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# The Minimum Dominating Set problem is polynomial for $(\textit{claw}, P_8)$ -free graphs

Valentin Bouquet\*      Christophe Picouleau \*

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## Abstract

We prove that the Minimum Dominating Set problem is polynomial for the class of  $(\textit{claw}, P_8)$ -free graphs.

**Keywords:** Minimum Dominating Set, polynomial time, claw-free graph,  $P_k$ -free graph.

## 1 Introduction

M. Yannakakis and F. Gavril [9] showed in 1980 that the Minimum Dominating Set problem restricted to claw-free graphs is  $NP$ -complete. Then in 1984, A. Bertossi [2] showed that the Minimum Dominating Set problem is also  $NP$ -complete for split graphs, a subclass of  $P_5$ -free graph. More recently, in 2016, D. Malyshev [8] proved that the Minimum Dominating Set problem is polynomial for  $(K_{1,4}, P_5)$ -free graphs, hence for  $(\textit{claw}, P_5)$ -free graphs. To our knowledge, the complexity of the Minimum Dominating Set problem is unknown for  $(\textit{claw}, P_k)$ -free graphs for any fixed  $k \geq 6$ . We show that the Minimum Dominating Set problem is polynomial for  $(\textit{claw}, P_8)$ -free graphs.

## Definitions and notations

We are only concerned with simple undirected graphs  $G = (V, E)$ . The reader is referred to [3] for definitions and notations in graph theory. For  $v \in V$ , we denote  $N(v)$  its neighborhood and  $N[v] = N(v) \cup \{v\}$  its closed neighborhood. A vertex  $v$  is *universal* if  $N[v] = V$ . For  $v \in V$  and  $A \subseteq V$ , we denote by  $N_A(v) = N(v) \cap A$  ( $N_A[v] = (N(v) \cap A) \cup \{v\}$ ) its (closed) neighborhood in  $A$ . For  $X \subseteq V, A \subseteq V$ , we denote  $N_A(X) = \bigcup_{x \in X} N_A(x)$  and  $N_A[X] = N_A(X) \cup X$ .

The *contraction* of an edge  $uv \in E$  removes the vertices  $u$  and  $v$  from  $V$ , and replaces them by a new vertex that is adjacent to the previous neighbors of  $u$  and  $v$  (neither introducing self-loops nor multiple edges). The following  $G/uv$  denotes the graph obtained from  $G$  after the contraction of  $uv$ .

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For  $S \subseteq V$ , let  $G[S]$  denote the subgraph of  $G$  induced by  $S$ , which has vertex-set  $S$  and edge-set  $\{uv \in E \mid u, v \in S\}$ . For  $v \in V$ , we write  $G - v = G[V \setminus \{v\}]$  and for a subset  $V' \subseteq V$  we write  $G - V' = G[V \setminus V']$ . For a fixed graph  $H$  we write  $H \subseteq_i G$  whenever  $G$  contains an induced subgraph isomorphic to  $H$ . For a set  $\{H_1, \dots, H_p\}$  of graphs,  $G$  is  $(H_1, \dots, H_p)$ -free if  $G$  has no induced subgraph isomorphic to a graph in  $\{H_1, \dots, H_p\}$ ; if  $p = 1$  we may write  $H_1$ -free instead of  $(H_1)$ -free. For two disjoint induced subgraphs  $G[A], G[B]$  of  $G$ ,  $G[A]$  is *complete* to  $G[B]$  if  $ab \in E$  for any  $a \in A, b \in B$ ,  $G[A]$  is *anticomplete* to  $G[B]$  if  $ab \notin E$  for any  $a \in A, b \in B$ .

For  $k \geq 1$ , the graph  $P_k = u_1 - u_2 - \dots - u_k$  denotes the *cordless path* on  $k$  vertices, that is,  $V(P_k) = \{u_1, \dots, u_k\}$  and  $E(P_k) = \{u_i u_{i+1} \mid 1 \leq i \leq k - 1\}$ . For  $k \geq 3$ , the graph  $C_k = u_1 - u_2 - \dots - u_k - u_1$  denotes the *cordless cycle* on  $k$  vertices, that is,  $V(C_k) = \{u_1, \dots, u_k\}$  and  $E(C_k) = \{u_i u_{i+1} \mid 1 \leq i \leq k - 1\} \cup \{u_k u_1\}$ . For  $k \geq 4$ ,  $C_k$  is called a *hole*. A graph without hole is *chordal*. The complete graph with  $p$  vertices is  $K_p$ , also called a *clique*. The graph  $C_3 = K_3$  is a *triangle*.  $K_{1,p}$  is the  $(p + 1)$ -vertices star, that is, the graph with vertices  $u, v_1, v_2, \dots, v_p$  and edges  $uv_1, uv_2, \dots, uv_p$ . The *claw* is  $K_{1,3}$ .

A set  $S \subseteq V$  is called a *stable set* or an *independent set* if any pairwise distinct vertices  $u, v \in S$  are non adjacent. The maximum cardinality of an independent set in  $G$  is denoted by  $\alpha(G)$ . A set  $S \subseteq V$  is called a *clique* if any pairwise distinct vertices  $u, v \in S$  are adjacent. When  $G[V]$  is a clique then  $G$  is a *complete graph*. The graph  $K_p, p \geq 1$ , is called a *clique* or a *complete graph* on  $p$  vertices.

A set  $S \subseteq V$  is a *dominating set* if every vertex  $v \in V$  is either an element of  $S$  or is adjacent to an element of  $S$ . The minimum cardinality of a dominating set in  $G$  is denoted by  $\gamma(G)$  and called the *dominating number* of  $G$ . A dominating set  $S$  with  $|S| = \gamma(G)$  is called a *Minimum Dominating Set*. Following [7] a Minimum Dominating Set is also called a  $\gamma$ -set. We denote  $V^+ \subseteq V$  the subset of vertices  $v$  of  $G$  such that  $\gamma(G - v) > \gamma(G)$ . If  $S \subseteq V$  is both a dominating and an independent set then  $S$  is an *independent dominating set*. The minimum cardinality of an independent dominating set in  $G$  is denoted by  $i(G)$ . Clearly we have  $\gamma(G) \leq i(G) \leq \alpha(G)$ . Note that a minimum independent dominating set is a *minimum maximal independent set*.

## Previous results

We give some results of the literature concerning the Minimum Dominating Set problem that will be useful in the following.

D. Bauer et al. showed in [4] that, for any non isolated vertex  $v$ , if  $v \in V^+$  then  $v$  is in any  $\gamma$ -set of  $G$ . For  $G$  a claw-free graph Allan et al. [1] proved that  $\gamma(G) = i(G)$ . From Yannakakis et al. [9] the Minimum Dominating Set problem restricted to claw-free graphs is *NP*-complete. From D. Malyshev [8] the Minimum Dominating Set problem is polynomial for  $(\text{claw}, P_5)$ -free graphs. From Farber [6] computing a minimum independent dominating set can be done in linear-time on chordal graphs. So the Minimum Dominating Set problem restricted to claw-free chordal graphs is polynomial. Cockayne et al. [5] showed that the Minimum Dominating Set problem restricted to trees is polynomial.

## Organization

The next section give some algorithmic properties. Two properties will allow us to make some simplifications on the graphs  $G$  that we consider. Two others will help us to conclude that computing  $\gamma(G)$  is polynomial when  $G$  have a specific structure relatively to a fixed size subgraph. Then we consider the case where the graph  $G$  has a long cycle. From there, we show our main result, starting from  $(\text{claw}, P_6)$ -free graphs and finishing with  $(\text{claw}, P_8)$ -free graphs. We conclude by some open questions regarding  $(\text{claw}, P_k)$ -free graphs for  $k \geq 9$ .

## 2 Algorithmic Properties

We give two properties that authorize us to make some assumptions and simplifications for the graphs we consider.

**Property 2.1** *Let  $G$  be a claw-free graph. If  $v, u$  are two vertices such that  $N[u] = N[v]$  then  $\gamma(G) = \gamma(G/uv)$ .*

**Proof:** Let  $\Gamma$  be a  $\gamma$ -set of  $G$ . Let  $u'$  be the vertex of  $G/uv$  resulting from the contraction of  $uv$ . At most one of  $u$  and  $v$  is in  $\Gamma$ . If  $u \in \Gamma$  then  $\Gamma' = (\Gamma - u) \cup \{u'\}$  is a  $\gamma$ -set of  $G/uv$ . If  $u, v \notin \Gamma$  then  $\Gamma' = \Gamma$  is a  $\gamma$ -set of  $G/uv$ .  $\square$

**Property 2.2** *Let  $G = (V, E)$  be a connected claw-free graph with  $uv \in E$  such that  $u$  is a leaf. The MINIMUM DOMINATING SET problem is polynomial for  $G' = G - N[v]$  if and only if it is polynomial for  $G$ .*

**Proof:** Since  $G$  is claw-free  $K = N(v) - u$  is a clique. Let  $k \in K$ , we show that  $k \notin V^+$ . For contradiction we assume that  $k \in V^+$ . As shown in [4]  $k$  is in every  $\gamma$ -set of  $G$ . Let  $\Gamma$  be a  $\gamma$ -set with  $v \in \Gamma$ . Let  $W = N(k) \setminus N(v)$ . If  $W = \emptyset$  then  $\Gamma - k$  is a dominating set, a contradiction. It follows  $W \neq \emptyset$  and since  $G$  is claw-free,  $W$  is a clique. Let  $w \in W$ .  $\Gamma' = (\Gamma - \{k\}) \cup \{w\}$  is another  $\gamma$ -set, a contradiction. Hence  $\gamma(G - K) \leq \gamma(G)$ . Since  $G - K$  consists of  $G'$  and the component  $uv$ , we have that  $\gamma(G - K) = \gamma(G') + 1$ . Then from  $\Gamma'$  a  $\gamma$ -set of  $G'$  we obtain  $\Gamma' \cup \{v\}$  a  $\gamma$ -set of  $G$  in polynomial time. Reciprocally, let  $\Gamma$  be a  $\gamma$ -set of  $G$ . Since  $u$  is a leaf we assume that  $v \in \Gamma$ . Then  $\Gamma - \{v\}$  is a  $\gamma$ -set for  $G'$ . Trivially it can be done in polynomial time from  $\Gamma$ .  $\square$

We show two conditions on the structure of  $G$  that authorize us to directly conclude that computing a  $\gamma$ -set for  $G$  can be done in polynomial time.

**Property 2.3** *Let  $k > 0$  be a fixed positive integer and  $G = (V, E)$  a graph. If there exists  $T \subseteq_i G$  of size  $|T| \leq k$  such that  $V = N[T]$  then computing a minimum dominating set for  $G$  is polynomial.*

**Proof:** Since  $|T| \leq k$  and  $V = N[T]$ , we have that  $\gamma(G) \leq k$ . So a minimum dominating set can be computed in  $O(n^k)$  which is polynomial.  $\square$

**Property 2.4** Let  $k, k' > 0$  two fixed positive integers and  $G = (V, E)$  a graph. If there exists  $T \subseteq_i G$  of size  $|T| \leq k$  such that  $W = V \setminus N[T]$  has a size  $|W| \leq k'$  then computing a minimum dominating set for  $G$  is polynomial.

**Proof:** It is clear that  $k$  vertices of  $T$  is enough to dominate  $N[T]$  and  $k'$  vertices of  $W$  is enough to dominate  $W$ . Hence we have  $\gamma(G) \leq k + k'$ . So a minimum dominating set can be computed in  $O(n^{k+k'})$  which is polynomial.  $\square$

### 3 $G$ has a long cycle

We give two lemmas that will authorize us to conclude that the Minimum Dominating Set problem is polynomial when  $G$ , a  $(\text{claw}, P_k)$ -free graph, contains a long induced cycle.

**Lemma 3.1** For any fixed  $k \geq 3$ , if  $G$  is a  $(\text{claw}, P_k)$ -free connected graph such that  $C_k \subseteq_i G$ , then a minimum dominating set of  $G$  can be given in polynomial time.

**Proof:** Let  $C_k = v_1 - \dots - v_k - v_1, C_k \subseteq_i G$ . Let  $v \notin C_k$  be such that  $N(v) \cap C_k \neq \emptyset$ . Since  $G$  is claw-free, we have  $2 \leq |N(v) \cap C_k| \leq 4$ . If  $|N(v) \cap C_k| = 2$ , the two neighbors of  $v$  in  $C_k$  must be adjacent, thus there is a  $P_k$  starting from  $v$ . In the second case,  $3 \leq |N(v) \cap C_k| \leq 4$ , there is no edge  $vw \in E, N(w) \cap C_k = \emptyset$ , else there is a claw centered onto  $v$ . Taking  $\Gamma = \{v_1, v_3, \dots, c_{\lfloor \frac{k}{2} \rfloor}\}$ ,  $\Gamma$  is a dominating set of  $G$ . It follows that  $\lfloor \frac{k}{2} \rfloor = |\Gamma| \geq \gamma(G)$ . Hence by brute force  $\gamma(G)$  and a  $\gamma$ -set can be computed with time  $O(n^{\lfloor \frac{k}{2} \rfloor})$  which is polynomial since  $k$  is fixed.  $\square$

**Lemma 3.2** For any fixed  $k \geq 3$ , if  $G$  is a  $(\text{claw}, P_k, C_k)$ -free connected graph such that  $C_{k-1} \subseteq_i G$ , then a minimum dominating set of  $G$  can be given in polynomial time.

