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The Minimum Dominating Set problem is polynomial for (\textit{claw}, P_8) -free graphs

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Abstract

We prove that the Minimum Dominating Set problem is polynomial for the class of (\textit{claw}, P_8) -free graphs.

Keywords: Minimum Dominating Set, polynomial time, claw-free graph, P_k -free graph.

1 Introduction

M. Yannakakis and F. Gavril [9] showed in 1980 that the Minimum Dominating Set problem restricted to claw-free graphs is NP -complete. Then in 1984, A. Bertossi [2] showed that the Minimum Dominating Set problem is also NP -complete for split graphs, a subclass of P_5 -free graph. More recently, in 2016, D. Malyshev [8] proved that the Minimum Dominating Set problem is polynomial for $(K_{1,4}, P_5)$ -free graphs, hence for (\textit{claw}, P_5) -free graphs. To our knowledge, the complexity of the Minimum Dominating Set problem is unknown for (\textit{claw}, P_k) -free graphs for any fixed $k \geq 6$. We show that the Minimum Dominating Set problem is polynomial for (\textit{claw}, P_8) -free graphs.

Definitions and notations

We are only concerned with simple undirected graphs $G = (V, E)$. The reader is referred to [3] for definitions and notations in graph theory. For $v \in V$, we denote $N(v)$ its neighborhood and $N[v] = N(v) \cup \{v\}$ its closed neighborhood. A vertex v is *universal* if $N[v] = V$. For $v \in V$ and $A \subseteq V$, we denote by $N_A(v) = N(v) \cap A$ ($N_A[v] = (N(v) \cap A) \cup \{v\}$) its (closed) neighborhood in A . For $X \subseteq V, A \subseteq V$, we denote $N_A(X) = \bigcup_{x \in X} N_A(x)$ and $N_A[X] = N_A(X) \cup X$.

The *contraction* of an edge $uv \in E$ removes the vertices u and v from V , and replaces them by a new vertex that is adjacent to the previous neighbors of u and v (neither introducing self-loops nor multiple edges). The following G/uv denotes the graph obtained from G after the contraction of uv .

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For $S \subseteq V$, let $G[S]$ denote the subgraph of G induced by S , which has vertex-set S and edge-set $\{uv \in E \mid u, v \in S\}$. For $v \in V$, we write $G - v = G[V \setminus \{v\}]$ and for a subset $V' \subseteq V$ we write $G - V' = G[V \setminus V']$. For a fixed graph H we write $H \subseteq_i G$ whenever G contains an induced subgraph isomorphic to H . For a set $\{H_1, \dots, H_p\}$ of graphs, G is (H_1, \dots, H_p) -free if G has no induced subgraph isomorphic to a graph in $\{H_1, \dots, H_p\}$; if $p = 1$ we may write H_1 -free instead of (H_1) -free. For two disjoint induced subgraphs $G[A], G[B]$ of G , $G[A]$ is *complete* to $G[B]$ if $ab \in E$ for any $a \in A, b \in B$, $G[A]$ is *anticomplete* to $G[B]$ if $ab \notin E$ for any $a \in A, b \in B$.

For $k \geq 1$, the graph $P_k = u_1 - u_2 - \dots - u_k$ denotes the *cordless path* on k vertices, that is, $V(P_k) = \{u_1, \dots, u_k\}$ and $E(P_k) = \{u_i u_{i+1} \mid 1 \leq i \leq k - 1\}$. For $k \geq 3$, the graph $C_k = u_1 - u_2 - \dots - u_k - u_1$ denotes the *cordless cycle* on k vertices, that is, $V(C_k) = \{u_1, \dots, u_k\}$ and $E(C_k) = \{u_i u_{i+1} \mid 1 \leq i \leq k - 1\} \cup \{u_k u_1\}$. For $k \geq 4$, C_k is called a *hole*. A graph without hole is *chordal*. The complete graph with p vertices is K_p , also called a *clique*. The graph $C_3 = K_3$ is a *triangle*. $K_{1,p}$ is the $(p + 1)$ -vertices star, that is, the graph with vertices u, v_1, v_2, \dots, v_p and edges uv_1, uv_2, \dots, uv_p . The *claw* is $K_{1,3}$.

A set $S \subseteq V$ is called a *stable set* or an *independent set* if any pairwise distinct vertices $u, v \in S$ are non adjacent. The maximum cardinality of an independent set in G is denoted by $\alpha(G)$. A set $S \subseteq V$ is called a *clique* if any pairwise distinct vertices $u, v \in S$ are adjacent. When $G[V]$ is a clique then G is a *complete graph*. The graph $K_p, p \geq 1$, is called a *clique* or a *complete graph* on p vertices.

A set $S \subseteq V$ is a *dominating set* if every vertex $v \in V$ is either an element of S or is adjacent to an element of S . The minimum cardinality of a dominating set in G is denoted by $\gamma(G)$ and called the *dominating number* of G . A dominating set S with $|S| = \gamma(G)$ is called a *Minimum Dominating Set*. Following [7] a Minimum Dominating Set is also called a γ -set. We denote $V^+ \subseteq V$ the subset of vertices v of G such that $\gamma(G - v) > \gamma(G)$. If $S \subset V$ is both a dominating and an independent set then S is an *independent dominating set*. The minimum cardinality of an independent dominating set in G is denoted by $i(G)$. Clearly we have $\gamma(G) \leq i(G) \leq \alpha(G)$. Note that a minimum independent dominating set is a *minimum maximal independent set*.

Previous results

We give some results of the literature concerning the Minimum Dominating Set problem that will be useful in the following.

D. Bauer et al. showed in [4] that, for any non isolated vertex v , if $v \in V^+$ then v is in any γ -set of G . For G a claw-free graph Allan et al. [1] proved that $\gamma(G) = i(G)$. From Yannakakis et al. [9] the Minimum Dominating Set problem restricted to claw-free graphs is *NP*-complete. From D. Malyshev [8] the Minimum Dominating Set problem is polynomial for (claw, P_5) -free graphs. From Farber [6] computing a minimum independent dominating set can be done in linear-time on chordal graphs. So the Minimum Dominating Set problem restricted to claw-free chordal graphs is polynomial. Cockayne et al. [5] showed that the Minimum Dominating Set problem restricted to trees is polynomial.

Organization

The next section give some algorithmic properties. Two properties will allow us to make some simplifications on the graphs G that we consider. Two others will help us to conclude that computing $\gamma(G)$ is polynomial when G have a specific structure relatively to a fixed size subgraph. Then we consider the case where the graph G has a long cycle. From there, we show our main result, starting from (claw, P_6) -free graphs and finishing with (claw, P_8) -free graphs. We conclude by some open questions regarding (claw, P_k) -free graphs for $k \geq 9$.

2 Algorithmic Properties

We give two properties that authorize us to make some assumptions and simplifications for the graphs we consider.

Property 2.1 *Let G be a claw-free graph. If v, u are two vertices such that $N[u] = N[v]$ then $\gamma(G) = \gamma(G/uv)$.*

Proof: Let Γ be a γ -set of G . Let u' be the vertex of G/uv resulting from the contraction of uv . At most one of u and v is in Γ . If $u \in \Gamma$ then $\Gamma' = (\Gamma - u) \cup \{u'\}$ is a γ -set of G/uv . If $u, v \notin \Gamma$ then $\Gamma' = \Gamma$ is a γ -set of G/uv . \square

Property 2.2 *Let $G = (V, E)$ be a connected claw-free graph with $uv \in E$ such that u is a leaf. The MINIMUM DOMINATING SET problem is polynomial for $G' = G - N[v]$ if and only if it is polynomial for G .*

Proof: Since G is claw-free $K = N(v) - u$ is a clique. Let $k \in K$, we show that $k \notin V^+$. For contradiction we assume that $k \in V^+$. As shown in [4] k is in every γ -set of G . Let Γ be a γ -set with $v \in \Gamma$. Let $W = N(k) \setminus N(v)$. If $W = \emptyset$ then $\Gamma - k$ is a dominating set, a contradiction. It follows $W \neq \emptyset$ and since G is claw-free, W is a clique. Let $w \in W$. $\Gamma' = (\Gamma - \{k\}) \cup \{w\}$ is another γ -set, a contradiction. Hence $\gamma(G - K) \leq \gamma(G)$. Since $G - K$ consists of G' and the component uv , we have that $\gamma(G - K) = \gamma(G') + 1$. Then from Γ' a γ -set of G' we obtain $\Gamma' \cup \{v\}$ a γ -set of G in polynomial time. Reciprocally, let Γ be a γ -set of G . Since u is a leaf we assume that $v \in \Gamma$. Then $\Gamma - \{v\}$ is a γ -set for G' . Trivially it can be done in polynomial time from Γ . \square

We show two conditions on the structure of G that authorize us to directly conclude that computing a γ -set for G can be done in polynomial time.

Property 2.3 *Let $k > 0$ be a fixed positive integer and $G = (V, E)$ a graph. If there exists $T \subseteq_i G$ of size $|T| \leq k$ such that $V = N[T]$ then computing a minimum dominating set for G is polynomial.*

Proof: Since $|T| \leq k$ and $V = N[T]$, we have that $\gamma(G) \leq k$. So a minimum dominating set can be computed in $O(n^k)$ which is polynomial. \square

Property 2.4 Let $k, k' > 0$ two fixed positive integers and $G = (V, E)$ a graph. If there exists $T \subseteq_i G$ of size $|T| \leq k$ such that $W = V \setminus N[T]$ has a size $|W| \leq k'$ then computing a minimum dominating set for G is polynomial.

Proof: It is clear that k vertices of T is enough to dominate $N[T]$ and k' vertices of W is enough to dominate W . Hence we have $\gamma(G) \leq k + k'$. So a minimum dominating set can be computed in $O(n^{k+k'})$ which is polynomial. \square

3 G has a long cycle

We give two lemmas that will authorize us to conclude that the Minimum Dominating Set problem is polynomial when G , a (claw, P_k) -free graph, contains a long induced cycle.

Lemma 3.1 For any fixed $k \geq 3$, if G is a (claw, P_k) -free connected graph such that $C_k \subseteq_i G$, then a minimum dominating set of G can be given in polynomial time.

Proof: Let $C_k = v_1 - \dots - v_k - v_1, C_k \subseteq_i G$. Let $v \notin C_k$ be such that $N(v) \cap C_k \neq \emptyset$. Since G is claw-free, we have $2 \leq |N(v) \cap C_k| \leq 4$. If $|N(v) \cap C_k| = 2$, the two neighbors of v in C_k must be adjacent, thus there is a P_k starting from v . In the second case, $3 \leq |N(v) \cap C_k| \leq 4$, there is no edge $vw \in E, N(w) \cap C_k = \emptyset$, else there is a claw centered onto v . Taking $\Gamma = \{v_1, v_3, \dots, c_{\lfloor \frac{k}{2} \rfloor}\}$, Γ is a dominating set of G . It follows that $\lfloor \frac{k}{2} \rfloor = |\Gamma| \geq \gamma(G)$. Hence by brute force $\gamma(G)$ and a γ -set can be computed with time $O(n^{\lfloor \frac{k}{2} \rfloor})$ which is polynomial since k is fixed. \square

Lemma 3.2 For any fixed $k \geq 3$, if G is a (claw, P_k, C_k) -free connected graph such that $C_{k-1} \subseteq_i G$, then a minimum dominating set of G can be given in polynomial time.

Proof: Let $C_k = v_1 - \dots - v_{k-1} - v_1, C_{k-1} \subseteq_i G$ and $v \notin C_{k-1}$ such that $N(v) \cap C_{k-1} \neq \emptyset$. We have $2 \leq |N(v) \cap C_{k-1}| \leq 4$. Let w be a neighbor of v such that $N(w) \cap C_{k-1} = \emptyset$. If $3 \leq |N(v) \cap C_{k-1}| \leq 4$, there is a claw. When $|N(v) \cap C_{k-1}| = 2$ there is a P_k starting from w . Taking $\Gamma = \{v_1, v_3, \dots, c_{\lfloor \frac{k-1}{2} \rfloor}\}$, Γ is a dominating set of G . It follows that $\lfloor \frac{k-1}{2} \rfloor = |\Gamma| \geq \gamma(G)$. Hence, by brute force $\gamma(G)$ and a γ -set can be computed with time $O(n^{\lfloor \frac{k-1}{2} \rfloor})$ which is polynomial since k is fixed. \square

4 G is $(\text{claw}, P_k, C_k, C_{k-1})$ -free, $C_{k-2} \subseteq_i G, k \leq 8$

In this section we prove that, for $k \leq 8$, if G is a $(\text{claw}, P_k, C_k, C_{k-1})$ -free graph such that $C_{k-2} \subseteq_i G$ then the Minimum Dominating Set problem is polynomial. The first lemma gives a structural property for G . We use this property to prove two other lemmas, the first one for $k = 6$, the second for $7 \leq k \leq 8$.

