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The Minimum Dominating Set problem is polynomial for (\textit{claw}, P_8) -free graphs

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Abstract

We prove that the Minimum Dominating Set problem is polynomial for the class of (\textit{claw}, P_8) -free graphs.

Keywords: Minimum Dominating Set, polynomial time, claw-free graph, P_k -free graph.

1 Introduction

M. Yannakakis and F. Gavril [9] showed in 1980 that the Minimum Dominating Set problem restricted to claw-free graphs is NP -complete. Then in 1984, A. Bertossi [2] showed that the Minimum Dominating Set problem is also NP -complete for split graphs, a subclass of P_5 -free graphs. More recently, in 2016, D. Malyshev [8] proved that the Minimum Dominating Set problem is polynomial for $(K_{1,4}, P_5)$ -free graphs, hence for (\textit{claw}, P_5) -free graphs. To our knowledge, the complexity of the Minimum Dominating Set problem is unknown for (\textit{claw}, P_k) -free graphs for every fixed $k \geq 6$. We show that the Minimum Dominating Set problem is polynomial for (\textit{claw}, P_8) -free graphs.

Definitions and notations

We are only concerned with simple undirected graphs $G = (V, E)$. The reader is referred to [3] for definitions and notations in graph theory. For $v \in V$, $N(v)$ denotes its neighborhood and $N[v] = N(v) \cup \{v\}$ its closed neighborhood. A vertex v is *universal* if $N[v] = V$. For $v \in V$ and $A \subseteq V$, we denote by $N_A(v) = N(v) \cap A$ ($N_A[v] = (N(v) \cap A) \cup \{v\}$) its (closed) neighborhood in A . For $X \subseteq V$, $A \subseteq V$, we denote $N_A(X) = \bigcup_{x \in X} N_A(x)$ and $N_A[X] = N_A(X) \cup X$.

The *contraction* of an edge $uv \in E$ removes the vertices u and v from V , and replaces them by a new vertex that is adjacent to the previous neighbors of u and v (neither introducing self-loops nor multiple edges). The graph obtained from G after the contraction of uv is denoted by G/uv .

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For $S \subseteq V$, let $G[S]$ denote the subgraph of G induced by S , which has vertex set S and edge set $\{uv \in E \mid u, v \in S\}$. For $v \in V$, we write $G - v = G[V \setminus \{v\}]$ and for a subset $V' \subseteq V$ we write $G - V' = G[V \setminus V']$. For a fixed graph H we write $H \subseteq_i G$ whenever G contains an induced subgraph isomorphic to H . For a set $\{H_1, \dots, H_p\}$ of graphs, G is (H_1, \dots, H_p) -free if G has no induced subgraph isomorphic to a graph in $\{H_1, \dots, H_p\}$; if $p = 1$ we may write H_1 -free instead of (H_1) -free. For two disjoint induced subgraphs $G[A], G[B]$ of G , $G[A]$ is *complete* to $G[B]$ if $ab \in E$ for every $a \in A, b \in B$, $G[A]$ is *anticomplete* to $G[B]$ if $ab \notin E$ for every $a \in A, b \in B$.

For $k \geq 1$, $P_k = u_1 - u_2 - \dots - u_k$ is the *cordless path* on k vertices, that is, $V(P_k) = \{u_1, \dots, u_k\}$ and $E(P_k) = \{u_i u_{i+1} \mid 1 \leq i \leq k - 1\}$. For $k \geq 3$, $C_k = u_1 - u_2 - \dots - u_k - u_1$ is the *cordless cycle* on k vertices, that is, $V(C_k) = \{u_1, \dots, u_k\}$ and $E(C_k) = \{u_i u_{i+1} \mid 1 \leq i \leq k - 1\} \cup \{u_k u_1\}$. For $k \geq 4$, C_k is called a *hole*. A graph without a hole is *chordal*.

A set $S \subseteq V$ is called a *stable set* or an *independent set* if $G[S]$ does not contain any edge. The maximum cardinality of an independent set in G is denoted by $\alpha(G)$. A set $S \subseteq V$ is called a *clique* if $G[S]$ is a *complete graph*, i.e., every pairwise distinct vertices $u, v \in S$ are adjacent. The graph $C_3 = K_3$ is a *triangle*. $K_{1,p}$ is the star on $p + 1$ vertices, that is, the graph with vertices u, v_1, v_2, \dots, v_p and edges uv_1, uv_2, \dots, uv_p . The *claw* is $K_{1,3}$.

A set $S \subseteq V$ is a *dominating set* if every vertex $v \in V$ is either an element of S or is adjacent to an element of S . The minimum cardinality of a dominating set in G is denoted by $\gamma(G)$ and called the *domination number* of G . A dominating set S with $|S| = \gamma(G)$ is called a *minimum dominating set*. Following [7] a minimum dominating set is also called a γ -set. We denote $V^+ \subseteq V$ the subset of vertices v of G such that $\gamma(G - v) > \gamma(G)$. If $S \subseteq V$ is both a dominating and an independent set then S is an *independent dominating set*. The minimum cardinality of an independent dominating set in G is denoted by $i(G)$. Clearly we have $\gamma(G) \leq i(G) \leq \alpha(G)$. Note that a minimum independent dominating set is a *minimum maximal independent set*.

Previous results

We give some results of the literature concerning the Minimum Dominating Set problem that will be useful in the following. D. Bauer et al. showed in [4] that for every non-isolated vertex v , if $v \in V^+$ then v is in every γ -set of G . Allan et al. [1] proved that $\gamma(G) = i(G)$ holds for every claw-free graph. Yannakakis et al. [9] proved that the Minimum Dominating Set problem restricted to claw-free graphs is *NP*-complete. D. Malyshev [8] proved that the Minimum Dominating Set problem is polynomial for $(K_{1,4}, P_5)$ -free graphs hence for (claw, P_5) -free graphs. As Farber [6] proved, a minimum independent dominating set can be determined in linear-time over the class of chordal graphs, the Minimum Dominating Set problem restricted to claw-free chordal graphs is polynomial.

Organization

The next section give some algorithmic properties. Two properties will allow us to make some simplifications on the graphs G that we consider. Two others will help

us to conclude that computing $\gamma(G)$ is polynomial when G have a specific structure relatively to a fixed size subgraph. Then we consider the case where the graph G has a long cycle. From there, we show our main result, starting from (claw, P_6) -free graphs and finishing with (claw, P_8) -free graphs. We conclude by some open questions regarding (claw, P_k) -free graphs for $k \geq 9$.

2 Algorithmic Properties

We give two properties that authorize us to make some assumptions and simplifications for the graphs we consider.

Property 2.1 *Let G be a graph. If u, v are two vertices such that $N[u] = N[v]$ then $\gamma(G) = \gamma(G/uv)$.*

Proof: Let u' be the vertex of G/uv resulting from the contraction of uv . Let Γ be a γ -set of G . At most one of u and v is in Γ . If $u \in \Gamma$ then let $\Gamma' = (\Gamma \setminus \{u\}) \cup \{u'\}$. If $u, v \notin \Gamma$ then let $\Gamma' = \Gamma$. In the two cases Γ' is a dominating set of G/uv , so $\gamma(G) = |\Gamma'| \geq \gamma(G/uv)$. Now suppose that $\gamma(G) > \gamma(G/uv)$. Let Γ' be a γ -set of G/uv . If $u' \in \Gamma'$ then $(\Gamma' \setminus \{u'\}) \cup \{u\}$ is a dominating set of G such that $|(\Gamma' \setminus \{u'\}) \cup \{u\}| = \gamma(G/uv) < \gamma(G)$, a contradiction. If $u' \notin \Gamma'$ then Γ' is a dominating set of G , a contradiction. Hence $\gamma(G) = \gamma(G/uv)$. \square

Property 2.2 *Let $G = (V, E)$ be a connected claw-free graph with $uv \in E$ such that u is a leaf. There exists Γ a minimum dominating set of G that consists of $\Gamma = \{v\} \cup \Gamma'$ where Γ' is a minimum dominating set of $G' = G - N[v]$.*

Proof: Since u is a leaf there exists Γ a minimum dominating set of G with $v \in \Gamma$. Let $w \in N(v) \setminus \{u\}$. Since G is claw-free then $N(w) \setminus N[v]$ is a clique. We can assume that $w \notin \Gamma$, otherwise replacing w by $w' \in N(w) \setminus N[v]$ we have another γ -set of G (note that if $N(w) \setminus N[v]$ is empty then Γ cannot be a minimum dominating set). We show that $\Gamma' = \Gamma \setminus N[v]$ is a minimum dominating set of $G' = G - N[v]$. Clearly Γ' dominates G' . If there exists S a γ -set of G' such that $|S| < |\Gamma'|$ then $S \cup \{v\}$ is a dominating set of G with $|S \cup \{v\}| < |\Gamma|$, a contradiction. \square

As a consequence if a minimum dominating set of $G' = G - N[v]$ can be determined in polynomial time then a minimum dominating set of G can be determined in polynomial time.

We show two conditions on the structure of G that authorize us to directly conclude that computing a γ -set for G can be done in polynomial time.

Property 2.3 *Let $k > 0$ be a fixed positive integer and $G = (V, E)$ be a graph. If there exists $T \subset V$ of size $|T| \leq k$ such that $V = N[T]$ then computing a minimum dominating set for G is polynomial.*

Proof: We have $\gamma(G) \leq k$. So a minimum dominating set can be computed in $O(n^k)$. \square

Property 2.4 *Let $k, k' > 0$ two fixed positive integers and $G = (V, E)$ be a graph. If there exists $T \subset V$ of size $|T| \leq k$ such that $W = V \setminus N[T]$ has a size $|W| \leq k'$ then computing a minimum dominating set for G is polynomial.*

Proof: We have $\gamma(G) \leq k + k'$. So a minimum dominating set can be computed in $O(n^{k+k'})$. \square

3 G has a long cycle

We give two lemmas that will authorize us to conclude that the Minimum Dominating Set problem is polynomial when G , a (claw, P_k) -free graph, contains a long induced cycle.

Lemma 3.1 *For every fixed $k \geq 6$, if G is a (claw, P_k) -free connected graph such that $C_k \subseteq_i G$, then a minimum dominating set of G can be given in polynomial time.*

Proof: Let $C_k = v_1 - \dots - v_k - v_1, C_k \subseteq_i G$. Let $v \notin V(C_k)$ be such that $N(v) \cap V(C_k) \neq \emptyset$. Since G is claw-free and $k \geq 6$, we have $2 \leq |N(v) \cap V(C_k)| \leq 4$. If $|N(v) \cap V(C_k)| = 2$, the two neighbors of v in C_k must be adjacent, thus there is an induced P_k -subgraph that is a contradiction. For $3 \leq |N(v) \cap V(C_k)| \leq 4$, let w be a neighbor of v . If $N(w) \cap V(C_k) = \emptyset$ then there is a claw centered onto v , a contradiction. Hence every neighbor of v has a neighbor in C_k and therefore $N[C_k] = V$. So, from Property 2.3 we can compute a γ -set of G in polynomial time. \square

Lemma 3.2 *For every fixed $k \geq 6$, if G is a (claw, P_k, C_k) -free connected graph such that $C_{k-1} \subseteq_i G$, then a minimum dominating set of G can be given in polynomial time.*

Proof: Let $C_{k-1} = v_1 - \dots - v_{k-1} - v_1, C_{k-1} \subseteq_i G$ and $v \notin V(C_{k-1})$ such that $N(v) \cap V(C_{k-1}) \neq \emptyset$. We have $2 \leq |N(v) \cap V(C_{k-1})| \leq 4$ for $k \geq 7$ and $2 \leq |N(v) \cap V(C_{k-1})| \leq 5$ for $k = 6$. Let w be a neighbor of v such that $N(w) \cap V(C_{k-1}) = \emptyset$. If $3 \leq |N(v) \cap V(C_{k-1})| \leq 5$, then there is a claw centered onto v , a contradiction. When $|N(v) \cap V(C_{k-1})| = 2$ there is an induced P_k -subgraph that is a contradiction. So $N[C_{k-1}] = V$ and therefore from Property 2.3 we can compute a γ -set of G in polynomial time. \square

4 G is $(\text{claw}, P_k, C_k, C_{k-1})$ -free, $C_{k-2} \subseteq_i G, k \leq 8$

In this section we prove that, for $k \leq 8$, if G is a $(\text{claw}, P_k, C_k, C_{k-1})$ -free graph such that $C_{k-2} \subseteq_i G$ then the Minimum Dominating Set problem is polynomial. The first lemma gives a structural property for G . We use this property to prove two other lemmas, the first one for $k = 6$, the second for $7 \leq k \leq 8$.