**Proof:** Let  $C_k = v_1 - \dots - v_{k-1} - v_1, C_{k-1} \subseteq_i G$  and  $v \notin C_{k-1}$  such that  $N(v) \cap C_{k-1} \neq \emptyset$ . We have  $2 \leq |N(v) \cap C_{k-1}| \leq 4$ . Let  $w$  be a neighbor of  $v$  such that  $N(w) \cap C_{k-1} = \emptyset$ . If  $3 \leq |N(v) \cap C_{k-1}| \leq 4$ , there is a claw. When  $|N(v) \cap C_{k-1}| = 2$  there is a  $P_k$  starting from  $w$ . Taking  $\Gamma = \{v_1, v_3, \dots, c_{\lfloor \frac{k-1}{2} \rfloor}\}$ ,  $\Gamma$  is a dominating set of  $G$ . It follows that  $\lfloor \frac{k-1}{2} \rfloor = |\Gamma| \geq \gamma(G)$ . Hence, by brute force  $\gamma(G)$  and a  $\gamma$ -set can be computed with time  $O(n^{\lfloor \frac{k-1}{2} \rfloor})$  which is polynomial since  $k$  is fixed.  $\square$

### 4 $G$ is $(\text{claw}, P_k, C_k, C_{k-1})$ -free, $C_{k-2} \subseteq_i G, k \leq 8$

In this section we prove that, for  $k \leq 8$ , if  $G$  is a  $(\text{claw}, P_k, C_k, C_{k-1})$ -free graph such that  $C_{k-2} \subseteq_i G$  then the Minimum Dominating Set problem is polynomial. The first lemma gives a structural property for  $G$ . We use this property to prove two other lemmas, the first one for  $k = 6$ , the second for  $7 \leq k \leq 8$ .

**Lemma 4.1** *For any fixed  $k \geq 6$ , if  $G$  is a (claw,  $P_k, C_k, C_{k-1}$ )-free connected graph such that  $C_{k-2} \subseteq_i G$ , then  $W = V \setminus C_{k-2}$  is an independent set.*

**Proof:** Let  $C = C_{k-2} = v_1 - \cdots - v_{k-2} - v_1, C \subseteq_i G$  and  $v \in N(C)$ . We have  $2 \leq |N_C(v)| \leq 5$  (note that  $|N_C(v)| = 5$  only for  $C = C_5$ ). Let us denote  $S = N(C)$ ,  $W = V \setminus (C \cup S)$ . Let  $w \in W$  be a neighbor of  $v$ . If  $3 \leq |N_C(v)| \leq 5$ , there is a claw. Hence,  $v$  is such that  $N_C(v) = \{v_i, v_{i+1}\}$ ,  $1 \leq i \leq k-2$  (for convenience, when  $i = k-2$ , we read  $v_{i+1} = v_1$ ). From Property 2.1, we can assume that all contractible vertices of  $G$  are contracted. Moreover, from Property 2.2 we can assume that  $G$  has no leaves.

Assume for contradiction that  $w$  has a neighbor  $w', w' \in W$ . When  $N_S(w') = \emptyset$  there is a  $P_k$  from  $w'$  to  $v_{k-3}$ . Hence  $w'$  has a neighbor in  $S$ . Since  $w$  and  $w'$  are not contracted, we have  $N[w] \neq N[w']$ . If  $vw' \notin E$  again there is a  $P_k$  from  $w'$  to  $v_{k-3}$ . Hence,  $w$  and  $w'$  have the same neighbors in  $S$  but not in  $W$ . So there exists  $r \in W$  with  $rw \in E, rw' \notin E$ . As above  $rv \in E$  but there exists  $v_i \in C$  such that  $G[\{r, v, v_i, w'\}]$  is a claw. Hence,  $W$  is independent.  $\square$

**Lemma 4.2** *If  $G$  is a (claw,  $P_6, C_6, C_5$ )-free connected graph such that  $C_4 \subseteq_i G$ , then a minimum dominating set of  $G$  can be given in polynomial time.*

**Proof:** Let  $C = C_4 = v_1 - \cdots - v_4 - v_1, C \subseteq_i G$  and  $v \in N(C)$ . We have  $2 \leq |N_C(v)| \leq 4$ . Let  $S = N(C)$ ,  $W = V \setminus (C \cup S)$  and  $w \in W$  a neighbor of  $v$ . If  $3 \leq |N_C(v)| \leq 4$  then  $G$  contains a claw. Hence,  $v$  is such that  $N_C(v) = \{v_i, v_{i+1}\}$ ,  $1 \leq i \leq 4$  (for convenience, when  $i = 4$ , we read  $v_{i+1} = v_1$ ).

From Property 2.4, we can assume that  $|W| \geq 2$  and from Lemma 4.1, we know that  $W$  is an independent set. We show that all vertices  $v \in N(W)$  have exactly the same neighbors in  $C$ .

Let  $w, w' \in W$ ,  $w \neq w'$ ,  $v \in N_S(w)$ ,  $v' \in N_S(w')$ . Since  $G$  is claw-free  $v \neq v'$ . W.l.o.g.  $N_C(v) = \{v_1, v_2\}$ . Assume that  $N_C(v) \neq N_C(v')$ . W.l.o.g.  $N_C(v') = \{v_2, v_3\}$  (note that  $N_C(v') = \{v_1, v_4\}$  is symmetric). If  $vv' \notin E$  then  $w - v - v_1 - v_4 - v' - w' = P_6$ , else  $v_1 - v - v' - v_3 - v_4 = C_5$ . Now it remains  $N_C(v') = \{v_3, v_4\}$ . We have  $vv' \notin E$  else there is a claw, but  $w - v - v_1 - v_4 - v' - w' = P_6$  a contradiction. Thus, w.l.o.g. any vertex  $w \in W$  has only neighbors  $v \in S$  such that  $N(v) = \{v_1, v_2\}$ .

Let  $|W| = q, q \geq 2$ . We show that  $\gamma(G) = q + 1$ . Since  $W$  is independent and for any distinct  $w, w' \in W$ , we have  $N[w] \cap N[w'] = \emptyset$ , we must take  $q$  vertices of  $N[W]$  to dominate the vertices of  $W$ . These vertices cannot dominate  $v_3$  nor  $v_4$ . Hence  $\gamma(G) \geq q + 1$ .

We construct a  $\gamma$ -set of  $G$  as follows. We set  $R$  by taking exactly one neighbor of each  $w, w \in W$ . Clearly,  $\Gamma = R \cup \{v_3\}$  dominates  $C \cup N[R]$ . Suppose that there exists  $s \in S$  such that  $s$  is not a neighbor of  $\Gamma$ . If  $N_C(s) = \{v_1, v_2\}$  then there exists  $r \in R$  such that  $G[\{r, s, v_1, v_4\}]$  is a claw. If  $N_C(s) = \{v_1, v_4\}$  then  $w - v - v_2 - v_3 - v_4 - s = P_6$ . If  $N_C(s) = \{v_1, v_2, v_4\}$  then there exists  $r \in R$  such that  $G[\{r, v, v_2, v_3\}]$  is a claw. Hence any  $s \notin N[R] \cup C$  is dominated by  $v_3$ . It follows that  $\Gamma$  is a  $\gamma$ -set of  $G$ .  $\square$

**Lemma 4.3** For  $k \in \{7, 8\}$ , if  $G$  is a (claw,  $P_k, C_k, C_{k-1}$ )-free connected graph such that  $C_{k-2} \subseteq_i G$ , then a minimum dominating set of  $G$  can be given in polynomial time.

**Proof:** Let  $C = C_{k-2} = v_1 - \dots - v_{k-2} - v_1, C \subseteq_i G$  and  $v \in N(C)$ . We have  $2 \leq |N_C(v)| \leq 5$  (note that  $|N_C(v)| = 5$  only for  $C = C_5$ ). Let  $S = N(C)$ ,  $W = V \setminus (C \cup S)$  and  $w \in W$  a neighbor of  $v$ . If  $3 \leq |N_C(v)| \leq 4$  then  $G$  contains a claw. Hence,  $v$  is such that  $N_C(v) = \{v_i, v_{i+1}\}$ ,  $1 \leq i \leq k-2$  (for convenience, when  $i = k-2$ , we read  $v_{i+1} = v_1$ ).

We show that for any  $w, w' \in W$ , there exists  $v, v' \in N(w)$  such that  $N_C(v) \cap N_C(v') = \emptyset$ . Let  $w \in W$  and  $v, v' \in N_S(w)$ ,  $v \neq v'$ .

First, we show that  $N_C(v) \neq N_C(v')$ . Suppose that  $N_C(v) = N_C(v')$ , w.l.o.g.  $N_C(v) = \{v_1, v_2\}$ .  $vv' \in E$  else  $G[\{v, v', v_1, v_{k-2}\}]$  is a claw. From Property 2.1, we have that  $N[v] \neq N[v']$ , else  $v, v'$  are contracted. Hence, there exists  $u \in V$  such that  $uv \in E$  and  $uv' \notin E$ . If  $u \in W$  then from Lemma 4.1  $uw \notin E$  but  $G[\{u, v, w, v_1\}]$  is a claw. So, we have  $u \in S$ . If  $N_C(u) = \{v_1, v_2\}$  then  $G[\{u, v', v_2, v_3\}]$  is a claw. So  $N_C(u) \neq N_C(v)$  and we can assume that  $wu \notin E$  or else we have  $u, v$  two neighbors of  $w$  with distinct neighborhoods in  $C$ . If  $N_C(u) \cap N_C(v) = \emptyset$  then  $G[\{u, v, v_1, w\}]$  is a claw. So W.l.o.g. we assume that  $N_C(u) \cap N_C(v) = \{v_1\}$  but  $G[\{u, v, v_2, w\}]$  is a claw. Hence  $N[v] = N[v']$  and  $v, v'$  can be contracted. Yet,  $w$  is a leaf and from Property 2.2 we can assume that  $G$  contains no leaves. So, for any  $w, w' \in W$ , there exists  $v, v' \in N_S(w)$ ,  $v \neq v'$  such that  $N_C(v) \neq N_C(v')$ .

Now we show that that  $N_C(v) \cap N_C(v') = \emptyset$ . W.l.o.g. assume that  $N_C(v) = \{v_1, v_2\}$  and  $N_C(v') = \{v_2, v_3\}$ . If  $vv' \in E$  then  $v_1 - v - v' - v_3 - \dots - v_{k-2} - v_1 = C_{k-1}$ , else  $v_1 - v - w - v' - v_3 - \dots - v_{k-2} - v_1 = C_k$ . Thus any  $w, w' \in W$ , has neighbors  $v, v' \in S$  such that  $N_C(v) \cap N_C(v') = \emptyset$ .

From Property 2.4, we can assume that  $|W| \geq 2$ . So let  $w, w' \in W$  (recall  $ww' \notin E$ ). Since both  $w$  and  $w'$  have two neighbors in  $S$  with non intersecting neighborhoods in  $C$ , let  $v \in N(w)$ ,  $v' \in N(w')$  such that  $N_C(v) \cap N_C(v') = \emptyset$ . W.l.o.g.  $N_C(v) = \{v_1, v_2\}$ . Assume that  $N_C(v') = \{v_3, v_4\}$  (note that  $N(v') = \{v_{k-2}, v_{k-3}\}$  is symmetric). If  $vv' \in E$  then  $G[\{v, v', v_1, w\}]$  is a claw, else  $w - v - v_1 - v_{k-2} - \dots - v_4 - v' - w' = P_k$ . Hence the two neighborhoods of  $N_C(v)$  and  $N_C(v')$  are not adjacent. It follows that for  $k = 7$ , since  $C_{k-2} = C_5$ , such a configuration is impossible. This yields to  $|W| \leq 1$  and from Property 2.4 a minimum dominating set can be computed in polynomial time.

Now, we focus on the remaining case of  $k = 8$ . Let  $|W| = q, q \geq 2$ . We show that  $\gamma(G) = q + 2$ . Since  $W$  is independent and that for any distinct vertices  $w, w' \in W$ , we have  $N[w] \cap N[w'] = \emptyset$ , we must take  $q$  vertices of  $N[W]$  to dominate the vertices of  $W$ . Let  $w, w' \in W$ . From above we can assume that  $w$  has a neighbor  $v$  such that  $N_C(v) = \{v_1, v_2\}$  and  $w'$  has a neighbor  $v'$  such that  $N_C(v') = \{v_4, v_5\}$  (each vertex of  $W$  has two neighbors whose are neighbors of respectively  $\{v_1, v_2\}$  and  $\{v_4, v_5\}$  since  $C = C_6$ ).  $G$  being claw-free we have  $vv' \notin E$ . The  $q$  vertices that dominates  $W$  cannot dominate  $v_3$  and  $v_6$ . Hence  $\gamma(G) \geq q + 1$ .

Suppose that  $\gamma(G) = q + 1$ . The mds of  $G$  must contain a vertex  $s \in S$  a neigh-

bor of both  $v_3$  and  $v_6$ . If  $vs \in E$ , respectively  $v's \in E$ , then  $G$  has a claw ( $s$  cannot be complete to  $N_C(v) \cup N_c(v')$ ). Also,  $s$  must have ( $v_1$  or  $v_5$ ) and ( $v_2$  or  $v_4$ ) as neighbors else there is a claw. We assume first that  $N(s) = \{v_1, v_2, v_3, v_6\}$ . Then  $w - v - v_1 - s - v_3 - v_4 - v' - w' = P_8$  (recall  $vv' \notin E$  or  $G$  contains a claw). The case where  $N(s) = \{v_3, v_4, v_5, v_6\}$  is symmetric. Now we assume that  $N(s) = \{v_1, v_3, v_4, v_6\}$  (note that  $N(s) = \{v_2, v_3, v_5, v_6\}$  is symmetric). Then  $w - v - v_2 - v_3 - s - v_6 - v_5 - v' = P_8$ . Hence  $\gamma(G) \geq q + 2$ .

