal{P}$. Then when $n \geq 0$ is large, there exists a neighborhood Γ'_p of z_p in Γ_p such that $f^n_p(\Gamma'_p)$ is a submanifold C^r-close to $W^u_\eta(\overleftrightarrow{k^n_p}, f_p)$ and $(f^n_p(\Gamma'_p))_{p\in\mathcal{P}}$ is C^r-close to $(W^u_\eta(\overleftrightarrow{k^n_p}, f_p))_p$ among any $\overleftrightarrow{k^n_0} \in \overleftrightarrow{K}_0$ such that $h_0(\overleftrightarrow{k^n_0}) = f^n_0(k_0)$.*

Proof. The result being semi-local, we may assume that f_p is a map from a small neighborhood U_p of K_p onto its image for every p. Then we extend f_p to a larger open set $U'_p \Supset U_p$ containing a

hyperbolic fixed point H_p such that Γ_p is a slice of the unstable manifold $W^u(H_p, f_p)$ and such that $f_p^{-n}(\Gamma_p)$ is disjoint from U_p for every $n \geq 0$. Then we consider the compact set $\hat{K}_p := K_p \cup \{f^n(z_p) : n \in \mathbb{Z}\} \cup \{H_p\}$. We notice that $\hat{K}_p$ is a hyperbolic compact set. Also $\Gamma_p^n = W_\eta^u(f_p^n(z_p), f_p)$. Thus the lemma follows from Theorem C.5. □

Proof of Theorem C.5. Let $\hat{\mathcal{P}}$ be an open set such that there exists a C^r-extension $(f_p)_{p\in\hat{\mathcal{P}}}$ of $(f_p)_{p\in\mathcal{P}}$ for which the compact hyperbolic set persists to $(K_p)_{p\in\hat{\mathcal{P}}}$. Take $p_0 \in \mathcal{P}$. Let $r > 0$ be small such that the balls B_r and B_{2r} centered at p_0 and of radii r and $2r$ respectively are included in $\hat{\mathcal{P}}$. As the statement to be proved is local, it suffices to show it for the family $(f_p)_{p\in B_r}$. Let

$$\rho : (x_1, ..., x_{d+1}) \in \mathbb{S}^d = \{(x_1, ..., x_{d+1}) \in \mathbb{R}^{d+1} : \sum x_i^2 = 1\} \mapsto p_0 + 2r(x_1, ..., x_d) \in B_{2r}$$

Note that $\rho^{-1}(B_r)$ is formed by two components which are diffeomorphically mapped to B_r by ρ. Thus it suffices to show the proposition for the family $(f_{\rho(s)})_{s\in\mathbb{S}^d}$. For every $\overleftrightarrow{k} \in \overleftrightarrow{K}_{p_0}$, let

$$\mathcal{L}(\overleftrightarrow{k}) := \{(s, h_{\rho(s)}(\overleftrightarrow{k})) : s \in \mathbb{S}^d\} \subset \mathbb{S}^d \times M.$$

By Theorem C.1, the sphere $\mathcal{L}(\overleftrightarrow{k})$ are of class C^r and depends continuously on $\overleftrightarrow{k} \in \overleftrightarrow{K_{p_0}}$ for this topology. Hence $\bigcup_{\overleftrightarrow{k}\in\overleftrightarrow{K}} \mathcal{L}(\overleftrightarrow{k})$ is the image of $\mathbb{S}^d \times \overleftrightarrow{K}$ by a C^r-immersion of this lamination. Moreover this lamination is r-normally hyperbolic in the sense of [Ber10, Definition 2.1]. Consequently, we can apply [Ber10, Proposition 9.1] which states that the strong stable and unstable local manifolds of a leaf form C^r-immersed laminations. This is equivalent to say that $(W_\eta^s(k_{\rho(s)}; f_{\rho(s)}))_{s\in\mathbb{S}^d}$ and $(W_\eta^u(\overleftrightarrow{k}_{\rho(s)}; f_{\rho(s)}))_{s\in\mathbb{S}^d}$ are C^r-families which depends continuously on $k_{p_0} \in K_{p_0}$ and $\overleftrightarrow{k}_{p_0} \in \overleftrightarrow{K}_{p_0}$. □

The following enables to extend the stable lamination to a C^1-foliation, by integrating the C^1-line field $\ell(p, \cdot)$.