Lemma 4.1 *For every fixed $k \geq 6$, if G is a $(\text{claw}, P_k, C_k, C_{k-1})$ -free connected graph such that $C_{k-2} \subseteq_i G$, then $W = V \setminus N[V(C_{k-2})]$ is an independent set.*

Proof: Let $C = C_{k-2} = v_1 - \dots - v_{k-2} - v_1$, $C \subseteq_i G$ and $v \in N[V(C)] \setminus V(C)$. We have $2 \leq |N_C(v)| \leq 5$ (note that $|N_C(v)| = 5$ only for $C = C_5$). Let $W = V \setminus N[V(C)]$ and let $w \in W$ be a neighbor of v . If $3 \leq |N_C(v)| \leq 5$, there is a claw, a contradiction. Hence, v is such that $N_C(v) = \{v_i, v_{i+1}\}$, $1 \leq i \leq k-2$ (for convenience, when $i = k-2$, we read $v_{i+1} = v_1$). By Property 2.1, we can assume that all contractible vertices of G are contracted. Moreover, from Property 2.2 we can assume that G has no leaves.

Assume for contradiction that w has a neighbor w' , $w' \in W$. When w' has no neighbor in $N(V(C))$, there is an induced P_k -subgraph that is a contradiction. Hence w' has a neighbor in $N(V(C))$. Recall that $N[w] \neq N[w']$. If $vw' \notin E$ then there is an induced P_k -subgraph, a contradiction. Hence, w and w' have the same neighbors in $N(V(C))$ but not in W . So there exists $r \in W$ with $rw \in E, rw' \notin E$. The arguments above implies $rv \in E$. But $G[\{r, v, v_i, w'\}]$ is a claw, a contradiction. Hence, $W = V \setminus N[V(C_{k-2})]$ is independent. \square

Lemma 4.2 *If G is a $(\text{claw}, P_6, C_6, C_5)$ -free connected graph such that $C_4 \subseteq_i G$, then a minimum dominating set of G can be given in polynomial time.*

Proof: Let $C = C_4 = v_1 - \dots - v_4 - v_1$, $C \subseteq_i G$ and $v \notin V(C)$ such that $N(v) \cap V(C) \neq \emptyset$. We have $2 \leq |N_C(v)| \leq 4$. Let $W = V \setminus N[C]$ and $w \in W$ be a neighbor of v . If $3 \leq |N_C(v)| \leq 4$ then G contains a claw, a contradiction. Hence, $N_C(v) = \{v_i, v_{i+1}\}$, $1 \leq i \leq 4$ (for convenience, when $i = 4$, we read $v_{i+1} = v_1$). We assume that all contractible vertices of G are contracted and G has no leaves.

By Property 2.4, if $|W| \leq 1$ then a minimum dominating set can be computed in polynomial time. So we assume that $|W| \geq 2$ and by Lemma 4.1, we know that W is an independent set. We show that all vertices $v \in N[W] \setminus W$ have exactly the same neighbors in C .

Let $w, w' \in W$, $w \neq w'$, be such that w has a neighbor $v \in N[C] \setminus V(C)$ and w' has a neighbor $v' \in N[C] \setminus V(C)$. Since G is claw-free $v \neq v'$. W.l.o.g. $N_C(v) = \{v_1, v_2\}$. Assume that $N_C(v) \neq N_C(v')$. W.l.o.g. $N_C(v') = \{v_2, v_3\}$ (note that $N_C(v') = \{v_1, v_4\}$ is symmetric). If $vv' \notin E$ then $w - v - v_1 - v_4 - v_3 - v' = P_6$, else $v_1 - v - v' - v_3 - v_4 = C_5$, a contradiction. Now it remains $N_C(v') = \{v_3, v_4\}$. We have $vv' \notin E$ else there is a claw, but $w - v - v_1 - v_4 - v' - w' = P_6$, a contradiction. Thus, w.l.o.g. every vertex $w \in W$ has only neighbors $v \in N[C] \setminus V(C)$ such that $N(v) = \{v_1, v_2\}$.

Let $|W| = q$, $q \geq 2$. We show that $\gamma(G) = q + 1$. Since W is independent and for every distinct $w, w' \in W$, we have $N[w] \cap N[w'] = \emptyset$, we must take q vertices of $N[W]$ to dominate the vertices of W . This vertices cannot dominate v_3 nor v_4 . Hence $\gamma(G) \geq q + 1$.

We construct a γ -set of G as follows. We set R by taking exactly one neighbor of each $w, w' \in W$. Clearly, $\Gamma = R \cup \{v_3\}$ dominates $V(C) \cup N[R]$. Suppose that there exists $s \in N[C] \setminus V(C)$ that is not dominated by Γ . If $N_C(s) = \{v_1, v_2\}$ then there exists $r \in R$ such that $G[\{r, s, v_1, v_4\}]$ is a claw, a contradiction. If $N_C(s) = \{v_1, v_4\}$ then $w - v - v_2 - v_3 - v_4 - s = P_6$, a contradiction. If $N_C(s) = \{v_1, v_2, v_4\}$ then

there exists $r \in R$ such that $G[\{r, s, v_2, v_3\}]$ is a claw, a contradiction. Hence every $s \notin N[R] \cup V(C)$ is dominated by v_3 . It follows that Γ is a γ -set of G . Clearly Γ can be constructed in polynomial time. \square

Lemma 4.3 *For $k \in \{7, 8\}$, if G is a $(\text{claw}, P_k, C_k, C_{k-1})$ -free connected graph such that $C_{k-2} \subseteq_i G$, then a minimum dominating set of G can be given in polynomial time.*

Proof: By Properties 2.1 and 2.2, we can assume that all contractible vertices of G are contracted and that G has no leaves. Let $C = C_{k-2} = v_1 - \dots - v_{k-2} - v_1$, $C \subseteq_i G$ and $v \in N[C] \setminus V(C)$. We have $2 \leq |N_C(v)| \leq 5$ (note that $|N_C(v)| = 5$ only for $C = C_5$). Let $S = N[C] \setminus V(C)$, $W = V \setminus N[C]$ and $w \in W$ a neighbor of v . If $3 \leq |N_C(v)| \leq 4$ then G has a claw, a contradiction. Hence, v is such that $N_C(v) = \{v_i, v_{i+1}\}$, $1 \leq i \leq k-2$ (for convenience, when $i = k-2$, we read $v_{i+1} = v_1$).

We show that for every $w \in W$, there exists $v, v' \in N(w)$ such that $N_C(v) \cap N_C(v') = \emptyset$. Let $w \in W$ and $v, v' \in N_S(w)$, $v \neq v'$.

First, we show that $N_C(v) \neq N_C(v')$. Suppose that $N_C(v) = N_C(v')$, w.l.o.g. $N_C(v) = \{v_1, v_2\}$. We have $vv' \in E$ else $G[\{v, v', v_1, v_{k-2}\}]$ is a claw. Since $N[v] \neq N[v']$ there exists $u \in V$ such that $uv \in E$ and $uv' \notin E$. If $u \in W$ then by Lemma 4.1 $uw \notin E$ but $G[\{u, v, w, v_1\}]$ is a claw, a contradiction. So, we have $u \in S$. If $N_C(u) = \{v_1, v_2\}$ then $G[\{u, v', v_2, v_3\}]$ is a claw, a contradiction. So $N_C(u) \neq N_C(v)$ and we can assume that $wu \notin E$, otherwise we have u, v two neighbors of w with distinct neighborhoods in C . If $N_C(u) \cap N_C(v) = \emptyset$ then $G[\{u, v, v_1, w\}]$ is a claw, a contradiction. So, w.l.o.g., we assume that $N_C(u) \cap N_C(v) = \{v_1\}$ but $G[\{u, v, v_2, w\}]$ is a claw, a contradiction. Hence $N[v] = N[v']$ and v, v' can be contracted implying that w is a leaf, a contradiction. Thus for every $w, w \in W$, there exists $v, v' \in N_S(w)$, $v \neq v'$ such that $N_C(v) \neq N_C(v')$.

Now we show that that $N_C(v) \cap N_C(v') = \emptyset$. W.l.o.g. assume that $N_C(v) = \{v_1, v_2\}$ and $N_C(v') = \{v_2, v_3\}$. If $vv' \in E$ then $v_1 - v - v' - v_3 - \dots - v_{k-2} - v_1 = C_{k-1}$, else $v_1 - v - w - v' - v_3 - \dots - v_{k-2} - v_1 = C_k$, a contradiction. Thus every $w \in W$, has two neighbors $v, v' \in S$ such that $N_C(v) \cap N_C(v') = \emptyset$.

It follows from Property 2.4 that we can assume that $|W| \geq 2$. So let $w, w' \in W$ (recall $ww' \notin E$). Since both w and w' have two neighbors in S with non intersecting neighborhoods in C , let $v \in N(w)$, $v' \in N(w')$ such that $N_C(v) \cap N_C(v') = \emptyset$. W.l.o.g. $N_C(v) = \{v_1, v_2\}$. Assume that $N_C(v') = \{v_3, v_4\}$ (note that $N(v') = \{v_{k-2}, v_{k-3}\}$ is symmetric). If $vv' \in E$ then $G[\{v, v', v_1, w\}]$ is a claw, else $w - v - v_1 - v_{k-2} - \dots - v_4 - v' - w' = P_k$, a contradiction. Hence the two neighborhoods of $N_C(v)$ and $N_C(v')$ are not adjacent. It follows that for $k = 7$, since $C_{k-2} = C_5$, such a configuration is impossible. This yields to $|W| \leq 1$ and by Property 2.4 a minimum dominating set can be computed in polynomial time.

Now, we focus on the remaining case of $k = 8$. Let $|W| = q, q \geq 2$. We show that $\gamma(G) = q+2$. Since W is independent and that for every distinct vertices $w, w' \in W$, we have $N[w] \cap N[w'] = \emptyset$, we must take q vertices of $N[W]$ to dominate the vertices

of W . Let $w, w' \in W$. From above we can assume that w has a neighbor v such that $N_C(v) = \{v_1, v_2\}$ and w' has a neighbor v' such that $N_C(v') = \{v_4, v_5\}$ (each vertex of W has two neighbors whose are neighbors of respectively $\{v_1, v_2\}$ and $\{v_4, v_5\}$ since $C = C_6$). G being claw-free we have $vv' \notin E$. The q vertices that dominates W cannot dominate v_3 and v_6 . Hence $\gamma(G) \geq q + 1$.

Suppose that $\gamma(G) = q + 1$. The minimum dominating set of G must contain a vertex $s \in S$ a neighbor of both v_3 and v_6 . If $vs \in E$, respectively $v's \in E$, then G has a claw (s cannot be complete to $N_C(v) \cup N_C(v')$), a contradiction. Also, s must have $(v_1$ or $v_5)$ and $(v_2$ or $v_4)$ as neighbors else there is a claw. We assume first that $N(s) = \{v_1, v_2, v_3, v_6\}$. Then $w - v - v_1 - s - v_3 - v_4 - v' - w' = P_8$ (recall $vv' \notin E$ since G is claw-free), a contradiction. The case where $N(s) = \{v_3, v_4, v_5, v_6\}$ is symmetric. Now we assume that $N(s) = \{v_1, v_3, v_4, v_6\}$ (note that $N(s) = \{v_2, v_3, v_5, v_6\}$ is symmetric). Then $w - v - v_2 - v_3 - s - v_6 - v_5 - v' = P_8$, a contradiction. Hence $\gamma(G) \geq q + 2$.

We show that $\Gamma = \{v_1, v_4\} \cup W$ is a γ -set of G . Clearly Γ dominates $N[W] \cup V(C)$. Let $s \notin N[W] \cup V(C)$. So $s \in S$. Suppose that $sv_1, sv_4 \notin E$. From above $ws \notin E$ and $vs \notin E$ else $G[\{v, s, v_1, w\}]$ is a claw. If $N(s) = \{v_2, v_3\}$ then $w - v - v_1 - v_6 - v_5 - v_4 - v_3 - s = P_8$, a contradiction. By symmetry $N(s) \neq \{v_5, v_6\}$. As shown before $N(s) = \{v_2, v_3, v_5, v_6\}$ is not possible. Hence every $s \notin N[W] \cup V(C)$ is dominated by v_1 or v_4 . It follows that $\Gamma = \{v_1, v_4\} \cup W$ is a γ -set of G . \square

By Lemmas 3.1, 3.2, 4.2, 4.3 we immediately obtain the corollary below.