Lemma 4.1 *For any fixed $k \geq 6$, if G is a (claw, P_k, C_k, C_{k-1})-free connected graph such that $C_{k-2} \subseteq_i G$, then $W = V \setminus C_{k-2}$ is an independent set.*

Proof: Let $C = C_{k-2} = v_1 - \dots - v_{k-2} - v_1, C \subseteq_i G$ and $v \in N(C)$. We have $2 \leq |N_C(v)| \leq 5$ (note that $|N_C(v)| = 5$ only for $C = C_5$). Let us denote $S = N(C)$, $W = V \setminus (C \cup S)$. Let $w \in W$ be a neighbor of v . If $3 \leq |N_C(v)| \leq 5$, there is a claw. Hence, v is such that $N_C(v) = \{v_i, v_{i+1}\}$, $1 \leq i \leq k-2$ (for convenience, when $i = k-2$, we read $v_{i+1} = v_1$). From Property 2.1, we can assume that all contractible vertices of G are contracted. Moreover, from Property 2.2 we can assume that G has no leaves.

Assume for contradiction that w has a neighbor $w', w' \in W$. When $N_S(w') = \emptyset$ there is a P_k from w' to v_{k-3} . Hence w' has a neighbor in S . Since w and w' are not contracted, we have $N[w] \neq N[w']$. If $vw' \notin E$ again there is a P_k from w' to v_{k-3} . Hence, w and w' have the same neighbors in S but not in W . So there exists $r \in W$ with $rw \in E, rw' \notin E$. As above $rv \in E$ but there exists $v_i \in C$ such that $G[\{r, v, v_i, w'\}]$ is a claw. Hence, W is independent. \square

Lemma 4.2 *If G is a (claw, P_6, C_6, C_5)-free connected graph such that $C_4 \subseteq_i G$, then a minimum dominating set of G can be given in polynomial time.*

Proof: Let $C = C_4 = v_1 - \dots - v_4 - v_1, C \subseteq_i G$ and $v \in N(C)$. We have $2 \leq |N_C(v)| \leq 4$. Let $S = N(C)$, $W = V \setminus (C \cup S)$ and $w \in W$ a neighbor of v . If $3 \leq |N_C(v)| \leq 4$ then G contains a claw. Hence, v is such that $N_C(v) = \{v_i, v_{i+1}\}$, $1 \leq i \leq 4$ (for convenience, when $i = 4$, we read $v_{i+1} = v_1$).

From Property 2.4, we can assume that $|W| \geq 2$ and from Lemma 4.1, we know that W is an independent set. We show that all vertices $v \in N(W)$ have exactly the same neighbors in C .

Let $w, w' \in W$, $w \neq w'$, $v \in N_S(w)$, $v' \in N_S(w')$. Since G is claw-free $v \neq v'$. W.l.o.g. $N_C(v) = \{v_1, v_2\}$. Assume that $N_C(v) \neq N_C(v')$. W.l.o.g. $N_C(v') = \{v_2, v_3\}$ (note that $N_C(v') = \{v_1, v_4\}$ is symmetric). If $vv' \notin E$ then $w - v - v_1 - v_4 - v' - w' = P_6$, else $v_1 - v - v' - v_3 - v_4 = C_5$. Now it remains $N_C(v') = \{v_3, v_4\}$. We have $vv' \notin E$ else there is a claw, but $w - v - v_1 - v_4 - v' - w' = P_6$ a contradiction. Thus, w.l.o.g. any vertex $w \in W$ has only neighbors $v \in S$ such that $N(v) = \{v_1, v_2\}$.

Let $|W| = q, q \geq 2$. We show that $\gamma(G) = q + 1$. Since W is independent and for any distinct $w, w' \in W$, we have $N[w] \cap N[w'] = \emptyset$, we must take q vertices of $N[W]$ to dominate the vertices of W . These vertices cannot dominate v_3 nor v_4 . Hence $\gamma(G) \geq q + 1$.

We construct a γ -set of G as follows. We set R by taking exactly one neighbor of each $w, w \in W$. Clearly, $\Gamma = R \cup \{v_3\}$ dominates $C \cup N[R]$. Suppose that there exists $s \in S$ such that s is not a neighbor of Γ . If $N_C(s) = \{v_1, v_2\}$ then there exists $r \in R$ such that $G[\{r, s, v_1, v_4\}]$ is a claw. If $N_C(s) = \{v_1, v_4\}$ then $w - v - v_2 - v_3 - v_4 - s = P_6$. If $N_C(s) = \{v_1, v_2, v_4\}$ then there exists $r \in R$ such that $G[\{r, v, v_2, v_3\}]$ is a claw. Hence any $s \notin N[R] \cup C$ is dominated by v_3 . It follows that Γ is a γ -set of G . \square

Lemma 4.3 For $k \in \{7, 8\}$, if G is a (claw, P_k, C_k, C_{k-1})-free connected graph such that $C_{k-2} \subseteq_i G$, then a minimum dominating set of G can be given in polynomial time.

Proof: Let $C = C_{k-2} = v_1 - \dots - v_{k-2} - v_1, C \subseteq_i G$ and $v \in N(C)$. We have $2 \leq |N_C(v)| \leq 5$ (note that $|N_C(v)| = 5$ only for $C = C_5$). Let $S = N(C)$, $W = V \setminus (C \cup S)$ and $w \in W$ a neighbor of v . If $3 \leq |N_C(v)| \leq 4$ then G contains a claw. Hence, v is such that $N_C(v) = \{v_i, v_{i+1}\}$, $1 \leq i \leq k-2$ (for convenience, when $i = k-2$, we read $v_{i+1} = v_1$).

We show that for any $w, w' \in W$, there exists $v, v' \in N(w)$ such that $N_C(v) \cap N_C(v') = \emptyset$. Let $w \in W$ and $v, v' \in N_S(w)$, $v \neq v'$.

First, we show that $N_C(v) \neq N_C(v')$. Suppose that $N_C(v) = N_C(v')$, w.l.o.g. $N_C(v) = \{v_1, v_2\}$. $vv' \in E$ else $G[\{v, v', v_1, v_{k-2}\}]$ is a claw. From Property 2.1, we have that $N[v] \neq N[v']$, else v, v' are contracted. Hence, there exists $u \in V$ such that $uv \in E$ and $uv' \notin E$. If $u \in W$ then from Lemma 4.1 $uw \notin E$ but $G[\{u, v, w, v_1\}]$ is a claw. So, we have $u \in S$. If $N_C(u) = \{v_1, v_2\}$ then $G[\{u, v', v_2, v_3\}]$ is a claw. So $N_C(u) \neq N_C(v)$ and we can assume that $wu \notin E$ or else we have u, v two neighbors of w with distinct neighborhoods in C . If $N_C(u) \cap N_C(v) = \emptyset$ then $G[\{u, v, v_1, w\}]$ is a claw. So W.l.o.g. we assume that $N_C(u) \cap N_C(v) = \{v_1\}$ but $G[\{u, v, v_2, w\}]$ is a claw. Hence $N[v] = N[v']$ and v, v' can be contracted. Yet, w is a leaf and from Property 2.2 we can assume that G contains no leaves. So, for any $w, w' \in W$, there exists $v, v' \in N_S(w)$, $v \neq v'$ such that $N_C(v) \neq N_C(v')$.

Now we show that that $N_C(v) \cap N_C(v') = \emptyset$. W.l.o.g. assume that $N_C(v) = \{v_1, v_2\}$ and $N_C(v') = \{v_2, v_3\}$. If $vv' \in E$ then $v_1 - v - v' - v_3 - \dots - v_{k-2} - v_1 = C_{k-1}$, else $v_1 - v - w - v' - v_3 - \dots - v_{k-2} - v_1 = C_k$. Thus any $w, w' \in W$, has neighbors $v, v' \in S$ such that $N_C(v) \cap N_C(v') = \emptyset$.

From Property 2.4, we can assume that $|W| \geq 2$. So let $w, w' \in W$ (recall $ww' \notin E$). Since both w and w' have two neighbors in S with non intersecting neighborhoods in C , let $v \in N(w)$, $v' \in N(w')$ such that $N_C(v) \cap N_C(v') = \emptyset$. W.l.o.g. $N_C(v) = \{v_1, v_2\}$. Assume that $N_C(v') = \{v_3, v_4\}$ (note that $N(v') = \{v_{k-2}, v_{k-3}\}$ is symmetric). If $vv' \in E$ then $G[\{v, v', v_1, w\}]$ is a claw, else $w - v - v_1 - v_{k-2} - \dots - v_4 - v' - w' = P_k$. Hence the two neighborhoods of $N_C(v)$ and $N_C(v')$ are not adjacent. It follows that for $k = 7$, since $C_{k-2} = C_5$, such a configuration is impossible. This yields to $|W| \leq 1$ and from Property 2.4 a minimum dominating set can be computed in polynomial time.

Now, we focus on the remaining case of $k = 8$. Let $|W| = q, q \geq 2$. We show that $\gamma(G) = q + 2$. Since W is independent and that for any distinct vertices $w, w' \in W$, we have $N[w] \cap N[w'] = \emptyset$, we must take q vertices of $N[W]$ to dominate the vertices of W . Let $w, w' \in W$. From above we can assume that w has a neighbor v such that $N_C(v) = \{v_1, v_2\}$ and w' has a neighbor v' such that $N_C(v') = \{v_4, v_5\}$ (each vertex of W has two neighbors whose are neighbors of respectively $\{v_1, v_2\}$ and $\{v_4, v_5\}$ since $C = C_6$). G being claw-free we have $vv' \notin E$. The q vertices that dominates W cannot dominate v_3 and v_6 . Hence $\gamma(G) \geq q + 1$.

Suppose that $\gamma(G) = q + 1$. The mds of G must contain a vertex $s \in S$ a neigh-

bor of both v_3 and v_6 . If $vs \in E$, respectively $v's \in E$, then G has a claw (s cannot be complete to $N_C(v) \cup N_c(v')$). Also, s must have (v_1 or v_5) and (v_2 or v_4) as neighbors else there is a claw. We assume first that $N(s) = \{v_1, v_2, v_3, v_6\}$. Then $w - v - v_1 - s - v_3 - v_4 - v' - w' = P_8$ (recall $vv' \notin E$ or G contains a claw). The case where $N(s) = \{v_3, v_4, v_5, v_6\}$ is symmetric. Now we assume that $N(s) = \{v_1, v_3, v_4, v_6\}$ (note that $N(s) = \{v_2, v_3, v_5, v_6\}$ is symmetric). Then $w - v - v_2 - v_3 - s - v_6 - v_5 - v' = P_8$. Hence $\gamma(G) \geq q + 2$.

We show that $\Gamma = \{v_1, v_4\} \cup W$ is a γ -set of G . Clearly Γ dominates $N[W] \cup C$. Let $s \notin N[W] \cup C$. So $s \in S$. Suppose that $sv_1, sv_4 \notin E$. From above $ws \notin E$ and $vs \notin E$ else $G[\{v, s, v_1, w\}]$ is a claw. If $N(s) = \{v_2, v_3\}$ then $w - v - v_1 - v_6 - v_5 - v_4 - v_3 - s = P_8$. By symmetry $N(s) \neq \{v_5, v_6\}$. As shown before $N(s) = \{v_2, v_3, v_5, v_6\}$ is not possible. Hence any $s \notin N[W] \cup C$ is dominated by v_1 or v_4 . It follows that $\Gamma = \{v_1, v_4\} \cup W$ is a γ -set of G . \square

From Lemmas 3.1, 3.2, 4.2, 4.3 we immediately obtain the corollary below.

Corollary 4.4 *Let G a (claw, P_k) -free graph, $6 \leq k \leq 8$. If $C_l \subseteq_i G$, $k - 2 \leq l \leq k$, then a minimum dominating set of G can be given in polynomial time.*

5 G is (claw, P_8) -free

Here we conclude by the main result proving that the Minimum Dominating Set problem is polynomial in the class of (claw, P_8) -free graphs. Starting from the result stating that the problem is polynomial when G is (claw, P_5) -free, we successively prove that the problem is polynomial for (claw, P_6) -free, (claw, P_7) -free graphs. Then we conclude for the class of (claw, P_8) -free graphs.

