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OPETOPIC ALGEBRAS III: PRESHEAF MODELS OF HOMOTOPY-COHERENT OPETOPIC ALGEBRAS

CÉDRIC HO THANH AND CHAITANYA LEENA SUBRAMANIAM

ABSTRACT. For Λ the category of free opetopic algebras, we construct a model structure *à la Cisinski* on the category of presheaves over Λ , and show that it is equivalent to opetopic complete Segal spaces. This generalizes the results of Joyal and Tierney in the simplicial case, and of Cisinski and Moerdijk in the planar dendroidal case.

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1. INTRODUCTION

This work is a continuation of [HTLS20a] and [HTLS20b]. In the former, we introduced the notion of opetopic algebra, which can be thought of as algebras whose operations have *higher dimensional arities*. In the latter, we lifted the notion of Segal and complete Segal space from the simplicial and dendroidal setting [Rez01, CM11] to our opetopic framework. In this paper, we provide a presheaf model for complete Segal spaces, namely *opetopic ∞ -algebras*. Much like in the simplicial case, where they are called *quasi-categories* [BV73, Joy08], ∞ -algebras are

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opetopic algebras where laws only hold up to homotopy, instead of strictly. Our strategy to prove this equivalence is largely adapted from that of Joyal and Tierney [JT07].

1.1. Related work. Complete Segal spaces, introduced in [Rez01], are simplicial spaces that have compositions and identities up to homotopy. In [JT07], Joyal and Tierney showed that they can be modeled by quasi-categories, that is, fibrant objects in a suitable model structure on the category $\mathcal{Psh}(\Delta)$ of simplicial sets. Specifically, they show that the discrete space functor is a left Quillen equivalence

$$(-)^{\text{disc}} : \mathcal{Psh}(\Delta)_{\text{Joyal}} \xleftrightarrow{\quad} \mathcal{Sp}(\Delta)_{\text{Rezk}} : (-)_{-,0}$$

where the right adjoint is the “first row” functor.

The notion of complete Segal space has been lifted to the dendroidal setting by Cisinski and Moerdijk in [CM13]. In this setting, they are dendroidal spaces having operadic compositions and identities up to homotopy. In [CM11], they show that ∞ -operads, model complete dendroidal Segal spaces in the same sense as before. Furthermore, since the category of simplicial sets (resp. spaces) can be recovered as a slice of the category of dendroidal sets (resp. spaces), the equivalence in the simplicial case can be recovered as a special case.

1.2. Plan. In section 2, we recall elements of Cisinski’s homotopical machinery from [Cis06], and specify the so-called Joyal–Tierney calculus [JT07] to an arbitrary category of simplicial presheaves. In section 3, we survey some results of [HTLS20a], and construct the folk model structure for opetopic algebras. In section 4, we consider presheaves over \mathbb{A} , the category of free opetopic algebras, we construct the model structure for ∞ -algebras following the methods of Cisinski presented in section 2.1. Lastly, in section 5, we establish a Quillen equivalence between ∞ -algebras and complete Segal spaces of [HTLS20b].

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2. PRELIMINARIES

2.1. Cisinski homotopy theory. We recall some results and constructions of [Cis06].

Definition 2.1.1 (Lifting property). Let \mathcal{C} be a category, and $l, r \in \mathcal{C}^{\rightarrow}$. We say that l has the *left lifting property against* r (equivalently, r has the *right lifting property against* l), written $l \pitchfork r$, if for any solid commutative square as follows, there exists a (non necessarily unique) dashed arrow making the two triangles commute:

$$\begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ l \downarrow & \nearrow \text{---} & \downarrow r \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \quad (2.1.2)$$

If \mathcal{C} has a terminal object 1 , then for all $X \in \mathcal{C}$, we write $l \pitchfork X$ for $l \pitchfork (X \rightarrow 1)$. Let \mathbf{L} and \mathbf{R} be two classes of morphisms of \mathcal{C} . We write $\mathbf{L} \pitchfork \mathbf{R}$ if for all $l \in \mathbf{L}$ and $r \in \mathbf{R}$ we have $l \pitchfork r$. The class of all morphisms r (resp. l) such that $\mathbf{L} \pitchfork r$ (resp. $l \pitchfork \mathbf{R}$) is denoted \mathbf{L}^{\pitchfork} (resp. ${}^{\pitchfork}\mathbf{R}$).

Definition 2.1.3 (Orthogonality). We say that l is *left orthogonal to* r (equivalently, r is *right orthogonal to* l), written $l \perp r$, if for any solid commutative square as in equation (2.1.2), there exists a *unique* dashed arrow making the two triangles commute. The relation \perp is also known as the *unique lifting property*. Note that notations of definition 2.1.1 also make sense when \pitchfork is replaced by \perp .

Notation 2.1.4. For \mathcal{C} a small category, let $\mathcal{Psh}(\mathcal{C}) := \text{Cat}(\mathcal{C}^{\text{op}}, \text{Set})$ be the category of Set-presheaves over \mathcal{C} , and $\mathcal{Sp}(\mathcal{C}) := \text{Cat}(\mathcal{C}^{\text{op}}, \mathcal{Psh}(\Delta))$ be the category of simplicial presheaves over \mathcal{C} .

Definition 2.1.5 (Cylinder object, [Cis06, definition 1.3.1]). Let \mathcal{C} be a small category, and $X \in \mathcal{Psh}(\mathcal{C})$ be a presheaf over \mathcal{C} . A *cylinder* of X is a factorization of the fold map

$$\begin{array}{ccc} X + X & \xrightarrow{\quad \nabla \quad} & X \\ & \searrow (i_0, i_1) & \swarrow \nabla \\ & & \mathbf{I}X, \end{array}$$

such that $(i_0, i_1) : X + X \rightarrow IX$ is a monomorphism. We write $X^{(e)}$ for the image of $i_e : X \rightarrow IX$.

Definition 2.1.6 (I-homotopy, [Cis06, definition 1.3.3, remark 1.3.4]). Let \mathcal{C} be a small category, $f, g : X \rightarrow Y$ be two parallel maps in $\mathcal{Psh}(\mathcal{C})$, and IX be a cylinder of X (definition 2.1.5). An *elementary I-homotopy* from f to g is a morphism $H : IA \rightarrow B$ such that the following triangle commutes:

$$\begin{array}{ccc} A + A & & \\ (i_0, i_1) \downarrow & \searrow (f, g) & \\ IA & \xrightarrow{H} & B. \end{array}$$

Let \simeq_1 (or just \simeq if the context is clear), the *I-homotopy relation*, be the equivalence relation spanned by this relation on $\mathcal{Psh}(\mathcal{C})(A, B)$.

One readily checks that \simeq is a congruence on the category $\mathcal{Psh}(\mathcal{C})$, and let $\mathcal{HcoPsh}(\mathcal{C})$ be the quotient category. A morphism $f : X \rightarrow Y$ is a *I-homotopy equivalence* (or just *homotopy equivalence* if the context is clear) if it is invertible in $\mathcal{HcoPsh}(\mathcal{C})$.

Definition 2.1.7 (Elementary homotopical data, [Cis06, definition 1.3.6]). Let \mathcal{C} be a small category. An *elementary homotopical data* on $\mathcal{Psh}(\mathcal{C})$ is a functorial cylinder $I : \mathcal{Psh}(\mathcal{C}) \rightarrow \mathcal{Psh}(\mathcal{C})$ (definition 2.1.5) such that

- (1) **(DH1)** I preserves small colimits and monomorphisms;
- (2) **(DH2)** for all monomorphism $f : X \rightarrow Y$ in $\mathcal{Psh}(\mathcal{C})$, and for $e = 0, 1$, the following square is a pullback:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_e \downarrow & & \downarrow i_e \\ IX & \xrightarrow{If} & IY. \end{array}$$

Definition 2.1.8 (Anodyne extension). Let \mathcal{C} be a small category, and $I : \mathcal{Psh}(\mathcal{C}) \rightarrow \mathcal{Psh}(\mathcal{C})$ be an elementary homotopical data on $\mathcal{Psh}(\mathcal{C})$ (definition 2.1.7). A *class of anodyne extensions* relative to I is a class $\mathbf{An} \subseteq \mathcal{Psh}(\mathcal{C})^{\rightarrow}$ such that

- (1) **(An0)** there exists a set $A \subseteq \mathcal{Psh}(\mathcal{C})^{\rightarrow}$ of monomorphisms such that $\mathbf{An} = \mathfrak{h}(A^{\mathfrak{h}})$ (definition 2.1.1);
- (2) **(An1)** for all monomorphism $m : X \hookrightarrow Y$ in \mathbf{An} , and $e = 0, 1$, the cocartesian gap map g is in \mathbf{An} :

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ i_e \downarrow & \lrcorner & \downarrow i_e \\ IX & \hookrightarrow & IX \cup Y^{(e)} \\ & & \dashrightarrow g \\ & & IY; \end{array}$$

- (3) **(An2)** for all $m : X \hookrightarrow Y$ in \mathbf{An} , the cocartesian gap map g is in \mathbf{An} :

$$\begin{array}{ccc} X + X & \xrightarrow{m+m} & Y + Y \\ \downarrow & \lrcorner & \downarrow \\ IX & \hookrightarrow & IX \cup (Y + Y) \\ & & \dashrightarrow g \\ & & IY. \end{array}$$

Definition 2.1.9 (Homotopical structure, [Cis06, definition 1.3.14]). Let \mathcal{C} be a small category. A *homotopical structure* on $\mathcal{Psh}(\mathcal{C})$ is a pair (I, \mathbf{An}) , where I is a functorial cylinder on $\mathcal{Psh}(\mathcal{C})$ (definition 2.1.5), and \mathbf{An} is a class of anodyne extension relative to I definition 2.1.8.

Definition 2.1.10 (Cisinski model category). A *Cisinski model category* is a model structure on a presheaf category over a small category, that is cofibrantly generated, and where cofibrations are the monomorphisms.

A notable source of such structures is the following theorem:

Theorem 2.1.11 ([Cis06, definition 1.3.21, theorem 1.3.22]). *Let \mathcal{C} be a small category, and (I, An) be a homotopical structure on $\mathcal{Psh}(\mathcal{C})$ (definition 2.1.9). There is a model structure on $\mathcal{Psh}(\mathcal{C})$ such that:*

- (1) *a morphism f is a naive fibration if $\text{An} \dashv f$; a presheaf $X \in \mathcal{Psh}(\mathcal{A})$ is fibrant if the terminal morphism $X \rightarrow 1$ is a naive fibration;*
- (2) *a morphism $f : X \rightarrow Y$ is a weak equivalence if for all fibrant object $P \in \mathcal{Psh}(\mathcal{A})$, the induced map $f^* : \mathcal{Hco}\mathcal{Psh}(\mathcal{A})(Y, P) \rightarrow \mathcal{Hco}\mathcal{Psh}(\mathcal{A})(X, P)$ is a bijection (definition 2.1.6);*
- (3) *a morphism f is a cofibrations if it is a monomorphisms, it is a acyclic cofibrations if it is a cofibration and a weak equivalence;*
- (4) *a morphism f is a fibration if it has the right lifting property with respect to acyclic cofibrations, it is an acyclic fibration if it has the right lifting property with respect to all cofibrations.*

This model structure is of Cisinski type, cellular, and proper.

Lemma 2.1.12 ([Cis06, proposition 1.3.36]). *Let \mathcal{C} be a small category, and $\mathcal{Psh}(\mathcal{C})$ be endowed with a model structure as in theorem 2.1.11. Let $f : X \rightarrow Y$ be a morphism in $\mathcal{Psh}(\mathcal{C})$, and assume that Y is fibrant. Then f is a fibration if and only if it is a naive fibration.*

Definition 2.1.13 (Skeletal category). A *skeletal category* [Cis06, definition 8.1.1] is a small category \mathcal{C} endowed with a map $\text{deg} : \text{ob } \mathcal{C} \rightarrow \mathbb{N}$ and two wide subcategories \mathcal{C}_+ and \mathcal{C}_- such that the following axioms are satisfied.

- **(Sq0)** (Invariance) Isomorphisms are in \mathcal{C}_+ and \mathcal{C}_- . Furthermore, if $c, c' \in \mathcal{C}$ are isomorphic, then $\text{deg } c = \text{deg } c'$.
- **(Sq1)** (Dimension) If $f : c \rightarrow c'$ is an arrow in \mathcal{C}_+ (resp. \mathcal{C}_-) that is not an isomorphism, then $\text{deg } c < \text{deg } c'$ (resp. $\text{deg } c > \text{deg } c'$).
- **(Sq2)** (Factorization) Every arrow f of \mathcal{C} can be essentially uniquely factored as $f = f_+ f_-$, with $f_+ \in \mathcal{C}_+$ and $f_- \in \mathcal{C}_-$.
- **(Sq3)** (Section) Every arrow in \mathcal{C}_- admits a section. Two arrows $f, f' \in \mathcal{C}_-$ are equal if and only if they have the same sections.