Corollary 4.4 *Let G a (claw, P_k) -free graph, $6 \leq k \leq 8$. If $C_l \subseteq_i G$, $k - 2 \leq l \leq k$, then a minimum dominating set of G can be given in polynomial time.*

5 G is (claw, P_8) -free

Here we conclude by the main result proving that the Minimum Dominating Set problem is polynomial in the class of (claw, P_8) -free graphs. Starting from the result stating that the problem is polynomial when G is (claw, P_5) -free, we successively prove that the problem is polynomial for (claw, P_6) -free, (claw, P_7) -free graphs. Then we conclude for the class of (claw, P_8) -free graphs.

In [8] D. Malyshev proved that the Minimum Dominating Set problem is polynomial for the class of $(K_{1,4}, P_5)$ -free graphs. Hence we obtain the following lemma.

Lemma 5.1 *Let G be a connected (claw, P_5) -free graph. Computing a minimum dominating set is polynomial-time solvable.*

Lemma 5.2 *Let G be a connected (claw, P_6) -free graph. Computing a minimum dominating set is polynomial-time solvable.*

Proof: It follows from Corollary 4.4, that if $C_l \subseteq_i G$, $4 \leq l \leq 6$, then computing a minimum dominating set is polynomial. When G is $(\text{claw}, C_4, C_5, C_6, P_6)$ -free then it is chordal. The Minimum Dominating Set problem is polynomial for claw-free chordal graphs. \square

Lemma 5.3 *Let G be a connected (claw, C_5, C_6, C_7, P_7)-free graph. Computing a minimum dominating set is polynomial-time solvable.*

Proof: By Properties 2.1 and 2.2, we can assume that all contractible vertices of G are contracted and that G has no leaves. By Lemma 5.2 we can assume that $P_6 \subseteq_i G$. Let $P = v_1 - v_2 - v_3 - v_4 - v_5 - v_6$.

Let $W = V \setminus N[V(P)]$. It follows from Property 2.3 that if $W = \emptyset$ then computing a minimum dominating set is polynomial. From now on $W \neq \emptyset$. Let $S = \{v \in V \setminus V(P) \text{ such that } 2 \leq |N_P(v)| \leq 4\}$, $S_i \subseteq S$ being the set of vertices v such that $|N_P(v)| = i$. Let $H_i = \{v \in S_2 : N_P(v) = \{v_i, v_{i+1}\}, 1 \leq i \leq 5\}$. Since G is claw-free each H_i is complete. If there is an edge $r_i r_{i+1}$ with $r_i \in H_i, r_{i+1} \in H_{i+1}$ then $P = v_1 - \dots - v_i - r_i - r_{i+1} - v_{i+2} - \dots - v_6 = P_7$, a contradiction. If there is an edge $r_i r_j$ with $r_i \in H_i, r_j \in H_j$ and $j \geq i + 3$ then $C_p \subseteq_i G$, $p \geq 5$. So H_1 is anticomplete to H_2, H_4, H_5 , the component H_2 is anticomplete to H_3, H_5 , and the component H_3 is anticomplete to H_4 .

We define R_i as the set of vertices of H_i having a neighbor in W , $R_i = \{v \in H_i : N_W(v) \neq \emptyset\}, 1 \leq i \leq 5$. Since G is P_7 -free $R_1 = R_5 = \emptyset$.

Let $r \in R_i, r' \in R_i, r \neq r', i \in \{2, 4\}$ be such that r , respectively r' , has a neighbor $w \in W$, respectively $w' \in W$. We show that $N_S(r) = N_S(r')$.

By contradiction we assume that there exists $s \in S$ such that $rs \in E, r's \notin E$. From above $s \notin R_i \cup H_{i-1} \cup H_{i+1}$. Let $i = 2$ (the case $i = 4$ is symmetric). Recall that H_2 is anticomplete to H_1, H_3, H_5 , thus $s \in H_4 \cup S_3 \cup S_4$. If $s \in H_4$ then $G[\{r, w, v_3, s\}]$ is a claw, a contradiction. Hence $s \in S_3 \cup S_4$. When $N_P(s) = \{v_1, v_2, v_3\}$ then $G[\{r', v_3, v_4, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_2, v_3, v_4\}$ then $G[\{r', v_1, v_2, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_3, v_4, v_5\}$ or $N_P(s) = \{v_4, v_5, v_6\}$ then $G[\{r, v_2, w, s\}]$ is a claw, a contradiction. So $s \in S_4$. When $N_P(s) = \{v_1, v_2, v_3, v_4\}$ then $G[\{r, v_1, v_4, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_2, v_3, v_4, v_5\}$ then $G[\{r', v_1, v_2, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_3, v_4, v_5, v_6\}$ then $G[\{r, v_4, v_6, s\}]$ is a claw, a contradiction. Now let $i = 3$. Recall that H_3 is anticomplete to H_2, H_4 , thus $s \in H_1 \cup H_5 \cup S_3 \cup S_4$. If $s \in H_1$ (the case $s \in H_5$ is symmetric) then $G[\{r, w, v_3, s\}]$ is a claw, a contradiction. Hence $s \in S_3 \cup S_4$. If $N_P(s) = \{v_1, v_2, v_3\}$ (the case $N_P(s) = \{v_4, v_5, v_6\}$ is symmetric) then $G[\{r, w, v_4, s\}]$ is a claw, a contradiction. If $N_P(s) = \{v_2, v_3, v_4\}$ (the case $N_P(s) = \{v_3, v_4, v_5\}$ is symmetric) then $G[\{r', v_4, v_5, s\}]$ is a claw, a contradiction. So $s \in S_4$. When $N_P(s) = \{v_1, v_2, v_4, v_5\}$ or $N_P(s) = \{v_1, v_2, v_5, v_6\}$ then $G[\{r, v_1, v_5, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_2, v_3, v_5, v_6\}$ then $G[\{r', v_3, v_4, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_1, v_2, v_3, v_4\}$ (the case $N_P(s) = \{v_3, v_4, v_5, v_6\}$ is symmetric) then $G[\{r', v_4, v_5, s\}]$ is a claw, a contradiction. Hence $N_P(s) = \{v_2, v_3, v_4, v_5\}$ but $G[\{r, v_2, v_5, s\}]$ is a claw, a contradiction. Thus $N_S(r) = N_S(r')$.

Let $r_2 \in R_2, r'_2 \in R_2, r_2 \neq r'_2$ be such that r_2 , respectively r'_2 , has a neighbor $w \in W$, respectively $w' \in W$. Let $r_4 \in R_4, r'_4 \in R_4, r_4 \neq r'_4$ be such that r_4 , respectively r'_4 , has w , respectively w' , as neighbor. We show that $N_{S \setminus H_4}(r_2) = N_{S \setminus H_4}(r'_2)$, respectively $N_{S \setminus H_2}(r_4) = N_{S \setminus H_2}(r'_4)$.

By contradiction we assume that there exists $s \in S$ such that $r_2s \in E, r'_2s \notin E$. From above $s \notin H_1 \cup H_2 \cup H_3$. When $s \in H_4$ we know that s is not a neighbor of w . If $s \in H_4 \cup H_5$ then $G[\{r_2, v_2, w, s\}]$ is a claw, a contradiction. Hence $s \in S_3 \cup S_4$. When $N_P(s) = \{v_1, v_2, v_3\}$ then $G[\{r'_2, v_3, v_4, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_2, v_3, v_4\}$ then $G[\{r'_2, v_1, v_2, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_3, v_4, v_5\}$ then $G[\{r_2, v_2, w, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_4, v_5, v_6\}$ then $G[\{r_2, v_4, v_6, s\}]$ is a claw, a contradiction. So $s \in S_4$. When $N_P(s) = \{v_1, v_2, v_4, v_5\}$ or $N_P(s) = \{v_1, v_2, v_5, v_6\}$ then $G[\{r, v_1, v_5, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_2, v_3, v_5, v_6\}$ then $G[\{r', v_3, v_4, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_1, v_2, v_3, v_4\}$ then $G[\{r_2, v_1, v_4, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_2, v_3, v_4, v_5\}$ then $G[\{r'_2, v_1, v_2, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_3, v_4, v_5, v_6\}$ then $G[\{r_2, v_4, v_6, s\}]$ is a claw, a contradiction. Thus $N_{S \setminus H_4}(r_2) = N_{S \setminus H_4}(r'_2)$ and by symmetry, for $r'_4 \in R_4, r'_4 \neq r_4$, we have $N_{S \setminus H_2}(r_4) = N_{S \setminus H_2}(r'_4)$.

Let $w \in W$. We show that w cannot have two neighbors r_i, r_{i+1} with $r_i \in R_i, r_{i+1} \in R_{i+1}$. Suppose for contradiction that these two neighbors exist. Then $v_1 - \dots - v_i - r_i - w - r_{i+1} - v_{i+2} - \dots - v_6 = P_8$, a contradiction. Now, since $R_1 = R_5 = \emptyset$, if w has two neighbors $r_i \in R_i, r_j \in R_j, i \neq j$, these two neighbors are $r_2 \in R_2, r_4 \in R_4$ and $r_2r_4 \in E$, else $w - r_4 - v_4 - v_3 - r_2 - w = C_5$. Moreover, when w has two neighbors $r_2 \in R_2, r_4 \in R_4$, then for each neighbor $w' \in N_W(w)$, w' has r_2 and r_4 as neighbors. Assume for contradiction that w has a neighbor $w' \in W$ such that $w'r_2 \notin E$ (by symmetry $w'r_4 \notin E$ is the same case). Then $w' - w - r_2 - v_3 - \dots - v_6 = P_7$, a contradiction. It follows that $N[w] = N[w']$, a contradiction. Hence setting $Z_{24} = \{w \in W : w \text{ has two neighbors } r_2 \in R_2, r_4 \in R_4\}$, Z_{24} is an independent set.

Let $w, w' \in Z_{24}, w \neq w'$. Since G is claw-free we have $N(w) \cap N(w') = \emptyset$. We show that $N_{R_2}(w)$ is anticomplete to $N_{R_4}(w')$ and $N_{R_4}(w)$ is anticomplete to $N_{R_2}(w')$. By contradiction if w has a neighbor $r_2 \in R_2, w'$ has a neighbor $r_4 \in R_4$, and $r_2r_4 \in E$ then $G[\{v_2, r_2, w, r_4\}]$ is a claw, a contradiction.

Let $Z_i = \{w \in W : w \text{ has a neighbor in } R_i \setminus (N_{R_i}(Z_{24}))\}, 2 \leq i \leq 4\}$.

We show that Z_2, Z_3, Z_4 are pairwise anticomplete. If there is an edge $w_2w_4, w_2 \in Z_2, w_4 \in Z_4$, with $r'_2 \in R_2, r'_4 \in R_4$ the neighbors of w_2, w_4 respectively, then $w_2 - r'_2 - v_3 - v_4 - r'_4 - w_4 - w_2 = C_6$ ($r'_2r'_4 \notin E$ else $G[\{v_2, r'_2, w_2, r'_4\}]$ is a claw). If there is an edge $w_2w_3, w_2 \in Z_2, w_3 \in Z_3$, with $r'_2 \in R_2, r'_3 \in R_3$ the neighbors of w_2, w_3 respectively, then $w_2 - r'_2 - v_3 - r'_3 - w_3 - w_2 = C_5$ (recall $r'_2r'_3 \notin E$). By symmetry there is no edge between Z_3 and Z_4 .

Let $Y = W \setminus (Z_2 \cup Z_3 \cup Z_4 \cup Z_{24})$. One can observe that for every $w \in Y$ we have $N_{Z_2}(w) = N_{Z_4}(w) = N_{Z_{24}}(w) = \emptyset$ else $P_7 \subseteq_i G$. Let $Y_3 = \{w \in Y : w \text{ has a neighbor in } Z_3\}$. If there exists $w' \in Y \setminus Y_3$ such that w' has a neighbor $w, w \in Y_3$, then $P_7 \subseteq_i G$. Hence $Y = Y_3$.