We show that  $\Gamma = \{v_1, v_4\} \cup W$  is a  $\gamma$ -set of  $G$ . Clearly  $\Gamma$  dominates  $N[W] \cup C$ . Let  $s \notin N[W] \cup C$ . So  $s \in S$ . Suppose that  $sv_1, sv_4 \notin E$ . From above  $ws \notin E$  and  $vs \notin E$  else  $G[\{v, s, v_1, w\}]$  is a claw. If  $N(s) = \{v_2, v_3\}$  then  $w - v - v_1 - v_6 - v_5 - v_4 - v_3 - s = P_8$ . By symmetry  $N(s) \neq \{v_5, v_6\}$ . As shown before  $N(s) = \{v_2, v_3, v_5, v_6\}$  is not possible. Hence any  $s \notin N[W] \cup C$  is dominated by  $v_1$  or  $v_4$ . It follows that  $\Gamma = \{v_1, v_4\} \cup W$  is a  $\gamma$ -set of  $G$ .  $\square$

From Lemmas 3.1, 3.2, 4.2, 4.3 we immediately obtain the corollary below.

**Corollary 4.4** *Let  $G$  a  $(\text{claw}, P_k)$ -free graph,  $6 \leq k \leq 8$ . If  $C_l \subseteq_i G$ ,  $k - 2 \leq l \leq k$ , then a minimum dominating set of  $G$  can be given in polynomial time.*

## 5 $G$ is $(\text{claw}, P_8)$ -free

Here we conclude by the main result proving that the Minimum Dominating Set problem is polynomial in the class of  $(\text{claw}, P_8)$ -free graphs. Starting from the result stating that the problem is polynomial when  $G$  is  $(\text{claw}, P_5)$ -free, we successively prove that the problem is polynomial for  $(\text{claw}, P_6)$ -free,  $(\text{claw}, P_7)$ -free graphs. Then we conclude for the class of  $(\text{claw}, P_8)$ -free graphs.

**Lemma 5.1** *Let  $G$  be a connected  $(\text{claw}, P_5)$ -free graph. Computing a minimum dominating set is polynomial-time solvable.*

**Proof:** From [8] we know that computing a minimum dominating set is polynomial when  $G$  is  $(K_{1,4}, P_5)$ -free. The result follows from the fact that  $((\text{claw}, P_5) - \text{free graphs}) \subset ((K_{1,4}, P_5) - \text{free graphs})$ .  $\square$

**Lemma 5.2** *Let  $G$  be a connected  $(\text{claw}, P_6)$ -free graph. Computing a minimum dominating set is polynomial-time solvable.*

**Proof:** From Corollary 4.4, if  $C_l \subseteq_i G$ ,  $4 \leq l \leq 6$ , then computing a minimum dominating set is polynomial. When  $G$  is  $(\text{claw}, C_4, C_5, C_6, P_6)$ -free then it is chordal. The Minimum Dominating Set problem is polynomial for  $\text{claw}$ -free chordal graphs.  $\square$

**Lemma 5.3** *Let  $G$  be a connected  $(\text{claw}, C_5, C_6, C_7, P_7)$ -free graph. Computing a minimum dominating set is polynomial-time solvable.*



**Proof:** From Lemma 5.2 we can assume that  $P_6 \subseteq_i G$ . Let  $P = v_1 - v_2 - v_3 - v_4 - v_5 - v_6$ .

Let  $W = V \setminus N[P]$ . From Property 2.3 if  $W = \emptyset$  computing a minimum dominating set is polynomial. From now on  $W \neq \emptyset$ .

Let  $S = \{v \in V \setminus P \text{ such that } 2 \leq |N_P(v)| \leq 4\}$ ,  $S_i \subseteq S$  being the set of vertices  $v$  such that  $|N_P(v)| = i$ .

Let  $H_i = \{v \in S_2 : N_P(v) = \{v_i, v_{i+1}\}, 1 \leq i \leq 5\}$ . Since  $G$  is claw-free each  $H_i$  is complete. If there is an edge  $r_i r_{i+1}$  with  $r_i \in H_i, r_{i+1} \in H_{i+1}$  then  $P = v_1 - \dots - v_i - r_i - r_{i+1} - v_{i+2} - \dots - v_6 = P_7$ . If there is an edge  $r_i r_j$  with  $r_i \in H_i, r_j \in H_j$  and  $j \geq i + 3$  then  $C_p \subseteq_i G$ ,  $p \geq 5$ . So  $H_1$  is anticomplete to  $H_2, H_4, H_5$ , the component  $H_2$  is anticomplete to  $H_3, H_5$ , and the component  $H_3$  is anticomplete to  $H_4$ .

We define  $R_i$  as the set of vertices of  $H_i$  having a neighbor in  $W$ ,  $R_i = \{v \in H_i : N_W(v) \neq \emptyset\}, 1 \leq i \leq 5$ . Since  $G$  is  $P_7$ -free  $R_1 = R_5 = \emptyset$ .

Let  $r \in R_i, r' \in R_i, r \neq r', i \in \{2, 4\}$  be such that  $r$ , respectively  $r'$ , has a neighbor  $w \in W$ , respectively  $w' \in W$ . We show that  $N_S(r) = N_S(r')$ .

By contradiction we assume that there exists  $s \in S$  such that  $rs \in E, r's \notin E$ . From above  $s \notin R_i \cup H_{i-1} \cup H_{i+1}$ . Let  $i = 2$  (the case  $i = 4$  is symmetric). Recall that  $H_2$  is anticomplete to  $H_1, H_3, H_5$ , thus  $s \in H_4 \cup S_3 \cup S_4$ . If  $s \in H_4$  then  $G[\{r, w, v_3, s\}]$  is a claw. Hence  $s \in S_3 \cup S_4$ . When  $N_P(s) = \{v_1, v_2, v_3\}$  then  $G[\{r', v_3, v_4, s\}]$  is a claw. When  $N_P(s) = \{v_2, v_3, v_4\}$  then  $G[\{r', v_1, v_2, s\}]$  is a claw. When  $N_P(s) = \{v_3, v_4, v_5\}$  or  $N_P(s) = \{v_4, v_5, v_6\}$  then  $G[\{r, v_2, w, s\}]$  is a claw. So  $s \in S_4$ . When  $N_P(s) = \{v_1, v_2, v_3, v_4\}$  then  $G[\{r, v_1, v_4, s\}]$  is a claw. When  $N_P(s) = \{v_2, v_3, v_4, v_5\}$  then  $G[\{r', v_1, v_2, s\}]$  is a claw. When  $N_P(s) = \{v_3, v_4, v_5, v_6\}$  then  $G[\{r, v_4, v_6, s\}]$  is a claw. Now let  $i = 3$ . Recall that  $H_3$  is anticomplete to  $H_2, H_4$ , thus  $s \in H_1 \cup H_5 \cup S_3 \cup S_4$ . If  $s \in H_1$  (the case  $s \in H_5$  is symmetric) then  $G[\{r, w, v_3, s\}]$  is a claw. Hence  $s \in S_3 \cup S_4$ . If  $N_P(s) = \{v_1, v_2, v_3\}$  (the case  $N_P(s) = \{v_4, v_5, v_6\}$  is symmetric) then  $G[\{r, w, v_4, s\}]$  is a claw. If  $N_P(s) = \{v_2, v_3, v_4\}$  (the case  $N_P(s) = \{v_3, v_4, v_5\}$  is symmetric) then  $G[\{r', v_4, v_5, s\}]$  is a claw. So  $s \in S_4$ . When  $N_P(s) = \{v_1, v_2, v_4, v_5\}$  or  $N_P(s) = \{v_1, v_2, v_5, v_6\}$  then  $G[\{r, v_1, v_5, s\}]$  is a claw. When  $N_P(s) = \{v_2, v_3, v_5, v_6\}$  then  $G[\{r', v_3, v_4, s\}]$  is a claw. When  $N_P(s) = \{v_1, v_2, v_3, v_4\}$  (the case  $N_P(s) = \{v_3, v_4, v_5, v_6\}$  is symmetric) then  $G[\{r', v_4, v_5, s\}]$  is a claw. Hence  $N_P(s) = \{v_2, v_3, v_4, v_5\}$  but  $G[\{r, v_2, v_5, s\}]$  is a claw. Thus  $N_S(r) = N_S(r')$ .

Let  $r_2 \in R_2, r'_2 \in R_2, r_2 \neq r'_2$  be such that  $r_2$ , respectively  $r'_2$ , has a neighbor  $w \in W$ , respectively  $w' \in W$ . Let  $r_4 \in R_4, r'_4 \in R_4, r_4 \neq r'_4$  be such that  $r_4$ , respectively  $r'_4$ , has  $w$ , respectively  $w'$ , as neighbor. We show that  $N_{S \setminus H_4}(r_2) = N_{S \setminus H_4}(r'_2)$ , respectively  $N_{S \setminus H_2}(r_4) = N_{S \setminus H_2}(r'_4)$ .

By contradiction we assume that there exists  $s \in S$  such that  $r_2 s \in E, r'_2 s \notin E$ . From above  $s \notin H_1 \cup H_2 \cup H_3$ . When  $s \in H_4$  we know that  $s$  is not a neighbor of  $w$ . If  $s \in H_4 \cup H_5$  then  $G[\{r_2, v_2, w, s\}]$  is a claw. Hence  $s \in S_3 \cup S_4$ . When  $N_P(s) = \{v_1, v_2, v_3\}$  then  $G[\{r'_2, v_3, v_4, s\}]$  is a claw. When  $N_P(s) = \{v_2, v_3, v_4\}$  then  $G[\{r'_2, v_1, v_2, s\}]$  is a claw. When  $N_P(s) = \{v_3, v_4, v_5\}$  then  $G[\{r_2, v_2, w, s\}]$  is a claw. When  $N_P(s) = \{v_4, v_5, v_6\}$  then  $G[\{r_2, v_4, v_6, s\}]$  is a claw. So  $s \in S_4$ .

When  $N_P(s) = \{v_1, v_2, v_4, v_5\}$  or  $N_P(s) = \{v_1, v_2, v_5, v_6\}$  then  $G[\{r, v_1, v_5, s\}]$  is a claw. When  $N_P(s) = \{v_2, v_3, v_5, v_6\}$  then  $G[\{r', v_3, v_4, s\}]$  is a claw. When  $N_P(s) = \{v_1, v_2, v_3, v_4\}$  then  $G[\{r_2, v_1, v_4, s\}]$  is a claw. When  $N_P(s) = \{v_2, v_3, v_4, v_5\}$  then  $G[\{r'_2, v_1, v_2, s\}]$  is a claw. When  $N_P(s) = \{v_3, v_4, v_5, v_6\}$  then  $G[\{r_2, v_4, v_6, s\}]$  is a claw. Thus  $N_{S \setminus H_4}(r_2) = N_{S \setminus H_4}(r'_2)$  and by symmetry, for  $r'_4 \in R_4, r'_4 \neq r_4$ , we have  $N_{S \setminus H_2}(r_4) = N_{S \setminus H_2}(r'_4)$ .

Let  $w \in W$ . We show that  $w$  cannot have two neighbors  $r_i, r_{i+1}$  with  $r_i \in R_i, r_{i+1} \in R_{i+1}$ . Suppose for contradiction that these two neighbors exist. Then  $v_1 - \dots - v_i - r_i - w - r_{i+1} - v_{i+2} - \dots - v_6 = P_8$ . Now, since  $R_1 = R_5 = \emptyset$ , if  $w$  has two neighbors  $r_i \in R_i, r_j \in R_j, i \neq j$ , these two neighbors are  $r_2 \in R_2, r_4 \in R_4$  and  $r_2 r_4 \in E$ , else  $w - r_4 - v_4 - v_3 - r_2 - w = C_5$ .

Moreover, when  $w$  has two neighbors  $r_2 \in R_2, r_4 \in R_4$ , then for each neighbor  $w' \in N_W(w)$ ,  $w'$  has  $r_2$  and  $r_4$  as neighbors. Assume for contradiction that  $w$  has a neighbor  $w' \in W$  such that  $w' r_2 \notin E$  (by symmetry  $w' r_4 \notin E$  is the same case). Then  $w' - w - r_2 - v_3 - \dots - v_6 = P_7$ . It follows that  $N[w] = N[w']$  and from Property 2.1  $ww'$  can be contracted.

Hence setting  $Z_{24} = \{w \in W : w \text{ has two neighbors } r_2 \in R_2, r_4 \in R_4\}$ ,  $Z_{24}$  is an independent set.

Let  $w, w' \in Z_{24}, w \neq w'$ . Since  $G$  is claw-free we have  $N(w) \cap N(w') = \emptyset$ . We show that  $N_{R_2}(w)$  is anticomplete to  $N_{R_4}(w')$  and  $N_{R_4}(w)$  is anticomplete to  $N_{R_2}(w')$ . By contradiction if  $w$  has a neighbor  $r_2 \in R_2$ ,  $w'$  has a neighbor  $r_4 \in R_4$ , and  $r_2 r_4 \in E$  then  $G[\{v_2, r_2, w, r_4\}]$  is a claw.

Let  $Z_i = \{w \in W : w \text{ has a neighbor in } R_i \setminus (N_{R_i}(Z_{24}))\}$ ,  $2 \leq i \leq 4$ .

We show that  $Z_2, Z_3, Z_4$  are pairwise anticomplete. If there is an edge  $w_2 w_4, w_2 \in Z_2, w_4 \in Z_4$ , with  $r'_2 \in R_2, r'_4 \in R_4$  the neighbors of  $w_2, w_4$  respectively, then  $w_2 - r'_2 - v_3 - v_4 - r'_4 - w_4 - w_2 = C_6$  ( $r'_2 r'_4 \notin E$  else  $G[\{v_2, r'_2, w_2, r'_4\}]$  is a claw). If there is an edge  $w_2 w_3, w_2 \in Z_2, w_3 \in Z_3$ , with  $r'_2 \in R_2, r'_3 \in R_3$  the neighbors of  $w_2, w_3$  respectively, then  $w_2 - r'_2 - v_3 - r'_3 - w_3 - w_2 = C_5$  (recall  $r'_2 r'_3 \notin E$ ). By symmetry there is no edge between  $Z_3$  and  $Z_4$ .