The skeletal category \mathcal{C} is *normal* [Cis06, definition 8.1.36, proposition 8.1.37] if \mathcal{C} is rigid, i.e. does not have non-trivial automorphisms. Note that in this case, \mathcal{C} is a Reedy category [Hir03, definition 15.1.12].

Definition 2.1.14 (Boundary). Let \mathcal{C} be a skeletal category. The *boundary* [Cis06, paragraph 8.1.30] $\partial c \in \mathcal{Psh}(\mathcal{C})$ of an object $c \in \mathcal{C}$ is defined as

$$\partial c := \underset{f \in \mathcal{C}_+, \text{ not an iso}}{\text{colim}}_{f: d \rightarrow c} d.$$

The canonical map $\mathbf{b}_c : \partial c \rightarrow c$ is a monomorphism and is called the *boundary inclusion* of c . Write $\mathbf{B}_{\mathcal{C}}$ (or just \mathbf{B} if the context is clear) for the set of boundary inclusions of \mathcal{C} .

Proposition 2.1.15 ([Cis06, propositions 8.1.35 and 8.1.37]). *Assume \mathcal{C} is a normal skeletal category. Then the class of monomorphisms of $\mathcal{Psh}(\mathcal{C})$ is $\text{Cell}(\mathbf{B}_{\mathcal{C}})$.*

2.2. Joyal–Tierney calculus. Consider the fully faithful functor $i_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathcal{C} \times \mathbb{A}$ mapping $c \in \mathcal{C}$ to the tuple $(c, [0])$. It induces an adjunction

$$(-)^{\text{disc}} : \mathcal{Psh}(\mathcal{C}) \rightleftarrows \text{Sp}(\mathcal{C}) : (-)_{-,0},$$

where $(-)^{\text{disc}}$ is the left Kan extension of the composite $\mathcal{C} \rightarrow \mathcal{C} \times \mathbb{A} \rightarrow \text{Sp}(\mathcal{C})$ along the Yoneda embedding, and where $(-)_{-,0}$ is the precomposition by $i_{\mathcal{C}}$, i.e. the “evaluation at 0”. We call $(-)^{\text{disc}}$ the *discrete space functor*, as for $X \in \mathcal{Psh}(\mathcal{C})$ and $c \in \mathcal{C}$, the space X_c^{disc} is discrete at X_c .

Dually, the projection $p_{\mathbb{A}} : \mathcal{C} \times \mathbb{A} \rightarrow \mathbb{A}$ induces an adjunction

$$(-)^{\text{const}} : \mathcal{Psh}(\mathbb{A}) \rightleftarrows \text{Sp}(\mathcal{C}) : r,$$

where r is the right Kan extension of $\mathcal{C} \times \Delta \rightarrow \Delta \rightarrow \mathcal{Psh}(\Delta)$ along the Yoneda embedding, and $(-)^{\text{const}}$ is the precomposition by p_Δ . We call $(-)^{\text{const}}$ the *constant space functor*, as for $Y \in \mathcal{Psh}(\Delta)$, the functor $Y^{\text{const}} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{Psh}(\Delta)$ is constant at Y . The functor $r : \text{Sp}(\mathcal{C}) \rightarrow \mathcal{Psh}(\Delta)$ provides a simplicial enrichment on $\text{Sp}(\mathcal{C})$ as follows:

$$\text{Map}(X, Y) := r(Y^X).$$

Note that $\text{Map}(X, Y)_n = \text{Sp}(\mathcal{C})(X \times \Delta[n]^{\text{const}}, Y)$.

Proposition 2.2.1 (Generalization of [JT07, proposition 2.4]). *The simplicially enriched category $\text{Sp}(\mathcal{C})$ is tensored and cotensored over $\mathcal{Psh}(\Delta)$: for $K \in \mathcal{Psh}(\Delta)$ and $X \in \text{Sp}(\mathcal{C})$, we define*

$$K \otimes X := K^{\text{const}} \times X, \quad X^K := X^{K^{\text{const}}}.$$

Proof. Let $Y \in \text{Sp}(\mathcal{C})$.

(1) We have

$$\begin{aligned} \text{Map}(K \otimes X, Y) &= \text{Map}(K^{\text{const}} \times X, Y) = \int_{c \in \mathcal{C}} \text{Map}(K \times X_c, Y_c) \cong \int_{c \in \mathcal{C}} \text{Map}(K, \text{Map}(X_c, Y_c)) \\ &\cong \text{Map}(K, \int_{c \in \mathcal{C}} \text{Map}(X_c, Y_c)) = \text{Map}(K, \text{Map}(X, Y)), \end{aligned}$$

naturally in all variables, and thus $- \otimes X$ is an enriched left adjoint to $\text{Map}(X, -)$.

(2) We have

$$\text{Map}(X, Y^K) = \text{Map}(X, Y^{K^{\text{const}}}) \cong \text{Map}(K^{\text{const}} \times X, Y) = \text{Map}(K \otimes X, Y)$$

naturally in all variables, and thus $(-)^K$ is an enriched right adjoint to $K \otimes -$. \square

We now specify the so-called Joyal–Tierney calculus [JT07, section 2] and [RV14, section 4] to the category $\text{Sp}(\mathcal{C})$ of simplicial presheaves over \mathcal{C} . This convenient formalism will be heavily used throughout this work.

Definition 2.2.2 (Box product). The *box product*¹ functor $\boxtimes : \mathcal{Psh}(\mathcal{C}) \times \mathcal{Psh}(\Delta) \rightarrow \text{Sp}(\mathcal{C})$ is defined as

$$(X \boxtimes Y)_{c,n} := X_c \times Y_n,$$

for $X \in \mathcal{Psh}(\mathcal{C})$, $Y \in \mathcal{Psh}(\Delta)$, $c \in \mathcal{C}$, and $[n] \in \Delta$.

This functor is divisible on both sides, meaning that it is left adjoint in each variable. The right adjoint to $X \boxtimes - : \mathcal{Psh}(\Delta) \rightarrow \text{Sp}(\mathcal{C})$ will be denoted by $X \setminus - : \text{Sp}(\mathcal{C}) \rightarrow \mathcal{Psh}(\Delta)$, and the right adjoint to $- \boxtimes Y : \mathcal{Psh}(\mathcal{C}) \rightarrow \text{Sp}(\mathcal{C})$ will be denoted by $- / Y : \text{Sp}(\mathcal{C}) \rightarrow \mathcal{Psh}(\mathcal{C})$. Note that $X \setminus -$ and $- / Y$ are contravariant in X and Y respectively. Consequently, for $W \in \text{Sp}(\mathcal{C})$, the functors $\setminus W : \mathcal{Psh}(\mathcal{C}) \rightarrow \mathcal{Psh}(\Delta)$ and $W / - : \mathcal{Psh}(\Delta) \rightarrow \mathcal{Psh}(\mathcal{C})$ are mutually right adjoint.

Lemma 2.2.3. (1) For $X \in \mathcal{Psh}(\mathcal{C})$, we have $X^{\text{disc}} = X \boxtimes \Delta[0]$. Dually, for $Y \in \mathcal{Psh}(\Delta)$, we have $Y^{\text{const}} = 1 \boxtimes Y$, where 1 is the terminal presheaf in $\mathcal{Psh}(\mathcal{C})$.

(2) Let $W \in \text{Sp}(\mathcal{C})$. For $c \in \mathcal{C}$, we have $c \setminus W = W_c \in \mathcal{Psh}(\Delta)$. Dually, for $[n] \in \Delta$, we have $W / \Delta[n] = W_{-,n}$.

(3) Let h be a morphism in $\text{Sp}(\mathcal{C})$, and $K \in \mathcal{Psh}(\mathcal{C})$. Then $K \setminus h = \langle (\emptyset \hookrightarrow K) \setminus h \rangle$. Similarly, if $L \in \mathcal{Psh}(\Delta)$, then $h / L = \langle h / (\emptyset \hookrightarrow L) \rangle$.

(4) Let $X \in \text{Sp}(\mathcal{C})$, and f be a morphism in $\mathcal{Psh}(\mathcal{C})$. Then $f \setminus X = \langle f \setminus (X \rightarrow 1) \rangle$. Likewise, if g is a morphism in $\mathcal{Psh}(\Delta)$, then $X / g = \langle (X \rightarrow 1) / g \rangle$.

Definition 2.2.4 (Leibniz construction, [RV14, definition 4.4]). Consider a functor $- \otimes - : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$, where \mathcal{C} has pushouts. Its *Leibniz construction* is the functor $- \hat{\otimes} - : \mathcal{A}^\rightarrow \times \mathcal{B}^\rightarrow \rightarrow \mathcal{C}^\rightarrow$ which maps an arrow $f : A_1 \rightarrow A_2$ in \mathcal{A} and $g : B_1 \rightarrow B_2$ in \mathcal{B} to the cocartesian gap map below:

$$\begin{array}{ccc} A_1 \otimes B_1 & \xrightarrow{A_1 \otimes g} & A_1 \otimes B_2 \\ \downarrow f \otimes B_1 & & \downarrow f' \\ A_2 \otimes B_1 & \xrightarrow{g'} \Gamma & P \\ & \searrow f \hat{\otimes} g & \downarrow f \otimes B_2 \\ & & A_2 \otimes B_2 \end{array} \quad P = A_1 \otimes B_2 \coprod_{A_1 \otimes B_1} A_2 \otimes B_1.$$

¹The box product is denoted by \square in [JT07, section 2]. In [RV14, notation 4.1], it is called the *exterior product* and written $\underline{\times}$.

The Leibniz construction $-\hat{\otimes}-$ essentially has the same properties as $-\otimes-$, see [RV14, section 4].

Definition 2.2.5 (Leibniz box product). The *Leibniz box product*² $-\hat{\boxtimes}- : \mathcal{Psh}(\mathcal{C})^\rightarrow \times \mathcal{Psh}(\Delta)^\rightarrow \longrightarrow \mathcal{Sp}(\mathcal{C})^\rightarrow$ is simply the Leibniz construction of definition 2.2.4 applied to the box product of definition 2.2.2. If \mathbf{K} and \mathbf{L} are classes of morphisms of $\mathcal{Psh}(\mathcal{C})$ and $\mathcal{Psh}(\Delta)$ respectively, let $\mathbf{K} \hat{\boxtimes} \mathbf{L} := \{k \hat{\boxtimes} l \mid k \in \mathbf{K}, l \in \mathbf{L}\}$.

Akin to the box product, the Leibniz box product is divisible on both sides. Specifically, if $h : W_1 \longrightarrow W_2$ is a morphism in $\mathcal{Sp}(\mathcal{C})$, let $\langle f \setminus h \rangle$, be the morphism induced by the universal property of the pullback:

$$\begin{array}{ccc}
 X_2 \setminus W_1 & \xrightarrow{f \setminus W_1} & X_1 \setminus W_1 \\
 \downarrow \langle f \setminus h \rangle & \searrow & \downarrow X_1 \setminus h \\
 P & \xrightarrow{\quad} & X_1 \setminus W_1 \\
 \downarrow & \lrcorner & \downarrow X_1 \setminus h \\
 X_2 \setminus W_2 & \xrightarrow{f \setminus W_2} & X_1 \setminus W_2
 \end{array}
 \quad P = (X_1 \setminus W_1) \prod_{X_1 \setminus W_2} (X_2 \setminus W_2).$$

The morphism $\langle f \setminus h \rangle$ is also called the *cartesian gap map* of the square above. Then the functor $\langle f \setminus - \rangle$ is right adjoint to $f \hat{\boxtimes} -$. Dually, let $\langle h / g \rangle$, on the left, be the cartesian gap map of the square on the right

$$\begin{array}{ccc}
 W_1 / Y_2 & \xrightarrow{h / Y_2} & W_2 / Y_2 \\
 \downarrow \langle h / g \rangle & \searrow & \downarrow W_2 / g \\
 P & \xrightarrow{\quad} & W_2 / Y_2 \\
 \downarrow & \lrcorner & \downarrow W_2 / g \\
 W_1 / Y_1 & \xrightarrow{h / Y_1} & W_2 / Y_1
 \end{array}
 \quad P = (W_1 / Y_1) \prod_{W_2 / Y_1} (W_2 / Y_2).$$

Then the functor $\langle - / g \rangle$ is right adjoint to $-\hat{\boxtimes} g$.

2.3. Application to simplicial presheaves. In this section, we lift some technical results of [JT07, section 2] from the settings of simplicial spaces to simplicial presheaves over a normal skeletal category \mathcal{C} (definition 2.1.13). Recall that by proposition 2.1.15, the set of boundary inclusions $\mathbf{B}_{\mathcal{C}} := \{\mathbf{b}_c : \partial c \hookrightarrow c \mid c \in \mathcal{C}\}$ generates the class of monomorphisms of $\mathcal{Psh}(\mathcal{C})$, in the sense that it is $\text{Cell}(\mathbf{B}_{\mathcal{C}})$.