We show that we can assume that Z_2, Z_4, Y_3 are three independent sets. The arguments are the same for the three sets, so we show that Z_2 is an independent set. For

contradiction, we assume that there are $w_1, w_2 \in Z_2$ such that $w_1w_2 \in E$. We prove that $N_{R_2}(w_1) = N_{R_2}(w_2)$. If $N_{R_2}(w_1) \neq N_{R_2}(w_2)$ then there exists $r_2 \in R_2$ which is a neighbor of w_1 but not a neighbor of w_2 . Then $w_2 - w_1 - r_2 - v_3 - \dots - v_6 = P_7$, a contradiction. If $N_{Z_2}(w_1) \neq N_{Z_2}(w_2)$ then there exists $w_3 \in Z_2$ such that $w_2w_3 \in E$, $w_1w_3 \notin E$, but $G[\{v_2, r_2, w_1, w_3\}]$ is a claw, a contradiction. Hence $N[w_1] = N[w_2]$, a contradiction. Hence Z_2, Z_4, Y_3 are three independent sets.

Since G is claw-free then for every two distinct vertices $w_1, w_2 \in Z_2 \cup Z_4 \cup Y_3$ we have $N(w_1) \cap N(w_2) = \emptyset$.

We prove that for every $w \in Y_3$, $N(w)$ is a clique. Let $w \in Z_3$. Suppose there are s, s' two non adjacent vertices in $N(w)$. Since G is claw-free s, s' cannot have a common neighbor in R_3 . Let $r \in R_3$ be a neighbor of s . Then $s' - w - s - r - v_3 - v_2 - v_1 = P_7$, a contradiction.

Since G is claw-free, if there are a vertex $r \in R_i$ with a neighbor $z \in Z_i$ and a vertex $s \in S$ such as $sz \notin E$ and $v_i \notin N(s)$ then G contains a claw, a contradiction, (note that $v_{i+1} \notin N(s)$ is symmetric). Hence $N(Z_i)$ is anticomplete to H_j , $j \neq i$.

We show that we can assume that $Z_2 = Z_4 = \emptyset$. The arguments are the same in the two cases, so we consider Z_2 . Let $r, r' \in R_2$ be two neighbors of $w \in Z_2$. We show that $N[r] = N[r']$. Since $N_R(w) = N_{R_2}(w)$ and $rr' \in E$ then, as proved above, $N_S(r) = N_S(r')$. For two distinct $w_1, w_2 \in Z_2$, $N(w_1) \cap N(w_2) = \emptyset$. Hence, $N[r] = N[r']$, a contradiction. Then w is a leaf, a contradiction.

Now we study the structure of Z_3 . For every distinct two vertices $w_1, w_2 \in Z_3$ such that $w_1w_2 \in E$, there cannot exist two distinct vertices $w'_1, w'_2 \in Z_3$ such that $w_1w'_1 \in E$, $w'_1w_2 \notin E$ and $w_2w'_2 \in E$, $w_1w'_2 \notin E$. For contradiction we suppose that such two vertices exist. We assume first that w_2 has a neighbor $r_2 \in R_3$ such that $r_2w_1 \notin E$. If $w'_1r_2 \notin E$ then $v_1 - v_2 - v_3 - r_2 - w_2 - w_1 - w'_1 = P_7$ else $G[\{v_4, r_2, w_2, w'_1\}]$ is a claw, a contradiction. So w_1, w_2 have a common neighbor $r_1 \in R_3$. If $w'_1r_1 \in E$ then $G[\{v_3, r_1, w_2, w'_1\}]$ is a claw, a contradiction. Thus $w'_1r_1 \notin E$ and w'_1 has a neighbor $r'_1 \in R_3, r'_1 \neq r_1$. If $r'_1w_2 \in E$ then $G[\{v_3, r'_1, w_2, w'_1\}]$ is a claw, a contradiction. So $r'_1w_2 \notin E$. If $r'_1w_1 \notin E$ then $v_1 - v_2 - v_3 - r'_1 - w'_1 - w_1 - w_2 = P_7$, a contradiction. Thus $r'_1w_1 \in E$. If $r'_1w'_2 \notin E$ then $v_1 - v_2 - v_3 - r'_1 - w_1 - w_2 - w'_2 = P_7$, a contradiction. So $r'_1w'_2 \in E$ but $G[\{v_4, r'_1, w_1, w'_2\}]$ is a claw, a contradiction.

As a consequence each connected component A_i of Z_3 has a universal vertex. Also, G being claw-free two distinct connected components cannot share a neighbor in R_3 . Moreover, by Property 2.2 we have assumed that each $w_3 \in Z_3$ is not a leaf.

We show that $w \in Y_3$ is connected to a universal vertex of a connected component A_i of Z_3 . We assume that the neighbors of w are not universal in A_i . Let $s \in A_i$ be a neighbor of w , let $u, u \neq s$, be a universal vertex of A_i . Since s is not universal there exists $v, v \in A_i$ such that $sv \notin E$ and $wv \in E$. Since $N(w)$ is complete $wv \notin E$. Let $r \in R_3$ be a neighbor of s . Since G is claw-free then $rv \notin E$. Let $r', r' \in R_3, r' \neq r$, be a neighbor of v . As just above $r's \notin E$. If $r'u \notin E$ then $v_1 - v_2 - v_3 - r' - v - u - s = P_7$ else $v_1 - v_2 - v_3 - r' - u - s - w = P_7$, a contradiction.

We are ready to show how to build a γ -set in polynomial time.

First, we treat the case where $Z_{24} \neq \emptyset$. Let $r_2 \in R_2$, $r_4 \in R_4$ be two neighbors of w , $w \in Z_{24}$.

We show that $R_3 = \emptyset$. Assume that there exists $w' \in W$ with a neighbor $r_3 \in R_3$. Since w' is not a neighbor of r_2 or r_4 we have $w' - r_3 - v_3 - r_2 - r_4 - v_5 - v_6 = P_7$, a contradiction. So $R_3 = \emptyset$ and since $Z_2 = Z_4 = \emptyset$ we have $W = Z_{24}$.

Recall that $W = Z_{24}$ is independent and that for every two distinct vertices $w, w' \in Z_{24}$ we have $N(w) \cap N(w') = \emptyset$.

The γ -set is build as follows:

By Property 2.4, we can assume that $|W| \geq 2$. We take $r_2 \in R_2$ a neighbor of w (recall that the neighbors of w in $R_i, i \in \{2, 4\}$, have the same neighborhood and that all vertices of R_i have the same neighbors in $S \setminus H_4$), and for each other $w' \in Z_{24}$ we take one adjacent vertex $r'_4 \in R_4$. These vertices dominate $Z_{24} \cup H_2 \cup H_4 \cup \{v_2, v_3, v_4, v_5\}$. At least one more vertex is necessary to dominate G since v_1 and v_6 are not dominated. Adding the three vertices v_2, v_4, v_6 we have a dominating set (not necessarily minimum). We check first if there exists s a neighbor of both v_1 and v_6 that dominates the rest of the graph. If such vertex s does not exist, checking for all the pairs s_1, s_6 where s_i is a neighbor of $v_i, i \in \{1, 6\}$, one can verify if there is a γ -set with only two more vertices (note that there are at most $O(n^2)$ of such pairs).

Now we deal with the case $Z_{24} = \emptyset$.

The γ -set is build as follows:

- $Y_3 \neq \emptyset$. For each $w \in Y_3$ we take one universal vertex in the connected component A_i of Z_3 connected to w . For each connected component A_i of Z_3 that is not connected to a vertex of Y_3 , we do as follows: if there exists $r_3 \in R_3$ which is complete to A_i (recall that such vertices have the same neighborhood) then we take r_3 , else we take one universal vertex of A_i . These vertices dominate $Y_3 \cup Z_3$. At least one more vertex is necessary to dominate G since v_1 and v_6 are not dominated. Adding the three vertices v_2, v_4, v_6 we have a dominating set (not necessarily minimum). We check first if there exists s a neighbor of both v_1 and v_6 that dominates the rest of the graph. If such vertex s is not found, checking for all the pairs s_1, s_6 where s_i is a neighbor of $v_i, i \in \{1, 6\}$, one can verify if there is a γ -set with only two more vertices (note that there are at most $O(n^2)$ such pairs).
- $Y_3 = \emptyset$. Thus $Z_3 = W$. For every connected component A_i of Z_3 , if there exists $r_3 \in R_3$ which is complete to A_i (recall that such vertices have the same neighborhood) then we take r_3 , else we take one universal vertex of A_i . These vertices dominate Z_3 . At least one more vertex is necessary to dominate G since v_1 and v_6 are not dominated. Adding the three vertices v_2, v_4, v_6 we have a dominating set (not necessarily minimum). We check first if there exists s a neighbor of both v_1 and v_6 that dominates the rest of the graph. If such vertex s is not found, checking for all the pairs s_1, s_6 where s_i is a neighbor of $v_i, i \in \{1, 6\}$, one can verify if there is a γ -set with only two more vertices (note that there are at most $O(n^2)$ such pairs).

Clearly the construction of the γ -set is polynomial. \square

Corollary 5.4 *The Minimum Dominating Set problem is polynomial for (claw, P_7)-free graphs.*

Lemma 5.5 *Let G be a connected (claw, C_6, C_7, C_8, P_8)-free graph. If $C_5 \subseteq_i G$ then computing a minimum dominating set is polynomial.*

Proof: By Properties 2.1 and 2.2, we can assume that all contractible vertices of G are contracted and that G has no leaves. Let $C = v_1 - v_2 - v_3 - v_4 - v_5 - v_1 = C_5 \subseteq_i G$. Let $W = V \setminus N[V(C)]$. It follows from Property 2.3 that if $W = \emptyset$ then computing a minimum dominating set is polynomial. From now on $W \neq \emptyset$.

Let $S = \{v \in V \setminus V(C) : 2 \leq |N(v) \cap V(C)| \leq 5\}$, and $S_i \subseteq S$ being the set of vertices v such that $|N(v) \cap V(C)| = i$. Let $H_i = \{v \in S_2 : N(v) \cap V(C) = \{v_i, v_{i+1}\}\}$, $1 \leq i \leq 5$ (for convenience v_{5+1} stands for v_1). Since G is claw-free, each H_i is complete. Moreover, if there is an edge $r_i r_{i+1}$ with $r_i \in H_i$, $r_{i+1} \in H_{i+1}$ then $r_i - v_i - v_{i-1} - \dots - v_{i+2} - r_{i+1} - r_i = C_6$, a contradiction. Hence H_i is anticomplete to H_{i+1} . We define R_i as the set of vertices of H_i having a neighbor in W , $R_i = \{v \in H_i : N_W(v) \neq \emptyset\}$, $1 \leq i \leq 5$, $R = R_1 \cup \dots \cup R_5$.

Since $W \neq \emptyset$, we assume that there exists $w_1 \in W$ such that w_1 has a neighbor $r_1 \in R_1$. Suppose that $R_2 \neq \emptyset$. There exists $w_2 \in W$ with a neighbor $r_2 \in R_2$. If $w_1 = w_2$ then $w_1 - r_2 - v_3 - v_4 - v_5 - v_1 - r_1 - w_1 = C_7$, a contradiction. So $w_1 \neq w_2$. If $w_1 w_2 \in E$ then $w_1 - w_2 - r_2 - v_3 - v_4 - v_5 - v_1 - r_1 - w_1 = C_8$ else $w_1 - r_1 - v_1 - v_5 - v_4 - v_3 - r_2 - w_2 - w_1 = P_8$, a contradiction. So, if $R_i \neq \emptyset$ then $R_{i-1} = R_{i+1} = \emptyset$. Hence $R_2 = R_5 = \emptyset$.

Let $r \in R_i$, $r' \in R_i$, $r \neq r'$, $i \in \{1, 3, 4\}$ be such that r , respectively r' , has a neighbor $w \in W$, respectively $w' \in W$. We show that $N_S(r) = N_S(r')$.