Lemma 5.1 *Let G be a connected (claw, P_5) -free graph. Computing a minimum dominating set is polynomial-time solvable.*

Proof: From [8] we know that computing a minimum dominating set is polynomial when G is $(K_{1,4}, P_5)$ -free. The result follows from the fact that $((\text{claw}, P_5) - \text{free graphs}) \subset ((K_{1,4}, P_5) - \text{free graphs})$. \square

Lemma 5.2 *Let G be a connected (claw, P_6) -free graph. Computing a minimum dominating set is polynomial-time solvable.*

Proof: From Corollary 4.4, if $C_l \subseteq_i G$, $4 \leq l \leq 6$, then computing a minimum dominating set is polynomial. When G is $(\text{claw}, C_4, C_5, C_6, P_6)$ -free then it is chordal. The Minimum Dominating Set problem is polynomial for claw -free chordal graphs. \square

Lemma 5.3 *Let G be a connected $(\text{claw}, C_5, C_6, C_7, P_7)$ -free graph. Computing a minimum dominating set is polynomial-time solvable.*

Proof: From Lemma 5.2 we can assume that $P_6 \subseteq_i G$. Let $P = v_1 - v_2 - v_3 - v_4 - v_5 - v_6$.

Let $W = V \setminus N[P]$. From Property 2.3 if $W = \emptyset$ computing a minimum dominating set is polynomial. From now on $W \neq \emptyset$.

Let $S = \{v \in V \setminus P \text{ such that } 2 \leq |N_P(v)| \leq 4\}$, $S_i \subseteq S$ being the set of vertices v such that $|N_P(v)| = i$.

Let $H_i = \{v \in S_2 : N_P(v) = \{v_i, v_{i+1}\}, 1 \leq i \leq 5\}$. Since G is claw-free each H_i is complete. If there is an edge $r_i r_{i+1}$ with $r_i \in H_i, r_{i+1} \in H_{i+1}$ then $P = v_1 - \dots - v_i - r_i - r_{i+1} - v_{i+2} - \dots - v_6 = P_7$. If there is an edge $r_i r_j$ with $r_i \in H_i, r_j \in H_j$ and $j \geq i + 3$ then $C_p \subseteq_i G$, $p \geq 5$. So H_1 is anticomplete to H_2, H_4, H_5 , the component H_2 is anticomplete to H_3, H_5 , and the component H_3 is anticomplete to H_4 .

We define R_i as the set of vertices of H_i having a neighbor in W , $R_i = \{v \in H_i : N_W(v) \neq \emptyset\}, 1 \leq i \leq 5$. Since G is P_7 -free $R_1 = R_5 = \emptyset$.

Let $r \in R_i, r' \in R_i, r \neq r', i \in \{2, 4\}$ be such that r , respectively r' , has a neighbor $w \in W$, respectively $w' \in W$. We show that $N_S(r) = N_S(r')$.

By contradiction we assume that there exists $s \in S$ such that $rs \in E, r's \notin E$. From above $s \notin R_i \cup H_{i-1} \cup H_{i+1}$. Let $i = 2$ (the case $i = 4$ is symmetric). Recall that H_2 is anticomplete to H_1, H_3, H_5 , thus $s \in H_4 \cup S_3 \cup S_4$. If $s \in H_4$ then $G[\{r, w, v_3, s\}]$ is a claw. Hence $s \in S_3 \cup S_4$. When $N_P(s) = \{v_1, v_2, v_3\}$ then $G[\{r', v_3, v_4, s\}]$ is a claw. When $N_P(s) = \{v_2, v_3, v_4\}$ then $G[\{r', v_1, v_2, s\}]$ is a claw. When $N_P(s) = \{v_3, v_4, v_5\}$ or $N_P(s) = \{v_4, v_5, v_6\}$ then $G[\{r, v_2, w, s\}]$ is a claw. So $s \in S_4$. When $N_P(s) = \{v_1, v_2, v_3, v_4\}$ then $G[\{r, v_1, v_4, s\}]$ is a claw. When $N_P(s) = \{v_2, v_3, v_4, v_5\}$ then $G[\{r', v_1, v_2, s\}]$ is a claw. When $N_P(s) = \{v_3, v_4, v_5, v_6\}$ then $G[\{r, v_4, v_6, s\}]$ is a claw. Now let $i = 3$. Recall that H_3 is anticomplete to H_2, H_4 , thus $s \in H_1 \cup H_5 \cup S_3 \cup S_4$. If $s \in H_1$ (the case $s \in H_5$ is symmetric) then $G[\{r, w, v_3, s\}]$ is a claw. Hence $s \in S_3 \cup S_4$. If $N_P(s) = \{v_1, v_2, v_3\}$ (the case $N_P(s) = \{v_4, v_5, v_6\}$ is symmetric) then $G[\{r, w, v_4, s\}]$ is a claw. If $N_P(s) = \{v_2, v_3, v_4\}$ (the case $N_P(s) = \{v_3, v_4, v_5\}$ is symmetric) then $G[\{r', v_4, v_5, s\}]$ is a claw. So $s \in S_4$. When $N_P(s) = \{v_1, v_2, v_4, v_5\}$ or $N_P(s) = \{v_1, v_2, v_5, v_6\}$ then $G[\{r, v_1, v_5, s\}]$ is a claw. When $N_P(s) = \{v_2, v_3, v_5, v_6\}$ then $G[\{r', v_3, v_4, s\}]$ is a claw. When $N_P(s) = \{v_1, v_2, v_3, v_4\}$ (the case $N_P(s) = \{v_3, v_4, v_5, v_6\}$ is symmetric) then $G[\{r', v_4, v_5, s\}]$ is a claw. Hence $N_P(s) = \{v_2, v_3, v_4, v_5\}$ but $G[\{r, v_2, v_5, s\}]$ is a claw. Thus $N_S(r) = N_S(r')$.

Let $r_2 \in R_2, r'_2 \in R_2, r_2 \neq r'_2$ be such that r_2 , respectively r'_2 , has a neighbor $w \in W$, respectively $w' \in W$. Let $r_4 \in R_4, r'_4 \in R_4, r_4 \neq r'_4$ be such that r_4 , respectively r'_4 , has w , respectively w' , as neighbor. We show that $N_{S \setminus H_4}(r_2) = N_{S \setminus H_4}(r'_2)$, respectively $N_{S \setminus H_2}(r_4) = N_{S \setminus H_2}(r'_4)$.

By contradiction we assume that there exists $s \in S$ such that $r_2 s \in E, r'_2 s \notin E$. From above $s \notin H_1 \cup H_2 \cup H_3$. When $s \in H_4$ we know that s is not a neighbor of w . If $s \in H_4 \cup H_5$ then $G[\{r_2, v_2, w, s\}]$ is a claw. Hence $s \in S_3 \cup S_4$. When $N_P(s) = \{v_1, v_2, v_3\}$ then $G[\{r'_2, v_3, v_4, s\}]$ is a claw. When $N_P(s) = \{v_2, v_3, v_4\}$ then $G[\{r'_2, v_1, v_2, s\}]$ is a claw. When $N_P(s) = \{v_3, v_4, v_5\}$ then $G[\{r_2, v_2, w, s\}]$ is a claw. When $N_P(s) = \{v_4, v_5, v_6\}$ then $G[\{r_2, v_4, v_6, s\}]$ is a claw. So $s \in S_4$.

When $N_P(s) = \{v_1, v_2, v_4, v_5\}$ or $N_P(s) = \{v_1, v_2, v_5, v_6\}$ then $G[\{r, v_1, v_5, s\}]$ is a claw. When $N_P(s) = \{v_2, v_3, v_5, v_6\}$ then $G[\{r', v_3, v_4, s\}]$ is a claw. When $N_P(s) = \{v_1, v_2, v_3, v_4\}$ then $G[\{r_2, v_1, v_4, s\}]$ is a claw. When $N_P(s) = \{v_2, v_3, v_4, v_5\}$ then $G[\{r'_2, v_1, v_2, s\}]$ is a claw. When $N_P(s) = \{v_3, v_4, v_5, v_6\}$ then $G[\{r_2, v_4, v_6, s\}]$ is a claw. Thus $N_{S \setminus H_4}(r_2) = N_{S \setminus H_4}(r'_2)$ and by symmetry, for $r'_4 \in R_4, r'_4 \neq r_4$, we have $N_{S \setminus H_2}(r_4) = N_{S \setminus H_2}(r'_4)$.

Let $w \in W$. We show that w cannot have two neighbors r_i, r_{i+1} with $r_i \in R_i, r_{i+1} \in R_{i+1}$. Suppose for contradiction that these two neighbors exist. Then $v_1 - \dots - v_i - r_i - w - r_{i+1} - v_{i+2} - \dots - v_6 = P_8$. Now, since $R_1 = R_5 = \emptyset$, if w has two neighbors $r_i \in R_i, r_j \in R_j, i \neq j$, these two neighbors are $r_2 \in R_2, r_4 \in R_4$ and $r_2 r_4 \in E$, else $w - r_4 - v_4 - v_3 - r_2 - w = C_5$.

Moreover, when w has two neighbors $r_2 \in R_2, r_4 \in R_4$, then for each neighbor $w' \in N_W(w)$, w' has r_2 and r_4 as neighbors. Assume for contradiction that w has a neighbor $w' \in W$ such that $w' r_2 \notin E$ (by symmetry $w' r_4 \notin E$ is the same case). Then $w' - w - r_2 - v_3 - \dots - v_6 = P_7$. It follows that $N[w] = N[w']$ and from Property 2.1 ww' can be contracted.

Hence setting $Z_{24} = \{w \in W : w \text{ has two neighbors } r_2 \in R_2, r_4 \in R_4\}$, Z_{24} is an independent set.

Let $w, w' \in Z_{24}, w \neq w'$. Since G is claw-free we have $N(w) \cap N(w') = \emptyset$. We show that $N_{R_2}(w)$ is anticomplete to $N_{R_4}(w')$ and $N_{R_4}(w)$ is anticomplete to $N_{R_2}(w')$. By contradiction if w has a neighbor $r_2 \in R_2$, w' has a neighbor $r_4 \in R_4$, and $r_2 r_4 \in E$ then $G[\{v_2, r_2, w, r_4\}]$ is a claw.

Let $Z_i = \{w \in W : w \text{ has a neighbor in } R_i \setminus (N_{R_i}(Z_{24}))\}$, $2 \leq i \leq 4$.

We show that Z_2, Z_3, Z_4 are pairwise anticomplete. If there is an edge $w_2 w_4, w_2 \in Z_2, w_4 \in Z_4$, with $r'_2 \in R_2, r'_4 \in R_4$ the neighbors of w_2, w_4 respectively, then $w_2 - r'_2 - v_3 - v_4 - r'_4 - w_4 - w_2 = C_6$ ($r'_2 r'_4 \notin E$ else $G[\{v_2, r'_2, w_2, r'_4\}]$ is a claw). If there is an edge $w_2 w_3, w_2 \in Z_2, w_3 \in Z_3$, with $r'_2 \in R_2, r'_3 \in R_3$ the neighbors of w_2, w_3 respectively, then $w_2 - r'_2 - v_3 - r'_3 - w_3 - w_2 = C_5$ (recall $r'_2 r'_3 \notin E$). By symmetry there is no edge between Z_3 and Z_4 .

Let $Y = W \setminus (Z_2 \cup Z_3 \cup Z_4 \cup Z_{24})$. One can observe that for any $w \in Y$ we have $N_{Z_2}(w) = N_{Z_4}(w) = N_{Z_{24}}(w) = \emptyset$ else $P_7 \subseteq_i G$.

Let $Y_3 = \{w \in Y : w \text{ has a neighbor in } Z_3\}$. If there exists $w' \in Y \setminus Y_3$ such that w' has a neighbor $w, w \in Y_3$, then $P_7 \subseteq_i G$. Hence $Y = Y_3$.