Let  $Y = W \setminus (Z_2 \cup Z_3 \cup Z_4 \cup Z_{24})$ . One can observe that for any  $w \in Y$  we have  $N_{Z_2}(w) = N_{Z_4}(w) = N_{Z_{24}}(w) = \emptyset$  else  $P_7 \subseteq_i G$ .

Let  $Y_3 = \{w \in Y : w \text{ has a neighbor in } Z_3\}$ . If there exists  $w' \in Y \setminus Y_3$  such that  $w'$  has a neighbor  $w, w \in Y_3$ , then  $P_7 \subseteq_i G$ . Hence  $Y = Y_3$ .

We show that we can assume that  $Z_2, Z_4, Y_3$  are three independent sets. The arguments are the same for the three sets, so we show that  $Z_2$  is an independent set. For contradiction, we assume that there are  $w_1, w_2 \in Z_2$  such that  $w_1 w_2 \in E$ . We prove that  $N_{R_2}(w_1) = N_{R_2}(w_2)$ . If  $N_{R_2}(w_1) \neq N_{R_2}(w_2)$  then there exists  $r_2 \in R_2$  which is a neighbor of  $w_1$  but not a neighbor of  $w_2$ . Then  $w_2 - w_1 - r_2 - v_3 - \dots - v_6 = P_7$ . If  $N_{Z_2}(w_1) \neq N_{Z_2}(w_2)$  then there exists  $w_3 \in Z_2$  such that  $w_2 w_3 \in E, w_1 w_3 \notin E$ , but  $G[\{v_2, r_2, w_1, w_3\}]$  is a claw. Hence  $N[w_1] = N[w_2]$  and we can apply the Property 2.1 to contract  $w_1 w_2$ . Hence  $Z_2, Z_4, Y_3$  are independent.

Since  $G$  is claw-free then for any two distinct vertices  $w_1, w_2 \in Z_2 \cup Z_4 \cup Y_3$  we have  $N(w_1) \cap N(w_2) = \emptyset$ .

We prove that for any  $w \in Y_3$ ,  $N(w)$  is a clique. Let  $w \in Z_3$ . Suppose there are  $s, s'$  two non adjacent vertices in  $N(w)$ . Since  $G$  is claw-free  $s, s'$  cannot have a common neighbor in  $R_3$ . Let  $r \in R_3$  be a neighbor of  $s$ . Then  $s' - w - s - r - v_3 - v_2 - v_1 = P_7$ .

Since  $G$  is claw-free, if there are a vertex  $r \in R_i$  with a neighbor  $z \in Z_i$  and a vertex  $s \in S$  such as  $sz \notin E$  and  $v_i \notin N(s)$  then  $G$  contains a claw (note that  $v_{i+1} \notin N(s)$  is symmetric). Hence  $N(Z_i)$  is anticomplete to  $H_j$ ,  $j \neq i$ .

We show that we can assume that  $Z_2 = Z_4 = \emptyset$ . The arguments are the same in the two cases, so we consider  $Z_2$ . Let  $r, r' \in R_2$  be two neighbors of  $w \in Z_2$ . We show that  $N[r] = N[r']$ . Since  $N_R(w) = N_{R_2}(w)$  and  $rr' \in E$  then, as proved above,  $N_S(r) = N_S(r')$ . For two distinct  $w_1, w_2 \in Z_2$ ,  $N(w_1) \cap N(w_2) = \emptyset$ . Hence,  $N[r] = N[r']$ . From Property 2.1 we can contract  $rr'$ . Then  $w$  is a leaf and by Property 2.2,  $w$  can be deleted from  $G$ .

Now we study the structure of  $Z_3$ . For any distinct two vertices  $w_1, w_2 \in Z_3$  such that  $w_1w_2 \in E$ , there cannot exist two distinct vertices  $w'_1, w'_2 \in Z_3$  such that  $w_1w'_1 \in E, w'_1w_2 \notin E$  and  $w_2w'_2 \in E, w_1w'_2 \notin E$ . For contradiction we suppose that such two vertices exist. We assume first that  $w_2$  has a neighbor  $r_2 \in R_3$  such that  $r_2w_1 \notin E$ . If  $w'_1r_2 \notin E$  then  $v_1 - v_2 - v_3 - r_2 - w_2 - w_1 - w'_1 = P_7$  else  $G[\{v_4, r_2, w_2, w'_1\}]$  is a claw. So  $w_1, w_2$  have a common neighbor  $r_1 \in R_3$ . If  $w'_1r_1 \in E$  then  $G[\{v_3, r_1, w_2, w'_1\}]$  is a claw. Thus  $w'_1r_1 \notin E$  and  $w'_1$  has a neighbor  $r'_1 \in R_3, r'_1 \neq r_1$ . If  $r'_1w_2 \in E$  then  $G[\{v_3, r'_1, w_2, w'_1\}]$  is a claw. So  $r'_1w_2 \notin E$ . If  $r'_1w_1 \notin E$  then  $v_1 - v_2 - v_3 - r'_1 - w'_1 - w_1 - w_2 = P_7$ . Thus  $r'_1w_1 \in E$ . If  $r'_1w'_2 \notin E$  then  $v_1 - v_2 - v_3 - r'_1 - w_1 - w_2 - w'_2 = P_7$ . So  $r'_1w'_2 \in E$  but  $G[\{v_4, r'_1, w_1, w'_2\}]$  is a claw.

As a consequence each connected component  $A_i$  of  $Z_3$  has a universal vertex. Also,  $G$  being claw-free two distinct connected components cannot share a neighbor in  $R_3$ . Moreover, from Property 2.2 we have assumed that each  $w_3 \in Z_3$  is not a leaf.

We show that  $w \in Y_3$  is connected to a universal vertex of a connected component  $A_i$  of  $Z_3$ . We assume that the neighbors of  $w$  are not universal in  $A_i$ . Let  $s \in A_i$  be a neighbor of  $w$ , let  $u, u \neq s$ , be a universal vertex of  $A_i$ . Since  $s$  is not universal there exists  $v, v \in A_i$  such that  $sv \notin E$  and  $uv \in E$ . Since  $N(w)$  is complete  $wv \notin E$ . Let  $r \in R_3$  be a neighbor of  $s$ . Since  $G$  is claw-free then  $rv \notin E$ . Let  $r', r' \in R_3, r' \neq r$ , be a neighbor of  $v$ . As just above  $r's \notin E$ . If  $r'u \notin E$  then  $v_1 - v_2 - v_3 - r' - v - u - s = P_7$  else  $v_1 - v_2 - v_3 - r' - u - s - w = P_7$ .

We are ready to show how to build a  $\gamma$ -set in polynomial time.

First, we treat the case where  $Z_{24} \neq \emptyset$ . Let  $r_2 \in R_2, r_4 \in R_4$  be two neighbors of  $w, w \in Z_{24}$ .

We show that  $R_3 = \emptyset$ . Assume that there exists  $w' \in W$  with a neighbor  $r_3 \in R_3$ . Since  $w'$  is not a neighbor of  $r_2$  or  $r_4$  we have  $w' - r_3 - v_3 - r_2 - r_4 - v_5 - v_6 = P_7$ . So  $R_3 = \emptyset$  and since  $Z_2 = Z_4 = \emptyset$  we have  $W = Z_{24}$ .

Recall that  $W = Z_{24}$  is independent and that for any two distinct vertices  $w', w' \in$

$Z_{24}$  we have  $N(w) \cap N(w') = \emptyset$ .

The  $\gamma$ -set is build as follows:

From Property 2.4, we can assume that  $|W| \geq 2$ . We take  $r_2 \in R_2$  a neighbor of  $w$  (recall that the neighbors of  $w$  in  $R_i, i \in \{2, 4\}$ , have the same neighborhood and that all vertices of  $R_i$  have the same neighbors in  $S \setminus H_4$ ), and for each other  $w' \in Z_{24}$  we take one adjacent vertex  $r'_4 \in R_4$ . These vertices dominate  $Z_{24} \cup H_2 \cup H_4 \cup \{v_2, v_3, v_4, v_5\}$ . At least one more vertex is necessary to dominate  $G$  since  $v_1$  and  $v_6$  are not dominated. Adding the three vertices  $v_2, v_4, v_6$  we have a dominating set (not necessarily minimum). We check first if there exists  $s$  a neighbor of both  $v_1$  and  $v_6$  that dominates the rest of the graph. If such vertex  $s$  is nowhere to be found, checking for all the pairs  $s_1, s_6$  where  $s_i$  is a neighbor of  $v_i, i \in \{1, 6\}$ , one can verify if there is a  $\gamma$ -set with only two more vertices (note that there are at most  $O(n^2)$  such pairs).

Now we deal with the case  $Z_{24} = \emptyset$ .

The  $\gamma$ -set is build as follows:

- $Y_3 \neq \emptyset$ . For each  $w \in Y_3$  we take one universal vertex in the connected component  $A_i$  of  $Z_3$  connected to  $w$ . For each connected component  $A_i$  of  $Z_3$  that is not connected to a vertex of  $Y_3$ , we do as follows: If there exists  $r_3 \in R_3$  which is complete to  $A_i$  (recall that such vertices have the same neighborhood) then we take  $r_3$ , else we take one universal vertex of  $A_i$ . These vertices dominate  $Y_3 \cup Z_3$ . At least one more vertex is necessary to dominate  $G$  since  $v_1$  and  $v_6$  are not dominated. Adding the three vertices  $v_2, v_4, v_6$  we have a dominating set (not necessarily minimum). We check first if there exists  $s$  a neighbor of both  $v_1$  and  $v_6$  that dominates the rest of the graph. If such vertex is  $s$  nowhere to be found, checking for all the pairs  $s_1, s_6$  where  $s_i$  is a neighbor of  $v_i, i \in \{1, 6\}$ , one can verify if there is a  $\gamma$ -set with only two more vertices (note that there are at most  $O(n^2)$  such pairs).
- $Y_3 = \emptyset$ . Thus  $Z_3 = W$ . For any connected component  $A_i$  of  $Z_3$ , if there exists  $r_3 \in R_3$  which is complete to  $A_i$  (recall that such vertices have the same neighborhood) then we take  $r_3$ , else else we take one universal vertex of  $A_i$ . These vertices dominate  $Z_3$ . At least one more vertex is necessary to dominate  $G$  since  $v_1$  and  $v_6$  are not dominated. Adding the three vertices  $v_2, v_4, v_6$  we have a dominating set (not necessarily minimum). We check first if there exists  $s$  a neighbor of both  $v_1$  and  $v_6$  that dominates the rest of the graph. If such vertex  $s$  is nowhere to be found, checking for all the pairs  $s_1, s_6$  where  $s_i$  is a neighbor of  $v_i, i \in \{1, 6\}$ , one can verify if there is a  $\gamma$ -set with only two more vertices (note that there are at most  $O(n^2)$  such pairs).

Clearly the construction of the  $\gamma$ -set is polynomial. □

**Corollary 5.4** *The Minimum Dominating Set problem is polynomial for (claw,  $P_7$ )-free graphs.*

**Lemma 5.5** *Let  $G$  be a connected (claw,  $C_6, C_7, C_8, P_8$ )-free graph. If  $C_5 \subseteq_i G$  then computing a minimum dominating set is polynomial.*

**Proof:** Let  $C = v_1 - v_2 - v_3 - v_4 - v_5 - v_1 = C_5 \subseteq_i G$ . Let  $W = V \setminus N[C]$ . From Property 2.3 if  $W = \emptyset$  computing a minimum dominating set is polynomial. From now on  $W \neq \emptyset$ .

Let  $S = \{v \in V \setminus C \text{ such that } 2 \leq |N(v) \cap C_5| \leq 5\}$ ,  $S_i \subseteq S$  being the set of vertices  $v$  such that  $|N(v) \cap C_5| = i$ . Let  $H_i = \{v \in S_2 : N(v) \cap C_5 = \{v_i, v_{i+1}\}\}$ ,  $1 \leq i \leq 5$  (for convenience  $v_{5+1}$  stands for  $v_1$ ). Since  $G$  is claw-free, each  $H_i$  is complete. Moreover, if there is an edge  $r_i r_{i+1}$  with  $r_i \in H_i, r_{i+1} \in H_{i+1}$  then  $r_i - v_i - v_{i-1} - \dots - v_{i+2} - r_{i+1} - r_i = C_6$ . Hence  $H_i$  is anticomplete to  $H_{i+1}$ . We define  $R_i$  as the set of vertices of  $H_i$  having a neighbor in  $W$ ,  $R_i = \{v \in H_i : N_W(v) \neq \emptyset\}$ ,  $1 \leq i \leq 5$ ,  $R = R_1 \cup \dots \cup R_5$ .

Since  $W \neq \emptyset$ , we assume that there exists  $w_1 \in W$  such that  $w_1$  has a neighbor  $r_1 \in R_1$ . Suppose that  $R_2 \neq \emptyset$ . There exists  $w_2 \in W$  with a neighbor  $r_2 \in R_2$ . If  $w_1 = w_2$  then  $w_1 - r_2 - v_3 - v_4 - v_5 - v_1 - r_1 - w_1 = C_7$ . So  $w_1 \neq w_2$ . If  $w_1 w_2 \in E$  then  $w_1 - w_2 - r_2 - v_3 - v_4 - v_5 - v_1 - r_1 - w_1 = C_8$  else  $w_1 - r_1 - v_1 - v_5 - v_4 - v_3 - r_2 - w_2 - w_1 = P_8$ . So, if  $R_i \neq \emptyset$  then  $R_{i-1} = R_{i+1} = \emptyset$ . Hence  $R_2 = R_5 = \emptyset$ .