Theorem C.7. *If $(f_p)_{p\in\mathcal{P}}$ is a C^2-family of diffeomorphisms of a surface M leaving invariant the continuation $(K_p)_p$ of a hyperbolic set, there exists a neighborhood $\hat{V} = \bigcup_{p\in\mathcal{P}}\{p\} \times V_p$ of $\bigcup_{p\in\mathcal{P}}\{p\} \times K_p$ and a C^1-function ℓ from $\hat{V}$ into $\mathbb{P}(\mathbb{R}^2)$ satisfying for every $p \in \mathcal{P}$:*

$$\ell(p, z) = T_z W_\eta^s(k) \quad \text{if } z \in W_\eta^s(k) \text{ with } k \in K_p \quad \text{and} \quad D_z f_p(\ell(p, z)) = \ell(p, f_p(z)) \quad \text{if } z \in V_p \cap f_p^{-1}(V_p).$$

Proof. The proof is the parametric counterpart of [BC16, Corollary 1.11]. Let $\mathbb{P}(TM)$ be the bundle over M whose fibers at $z \in M$ is the Grassmanian $\mathbb{P}(T_zM)$ of lines in T_zM. We consider:

$$\hat{f} : (p, z, \ell) \in \hat{\mathcal{P}} \times \mathbb{P}(TM) \mapsto (p, f_p(z), D_z f_p(\ell)) \in \hat{\mathcal{P}} \times \mathbb{P}(TM),$$

where $\hat{\mathcal{P}}$ is the open neighborhood of $\mathcal{P}$ to which the family extends. We remark that $\hat{f}$ is of class C^1 on $\hat{M} := \hat{\mathcal{P}} \times \mathbb{P}(TM)$. We denote $E_p^s(k) := T_k W^s(k, f_p)$ and $E_p^u(k) := T_k W^u(k, f_p)$ for every $k \in K_p$ and $p \in \mathcal{P}$. Let $\hat{K} := \{(p, k, E_p^s(k)) : p \in \mathcal{P} \quad \text{and} \quad k \in K_p\}$. It is a partially hyperbolic compact set for $\hat{f}$ with strong unstable direction at $(p, k, E_p^s(k))$ equal to $\{(0, 0)\} \times T_{E_p^s(k)}\mathbb{P}(T_kM)$. The strong unstable manifold of a point $(p, k, E_p^s(k))$ is $(p, k, \mathbb{P}(T_kM) \setminus \{E_p^u(k)\})$. It intersects $\hat{K}$ only at one point. Thus we can apply the following:

Theorem C.8 ([BC16, Main result])**.** *Let $\hat{f}$ be a C^1-diffeomorphism of a manifold $\hat{M}$ and $\hat{K}$ a partially hyperbolic compact invariant set such that $T\hat{M}|\hat{K} = E^c \oplus E^{uu}$. Then, the next two properties are equivalent.*

(1) There exists a compact C^1-submanifold S with boundary which:
- *contains $\hat{K}$ in its interior,*
- *is tangent to E^c at each point of $\hat{K}$ (i.e. $T_xS = E^c(x)$ for each $x \in \hat{K}$),*
- *is locally invariant: $\hat{f}(S)$ contains a neighborhood of $\hat{K}$ in S,*

(2) The strong unstable manifold of any $x \in \hat{K}$ intersect $\hat{K}$ only at x (i.e. $W^{uu}(x) \cap \hat{K} = \{x\}$).

Remark C.9. *By* [BC16, Corollary 1.5]*, if $\hat{f}$ is of class C^{1+}, then S can be chosen of class C^{1+}.*

As the strong unstable direction is the last coordinate, up to restricting S, there exists a function from a neighborhood $\hat{V}$ of $\hat{K}$ and a C^1-function ℓ from $\hat{V}$ into $\mathbb{P}(\mathbb{R}^2)$ whose graph is equal to S. Thus by invariance and since the action of $\hat{f}$ on the two first coordinates is $(p, z) \mapsto (p, f_p(z))$, we have:

$$\hat{f}(p, z, \ell(p, z)) = (p, f_p(z), \ell(p, f_p(z))), \quad \forall p \in \mathcal{P} \text{ and } z \in V_p \cap f_p^{-1}(V_p).$$

As by definition $\hat{f}(p, z, \ell(p, z)) = (p, f_p(z), D_z f_p(\ell(p, z)))$, we obtain $D_z f_p(\ell(p, z)) = \ell(p, f_p(z))$. Given $p \in \mathcal{P}$, $k \in K_p$ and $z \in W^s_\eta(k, f_p)$, if $\ell(p, z) \neq T_z W^s_\eta(k, f_p)$, then by invariance $\ell(p, f_p^n(z)) = D_z f_p^n(\ell(p, z)) \neq D_z f_p^n(T_z W^s_\eta(k, f_p)) = T_{f_p^n(z)} W^s_\eta(f_p^n(k), f_p)$ for every $n \geq 0$. Moreover, by hyperbolicity the angle between $D_z f_p^n(\ell(p, z))$ and $D_z f_p^n(T_z W^s_\eta(k, f_p))$ is bounded from below. This contradicts that the angle between $\ell(p, f_p^n(z))$, $\ell(p, f_p^n(k))$ and $T_{f_p^n(z)} W^s_\eta(f_p^n(k), f_p)$ goes to 0 as $d(f_p^n(z), f_p^n(k)) \to 0$. □

Using Remark C.9 we obtain:

Corollary C.10. *If under the assumptions of Theorem C.7, the family $(f_p)_{p \in \mathcal{P}}$ is of class C^{2+}, then the function ℓ given by Theorem C.7 can be assumed of class C^{1+}.*