By contradiction we assume that there exists $s \in S$ such that $rs \in E$, $r's \notin E$. From above $s \notin H_i \cup H_{i-1} \cup H_{i+1}$. Let $i = 1$. If $s \in H_3 \cup H_4$ then $G[\{r, v_1, w, s\}]$ is a claw, a contradiction. Thus $s \in S_3 \cup S_4 \cup S_5$. When $N_C(s) = \{v_1, v_2, v_3\}$ (the case $N_C(s) = \{v_1, v_2, v_5\}$ is symmetric) then $G[\{r', v_5, w, s\}]$ is a claw, a contradiction. When $N_C(s) = \{v_2, v_3, v_4\}$ (the case $N_C(s) = \{v_1, v_4, v_5\}$ is symmetric) then $G[\{r, v_1, w, s\}]$ is a claw, a contradiction. When $N_C(s) = \{v_3, v_4, v_5\}$ then $G[\{r, v_3, v_5, s\}]$ is a claw, a contradiction. So $s \in S_4 \cup S_5$. When $N_C(s) = \{v_1, v_2, v_3, v_4\}$ (the case $N_C(s) = \{v_1, v_2, v_4, v_5\}$ is symmetric) then $G[\{r', v_1, v_5, s\}]$ is a claw, a contradiction. When $N_C(s) = \{v_1, v_3, v_4, v_5\}$ or $N_C(s) = \{v_2, v_3, v_4, v_5\}$ or $N_C(s) = \{v_1, v_2, v_3, v_5\}$ or $s \in S_5$ then $G[\{r, v_3, v_5, s\}]$ is a claw, a contradiction. For $i = 3$ and $i = 4$ the arguments are the same. Thus $N_S(r) = N_S(r')$.

Let $r_1 \in R_1$, $r'_1 \in R_1$, $r_1 \neq r'_1$ be such that r_1 , respectively r'_1 , has a neighbor $w \in W$, respectively $w' \in W$. Let $r_3 \in R_3$, $r'_3 \in R_3$, $r_3 \neq r'_3$ be such that r_3 , respectively r'_3 , has w , respectively w' , as neighbor. We show that $N_{S \setminus H_3}(r_1) = N_{S \setminus H_3}(r'_1)$, respectively $N_{S \setminus H_1}(r_3) = N_{S \setminus H_1}(r'_3)$.

Let $i = 1$. By contradiction we assume that there exists $s \in S \setminus R_3$ such that $r_1 s \in E$, $r'_1 s \notin E$. From above $s \notin H_1 \cup H_2 \cup H_5$. If $s \in H_4$ then $G[\{r_1, v_1, w, s\}]$

is a claw, a contradiction. So $s \in S_3 \cup S_4 \cup S_5$. When $N_C(s) = \{v_1, v_2, v_3\}$ (the case $N_C(s) = \{v_1, v_2, v_5\}$ is symmetric) then $G[\{r'_1, v_1, v_5, s\}]$ is a claw, a contradiction. When $N_C(s) = \{v_2, v_3, v_4\}$ (the case $N_C(s) = \{v_1, v_4, v_5\}$ is symmetric) then $G[\{r_1, v_1, w, s\}]$ is a claw, a contradiction. When $N_C(s) = \{v_3, v_4, v_5\}$ then $G[\{r_1, v_3, v_5, s\}]$ is a claw, a contradiction. So $s \in S_4 \cup S_5$. When $N_C(s) = \{v_1, v_2, v_3, v_4\}$ (the case $N_C(s) = \{v_1, v_2, v_4, v_5\}$ is symmetric) then $G[\{r'_1, v_1, v_5, s\}]$ is a claw, a contradiction. When $N_C(s) = \{v_1, v_3, v_4, v_5\}$ or $N_C(s) = \{v_2, v_3, v_4, v_5\}$ or $N_C(s) = \{v_1, v_2, v_3, v_5\}$ or $s \in S_5$ then $G[\{r_1, v_3, v_5, s\}]$ is a claw, a contradiction. By symmetry the arguments are the same for $i = 3$. Hence $N_{S \setminus H_3}(r_1) = N_{S \setminus H_3}(r'_1)$ and $N_{S \setminus H_1}(r_3) = N_{S \setminus H_1}(r'_3)$.

We study the case where w_1 has a neighbor r_i , $r_i \in R_i$, $i \in \{3, 4\}$. Since both cases are symmetric, let $r_3, r_3 \in R_3$, be a neighbor of w_1 . If $r_1 r_3 \notin E$ then $w_1 - r_1 - v_1 - v_5 - v_4 - r_3 - w_1 = C_6$, a contradiction. Hence $N_{R_1}(w_1)$ is complete to $N_{R_3}(w_1)$. Since $R_3 \neq \emptyset$, we have $R_4 = \emptyset$ and $N_R(w_1) \subseteq R_1 \cup R_3$. Hence, we define the following subsets of W :

- $Z = \{w \in W : N_R(w) \neq \emptyset\}$;
- $Z_i = \{z \in Z : N_{R_i}(z) \neq \emptyset, N_{R_j}(z) = \emptyset, 1 \leq i \leq 5, i \neq j\}$;
- $Z_{ij} = \{z \in Z : N_{R_i}(z) \neq \emptyset, N_{R_j}(z) \neq \emptyset, 1 \leq i < j \leq 5\}$;
- $Y = W \setminus Z$.

First, we show that Z_i is anticomplete to Z_{ij} , then we show that Z_i consists of leaves (so is empty). We conclude that $Z_{ij} \neq \emptyset$ implies $Z = Z_{ij}$. We set $w_1 \in Z_{13}$, and since all cases are symmetric, we focus on $Z_1 \neq \emptyset$.

Let $w'_1 \in Z_1$ with a neighbor r'_1 , $r'_1 \in R_1$, $r'_1 \neq r_1$. Note that $r'_1 r_3 \notin E$ else G contains a claw. If $w_1 w'_1 \in E$, then $r'_1 w_1 \in E$, else $w_1 - w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 = C_7$, but $w_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 = C_6$, a contradiction. Hence, $w_1 w'_1 \notin E$ (by symmetry, for every $w_3 \in Z_3$, $w_1 w_3 \notin E$). Thus Z_1 and Z_3 are anticomplete to Z_{13} .

Now, we show that the vertices of Z_1 are leaves. Assume that there exists $v \in N(w'_1)$, $v \neq r'_1$ such that $N[v] \neq N[w'_1]$. If $v \in Z_3$ then $v - w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 = P_8$, a contradiction. If $v \in Y$ then either $v - w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 = P_8$ or $v - w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 - v = C_8$, a contradiction. If $v \in Z_1$ and $r'_1 v \notin E$ then $v - w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 = P_8$, a contradiction. Hence $N_{R_1}(w'_1) = N_{R_1}(v)$. Since $N[v] \neq N[w'_1]$ we can assume that there exists $v' \in W$ such that $vv' \in E$ but $v'w'_1 \notin E$. Yet with the same arguments as before we have $N_{R_1}(v) = N_{R_1}(v')$ and since $w'_1 v' \notin E$ then $G[\{r'_1, v', v_1, w'_1\}]$ is a claw, a contradiction. Thus Z_1 consists of leaves, a contradiction. Thus $Z_1 = \emptyset$ and by symmetry $Z_3 = \emptyset$. So $Z = Z_{13}$.

We show that every pair $v, v' \in Z_{13}$ with $vv' \in E$ satisfy $N_{R_1 \cup R_3}(v) = N_{R_1 \cup R_3}(v')$. Let $w'_1 \in Z_{13}$ be a neighbor of w_1 . Suppose that there exists $r'_1 \in N_{R_1}(w'_1)$ such that $r'_1 w_1 \notin E$. If $r'_1 r_3 \in E$ then $G[\{r'_1, r_3, v_3, w_1\}]$ is a claw, a contradiction. If $r_3 w'_1 \notin E$ then $w_1 - r_3 - v_4 - v_5 - v_1 - r'_1 - w'_1 - w_1 = C_7$, else $w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w'_1 = C_6$,

a contradiction. Hence, $N_{R_1}(w_1) = N_{R_1}(w'_1)$ and by symmetry $N_{R_3}(w_1) = N_{R_3}(w'_1)$.

Suppose that $Y \neq \emptyset$. Let $y \in Y$ be a neighbor of w_1 . We show that Z_{13} is a clique. Let $w'_1 \in Z_{13}$ such that $w_1 w'_1 \notin E$. We have $N_{R_1 \cup R_3}(w_1) \cap N_{R_1 \cup R_3}(w'_1) = \emptyset$ else G contains a claw. Yet, there exists r'_1 a neighbor of w'_1 in R_1 such that either $y - w_1 - r_3 - v_4 - v_5 - v_1 - r'_1 - w'_1 = P_8$ or $y - w_1 - r_3 - v_4 - v_5 - v_1 - r'_1 - w'_1 - y = C_8$, a contradiction, (note that $r'_1 r_3 \notin E$ else G contains a claw). Hence Z_{13} is a clique.

We show that the vertices of Y are leaves. Suppose that y has a neighbor $y' \in Y$. If $y' w_1 \notin E$ then $y' - y - w_1 - r_1 - v_1 - v_5 - v_4 - v_3 = P_8$, a contradiction. Hence $N_Z(y) = N_Z(y')$. Since we assume that $N[y] \neq N[y']$, there exists $v, v \in Y$, such that $vy \in E$, $vy' \notin E$. From above $w_1 v \in E$ but $G[\{r_1, w_1, y', v\}]$ is a claw, a contradiction. Thus, Y is an independent set. Now, $N(y) \subseteq Z_{13}$ is a clique. Since for every two vertices $w_1, w'_1 \in Z_{13}$ we have $N_{R_1 \cup R_3}(w_1) = N_{R_1 \cup R_3}(w'_1)$ we can assume that $N(y)$ can be contracted into a unique vertex. Thus, Y consists of leaves, a contradiction. Hence $Y = \emptyset$.

As shown before, every two neighbors of Z_{13} have the same neighbors in R , so they can be contracted and we can assume that Z_{13} is an independent set. Moreover, since G is *claw-free*, for every two distinct $z, z' \in Z_{13}$, $N[z] \cap N[z'] = \emptyset$. Also, recall that the neighbors of each $z, z \in Z_{13}$ induce a clique.

We show how to build a γ -set of G . Recall that $W = Z_{13}$. By Property 2.4 we can assume that $|W| \geq 2$. So there are $w_1, w'_1 \in Z_{13}$ with neighbors $r_1, r'_1 \in R_1$ and $r_3, r'_3 \in R_3$, respectively. Let $q = |Z_{13}|$. Clearly, to dominate Z_{13} we must take q vertices. We take r_1 and r'_3 . Recall that the vertices of R_1 and R_3 have the same neighbors in $S \cup V(C)$. Then, we take the $q - 2$ vertices of $w \in Z_{13}$, $w \neq w_1, w'_1$. These q vertices dominate $\{v_1, v_2, v_3, v_4\} \cup H_1 \cup H_3 \cup Z_{13}$. It remains to dominate some vertices of $H_2 \cup H_4 \cup H_5 \cup S_3 \cup S_4 \cup \{v_5\}$. If there exists a vertex $v, v \in S \cup \{v_5\}$, which is universal to these non dominated vertices we take v , else we take the vertices $\{v_2, v_5\}$.

Now, we assume that $Z_{ij} = \emptyset$. Hence let $w_1 \in Z_1$. We study the case $R_3 \neq \emptyset$. Recall that $R_2 = R_4 = R_5 = \emptyset$. Let $w_3 \in W$ such that w_3 has a neighbor $r_3 \in R_3$. If $w_1 w_3 \in E$ then $w_1 - w_3 - r_3 - v_4 - v_5 - v_1 - r_1 - w_1 = C_7$ ($r_1 r_3 \notin E$ else G contains a claw), a contradiction. So Z_1 is anticomplete to Z_3 . We assume that w_1 has a neighbor $v \in Y$. If $vw_3 \in E$ then $v - w_3 - r_3 - v_4 - v_5 - v_1 - r_1 - w_1 - v = C_8$ else $v - w_1 - r_1 - v_1 - v_5 - v_4 - r_3 - w_3 = P_8$, a contradiction. Hence every neighbor $w'_1, w'_1 \in W$, of w_1 is in Z_1 . If $w'_1 r_1 \notin E$ then $w'_1 - w_1 - r_1 - v_1 - v_5 - v_4 - r_3 - w_3 = P_8$, a contradiction. Hence $N_R(w_1) = N_R(w'_1)$. Since G is *claw-free*, for every $r \in R$, $N_W(r)$ is a clique, thus $N[w_1] = N[w'_1]$, a contradiction. So Z_1 is an independent set. Now, recall that for every pair of vertices $r, r' \in R_i$, $1 \leq i \leq 5$, $N_S(r) = N_S(r')$. Hence, when $r, r' \in R_1$ have a common neighbor in Z_1 , we have $N[r] = N[r']$, a contradiction. Hence Z_1 consists of leaves, a contradiction. Also, by symmetry, $W = Z_1 \cup Z_3 = \emptyset$, a contradiction.