Proposition 2.3.1. *The class of monomorphisms of $\mathcal{Sp}(\mathcal{C})$ is $\text{Cell}(\mathbf{B}_{\mathcal{C}} \hat{\boxtimes} \mathbf{B}_{\Delta})$.*

Proof. Observe that Δ is normal skeletal, and thus so is the product $\mathcal{C} \times \Delta$ in an evident way [Cis06, remark 8.1.7]. In particular, for $(c, [n]) \in \mathcal{C} \times \Delta$, maps $f \in (\mathcal{C} \times \Delta)_+$ with codomain $(c, [n])$ are adequate pairs or morphisms $f = (f_{\mathcal{C}}, f_{\Delta}) \in \mathcal{C}_+ \times \Delta_+$, and f is not an isomorphism if and only if $f_{\mathcal{C}}$ or f_{Δ} is not³. Thus it is easy to see that the boundary and boundary inclusion of $(c, [n])$ are given by

$$(c \boxtimes \partial \Delta[n]) \coprod_{\partial c \boxtimes \partial \Delta[n]} (\partial c \boxtimes \Delta[n]) \xrightarrow{\mathbf{b}_c \hat{\boxtimes} \mathbf{b}_n} c \boxtimes \Delta[n].$$

We apply proposition 2.1.15 to conclude. □

Definition 2.3.2 (Trivial fibration). We say that a morphism f (in some category) is a *trivial fibration* if it has the right lifting property with respect to all monomorphisms.

In model category theory, a trivial fibration is usually a fibration that is also a weak equivalence. In fact, the motivation for the terminology of definition 2.3.2 is that in the familiar Quillen model structure on $\mathcal{Psh}(\Delta)$, both notions coincide. More generally, they coincide in all Cisinski model category (definition 2.1.10). In general however, there is no reason for it to be the case, and both terminologies clash. To remedy this, we resort to a classical alternative: fibrations that are weak equivalences will be called *acyclic fibrations* throughout this paper. Likewise, we will favor the name *acyclic cofibration* instead of trivial cofibration.

²We follow the terminology and notation of [RV14, Rie19]. In [JT07, section 2], it is denoted by \square' .

³This observation is called *Leibniz's formula* in [RV14, observation 4.2].

Proposition 2.3.3 (Generalization of [JT07, proposition 2.3]). *Let $f : X \rightarrow Y$ be a morphism in $\mathrm{Sp}(\mathcal{C})$. The following are equivalent:*

- (1) f is a trivial fibration;
- (2) the map $\langle \mathbf{b}_c \setminus f \rangle$ is a trivial fibration, for all $c \in \mathcal{C}$;
- (3) the map $\langle u \setminus f \rangle$ is a acyclic fibration, for all monomorphism u in $\mathcal{Psh}(\mathcal{C})$;
- (4) the map $\langle f / \mathbf{b}_n \rangle$ is a trivial fibration, for all $n \in \mathbb{N}$;
- (5) the map $\langle f / v \rangle$ is a trivial fibration, for all monomorphism v in $\mathcal{Psh}(\Delta)$.

Proof. Simple consequence of proposition 2.3.1 and the adjunctions $u \hat{\boxtimes} - \dashv \langle u \setminus - \rangle$ and $-\hat{\boxtimes} v \dashv \langle - / v \rangle$ of section 2.2. \square

Lemma 2.3.4. *Let \mathcal{D} be a normal skeletal category (particular, it is Reedy), and $\mathcal{Psh}(\mathcal{E})$ be endowed with a model structure. Consider the Reedy model structure on $\mathcal{Psh}(\mathcal{E})^{\mathcal{D}^{\mathrm{op}}}$.*

- (1) For $X \in \mathcal{Psh}(\mathcal{E})^{\mathcal{D}^{\mathrm{op}}}$ and $d \in \mathcal{D}$, the matching object of X at d is $\partial d \setminus X$.
- (2) For $f : X \rightarrow Y$ in $\mathcal{Psh}(\mathcal{E})^{\mathcal{D}^{\mathrm{op}}}$ and $d \in \mathcal{D}$, the relative matching map of f at d is $\langle \mathbf{b}_d \setminus f \rangle$. In particular, f is a Reedy fibration if and only if for all $d \in \mathcal{D}$, the map $\langle \mathbf{b}_d \setminus f \rangle$ is a fibration in $\mathcal{Psh}(\mathcal{E})$.
- (3) A map $f : X \rightarrow Y$ in $\mathcal{Psh}(\mathcal{E})^{\mathcal{D}^{\mathrm{op}}}$ is an acyclic Reedy fibration if and only for all $d \in \mathcal{D}$, the relative matching map $\langle \mathbf{b}_d \setminus f \rangle$ is an acyclic fibration in $\mathcal{Psh}(\mathcal{E})$.

Proof. For the first point, observe that

$$\partial d \setminus X = \left(\operatorname{colim}_{\substack{d' \rightarrow d \\ \text{in } \mathcal{D}_+, \text{not iso}}} d' \right) \setminus X \cong \lim_{\substack{d' \rightarrow d \\ \text{in } \mathcal{D}_+, \text{not iso}}} (d' \setminus X) \cong \lim_{\substack{d' \rightarrow d \\ \text{in } \mathcal{D}_+, \text{not iso}}} X_{d'},$$

which is the matching object of X at d . The second claim is by definition, and the third is [Hir03, theorem 15.3.15]. \square

For the the rest of this section, we assume that $\mathcal{Psh}(\mathcal{C})$ is endowed with a Cisinski model structure (definition 2.1.10).

Definition 2.3.5. Since \mathcal{C} is normal skeletal, it is a Reedy category. Let $\mathrm{Sp}(\mathcal{C})_v$, the *vertical model structure* on $\mathrm{Sp}(\mathcal{C}) \cong \mathrm{Sp}(\Delta)^{\mathcal{C}^{\mathrm{op}}}$, be the Reedy structure induced by the Quillen model structure on $\mathcal{Psh}(\Delta)$. In this structure, a map $f : X \rightarrow Y$ is a weak equivalence (also called *column-wise weak equivalence*) if for all $c \in \mathcal{C}$, the map of simplicial sets $c \setminus f = f_c : X_c \rightarrow Y_c$ is a weak equivalence. It is a fibration (also called *vertical fibration*) if for all $c \in \mathcal{C}$, the relative matching map $\langle \mathbf{b}_c \setminus f \rangle$ is a Kan fibration, where $\mathbf{b}_c : \partial c \rightarrow c$ is the boundary inclusion of c . Fibrant spaces in $\mathrm{Sp}(\mathcal{C})_v$ are also called *vertically fibrant*.

Dually, let $\mathrm{Sp}(\mathcal{C})_h$, the *horizontal model structure* on $\mathrm{Sp}(\mathcal{C}) \cong \mathcal{Psh}(\mathcal{C})^{\Delta^{\mathrm{op}}}$, be the Reedy structure induced by the model structure on $\mathcal{Psh}(\mathcal{C})$. The description of weak equivalence and fibrations transpose from the vertical model structure *mutadis mutandis*.

Proposition 2.3.6. *The model structures $\mathrm{Sp}(\mathcal{C})_v$ and $\mathrm{Sp}(\mathcal{C})_h$ are of Cisinsky type (definition 2.1.10).*

Proof. By [Hir03, theorem 15.6.27], both are cofibrantly generated. A map $f \in \mathrm{Sp}(\mathcal{C})$ is a vertical (resp. horizontal) acyclic fibration if and only if for all $c \in \mathcal{C}$ (resp. $n \in \mathbb{N}$), the matching map $\langle \mathbf{b}_c \setminus f \rangle$ (resp. $\langle f / \mathbf{b}_n \rangle$) is a acyclic fibration in $\mathcal{Psh}(\Delta)_{\mathrm{Quillen}}$ (resp. in the model structure on $\mathcal{Psh}(\mathcal{C})$, which is assumed to be of Cisinsky type), i.e. a trivial fibration. By proposition 2.3.3, f is a trivial fibration. Finally, f is a vertical or a horizontal acyclic fibration if and only if it is a trivial fibration. Therefore, vertical and horizontal cofibrations are the monomorphisms. \square

Proposition 2.3.7 (Generalization of [JT07, proposition 2.5]). *Let \mathcal{C} be a normal skeletal category, and $f : X \rightarrow Y$ be a morphism in $\mathrm{Sp}(\mathcal{C})$. The following are equivalent:*

- (1) f is a vertical fibration, i.e. the map $\langle \mathbf{b}_c \setminus f \rangle$ is a Kan fibration, for $c \in \mathcal{C}$;
- (2) the map $\langle u \setminus f \rangle$ is a Kan fibration, for all monomorphism $u \in \mathcal{Psh}(\mathcal{C})$;
- (3) the map $\langle f / \mathbf{h}_n^k \rangle$ is a trivial fibration, where $\mathbf{h}_n^k : \Lambda^k[n] \rightarrow \Delta[n]$ is the k -th horn inclusion of $[n]$, for all $n \in \mathbb{N}$ and $0 \leq k \leq n$;
- (4) the map $\langle f / v \rangle$ is a trivial fibration, for all anodyne extension $v \in \mathcal{Psh}(\Delta)$.

Definition 2.3.8 (Homotopically constant space [RSS01, section 3]). A space $X \in \mathbb{S}p(\mathbb{C})$ is *homotopically constant* if for all map $f : [k] \rightarrow [l]$ in Δ , the structure map $X/f : X_{-,l} \rightarrow X_{-,k}$ is a weak equivalence⁴.

Lemma 2.3.9. *Let $X \in \mathbb{S}p(\mathbb{C})$. The following are equivalent:*

- (1) X is homotopically constant;
- (2) for all $k \in \mathbb{N}$, writing $s : [k] \rightarrow [0]$ the terminal map in Δ , the structure map $X/s : X_{-,0} \rightarrow X_{-,k}$ a weak equivalence;
- (3) for all codegeneracy $s^i : [k] \rightarrow [k-1]$ in Δ , the structure map $X/s^i : X_{-,k-1} \rightarrow X_{-,k}$ is a weak equivalence;
- (4) for all $k \in \mathbb{N}$ and all map $d : [0] \rightarrow [k]$ in Δ , the structure map $X/d : X_{-,k} \rightarrow X_{-,0}$ is a weak equivalence;
- (5) for all coface map $d^i : [k] \rightarrow [k+1]$ in Δ , the structure map $X/d^i : X_{-,k+1} \rightarrow X_{-,k}$ is a weak equivalence.

Proof. • (1) \implies (2) \implies (3) and (1) \implies (4) \implies (5) are trivial.

- (2) \implies (1) Take a map $f : [k] \rightarrow [l]$ in Δ . Clearly, $s = sf$, so $X/s = (X/f)(X/s)$. By 3-for-2, X/f is a weak equivalence.
- (3) \implies (2) Note that the terminal map $s : [k] \rightarrow [0]$ is a composite of codegeneracies $[k] \rightarrow [k-1] \rightarrow \dots \rightarrow [1] \rightarrow [0]$.
- (4) \implies (2) The terminal map $s : [k] \rightarrow [0]$ is a retraction of any map $d : [0] \rightarrow [k]$, thus X/s is a section of X/d . By 3-for-2, X/s is a weak equivalence.
- (5) \implies (4) Note that all map $d : [0] \rightarrow [k]$ is a composite of coface maps. □

Proposition 2.3.10 (Generalization of [JT07, proposition 2.8]). *A vertically fibrant space $X \in \mathbb{S}p(\mathbb{C})$ (definition 2.3.5) is homotopically constant.*

Proof. Take $d : [0] \rightarrow [n]$ in Δ . Since $d : \Delta[0] \rightarrow \Delta[n]$ is a trivial cofibration in $\mathcal{P}sh(\Delta)_{\text{Quillen}}$, by proposition 2.3.7, the map $X/d = \langle (X \rightarrow 1)/d \rangle$ is a trivial fibration. Apply lemma 2.3.9 to conclude. □

Proposition 2.3.11 (Generalization of [JT07, proposition 2.9]). *A map $f : X \rightarrow Y$ between vertically fibrant spaces is a weak equivalence in $\mathbb{S}p(\mathbb{C})_h$ if and only if $f_{-,0} : X_{-,0} \rightarrow Y_{-,0}$ is a weak equivalence.*

Proof. Let $n \in \mathbb{N}$ and $s : [n] \rightarrow [0]$ be the terminal map in Δ . We have a commutative square

$$\begin{array}{ccc} X_{-,0} & \xrightarrow{f_{-,0}} & Y_{-,0} \\ X/s \downarrow & & \downarrow Y/s \\ X_{-,n} & \xrightarrow{f_{-,n}} & Y_{-,n} \end{array}$$

where by proposition 2.3.10, the vertical morphisms are weak equivalences. The result follows by 3-for-2. □