Appendix D. Modeling a horseshoe by a hyperbolic map of type **A**

Proof of Proposition 2.25. Construction of the maps H_p and verification of property (1). Let $\mathcal{P}$ be a closed ball centered at 0 included in $\hat{\mathcal{P}}$. Hyperbolicity implies the existence of a splitting $TM|K_p = E^s_p \oplus E^u_p$ of two Df_p-invariant directions which are respectively contracted and expanded by Df_p. Let $\hat{K} = \bigcup_{p \in \mathcal{P}} \{p\} \times K_p$. By Theorem C.7 Page 75, we have:

Theorem D.1. *There exist a neighborhood $\hat{U} = \bigcup_{p \in \mathcal{P}} \{p\} \times U_p$ of $\hat{K}$ and two C^1-families $(\ell^s_p)_p$ and $(\ell^u_p)_p$ of line fields ℓ^s_p and ℓ^u_p on U_p such that:*

(1) for every $z \in K_p$, the restriction $\ell^s_p | W^s_{loc}(z, f_p) \cap U_p$ coincides with $TW^s_{loc}(z, f_p) \cap U_p$ and $\ell^u_p | W^u_{loc}(z, f_p) \cap U_p$ coincides with $TW^u_{loc}(z, f_p) \cap U_p$,

(2) for every $z \in U_p \cap f_p^{-1}(U_p)$, the line $D_z f_p(\ell^s_p(z))$ is equal to $\ell^s_p(f_p(z))$ and the line $D_z f_p(\ell^u_p(z))$ is equal to $\ell^u_p(f_p(z))$.

For every $p \in \mathcal{P}$, let $\mathcal{F}^s_p$ and $\mathcal{F}^u_p$ be the foliations on U_p whose tangent spaces to the leaves are respectively ℓ^s_p and ℓ^u_p. For $\delta > 0$, recall that the δ-$\mathcal{F}^s_p$-plaque of $z \in U_p$ is the connected component of z in $\mathcal{F}^s_p(z) \cap B(z, \delta)$, where $\mathcal{F}^s_p(z)$ is the leaf of $\mathcal{F}^s_p$ containing z and $B(z, \delta)$ the closed δ-ball about z. The δ-$\mathcal{F}^u_p$-plaque of $z \in U_p$ is defined similarly. Note that the δ-local stable manifold $W^s_\delta(z; f_p)$ of $z \in K_p$ is a δ- $\mathcal{F}^s_p$-plaque.

Up to shrinking $\hat{U} = \bigcup_p \{p\} \times U_p$, by hyperbolicity of K_p, we can assume that any pair of leaves of $\mathcal{F}_p^s$ and $\mathcal{F}_p^u$ are transverse. Observe that for $\delta > 0$ sufficiently small, the intersection of every pair of δ-plaques of $\mathcal{F}_p^u$ and $\mathcal{F}_p^s$ consists of at most one point. Then for every $\eta > 0$ sufficiently small, the following property holds true: for any $p \in \mathcal{P}$, $z \in K_p$, $x \in W_\eta^u(z; f_p)$ and $y \in W_\eta^s(z; f_p)$, the δ-$\mathcal{F}_p^s$-plaque of x intersects the δ-$\mathcal{F}_p^u$-plaque of y at exactly one point denoted $[x, y]$.

By identifying both $W_\eta^u(z; f_p)$ and $W_\eta^s(z; f_p)$ to $(-\eta, \eta)$ using the relative distance to z, we obtain a map:

$$\phi_{p,z} : (x, y) \in (-\eta, \eta)^2 \equiv W_\eta^u(z; f_p) \times W_\eta^s(z; f_p) \mapsto [x, y] \in M$$

Note that $\phi_{p,z}$ is a local C^1-diffeomorphism by transversality. It is injective by the uniqueness of the intersection point between the δ-plaques. Thus $\phi_{p,z}$ is a diffeomorphism onto its image which is an open neighborhood of z. As $(\ell_p^s)_p$ and $(\ell_p^u)_p$ are C^1-families, by construction, the family of maps $(\phi_{p,z})_{p\in\mathcal{P}, z\in K_p}$ is C^1. So by compactness of $\mathcal{P}$ and K_p, there is $\delta' > 0$, such that the image of $\phi_{p,z}$ contains the δ'-neighborhood of z for any $p \in \mathcal{P}$ and $z \in K_p$.