Let  $r \in R_i, r' \in R_i, r \neq r', i \in \{1, 3, 4\}$  be such that  $r$ , respectively  $r'$ , has a neighbor  $w \in W$ , respectively  $w' \in W$ . We show that  $N_S(r) = N_S(r')$ .

By contradiction we assume that there exists  $s \in S$  such that  $rs \in E, r's \notin E$ . From above  $s \notin H_i \cup H_{i-1} \cup H_{i+1}$ . Let  $i = 1$ . If  $s \in H_3 \cup H_4$  then  $G[\{r, v_1, w, s\}]$  is a claw. Thus  $s \in S_3 \cup S_4 \cup S_5$ . When  $N_C(s) = \{v_1, v_2, v_3\}$  (the case  $N_C(s) = \{v_1, v_2, v_5\}$  is symmetric) then  $G[\{r', v_5, w, s\}]$  is a claw. When  $N_C(s) = \{v_2, v_3, v_4\}$  (the case  $N_C(s) = \{v_1, v_4, v_5\}$  is symmetric) then  $G[\{r, v_1, w, s\}]$  is a claw. When  $N_C(s) = \{v_3, v_4, v_5\}$  then  $G[\{r, v_3, v_5, s\}]$  is a claw. So  $s \in S_4 \cup S_5$ . When  $N_C(s) = \{v_1, v_2, v_3, v_4\}$  (the case  $N_C(s) = \{v_1, v_2, v_4, v_5\}$  is symmetric) then  $G[\{r', v_1, v_5, s\}]$  is a claw. When  $N_C(s) = \{v_1, v_3, v_4, v_5\}$  or  $N_C(s) = \{v_2, v_3, v_4, v_5\}$  or  $N_C(s) = \{v_1, v_2, v_3, v_5\}$  or  $s \in S_5$  then  $G[\{r, v_3, v_5, s\}]$  is a claw. For  $i = 3$  and  $i = 4$  the arguments are the same. Thus  $N_S(r) = N_S(r')$ .

Let  $r_1 \in R_1, r'_1 \in R_1, r_1 \neq r'_1$  be such that  $r_1$ , respectively  $r'_1$ , has a neighbor  $w \in W$ , respectively  $w' \in W$ . Let  $r_3 \in R_3, r'_3 \in R_3, r_3 \neq r'_3$  be such that  $r_3$ , respectively  $r'_3$ , has  $w$ , respectively  $w'$ , as neighbor. We show that  $N_{S \setminus H_3}(r_1) = N_{S \setminus H_3}(r'_1)$ , respectively  $N_{S \setminus H_1}(r_3) = N_{S \setminus H_1}(r'_3)$ .

Let  $i = 1$ . By contradiction we assume that there exists  $s \in S \setminus R_3$  such that  $r_1 s \in E, r'_1 s \notin E$ . From above  $s \notin H_1 \cup H_2 \cup H_5$ . If  $s \in H_4$  then  $G[\{r_1, v_1, w, s\}]$  is a claw. So  $s \in S_3 \cup S_4 \cup S_5$ . When  $N_C(s) = \{v_1, v_2, v_3\}$  (the case  $N_C(s) = \{v_1, v_2, v_5\}$  is symmetric) then  $G[\{r'_1, v_1, v_5, s\}]$  is a claw. When  $N_C(s) = \{v_2, v_3, v_4\}$  (the case  $N_C(s) = \{v_1, v_4, v_5\}$  is symmetric) then  $G[\{r_1, v_1, w, s\}]$  is a claw. When  $N_C(s) = \{v_3, v_4, v_5\}$  then  $G[\{r_1, v_3, v_5, s\}]$  is a claw. So  $s \in S_4 \cup S_5$ . When  $N_C(s) = \{v_1, v_2, v_3, v_4\}$  (the case  $N_C(s) = \{v_1, v_2, v_4, v_5\}$  is symmetric) then  $G[\{r'_1, v_1, v_5, s\}]$  is a claw. When  $N_C(s) = \{v_1, v_3, v_4, v_5\}$  or  $N_C(s) = \{v_2, v_3, v_4, v_5\}$  or  $N_C(s) = \{v_1, v_2, v_3, v_5\}$  or  $s \in S_5$  then  $G[\{r_1, v_3, v_5, s\}]$  is a claw. By symmetry the arguments are the same

for  $i = 3$ . Hence  $N_{S \setminus H_3}(r_1) = N_{S \setminus H_3}(r'_1)$  and  $N_{S \setminus H_1}(r_3) = N_{S \setminus H_1}(r'_3)$ .

We study the case where  $w_1$  has a neighbor  $r_i, r_i \in R_i, i \in \{3, 4\}$ . Since both cases are symmetric, let  $r_3, r_3 \in R_3$ , be a neighbor of  $w_1$ . If  $r_1 r_3 \notin E$  then  $w_1 - r_1 - v_1 - v_5 - v_4 - r_3 - w_1 = C_6$ . Hence  $N_{R_1}(w_1)$  is complete to  $N_{R_3}(w_1)$ . Since  $R_3 \neq \emptyset$ , we have  $R_4 = \emptyset$  and  $N_R(w_1) \subseteq R_1 \cup R_3$ . Hence, we define the following subsets of  $W$ :

- $Z = \{w \in W : N_R(w) \neq \emptyset\}$ ;
- $Z_i = \{z \in Z : N_{R_i}(z) \neq \emptyset, N_{R_j}(z) = \emptyset, 1 \leq i \leq 5, i \neq j\}$ ;
- $Z_{ij} = \{z \in Z : N_{R_i}(z) \neq \emptyset, N_{R_j}(z) \neq \emptyset, 1 \leq i < j \leq 5\}$ ;
- $Y = W \setminus Z$ .

First, we show that  $Z_i$  and  $Z_{ij}$  are anticomplete, then we show that  $Z_i$  consists of leaves. We conclude that  $Z_{ij} \neq \emptyset$  implies  $Z = Z_{ij}$ . We set  $w_1 \in Z_{13}$ , and since all cases are symmetric, we focus on  $Z_1 \neq \emptyset$ .

Let  $w'_1 \in Z_1$  with a neighbor  $r'_1, r'_1 \in R_1, r'_1 \neq r_1$ . Note that  $r'_1 r_3 \notin E$  else  $G$  contains a claw. If  $w_1 w'_1 \in E$ , then  $r'_1 w_1 \in E$ , else  $w_1 - w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 = C_7$ , but  $w_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 = C_6$ . Hence,  $w_1 w'_1 \notin E$  (by symmetry, for any  $w_3 \in Z_3, w_1 w_3 \notin E$ ). Thus  $Z_1$  and  $Z_3$  are anticomplete to  $Z_{13}$ .

Now, we show that the vertices of  $Z_1$  are leaves. Assume that there exists  $v \in N(w'_1), v \neq r'_1$  such that  $N[v] \neq N[w'_1]$ . If  $v \in Z_3$  then  $v - w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 = P_8$ . If  $v \in Y$  then either  $v - w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 = P_8$  or  $v - w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 - v = C_8$ . If  $v \in Z_1$  and  $r'_1 v \notin E$  then  $v - w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w_1 = P_8$ . Hence  $N_{R_1}(w'_1) = N_{R_1}(v)$ . Since  $N[v] \neq N[w'_1]$  we can assume that there exists  $v' \in W$  such that  $vv' \in E$  but  $v'w'_1 \notin E$ . Yet with the same arguments as before we have  $N_{R_1}(v) = N_{R_1}(v')$  and since  $w'_1 v' \notin E$  then  $G[\{r'_1, v', v_1, w'_1\}]$  is a claw. Thus  $Z_1$  consists of leaves. Now, from Property 2.2 we can suppose that  $Z_1 = \emptyset$ . By symmetry  $Z_3 = \emptyset$ . So  $Z = Z_{13}$ .

We show that any two vertices  $v, v' \in Z_{13}$  with  $vv' \in E$  satisfy  $N_{R_1 \cup R_3}(v) = N_{R_1 \cup R_3}(v')$ . Let  $w'_1 \in Z_{13}$  be a neighbor of  $w_1$ . Suppose that there exists  $r'_1 \in N_{R_1}(w'_1)$  such that  $r'_1 w_1 \notin E$ . If  $r'_1 r_3 \in E$  then  $G[\{r'_1, r_3, v_3, w_1\}]$  is a claw. If  $r_3 w'_1 \notin E$  then  $w_1 - r_3 - v_4 - v_5 - v_1 - r'_1 - w'_1 - w_1 = C_7$ , else  $w'_1 - r'_1 - v_1 - v_5 - v_4 - r_3 - w'_1 = C_6$ . Hence,  $N_{R_1}(w_1) = N_{R_1}(w'_1)$  and by symmetry  $N_{R_3}(w_1) = N_{R_3}(w'_1)$ .

Suppose that  $Y \neq \emptyset$ . Let  $y \in Y$  be a neighbor of  $w_1$ . We show that  $Z_{13}$  is a clique. Let  $w'_1 \in Z_{13}$  such that  $w_1 w'_1 \notin E$ . We have  $N_{R_1 \cup R_3}(w_1) \cap N_{R_1 \cup R_3}(w'_1) = \emptyset$  else  $G$  contains a claw. Yet, there exists  $r'_1$  a neighbor of  $w'_1$  in  $R_1$  such that either  $y - w_1 - r_3 - v_4 - v_5 - v_1 - r'_1 - w'_1 = P_8$  or  $y - w_1 - r_3 - v_4 - v_5 - v_1 - r'_1 - w'_1 - y = C_8$  (note that  $r'_1 r_3 \notin E$  else  $G$  contains a claw). Hence  $Z_{13}$  is a clique.

We show that the vertices of  $Y$  are leaves. Suppose that  $y$  has a neighbor  $y' \in Y$ . If  $y'w_1 \notin E$  then  $y' - y - w_1 - r_1 - v_1 - v_5 - v_4 - v_3 = P_8$ . Hence  $N_Z(y) = N_Z(y')$ . Since we assume that  $N[y] \neq N[y']$ , there exists  $v, v \in Y$ , such that  $vy \in E, vy' \notin E$ .

From above  $w_1v \in E$  but  $G[\{r_1, w_1, y', v\}]$  is a claw. Thus,  $Y$  is an independent set. Now,  $N(y) \subseteq Z_{13}$  is a clique. Since for any two vertices  $w_1, w'_1 \in Z_{13}$  we have  $N_{R_1 \cup R_3}(w_1) = N_{R_1 \cup R_3}(w'_1)$  we can assume that  $N(y)$  can be contracted into a unique vertex. Thus,  $Y$  consists of leaves. Hence from property 2.2, we can assume that  $Y = \emptyset$ .

From now on  $Y = \emptyset$ . As shown before any two neighbors of  $Z_{13}$  have the same neighbors in  $R$ , so they can be contracted and we can assume that  $Z_{13}$  is an independent set. Moreover, since  $G$  is claw-free, for any two distinct  $z, z' \in Z_{13}$ ,  $N[z] \cap N[z'] = \emptyset$ . Also, recall that the neighbors of each  $z, z \in Z_{13}$  induce a clique.

We show how to build a  $\gamma$ -set of  $G$ . Recall that  $W = Z_{13}$ . From Property 2.4 we can assume that  $|W| \geq 2$ . So there are  $w_1, w'_1 \in Z_{13}$  with neighbors  $r_1, r'_1 \in R_1$  and  $r_3, r'_3 \in R_3$ , respectively. Let  $q = |Z_{13}|$ . Clearly, to dominate  $Z_{13}$  we must take  $q$  vertices. We take  $r_1$  and  $r'_3$ . Recall that the vertices of  $R_1$  and  $R_3$  have the same neighbors in  $S \cup C$ . Then, we take the  $q - 2$  vertices of  $w \in Z_{13}, w \neq w_1, w'_1$ . These  $q$  vertices dominate  $\{v_1, v_2, v_3, v_4\} \cup H_1 \cup H_3 \cup Z_{13}$ . It remains to dominate some vertices of  $H_2 \cup H_4 \cup H_5 \cup S_3 \cup S_4 \cup \{v_5\}$ . If there exists a vertex  $v, v \in S \cup \{v_5\}$ , which is universal to these non dominated vertices we take  $v$ , else we take the vertices  $\{v_2, v_5\}$ .

Now, we assume that  $Z_{ij} = \emptyset$ . Hence let  $w_1 \in Z_1$ . We study the case  $R_3 \neq \emptyset$ . Recall then  $R_2 = R_4 = R_5 = \emptyset$ . Let  $w_3 \in W$  such that  $w_3$  has a neighbor  $r_3 \in R_3$ . If  $w_1w_3 \in E$  then  $w_1 - w_3 - r_3 - v_4 - v_5 - v_1 - r_1 - w_1 = C_7$  ( $r_1r_3 \notin E$  else  $G$  contains a claw). So  $Z_1$  is anticomplete to  $Z_3$ . We assume that  $w_1$  has a neighbor  $v \in Y$ . If  $vw_3 \in E$  then  $v - w_3 - r_3 - v_4 - v_5 - v_1 - r_1 - w_1 - v = C_8$  else  $v - w_1 - r_1 - v_1 - v_5 - v_4 - r_3 - w_3 = P_8$ . Hence any neighbor  $w'_1, w'_1 \in W$ , of  $w_1$  is in  $Z_1$ . From Property 2.1 we can suppose  $N[w_1] \neq N[w'_1]$ . If  $w'_1r_1 \notin E$  then  $w'_1 - w_1 - r_1 - v_1 - v_5 - v_4 - r_3 - w_3 = P_8$ . Hence  $N_R(w_1) = N_R(w'_1)$ . Since  $G$  is claw-free, for any  $r \in R$ ,  $N_W(r)$  is a clique, thus  $N[w_1] = N[w'_1]$ . So  $Z_1$  is an independent set. Now, recall that for any pair of vertices  $r, r' \in R_i, 1 \leq i \leq 5$ ,  $N_S(r) = N_S(r')$ . Hence, when  $r, r' \in R_1$  have a common neighbor in  $Z_1$ , we have  $N[r] = N[r']$ , and so  $r, r'$  can be contracted. Hence  $Z_1$  consists of leaves. Also, by symmetry,  $W = Z_1 \cup Z_3$  consists of leaves and then from Property 2.2 we have  $W = \emptyset$ , a contradiction.