3. OPETOPIC ALGEBRAS

3.1. Reminders. In this section, we fix once and for all a parameter $n \geq 1$ and $k \geq 0$, and write $\mathfrak{3}$ for $\mathfrak{3}^n$, $\mathcal{A}lg$ for $\mathcal{A}lg^k(\mathfrak{3}^n)$, \mathbf{A} for $\mathbf{A}_{k,n}$, etc. Recall the main construction of [HTLS19], namely the reflective adjunction $h : \mathcal{P}sh(\mathbb{O}_{\geq n-k}) \rightleftarrows \mathcal{A}lg : N$ between the category of truncated opetopic sets (i.e. trivial below dimension $n-k$), and k -colored n -opetopic algebras. It exhibits $\mathcal{A}lg$ as the localization $\mathbf{A}_{k,n}^{-1} \mathcal{P}sh(\mathbb{O})$, or equivalently, as the orthogonality class $\mathbf{A}_{k,n}^\perp$ (definition 2.1.3). Let \mathfrak{A} be the full subcategory of $\mathcal{A}lg$ spanned by the image of h , i.e. the full subcategory of free algebras. Taking $h_! : \mathcal{P}sh(\mathbb{O}_{\geq n-k}) \rightarrow \mathcal{P}sh(\mathfrak{A})$ to be the left Kan extension of $\mathbb{O}_{\geq n-k} \xrightarrow{h} \mathfrak{A} \rightarrow \mathcal{P}sh(\mathfrak{A})$ along the Yoneda embedding, and $v : \mathcal{P}sh(\mathfrak{A}) \rightarrow \mathcal{A}lg$ to be the left Kan extension of the inclusion $\mathfrak{A} \hookrightarrow \mathcal{A}lg$ along the Yoneda embedding, the reflection h factors as $h \cong v h_!$:

$$\begin{array}{ccc} & \mathcal{A}lg & \\ h \nearrow & & \nwarrow v \\ \mathcal{P}sh(\mathbb{O}_{\geq n-k}) & \xrightleftharpoons[N]{h_!} \mathcal{P}sh(\mathfrak{A}) & \\ h^* \longleftarrow & & \longrightarrow \end{array} \quad (3.1.1)$$

⁴A vertically fibrant and homotopically constant space is called a *simplicial resolution* in [Dug01, definition 4.7].

Proposition 3.1.2. *The adjunction $v : \mathcal{Psh}(\mathbb{A}) \xrightarrow{\leftarrow} \mathcal{Alg} : M$ is reflective, i.e. M is an embedding of categories.*

Proof. Recall that N is an embedding, i.e. fully faithful and injective on objects. In particular, M must be injective on objects as well. By [Web07, theorem 4.10], M is fully faithful. \square

Definition 3.1.3 (Spine). Let $\omega \in \mathbb{O}_{n+1}$, let $\mathcal{S}_\omega := \mathbb{O}_{n-k,n}/S[\omega]$ be the category of elements of the spine $S[\omega] \in \mathcal{Psh}(\mathbb{O}_{n-k,n})$ of ω . Define the *spine* of $h\omega$ to be the colimit

$$S[h\omega] := \operatorname{colim} \left(\mathcal{S}_\omega \longrightarrow \mathbb{O}_{n-k,n} \xrightarrow{h} \mathbb{A} \longrightarrow \mathcal{Psh}(\mathbb{A}) \right).$$

We have an inclusion $\mathfrak{s}_{h\omega} : S[h\omega] \longrightarrow h\omega$ called the *spine inclusion* of $h\omega$ and let $\mathcal{S} := \{\mathfrak{s}_{h\omega} : S[h\omega] \longrightarrow h\omega \mid \omega \in \mathbb{O}_{n+1}\}$ be the set of spine inclusions of $\mathcal{Psh}(\mathbb{A})$.

Lemma 3.1.4. *The functor $h_! : \mathcal{Psh}(\mathbb{O}) \longrightarrow \mathcal{Psh}(\mathbb{A})$ maps $\mathcal{S}_{n+1} \subseteq \mathcal{Psh}(\mathbb{O})$ to $\mathcal{S} \subseteq \mathcal{Psh}(\mathbb{A})$, and morphisms in $\mathcal{S}_{>n+1} \subseteq \mathcal{Psh}(\mathbb{O})$ to \mathcal{S} -local isomorphisms.*

Proof. Recall from [HTLS19, lemma 4.5.1] that $h_! : \mathcal{Psh}(\mathbb{O}_{n-k,n+2}) \longrightarrow \mathcal{Psh}(\mathbb{A})$ maps \mathcal{S}_{n+1} to \mathcal{S} and \mathcal{S}_{n+2} to \mathcal{S} -local isomorphisms. Thus, so does $h_! : \mathcal{Sp}(\mathbb{O}_{\geq n-k}) \longrightarrow \mathcal{Sp}(\mathbb{A})$. Take $\omega \in \mathbb{O}_m$ with $m \geq n+3$ and consider the following square in $\mathcal{Psh}(\mathbb{O})$:

$$\begin{array}{ccc} S[\mathfrak{t}\omega] & \xrightarrow{i} & S[\omega] \\ \mathfrak{s}_{\mathfrak{t}\omega} \downarrow & & \downarrow \mathfrak{s}_\omega \\ O[\mathfrak{t}\omega] & \xrightarrow{\mathfrak{t}} & O[\omega]. \end{array}$$

By [HTLS19, lemma 3.4.9], $i \in \operatorname{Cell}(\mathcal{S}_{n+1})$, thus a \mathcal{S} -local isomorphism, and by induction, $\mathfrak{s}_{\mathfrak{t}\omega}$ is a \mathcal{S} -local isomorphism. By [HTLS19, corollary 3.4.10] the bottom target embedding is a $\mathcal{S}_{m-1,m}$ -local isomorphism, thus a \mathcal{S} -local isomorphism by induction. Consequently, $\mathfrak{s}_{h\omega}$ is a \mathcal{S} -local isomorphism. \square

Theorem 3.1.5. *The left adjoint $v : \mathcal{Psh}(\mathbb{A}) \longrightarrow \mathcal{Alg}$ is the Gabriel–Ulmer localization with respect to the set \mathcal{S} of spine inclusions. Equivalently, $M : \mathcal{Alg} \longrightarrow \mathcal{Psh}(\mathbb{A})$ corestricts as an isomorphism $\mathcal{Alg} \longrightarrow \mathcal{S}^\perp$.*

Proof. By [Web07, theorem 4.10], a presheaf $X \in \mathcal{Psh}(\mathbb{A})$ is in the essential image of M if and only if h^*X is in the essential image of N . In other words, X is an algebra if and only if $\mathcal{S}_{\geq n+1} \perp h^*X$, which under the adjunction $h_! \dashv h^*$, is equivalent to $\mathcal{S} \perp X$. \square

3.2. Free opetopic algebras. Recall that $\mathbb{A} = \mathbb{A}_{1,n}$ is the full subcategory of \mathcal{Alg} spanned by free algebras on opetopes in $\mathbb{O}_{n-1,n+1}$. However, if $\phi \in \mathbb{O}_{n-1}$, then $h\phi = h\mathbb{1}_\phi$, and if $\psi \in \mathbb{O}_n$, then $h\psi = h\mathbb{Y}_\psi$. So we may consider \mathbb{A} to be the full subcategory of \mathcal{Alg} spanned by free algebras over $(n+1)$ -opetopes.

Definition 3.2.1 (Diagram). For $\omega, \omega' \in \mathbb{O}_{n+1}$ and $f : h\omega \longrightarrow h\omega'$ in \mathbb{A} , a *diagram* [HTLS19] of f is the datum of a cospan

$$\omega \xrightarrow{\mathfrak{s}_{[p]}} \xi \xleftarrow{\mathfrak{t}} \omega',$$

with $\xi \in \mathbb{O}_{n+2}$, such that $f = (h\mathfrak{t})^{-1}(h\mathfrak{s}_{[p]})$. The situation is summarized as follows:

$$\begin{array}{ccc} & & \xi \\ & \nearrow \mathfrak{s}_{[p]} & \uparrow \mathfrak{t} \\ \omega & & \omega' \\ \hline h\omega & \xrightarrow{f} & h\omega' \end{array} \quad (3.2.2)$$

The morphism f is said to be *diagrammatic* if it admits a diagram.

Lemma 3.2.3 (Diagrammatic lemma, [HTLS19]). *Let $f : h\omega \longrightarrow h\omega'$ be a morphism in \mathbb{A} . If ω is not degenerate, then f is diagrammatic.*

Lemma 3.2.4. *Let $\omega \in \mathbb{O}_{n+1} = \operatorname{tr} \mathfrak{Z}^{n-1}$ and $\psi \in \mathbb{O}_n$. Then $h\omega_\psi = (Nh\omega)_\psi$ is the set of sub- \mathfrak{Z}^{n-1} -trees ν of ω such that $\mathfrak{t}\nu = \psi$.*

Proof. By definition,

$$h\omega_\psi = \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathfrak{t}\nu = \psi}} \mathcal{Psh}(\mathbb{O})(S[\nu], O[\omega]),$$

and morphisms $S[\nu] \rightarrow O[\omega]$ are precisely the \mathfrak{Z}^{n-1} -tree embeddings of ν in ω . \square

Lemma 3.2.5. *Let $f_1, f_2 : h\omega \rightarrow h\omega'$ be two morphism in \mathbb{A} , with $\omega, \omega' \in \mathbb{O}_{n+1}$. Then $f_1 = f_2$ if and only if for all $[p] \in \omega^\bullet$, $f_1 c_{\omega, [p]} = f_2 c_{\omega, [p]}$. In other words, $f_1 = f_2$ if and only if f_1 and f_2 agree on all one-node subtrees of ω .*

Proof. Under the adjunction $\mathcal{Psh}(\mathbb{O}_{\geq n-1}) \rightleftarrows \mathcal{Alg}$, f_i corresponds to a morphism $\bar{f}_i : O[\omega] \rightarrow h\omega'$, and since $S_{\geq n+1} \perp h\omega'$, it is uniquely determined by its restriction $\bar{f}_i : S[h\omega] \rightarrow h\omega'$. \square

Notation 3.2.6. As a consequence of lemma 3.2.5, if $\omega \in \mathbb{O}_{n+1}$ is not degenerate, the algebra $h\omega$ is freely generated by its one-node subtrees. We denote these generators by $c_{\omega, [p]}$, where $[p]$ ranges over ω^\bullet .

3.3. Quotients. Let $X \in \mathcal{Alg}$. Recall that X can be seen as an opetopic set $X \in \mathcal{Psh}(\mathbb{O})$ such that $\mathbf{A}_{k,n} \perp X$. We now present a notation that shows how solutions of lifting problems $S[h\omega] \rightarrow X$ can really be understood as compositions of ‘‘tree-shaped arities’’.

For $\psi \in \mathbb{O}_{n-1}$ and $x \in X_\psi$, write $\text{id}_x \in X_{Y_\psi}$ for the target of the solution of the lifting problem

$$\begin{array}{ccc} S[I_\psi] = O[\psi] & \xrightarrow{x} & X \\ \downarrow & \searrow \exists! f & \\ O[I_\psi] & & \end{array}$$

Explicitly, if f is the solution, let id_x be the cell of X_{Y_ψ} selected by

$$O[Y_\psi] \xrightarrow{\mathfrak{t}} O[I_\psi] \xrightarrow{f} X.$$

Let $\omega, \omega' \in \mathbb{O}_n$, and $[p] \in \omega^\bullet$ such that $\mathfrak{s}_{[p]}\omega = \mathfrak{t}\omega'$, so that $Y_\omega \circ_{[[p]]} Y_{\omega'}$ is a well defined $(n+1)$ -opetope. For $x \in X_\omega$, $x' \in X_{\omega'}$ such that $\mathfrak{s}_{[p]}x = \mathfrak{t}x'$, let $x \circ_{[p]} x'$ be the target of the solution of the following lifting problem:

$$S[Y_\omega \circ_{[[p]]} Y_{\omega'}] \xrightarrow{[\] \mapsto x, [[p]] \mapsto x'} X.$$

An iterated composition as on the right can be concisely written as on the left:

$$x \bigcirc_{[p_i]} y_i := (\cdots (x \circ_{[p_1]} y_1) \circ_{[p_2]} y_2 \cdots) \circ_{[p_k]} y_k,$$

where $x \in X_\omega$, $\omega^\bullet = \{[p_1], \dots, [p_k]\}$, $y_i \in X_{\omega_i}$, and $\mathfrak{t}\omega_i = \mathfrak{s}_{[p_i]}\omega$.