Now we focus on $R_3 = R_4 = \emptyset$ (note that $Z = Z_1$).

We study the case where $H_2 \neq \emptyset$ or $H_5 \neq \emptyset$. Let $v \in H_2$ (the case $v \in H_5$ is symmetric). We have $Y = \emptyset$, else there are $y \in Y$, $z \in Z_1$, $r \in R_1$ such that $v - v_3 - v_4 - v_5 - v_1 - r - z - y = P_8$ (recall that $vr \notin E$). So $W = Z = Z_1$. Let $w_1, w_2 \in W$. We assume that $w_1 w_2 \in E$. Recall that $N[w_1] \neq N[w_2]$. Let $w_1 r_1, w_2 r_2 \in E$, $r_1 \neq r_2$, such that $w_1 r_2 \notin E$. We have $v - v_3 - v_4 - v_5 - v_1 - r_2 - w_2 - w_1 = P_8$, a contradiction. Recall that for every $r \in R$, $N_W(r)$ is a clique, thus $N[w_1] = N[w_2]$, a contradiction. Hence W is an independent set. Moreover, for every $w \in W$ and $r, r' \in N(w)$ we know that r and r' share the same neighbors in $V(C) \cup S$. Hence W is composed exclusively of leaves, so $W = \emptyset$, a contradiction.

Now we can assume that $H_2 = H_5 = \emptyset$. Let $Z_A \subset Z$, $Z_A = \{w \in W : N_Y(w) = \emptyset\}$. We show that each connected component A_i of $G[Z_A]$ contains a universal vertex relatively to A_i . For contradiction we suppose that there exists A_i , $A_i \subseteq Z_A$ with no universal vertex in it. Assume that $z_1 - z_2 - z_3 - z_4 = P_4 \subseteq_i A_i$. Let $r, r \in R_1$, be a neighbor of z_1 (note that there is a P_5 from v_3 to r).

Since G is claw-free $rz_3, rz_4 \notin E$. If $rz_2 \in E$ then there is a P_8 from v_3 to z_4 else there is a P_8 from v_3 to z_3 , a contradiction. Now, we assume that $z_1 - z_2 - z_3 - z_4 - z_1 = C_4 \subseteq_i A_i$. Let $r, r \in R_1$, be a neighbor of z_1 . Since G is claw-free we have $rz_3 \notin E$. If $rz_2 \in E$ then $rz_4 \notin E$ else G contains a claw, but $v_3 - v_4 - v_5 - v_1 - r - z_2 - z_3 - z_4 = P_8$, a contradiction. If $rz_2 \notin E$ then $v_3 - v_4 - v_5 - v_1 - r - z_1 - z_2 - z_3 = P_8$, a contradiction. So A_i is (C_4, P_4) -free. It follows that there are $z_1 - z_2 - z_3 = P_3 \subseteq_i A_i$ and $z_4 \in A_i$ such that $z_4 z_1, z_4 z_2, z_4 z_3 \notin E$. Also there exists $z \in A_i$ such that $z_2 - z - z_4$ and $zz_1, zz_3 \in E$ but $A_i[\{z, z_1, z_3, z_4\}]$ is a claw, a contradiction. So each A_i has a universal vertex. Clearly, for two distinct components A_i, A_j we have $N_{R_1}(A_i) \cap N_{R_1}(A_j) = \emptyset$ else there is a claw.

Suppose that $Y \neq \emptyset$. We show that Y is an independent set. Suppose that there are $y, y' \in Y$ with $yy' \in E$. Recall that $N[y] \neq N[y']$. If $N_{Z_1}(y) \neq N_{Z_1}(y')$ then, w.l.o.g, $yw_1 \in E$, $y'w_1 \notin E$, but $v_3 - v_4 - v_5 - v_1 - r_1 - w_1 - y - y' = P_8$, a contradiction. So $N_{Z_1}(y) = N_{Z_1}(y')$. There is no vertex $y'' \in Y$ such that $yy'' \in E$, $y'y'' \notin E$, else G contains a claw. Hence Y is an independent set and for every pair of vertices $y, y' \in Y$ we have $N(y) \cap N(y') = \emptyset$.

We show that for every $y \in Y$ its neighborhood $N(y)$ is a clique. For contradiction we assume that y has two neighbors $z_1, z_2 \in Z$, $z_1 z_2 \notin E$. Since G is claw-free z_1 and z_2 cannot have a common neighbor in R_1 . Let $r, r \in R_1$, be a neighbor of z_1 . Then $v_3 - v_4 - v_5 - v_1 - r - z_1 - y - z_2 = P_8$, a contradiction. Hence, Y is an independent set, for each $y, y \in Y$, $N(y)$ is a clique. So we suppose $|N(y)| \geq 2$, else y is a leaf.

We show that we can assume that each connected component A_i of $G[Z_A]$ is anti-complete to $N(Y)$. Since Y has no leaves, let $y \in Y$ with two neighbors $z, z' \in Z_1$ such that $N[z] \neq N[z']$. Suppose that there exists $u \in Z_A$ a neighbor of z . First, we assume that $N_R(z) \neq N_R(z')$. W.l.o.g. let $r, r' \in R_1$ be respectively the neighbors of z, z' such that $r'z, rz' \notin E$. If $uz' \notin E$ then $ur' \notin E$ else G contains a claw, but then $u - z - z' - r' - v_1 - v_5 - v_4 - v_3 = P_8$, a contradiction. Hence $uz', r'u \in E$ but $y - z - u - r' - v_1 - v_5 - v_4 - v_3 = P_8$, a contradiction. So $N_R(z) = N_R(z')$. Second, we assume that $N_Z[z] \neq N_Z[z']$. W.l.o.g. $uz' \notin E$. Let $r \in R_1$ a neighbor

of both z, z' . Clearly $ru \notin E$ else G contains a claw, but $G[\{r, u, y, z\}]$ is a claw, a contradiction. So we can assume that each A_i is anticomplete to $N(Y)$.

We construct a γ -set as follows:

Let $q = |Y|$ and k be the number of connected components of Z_A . Clearly, q vertices are necessary to dominate Y . So for each $y_i \in Y$ we will take one of its neighbors as follows. Let us denote $R_1(y_i) = N_{R_1}(N(y_i))$. If y_i has a neighbor z_i which is complete to $R_1(y_i)$ then we take z_i , else we take every arbitrary neighbor of y_i (recall that in both cases these y_i have the same neighbors in Z). These q vertices dominate $Y \cup (Z \setminus Z_A)$ and some of the vertices in $R_1(Y)$.

Now k vertices are necessary to dominate Z_A . For each component $A_i \subset Z_A$ we do as follows. If there exists $r \in R_1$ which is complete to A_i we take r into the γ -set (case a), else we take one universal vertex of A_i (case b) (recall that in both cases these r have the same neighbors in S).

These k vertices dominate $Z_A \cup H_1 \cup \{v_1, v_2\}$ if at least one vertex is chosen in the case a , else they dominate Z_A .

Case where at least one vertex is chosen with the case a : v_3, v_4, v_5 are not dominated with the $q + k$ already chosen vertices (H_1 is complete thus $r \in R_1$ dominates $H_1 \cup \{v_1, v_2\}$). So a dominating set of G has size at least $q + k + 1$. Adding the two vertices v_3 and v_5 , we have a dominating set (not necessarily minimum). Checking if there exists a vertex $v \in V(C) \cup S$, that is universal to the remaining non-dominated vertices, can be done in polynomial-time.

Case where all the vertices are chosen with the case b : it remains to dominate C and some vertices of $S_2 \cup S_3 \cup S_4$. So a dominating set of G has a size at least $q + k + 1$. Adding the three vertices v_1, v_3, v_5 , we have a dominating set (not necessarily minimum). If there exists a vertex $v \in S_5$ that is universal to the remaining non-dominated vertices we take it. If no such vertex exists, checking for all the pairs $\{v, v'\} \subset N[V(C)]$, one can verify if there exists a γ -set with $q + k + 2$ vertices (note that there are at most $O(n^2)$ of such pairs). \square

Lemma 5.6 *Let G be a connected (claw, C_5, C_6, C_7, C_8, P_8)-free graph. Computing a minimum dominating set is polynomial-time solvable.*

Proof: By Lemma 5.4 we can assume that $P_7 \subseteq_i G$. Let $P = v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7$. By Properties 2.1 and 2.2, we can assume that all contractible vertices of G are contracted and that G has no leaves.

Let $W = V \setminus N[V(P)]$. By Property 2.3 if $W = \emptyset$ then computing a minimum dominating set is polynomial. From now on $W \neq \emptyset$. Let $S = \{v \in V \setminus V(P) : 2 \leq |N(v) \cap P| \leq 4\}$, and $S_i \subseteq S$ being the set of vertices v such that $|N(v) \cap V(P)| = i$. Let $H_i = \{v \in S_2 : N(v) \cap V(P) = \{v_i, v_{i+1}\}, 1 \leq i \leq 6\}$. Since G is claw-free each H_i is complete. If there is an edge $r_i r_{i+1}$ with $r_i \in H_i, r_{i+1} \in H_{i+1}$ then $P = v_1 - \dots - v_i - r_i - r_{i+1} - v_{i+2} - \dots - v_7 = P_8$, a contradiction. If there is an edge $r_i r_j$ with $r_i \in H_i, r_j \in H_j$ and $j \geq i + 3$ then $C_p \subseteq_i G, p \geq 5$, a contradiction. So H_1 is anticomplete to H_2, H_4, H_5, H_6 , and H_2 is anticomplete to H_3, H_5, H_6 , and

H_3 is anticomplete to H_4, H_6 .

We define R_i as the set of vertices of H_i having a neighbor in W , that is, $R_i = \{v \in H_i : N(v) \cap W \neq \emptyset\}$, $1 \leq i \leq 6$. Since G is P_8 -free $R_1 = R_6 = \emptyset$.

Let $w \in W$. We show that there cannot exist three indices $1 \leq i < j < k \leq 6$ such that w has three neighbors $r_i \in R_i, r_j \in R_j, r_k \in R_k$. Suppose for contradiction that these three neighbors of w exist. Since $R_1 = R_6 = \emptyset$ then $2 \leq i < j < k \leq 5$. Since G is claw-free and H_p is anticomplete to H_{p+1} , these three indices cannot be successive. So w.l.o.g. we can assume that $i = 2, j = 4, k = 5$. Now H_2 is anticomplete to H_5 , but $v_3 - r_2 - w - r_5 - v_5 - v_4 - v_3 = C_6$, a contradiction. Hence for every $w \in W$ there is at most two neighbors r_i, r_j such that $r_i \in R_i, r_j \in R_j, i \neq j$.

If w has two neighbors $r_i \in R_i, r_j \in R_j, i < j$, then either $r_i \in R_2, r_j \in R_4$ or $r_i \in R_3, r_j \in R_5$ (recall that H_i is anticomplete to $H_{i+1}, H_p, p \geq i + 3$ and $R_1 = R_6 = \emptyset$).

If w has two neighbors $r_i \in R_2, r_j \in R_4$, respectively $r_i \in R_3, r_j \in R_5$, then $r_i r_j \in E$, else $w - r_j - v_4 - v_3 - r_i - w = C_5 \subseteq_i G$, respectively $w - r_j - v_5 - v_4 - r_i - w = C_5 \subseteq_i G$, a contradiction.

Let $Z_{24} = \{w \in W : N_{R_2}(w) \neq \emptyset, N_{R_4}(w) \neq \emptyset\}$ and $Z_{35} = \{w \in W : N_{R_3}(w) \neq \emptyset, N_{R_5}(w) \neq \emptyset\}$. We show that Z_{24} is anticomplete to Z_{35} . For contradiction we suppose that there are $w_1 \in Z_{24}, w_2 \in Z_{35}$ with $w_1 w_2 \in E$. Let $r_1 \in R_2$ be a neighbor of w_1 and $r_2 \in R_5$ be a neighbor of w_2 . Since $r_1 r_2 \notin E$ we have $w_1 - r_1 - v_3 - v_4 - v_5 - v_6 - r_2 - w_2 - w_1 = C_8$, a contradiction.