We show that we can assume that Z_2, Z_4, Y_3 are three independent sets. The arguments are the same for the three sets, so we show that Z_2 is an independent set. For contradiction, we assume that there are $w_1, w_2 \in Z_2$ such that $w_1 w_2 \in E$. We prove that $N_{R_2}(w_1) = N_{R_2}(w_2)$. If $N_{R_2}(w_1) \neq N_{R_2}(w_2)$ then there exists $r_2 \in R_2$ which is a neighbor of w_1 but not a neighbor of w_2 . Then $w_2 - w_1 - r_2 - v_3 - \dots - v_6 = P_7$. If $N_{Z_2}(w_1) \neq N_{Z_2}(w_2)$ then there exists $w_3 \in Z_2$ such that $w_2 w_3 \in E, w_1 w_3 \notin E$, but $G[\{v_2, r_2, w_1, w_3\}]$ is a claw. Hence $N[w_1] = N[w_2]$ and we can apply the Property 2.1 to contract $w_1 w_2$. Hence Z_2, Z_4, Y_3 are independent.

Since G is claw-free then for any two distinct vertices $w_1, w_2 \in Z_2 \cup Z_4 \cup Y_3$ we have $N(w_1) \cap N(w_2) = \emptyset$.

We prove that for any $w \in Y_3$, $N(w)$ is a clique. Let $w \in Z_3$. Suppose there are s, s' two non adjacent vertices in $N(w)$. Since G is claw-free s, s' cannot have a common neighbor in R_3 . Let $r \in R_3$ be a neighbor of s . Then $s' - w - s - r - v_3 - v_2 - v_1 = P_7$.

Since G is claw-free, if there are a vertex $r \in R_i$ with a neighbor $z \in Z_i$ and a vertex $s \in S$ such as $sz \notin E$ and $v_i \notin N(s)$ then G contains a claw (note that $v_{i+1} \notin N(s)$ is symmetric). Hence $N(Z_i)$ is anticomplete to H_j , $j \neq i$.

We show that we can assume that $Z_2 = Z_4 = \emptyset$. The arguments are the same in the two cases, so we consider Z_2 . Let $r, r' \in R_2$ be two neighbors of $w \in Z_2$. We show that $N[r] = N[r']$. Since $N_R(w) = N_{R_2}(w)$ and $rr' \in E$ then, as proved above, $N_S(r) = N_S(r')$. For two distinct $w_1, w_2 \in Z_2$, $N(w_1) \cap N(w_2) = \emptyset$. Hence, $N[r] = N[r']$. From Property 2.1 we can contract rr' . Then w is a leaf and by Property 2.2, w can be deleted from G .

Now we study the structure of Z_3 . For any distinct two vertices $w_1, w_2 \in Z_3$ such that $w_1w_2 \in E$, there cannot exist two distinct vertices $w'_1, w'_2 \in Z_3$ such that $w_1w'_1 \in E, w'_1w_2 \notin E$ and $w_2w'_2 \in E, w_1w'_2 \notin E$. For contradiction we suppose that such two vertices exist. We assume first that w_2 has a neighbor $r_2 \in R_3$ such that $r_2w_1 \notin E$. If $w'_1r_2 \notin E$ then $v_1 - v_2 - v_3 - r_2 - w_2 - w_1 - w'_1 = P_7$ else $G[\{v_4, r_2, w_2, w'_1\}]$ is a claw. So w_1, w_2 have a common neighbor $r_1 \in R_3$. If $w'_1r_1 \in E$ then $G[\{v_3, r_1, w_2, w'_1\}]$ is a claw. Thus $w'_1r_1 \notin E$ and w'_1 has a neighbor $r'_1 \in R_3, r'_1 \neq r_1$. If $r'_1w_2 \in E$ then $G[\{v_3, r'_1, w_2, w'_1\}]$ is a claw. So $r'_1w_2 \notin E$. If $r'_1w_1 \notin E$ then $v_1 - v_2 - v_3 - r'_1 - w'_1 - w_1 - w_2 = P_7$. Thus $r'_1w_1 \in E$. If $r'_1w'_2 \notin E$ then $v_1 - v_2 - v_3 - r'_1 - w_1 - w_2 - w'_2 = P_7$. So $r'_1w'_2 \in E$ but $G[\{v_4, r'_1, w_1, w'_2\}]$ is a claw.

As a consequence each connected component A_i of Z_3 has a universal vertex. Also, G being claw-free two distinct connected components cannot share a neighbor in R_3 . Moreover, from Property 2.2 we have assumed that each $w_3 \in Z_3$ is not a leaf.

We show that $w \in Y_3$ is connected to a universal vertex of a connected component A_i of Z_3 . We assume that the neighbors of w are not universal in A_i . Let $s \in A_i$ be a neighbor of w , let $u, u \neq s$, be a universal vertex of A_i . Since s is not universal there exists $v, v \in A_i$ such that $sv \notin E$ and $uv \in E$. Since $N(w)$ is complete $wv \notin E$. Let $r \in R_3$ be a neighbor of s . Since G is claw-free then $rv \notin E$. Let $r', r' \in R_3, r' \neq r$, be a neighbor of v . As just above $r's \notin E$. If $r'u \notin E$ then $v_1 - v_2 - v_3 - r' - v - u - s = P_7$ else $v_1 - v_2 - v_3 - r' - u - s - w = P_7$.

We are ready to show how to build a γ -set in polynomial time.

First, we treat the case where $Z_{24} \neq \emptyset$. Let $r_2 \in R_2, r_4 \in R_4$ be two neighbors of $w, w \in Z_{24}$.

We show that $R_3 = \emptyset$. Assume that there exists $w' \in W$ with a neighbor $r_3 \in R_3$. Since w' is not a neighbor of r_2 or r_4 we have $w' - r_3 - v_3 - r_2 - r_4 - v_5 - v_6 = P_7$. So $R_3 = \emptyset$ and since $Z_2 = Z_4 = \emptyset$ we have $W = Z_{24}$.

Recall that $W = Z_{24}$ is independent and that for any two distinct vertices $w', w' \in$

Z_{24} we have $N(w) \cap N(w') = \emptyset$.

The γ -set is build as follows:

From Property 2.4, we can assume that $|W| \geq 2$. We take $r_2 \in R_2$ a neighbor of w (recall that the neighbors of w in $R_i, i \in \{2, 4\}$, have the same neighborhood and that all vertices of R_i have the same neighbors in $S \setminus H_4$), and for each other $w' \in Z_{24}$ we take one adjacent vertex $r'_4 \in R_4$. These vertices dominate $Z_{24} \cup H_2 \cup H_4 \cup \{v_2, v_3, v_4, v_5\}$. At least one more vertex is necessary to dominate G since v_1 and v_6 are not dominated. Adding the three vertices v_2, v_4, v_6 we have a dominating set (not necessarily minimum). We check first if there exists s a neighbor of both v_1 and v_6 that dominates the rest of the graph. If such vertex s is nowhere to be found, checking for all the pairs s_1, s_6 where s_i is a neighbor of $v_i, i \in \{1, 6\}$, one can verify if there is a γ -set with only two more vertices (note that there are at most $O(n^2)$ such pairs).

Now we deal with the case $Z_{24} = \emptyset$.

The γ -set is build as follows:

- $Y_3 \neq \emptyset$. For each $w \in Y_3$ we take one universal vertex in the connected component A_i of Z_3 connected to w . For each connected component A_i of Z_3 that is not connected to a vertex of Y_3 , we do as follows: If there exists $r_3 \in R_3$ which is complete to A_i (recall that such vertices have the same neighborhood) then we take r_3 , else we take one universal vertex of A_i . These vertices dominate $Y_3 \cup Z_3$. At least one more vertex is necessary to dominate G since v_1 and v_6 are not dominated. Adding the three vertices v_2, v_4, v_6 we have a dominating set (not necessarily minimum). We check first if there exists s a neighbor of both v_1 and v_6 that dominates the rest of the graph. If such vertex is s nowhere to be found, checking for all the pairs s_1, s_6 where s_i is a neighbor of $v_i, i \in \{1, 6\}$, one can verify if there is a γ -set with only two more vertices (note that there are at most $O(n^2)$ such pairs).
- $Y_3 = \emptyset$. Thus $Z_3 = W$. For any connected component A_i of Z_3 , if there exists $r_3 \in R_3$ which is complete to A_i (recall that such vertices have the same neighborhood) then we take r_3 , else we take one universal vertex of A_i . These vertices dominate Z_3 . At least one more vertex is necessary to dominate G since v_1 and v_6 are not dominated. Adding the three vertices v_2, v_4, v_6 we have a dominating set (not necessarily minimum). We check first if there exists s a neighbor of both v_1 and v_6 that dominates the rest of the graph. If such vertex s is nowhere to be found, checking for all the pairs s_1, s_6 where s_i is a neighbor of $v_i, i \in \{1, 6\}$, one can verify if there is a γ -set with only two more vertices (note that there are at most $O(n^2)$ such pairs).

Clearly the construction of the γ -set is polynomial. □

Corollary 5.4 *The Minimum Dominating Set problem is polynomial for (claw, P_7)-free graphs.*

Lemma 5.5 *Let G be a connected (claw, C_6, C_7, C_8, P_8)-free graph. If $C_5 \subseteq_i G$ then computing a minimum dominating set is polynomial.*

Proof: Let $C = v_1 - v_2 - v_3 - v_4 - v_5 - v_1 = C_5 \subseteq_i G$. Let $W = V \setminus N[C]$. From Property 2.3 if $W = \emptyset$ computing a minimum dominating set is polynomial. From now on $W \neq \emptyset$.

Let $S = \{v \in V \setminus C \text{ such that } 2 \leq |N(v) \cap C_5| \leq 5\}$, $S_i \subseteq S$ being the set of vertices v such that $|N(v) \cap C_5| = i$. Let $H_i = \{v \in S_2 : N(v) \cap C_5 = \{v_i, v_{i+1}\}\}$, $1 \leq i \leq 5$ (for convenience v_{5+1} stands for v_1). Since G is claw-free, each H_i is complete. Moreover, if there is an edge $r_i r_{i+1}$ with $r_i \in H_i, r_{i+1} \in H_{i+1}$ then $r_i - v_i - v_{i-1} - \dots - v_{i+2} - r_{i+1} - r_i = C_6$. Hence H_i is anticomplete to H_{i+1} . We define R_i as the set of vertices of H_i having a neighbor in W , $R_i = \{v \in H_i : N_W(v) \neq \emptyset\}$, $1 \leq i \leq 5$, $R = R_1 \cup \dots \cup R_5$.

Since $W \neq \emptyset$, we assume that there exists $w_1 \in W$ such that w_1 has a neighbor $r_1 \in R_1$. Suppose that $R_2 \neq \emptyset$. There exists $w_2 \in W$ with a neighbor $r_2 \in R_2$. If $w_1 = w_2$ then $w_1 - r_2 - v_3 - v_4 - v_5 - v_1 - r_1 - w_1 = C_7$. So $w_1 \neq w_2$. If $w_1 w_2 \in E$ then $w_1 - w_2 - r_2 - v_3 - v_4 - v_5 - v_1 - r_1 - w_1 = C_8$ else $w_1 - r_1 - v_1 - v_5 - v_4 - v_3 - r_2 - w_2 - w_1 = P_8$. So, if $R_i \neq \emptyset$ then $R_{i-1} = R_{i+1} = \emptyset$. Hence $R_2 = R_5 = \emptyset$.

Let $r \in R_i, r' \in R_i, r \neq r', i \in \{1, 3, 4\}$ be such that r , respectively r' , has a neighbor $w \in W$, respectively $w' \in W$. We show that $N_S(r) = N_S(r')$.

By contradiction we assume that there exists $s \in S$ such that $rs \in E, r's \notin E$. From above $s \notin H_i \cup H_{i-1} \cup H_{i+1}$. Let $i = 1$. If $s \in H_3 \cup H_4$ then $G[\{r, v_1, w, s\}]$ is a claw. Thus $s \in S_3 \cup S_4 \cup S_5$. When $N_C(s) = \{v_1, v_2, v_3\}$ (the case $N_C(s) = \{v_1, v_2, v_5\}$ is symmetric) then $G[\{r', v_5, w, s\}]$ is a claw. When $N_C(s) = \{v_2, v_3, v_4\}$ (the case $N_C(s) = \{v_1, v_4, v_5\}$ is symmetric) then $G[\{r, v_1, w, s\}]$ is a claw. When $N_C(s) = \{v_3, v_4, v_5\}$ then $G[\{r, v_3, v_5, s\}]$ is a claw. So $s \in S_4 \cup S_5$. When $N_C(s) = \{v_1, v_2, v_3, v_4\}$ (the case $N_C(s) = \{v_1, v_2, v_4, v_5\}$ is symmetric) then $G[\{r', v_1, v_5, s\}]$ is a claw. When $N_C(s) = \{v_1, v_3, v_4, v_5\}$ or $N_C(s) = \{v_2, v_3, v_4, v_5\}$ or $N_C(s) = \{v_1, v_2, v_3, v_5\}$ or $s \in S_5$ then $G[\{r, v_3, v_5, s\}]$ is a claw. For $i = 3$ and $i = 4$ the arguments are the same. Thus $N_S(r) = N_S(r')$.