By [Bow08, Theorem 3.12], there is a *Markov partition* of K_0 by rectangles of η-small diameter. This is a finite partition $(R_0^{\mathbf{v}})_{\mathbf{v}\in\mathbf{V}}$ by clopen subsets $R_0^{\mathbf{v}} \subset K_0$ satisfying:

— Each subset $R_0^{\mathbf{v}}$ is a rectangle: there exist two segments $I_0^{\mathbf{v}}, J_0^{\mathbf{v}} \subset (-\eta, \eta)$ satisfying

$$R_0^{\mathbf{v}} = \breve{\phi}_0^{\mathbf{v}}(I_0^{\mathbf{v}} \times J_0^{\mathbf{v}}) \cap K_0 \qquad \text{with} \quad z_0^{\mathbf{v}} \in R_0^{\mathbf{v}} \text{ and } \breve{\phi}_0^{\mathbf{v}} := \phi_{p,z_0^{\mathbf{v}}} \,. \tag{D.1}$$

— For every pair $(\mathbf{v}, \mathbf{w}) \in \mathbf{V}^2$ such that $R_0^{\mathbf{vw}} := R_0^{\mathbf{v}} \cap f_0^{-1}(R_0^{\mathbf{w}}) \neq \emptyset$, there are two[12] segments $I_0^{\mathbf{vw}} \subset I_0^{\mathbf{v}}$ and $J_0^{\mathbf{vw}} \subset J_0^{\mathbf{w}}$ such that:

$$R_0^{\mathbf{vw}} = \breve{\phi}_0^{\mathbf{v}}(I_0^{\mathbf{vw}} \times J_0^{\mathbf{v}}) \cap K_0 \quad \text{and} \quad f_0 \circ \breve{\phi}_0^{\mathbf{v}}(I_0^{\mathbf{vw}} \times J_0^{\mathbf{v}}) = \breve{\phi}_0^{\mathbf{w}}(I_0^{\mathbf{w}} \times J_0^{\mathbf{vw}}) \,. \tag{D.2}$$

We assume the segments $I_0^{\mathbf{v}}$ and $J_0^{\mathbf{v}}$ minimal such that Eq. (D.1) holds true. Then $I_0^{\mathbf{vw}}$ and $J_0^{\mathbf{vw}}$ are uniquely defined by Eq. (D.2). Let $(R_p^{\mathbf{v}})_{p\in\mathcal{P}}$, $(R_p^{\mathbf{vw}})_{p\in\mathcal{P}}$ and $(z_p^{\mathbf{v}})_{p\in\mathcal{P}}$ be the hyperbolic continuations of $R_0^{\mathbf{v}}$, $R_0^{\mathbf{vw}}$ and $z_0^{\mathbf{v}}$ (see Theorem C.1 Page 73). For $\mathcal{P}$ sufficiently small, for every $p \in \mathcal{P}$, there are minimal segments $I_p^{\mathbf{v}}$ and $J_p^{\mathbf{v}}$ included in $(-\eta, \eta)$ satisfying

$$R_p^{\mathbf{v}} = \breve{\phi}_p^{\mathbf{v}}(I_p^{\mathbf{v}} \times J_p^{\mathbf{v}}) \cap K_p \qquad \text{with} \quad \breve{\phi}_p^{\mathbf{v}} := \phi_{p,z_p^{\mathbf{v}}} \tag{D.3}$$

and there are segments $I_p^{\mathbf{vw}} \subset I_p^{\mathbf{v}}$ and $J_p^{\mathbf{vw}} \subset J_p^{\mathbf{w}}$ such that:

$$R_p^{\mathbf{vw}} = \breve{\phi}_p^{\mathbf{v}}(I_p^{\mathbf{vw}} \times J_p^{\mathbf{v}}) \cap K_p \quad \text{and} \quad f_p \circ \breve{\phi}_p^{\mathbf{v}}(I_p^{\mathbf{vw}} \times J_p^{\mathbf{v}}) = \breve{\phi}_p^{\mathbf{w}}(I_p^{\mathbf{w}} \times J_p^{\mathbf{vw}}) \,. \tag{D.4}$$

Fact D.2. *The segments $I_p^{\mathbf{v}}$, $J_p^{\mathbf{v}}$, $I_p^{\mathbf{vw}}$ and $J_p^{\mathbf{vw}}$ vary C^1 with $p \in \mathcal{P}$, that is each of them has its two boundary points varying C^1 with $p \in \mathcal{P}$.*

Proof. Let us prove the fact for $(I_p^{\mathbf{v}})_p$, the proof is the same in the other cases. By the local product structure of K_p, by hyperbolic continuation of $(K_p)_p$ and by definition of $I_p^{\mathbf{v}}$ as the minimal segment satisfying Eq. (D.3), there exist C^1-families of points $k_p^- \in K_p$ and $k_p^+ \in K_p$ such that $\{k_p^-, k_p^+\} = \breve{\phi}_p^{\mathbf{v}}(\partial I_p^{\mathbf{v}})$. As $(\breve{\phi}_p^{\mathbf{v}})_p = (\phi_{p,z_p^{\mathbf{v}}})_p$ is of class C^1, the endpoints $\partial I_p^{\mathbf{v}}$ vary C^1 with $p \in \mathcal{P}$. The proof of the regularity of the other endpoints is similar. □