We show that we can assume that Z_{24} and Z_{35} are two independent sets. The two sets being symmetric we show that Z_{24} is an independent set. For contradiction we assume that there are $w_1, w_2 \in Z_{24}$ such that $w_1 w_2 \in E$. We prove that $N_{R_2}(w_1) = N_{R_2}(w_2)$. If $N_{R_2}(w_1) \neq N_{R_2}(w_2)$ then there exists $r_2 \in R_2$ which is a neighbor of w_1 but not a neighbor of w_2 . Then $w_2 - w_1 - r_2 - v_3 - \dots - v_7 = P_8$, a contradiction. We prove that $N_{R_4}(w_1) = N_{R_4}(w_2)$. If $N_{R_4}(w_1) \neq N_{R_4}(w_2)$ then there exists $r_4 \in R_4$ which is a neighbor of w_1 but not a neighbor of w_2 . There exists $r_2 \in R_2$ a neighbor of w_1 and w_2 . We know that $r_2 r_4 \in E$. It follows that $G[\{v_2, r_2, r_4, w_2\}]$ is a claw, a contradiction. Hence $N_{R_2}(w_1) = N_{R_2}(w_2)$ and $N_{R_4}(w_1) = N_{R_4}(w_2)$. By Property 2.1 there exists $s \notin R_2 \cup R_4$ such that s is a neighbor of w_1 but not a neighbor of w_2 . Let $r_2 \in R_2$ a neighbor of w_1 and w_2 . If $s r_2 \notin E$ then $s - w_1 - r_2 - v_3 - \dots - v_7 = P_8$, a contradiction. When $s r_2 \in E$ then $G[\{v_2, r_2, s, w_2\}]$ is a claw, a contradiction. Hence Z_{24} is an independent and by symmetry Z_{35} is also independent. Moreover, since G is claw-free for every two distinct $w, w' \in Z_{24} \cup Z_{35}$ we have $N(w) \cap N(w') = \emptyset$.

For every two distinct $w, w' \in Z_{24}$, respectively $w, w' \in Z_{35}$ we have that $N_{R_2}(w)$ is anticomplete to $N_{R_4}(w')$ and $N_{R_4}(w)$ is anticomplete to $N_{R_2}(w')$, respectively $N_{R_3}(w)$ is anticomplete to $N_{R_5}(w')$ and $N_{R_5}(w)$ is anticomplete to $N_{R_3}(w')$. For contradiction we assume that w has a neighbor $r_2 \in R_2, w'$ has a neighbor $r_4 \in R_4$, and $r_2 r_4 \in E$. Then $G[\{v_2, r_2, w, r_4\}]$ is a claw, a contradiction.

Let $Z_i = \{w \in W : N(w) \cap R_i \setminus (N_{R_i}(Z_{24} \cup Z_{35})) \neq \emptyset\}$, $2 \leq i \leq 5$. We show that Z_2, Z_3, Z_4, Z_5 are pairwise anticomplete. If there is an edge $w_2 w_4, w_2 \in$

$Z_2, w_4 \in Z_4$, with $r'_2 \in R_2, r'_4 \in R_4$ the neighbors of w_2, w_4 respectively, then $w_2 - r'_2 - v_3 - v_4 - r'_4 - w_4 - w_2 = C_6$ ($r'_2 r'_4 \notin E$ else $G[\{v_2, r'_2, w_2, r'_4\}]$ is a claw), a contradiction. By symmetry there is no edges between Z_3, Z_5 . If there is an edge $w_2 w_5, w_2 \in Z_2, w_5 \in Z_5$, with $r'_2 \in R_2, r'_5 \in R_5$ the neighbors of w_2, w_5 respectively, then $w_2 - r'_2 - v_3 - v_4 - v_5 - r'_5 - w_5 - w_2 = C_7$ (remember $r'_2 r'_5 \notin E$), a contradiction. If there is an edge $w_4 w_5, w_4 \in Z_4, w_5 \in Z_5$, with $r'_4 \in R_4, r'_5 \in R_5$ the neighbors of w_4, w_5 respectively, then $w_4 - r'_4 - v_5 - r'_5 - w_5 - w_4 = C_5$ (recall $r'_4 r'_5 \notin E$), a contradiction. By symmetry there is no edges between Z_2, Z_3 .

Let $Y = W \setminus (Z_2 \cup Z_3 \cup Z_4 \cup Z_5 \cup Z_{24} \cup Z_{35})$. One can observe that for every $w \in Y$ we have $N_{Z_2}(w) = N_{Z_5}(w) = N_{Z_{24}}(w) = N_{Z_{35}}(w) = \emptyset$ else $P_8 \subseteq_i G$. Now, if $w \in Y$ has two neighbors $w_3 \in Z_3, w_4 \in Z_4$ then $C_6 \subseteq_i G$, a contradiction. Hence $Y = Y_3 \cup Y_4$ with $Y_3 \cap Y_4 = \emptyset, Y_3 = \{w \in Y : N_{Z_3}(w) \neq \emptyset\}, Y_4 = \{w \in Y : N_{Z_4}(w) \neq \emptyset\}$. Moreover Y_3 is anticomplete to Y_4 else $C_7 \subseteq_i G$.

We show that we can assume that Z_2, Z_5, Y_3, Y_4 are four independent sets. The arguments are the same for the four sets, so we show that the statement is true for Z_2 . For contradiction we assume that there are $w_1, w_2 \in Z_2$ such that $w_1 w_2 \in E$. We prove that $N_{R_2}(w_1) = N_{R_2}(w_2)$. If $N_{R_2}(w_1) \neq N_{R_2}(w_2)$ then there exists $r_2 \in R_2$ which is a neighbor of w_1 but not a neighbor of w_2 . Then $w_2 - w_1 - r_2 - v_3 - \dots - v_7 = P_8$, a contradiction. Since $N(w_1), N(w_2) \subseteq Z_2 \cup R_2$ the result is obtained by Property 2.1. Hence Z_2, Z_5, Y_3, Y_4 are four independent sets.

Since G is claw-free then for every two distinct vertices $w_1, w_2 \in Z_2 \cup Z_5 \cup Y_3 \cup Y_4$ we have $N(w_1) \cap N(w_2) = \emptyset$.

We prove that for every $w \in Y_3 \cup Y_4, N(w)$ is a clique. The two cases being symmetric, let $w \in Y_4$. Suppose that there are s, s' two non adjacent vertices in $N(w)$. Since G is claw-free, s, s' cannot have a common neighbor in R_4 . Let $r \in R_4$ be a neighbor of s . Then $s' - w - s - r - v_4 - v_3 - v_2 - v_1 = P_8$, a contradiction.

Since G is claw-free, if there is a vertex $r \in R_i$ with a neighbor $z \in Z_i$ and a vertex $s \in S$ such as $sz \notin E$ and $v_i \notin N(s)$ then G contains a claw (note that $v_{i+1} \notin N(s)$ is symmetric). Hence $N(Z_i)$ is anticomplete to $H_j, j \neq i$.

We show that we can assume that $Z_2 = Z_5 = \emptyset$. The arguments are the same in the two cases, so we consider Z_2 . Let $r, r' \in R_2$ be two neighbors of $w \in Z_2$. By previous arguments, $N(w)$ is complete to H_2 but anticomplete to H_1, H_3, H_4, H_5, H_6 . Hence, it remains the case where $N_{S_3 \cup S_4}(r) \neq N_{S_3 \cup S_4}(r')$. Suppose that $N_{S_3 \cup S_4}(r) \neq N_{S_3 \cup S_4}(r')$. Let $s \in S_3 \cup S_4$ such as $rs \in E$ but $r's \notin E$. If $\{v_2, v_3\} \not\subseteq N_P(s)$ then $G[\{r, s, v_2, w\}]$ or $G[\{r, s, v_3, w\}]$ is a claw, a contradiction. So $\{v_2, v_3\} \subseteq N_P(s)$. If $v_1 \notin N_P(s)$, respectively $v_4 \notin N_P(s)$, then $G[\{r', s, v_1, v_2\}]$, respectively $G[\{r', s, v_3, v_4\}]$, is a claw, a contradiction. Hence $N_P(s) = \{v_1, v_2, v_3, v_4\}$ but $G[\{r, s, v_1, v_4\}]$ is a claw, a contradiction. Hence, $N[r] = N[r']$, a contradiction. Then w is a leaf, a contradiction.

Now we study the structures of Z_3 and Z_4 . The two cases being symmetric we deal with Z_4 . For every distinct vertices $w_1, w_2 \in Z_4$ such that $w_1 w_2 \in E$, then there cannot exist two distinct vertices $w'_1, w'_2 \in Z_4$ such that $w'_1 w_1 \in E, w'_1 w_2 \notin E$ and

$w'_2w_2 \in E$, $w'_2w_1 \notin E$. For contradiction we suppose that such two vertices exist. First, we suppose that w_1, w_2 have two distinct neighbors $r_1, r_2 \in R_4$, respectively. If $w'_1r_2 \notin E$ then $v_1 - v_2 - v_3 - v_4 - r_2 - w_2 - w_1 - w'_1 = P_8$, a contradiction. If $w'_1r_2 \in E$ then $G[\{v_4, r_2, w_2, w'_1\}]$ is a claw, a contradiction. Second, w.l.o.g., $r_1 \in R_4$ is a common neighbor of w_1, w_2 and $r_2 \in R_4$ is a neighbor of w_2 but not w_1 . If $w'_1r_2 \notin E$ then $v_1 - v_2 - v_3 - v_4 - r_2 - w_2 - w_1 - w'_1 = P_8$ else $G[\{v_4, r_2, w_2, w'_1\}]$ is a claw, a contradiction. Finally, $r_1, r_2 \in R_4$ are two common neighbors of w_1, w_2 (r_1, r_2 are not necessarily distinct). If, w.l.o.g., $w'_1r_1 \in E$ then $G[\{v_4, r_1, w_2, w'_1\}]$ is a claw, a contradiction. In the case where $w'_1r_1, w'_1r_2 \notin E$ then w'_1 has a neighbor $r'_1 \in R_4$, $r'_1 \neq r_1, r_2$. If $r'_1w_2 \in E$ then $G[\{v_4, r'_1, w_2, w'_1\}]$ is a claw, a contradiction. So $r'_1w_2 \notin E$. If $r'_1w_1 \notin E$ then $v_1 - v_2 - v_3 - v_4 - r'_1 - w'_1 - w_1 - w_2 = P_8$, a contradiction. Thus $r'_1w_1 \in E$. If $r'_1w'_2 \notin E$ then $v_1 - v_2 - v_3 - v_4 - r'_1 - w_1 - w_2 - w'_2 = P_8$, a contradiction. So $r'_1w'_2 \in E$ but $G[\{v_4, r'_1, w_1, w'_2\}]$ is a claw, a contradiction. As a consequence each connected component A_i of $Z_3 \cup Z_4$ has a universal vertex. Also, G being claw-free two distinct components cannot share a neighbor in $R_3 \cup R_4$. Moreover each $w_4 \in Z_3 \cup Z_4$ is not a leaf.

We show that $w \in Y_3 \cup Y_4$ is connected to a universal vertex of a connected component A_i of $Z_3 \cup Z_4$. The two cases being symmetric, we deal with Z_4 . Let $w \in Z_4$. We assume that the neighbors of w are not universal in A_i . Let $s \in A_i$ be a neighbor of w , let u , $u \neq s$, be a universal vertex of A_i . Since s is not universal there exists v , $v \in A_i$ such that $sv \notin E$ and $uv \in E$. Since $N(w)$ is complete $wv \notin E$. Let $r \in R_4$ be a neighbor of s . Since G is claw-free then $rv \notin E$. Let r' , $r' \in R_4$, $r' \neq r$, be a neighbor of v . As just above $r's \notin E$. If $r'u \notin E$ then $v_1 - v_2 - v_3 - v_4 - r' - v - u - s = P_8$ else $v_1 - v_2 - v_3 - v_4 - r' - u - s - w = P_8$, a contradiction.

We are ready to show how to build a γ -set in polynomial time.