Let $r_1 \in R_1, r'_1 \in R_1, r_1 \neq r'_1$ be such that r_1 , respectively r'_1 , has a neighbor $w \in W$, respectively $w' \in W$. Let $r_3 \in R_3, r'_3 \in R_3, r_3 \neq r'_3$ be such that r_3 , respectively r'_3 , has w , respectively w' , as neighbor. We show that $N_{S \setminus H_3}(r_1) = N_{S \setminus H_3}(r'_1)$, respectively $N_{S \setminus H_1}(r_3) = N_{S \setminus H_1}(r'_3)$.

Let $i = 1$. By contradiction we assume that there exists $s \in S \setminus R_3$ such that $r_1 s \in E, r'_1 s \notin E$. From above $s \notin H_1 \cup H_2 \cup H_5$. If $s \in H_4$ then $G[\{r_1, v_1, w, s\}]$ is a claw. So $s \in S_3 \cup S_4 \cup S_5$. When $N_C(s) = \{v_1, v_2, v_3\}$ (the case $N_C(s) = \{v_1, v_2, v_5\}$ is symmetric) then $G[\{r'_1, v_1, v_5, s\}]$ is a claw. When $N_C(s) = \{v_2, v_3, v_4\}$ (the case $N_C(s) = \{v_1, v_4, v_5\}$ is symmetric) then $G[\{r_1, v_1, w, s\}]$ is a claw. When $N_C(s) = \{v_3, v_4, v_5\}$ then $G[\{r_1, v_3, v_5, s\}]$ is a claw. So $s \in S_4 \cup S_5$. When $N_C(s) = \{v_1, v_2, v_3, v_4\}$ (the case $N_C(s) = \{v_1, v_2, v_4, v_5\}$ is symmetric) then $G[\{r'_1, v_1, v_5, s\}]$ is a claw. When $N_C(s) = \{v_1, v_3, v_4, v_5\}$ or $N_C(s) = \{v_2, v_3, v_4, v_5\}$ or $N_C(s) = \{v_1, v_2, v_3, v_5\}$ or $s \in S_5$ then $G[\{r_1, v_3, v_5, s\}]$ is a claw. By symmetry the arguments are the same

for $i = 3$. Hence $N_{S \setminus H_3}(r_1) = N_{S \setminus H_3}(r'_1)$ and $N_{S \setminus H_1}(r_3) = N_{S \setminus H_1}(r'_3)$.

We study the case where w_1 has a neighbor $r_i, r_i \in R_i, i \in \{3, 4\}$. Since both cases are symmetric, let $r_3, r_3 \in R_3$, be a neighbor of w_1 . If $r_1 r_3 \notin E$ then $w_1 - r_1 - v_1 - v_5 - v_4 - r_3 - w_1 = C_6$. Hence $N_{R_1}(w_1)$ is complete to $N_{R_3}(w_1)$. Since $R_3 \neq \emptyset$, we have $R_4 = \emptyset$ and $N_R(w_1) \subseteq R_1 \cup R_3$. Hence, we define the following subsets of W :

- $Z = \{w \in W : N_R(w) \neq \emptyset\}$;
- $Z_i = \{z \in Z : N_{R_i}(z) \neq \emptyset, N_{R_j}(z) = \emptyset, 1 \leq i \leq 5, i \neq j\}$;
- $Z_{ij} = \{z \in Z : N_{R_i}(z) \neq \emptyset, N_{R_j}(z) \neq \emptyset, 1 \leq i < j \leq 5\}$;
- $Y = W \setminus Z$.

First, we show that Z_i and Z_{ij} are anticomplete, then we show that Z_i consists of leaves. We conclude that $Z_{ij} \neq \emptyset$ implies $Z = Z_{ij}$. We set $w_1 \in Z_{13}$, and since all cases are symmetric, we focus on $Z_1 \neq \emptyset$.

Let $w'_1 \in Z_1$ with a neighbor $r'_1, r'_1 \in R_1, r'_1 \neq r_1$. Note that $r'_1 r_3 \notin E$ else G contains a claw. If $w_1 w'_1 \in E$, then $r'_1 w_1 \in E$, else $w_1 - w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 = C_7$, but $w_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 = C_6$. Hence, $w_1 w'_1 \notin E$ (by symmetry, for any $w_3 \in Z_3, w_1 w_3 \notin E$). Thus Z_1 and Z_3 are anticomplete to Z_{13} .

Now, we show that the vertices of Z_1 are leaves. Assume that there exists $v \in N(w'_1), v \neq r'_1$ such that $N[v] \neq N[w'_1]$. If $v \in Z_3$ then $v - w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 = P_8$. If $v \in Y$ then either $v - w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 = P_8$ or $v - w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 - v = C_8$. If $v \in Z_1$ and $r'_1 v \notin E$ then $v - w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 = P_8$. Hence $N_{R_1}(w'_1) = N_{R_1}(v)$. Since $N[v] \neq N[w'_1]$ we can assume that there exists $v' \in W$ such that $vv' \in E$ but $v'w'_1 \notin E$. Yet with the same arguments as before we have $N_{R_1}(v) = N_{R_1}(v')$ and since $w'_1 v' \notin E$ then $G[\{r'_1, v', v_1, w'_1\}]$ is a claw. Thus Z_1 consists of leaves. Now, from Property 2.2 we can suppose that $Z_1 = \emptyset$. By symmetry $Z_3 = \emptyset$. So $Z = Z_{13}$.

We show that any two vertices $v, v' \in Z_{13}$ with $vv' \in E$ satisfy $N_{R_1 \cup R_3}(v) = N_{R_1 \cup R_3}(v')$. Let $w'_1 \in Z_{13}$ be a neighbor of w_1 . Suppose that there exists $r'_1 \in N_{R_1}(w'_1)$ such that $r'_1 w_1 \notin E$. If $r'_1 r_3 \in E$ then $G[\{r'_1, r_3, v_3, w_1\}]$ is a claw. If $r_3 w'_1 \notin E$ then $w_1 - r_3 - v_4 - v_5 - v_1 - r'_1 - w'_1 - w_1 = C_7$, else $w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w'_1 = C_6$. Hence, $N_{R_1}(w_1) = N_{R_1}(w'_1)$ and by symmetry $N_{R_3}(w_1) = N_{R_3}(w'_1)$.

Suppose that $Y \neq \emptyset$. Let $y \in Y$ be a neighbor of w_1 . We show that Z_{13} is a clique. Let $w'_1 \in Z_{13}$ such that $w_1 w'_1 \notin E$. We have $N_{R_1 \cup R_3}(w_1) \cap N_{R_1 \cup R_3}(w'_1) = \emptyset$ else G contains a claw. Yet, there exists r'_1 a neighbor of w'_1 in R_1 such that either $y - w_1 - r_3 - v_4 - v_5 - v_1 - r'_1 - w'_1 = P_8$ or $y - w_1 - r_3 - v_4 - v_5 - v_1 - r'_1 - w'_1 - y = C_8$ (note that $r'_1 r_3 \notin E$ else G contains a claw). Hence Z_{13} is a clique.

We show that the vertices of Y are leaves. Suppose that y has a neighbor $y' \in Y$. If $y'w_1 \notin E$ then $y' - y - w_1 - r_1 - v_1 - v_5 - v_4 - v_3 = P_8$. Hence $N_Z(y) = N_Z(y')$. Since we assume that $N[y] \neq N[y']$, there exists $v, v \in Y$, such that $vy \in E, vy' \notin E$.

From above $w_1v \in E$ but $G[\{r_1, w_1, y', v\}]$ is a claw. Thus, Y is an independent set. Now, $N(y) \subseteq Z_{13}$ is a clique. Since for any two vertices $w_1, w'_1 \in Z_{13}$ we have $N_{R_1 \cup R_3}(w_1) = N_{R_1 \cup R_3}(w'_1)$ we can assume that $N(y)$ can be contracted into a unique vertex. Thus, Y consists of leaves. Hence from property 2.2, we can assume that $Y = \emptyset$.

From now on $Y = \emptyset$. As shown before any two neighbors of Z_{13} have the same neighbors in R , so they can be contracted and we can assume that Z_{13} is an independent set. Moreover, since G is claw-free, for any two distinct $z, z' \in Z_{13}$, $N[z] \cap N[z'] = \emptyset$. Also, recall that the neighbors of each $z, z \in Z_{13}$ induce a clique.

We show how to build a γ -set of G . Recall that $W = Z_{13}$. From Property 2.4 we can assume that $|W| \geq 2$. So there are $w_1, w'_1 \in Z_{13}$ with neighbors $r_1, r'_1 \in R_1$ and $r_3, r'_3 \in R_3$, respectively. Let $q = |Z_{13}|$. Clearly, to dominate Z_{13} we must take q vertices. We take r_1 and r'_3 . Recall that the vertices of R_1 and R_3 have the same neighbors in $S \cup C$. Then, we take the $q - 2$ vertices of $w \in Z_{13}, w \neq w_1, w'_1$. These q vertices dominate $\{v_1, v_2, v_3, v_4\} \cup H_1 \cup H_3 \cup Z_{13}$. It remains to dominate some vertices of $H_2 \cup H_4 \cup H_5 \cup S_3 \cup S_4 \cup \{v_5\}$. If there exists a vertex $v, v \in S \cup \{v_5\}$, which is universal to these non dominated vertices we take v , else we take the vertices $\{v_2, v_5\}$.

Now, we assume that $Z_{ij} = \emptyset$. Hence let $w_1 \in Z_1$. We study the case $R_3 \neq \emptyset$. Recall then $R_2 = R_4 = R_5 = \emptyset$. Let $w_3 \in W$ such that w_3 has a neighbor $r_3 \in R_3$. If $w_1w_3 \in E$ then $w_1 - w_3 - r_3 - v_4 - v_5 - v_1 - r_1 - w_1 = C_7$ ($r_1r_3 \notin E$ else G contains a claw). So Z_1 is anticomplete to Z_3 . We assume that w_1 has a neighbor $v \in Y$. If $vw_3 \in E$ then $v - w_3 - r_3 - v_4 - v_5 - v_1 - r_1 - w_1 - v = C_8$ else $v - w_1 - r_1 - v_1 - v_5 - v_4 - r_3 - w_3 = P_8$. Hence any neighbor $w'_1, w'_1 \in W$, of w_1 is in Z_1 . From Property 2.1 we can suppose $N[w_1] \neq N[w'_1]$. If $w'_1r_1 \notin E$ then $w'_1 - w_1 - r_1 - v_1 - v_5 - v_4 - r_3 - w_3 = P_8$. Hence $N_R(w_1) = N_R(w'_1)$. Since G is claw-free, for any $r \in R$, $N_W(r)$ is a clique, thus $N[w_1] = N[w'_1]$. So Z_1 is an independent set. Now, recall that for any pair of vertices $r, r' \in R_i, 1 \leq i \leq 5$, $N_S(r) = N_S(r')$. Hence, when $r, r' \in R_1$ have a common neighbor in Z_1 , we have $N[r] = N[r']$, and so r, r' can be contracted. Hence Z_1 consists of leaves. Also, by symmetry, $W = Z_1 \cup Z_3$ consists of leaves and then from Property 2.2 we have $W = \emptyset$, a contradiction.

Now we focus on $R_3 = R_4 = \emptyset$ (note that $Z = Z_1$).