Let $X \in \mathcal{Alg}$ be an opetopic algebra, $\omega \in \mathbb{O}_n$, and $x, y \in X_\omega$. We say that x and y are *parallel* if the following two composites are equal:

$$\partial O[\omega] \xrightarrow{\mathfrak{b}_\omega} O[\omega] \xrightarrow[x]{x} X.$$

In that case, let the *quotient* of X by the equality $x = y$ be the following coequalizer in \mathcal{Alg} :

$$h\omega \xrightarrow[y]{x} X \longrightarrow X / \{x = y\}.$$

For each $\omega \in \mathbb{O}_n$, take a set $K_\omega \subseteq X_n \times X_n$ of pairs of parallel cells, and let $K := \sum_\omega K_\omega$. Then the quotient of X by K is the following coequalizer in \mathcal{Alg} :

$$\sum_{\omega \in \mathbb{O}_n} \sum_{(x,y) \in K_\omega} h\omega \xrightarrow[p_2]{p_1} X \longrightarrow X/K,$$

where $p_1(x, y) := x$ and $p_2(x, y) := y$.

3.4. The folk model structure. In this section, fix a parameter $n \geq 1$, and take $k = 1$. As usual, we omit n and k from most notations, e.g. $\mathcal{Alg} = \mathcal{Alg}^1(\mathfrak{Z}^n)$. Let $A \in \mathcal{Alg}$ be an algebra, $\psi \in \mathbb{O}_n$, and $d : \partial h\psi \rightarrow A$, and consider the following pullback

$$\begin{array}{ccc} A_d & \hookrightarrow & A_{h\psi} = \mathcal{Psh}(\mathbb{A})(h\psi, A) \\ \downarrow & \lrcorner & \downarrow \mathfrak{b}_{h\psi}^* \\ * & \xrightarrow{d} & \mathcal{Psh}(\mathbb{A})(\partial h\psi, A). \end{array}$$

Explicitly, A_d is the subset of $A_{h\psi}$ of all cell a such that $\mathfrak{s}_{[q]}a = d(\mathfrak{s}_{[q]}\psi)$ for all $[q] \in \psi^\bullet$, and $\mathfrak{t}a = d(\mathfrak{t}\psi)$.

Example 3.4.1. Take $n = 2$, so that \mathcal{Alg} is the category of planar colored operads. For $P \in \mathcal{Alg}$ and $\mathbf{k} \in \mathbb{O}_2$, a morphism $d : \partial h\mathbf{k} \rightarrow P$ is just a $(k+1)$ -tuple of colors $(d_0, \dots, d_{k-1}; d_k)$ of P , where $d_i = d\mathfrak{s}_{[*^i]}\mathbf{k}$ for $0 \leq i < k$, and $d_k = d\mathfrak{t}\mathbf{k}$. The set P_d is simply $P(d_0, \dots, d_{k-1}; d_k)$.

Definition 3.4.2 (Internal isomorphism). Let $f : A \rightarrow B$ be a morphism of algebras. Let $\phi \in \mathbb{O}_{n-1}$ and $a \in A_{h\mathcal{Y}_\phi}$. If $x = \mathfrak{s}_{\square}a$ and $y = \mathfrak{t}a$, we write $a : x \rightarrow y$. We say that a is an *internal isomorphism* (or just *isomorphism*) if it is invertible, i.e. if there exists a cell $a^{-1} \in A_{h\mathcal{Y}_\phi}$ such that

$$a \circ_{\square} a^{-1} = \text{id}_y, \quad a^{-1} \circ_{\square} a = \text{id}_x.$$

In this case, we also say that x and y are *isomorphic*, and write $x \cong y$.

Definition 3.4.3 (Natural transformation). Let $f, g : A \rightarrow B$ be two parallel morphisms of algebras. A *natural transformation* $\alpha : f \rightarrow g$ is a collection of cells $\alpha_a : f(a) \rightarrow g(a)$ (called *components*, see definition 3.4.2 for notation) such that for all $\psi \in \mathbb{O}_n$ and $x \in A_{h\psi}$, the following relation holds:

$$g(x) \bigcirc_{[q]} \alpha_{\mathfrak{s}_{[q]}a} = \alpha_{\mathfrak{t}a} \circ_{\square} f(x).$$

Natural transformations can be composed in the obvious way, and the adequate exchange law holds. A *natural isomorphism* is an invertible natural transformation, or equivalently, on whose components are all isomorphisms.

Definition 3.4.4 (Algebraic equivalence). Let $f : A \rightarrow B$ be a morphism of algebras.

- (1) We say that f is *fully faithful* if for all $\psi \in \mathbb{O}_n$ and for all morphism $d : \partial h\psi \rightarrow A$, the induced map $f_d : A_d \rightarrow B_{fd}$ is a bijection.
- (2) We say that f is *essentially surjective* if for all $b \in B_{n-1}$, there exist an $a \in A_{n-1}$ such that $f(a) \cong b$.
- (3) We say that f is an *algebraic equivalence* (or *equivalence of algebras*, or simply *equivalence*) if it is fully faithful and essentially surjective. Clearly, if $n = 1$, we recover the notion of fully faithful functor between categories, whereas if $n = 2$, this matches the definition of operadic equivalence of [MW07].

Proposition 3.4.5. *A morphism $f : A \rightarrow B$ of algebras is an equivalence if and only if it is invertible up to natural isomorphism, i.e. there exists $g : B \rightarrow A$ and natural isomorphisms $\varepsilon : gf \rightarrow \text{id}_A$ and $\eta : \text{id}_B \rightarrow fg$.*

Proof. This is similar to [ML98, theorem IV.4.1]. Necessity is easy. Assume that f is an equivalence. For $b \in B_{n-1}$, choose an $a = g(b) \in A_{n-1}$ and an isomorphism $\eta_b : b \rightarrow f(g(b))$. Let $\psi \in \mathbb{O}_n$ and $y \in B_{h\psi}$ be such that $\mathfrak{t}y$ is in the image of f , say $\mathfrak{t}y = f(a)$ for some $a \in A_{n-1}$. Then

$$z := y \bigcirc_{[q]} \eta_{\mathfrak{s}_{[q]}y}^{-1}$$

where $[q]$ ranges over ψ^\bullet , has all its faces in the image of f , as $\mathfrak{s}_{[q]}z = \mathfrak{s}_{\square} \eta_{\mathfrak{s}_{[q]}y}^{-1} = f(g(\mathfrak{s}_{[q]}y))$. Since f is fully faithful, there exist a unique $g(y)$ in $A_{h\psi}$ such that $f(g(y)) = z$. This defines a morphism $g : B \rightarrow A$, and a natural isomorphism $\eta : \text{id}_B \rightarrow fg$.

Let $a \in A_{n-1}$, and consider the isomorphism $e := \eta_{f(a)}^{-1}$. Note that $e : fgf(a) \rightarrow f(a)$, and since f is fully faithful, there exists a unique $\varepsilon_a : gf(a) \rightarrow a$ such that $e = f(\varepsilon_a)$. It is straightforward to check that the ε_a s are the component of a natural isomorphism $\varepsilon : gf \rightarrow \text{id}_A$. \square

Lemma 3.4.6. *Let $f : A \rightarrow B$ be an equivalence that is injective on $(n-1)$ -cells. Then f admits a retract up to isomorphism, i.e. a weak inverse $g : B \rightarrow A$ together with natural isomorphisms $\varepsilon : gf \rightarrow \text{id}_A$ and $\eta : \text{id}_B \rightarrow fg$ as in proposition 3.4.5, but where $gf = \text{id}_A$ and ε is an identity (i.e. all its components are identities), and where $f = fgf$, and for $a \in A_{n-1}$, $\eta_{f(a)} = \text{id}_{f(a)}$*

Proof. It suffices to amend the proof of proposition 3.4.5 so that g , ε and η have the desired properties. For $a \in A_{n-1}$, since f is injective on objects, we may choose $g(f(a))$ to be a , and $\eta_{f(a)}$ to be $\text{id}_{f(a)}$. It follows that, after extending g to a morphism $B \rightarrow A$, we have $gf = \text{id}_A$. Further, for $a \in A_{n-1}$, $\varepsilon_a : gf(a) \rightarrow a$ is the only n -cell of A such that $f(\varepsilon_a) = \eta_{f(a)}^{-1} = \text{id}_{f(a)}$, whence ε is the identity. \square

Definition 3.4.7 (IsOfibration). A morphism $f : A \rightarrow B$ of algebras is an *isofibration* if for all $a \in A_{n-1}$, all isomorphism $g : f(a) \rightarrow b$ in B , there exists an isomorphism $g' : a \rightarrow a'$ in A such that $f(g') = g$.

Theorem 3.4.8 (Generalization of [JT07, theorem 1.4]). *The category $\text{Alg} = \text{Alg}^1(\mathbf{3}^n)$ of 1-colored algebras admits a model structure where the weak equivalences are the algebraic equivalences (definition 3.4.4), the cofibrations are those morphism that are injective on $(n-1)$ -cells, and where the fibrations are the isofibrations (definition 3.4.7). We call this structure the folk model structure, and denote it by Alg_{folk} .*

Furthermore, acyclic fibrations are the algebraic equivalences that are surjective on $(n-1)$ -cells, and every object is both fibrant and cofibrant.

The second claim can easily be checked once the model structure is established. The rest of this section is dedicated to proving this. To that end, we verify each of Quillen's axioms, cf [Hir03, definition 7.1.3].

Proof of theorem 3.4.8, (M1): limit axiom. Since Alg is the category of models of a small projective sketch [HTLS19], it is locally presentable [AK80, corollary 1.52], and therefore has all small limits and colimits. \square

Proof of theorem 3.4.8, (M2): 3-for-2 axiom. This is clear. \square

Proof of theorem 3.4.8, (M3): retract axiom. Straightforward verifications. \square

Proof of theorem 3.4.8, (M4): lifting axiom. Consider a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow i & & \downarrow p \\ C & \xrightarrow{g} & D, \end{array}$$

where i is a cofibration and p is a fibration. We show that a lift exists whenever i or p is a weak equivalence.

- (1) Assume that i is a weak equivalence, and let $r : C \rightarrow A$ be a weak retract of i as in lemma 3.4.6, together with the natural isomorphism $\eta : \text{id}_C \rightarrow ir$. Let $c \in C_{n-1}$, and consider $gir(c) = pfr(c) \in D_{n-1}$. Since p is an isofibration, there exist an isomorphism $\beta_c : fr(c) \rightarrow b$ in B such that $p(\beta_c) = g(\eta_c^{-1}) : gir(c) = pfr(c) \rightarrow g(c)$. In particular, $p(b) = g(c)$, and define $l(c) := b$. This defined a lift $l : C_{n-1} \rightarrow B_{n-1}$.

The construction above also provides a natural isomorphism $\beta : fr \rightarrow l$. Without loss of generality, we choose l and the components of β such that for all $a \in A_{n-1}$, $li(a) = f(a)$, and $\beta_{i(a)} = \text{id}_{f(a)}$. For $\psi \in \mathbb{O}_n$ and $x \in C_{h\psi}$, let

$$l(x) := \beta_{ix} \circ fr(x) \circ \beta_{[g]x}^{-1}.$$

It can easily be checked that l is then a morphism of algebras $C \rightarrow B$, and finally the desired lift.

- (2) Assume that p is a weak equivalence. In particular, on $(n-1)$ -cells, i is an injection, and p a surjection, and so a lift $l : C_{n-1} \rightarrow B_{n-1}$ can be found. We now extend l to a morphism of algebras $l : C \rightarrow B$.

Let $\psi \in \mathbb{O}_n$, $x \in C_\psi$, and d be the composite

$$\partial h\psi \xrightarrow{b_{h\psi}} h\psi \xrightarrow{x} C.$$

Since p is fully faithful, it induces a bijection

$$p_d : C_{ld} \xrightarrow{\cong} D_{gd},$$

and letting $l(x) := p_d^{-1}g(x)$ extends l to an algebra morphism which is the desired lift. \square

Proof of theorem 3.4.8, (M5): factorization axiom. Let $f : A \rightarrow B$ be a morphism of algebra.

- (1) We decompose f as $f = pi$, where i is an acyclic cofibration, and where p is fibration. Define $C \in \mathcal{Psh}(\Lambda)$ as follows. For $\phi \in \mathbb{O}_{n-1}$, let

$$C_{h\phi} := \{(a, v, b) \mid a \in A_{h\phi}, b \in B_{h\phi}, v \in B_{Y_\phi} \text{ isomorphism } f(a) \longrightarrow b\}.$$

There is an obvious projection $\text{proj} : C_{n-1} \longrightarrow A_{n-1}$, and if $\psi \in \mathbb{O}_n$ and $d : \partial h\psi \longrightarrow C_{n-1}$, let $C_d := A_{\text{proj}d}$. At this stage, C clearly extends as an algebra, essentially inheriting the same law as A .

Let $i : A \longrightarrow C$ map an $(n-1)$ -cell a to $(a, \text{id}_{f(a)}, f(a))$. This completely determines i , as indeed, for an n -cell $x \in A_{\geq n}$ we necessarily have $i(x) = x$. Clearly, i is a fully faithful cofibration. It remains to show that it is essentially surjective. If $(a, v, b) \in C_{n-1}$, note that id_a exhibits an isomorphism $i(a) = (a, \text{id}_{f(a)}, f(a)) \longrightarrow (a, v, b)$ in C . Therefore, i is an acyclic cofibration.