First, we treat the case where $Z_{24} \neq \emptyset$ (the case $Z_{35} \neq \emptyset$ is the same). Let $r_2 \in R_2, r_4 \in R_4$ be the two neighbors of w , $w \in Z_{24}$. Recall that $N(Z_{24}) \subseteq R_2 \cup R_4$. We show that $R_3 = \emptyset$. Assume that there exists $w' \in W$ with a neighbor $r_3 \in H_3$ (thus $R_3 \neq \emptyset$). Note that w' cannot be a neighbor of r_2 or r_4 . Then $w' - r_3 - v_3 - r_2 - r_4 - v_5 - v_6 - v_7 = P_8$, a contradiction. An immediate consequence is that $Z_{35} = \emptyset$. There is no vertex $w', w' \in W$, with r_2 as a neighbor else $G[\{v_2, r_2, r_4, w'\}]$ is a claw. By symmetry, there is no vertex $w', w' \in W$, with r_4 as a neighbor.

Let $r_2 \in R_2, r'_2 \in R_2, r_2 \neq r'_2$ be such that r_2 , respectively r'_2 , has a neighbor $w \in Z_{24}$, respectively $w' \in Z_{24}$. Let $r_4 \in R_4, r'_4 \in R_4, r_4 \neq r'_4$ be such that r_4 , respectively r'_4 , has w , respectively w' , as neighbor. We show that $N_{S \setminus H_4}(r_2) = N_{S \setminus H_4}(r'_2)$, respectively $N_{S \setminus H_2}(r_4) = N_{S \setminus H_2}(r'_4)$.

Let $i = 2$ (the case $i = 4$ is symmetric). By contradiction, we assume that there exists $s \in S \setminus H_4$ such that $r_2s \in E$, $r'_2s \notin E$. From above $s \notin S_2$. So $s \in S_3 \cup S_4$. If $N_P(s) = \{v_1, v_2, v_3\}$ then $G[\{r'_2, v_3, v_4, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_2, v_3, v_4\}$ then $G[\{r'_2, v_1, v_2, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_j, v_{j+1}, v_{j+2}\}, j \geq 3$, then $G[\{r_2, w, v_2, s\}]$ is a claw, a contradiction. Thus $s \in S_4$. When $N_P(s) = \{v_1, v_2, v_3, v_4\}$ then $G[\{r_2, v_1, v_4, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_1, v_2, v_j, v_{j+1}\}, j \geq 4$, then $G[\{r_2, v_1, v_j, s\}]$ is a

claw, a contradiction. When $N_P(s) = \{v_2, v_3, v_j, v_{j+1}\}, j \geq 4$, then $G[\{r'_2, v_1, v_2, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_j, v_{j+1}, v_{j'}, v_{j'+1}\}, j \geq 3, j' \geq j + 2$ then $G[\{r_2, w, v_2, s\}]$ is a claw, a contradiction. Hence $N_{S \setminus H_4}(r_2) = N_{S \setminus H_4}(r'_2)$ and $N_{S \setminus H_2}(r_4) = N_{S \setminus H_2}(r'_4)$.

Let $r_4 \in R_4, r'_4 \in R_4, r_4 \neq r'_4$ be such that r_4 , respectively r'_4 , has a neighbor $w \in Z_4$, respectively $w' \in Z_4$. We show that $N_S(r_4) = N_S(r'_4)$.

By contradiction, we assume that there exists $s \in S$ such that $r_4 s \in E, r'_4 s \notin E$. From above $s \notin H_1 \cup H_3 \cup H_4 \cup H_5$. So $s \in H_2 \cup H_6 \cup S_3 \cup S_4$. If $s \in H_2$ or $s \in H_6$ then $G[\{v_4, w, r_4, s\}]$ is a claw, a contradiction. So $s \in S_3 \cup S_4$. If $N_P(s) = \{v_1, v_2, v_3\}$ then $G[\{v_1, v_3, r_4, s\}]$ is a claw, a contradiction. If $N_P(s) = \{v_2, v_3, v_4\}$ then $G[\{v_5, w, r_4, s\}]$ is a claw, a contradiction. If $N_P(s) = \{v_3, v_4, v_5\}$ then $G[\{v_5, v_6, r'_4, s\}]$ is a claw, a contradiction. If $N_P(s) = \{v_4, v_5, v_6\}$ then $G[\{v_3, v_4, r'_4, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_5, v_6, v_7\}$ then $G[\{v_4, w, r_4, s\}]$ is a claw, a contradiction. Thus $s \in S_4$. When $N_P(s) = \{v_1, v_2, v_3, v_4\}$ then $G[\{r_4, v_1, v_3, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_2, v_3, v_4, v_5\}$ then $G[\{r'_4, v_5, v_6, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_3, v_4, v_5, v_6\}$ then $G[\{r_4, v_3, v_6, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_4, v_5, v_6, v_7\}$ then $G[\{r'_4, v_3, v_4, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_1, v_2, v_6, v_7\}$ or $N_P(s) = \{v_2, v_3, v_6, v_7\}$ then $G[\{r_4, v_2, v_6, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_3, v_4, v_6, v_7\}$ then $G[\{r_4, v_3, v_6, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_1, v_2, v_5, v_6\}$ or $N_P(s) = \{v_2, v_3, v_5, v_6\}$ then $G[\{r_4, v_2, v_6, s\}]$ is a claw, a contradiction. When $N_P(s) = \{v_1, v_2, v_4, v_5\}$ then $G[\{r'_4, v_5, v_6, s\}]$ is a claw, a contradiction. Hence $N_S(r_4) = N_S(r'_4)$. By symmetry, for $r_2 \in R_2, r'_2 \in R_2, r_2 \neq r'_2$ such that r_2 , respectively r'_2 , has a neighbor $w \in Z_2$, respectively $w' \in Z_2$ we have $N_S(r_2) = N_S(r'_2)$.

The γ -set is build as follows:

- $|Z_{24}| \geq 2$. We take $r_2 \in R_2$ a neighbor of w , and for each other $w' \in Z_{24}$ we take one adjacent vertex $r'_4 \in R_4$. For each $w' \in Y_4$ we take one universal vertex in the connected component A_i of Z_4 connected to w' . For each connected component A_i of Z_4 that is not connected with Y_4 , we take one universal vertex of A_i . These vertices dominate $Z_{24} \cup Y_4 \cup Z_4 \cup H_2 \cup H_4 \cup \{v_2, v_3, v_4, v_5\}$. Since v_1, v_7 have no common neighbor at least two more vertices are needed. Adding the three vertices v_2, v_4, v_6 we have a dominating set (not necessarily minimum). Checking for all the pairs s_1, s_7 where s_i is a neighbor of $v_i, i \in \{1, 7\}$, one can verify if there is a γ -set with only two more vertices (note that there are at most $O(n^2)$ such pairs).
- $|Z_{24}| = 1$. For each $w' \in Y_4$ we take one universal vertex in the connected component A_i of Z_4 connected to w' . If there exists a vertex $r \in R_4$ complete to a component A_i of Z_4 that is not connected to Y_4 then we take r . For each remaining component A_i of Z_4 that is not connected to Y_4 , we take one universal vertex of A_i . These vertices dominate $Y_4 \cup Z_4$ (note that H_2, H_4 are not necessarily dominated). Since v_1, v_7, w have no common neighbor at least three more vertices are needed. Adding the four vertices v_2, v_4, v_6, w we have a dominating set (not necessarily minimum). Checking for all the pairs s_1, s_7

where s_i is a neighbor of $v_i, i \in \{1, 7\}$, if there is a dominating set by adding s_1, s_7, r_4 or s_1, s_7, r_2 , one can verify if there is a γ -set with only three more vertices (note that there are at most $O(n^2)$ such pairs).

In the case of $Z_{24} = Z_{35} = \emptyset$, we build the γ -set as follows:

- $Y_3, Y_4 \neq \emptyset$. For each $w \in Y_3 \cup Y_4$ we take one universal vertex in the connected component A_i of $Z_3 \cup Z_4$ connected to w . If there exists $r_4 \in R_4$ which is complete to a component A_i of Z_4 that is not connected to $Y_3 \cup Y_4$ then, we take r_4 . We do the same for the component of Z_3 with no neighbors in Y_3 . For each remaining connected component A_i of $Z_3 \cup Z_4$ that is not connected to $Y_3 \cup Y_4$, we take one universal vertex of A_i . These vertices dominate $Y_3 \cup Z_3 \cup Y_4 \cup Z_4$ (note that H_2, H_4 are not necessarily dominated). Since v_1, v_7 have no common neighbor at least two more vertices are needed. Adding the three vertices v_2, v_4, v_6 we have a dominating set (not necessarily minimum). Checking for all the pairs s_1, s_7 where s_i is a neighbor of $v_i, i \in \{1, 7\}$, one can verify if there is a γ -set with only two more vertices (note that there are at most $O(n^2)$ such pairs).
- $Y_3 \neq \emptyset, Y_4 = \emptyset$ or $Y_4 \neq \emptyset, Y_3 = \emptyset$. The two cases being symmetric, let $Y_4 = \emptyset$.
 - $Z_4 \neq \emptyset$. For each $w \in Y_3$ we take one universal vertex in the connected component A_i of Z_3 connected to w . If there exists $r_4 \in R_4$ which is complete to A_i , a connected component of Z_4 , then we take r_4 . If there exists $r_3 \in R_3$ which is complete to a connected component A_j of Z_3 with no neighbors in Y_3 , then we take r_3 . Now, we take one universal vertex for each other component $A_l, A_l \neq A_i, A_j$, of $Z_3 \cup Z_4$. These vertices dominate $Y_3 \cup Z_3 \cup Z_4$. Since v_1, v_7 have no common neighbor at least two more vertices are needed. Adding the three vertices v_2, v_4, v_6 we have a dominating set (not necessarily minimum). Checking for all the pairs s_1, s_7 where s_i is a neighbor of $v_i, i \in \{1, 7\}$, one can verify if there is a γ -set with only two more vertices (note that there are at most $O(n^2)$ such pairs).
 - $Z_4 = \emptyset$. For each $w \in Y_3$ we take one universal vertex in the connected component A_i of Z_3 connected to w . If there exists $r_3 \in R_3$ which is complete to a connected component A_i of Z_3 with no neighbors in Y_3 , then we take r_3 . Now, we take one universal vertex for each other component A_i of Z_3 . Adding the vertices v_2, v_4, v_6 we have a dominating set (not necessarily minimum). Checking for all the pairs s_1, s_7 where s_i is a neighbor of $v_i, i \in \{1, 7\}$, one can verify if there is a γ -set with only two more vertices.
- $Y_3, Y_4 = \emptyset$.
 - $Z_3, Z_4 \neq \emptyset$. If there exists $r_4 \in R_4$, respectively $r_3 \in R_3$, which is complete to A_i , a connected component of Z_4 , respectively Z_3 , then we take r_4 , respectively r_3 . For each remaining component of $Z_3 \cup Z_4$ we take one universal vertex. Adding the vertices v_2, v_4, v_6 we have a dominating

set (not necessarily minimum). Checking for all the pairs s_1, s_7 where s_i is a neighbor of $v_i, i \in \{1, 7\}$, one can verify if there is a γ -set with only two more vertices.

- $Z_3 \neq \emptyset, Z_4 = \emptyset$ or $Z_4 \neq \emptyset, Z_3 = \emptyset$. Let $Z_3 \neq \emptyset$. If there exists $r_3 \in R_3$ which is complete to a connected component of Z_3 , then we take r_3 . We add one universal vertex for each remaining component of Z_3 . Now, adding the vertices v_2, v_4, v_6 we have a dominating set (not necessarily minimum). Checking for all the pairs s_1, s_7 where s_i is a neighbor of $v_i, i \in \{1, 7\}$, one can verify if there is a γ -set with only two more vertices.
- $Z_3 = Z_4 = \emptyset$. Then $V = N[V(C)]$ and by Property 2.3 computing a minimum dominating set is polynomial.

□

From Lemmas 3.1, 3.2, 4.3, 5.5, 5.6, we obtain the main result of this paper.

Theorem 5.7 *The Minimum Dominating Set problem is polynomial for (claw, P_8) -free graphs.*

6 Conclusion

We have shown that the Minimum Dominating Set problem is polynomial for (claw, P_8) -free graphs. We left open the following problem: is there a positive integer $k, k \geq 9$, such that the Minimum Dominating Set problem is NP -complete for the class of (claw, P_k) -free graphs? If the the answer is positive, a challenge should be to show a dichotomy: find the minimum integer k such that the Minimum Dominating Set problem is NP -complete for (claw, P_k) -free graphs and polynomial for (claw, P_{k-1}) -free graphs.

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