We study the case where $H_2 \neq \emptyset$ or $H_5 \neq \emptyset$. Let $v \in H_2$ (the case $v \in H_5$ is symmetric). We have $Y = \emptyset$, else there are $y \in Y, z \in Z_1, r \in R_1$ such that $v - v_3 - v_4 - v_5 - v_1 - r - z - y = P_8$ (recall $vr \notin E$). So $W = Z = Z_1$. Let $w_1, w_2 \in W$. We assume that $w_1w_2 \in E$. From Property 2.1 we can suppose that $N[w_1] \neq N[w_2]$. Let $w_1r_1, w_2r_2 \in E, r_1 \neq r_2$, such that $w_1r_2 \notin E$. We have $v - v_3 - v_4 - v_5 - v_1 - r_2 - w_2 - w_1 = P_8$. Recall that for any $r \in R$, $N_W(r)$ is a clique, thus $N[w_1] = N[w_2]$. Hence W is an independent set. Moreover, for any $w \in W$ and $r, r' \in N(w)$ we know that r and r' share the same neighbors in $C \cup S$. Hence W is composed exclusively of leaves. So $W = \emptyset$, a contradiction.

Now we can assume that $H_2 = H_5 = \emptyset$. Let $Z_A \subset Z, Z_A = \{w \in W : N_Y(w) = \emptyset\}$.

We show that each connected component A_i of $G[Z_A]$ contains a universal vertex relatively to A_i . For contradiction we suppose that there exists $A_i, A_i \subseteq Z_A$ with no universal vertex in it. Assume that $z_1 - z_2 - z_3 - z_4 = P_4 \subseteq_i A_i$. Let $r, r \in R_1$, be a neighbor of z_1 (note that there is a P_5 from v_3 to r).

Since G is claw-free $rz_3, rz_4 \notin E$. If $rz_2 \in E$ then there is a P_8 from v_3 to z_4 else there is a P_8 from v_3 to z_3 . Now, we assume that $z_1 - z_2 - z_3 - z_4 - z_1 = C_4 \subseteq_i A_i$. Let $r, r \in R_1$, be a neighbor of z_1 . Since G is claw-free we have $rz_3 \notin E$. If $rz_2 \in E$ then $rz_4 \notin E$ else G contains a claw, but $v_3 - v_4 - v_5 - v_1 - r - z_2 - z_3 - z_4 = P_8$. If $rz_2 \notin E$ then $v_3 - v_4 - v_5 - v_1 - r - z_1 - z_2 - z_3 = P_8$. So A_i is (C_4, P_4) -free. It follows that there are $z_1 - z_2 - z_3 = P_3 \subseteq_i A_i$ and $z_4 \in A_i$ such that $z_4z_1, z_4z_2, z_4z_3 \notin E$. Also there exists $z \in A_i$ such that $z_2 - z - z_4$ and $zz_1, zz_3 \in E$ but $A_i[\{z, z_1, z_3, z_4\}]$ is a claw, a contradiction. So each A_i has a universal vertex. Clearly, for two distinct components A_i, A_j we have $N_{R_1}(A_i) \cap N_{R_1}(A_j) = \emptyset$ else there is a claw.

Suppose that $Y \neq \emptyset$. We show that Y is an independent set. Suppose that there are $y, y' \in Y$ with $yy' \in E$. From Property 2.1 we assume that $N[y] \neq N[y']$. If $N_{Z_1}(y) \neq N_{Z_1}(y')$ then, w.l.o.g, $yw_1 \in E, y'w_1 \notin E$, but $v_3 - v_4 - v_5 - v_1 - r_1 - w_1 - y - y' = P_8$. So $N_{Z_1}(y) = N_{Z_1}(y')$. There is no vertex $y'' \in Y$ such that $yy'' \in E, y'y'' \notin E$, else G contains a claw. Hence Y is an independent set and for any pair of vertices $y, y' \in Y$ we have $N(y) \cap N(y') = \emptyset$.

We show that for any $y \in Y$ its neighborhood $N(y)$ is a clique. For contradiction we assume that y has two neighbors $z_1, z_2 \in Z, z_1z_2 \notin E$. Since G is claw-free z_1 and z_2 cannot have a common neighbor in R_1 . Let $r, r \in R_1$, be a neighbor of z_1 . Then $v_3 - v_4 - v_5 - v_1 - r - z_1 - y - z_2 = P_8$. Hence, Y is an independent set, for each $y, y \in Y, N(y)$ is a clique. So we suppose $|N(y)| \geq 2$, else y is a leaf.

We show that we can assume that each connected component A_i of $G[Z_A]$ is anti-complete to $N(Y)$. Since Y has no leaves, let $y \in Y$ with two neighbors $z, z' \in Z_1$ such that $N[z] \neq N[z']$. Suppose that there exists $u \in Z_A$ a neighbor of z . First, we assume that $N_R(z) \neq N_R(z')$. W.l.o.g. let $r, r' \in R_1$ be respectively the neighbors of z, z' such that $r'z, rz' \notin E$. If $uz' \notin E$ then $ur' \notin E$ else G contains a claw, but then $u - z - z' - r' - v_1 - v_5 - v_4 - v_3 = P_8$. Hence $uz', r'u \in E$ but $y - z - u - r' - v_1 - v_5 - v_4 - v_3 = P_8$. So $N_R(z) = N_R(z')$. Second, we assume that $N_Z[z] \neq N_Z[z']$. W.l.o.g. $uz' \notin E$. Let $r \in R_1$ a neighbor of both z, z' . Clearly $ru \notin E$ else G contains a claw, but $G[\{r, u, y, z\}]$ is a claw. So we can assume that each A_i is anticomplete to $N(Y)$.

We construct a γ -set as follows:

Let $q = |Y|$ and k be the number of connected components of Z_A . Clearly, q vertices are necessary to dominate Y . So for each $y_i \in Y$ we will take one of its neighbor as follows. Let us denote $R_1(y_i) = N_{R_1}(N(y_i))$. If y_i has a neighbor z_i which is complete to $R_1(y_i)$ then we take z_i , else we take any arbitrary neighbor of y_i (recall that in both cases these y_i have the same neighbors in Z). These q vertices dominate $Y \cup (Z \setminus Z_A)$ and some of the vertices in $R_1(Y)$.

Now k vertices are necessary to dominate Z_A . For each component $A_i \subset Z_A$ we do as follows. If there exists $r \in R_1$ which is complete to A_i we take r into the γ -set (case *a*), else we take one universal vertex of A_i (case *b*) (recall that in both cases

these r have the same neighbors in S).

These k vertices dominate $Z_A \cup H_1 \cup \{v_1, v_2\}$ if at least one vertex is chosen in the case a , else they dominate Z_A .

Case where at least one vertex is chosen with the case a : v_3, v_4, v_5 are not dominated with the $q + k$ already chosen vertices (H_1 is complete thus $r \in R_1$ dominates $H_1 \cup \{v_1, v_2\}$). So a dominating set of G has size at least $q + k + 1$. Adding the two vertices v_3 and v_5 , we have a dominating set (not necessarily minimum). Checking if there exists a vertex $v \in C \cup S$, that is universal to the remaining non-dominated vertices, can be done in polynomial-time.

Case where all the vertices are chosen with the case b : it remains to dominate C_5 and some vertices of $S_2 \cup S_3 \cup S_4$. So a dominating set of G has a size at least $q + k + 1$. Adding the three vertices v_1, v_3, v_5 , we have a dominating set (not necessarily minimum). If there exists a vertex $v \in S_5$ that is universal to the remaining non-dominated vertices we take it. If no such vertex exists, checking for all the pairs $\{v, v'\} \subset N[C_5]$, one can verify if there exists a γ -set with $q + k + 2$ vertices (note that there are at most $O(n^2)$ of such pairs). \square

Lemma 5.6 *Let G be a connected ($claw, C_5, C_6, C_7, C_8, P_8$)-free graph. Computing a minimum dominating set is polynomial-time solvable.*

Proof: From Lemma 5.4 we can assume that $P_7 \subseteq_i G$. Let $P = v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7$.

Let $W = V \setminus N[P]$. From Property 2.3 if $W = \emptyset$ computing a minimum dominating set is polynomial. From now on $W \neq \emptyset$.

Let $S = \{v \in V \setminus P \text{ such that } 2 \leq |N(v) \cap P| \leq 4\}$, $S_i \subseteq S$ being the set of vertices v such that $|N(v) \cap P| = i$.

Let $H_i = \{v \in S_2 : N(v) \cap P = \{v_i, v_{i+1}\}, 1 \leq i \leq 6\}$. Since G is claw-free each H_i is complete. If there is an edge $r_i r_{i+1}$ with $r_i \in H_i, r_{i+1} \in H_{i+1}$ then $P = v_1 - \dots - v_i - r_i - r_{i+1} - v_{i+2} - \dots - v_7 = P_8$. If there is an edge $r_i r_j$ with $r_i \in H_i, r_j \in H_j$ and $j \geq i + 3$ then $C_p \subseteq_i G$, $p \geq 5$. So H_1 is anticomplete to H_2, H_4, H_5, H_6 , and H_2 is anticomplete to H_3, H_5, H_6 , and H_3 is anticomplete to H_4, H_6 .

We define R_i as the set of vertices of H_i having a neighbor in W , $R_i = \{v \in H_i : N(v) \cap W \neq \emptyset\}, 1 \leq i \leq 6$. Since G is P_8 -free $R_1 = R_6 = \emptyset$.

Let $w \in W$. We show that there cannot exist three indices $1 \leq i < j < k \leq 6$ such that w has three neighbors $r_i \in R_i, r_j \in R_j, r_k \in R_k$. Suppose for contradiction that these three neighbors of w exist. Since $R_1 = R_6 = \emptyset$ then $2 \leq i < j < k \leq 5$. Since G is claw-free and H_p is anticomplete to H_{p+1} , these three indices cannot be successive. So w.l.o.g. we can assume that $i = 2, j = 4, k = 5$. Now H_2 is anticomplete to H_5 , but $v_3 - r_2 - w - r_5 - v_5 - v_4 - v_3 = C_6$. Hence for any $w \in W$ there is at most two neighbors r_i, r_j such that $r_i \in R_i, r_j \in R_j, i \neq j$.

If w has two neighbors $r_i \in R_i, r_j \in R_j, i < j$, then either $r_i \in R_2, r_j \in R_4$ or $r_i \in R_3, r_j \in R_5$ (recall that H_i is anticomplete to H_{i+1} , $H_p, p \geq i + 3$ and $R_1 = R_6 = \emptyset$).

If w has two neighbors $r_i \in R_2, r_j \in R_4$, respectively $r_i \in R_3, r_j \in R_5$, then $r_i r_j \in E$, else $w - r_j - v_4 - v_3 - r_i - w = C_5 \subseteq_i G$, respectively $w - r_j - v_5 - v_4 - r_i - w = C_5 \subseteq_i G$. Let $Z_{24} = \{w \in W : w \text{ has two neighbors } r_2 \in R_2, r_4 \in R_4\}$ and $Z_{35} = \{w \in W : w \text{ has two neighbors } r_3 \in R_3, r_5 \in R_5\}$.

We show that Z_{24} is anticomplete to Z_{35} . For contradiction we suppose that there are $w_1 \in Z_{24}, w_2 \in Z_{35}$ with $w_1 w_2 \in E$. Let $r_1 \in R_2$ be a neighbor of w_1 and $r_2 \in R_5$ be a neighbor of w_2 . Since $r_1 r_2 \notin E$ we have $w_1 - r_1 - v_3 - v_4 - v_5 - v_6 - r_2 - w_2 - w_1 = C_8$. We show that we can assume that Z_{24} and Z_{35} are two independent sets. The two sets being symmetric we show that Z_{24} is an independent set. For contradiction we assume that there are $w_1, w_2 \in Z_{24}$ such that $w_1 w_2 \in E$. We prove that $N_{R_2}(w_1) = N_{R_2}(w_2)$. If $N_{R_2}(w_1) \neq N_{R_2}(w_2)$ then there exist $r_2 \in R_2$ which is a neighbor of w_1 but not a neighbor of w_2 . Then $w_2 - w_1 - r_2 - v_3 - \dots - v_7 = P_8$. We prove that $N_{R_4}(w_1) = N_{R_4}(w_2)$. If $N_{R_4}(w_1) \neq N_{R_4}(w_2)$ then there exists $r_4 \in R_4$ which is a neighbor of w_1 but not a neighbor of w_2 . There exists $r_2 \in R_2$ a neighbor of w_1 and w_2 . We know that $r_2 r_4 \in E$. It follows that $G[\{v_2, r_2, r_4, w_2\}]$ is a claw. Hence $N_{R_2}(w_1) = N_{R_2}(w_2)$ and $N_{R_4}(w_1) = N_{R_4}(w_2)$. From Property 2.1 we can assume that there exists $s \notin R_2 \cup R_4$ such that s is a neighbor of w_1 but not a neighbor of w_2 . Let $r_2 \in R_2$ a neighbor of w_1 and w_2 . If $s r_2 \notin E$ then $s - w_1 - r_2 - v_3 - \dots - v_7 = P_8$. When $s r_2 \in E$ then $G[\{v_2, r_2, s, w_2\}]$ is a claw. Hence Z_{24} is an independent and by symmetry Z_{35} is also independent.