12. When the diameters of the rectangle are not small, one has to consider more segments.

Let:

$$Q_p^{\mathbf{v}} := \breve{\phi}_p^{\mathbf{v}}(I_p^{\mathbf{v}} \times J_p^{\mathbf{v}}) \ , \quad \partial^s Q_p^{\mathbf{v}} := \breve{\phi}_p^{\mathbf{v}}(\partial(I_p^{\mathbf{v}}) \times J_p^{\mathbf{v}}) \quad \text{and} \quad \partial^u Q_p^{\mathbf{v}} := \breve{\phi}_p^{\mathbf{v}}(I_p^{\mathbf{v}} \times \partial(J_p^{\mathbf{v}})) \, .$$

Fact D.3. *The compact sets $(Q_p^{\mathbf{v}})_{\mathbf{v}}$ are disjoint for every $p \in \mathcal{P}$.*

Proof. For every $\mathbf{v} \in \mathbf{V}$, by minimality of $I_p^{\mathbf{v}}$ and Eq. (D.3), each of the two components of $\partial^s Q_p^{\mathbf{v}}$ contains a point of K_p. Hence, by Theorem D.1 (1), each of the two components of $\partial^s Q_p^{\mathbf{v}}$ is equal to a local stable manifold of it. Likewise by minimality of $J_p^{\mathbf{v}}$, Eq. (D.3) and Theorem D.1 (1), each of the two components of $\partial^u Q_p^{\mathbf{v}}$ is equal to a local unstable manifold of a point of K_p. Hence by local maximality of K_p, the corners $\partial^u Q_p^{\mathbf{v}} \cap \partial^s Q_p^{\mathbf{v}}$ of $Q_p^{\mathbf{v}}$ are in K_p. Hence for $\mathbf{v}, \mathbf{w} \in \mathbf{V}$, if $Q_p^{\mathbf{v}}$ intersects $Q_p^{\mathbf{w}}$ then the sets $Q_p^{\mathbf{v}}$ and $Q_p^{\mathbf{w}}$ are nested or a stable boundary of $Q_p^{\mathbf{v}}$ (resp. $Q_p^{\mathbf{w}}$) intersects an unstable boundary of $Q_p^{\mathbf{w}}$ (resp. $Q_p^{\mathbf{v}}$). Thus if $Q_p^{\mathbf{v}}$ intersects $Q_p^{\mathbf{w}}$, then $R_p^{\mathbf{v}} = Q_p^{\mathbf{v}} \cap K_p$ intersects $R_p^{\mathbf{w}} = Q_p^{\mathbf{w}} \cap K_p$ and so $\mathbf{v} = \mathbf{w}$. □

Let $k \geq 1$ be a large integer so that $f_p^k | K_p$ is still transitive. Put:

$$\mathbf{A} := \left\{ (\mathbf{v}_0, \dots, \mathbf{v}_k) \in \mathbf{V}^{k+1} : \bigcap_{j=0}^{k} f_p^{-j}(R_p^{\mathbf{v}_j}) \neq \emptyset \right\} \ , \quad \mathbf{o}(\mathbf{v}_0, \dots, \mathbf{v}_k) = \mathbf{v}_0 \quad \text{and} \quad \mathbf{t}(\mathbf{v}_0, \dots, \mathbf{v}_k) = \mathbf{v}_k \ .$$

Put $R_p^{\mathbf{a}} := \bigcap_{j=0}^{k} f_p^{-j}(R_p^{\mathbf{v}_j})$ for every $\mathbf{a} \in \mathbf{A}$ and $p \in \mathcal{P}$. We recall that each $\breve{\phi}_p^{\mathbf{w}}$ has its range in $U_p \cap f_p^{-1}(U_p)$, and so each map $(\breve{\phi}_p^{\mathbf{w}})^{-1} \circ f_p \circ \breve{\phi}_p^{\mathbf{v}}$ preserves the horizontal and vertical directions, and respectively expands and contracts them. Thus by Eq. (D.4), for every $\mathbf{a} \in \mathbf{A}$, there exist unique $I_p^{\mathbf{a}} \subset I_p^{\mathbf{o}(\mathbf{a})}$ and $J_p^{\mathbf{a}} \subset J_p^{\mathbf{t}(\mathbf{a})}$ such that:

$$R_p^{\mathbf{a}} = \breve{\phi}_p^{\mathbf{v}}(I_p^{\mathbf{a}} \times J_p^{\mathbf{o}(\mathbf{a})}) \cap K_p \quad \text{and} \quad f_p^k \circ \breve{\phi}_p^{\mathbf{o}(\mathbf{a})}(I_p^{\mathbf{a}} \times J_p^{\mathbf{o}(\mathbf{a})}) = \breve{\phi}_p^{\mathbf{t}(\mathbf{a})}(I_p^{\mathbf{t}(\mathbf{a})} \times J_p^{\mathbf{a}}) \ . \tag{D.5}$$

For every $\mathbf{a} \in \mathbf{A}$ and $p \in \mathcal{P}$, the following map preserves again the horizontal and vertical directions

$$\breve{f}_p^{\mathbf{a}} := (\breve{\phi}_p^{\mathbf{t}(\mathbf{a})})^{-1} \circ f_p^k \circ \breve{\phi}_p^{\mathbf{o}(\mathbf{a})} | I_p^{\mathbf{a}} \times J_p^{\mathbf{o}(\mathbf{a})} \, . \tag{D.6}$$