Let $p : C \longrightarrow B$ be the obvious projection $(a, v, b) \longmapsto b$ on $(n-1)$ -cells. Let $\psi \in \mathbb{O}_n$ and $x \in C_{h\psi}$. Write $\mathbf{t}x = (a_{\mathbf{t}}, v_{\mathbf{t}}, b_{\mathbf{t}})$, for a source address $[q] \in \psi^\bullet$, write $s_{[q]}x = (a_{[q]}, v_{[q]}, b_{[q]})$, and define

$$p(x) := v_{\mathbf{t}} \circ_{\square} f(x) \circ_{\square} v_{[q]}^{-1}.$$

This defines a morphism $p : C \longrightarrow B$ which we claim to be a fibration. Indeed, if $(a, v, b) \in C_{n-1}$ and $w : b \longrightarrow b'$ is an isomorphism in C , we have an isomorphism $\text{id}_a : (a, v, b) \longrightarrow (a, w \circ_{\square} v, b')$ in C , and

$$p(\text{id}_{f(a)}) = (w \circ_{\square} v) \circ_{\square} f(\text{id}_a) \circ_{\square} v = w.$$

Lastly, it is clear that $f = pi$, and thus f decomposes as a acyclic cofibration followed by a fibration.

- (2) We decompose f as $f = pi$, where i is a cofibration, and where p is an acyclic fibration. Define $D \in \mathcal{Psh}(\Lambda)$ as follows. On $(n-1)$ -cells, it is given as on the left, and let \bar{f} be defined as on the right:

$$D_{n-1} := A_{n-1} + B_{n-1}, \quad \bar{f} := (f, \text{id}_{B_{n-1}}) : D_{n-1} \longrightarrow B_{n-1}.$$

Explicitly, \bar{f} maps $a \in A_{n-1}$ to $f(a)$, and $b \in B_{n-1}$ to b . For $\psi \in \mathbb{O}_n$ and $d : \partial h\psi \longrightarrow D_{n-1}$, let the fiber D_d be simply $B_{\bar{f}d}$. At this stage, D clearly extends as an algebra, essentially inheriting the same law as B .

Let $i : A \longrightarrow D$ map an $(n-1)$ -cell a to a , and an n -cell x to $f(x)$. Obviously, this is a cofibration. Let $p : D \longrightarrow A$ map an $(n-1)$ -cell d to $\bar{f}(d)$, and an n -cell x to x . This can easily be seen to be an acyclic fibration. Lastly, $f = pi$, so that f can be decomposed into a cofibration followed by an acyclic fibration. \square

Definition 3.4.9 (Rezk interval). Let $\phi \in \mathbb{O}_{n-1}$. The *Rezk interval of shape ϕ* is the algebra $\mathfrak{J}_\phi \in \mathcal{Alg}$ generated by one invertible operation j_ϕ of shape Y_ϕ . Explicitly, \mathfrak{J}_ϕ has two $(n-1)$ -cells $0_\phi, 1_\phi \in (\mathfrak{J}_\phi)_\phi$, and four n -cells $j_\phi, j_\phi^{-1}, \text{id}_0, \text{id}_1 \in (\mathfrak{J}_\phi)_{Y_\phi}$, satisfying the following equalities

$$s_{\square} j_\phi = \mathbf{t} j_\phi^{-1} = 0_\phi, \quad \mathbf{t} j_\phi = s_{\square} j_\phi^{-1} = 1_\phi, \quad j_\phi \circ_{\square} j_\phi^{-1} = \text{id}_1, \quad j_\phi^{-1} \circ_{\square} j_\phi = \text{id}_0.$$

Note that up to isomorphism, there is a unique endpoint inclusion $j_\phi : h\phi \longrightarrow \mathfrak{J}_\phi$, and let $\mathbf{E}_{\mathfrak{J}} := \{j_\phi \mid \phi \in \mathbb{O}_{n-1}\}$.

If $X = (X_\psi \mid \psi \in \mathbb{O}_{n-1})$ is a set over \mathbb{O}_{n-1} , let $\mathfrak{J}_X := \sum_\psi \sum_{x \in X_\psi} \mathfrak{J}_\psi$. In the $x \in X_\psi$ component, we write j_x instead of j_ψ , and similarly, j_x^{-1} , 0_x , and 1_x . The *Rezk interval* (without any mention of shape) is the sum $\mathfrak{J} = \mathfrak{J}_{\mathbb{O}_{n-1}} := \sum_\psi \mathfrak{J}_\psi$ in \mathcal{Alg} .

Theorem 3.4.10. *The model structure $\mathcal{Alg}_{\text{folk}}$ is cofibrantly generated, and $\mathbf{E}_{\mathfrak{J}}$ can be taken as a set of generating acyclic cofibrations.*

Proof. Clearly, a morphism f is an isofibration if and only if $\mathbf{E}_{\mathfrak{J}} \pitchfork f$. It is straightforward to check that f is an acyclic fibration if and only if $\mathbf{J} \pitchfork f$, where \mathbf{J} is

$$\left\{ \emptyset \longmapsto h\phi \mid \phi \in \mathbb{O}_{n-1} \right\} \cup v\mathbf{B} \cup \left\{ h\psi \prod_{\partial h\psi} h\psi \longrightarrow h\psi \mid \psi \in \mathbb{O}_n \right\},$$

(see definition 4.2.1 for the definition of \mathbf{B}). \square

3.5. Homotopy equivalences.

Definition 3.5.1 (Rezk cylinder of an algebra). Let $A \in \mathcal{Alg}$, and consider the following pushout in \mathcal{Alg} :

$$\begin{array}{ccc} hA_{n-1} + hA_{n-1} & \longrightarrow & A + A \\ \downarrow & & \downarrow (i_0, i_1) \\ \mathfrak{J}A_{n-1} & \xrightarrow{\quad \Gamma \quad} & A'. \end{array}$$

For a a cell of A , write $a^{(e)} := i_e(a)$, for $e = 0, 1$. Explicitly, A' is generated by two copies of A , and for each $a \in A_\psi$, $\psi \in \mathbb{O}_{n-1}$, an additional cell j_a of shape \mathbf{Y}_ψ with $\mathfrak{s}_\square j_a = a^{(0)}$ and $\mathfrak{t}_\square j_a = a^{(1)}$, which can informally be written $j_a : a^{(0)} \longrightarrow a^{(1)}$.

The *Rezk cylinder* of A is the quotient $\mathfrak{J}A := A'/K$, where

$$K := \left\{ a^{(1)} \circlearrowleft_{[p_i]} j_{\mathfrak{s}_{[p_i]} a} = j_{\mathfrak{t}_a} \circlearrowright_{\square} a^{(0)} \mid a \in A_\omega, \omega \in \mathbb{O}_n, \omega^\bullet = \{[p_1], \dots\} \right\}. \quad (3.5.2)$$

See section 3.3 for the \circ notation. It is a cylinder object in the sense of definition 2.1.5, i.e. we have a canonical factorization of the codiagonal map

$$\begin{array}{ccc} A + A & \xrightarrow{\quad \nabla \quad} & \mathfrak{J}A \\ \downarrow (i_0, i_1) & & \uparrow \nabla \\ & A & \end{array}$$

Explicitly, $\nabla : \mathfrak{J}A \longrightarrow A$ maps $a^{(e)}$ to a , for a cell $a \in A$, and j_a to id_a , if a is $(n-1)$ -dimensional. Note that $(i_0, i_1) : A + A \longrightarrow \mathfrak{J}A$ is a monomorphism, since the relation K of equation (3.5.2) does not identify cells of eA , for $e = 0, 1$. We write $A^{(e)}$ the image of i_e , and by abuse of notation, $i_e : A^{(e)} \hookrightarrow \mathfrak{J}A$ the obvious inclusion.

Example 3.5.3. Let $\psi \in \mathbb{O}_n$. Then $\mathfrak{J}h\psi$ is generated by

- (1) $\psi^{(0)}, \psi^{(1)} \in (\mathfrak{J}h\psi)_\psi$, i.e. two copies of $h\psi$,
- (2) for each $[p] \in \psi^\bullet$, and writing $\phi := \mathfrak{s}_{[p]} \psi$, two cells $j_\phi, j_\phi^{-1} \in (\mathfrak{J}h\psi)_{\mathbf{Y}_\phi}$, i.e. one copy of $\mathfrak{J}\phi$,
- (3) $j_{\mathfrak{t}_\psi}, j_{\mathfrak{t}_\psi}$, i.e. one copy of $\mathfrak{J}\mathfrak{t}_\psi$,

subject to the relation equation (3.5.2). Likewise, for $\omega \in \mathbb{O}_{n+1}$, the cylinder $\mathfrak{J}h\omega$ is generated by two copies of $h\omega$, and one copy $\mathfrak{J}_{e[q]}\omega$ for all edge address $[q]$ of ω .

Remark 3.5.4. The cylinder $\mathfrak{J}A$ of an algebra A can be thought of the *Boardman–Vogt tensor product* [Wei11, BV73, May72] $\mathfrak{J} \otimes_{\text{BV}} A$. Unfortunately, in the planar case, a general such construction is not possible, as there is no way to “shuffle the inputs”. Since this is the only obstruction, one could still define a $A \otimes_{\text{BV}} B$ as long as either A or B only have unary (i.e. endotopic) operations.

Lemma 3.5.5. *Let $f, g : A \longrightarrow B$ be two parallel morphisms of algebras. The following are equivalent:*

- (1) $f \simeq g$ (definition 2.1.6);
- (2) there is an elementary \mathfrak{J} -homotopy from f to g ;
- (3) there exist a natural isomorphism $f \longrightarrow g$.

Proof. • (1) \implies (3). A homotopy H from f to g as in definition 2.1.6 induces a natural isomorphism \hat{H} with components $\hat{H}_a = H(j_a)$ (see definition 3.5.1 for notations), for $a \in A_{n-1}$.

- (3) \implies (2). A natural isomorphism $\alpha : f \longrightarrow g$ induces a homotopy $\check{\alpha} : \mathfrak{J}A \longrightarrow B$ from f to g , where for $a \in A$, $\check{\alpha}(a^{(0)}) := f(a)$ (see definition 3.5.1 for notations), $\check{\alpha}(a^{(1)}) := g(a)$, $\check{\alpha}(j_a) = \alpha_a$, and $\check{\alpha}(j_a^{-1}) := \alpha_a^{-1}$.
- (2) \implies (1). By definition. \square

Proposition 3.5.6. *A morphism $f : A \longrightarrow B$ is a weak equivalence (for the folk structure of theorem 3.4.8) if and only if it is an isomorphism in $\mathcal{H}\text{CoAlg}$ (definition 2.1.6). Therefore, the category $\mathcal{H}\text{CoAlg}$ is the localization of \mathcal{Alg} at the algebraic equivalences.*

Proof. Follows from proposition 3.4.5 and lemma 3.5.5. \square

4. THE HOMOTOPY THEORY OF ∞ -OPETOPIC ALGEBRA

We fix a parameter $n \geq 1$, and assume that $k = 1$. In [HTLS19], we built a reflective adjunction

$$h : \mathcal{Psh}(\mathbb{O}_{\geq n-1}) \xleftrightarrow{\quad} \mathcal{Alg} : N.$$

We consider N to be the inclusion of a full subcategory $\mathcal{Alg} \hookrightarrow \mathcal{Psh}(\mathbb{O}_{\geq n-1})$, and omit it if unambiguous.

4.1. A skeletal structure on \mathbb{A} . In this section, we endow $\mathbb{A} = \mathbb{A}_{1,n}$ with the structure of a skeletal category (cf. section 2.1). We first need to assign a notion of *degree* to any objects $\lambda \in \mathbb{A}$, denoted by $\deg \lambda \in \mathbb{N}$. Next, we need to specify two classes of morphisms \mathbb{A}_+ and \mathbb{A}_- satisfying axioms **(Sq0)** to **(Sq3)**.

Definition 4.1.1. Recall that the objects of \mathbb{A} are the free algebras on $(n+1)$ -opetopes. For $\omega \in \mathbb{O}_{n+1}$, let $\deg h\omega := \#\omega^\bullet$.

Definition 4.1.2. Let $f : h\omega \rightarrow h\omega'$ be a morphism in \mathbb{A} , with $\omega, \omega' \in \mathbb{O}_{n+1}$. In particular, it induces a set map between $(n-1)$ -cells: $f_{n-1} : h\omega_{n-1} \rightarrow h\omega'_{n-1}$. Let \mathbb{A}_+ (resp. \mathbb{A}_-) be the wide subcategory spanned by those morphisms f such that f_{n-1} is a monomorphism (resp. epimorphism).

The rest of this section is dedicated to prove the following result.