Moreover, since G is claw-free for any two distinct $w, w' \in Z_{24} \cup Z_{35}$ we have $N(w) \cap N(w') = \emptyset$.

For any two distinct $w, w' \in Z_{24}$, respectively $w, w' \in Z_{35}$ we have that $N_{R_2}(w)$ is anticomplete to $N_{R_4}(w')$ and $N_{R_4}(w)$ is anticomplete to $N_{R_2}(w')$, respectively $N_{R_3}(w)$ is anticomplete to $N_{R_5}(w')$ and $N_{R_5}(w)$ is anticomplete to $N_{R_3}(w')$. For contradiction we assume that w has a neighbor $r_2 \in R_2$, w' has a neighbor $r_4 \in R_4$, and $r_2 r_4 \in E$. Then $G[\{v_2, r_2, w, r_4\}]$ is a claw.

Let $Z_i = \{w \in W : w \text{ has a neighbor in } R_i \setminus (N_{R_i}(Z_{24} \cup Z_{35}))\}, 2 \leq i \leq 5$.

We show that Z_2, Z_3, Z_4, Z_5 are pairwise anticomplete. If there is an edge $w_2 w_4, w_2 \in Z_2, w_4 \in Z_4$, with $r'_2 \in R_2, r'_4 \in R_4$ the neighbors of w_2, w_4 respectively, then $w_2 - r'_2 - v_3 - v_4 - r'_4 - w_4 - w_2 = C_6$ ($r'_2 r'_4 \notin E$ else $G[\{v_2, r'_2, w_2, r'_4\}]$ is a claw). By symmetry there is no edges between Z_3, Z_5 . If there is an edge $w_2 w_5, w_2 \in Z_2, w_5 \in Z_5$, with $r'_2 \in R_2, r'_5 \in R_5$ the neighbors of w_2, w_5 respectively, then $w_2 - r'_2 - v_3 - v_4 - v_5 - r'_5 - w_5 - w_2 = C_7$ (remember $r'_2 r'_5 \notin E$). If there is an edge $w_4 w_5, w_4 \in Z_4, w_5 \in Z_5$, with $r'_4 \in R_4, r'_5 \in R_5$ the neighbors of w_4, w_5 respectively, then $w_4 - r'_4 - v_5 - r'_5 - w_5 - w_4 = C_5$ (recall $r'_4 r'_5 \notin E$). By symmetry there is no edges between Z_2, Z_3 .

Let $Y = W \setminus (Z_2 \cup Z_3 \cup Z_4 \cup Z_5 \cup Z_{24} \cup Z_{35})$. One can observe that for any $w \in Y$ we have $N_{Z_2}(w) = N_{Z_5}(w) = N_{Z_{24}}(w) = N_{Z_{35}}(w) = \emptyset$ else $P_8 \subseteq_i G$. Now, if $w \in Y$ has two neighbors $w_3 \in Z_3, w_4 \in Z_4$ then $C_6 \subseteq_i G$.

Hence $Y = Y_3 \cup Y_4$ with $Y_3 \cap Y_4 = \emptyset, Y_3 = \{w \in Y : w \text{ has a neighbor in } Z_3\}, Y_4 = \{w \in Y : w \text{ has a neighbor in } Z_4\}$. Moreover Y_3 is anticomplete to Y_4 else $C_7 \subseteq_i G$.

We show that we can assume that Z_2, Z_5, Y_3, Y_4 are four independent sets. The arguments are the same for the four sets, so we show that Z_2 is an independent set.

For contradiction we assume that there are $w_1, w_2 \in Z_2$ such that $w_1 w_2 \in E$. We prove that $N_{R_2}(w_1) = N_{R_2}(w_2)$. If $N_{R_2}(w_1) \neq N_{R_2}(w_2)$ then there exists $r_2 \in R_2$ which is a neighbor of w_1 but not a neighbor of w_2 . Then $w_2 - w_1 - r_2 - v_3 - \dots - v_7 = P_8$. Since $N(w_1), N(w_2) \subseteq Z_2 \cup R_2$ the result is obtained from Property 2.1. Hence Z_2, Z_5, Y_3, Y_4 are independent.

Since G is claw-free then for any two distinct vertices $w_1, w_2 \in Z_2 \cup Z_5 \cup Y_3 \cup Y_4$ we have $N(w_1) \cap N(w_2) = \emptyset$.

We prove that for any $w \in Y_3 \cup Y_4$, $N(w)$ is a clique. The two cases being symmetric, let $w \in Y_4$. Suppose there are s, s' two non adjacent vertices in $N(w)$. Since G is claw-free s, s' cannot have a common neighbor in R_4 . Let $r \in R_4$ be a neighbor of s . Then $s' - w - s - r - v_4 - v_3 - v_2 - v_1 = P_8$.

Since G is claw-free, if there are a vertex $r \in R_i$ with a neighbor $z \in Z_i$ and a vertex $s \in S$ such as $sz \notin E$ and $v_i \notin N(s)$ then G contains a claw (note that $v_{i+1} \notin N(s)$ is symmetric). Hence $N(Z_i)$ is anticomplete to H_j , $j \neq i$.

We show that we can assume that $Z_2 = Z_5 = \emptyset$. The arguments are the same in the two cases, so we consider Z_2 . Let $r, r' \in R_2$ be two neighbors of $w \in Z_2$. From previous arguments, $N(w)$ is complete to H_2 but anticomplete to H_1, H_3, H_4, H_5, H_6 . Hence, it remains the case where $N_{S_3 \cup S_4}(r) \neq N_{S_3 \cup S_4}(r')$. Suppose that $N_{S_3 \cup S_4}(r) \neq N_{S_3 \cup S_4}(r')$. Let $s \in S_3 \cup S_4$ such as $rs \in E$ but $r's \notin E$. If $\{v_2, v_3\} \not\subseteq N_P(s)$ then $G[\{r, s, v_2, w\}]$ or $G[\{r, s, v_3, w\}]$ is a claw. So $\{v_2, v_3\} \subset N_P(s)$. If $v_1 \notin N_P(s)$, respectively $v_4 \notin N_P(s)$, then $G[\{r', s, v_1, v_2\}]$, respectively $G[\{r', s, v_3, v_4\}]$, is a claw. Hence $N_P(s) = \{v_1, v_2, v_3, v_4\}$ but $G[\{r, s, v_1, v_4\}]$ is a claw. Hence, $N[r] = N[r']$. From Property 2.1 we can contract rr' . Then w is a leaf and by Property 2.2, w can be deleted from G .

Now we study the structures of Z_3 and Z_4 . The two cases being symmetric we deal with Z_4 . For any distinct vertices $w_1, w_2 \in Z_4$ such that $w_1 w_2 \in E$, then there cannot exist two distinct vertices $w'_1, w'_2 \in Z_4$ such that $w'_1 w_1 \in E, w'_1 w_2 \notin E$ and $w'_2 w_2 \in E, w'_2 w_1 \notin E$. For contradiction we suppose that such two vertices exist. Firstly, we suppose that w_1, w_2 have two distinct neighbors $r_1, r_2 \in R_4$, respectively. If $w'_1 r_2 \notin E$ then $v_1 - v_2 - v_3 - v_4 - r_2 - w_2 - w_1 - w'_1 = P_8$. If $w'_1 r_2 \in E$ then $G[\{v_4, r_2, w_2, w'_1\}]$ is a claw. Secondly, w.l.o.g., $r_1 \in R_4$ is a common neighbor of w_1, w_2 and $r_2 \in R_4$ is a neighbor of w_2 but not w_1 . If $w'_1 r_2 \notin E$ then $v_1 - v_2 - v_3 - v_4 - r_2 - w_2 - w_1 - w'_1 = P_8$ else $G[\{v_4, r_2, w_2, w'_1\}]$ is a claw. Finally, $r_1, r_2 \in R_4$ are two common neighbors of w_1, w_2 (r_1, r_2 are not necessarily distinct). If, w.l.o.g., $w'_1 r_1 \in E$ then $G[\{v_4, r_1, w_2, w'_1\}]$ is a claw. In the case where $w'_1 r_1, w'_1 r_2 \notin E$ then w'_1 has a neighbor $r'_1 \in R_4, r'_1 \neq r_1, r_2$. If $r'_1 w_2 \in E$ then $G[\{v_4, r'_1, w_2, w'_1\}]$ is a claw. So $r'_1 w_2 \notin E$. If $r'_1 w_1 \notin E$ then $v_1 - v_2 - v_3 - v_4 - r'_1 - w'_1 - w_1 - w_2 = P_8$. Thus $r'_1 w_1 \in E$. If $r'_1 w'_2 \notin E$ then $v_1 - v_2 - v_3 - v_4 - r'_1 - w_1 - w_2 - w'_2 = P_8$. So $r'_1 w'_2 \in E$ but $G[\{v_4, r'_1, w_1, w'_2\}]$ is a claw.

As a consequence each connected component A_i of $Z_3 \cup Z_4$ has a universal vertex. Also, G being claw-free two distinct components cannot share a neighbor in $R_3 \cup R_4$. Moreover, from Property 2.2 we have assumed that each $w_4 \in Z_3 \cup Z_4$ is not a leaf.

We show that $w \in Y_3 \cup Y_4$ is connected to a universal vertex of a connected compo-

ment A_i of $Z_3 \cup Z_4$. The two cases being symmetric, we deal with Z_4 . Let $w \in Z_4$. We assume that the neighbors of w are not universal in A_i . Let $s \in A_i$ be a neighbor of w , let $u, u \neq s$, be a universal vertex of A_i . Since s is not universal there exists $v, v \in A_i$ such that $sv \notin E$ and $uv \in E$. Since $N(w)$ is complete $wv \notin E$. Let $r \in R_4$ be a neighbor of s . Since G is claw-free then $rv \notin E$. Let $r', r' \in R_4, r' \neq r$, be a neighbor of v . As just above $r's \notin E$. If $r'u \notin E$ then $v_1 - v_2 - v_3 - v_4 - r' - v - u - s = P_8$ else $v_1 - v_2 - v_3 - v_4 - r' - u - s - w = P_8$.

We are ready to show how to build a γ -set in polynomial time.

First, we treat the case where $Z_{24} \neq \emptyset$ (the case $Z_{35} \neq \emptyset$ is the same). Let $r_2 \in R_2, r_4 \in R_4$ be the two neighbors of $w, w \in Z_{24}$. Recall that $N(Z_{24}) \subseteq R_2 \cup R_4$.

We show that $R_3 = \emptyset$. Assume that there exists $w' \in W$ with a neighbor $r_3 \in H_3$ (that is $R_3 \neq \emptyset$). w' cannot be a neighbor of r_2 or r_4 . Then $w' - r_3 - v_3 - r_2 - r_4 - v_5 - v_6 - v_7 = P_8$.

An immediate consequence is that $Z_{35} = \emptyset$.

There is no vertex $w', w' \in W$, with r_2 as a neighbor: else $G[\{v_2, r_2, r_4, w'\}]$ is a claw. By symmetry, there is no vertex $w', w' \in W$, with r_4 as a neighbor.

Let $r_2 \in R_2, r'_2 \in R_2, r_2 \neq r'_2$ be such that r_2 , respectively r'_2 , has a neighbor $w \in Z_{24}$, respectively $w' \in Z_{24}$. Let $r_4 \in R_4, r'_4 \in R_4, r_4 \neq r'_4$ be such that r_4 , respectively r'_4 , has w , respectively w' , as neighbor. We show that $N_{S \setminus H_4}(r_2) = N_{S \setminus H_4}(r'_2)$, respectively $N_{S \setminus H_2}(r_4) = N_{S \setminus H_2}(r'_4)$.