We assume k large enough so that it satisfies moreover:

$$\min |\partial_x \breve{f}_p^{\mathbf{a}}| > \frac{C \cdot \lambda}{\theta^2} \ , \quad \max |\partial_y \breve{f}_p^{\mathbf{a}}| < \frac{\theta^2}{C \cdot \lambda} \quad \text{where } C := \max_{(\mathbf{v},\mathbf{w}) \in \mathbf{V}^2} \left\{ \frac{|I_p^{\mathbf{v}}|}{|I_p^{\mathbf{w}}|}, \frac{|J_p^{\mathbf{v}}|}{|J_p^{\mathbf{w}}|} \right\} \ . \tag{D.7}$$

Thus there are $\theta^2/(C\lambda)$-contracting maps $\breve{x}_p^{\mathbf{a}} : I_p^{\mathbf{t}(\mathbf{a})} \to I_p^{\mathbf{a}}$ and $\breve{y}_p^{\mathbf{a}} : J_p^{\mathbf{o}(\mathbf{a})} \to J_p^{\mathbf{a}}$ such that:

$$\breve{f}_p^{\mathbf{a}}(\breve{x}_p^{\mathbf{a}}(x_1), y_0) = (x_1, \breve{y}_p^{\mathbf{a}}(y_0)) \ , \tag{D.8}$$

for any $(x_1, y_0) \in I_p^{\mathbf{t}(\mathbf{a})} \times J_p^{\mathbf{o}(\mathbf{a})}$. Note that $\breve{x}_p^{\mathbf{a}}$ and $\breve{y}_p^{\mathbf{a}}$ are defined via Eqs. (D.6) and (D.8) on some η'-neighborhoods $V_p^{\mathbf{t}(\mathbf{a})}$ and $W^{\mathbf{o}(\mathbf{a})}$ of $I_p^{\mathbf{t}(\mathbf{a})}$ and $J_p^{\mathbf{o}(\mathbf{a})}$ in such a way that Eq. (D.8) remains valid. If η' is sufficiently small, these maps are still $\frac{\theta^2}{C \cdot \lambda}$-contracting by Eq. (D.7). Let $\eta' > 0$ be small enough so that these properties are valid for every $\mathbf{a}$ in the finite set $\mathbf{A}$ and p in the compact set $\mathcal{P}$. By contraction, it holds:

$$\breve{x}_p^{\mathbf{a}}(V_p^{\mathbf{t}(\mathbf{a})}) \Subset V_p^{\mathbf{o}(\mathbf{a})} \quad \text{and} \quad \breve{y}_p^{\mathbf{a}}(W_p^{\mathbf{o}(\mathbf{a})}) \Subset W_p^{\mathbf{t}(\mathbf{a})} \ . \tag{D.9}$$

For every $\mathbf{v} \in \mathbf{V}$, let $\Delta_p^{\mathbf{v}}$ be the affine map with diagonal and positive linear part which sends $Y^{\mathbf{e}}$ to $V_p^{\mathbf{v}} \times W_p^{\mathbf{v}}$. We define the C^1-family $(\breve{H}_p)_{p \in \mathcal{P}}$ by:

$$\breve{H}_p : (z, \mathbf{v}) \in Y^{\mathbf{e}} \times \mathbf{V} \mapsto \breve{H}_p^{\mathbf{v}}(z) \in M \ , \quad \text{with } \breve{H}_p^{\mathbf{v}} : z \in Y^{\mathbf{e}} \mapsto \breve{\phi}_p^{\mathbf{v}} \circ \Delta_p^{\mathbf{v}}(z) \ .$$

Let $\breve{Y}_p^{\mathbf{a}}$ be the preimage by $\Delta_p^{\mathbf{o}(\mathbf{a})}$ of $\breve{x}_p^{\mathbf{a}}(V_p^{\mathbf{t}(\mathbf{a})}) \times W^{\mathbf{o}(\mathbf{a})}$. By Eq. (D.9), $\breve{Y}_p^{\mathbf{a}}$ is included in $Y^{\mathbf{e}} \setminus \partial^s Y^{\mathbf{e}}$. Also, $\breve{Y}_p^{\mathbf{a}}$ is sent by $\breve{f}_p^{\mathbf{a}} \circ \Delta_p^{\mathbf{o}(\mathbf{a})}$ onto $V_p^{\mathbf{t}(\mathbf{a})} \times \breve{y}_p^{\mathbf{a}}(W^{\mathbf{o}(\mathbf{a})}) \subset \Delta_p^{\mathbf{t}(\mathbf{a})}(Y^{\mathbf{e}} \setminus \partial^u Y^{\mathbf{e}})$. Put:

$$\breve{F}_p^{\mathbf{a}} := (\Delta_p^{\mathbf{t}(\mathbf{a})})^{-1} \circ \breve{f}_p^{\mathbf{a}} \circ \Delta_p^{\mathbf{o}(\mathbf{a})} : \breve{Y}_p^{\mathbf{a}} \to Y^{\mathbf{e}} \setminus \partial^u Y^{\mathbf{e}} .$$