Theorem 4.1.3. *With the data above, \mathbb{A} is a skeletal category (definition 2.1.13).*

Proof of theorem 4.1.3, axiom (Sq0). Since \mathbb{A} is rigid (i.e. has no isomorphisms beside the identities), this axiom holds trivially. \square

Lemma 4.1.4. *Let $f : h\omega \rightarrow h\omega'$ be a morphism in \mathbb{A} , with $\omega, \omega' \in \mathbb{O}_{n+1}$. The following are equivalent:*

- (1) $f \in \mathbb{A}_+$ (resp. \mathbb{A}_-);
- (2) the set map between n -cells $f_n : h\omega_n \rightarrow h\omega'_n$ is a monomorphism (resp. epimorphism);
- (3) f is a monomorphism in $\mathcal{Psh}(\mathbb{O}_{\geq n-1})$ (resp. epimorphism).

Proof. Recall that $h : \mathcal{Psh}(\mathbb{O}_{\geq n-1}) \rightarrow \mathcal{Alg}$ is the localization with respect to the set $\mathfrak{s}_{\geq n+1}$. Therefore, $f_{\geq n}$ is a monomorphism (resp. epimorphism) if and only if f_n is. It remains to show that f_{n-1} is a monomorphism (resp. epimorphism) if and only if f_n is. But this is a direct consequence of lemma 3.2.4, stating that $h\omega_n$ is the set of sub- \mathfrak{J}^{n-1} -trees of ω . \square

Corollary 4.1.5. *Let $f : h\omega \rightarrow h\omega'$ be a morphism in \mathbb{A} , with $\omega, \omega' \in \mathbb{O}_{n+1}$. The following are equivalent:*

- (1) $f \in \mathbb{A}_+ \cap \mathbb{A}_-$, i.e. f_{n-1} is an isomorphism;
- (2) f_n is an isomorphism;
- (3) f is an isomorphism;

Proof. Recall that in any presheaf category, the epi-monos are exactly the isomorphisms. \square

Proof of theorem 4.1.3, axiom (Sq1). Let $f : h\omega \rightarrow h\omega'$ be a morphism in \mathbb{A} that is not an isomorphism, with $\omega, \omega' \in \mathbb{O}_{n+1}$. If $f \in \mathbb{A}_+$, then by lemma 4.1.4 and corollary 4.1.5, f_n is a monomorphism that is not an isomorphism, thus $\#h\omega_n < \#h\omega'_n$, i.e. the number of subtrees of ω is strictly less than that of ω' . In particular, the number of nodes of ω is strictly less than that of ω' , i.e. $\deg h\omega < \deg h\omega'$.

The case where $f \in \mathbb{A}_-$ is treated similarly. \square

Proof of theorem 4.1.3, axiom (Sq2). Let $f : h\omega \rightarrow h\omega'$ be a morphism in \mathbb{A} , with $\omega, \omega' \in \mathbb{O}_{n+1}$. It maps the maximal subtree $\omega \subseteq \omega$ to a subtree of ω' , say $\nu \subseteq \omega'$. Then the following is the desired factorization:

$$\begin{array}{ccc} h\omega & \xrightarrow{f} & h\omega' \\ & \searrow f & \nearrow \\ & h\nu & \end{array}$$

Uniqueness comes from the fact that \mathbb{A}_+ only contains inclusions (of $\mathcal{Psh}(\mathbb{O}_{\geq n-1})$). \square

Proof of theorem 4.1.3, axiom (Sq3). Let $f : h\omega \rightarrow h\omega'$ be a morphism in \mathbb{A}_- . We define a section g of f . By adjointness, this is equivalent to specifying a map $\bar{g} : O[\omega'] \rightarrow h\omega$ in $\mathcal{Psh}(\mathbb{O}_{\geq n-1})$. Since $\mathbb{S}_{\geq n+1} \perp h\omega$, it is enough to define \bar{g} on the spine $S[\omega']$ of ω' . For $[p] \in (\omega')^\bullet$, let

$$\bar{g}(\mathfrak{s}_{[p]}\omega') := \mathfrak{s}_{[q]}\omega, \quad [q] := \min \{ [r] \in \omega^\bullet \mid f(\mathfrak{s}_{[r]}\omega) = \mathfrak{s}_{[p]}\omega' \}.$$

In other words, g maps the node $\mathfrak{s}_{[p]}\omega'$ to the lexicographically minimal node in the fiber $f^{-1}(\mathfrak{s}_{[p]}\omega')$. The composite

$$S[\omega'] \xrightarrow{\bar{g}} h\omega \xrightarrow{f} h\omega'$$

maps a node $\mathfrak{s}_{[p]}\omega'$ to $\mathfrak{s}_{[p]}\omega'$, thus g is a section of f .

Let now $f_1, f_2 : h\omega \rightarrow h\omega'$ be a morphism in \mathbb{A}_- having the same sections. In particular, for g_i the section of f_i constructed as above, where $i = 1, 2$, we have $f_1 g_2 = \text{id}_{h\omega'}$. Thus for $[p] \in (\omega')^\bullet$, we have $g_2(\mathfrak{s}_{[p]}\omega') \leq g_1(\mathfrak{s}_{[p]}\omega')$, meaning that the node $g_2(\mathfrak{s}_{[p]}\omega')$ is lexicographically inferior to (or “below”) $g_1(\mathfrak{s}_{[p]}\omega')$ in ω . Conversely, since $f_2 g_1 = \text{id}_{h\omega'}$, we have $g_1(\mathfrak{s}_{[p]}\omega') \leq g_2(\mathfrak{s}_{[p]}\omega')$, and finally, $g_1(\mathfrak{s}_{[p]}\omega') = g_2(\mathfrak{s}_{[p]}\omega')$. By lemma 3.2.5, $g_1 = g_2$, so clearly, $f_1 = f_2$. \square

4.2. Anodyne extensions.

Definition 4.2.1 (Boundary, [Cis06, 8.1.30]). Let $\lambda \in \mathbb{A}$. The *boundary* $\partial\lambda \in \mathcal{Psh}(\mathbb{A})$ of λ (see definition 2.1.14) is the colimit in the middle, which can equivalently be defined as the union (in $\mathcal{Psh}(\mathbb{A})$) on the right

$$\partial\lambda := \text{colim}_{\substack{f: \lambda' \rightarrow \lambda \text{ in } \mathbb{A}_+, \\ f \text{ not an iso.}}} \lambda' = \bigcup_{\substack{f: \lambda' \rightarrow \lambda \text{ in } \mathbb{A}_+, \\ \text{deg } \lambda' = \text{deg } \lambda - 1}} \text{im } f.$$

Explicitly, if $\lambda = h\omega$, for $\omega \in \mathbb{O}_{n+1}$, then $\partial h\omega$ is the subpresheaf of $h\omega$ spanned by its $(n-1)$ -cells.

We write $\mathfrak{b}_\lambda : \partial\lambda \rightarrow \lambda$ the *boundary inclusion* of λ , and $\mathbb{B} := \{\mathfrak{b}_\lambda \mid \lambda \in \mathbb{A}\}$ the set of boundary inclusions.

Proposition 4.2.2. *The class of monomorphisms in $\mathcal{Psh}(\mathbb{A})$ is exactly the class of \mathbb{B} -cell complexes $\text{Cell}(\mathbb{B})$. Thus in the terminology of [Cis06, definition 1.2.26], \mathbb{B} is a cellular model of $\mathcal{Psh}(\mathbb{A})$.*

Proof. Since \mathbb{A} is a normal skeletal category, proposition 2.1.15 applies. \square

Definition 4.2.3 (Elementary face). Let $\lambda \in \mathbb{A}$.

- (1) A *elementary face* of λ is a morphism $f : \lambda' \rightarrow \lambda$ in \mathbb{A}_+ , where $\text{deg } \lambda' = \text{deg } \lambda - 1$.
- (2) Let $f : \lambda' \rightarrow \lambda$ be an elementary face of λ , and write $\lambda = h\omega$ and $\lambda' = h\omega'$, with $\omega, \omega' \in \mathbb{O}_{n+1}$. The face f is *inner* if f_{n-1} exhibits a bijection between the leaves of ω' (seen as $(n-1)$ -cells of $h\omega'$) and the leaves of ω , and if it maps the root edge $e_{\square}[\omega]$ to $e_{\square}[\omega']$. In other words, f_{n-1} is a bijection $h\partial O[\mathfrak{t}\omega] \rightarrow h\partial O[\mathfrak{t}\omega']$.

Remark 4.2.4. If $f : h\omega' \rightarrow h\omega$ is an inner face of $h\omega$, then a counting argument on the number of nodes of ω' and ω , there exists a unique $[p] \in (\omega')^\bullet$ such that $f_{C_{\omega', [p]}}$ (see notation 3.2.6) is a subtree of ω with two nodes. If $[p] \neq [p'] \in (\omega')^\bullet$, then $f_{C_{\omega', [p'()]}}$ is a generator of $h\omega$, i.e. a one-node subtree. Thus f exhibits a subtree $\nu := f_{C_{\omega, [p]}}$ of ω with two nodes, or equivalently, an inner edge.

This remark motivates the following terminology:

Definition 4.2.5 (Inner horn). Let $\omega \in \mathbb{O}_{n+1}$.

- (1) If ω is degenerate (resp. an endotope), say $\omega = \mathfrak{l}_\phi$ for some $\phi \in \mathbb{O}_{n+1}$ (resp. \mathfrak{Y}_ϕ for some $\phi \in \mathbb{O}_n$), then the inner horn of $h\omega$ is simply $\Lambda h\omega := h\phi$. Write $\mathfrak{h} : \Lambda h\omega \rightarrow h\omega$ for the horn inclusion of $h\omega$.
- (2) Otherwise, for f an inner face of λ , define $\Lambda^f \lambda$, the *inner horn* of λ at f , as the colimit on the left, or equivalently, as the union on the right

$$\Lambda^f \lambda := \text{colim}_{\substack{g: \lambda' \rightarrow \lambda \text{ elem. face} \\ g \neq f}} \lambda' = \bigcup_{\substack{g: \lambda' \rightarrow \lambda \text{ elem. face} \\ g \neq f}} \text{im } g.$$

Let $\mathfrak{h}_\lambda^f : \Lambda^f \lambda \rightarrow \lambda$ be the *inner horn inclusion* λ at f .

Let $\mathbb{H}_{\text{inner}}$ be the set of inner horn inclusions.

Definition 4.2.6 (Inner anodyne extension). The class An_{inner} of *inner anodyne extensions* is defined as $\text{An}_{\text{inner}} := \mathfrak{h}(\mathbb{H}_{\text{inner}}^{\mathfrak{h}})$. An *inner fibration* is a morphism $f \in \mathcal{Psh}(\mathbb{A})$ such that $\mathbb{H}_{\text{inner}} \mathfrak{h} f$, or equivalently, such that $\text{An}_{\text{inner}} \mathfrak{h} f$.

Definition 4.2.7. Generalizing definition 4.2.5, if $\omega \in \mathbb{O}_{n+1}$, and I is a set of (not necessarily inner) faces of $h\omega$, we define

$$\Lambda^I \lambda := \operatorname{colim}_{\substack{g: \lambda' \rightarrow \lambda \\ \text{elem. face} \\ g \notin I}} \lambda' = \bigcup_{\substack{g: \lambda' \rightarrow \lambda \\ \text{elem. face} \\ g \notin I}} \operatorname{im} g,$$

and write $h_{h\omega}^I : \Lambda^I h\omega \rightarrow h\omega$ the canonical inclusion.

Lemma 4.2.8. *Let $\omega \in \mathbb{O}_{n+1}$ have $d \geq 2$ nodes, and $\emptyset \neq I \subseteq J$ be two non empty sets of inner faces of $h\omega$. Then the inclusion $h_{h\omega}^I$ factors as*

$$\Lambda^J h\omega \xrightarrow{u} \Lambda^I h\omega \xrightarrow{h_{h\omega}^J} h\omega,$$

and u is a cell complex of inner horn inclusions of opetopes with at most $d-1$ nodes.

Proof. We proceed by induction on $m := \#(J - I)$. If $m = 0$, then u is an identity, and the result holds trivially. Assume the lemma holds up to $m-1$. Take $f \in J - I$, and let $J' := J - \{f\}$. The inclusion u decomposes as

$$\Lambda^J h\omega \xrightarrow{v} \Lambda^{J'} h\omega \xrightarrow{w} \Lambda^I h\omega$$

and by induction, w is a cell complex of inner horn inclusions of opetopes with at most $d-1$ nodes. It remains to show that v is too.