Let $i = 2$ (the case $i = 4$ is symmetric). By contradiction, we assume that there exists $s \in S \setminus H_4$ such that $r_2s \in E, r'_2s \notin E$. From above $s \notin S_2$. So $s \in S_3 \cup S_4$. If $N_P(s) = \{v_1, v_2, v_3\}$ then $G[\{r'_2, v_3, v_4, s\}]$ is a claw. When $N_P(s) = \{v_2, v_3, v_4\}$ then $G[\{r'_2, v_1, v_2, s\}]$ is a claw. When $N_P(s) = \{v_j, v_{j+1}, v_{j+2}\}, j \geq 3$, then $G[\{r_2, w, v_2, s\}]$ is a claw. Thus $s \in S_4$. When $N_P(s) = \{v_1, v_2, v_3, v_4\}$ then $G[\{r_2, v_1, v_4, s\}]$ is a claw. When $N_P(s) = \{v_1, v_2, v_j, v_{j+1}\}, j \geq 4$, then $G[\{r_2, v_1, v_j, s\}]$ is a claw. When $N_P(s) = \{v_2, v_3, v_j, v_{j+1}\}, j \geq 4$, then $G[\{r'_2, v_1, v_2, s\}]$ is a claw. When $N_P(s) = \{v_j, v_{j+1}, v_{j'}, v_{j'+1}\}, j \geq 3, j' \geq j + 2$ then $G[\{r_2, w, v_2, s\}]$ is a claw. Hence $N_{S \setminus H_4}(r_2) = N_{S \setminus H_4}(r'_2)$ and $N_{S \setminus H_2}(r_4) = N_{S \setminus H_2}(r'_4)$.

Let $r_4 \in R_4, r'_4 \in R_4, r_4 \neq r'_4$ be such that r_4 , respectively r'_4 , has a neighbor $w \in Z_4$, respectively $w' \in Z_4$. We show that $N_S(r_4) = N_S(r'_4)$.

By contradiction, we assume that there exists $s \in S$ such that $r_4s \in E, r'_4s \notin E$. From above $s \notin H_1 \cup H_3 \cup H_4 \cup H_5$. So $s \in H_2 \cup H_6 \cup S_3 \cup S_4$. If $s \in H_2$ or $s \in H_6$ then $G[\{v_4, w, r_4, s\}]$ is a claw. So $s \in S_3 \cup S_4$. If $N_P(s) = \{v_1, v_2, v_3\}$ then $G[\{v_1, v_3, r_4, s\}]$ is a claw. If $N_P(s) = \{v_2, v_3, v_4\}$ then $G[\{v_5, w, r_4, s\}]$ is a claw. If $N_P(s) = \{v_3, v_4, v_5\}$ then $G[\{v_5, v_6, r'_4, s\}]$ is a claw. If $N_P(s) = \{v_4, v_5, v_6\}$ then $G[\{v_3, v_4, r'_4, s\}]$ is a claw. When $N_P(s) = \{v_5, v_6, v_7\}$ then $G[\{v_4, w, r_4, s\}]$ is a claw. Thus $s \in S_4$. When $N_P(s) = \{v_1, v_2, v_3, v_4\}$ then $G[\{r_4, v_1, v_3, s\}]$ is a claw. When $N_P(s) = \{v_2, v_3, v_4, v_5\}$ then $G[\{r'_4, v_5, v_6, s\}]$ is a claw. When $N_P(s) = \{v_3, v_4, v_5, v_6\}$ then $G[\{r_4, v_3, v_6, s\}]$ is a claw. When $N_P(s) = \{v_4, v_5, v_6, v_7\}$ then $G[\{r'_4, v_3, v_4, s\}]$ is a claw. When $N_P(s) = \{v_1, v_2, v_6, v_7\}$ or $N_P(s) = \{v_2, v_3, v_6, v_7\}$ then $G[\{r_4, v_2, v_6, s\}]$ is a claw. When $N_P(s) = \{v_3, v_4, v_6, v_7\}$ then $G[\{r_4, v_3, v_6, s\}]$ is a claw. When $N_P(s) = \{v_1, v_2, v_5, v_6\}$ or $N_P(s) = \{v_2, v_3, v_5, v_6\}$ then $G[\{r_4, v_2, v_6, s\}]$ is a claw. When $N_P(s) = \{v_1, v_2, v_4, v_5\}$ then $G[\{r'_4, v_5, v_6, s\}]$ is a claw. Hence

$N_S(r_4) = N_S(r'_4)$. By symmetry, for $r_2 \in R_2, r'_2 \in R_2, r_2 \neq r'_2$ such that r_2 , respectively r'_2 , has a neighbor $w \in Z_2$, respectively $w' \in Z_2$ we have $N_S(r_2) = N_S(r'_2)$.

The γ -set is build as follows:

- $|Z_{24}| \geq 2$. We take $r_2 \in R_2$ a neighbor of w , and for each other $w' \in Z_{24}$ we take one adjacent vertex $r'_4 \in R_4$. For each $w' \in Y_4$ we take one universal vertex in the connected component A_i of Z_4 connected to w' . For each connected component A_i of Z_4 that is not connected with Y_4 , we take one universal vertex of A_i . These vertices dominate $Z_{24} \cup Y_4 \cup Z_4 \cup H_2 \cup H_4 \cup \{v_2, v_3, v_4, v_5\}$. Since v_1, v_7 have no common neighbor at least two more vertices are needed. Adding the three vertices v_2, v_4, v_6 we have a dominating set (not necessarily minimum). Checking for all the pairs s_1, s_7 where s_i is a neighbor of $v_i, i \in \{1, 7\}$, one can verify if there is a γ -set with only two more vertices (note that there are at most $O(n^2)$ such pairs).
- $|Z_{24}| = 1$. For each $w' \in Y_4$ we take one universal vertex in the connected component A_i of Z_4 connected to w' . If there exists a vertex $r \in R_4$ complete to a component A_i of Z_4 that is not connected to Y_4 then we take r . For each remaining component A_i of Z_4 that is not connected to Y_4 , we take one universal vertex of A_i . These vertices dominate $Y_4 \cup Z_4$ (note that H_2, H_4 are not necessarily dominated). Since v_1, v_7, w have no common neighbor at least three more vertices are needed. Adding the four vertices v_2, v_4, v_6, w we have a dominating set (not necessarily minimum). Checking for all the pairs s_1, s_7 where s_i is a neighbor of $v_i, i \in \{1, 7\}$, if there is a dominating set by adding s_1, s_7, r_4 or s_1, s_7, r_2 , one can verify if there is a γ -set with only three more vertices (note that there are at most $O(n^2)$ such pairs).

In the case of $Z_{24} = Z_{35} = \emptyset$, we build the γ -set as follows:

- $Y_3, Y_4 \neq \emptyset$. For each $w \in Y_3 \cup Y_4$ we take one universal vertex in the connected component A_i of $Z_3 \cup Z_4$ connected to w . If there exists $r_4 \in R_4$ which is complete to a component A_i of Z_4 that is not connected to $Y_3 \cup Y_4$ then, we take r_4 . We do the same for the component of Z_3 with no neighbors in Y_3 . For each remaining connected component A_i of $Z_3 \cup Z_4$ that is not connected to $Y_3 \cup Y_4$, we take one universal vertex of A_i . These vertices dominate $Y_3 \cup Z_3 \cup Y_4 \cup Z_4$ (note that H_2, H_4 are not necessarily dominated). Since v_1, v_7 have no common neighbor at least two more vertices are needed. Adding the three vertices v_2, v_4, v_6 we have a dominating set (not necessarily minimum). Checking for all the pairs s_1, s_7 where s_i is a neighbor of $v_i, i \in \{1, 7\}$, one can verify if there is a γ -set with only two more vertices (note that there are at most $O(n^2)$ such pairs).
- $Y_3 \neq \emptyset, Y_4 = \emptyset$ or $Y_4 \neq \emptyset, Y_3 = \emptyset$. The two cases being symmetric, let $Y_4 = \emptyset$.
 - $Z_4 \neq \emptyset$. For each $w \in Y_3$ we take one universal vertex in the connected component A_i of Z_3 connected to w . If there exists $r_4 \in R_4$ which is complete to A_i , a connected component of Z_4 , then we take r_4 . If there exists $r_3 \in R_3$ which is complete to a connected component A_j of Z_3 with

no neighbors in Y_3 , then we take r_3 . Now, we take one universal vertex for each other component $A_l, A_l \neq A_i, A_j$, of $Z_3 \cup Z_4$. These vertices dominate $Y_3 \cup Z_3 \cup Z_4$. Since v_1, v_7 have no common neighbor at least two more vertices are needed. Adding the three vertices v_2, v_4, v_6 we have a dominating set (not necessarily minimum). Checking for all the pairs s_1, s_7 where s_i is a neighbor of $v_i, i \in \{1, 7\}$, one can verify if there is a γ -set with only two more vertices (note that there are at most $O(n^2)$ such pairs).

- $Z_4 = \emptyset$. For each $w \in Y_3$ we take one universal vertex in the connected component A_i of Z_3 connected to w . If there exists $r_3 \in R_3$ which is complete to a connected component A_i of Z_3 with no neighbors in Y_3 , then we take r_3 . Now, we take one universal vertex for each other component A_i of Z_3 . Adding the vertices v_2, v_4, v_6 we have a dominating set (not necessarily minimum). Checking for all the pairs s_1, s_7 where s_i is a neighbor of $v_i, i \in \{1, 7\}$, one can verify if there is a γ -set with only two more vertices.

- $Y_3, Y_4 = \emptyset$.

- $Z_3, Z_4 \neq \emptyset$. If there exists $r_4 \in R_4$, respectively $r_3 \in R_3$, which is complete to A_i , a connected component of Z_4 , respectively Z_3 , then we take r_4 , respectively r_3 . For each remaining component of $Z_3 \cup Z_4$ we take one universal vertex. Adding the vertices v_2, v_4, v_6 we have a dominating set (not necessarily minimum). Checking for all the pairs s_1, s_7 where s_i is a neighbor of $v_i, i \in \{1, 7\}$, one can verify if there is a γ -set with only two more vertices.
- $Z_3 \neq \emptyset, Z_4 = \emptyset$ or $Z_4 \neq \emptyset, Z_3 = \emptyset$. Let $Z_3 \neq \emptyset$. If there exists $r_3 \in R_3$ which is complete to a connected component of Z_3 , then we take r_3 . We add one universal vertex for each remaining component of Z_3 . Now, adding the vertices v_2, v_4, v_6 we have a dominating set (not necessarily minimum). Checking for all the pairs s_1, s_7 where s_i is a neighbor of $v_i, i \in \{1, 7\}$, one can verify if there is a γ -set with only two more vertices.
- $Z_3 = Z_4 = \emptyset$. Then $V = N[C]$ and from Property 2.3 computing a minimum dominating set is polynomial.

□

From Lemmas 3.1, 3.2, 4.3, 5.5, 5.6 we have the following.

Theorem 5.7 *The Minimum Dominating Set problem is polynomial for (claw, P_8) -free graphs.*

6 Conclusion

We have shown that the Minimum Dominating Set problem is polynomial for (claw, P_8) -free graphs. We left open the following problem: is there a positive integer $k, k \geq 9$,

such that the Minimum Dominating Set problem is NP -complete for the class of $(claw, P_k)$ -free graphs? If the the answer is positive, a challenge should be to show a dichotomy: find the minimum integer k such that the Minimum Dominating Set problem is NP -complete for $(claw, P_k)$ -free graphs and polynomial for $(claw, P_{k-1})$ -free graphs.

References

- [1] B. Allan, R. Laskar (1978), *On domination and independent domination numbers of a graph*, Discrete Mathematics 23, 73-76.
- [2] A. Bertossi (1984), *Dominating sets for split and bipartite graphs*, Information Processing Letters 19, 37-40.
- [3] J.A. Bondy, U. S. R. Murty, *Graph Theory*, Springer, (2008).
- [4] D. Bauer, F. Harary, J. Nieminen, and C. L. Suffel (1983), *Domination alteration sets in graphs*. Discrete Mathematics, 47:153-161.
- [5] E. Cockayne, S. Goodman, S. Hedetniemi (1975), *A linear algorithm for the domination number of a tree*, Information Processing Letters 4 (2), 41-44.
- [6] M. Farber (1982), *Independent domination in chordal graphs*, Operations Research Letters 1, 134-138.
- [7] Teresa W. Haynes, Stephen T. Hedetniemi, Peter J. Slater *Fundamentals of Domination in Graphs*, Marcel Dekker Inc., (1998).
- [8] D. Malyshev (2016), *A complexity dichotomy and a new boundary class for the dominating set problem*, J. Comb. Optim. 32, 226-243.
- [9] M. Yannakakis, F. Gavril (1980) *Edge dominating sets in graphs*, SIAM J. Appl. Math. 38 (3), 364-372.