Let $(\breve{F}_p^{\mathbf{A}})_{p\in\mathcal{P}}$ be the C^1-family of maps $\breve{F}_p^{\mathbf{A}}$ from $\breve{D}_p(\mathbf{A}) := \bigsqcup_{\mathbf{a}\in\mathbf{A}} \breve{Y}_p^{\mathbf{a}} \times \{\mathbf{o}(\mathbf{a})\}$ equal to $(\breve{F}_p^{\mathbf{a}}, \mathbf{t}(\mathbf{a}))$ at $\breve{Y}_p^{\mathbf{a}} \times \{\mathbf{o}(\mathbf{a})\}$. Then Proposition 2.25 (0) holds true since $\breve{F}_p^{\mathbf{A}}$ leaves invariant the vertical and horizontal directions, and by Eq. (D.6):

$$f_p^k \circ \breve{H}_p^{\mathbf{o}(\mathbf{a})}|\breve{Y}_p^{\mathbf{a}} = \breve{\phi}_p^{\mathbf{t}(\mathbf{a})} \circ \breve{f}_p^{\mathbf{a}} \circ \Delta_p^{\mathbf{o}(\mathbf{a})} = \breve{H}_p^{\mathbf{t}(\mathbf{a})} \circ \breve{F}_p^{\mathbf{a}} . \tag{D.10}$$

Let $(H_p)_{p\in\mathcal{P}}$ be a C^r-family which is C^1-close to $(\breve{H}_p)_{p\in\mathcal{P}}$. For every $\mathbf{v} \in \mathbf{V}$ and $z \in Y^{\mathbf{e}}$, let $H_p^{\mathbf{v}}(z) := H_p(z, \mathbf{v})$.

Fact D.4. *Every nonzero vector in the complement of χ_v is sent by $D\breve{F}_p^{\mathbf{A}}$ into χ_h and λ-expands its horizontal component, also $D(\breve{F}_p^{\mathbf{A}})^{-1}$ sends every nonzero vector in the complement of χ_h into χ_v and expands its vertical component.*

Proof. This is a consequence of Eq. (D.7) which implies $|\partial_x \breve{F}_p^{\mathbf{A}}| > \lambda/\theta^2$ and $|\partial_y \breve{F}_p^{\mathbf{A}}| < \theta^2/\lambda$. □

Also by Eq. (D.10) and then Eqs. (D.8) and (D.9), for every $(x_1, y_0) \in I^2$, there exist $\breve{\mathcal{X}}_p^{\mathbf{a}}(x_1)$ and $\breve{\mathcal{Y}}_p^{\mathbf{a}}(y_0)$ in the interior of I such that:

$$f_p^k \circ \breve{H}_p^{\mathbf{o}(\mathbf{a})}(\breve{\mathcal{X}}_p^{\mathbf{a}}(x_1), y_0) = \breve{H}_p^{\mathbf{t}(\mathbf{a})} \circ \breve{F}_p^{\mathbf{a}}(\breve{\mathcal{X}}_p^{\mathbf{a}}(x_1), y_0) = \breve{H}_p^{\mathbf{t}(\mathbf{a})}(x_1, \breve{\mathcal{Y}}_p^{\mathbf{a}}(y_0)) .$$

Thus, as $H_p^{\mathbf{o}(\mathbf{a})}$ and $H_p^{\mathbf{t}(\mathbf{a})}$ are C^1-close to $\breve{H}_p^{\mathbf{o}(\mathbf{a})}$ and $\breve{H}_p^{\mathbf{t}(\mathbf{a})}$, by hyperbolicity of $\breve{F}_p^{\mathbf{a}}$ and the implicit function theorem, there exist unique points $\mathcal{X}_p^{\mathbf{a}}(x_1, y_0)$ and $\mathcal{Y}_p^{\mathbf{a}}(x_1, y_0)$ close to $\breve{\mathcal{X}}_p^{\mathbf{a}}(x_1)$ and $\breve{\mathcal{Y}}_p^{\mathbf{a}}(y_0)$ satisfying :

$$f_p^k \circ H_p^{\mathbf{o}(\mathbf{a})}(\mathcal{X}_p^{\mathbf{a}}(x_1, y_0), y_0) = H_p^{\mathbf{t}(\mathbf{a})}(x_1, \mathcal{Y}_p^{\mathbf{a}}(x_1, y_0)) .$$