Now we focus on  $R_3 = R_4 = \emptyset$  (note that  $Z = Z_1$ ).

We study the case where  $H_2 \neq \emptyset$  or  $H_5 \neq \emptyset$ . Let  $v \in H_2$  (the case  $v \in H_5$  is symmetric). We have  $Y = \emptyset$ , else there are  $y \in Y, z \in Z_1, r \in R_1$  such that  $v - v_3 - v_4 - v_5 - v_1 - r - z - y = P_8$  (recall  $vr \notin E$ ). So  $W = Z = Z_1$ . Let  $w_1, w_2 \in W$ . We assume that  $w_1w_2 \in E$ . From Property 2.1 we can suppose that  $N[w_1] \neq N[w_2]$ . Let  $w_1r_1, w_2r_2 \in E, r_1 \neq r_2$ , such that  $w_1r_2 \notin E$ . We have  $v - v_3 - v_4 - v_5 - v_1 - r_2 - w_2 - w_1 = P_8$ . Recall that for any  $r \in R$ ,  $N_W(r)$  is a clique, thus  $N[w_1] = N[w_2]$ . Hence  $W$  is an independent set. Moreover, for any  $w \in W$  and  $r, r' \in N(w)$  we know that  $r$  and  $r'$  share the same neighbors in  $C \cup S$ . Hence  $W$  is composed exclusively of leaves. So  $W = \emptyset$ , a contradiction.

Now we can assume that  $H_2 = H_5 = \emptyset$ . Let  $Z_A \subset Z, Z_A = \{w \in W : N_Y(w) = \emptyset\}$ .

We show that each connected component  $A_i$  of  $G[Z_A]$  contains a universal vertex relatively to  $A_i$ . For contradiction we suppose that there exists  $A_i, A_i \subseteq Z_A$  with no universal vertex in it. Assume that  $z_1 - z_2 - z_3 - z_4 = P_4 \subseteq_i A_i$ . Let  $r, r \in R_1$ , be a neighbor of  $z_1$  (note that there is a  $P_5$  from  $v_3$  to  $r$ ).

Since  $G$  is claw-free  $rz_3, rz_4 \notin E$ . If  $rz_2 \in E$  then there is a  $P_8$  from  $v_3$  to  $z_4$  else there is a  $P_8$  from  $v_3$  to  $z_3$ . Now, we assume that  $z_1 - z_2 - z_3 - z_4 - z_1 = C_4 \subseteq_i A_i$ . Let  $r, r \in R_1$ , be a neighbor of  $z_1$ . Since  $G$  is claw-free we have  $rz_3 \notin E$ . If  $rz_2 \in E$  then  $rz_4 \notin E$  else  $G$  contains a claw, but  $v_3 - v_4 - v_5 - v_1 - r - z_2 - z_3 - z_4 = P_8$ . If  $rz_2 \notin E$  then  $v_3 - v_4 - v_5 - v_1 - r - z_1 - z_2 - z_3 = P_8$ . So  $A_i$  is  $(C_4, P_4)$ -free. It follows that there are  $z_1 - z_2 - z_3 = P_3 \subseteq_i A_i$  and  $z_4 \in A_i$  such that  $z_4z_1, z_4z_2, z_4z_3 \notin E$ . Also there exists  $z \in A_i$  such that  $z_2 - z - z_4$  and  $zz_1, zz_3 \in E$  but  $A_i[\{z, z_1, z_3, z_4\}]$  is a claw, a contradiction. So each  $A_i$  has a universal vertex. Clearly, for two distinct components  $A_i, A_j$  we have  $N_{R_1}(A_i) \cap N_{R_1}(A_j) = \emptyset$  else there is a claw.

Suppose that  $Y \neq \emptyset$ . We show that  $Y$  is an independent set. Suppose that there are  $y, y' \in Y$  with  $yy' \in E$ . From Property 2.1 we assume that  $N[y] \neq N[y']$ . If  $N_{Z_1}(y) \neq N_{Z_1}(y')$  then, w.l.o.g,  $yw_1 \in E, y'w_1 \notin E$ , but  $v_3 - v_4 - v_5 - v_1 - r_1 - w_1 - y - y' = P_8$ . So  $N_{Z_1}(y) = N_{Z_1}(y')$ . There is no vertex  $y'' \in Y$  such that  $yy'' \in E, y'y'' \notin E$ , else  $G$  contains a claw. Hence  $Y$  is an independent set and for any pair of vertices  $y, y' \in Y$  we have  $N(y) \cap N(y') = \emptyset$ .

We show that for any  $y \in Y$  its neighborhood  $N(y)$  is a clique. For contradiction we assume that  $y$  has two neighbors  $z_1, z_2 \in Z, z_1z_2 \notin E$ . Since  $G$  is claw-free  $z_1$  and  $z_2$  cannot have a common neighbor in  $R_1$ . Let  $r, r \in R_1$ , be a neighbor of  $z_1$ . Then  $v_3 - v_4 - v_5 - v_1 - r - z_1 - y - z_2 = P_8$ . Hence,  $Y$  is an independent set, for each  $y, y \in Y, N(y)$  is a clique. So we suppose  $|N(y)| \geq 2$ , else  $y$  is a leaf.

We show that we can assume that each connected component  $A_i$  of  $G[Z_A]$  is anti-complete to  $N(Y)$ . Since  $Y$  has no leaves, let  $y \in Y$  with two neighbors  $z, z' \in Z_1$  such that  $N[z] \neq N[z']$ . Suppose that there exists  $u \in Z_A$  a neighbor of  $z$ . First, we assume that  $N_R(z) \neq N_R(z')$ . W.l.o.g. let  $r, r' \in R_1$  be respectively the neighbors of  $z, z'$  such that  $r'z, rz' \notin E$ . If  $uz' \notin E$  then  $ur' \notin E$  else  $G$  contains a claw, but then  $u - z - z' - r' - v_1 - v_5 - v_4 - v_3 = P_8$ . Hence  $uz', r'u \in E$  but  $y - z - u - r' - v_1 - v_5 - v_4 - v_3 = P_8$ . So  $N_R(z) = N_R(z')$ . Second, we assume that  $N_Z[z] \neq N_Z[z']$ . W.l.o.g.  $uz' \notin E$ . Let  $r \in R_1$  a neighbor of both  $z, z'$ . Clearly  $ru \notin E$  else  $G$  contains a claw, but  $G[\{r, u, y, z\}]$  is a claw. So we can assume that each  $A_i$  is anticomplete to  $N(Y)$ .

We construct a  $\gamma$ -set as follows:

Let  $q = |Y|$  and  $k$  be the number of connected components of  $Z_A$ . Clearly,  $q$  vertices are necessary to dominate  $Y$ . So for each  $y_i \in Y$  we will take one of its neighbor as follows. Let us denote  $R_1(y_i) = N_{R_1}(N(y_i))$ . If  $y_i$  has a neighbor  $z_i$  which is complete to  $R_1(y_i)$  then we take  $z_i$ , else we take any arbitrary neighbor of  $y_i$  (recall that in both cases these  $y_i$  have the same neighbors in  $Z$ ). These  $q$  vertices dominate  $Y \cup (Z \setminus Z_A)$  and some of the vertices in  $R_1(Y)$ .

Now  $k$  vertices are necessary to dominate  $Z_A$ . For each component  $A_i \subset Z_A$  we do as follows. If there exists  $r \in R_1$  which is complete to  $A_i$  we take  $r$  into the  $\gamma$ -set (case *a*), else we take one universal vertex of  $A_i$  (case *b*) (recall that in both cases



these  $r$  have the same neighbors in  $S$ ).

These  $k$  vertices dominate  $Z_A \cup H_1 \cup \{v_1, v_2\}$  if at least one vertex is chosen in the case  $a$ , else they dominate  $Z_A$ .

Case where at least one vertex is chosen with the case  $a$ :  $v_3, v_4, v_5$  are not dominated with the  $q + k$  already chosen vertices ( $H_1$  is complete thus  $r \in R_1$  dominates  $H_1 \cup \{v_1, v_2\}$ ). So a dominating set of  $G$  has size at least  $q + k + 1$ . Adding the two vertices  $v_3$  and  $v_5$ , we have a dominating set (not necessarily minimum). Checking if there exists a vertex  $v \in C \cup S$ , that is universal to the remaining non-dominated vertices, can be done in polynomial-time.

Case where all the vertices are chosen with the case  $b$ : it remains to dominate  $C_5$  and some vertices of  $S_2 \cup S_3 \cup S_4$ . So a dominating set of  $G$  has a size at least  $q + k + 1$ . Adding the three vertices  $v_1, v_3, v_5$ , we have a dominating set (not necessarily minimum). If there exists a vertex  $v \in S_5$  that is universal to the remaining non-dominated vertices we take it. If no such vertex exists, checking for all the pairs  $\{v, v'\} \subset N[C_5]$ , one can verify if there exists a  $\gamma$ -set with  $q + k + 2$  vertices (note that there are at most  $O(n^2)$  of such pairs).  $\square$

**Lemma 5.6** *Let  $G$  be a connected ( $claw, C_5, C_6, C_7, C_8, P_8$ )-free graph. Computing a minimum dominating set is polynomial-time solvable.*

**Proof:** From Lemma 5.4 we can assume that  $P_7 \subseteq_i G$ . Let  $P = v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7$ .

Let  $W = V \setminus N[P]$ . From Property 2.3 if  $W = \emptyset$  computing a minimum dominating set is polynomial. From now on  $W \neq \emptyset$ .

Let  $S = \{v \in V \setminus P \text{ such that } 2 \leq |N(v) \cap P| \leq 4\}$ ,  $S_i \subseteq S$  being the set of vertices  $v$  such that  $|N(v) \cap P| = i$ .

Let  $H_i = \{v \in S_2 : N(v) \cap P = \{v_i, v_{i+1}\}, 1 \leq i \leq 6\}$ . Since  $G$  is claw-free each  $H_i$  is complete. If there is an edge  $r_i r_{i+1}$  with  $r_i \in H_i, r_{i+1} \in H_{i+1}$  then  $P = v_1 - \dots - v_i - r_i - r_{i+1} - v_{i+2} - \dots - v_7 = P_8$ . If there is an edge  $r_i r_j$  with  $r_i \in H_i, r_j \in H_j$  and  $j \geq i + 3$  then  $C_p \subseteq_i G$ ,  $p \geq 5$ . So  $H_1$  is anticomplete to  $H_2, H_4, H_5, H_6$ , and  $H_2$  is anticomplete to  $H_3, H_5, H_6$ , and  $H_3$  is anticomplete to  $H_4, H_6$ .

We define  $R_i$  as the set of vertices of  $H_i$  having a neighbor in  $W$ ,  $R_i = \{v \in H_i : N(v) \cap W \neq \emptyset\}, 1 \leq i \leq 6$ . Since  $G$  is  $P_8$ -free  $R_1 = R_6 = \emptyset$ .

Let  $w \in W$ . We show that there cannot exist three indices  $1 \leq i < j < k \leq 6$  such that  $w$  has three neighbors  $r_i \in R_i, r_j \in R_j, r_k \in R_k$ . Suppose for contradiction that these three neighbors of  $w$  exist. Since  $R_1 = R_6 = \emptyset$  then  $2 \leq i < j < k \leq 5$ . Since  $G$  is claw-free and  $H_p$  is anticomplete to  $H_{p+1}$ , these three indices cannot be successive. So w.l.o.g. we can assume that  $i = 2, j = 4, k = 5$ . Now  $H_2$  is anticomplete to  $H_5$ , but  $v_3 - r_2 - w - r_5 - v_5 - v_4 - v_3 = C_6$ . Hence for any  $w \in W$  there is at most two neighbors  $r_i, r_j$  such that  $r_i \in R_i, r_j \in R_j, i \neq j$ .

If  $w$  has two neighbors  $r_i \in R_i, r_j \in R_j, i < j$ , then either  $r_i \in R_2, r_j \in R_4$  or  $r_i \in R_3, r_j \in R_5$  (recall that  $H_i$  is anticomplete to  $H_{i+1}$ ,  $H_p, p \geq i + 3$  and  $R_1 = R_6 = \emptyset$ ).

If  $w$  has two neighbors  $r_i \in R_2, r_j \in R_4$ , respectively  $r_i \in R_3, r_j \in R_5$ , then  $r_i r_j \in E$ , else  $w - r_j - v_4 - v_3 - r_i - w = C_5 \subseteq_i G$ , respectively  $w - r_j - v_5 - v_4 - r_i - w = C_5 \subseteq_i G$ . Let  $Z_{24} = \{w \in W : w \text{ has two neighbors } r_2 \in R_2, r_4 \in R_4\}$  and  $Z_{35} = \{w \in W : w \text{ has two neighbors } r_3 \in R_3, r_5 \in R_5\}$ .