Still by the implicit function theorem, the map $(p, x_1, y_0) \mapsto (\mathcal{X}_p^{\mathbf{a}}, \mathcal{Y}_p^{\mathbf{a}})(x_1, y_0)$ is of class C^r and is C^1-close to $(p, x_1, y_0) \mapsto (\breve{\mathcal{X}}_p^{\mathbf{a}}(x_1), \mathcal{Y}_p^{\mathbf{a}}(y_0))$. Hence the following set:

$$Y_p^{\mathbf{a}} := \{(\mathcal{X}_p^{\mathbf{a}}(x_1, y_0), y_0) : (x_1, y_0) \in I^2\}$$

is a box included in $Y^{\mathbf{e}} \setminus \partial^s Y^{\mathbf{e}}$. Moreover it is sent by $F_p^{\mathbf{a}} := (H_p^{\mathbf{t}(\mathbf{a})})^{-1} \circ f_p^k \circ H_p^{\mathbf{o}(\mathbf{a})}|Y_p^{\mathbf{a}}$ into $\{(x_1, \mathcal{Y}_p^{\mathbf{a}}(x_1, y_0)) : (x_1, y_0) \in I^2\} \subset Y^{\mathbf{e}} \setminus \partial^u Y^{\mathbf{e}}$. Hence by Fact D.4, $(F_p^{\mathbf{a}}, Y_p^{\mathbf{a}})$ is a hyperbolic transformation. Also the C^r-family of maps $F_p^{\mathbf{A}}$ from $D_p(\mathbf{A}) := \bigsqcup_{\mathbf{a}\in\mathbf{A}} Y_p^{\mathbf{a}} \times \{\mathbf{o}(\mathbf{a})\}$ equal to $(F_p^{\mathbf{a}}, \mathbf{t}(\mathbf{a}))$ at $Y_p^{\mathbf{a}} \times \{\mathbf{o}(\mathbf{a})\}$ satisfies Proposition 2.25 (1).

Verification of property (2). If $\limsup_{n\to\infty} \|\det Df_0^n|K_0\|^{1/\epsilon} \cdot \|Df_0^n|K_0\| < 1$, then for k large enough, $\breve{F}_0^{\mathbf{A}}$ and so $F_0^{\mathbf{A}}$ are moderately dissipative. So are any $F_p^{\mathbf{A}}$ for p in a sufficiently small $\mathcal{P}$.

Verification of property (3). We notice that the pull back by H_p of the vector field e_p^s is C^1 close to be tangent to $\{0\} \times \mathbb{R}$ and is defined on $Y^{\mathbf{e}} \times \mathbf{V}$. Thus we can extend it to a C^1-nonzero-vector field on $\mathbb{R} \times I \times \mathbf{V}$ with values in χ_v. Then the holonomy π_p of this vector field restricted to the transverse section $\mathbb{R} \times \{0\} \times \mathbf{V}$ defines a family of projections $(\pi_p)_{p\in\mathcal{P}}$ satisfying (0)-(1)-(2)-(3) of Definition 1.23, which shows property (3) of Proposition 2.25 and concludes the proof. □

Proof of Corollary 2.26. We use Corollary C.10 while using Theorem C.7 in the above proof. □

Appendix E. A general result on Emergence

The aim of this section is to prove Theorem 0.9 Page 8. Let K is a horseshoe of a C^{1+}-surface diffeomorphism f. We want to show that the covering number $\mathcal{N}(\epsilon)$ at scale ϵ of $\mathcal{M}_f(K)$ for the Wasserstein distance (see Section 0.3) has order at least the unstable dimension d_u of K:

$$\liminf_{\epsilon\to 0} \frac{\log\log\mathcal{N}(\epsilon)}{-\log\epsilon} \geq d_u \ .$$

Proof of Theorem 0.9. First we can assume f of class C^∞. Indeed, we can consider a smooth approximation $\tilde{f}$ of f. Then by Remark C.3, the restriction of $\tilde{f}$ to the continuation $\tilde{K}$ of K is conjugated to $f|K$ via a bi-Hölder conjugacy with exponent close to 1. Thus the unstable dimension of $\tilde{K}$ is close to the one of K and the space of invariant probability measures $\mathcal{M}_{\tilde{f}}(\tilde{K})$ of $\tilde{f}|K$ is in bijection to $\mathcal{M}_f(K)$ via a bi-Hölder map with exponent close to 1. So the orders of their covering numbers are close.

Thus we assume that f is of class C^∞. Then by Corollary 2.26 of Proposition 2.25 (for $d=0$), the map $f|K$ is C^{1+}-semi-conjugate (via π) to a C^{1+}-function g on a disjoint union of segments of the real line, which leaves invariant an expanding Cantor set Λ_g of Hausdorff dimension equal to the unstable dimension d^u of K.

Then by [BB21, Theorem A] the covering number $\mathcal{N}_g$ of the space of invariant measures of $g|\Lambda_g$ has order d^u:

$$\log\log\mathcal{N}_g(\eta) \sim -d^u\cdot\log\eta, \quad \text{as } \eta\to 0$$

The push forward by the Lipschitz semi-conjugacy is a Lipschitz map from the space of invariant measure of $F^{\mathbf{A}}$ to those of g. By the same reasoning as in Lemma 4.35, this pushforward is a bijection between the spaces of invariant measures of g and $F^{\mathbf{A}}$. Thus the covering number of the space of invariant measure of $F^{\mathbf{A}}$ has the sought property. □

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