We show that  $Z_{24}$  is anticomplete to  $Z_{35}$ . For contradiction we suppose that there are  $w_1 \in Z_{24}, w_2 \in Z_{35}$  with  $w_1 w_2 \in E$ . Let  $r_1 \in R_2$  be a neighbor of  $w_1$  and  $r_2 \in R_5$  be a neighbor of  $w_2$ . Since  $r_1 r_2 \notin E$  we have  $w_1 - r_1 - v_3 - v_4 - v_5 - v_6 - r_2 - w_2 - w_1 = C_8$ . We show that we can assume that  $Z_{24}$  and  $Z_{35}$  are two independent sets. The two sets being symmetric we show that  $Z_{24}$  is an independent set. For contradiction we assume that there are  $w_1, w_2 \in Z_{24}$  such that  $w_1 w_2 \in E$ . We prove that  $N_{R_2}(w_1) = N_{R_2}(w_2)$ . If  $N_{R_2}(w_1) \neq N_{R_2}(w_2)$  then there exist  $r_2 \in R_2$  which is a neighbor of  $w_1$  but not a neighbor of  $w_2$ . Then  $w_2 - w_1 - r_2 - v_3 - \dots - v_7 = P_8$ . We prove that  $N_{R_4}(w_1) = N_{R_4}(w_2)$ . If  $N_{R_4}(w_1) \neq N_{R_4}(w_2)$  then there exists  $r_4 \in R_4$  which is a neighbor of  $w_1$  but not a neighbor of  $w_2$ . There exists  $r_2 \in R_2$  a neighbor of  $w_1$  and  $w_2$ . We know that  $r_2 r_4 \in E$ . It follows that  $G[\{v_2, r_2, r_4, w_2\}]$  is a claw. Hence  $N_{R_2}(w_1) = N_{R_2}(w_2)$  and  $N_{R_4}(w_1) = N_{R_4}(w_2)$ . From Property 2.1 we can assume that there exists  $s \notin R_2 \cup R_4$  such that  $s$  is a neighbor of  $w_1$  but not a neighbor of  $w_2$ . Let  $r_2 \in R_2$  a neighbor of  $w_1$  and  $w_2$ . If  $s r_2 \notin E$  then  $s - w_1 - r_2 - v_3 - \dots - v_7 = P_8$ . When  $s r_2 \in E$  then  $G[\{v_2, r_2, s, w_2\}]$  is a claw. Hence  $Z_{24}$  is an independent and by symmetry  $Z_{35}$  is also independent.

Moreover, since  $G$  is claw-free for any two distinct  $w, w' \in Z_{24} \cup Z_{35}$  we have  $N(w) \cap N(w') = \emptyset$ .

For any two distinct  $w, w' \in Z_{24}$ , respectively  $w, w' \in Z_{35}$  we have that  $N_{R_2}(w)$  is anticomplete to  $N_{R_4}(w')$  and  $N_{R_4}(w)$  is anticomplete to  $N_{R_2}(w')$ , respectively  $N_{R_3}(w)$  is anticomplete to  $N_{R_5}(w')$  and  $N_{R_5}(w)$  is anticomplete to  $N_{R_3}(w')$ . For contradiction we assume that  $w$  has a neighbor  $r_2 \in R_2$ ,  $w'$  has a neighbor  $r_4 \in R_4$ , and  $r_2 r_4 \in E$ . Then  $G[\{v_2, r_2, w, r_4\}]$  is a claw.

Let  $Z_i = \{w \in W : w \text{ has a neighbor in } R_i \setminus (N_{R_i}(Z_{24} \cup Z_{35}))\}, 2 \leq i \leq 5$ .

We show that  $Z_2, Z_3, Z_4, Z_5$  are pairwise anticomplete. If there is an edge  $w_2 w_4, w_2 \in Z_2, w_4 \in Z_4$ , with  $r'_2 \in R_2, r'_4 \in R_4$  the neighbors of  $w_2, w_4$  respectively, then  $w_2 - r'_2 - v_3 - v_4 - r'_4 - w_4 - w_2 = C_6$  ( $r'_2 r'_4 \notin E$  else  $G[\{v_2, r'_2, w_2, r'_4\}]$  is a claw). By symmetry there is no edges between  $Z_3, Z_5$ . If there is an edge  $w_2 w_5, w_2 \in Z_2, w_5 \in Z_5$ , with  $r'_2 \in R_2, r'_5 \in R_5$  the neighbors of  $w_2, w_5$  respectively, then  $w_2 - r'_2 - v_3 - v_4 - v_5 - r'_5 - w_5 - w_2 = C_7$  (remember  $r'_2 r'_5 \notin E$ ). If there is an edge  $w_4 w_5, w_4 \in Z_4, w_5 \in Z_5$ , with  $r'_4 \in R_4, r'_5 \in R_5$  the neighbors of  $w_4, w_5$  respectively, then  $w_4 - r'_4 - v_5 - r'_5 - w_5 - w_4 = C_5$  (recall  $r'_4 r'_5 \notin E$ ). By symmetry there is no edges between  $Z_2, Z_3$ .

Let  $Y = W \setminus (Z_2 \cup Z_3 \cup Z_4 \cup Z_5 \cup Z_{24} \cup Z_{35})$ . One can observe that for any  $w \in Y$  we have  $N_{Z_2}(w) = N_{Z_5}(w) = N_{Z_{24}}(w) = N_{Z_{35}}(w) = \emptyset$  else  $P_8 \subseteq_i G$ . Now, if  $w \in Y$  has two neighbors  $w_3 \in Z_3, w_4 \in Z_4$  then  $C_6 \subseteq_i G$ .

Hence  $Y = Y_3 \cup Y_4$  with  $Y_3 \cap Y_4 = \emptyset, Y_3 = \{w \in Y : w \text{ has a neighbor in } Z_3\}, Y_4 = \{w \in Y : w \text{ has a neighbor in } Z_4\}$ . Moreover  $Y_3$  is anticomplete to  $Y_4$  else  $C_7 \subseteq_i G$ .

We show that we can assume that  $Z_2, Z_5, Y_3, Y_4$  are four independent sets. The arguments are the same for the four sets, so we show that  $Z_2$  is an independent set.

For contradiction we assume that there are  $w_1, w_2 \in Z_2$  such that  $w_1 w_2 \in E$ . We prove that  $N_{R_2}(w_1) = N_{R_2}(w_2)$ . If  $N_{R_2}(w_1) \neq N_{R_2}(w_2)$  then there exists  $r_2 \in R_2$  which is a neighbor of  $w_1$  but not a neighbor of  $w_2$ . Then  $w_2 - w_1 - r_2 - v_3 - \dots - v_7 = P_8$ . Since  $N(w_1), N(w_2) \subseteq Z_2 \cup R_2$  the result is obtained from Property 2.1. Hence  $Z_2, Z_5, Y_3, Y_4$  are independent.

Since  $G$  is claw-free then for any two distinct vertices  $w_1, w_2 \in Z_2 \cup Z_5 \cup Y_3 \cup Y_4$  we have  $N(w_1) \cap N(w_2) = \emptyset$ .

We prove that for any  $w \in Y_3 \cup Y_4$ ,  $N(w)$  is a clique. The two cases being symmetric, let  $w \in Y_4$ . Suppose there are  $s, s'$  two non adjacent vertices in  $N(w)$ . Since  $G$  is claw-free  $s, s'$  cannot have a common neighbor in  $R_4$ . Let  $r \in R_4$  be a neighbor of  $s$ . Then  $s' - w - s - r - v_4 - v_3 - v_2 - v_1 = P_8$ .

Since  $G$  is claw-free, if there are a vertex  $r \in R_i$  with a neighbor  $z \in Z_i$  and a vertex  $s \in S$  such as  $sz \notin E$  and  $v_i \notin N(s)$  then  $G$  contains a claw (note that  $v_{i+1} \notin N(s)$  is symmetric). Hence  $N(Z_i)$  is anticomplete to  $H_j$ ,  $j \neq i$ .

We show that we can assume that  $Z_2 = Z_5 = \emptyset$ . The arguments are the same in the two cases, so we consider  $Z_2$ . Let  $r, r' \in R_2$  be two neighbors of  $w \in Z_2$ . From previous arguments,  $N(w)$  is complete to  $H_2$  but anticomplete to  $H_1, H_3, H_4, H_5, H_6$ . Hence, it remains the case where  $N_{S_3 \cup S_4}(r) \neq N_{S_3 \cup S_4}(r')$ . Suppose that  $N_{S_3 \cup S_4}(r) \neq N_{S_3 \cup S_4}(r')$ . Let  $s \in S_3 \cup S_4$  such as  $rs \in E$  but  $r's \notin E$ . If  $\{v_2, v_3\} \not\subseteq N_P(s)$  then  $G[\{r, s, v_2, w\}]$  or  $G[\{r, s, v_3, w\}]$  is a claw. So  $\{v_2, v_3\} \subset N_P(s)$ . If  $v_1 \notin N_P(s)$ , respectively  $v_4 \notin N_P(s)$ , then  $G[\{r', s, v_1, v_2\}]$ , respectively  $G[\{r', s, v_3, v_4\}]$ , is a claw. Hence  $N_P(s) = \{v_1, v_2, v_3, v_4\}$  but  $G[\{r, s, v_1, v_4\}]$  is a claw. Hence,  $N[r] = N[r']$ . From Property 2.1 we can contract  $rr'$ . Then  $w$  is a leaf and by Property 2.2,  $w$  can be deleted from  $G$ .

Now we study the structures of  $Z_3$  and  $Z_4$ . The two cases being symmetric we deal with  $Z_4$ . For any distinct vertices  $w_1, w_2 \in Z_4$  such that  $w_1 w_2 \in E$ , then there cannot exist two distinct vertices  $w'_1, w'_2 \in Z_4$  such that  $w'_1 w_1 \in E, w'_1 w_2 \notin E$  and  $w'_2 w_2 \in E, w'_2 w_1 \notin E$ . For contradiction we suppose that such two vertices exist. Firstly, we suppose that  $w_1, w_2$  have two distinct neighbors  $r_1, r_2 \in R_4$ , respectively. If  $w'_1 r_2 \notin E$  then  $v_1 - v_2 - v_3 - v_4 - r_2 - w_2 - w_1 - w'_1 = P_8$ . If  $w'_1 r_2 \in E$  then  $G[\{v_4, r_2, w_2, w'_1\}]$  is a claw. Secondly, w.l.o.g.,  $r_1 \in R_4$  is a common neighbor of  $w_1, w_2$  and  $r_2 \in R_4$  is a neighbor of  $w_2$  but not  $w_1$ . If  $w'_1 r_2 \notin E$  then  $v_1 - v_2 - v_3 - v_4 - r_2 - w_2 - w_1 - w'_1 = P_8$  else  $G[\{v_4, r_2, w_2, w'_1\}]$  is a claw. Finally,  $r_1, r_2 \in R_4$  are two common neighbors of  $w_1, w_2$  ( $r_1, r_2$  are not necessarily distinct). If, w.l.o.g.,  $w'_1 r_1 \in E$  then  $G[\{v_4, r_1, w_2, w'_1\}]$  is a claw. In the case where  $w'_1 r_1, w'_1 r_2 \notin E$  then  $w'_1$  has a neighbor  $r'_1 \in R_4, r'_1 \neq r_1, r_2$ . If  $r'_1 w_2 \in E$  then  $G[\{v_4, r'_1, w_2, w'_1\}]$  is a claw. So  $r'_1 w_2 \notin E$ . If  $r'_1 w_1 \notin E$  then  $v_1 - v_2 - v_3 - v_4 - r'_1 - w'_1 - w_1 - w_2 = P_8$ . Thus  $r'_1 w_1 \in E$ . If  $r'_1 w'_2 \notin E$  then  $v_1 - v_2 - v_3 - v_4 - r'_1 - w_1 - w_2 - w'_2 = P_8$ . So  $r'_1 w'_2 \in E$  but  $G[\{v_4, r'_1, w_1, w'_2\}]$  is a claw.

As a consequence each connected component  $A_i$  of  $Z_3 \cup Z_4$  has a universal vertex. Also,  $G$  being claw-free two distinct components cannot share a neighbor in  $R_3 \cup R_4$ . Moreover, from Property 2.2 we have assumed that each  $w_4 \in Z_3 \cup Z_4$  is not a leaf.

We show that  $w \in Y_3 \cup Y_4$  is connected to a universal vertex of a connected compo-

ment  $A_i$  of  $Z_3 \cup Z_4$ . The two cases being symmetric, we deal with  $Z_4$ . Let  $w \in Z_4$ . We assume that the neighbors of  $w$  are not universal in  $A_i$ . Let  $s \in A_i$  be a neighbor of  $w$ , let  $u, u \neq s$ , be a universal vertex of  $A_i$ . Since  $s$  is not universal there exists  $v, v \in A_i$  such that  $sv \notin E$  and  $uv \in E$ . Since  $N(w)$  is complete  $wv \notin E$ . Let  $r \in R_4$  be a neighbor of  $s$ . Since  $G$  is claw-free then  $rv \notin E$ . Let  $r', r' \in R_4, r' \neq r$ , be a neighbor of  $v$ . As just above  $r's \notin E$ . If  $r'u \notin E$  then  $v_1 - v_2 - v_3 - v_4 - r' - v - u - s = P_8$  else  $v_1 - v_2 - v_3 - v_4 - r' - u - s - w = P_8$ .

We are ready to show how to build a  $\gamma$ -set in polynomial time.

First, we treat the case where  $Z_{24} \neq \emptyset$  (the case  $Z_{35} \neq \emptyset$  is the same). Let  $r_2 \in R_2, r_4 \in R_4$  be the two neighbors of  $w, w \in Z_{24}$ . Recall that  $N(Z_{24}) \subseteq R_2 \cup R_4$ .

We show that  $R_3 = \emptyset$ . Assume that there exists  $w' \in W$  with a neighbor  $r_3 \in H_3$  (that is  $R_3 \neq \emptyset$ ).  $w'$  cannot be a neighbor of  $r_2$  or  $r_4$ . Then  $w' - r_3 - v_3 - r_2 - r_4 - v_5 - v_6 - v_7 = P_8$ .

An immediate consequence is that  $Z_{35} = \emptyset$ .

There is no vertex  $w', w' \in W$ , with  $r_2$  as a neighbor: else  $G[\{v_2, r_2, r_4, w'\}]$  is a claw. By symmetry, there is no vertex  $w', w' \in W$ , with  $r_4$  as a neighbor.

Let  $r_2 \in R_2, r'_2 \in R_2, r_2 \neq r'_2$  be such that  $r_2$ , respectively  $r'_2$ , has a neighbor  $w \in Z_{24}$ , respectively  $w' \in Z_{24}$ . Let  $r_4 \in R_4, r'_4 \in R_4, r_4 \neq r'_4$  be such that  $r_4$ , respectively  $r'_4$ , has  $w$ , respectively  $w'$ , as neighbor. We show that  $N_{S \setminus H_4}(r_2) = N_{S \setminus H_4}(r'_2)$ , respectively  $N_{S \setminus H_2}(r_4) = N_{S \setminus H_2}(r'_4)$ .

Let  $i = 2$  (the case  $i = 4$  is symmetric). By contradiction, we assume that there exists  $s \in S \setminus H_4$  such that  $r_2s \in E, r'_2s \notin E$ . From above  $s \notin S_2$ . So  $s \in S_3 \cup S_4$ . If  $N_P(s) = \{v_1, v_2, v_3\}$  then  $G[\{r'_2, v_3, v_4, s\}]$  is a claw. When  $N_P(s) = \{v_2, v_3, v_4\}$  then  $G[\{r'_2, v_1, v_2, s\}]$  is a claw. When  $N_P(s) = \{v_j, v_{j+1}, v_{j+2}\}, j \geq 3$ , then  $G[\{r_2, w, v_2, s\}]$  is a claw. Thus  $s \in S_4$ . When  $N_P(s) = \{v_1, v_2, v_3, v_4\}$  then  $G[\{r_2, v_1, v_4, s\}]$  is a claw. When  $N_P(s) = \{v_1, v_2, v_j, v_{j+1}\}, j \geq 4$ , then  $G[\{r_2, v_1, v_j, s\}]$  is a claw. When  $N_P(s) = \{v_2, v_3, v_j, v_{j+1}\}, j \geq 4$ , then  $G[\{r'_2, v_1, v_2, s\}]$  is a claw. When  $N_P(s) = \{v_j, v_{j+1}, v_{j'}, v_{j'+1}\}, j \geq 3, j' \geq j + 2$  then  $G[\{r_2, w, v_2, s\}]$  is a claw. Hence  $N_{S \setminus H_4}(r_2) = N_{S \setminus H_4}(r'_2)$  and  $N_{S \setminus H_2}(r_4) = N_{S \setminus H_2}(r'_4)$ .

Let  $r_4 \in R_4, r'_4 \in R_4, r_4 \neq r'_4$  be such that  $r_4$ , respectively  $r'_4$ , has a neighbor  $w \in Z_4$ , respectively  $w' \in Z_4$ . We show that  $N_S(r_4) = N_S(r'_4)$ .

By contradiction, we assume that there exists  $s \in S$  such that  $r_4s \in E, r'_4s \notin E$ . From above  $s \notin H_1 \cup H_3 \cup H_4 \cup H_5$ . So  $s \in H_2 \cup H_6 \cup S_3 \cup S_4$ . If  $s \in H_2$  or  $s \in H_6$  then  $G[\{v_4, w, r_4, s\}]$  is a claw. So  $s \in S_3 \cup S_4$ . If  $N_P(s) = \{v_1, v_2, v_3\}$  then  $G[\{v_1, v_3, r_4, s\}]$  is a claw. If  $N_P(s) = \{v_2, v_3, v_4\}$  then  $G[\{v_5, w, r_4, s\}]$  is a claw. If  $N_P(s) = \{v_3, v_4, v_5\}$  then  $G[\{v_5, v_6, r'_4, s\}]$  is a claw. If  $N_P(s) = \{v_4, v_5, v_6\}$  then  $G[\{v_3, v_4, r'_4, s\}]$  is a claw. When  $N_P(s) = \{v_5, v_6, v_7\}$  then  $G[\{v_4, w, r_4, s\}]$  is a claw. Thus  $s \in S_4$ . When  $N_P(s) = \{v_1, v_2, v_3, v_4\}$  then  $G[\{r_4, v_1, v_3, s\}]$  is a claw. When  $N_P(s) = \{v_2, v_3, v_4, v_5\}$  then  $G[\{r'_4, v_5, v_6, s\}]$  is a claw. When  $N_P(s) = \{v_3, v_4, v_5, v_6\}$  then  $G[\{r_4, v_3, v_6, s\}]$  is a claw. When  $N_P(s) = \{v_4, v_5, v_6, v_7\}$  then  $G[\{r'_4, v_3, v_4, s\}]$  is a claw. When  $N_P(s) = \{v_1, v_2, v_6, v_7\}$  or  $N_P(s) = \{v_2, v_3, v_6, v_7\}$  then  $G[\{r_4, v_2, v_6, s\}]$  is a claw. When  $N_P(s) = \{v_3, v_4, v_6, v_7\}$  then  $G[\{r_4, v_3, v_6, s\}]$  is a claw. When  $N_P(s) = \{v_1, v_2, v_5, v_6\}$  or  $N_P(s) = \{v_2, v_3, v_5, v_6\}$  then  $G[\{r_4, v_2, v_6, s\}]$  is a claw. When  $N_P(s) = \{v_1, v_2, v_4, v_5\}$  then  $G[\{r'_4, v_5, v_6, s\}]$  is a claw. Hence

$N_S(r_4) = N_S(r'_4)$ . By symmetry, for  $r_2 \in R_2, r'_2 \in R_2, r_2 \neq r'_2$  such that  $r_2$ , respectively  $r'_2$ , has a neighbor  $w \in Z_2$ , respectively  $w' \in Z_2$  we have  $N_S(r_2) = N_S(r'_2)$ .

The  $\gamma$ -set is build as follows:

- $|Z_{24}| \geq 2$ . We take  $r_2 \in R_2$  a neighbor of  $w$ , and for each other  $w' \in Z_{24}$  we take one adjacent vertex  $r'_4 \in R_4$ . For each  $w' \in Y_4$  we take one universal vertex in the connected component  $A_i$  of  $Z_4$  connected to  $w'$ . For each connected component  $A_i$  of  $Z_4$  that is not connected with  $Y_4$ , we take one universal vertex of  $A_i$ . These vertices dominate  $Z_{24} \cup Y_4 \cup Z_4 \cup H_2 \cup H_4 \cup \{v_2, v_3, v_4, v_5\}$ . Since  $v_1, v_7$  have no common neighbor at least two more vertices are needed. Adding the three vertices  $v_2, v_4, v_6$  we have a dominating set (not necessarily minimum). Checking for all the pairs  $s_1, s_7$  where  $s_i$  is a neighbor of  $v_i, i \in \{1, 7\}$ , one can verify if there is a  $\gamma$ -set with only two more vertices (note that there are at most  $O(n^2)$  such pairs).
- $|Z_{24}| = 1$ . For each  $w' \in Y_4$  we take one universal vertex in the connected component  $A_i$  of  $Z_4$  connected to  $w'$ . If there exists a vertex  $r \in R_4$  complete to a component  $A_i$  of  $Z_4$  that is not connected to  $Y_4$  then we take  $r$ . For each remaining component  $A_i$  of  $Z_4$  that is not connected to  $Y_4$ , we take one universal vertex of  $A_i$ . These vertices dominate  $Y_4 \cup Z_4$  (note that  $H_2, H_4$  are not necessarily dominated). Since  $v_1, v_7, w$  have no common neighbor at least three more vertices are needed. Adding the four vertices  $v_2, v_4, v_6, w$  we have a dominating set (not necessarily minimum). Checking for all the pairs  $s_1, s_7$  where  $s_i$  is a neighbor of  $v_i, i \in \{1, 7\}$ , if there is a dominating set by adding  $s_1, s_7, r_4$  or  $s_1, s_7, r_2$ , one can verify if there is a  $\gamma$ -set with only three more vertices (note that there are at most  $O(n^2)$  such pairs).

In the case of  $Z_{24} = Z_{35} = \emptyset$ , we build the  $\gamma$ -set as follows:

- $Y_3, Y_4 \neq \emptyset$ . For each  $w \in Y_3 \cup Y_4$  we take one universal vertex in the connected component  $A_i$  of  $Z_3 \cup Z_4$  connected to  $w$ . If there exists  $r_4 \in R_4$  which is complete to a component  $A_i$  of  $Z_4$  that is not connected to  $Y_3 \cup Y_4$  then, we take  $r_4$ . We do the same for the component of  $Z_3$  with no neighbors in  $Y_3$ . For each remaining connected component  $A_i$  of  $Z_3 \cup Z_4$  that is not connected to  $Y_3 \cup Y_4$ , we take one universal vertex of  $A_i$ . These vertices dominate  $Y_3 \cup Z_3 \cup Y_4 \cup Z_4$  (note that  $H_2, H_4$  are not necessarily dominated). Since  $v_1, v_7$  have no common neighbor at least two more vertices are needed. Adding the three vertices  $v_2, v_4, v_6$  we have a dominating set (not necessarily minimum). Checking for all the pairs  $s_1, s_7$  where  $s_i$  is a neighbor of  $v_i, i \in \{1, 7\}$ , one can verify if there is a  $\gamma$ -set with only two more vertices (note that there are at most  $O(n^2)$  such pairs).
- $Y_3 \neq \emptyset, Y_4 = \emptyset$  or  $Y_4 \neq \emptyset, Y_3 = \emptyset$ . The two cases being symmetric, let  $Y_4 = \emptyset$ .
  - $Z_4 \neq \emptyset$ . For each  $w \in Y_3$  we take one universal vertex in the connected component  $A_i$  of  $Z_3$  connected to  $w$ . If there exists  $r_4 \in R_4$  which is complete to  $A_i$ , a connected component of  $Z_4$ , then we take  $r_4$ . If there exists  $r_3 \in R_3$  which is complete to a connected component  $A_j$  of  $Z_3$  with

no neighbors in  $Y_3$ , then we take  $r_3$ . Now, we take one universal vertex for each other component  $A_l, A_l \neq A_i, A_j$ , of  $Z_3 \cup Z_4$ . These vertices dominate  $Y_3 \cup Z_3 \cup Z_4$ . Since  $v_1, v_7$  have no common neighbor at least two more vertices are needed. Adding the three vertices  $v_2, v_4, v_6$  we have a dominating set (not necessarily minimum). Checking for all the pairs  $s_1, s_7$  where  $s_i$  is a neighbor of  $v_i, i \in \{1, 7\}$ , one can verify if there is a  $\gamma$ -set with only two more vertices (note that there are at most  $O(n^2)$  such pairs).

- $Z_4 = \emptyset$ . For each  $w \in Y_3$  we take one universal vertex in the connected component  $A_i$  of  $Z_3$  connected to  $w$ . If there exists  $r_3 \in R_3$  which is complete to a connected component  $A_i$  of  $Z_3$  with no neighbors in  $Y_3$ , then we take  $r_3$ . Now, we take one universal vertex for each other component  $A_i$  of  $Z_3$ . Adding the vertices  $v_2, v_4, v_6$  we have a dominating set (not necessarily minimum). Checking for all the pairs  $s_1, s_7$  where  $s_i$  is a neighbor of  $v_i, i \in \{1, 7\}$ , one can verify if there is a  $\gamma$ -set with only two more vertices.

- $Y_3, Y_4 = \emptyset$ .

- $Z_3, Z_4 \neq \emptyset$ . If there exists  $r_4 \in R_4$ , respectively  $r_3 \in R_3$ , which is complete to  $A_i$ , a connected component of  $Z_4$ , respectively  $Z_3$ , then we take  $r_4$ , respectively  $r_3$ . For each remaining component of  $Z_3 \cup Z_4$  we take one universal vertex. Adding the vertices  $v_2, v_4, v_6$  we have a dominating set (not necessarily minimum). Checking for all the pairs  $s_1, s_7$  where  $s_i$  is a neighbor of  $v_i, i \in \{1, 7\}$ , one can verify if there is a  $\gamma$ -set with only two more vertices.
- $Z_3 \neq \emptyset, Z_4 = \emptyset$  or  $Z_4 \neq \emptyset, Z_3 = \emptyset$ . Let  $Z_3 \neq \emptyset$ . If there exists  $r_3 \in R_3$  which is complete to a connected component of  $Z_3$ , then we take  $r_3$ . We add one universal vertex for each remaining component of  $Z_3$ . Now, adding the vertices  $v_2, v_4, v_6$  we have a dominating set (not necessarily minimum). Checking for all the pairs  $s_1, s_7$  where  $s_i$  is a neighbor of  $v_i, i \in \{1, 7\}$ , one can verify if there is a  $\gamma$ -set with only two more vertices.
- $Z_3 = Z_4 = \emptyset$ . Then  $V = N[C]$  and from Property 2.3 computing a minimum dominating set is polynomial.

□

From Lemmas 3.1, 3.2, 4.3, 5.5, 5.6 we have the following.

**Theorem 5.7** *The Minimum Dominating Set problem is polynomial for  $(\text{claw}, P_8)$ -free graphs.*

## 6 Conclusion

We have shown that the Minimum Dominating Set problem is polynomial for  $(\text{claw}, P_8)$ -free graphs. We left open the following problem: is there a positive integer  $k, k \geq 9$ ,

such that the Minimum Dominating Set problem is  $NP$ -complete for the class of  $(claw, P_k)$ -free graphs? If the the answer is positive, a challenge should be to show a dichotomy: find the minimum integer  $k$  such that the Minimum Dominating Set problem is  $NP$ -complete for  $(claw, P_k)$ -free graphs and polynomial for  $(claw, P_{k-1})$ -free graphs.

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