The inner face f of $h\omega$ exhibits a subtree $\nu \subseteq \omega$ with two nodes, or equivalently, an inner edge of ω , say at address $[e]$. Let $\omega_{/[e]}$ be ω where this inner edge has been contracted. Explicitly, decomposing ω on the left so as to exhibit the subtree ν , the opetope $\omega_{/[e]}$ is defined on the right:

$$\omega = \alpha \circ_{[p]} \nu \bigcirc_{[l_i]} \beta_i, \quad \omega_{/[e]} := \alpha \circ_{[p]} \mathbf{Y}_{\mathbf{t}\nu} \bigcirc_{[\emptyset\nu[l_i]]} \beta_i,$$

where $[l_i]$ ranges over ν^l . Then the inclusion $w : \Lambda^{J'} h\omega \rightarrow \Lambda^I h\omega$ above is obtained as the following pushout

$$\begin{array}{ccc} \Lambda^{J'} h\omega_{/[e]} & \longrightarrow & \Lambda^{J'} h\omega \\ h_{\omega_{/[e]}}^{J'} \downarrow & & \downarrow w \\ h\omega_{/[e]} & \longrightarrow & \Lambda^{J'} h\omega. \end{array}$$

□

Lemma 4.2.9 (Generalization of [MW09, lemma 5.1]). *Let $\omega \in \mathbb{O}_{n+1}$, and I be a non empty set of inner faces of $h\omega$. Then the inclusion $h_{h\omega}^I : \Lambda^I h\omega \rightarrow h\omega$ is an inner anodyne extension.*

Proof. We proceed by induction $d := \deg h\omega$ and on $m := \#I$. Since I is non empty, $d \geq 2$. If $d = 2$, then $h\omega$ has a unique inner face, and the corresponding inner horn inclusion is just the spine inclusion $S[h\omega] \hookrightarrow h\omega$.

Assume now that $d \geq 3$. If $m = 1$, then $\Lambda^I h\omega$ is just an inner horn, and the claim holds. If $m \geq 2$, take $f \in I$, and let $J := I - \{f\}$. The inclusion $h_{h\omega}^I$ decomposes as

$$\Lambda^I h\omega \xrightarrow{u} \Lambda^J h\omega \xrightarrow{h_{h\omega}^J} h\omega.$$

By induction on m , $h_{h\omega}^J$ is an inner anodyne extension. By lemma 4.2.8, u is a cell complex of inner horn inclusions of opetopes of at most $d-1$ nodes, and so by induction on d , u is an inner anodyne extension as well. □

Lemma 4.2.10. *Let $\omega \in \mathbb{O}_{n+1}$ have $d \geq 2$ nodes, and I be the set of all inner faces of ω . Note that $\Lambda^I h\omega$ contains all the generators $c_{\omega, [p]}$, and thus the spine inclusion $s_{h\omega}$ decomposes as*

$$S[h\omega] \xrightarrow{u} \Lambda^I h\omega \xrightarrow{h_{h\omega}^I} h\omega.$$

Then the inclusion u is a cell complex of spine inclusions of opetopes of at most $d-1$ nodes.

Proof. Note that an outer face (all elementary faces that are not inner) of $h\omega$ are inclusions $\omega' \subseteq \omega$ of subtrees of ω of $d-1$ nodes. Hence $\Lambda^I h\omega$ is the union on the left, while $S[h\omega]$ can be expressed as on the right

$$\Lambda^I h\omega = \bigcup_{\substack{\omega' \subseteq \omega \\ \deg h\omega' = d-1}} h\omega', \quad S[h\omega] = \bigcup_{\substack{\omega' \subseteq \omega \\ \deg h\omega' = d-1}} S[h\omega'].$$

Thus

$$u : \bigcup_{\substack{\omega' \subseteq \omega \\ \deg h\omega' = d-1}} S[h\omega'] \hookrightarrow \bigcup_{\substack{\omega' \subseteq \omega \\ \deg h\omega' = d-1}} h\omega'$$

can be obtained as a composition of pushouts of the $s_{\omega'}$'s. Explicitly, write $\omega_1, \dots, \omega_m \subseteq \omega$ be the subtrees of ω with $d-1$ nodes, and g_1, \dots, g_m the associated outer face inclusions of $h\omega$. Define $X_0 := S[\omega]$, ι_0 be the identity on X_0 , and for $1 \leq i \leq m$, let X_i be the pushout

$$\begin{array}{ccc} S[\omega_i] & \xrightarrow{\iota_{i-1} g_i s_{\omega_i}} & X_{i-1} \\ \downarrow & \lrcorner & \downarrow v \\ h\omega_i & \longrightarrow & X_i, \end{array}$$

and $\iota_i := v \iota_{i-1}$. Then $\Lambda^I h\omega = X_m$. □

Proposition 4.2.11 (Generalization of [CM13, proposition 2.4]). *We have an inclusion $\mathfrak{h}(\mathfrak{S}^{\mathfrak{h}}) \subseteq \text{An}_{\text{inner}}$.*

Proof. It is enough to show that spine inclusions $s_{h\omega} : S[h\omega] \hookrightarrow h\omega$ are inner anodyne extensions.

- (1) If $\deg h\omega = \#\omega^\bullet = 0, 1$, then $s_{h\omega}$ is an identity, thus an inner anodyne extension.
- (2) If $\deg h\omega = 2$, then $h\omega$ admits a unique inner face $f : hY_{t\omega} \rightarrow h\omega$, and note that $\Lambda^f h\omega = S[h\omega]$. Thus in this case, the spine inclusion is an inner horn inclusion.
- (3) Assume $d = \deg h\omega \geq 3$, and let I be the set of all inner faces of $h\omega$. Note that since $\Lambda^I h\omega$ contains all generators of $h\omega$ (i.e. subtrees of one node), the spine inclusion $s_{h\omega}$ decomposes as

$$S[h\omega] \xhookrightarrow{u} \Lambda^I h\omega \xhookrightarrow{h_{h\omega}^f} h\omega,$$

and in order to show that $s_{h\omega}$ is an inner anodyne extension, it suffices to show that u is. By lemma 4.2.10, it is a cell complex of spine inclusions of opetopes of at most $d-1$ nodes, thus by induction, it is an inner anodyne extension. □

Proposition 4.2.12. *Inner anodyne extensions are S-local isomorphisms.*

Proof. It is enough to show that inner horn inclusions are S-local isomorphisms. Let $\omega \in \mathbb{O}_{n+1}$ have $d \geq 2$ nodes. If $d = 2$, then it has a unique inner face, and the corresponding inner horn inclusion is simply the spine inclusion $S[h\omega] \hookrightarrow h\omega$.

Assume $d \geq 3$, let I be the set of all inner faces of $h\omega$, and $f \in I$. Then the spine inclusion $s_{h\omega}$ decomposes as

$$S[h\omega] \xhookrightarrow{u} \Lambda^I h\omega \xhookrightarrow{v} \Lambda^f h\omega \xhookrightarrow{h_{h\omega}^f} h\omega.$$

By lemma 4.2.10, u is a spine complex, thus an S local-isomorphism. By lemma 4.2.8, v is a cell complex of inner horn inclusions of opetopes with at most $d-1$ nodes. By induction, v is an S local-isomorphism. Thus vu and $s_{h\omega} = h_{h\omega}^f \cdot (vu)$ are S local-isomorphisms, and by 3-for-2, so is $h_{h\omega}^f$. □

4.3. Cylinder objects. The Resk cylinder construction of definition 3.5.1 extends to $\mathcal{Psh}(\mathfrak{A})$.

Definition 4.3.1 (Rezk interval). For $\phi \in \mathbb{O}_{n-1}$, recall the definition of the Rezk interval $\mathfrak{J}_\phi \in \mathcal{Alg}$. Define $\mathfrak{J}_\phi := M\mathfrak{J}_\phi$. Write $i_\phi : h\phi \rightarrow \mathfrak{J}_\phi$ for the endpoint inclusion, and $\text{E}_{\mathfrak{J}} := \{i_\phi \mid \phi \in \mathbb{O}_{n-1}\}$.

Definition 4.3.2 (Rezk cylinder). For $\omega \in \mathbb{O}_{\geq n-1}$, let $\mathcal{J}h\omega := M(\mathcal{J}h\omega)$. Extend \mathcal{J} by colimits to obtain a functor $\mathcal{J} : \mathcal{Psh}(\mathbb{A}) \rightarrow \mathcal{Psh}(\mathbb{A})$. The *Rezk cylinder* $\mathcal{J}X$ of a presheaf $X \in \mathcal{Psh}(\mathbb{A})$ is a cylinder object in the sense of definition 2.1.5, i.e. we have a canonical factorization of the codiagonal map

$$\begin{array}{ccc} X + X & \xrightarrow{\nabla} & \mathcal{J}X \\ & \searrow^{(i_0, i_1)} & \nearrow_{\nabla} \\ & X & \end{array}$$

Explicitly, for $X \in \mathcal{Psh}(\mathbb{A})$,

$$\mathcal{J}X := \operatorname{colim}_{h\omega \rightarrow X} M(\mathcal{J}h\omega).$$

In dimension $(n-1)$ and n , $\mathcal{J}X$ is the following pushout

$$\begin{array}{ccc} X_{n-1} + X_{n-1} & \longrightarrow & X_{n-1,n} + X_{n-1,n} \\ \downarrow b & & \downarrow \\ M\mathcal{J}_{X_{n-1}} & \longrightarrow & (\mathcal{J}X)_{n-1,n}, \end{array}$$

where b maps a $x \in X_\psi$ in the first (resp. second) component to 0_x (resp. 1_x) in $M\mathcal{J}_x \subseteq M\mathcal{J}_{X_{n-1}}$. The $(n+1)$ -cells of $\mathcal{J}X$ are so that in $v\mathcal{J}X$, the following relation (analogous to equation (3.5.2)) holds:

$$x^{(1)} \bigcirc_{[p_i]} j_{s_{[p_i]}x} = j_{t_x} \circ x^{(0)}$$

for a cell $x \in X_\omega$, $\omega \in \mathbb{O}_n$, and $\omega^\bullet = \{[p_1], \dots\}$. We readily deduce the following:

Lemma 4.3.3. For $X \in \mathcal{Psh}(\mathbb{A})$ we have a canonical isomorphism $\mathcal{J}vX \cong v\mathcal{J}X$.

Proposition 4.3.4. The functorial cylinder \mathcal{J} is an elementary homotopical data (definition 2.1.7).

Proof. Straightforward unpacking of the definition of \mathcal{J} . □

4.4. Homotopical structure. Recall from definition 4.2.5 that the class $\mathbf{An}_{\text{inner}}$ of inner anodyne extensions of $\mathcal{Psh}(\mathbb{A})$ is the class ${}^{\mathfrak{h}}(\mathbf{H}_{\text{inner}}^{\mathfrak{h}})$ of retracts of cell complexes of inner horn inclusions.

Definition 4.4.1 (\mathcal{J} -anodyne extension). Recall from definition 4.3.1 the set $\mathbf{E}_{\mathcal{J}}$ of endpoint inclusions of the Rezk intervals in $\mathcal{Psh}(\mathbb{A})$. Let $\mathbf{An}_{\mathcal{J}}$, the class of \mathcal{J} -anodyne extensions, be the class ${}^{\mathfrak{h}}((\mathbf{H}_{\text{inner}} \cup \mathbf{E}_{\mathcal{J}})^{\mathfrak{h}})$.

Definition 4.4.2 (Lifting problem). Let $k : K \hookrightarrow h\omega$ be a subpresheaf of a representable presheaf, and $f : K \rightarrow X$ a morphism. We say that f is a *lifting problem* of degree d (or *d-lifting problem*), where $d = \deg h\omega$. We say that f is *k-unsolved* (or just *unsolved* if k is clear from the context) if f does not factor through k .

Lemma 4.4.3. Let $X \in \mathcal{Psh}(\mathbb{A})$, be such that the unit map $\eta_X : X \rightarrow MvX$ is a monomorphism. Then η_X is an inner horn complex, and in particular an inner anodyne extension.

Proof. Let $X^{(0)} := X$ and $\iota^{(0)} := \eta_X$. If $x \in X^{(0)}$, we write $\deg x = 0$. Assume by induction that we have an inclusion $\iota^{(\alpha)} : X^{(\alpha)} \hookrightarrow MvX$ for all ordinal $\alpha < \beta$, and a degree function $\deg : X^{(\alpha)} \rightarrow \alpha + 1$.

If β is a limit ordinal, simply set $X^{(\beta)} := \bigcup_{\alpha < \beta} X^{(\alpha)}$, and $\iota^{(\beta)}$ to be the induced inclusion. Assume that β is a successor ordinal, say $\beta = \alpha + 1$, and choose a horn lifting problem $l : \Lambda^f h\omega \rightarrow X^{(\alpha)}$ such that

- (1) **(L1)** l is unsolved in $X^{(\alpha)}$,
- (2) **(L2)** l has degree ≤ 2 only if there is no unsolved horn lifting problem of degree ≥ 3 .

If such an l does not exist, simply set $X^{(\alpha+1)} := X^{(\alpha)}$ and $\iota^{(\alpha+1)} := \iota^{(\alpha)}$. If it does, set $X^{(\alpha+1)}$ to be the following pushout

$$\begin{array}{ccc} \Lambda^f h\omega & \xrightarrow{l} & X^{(\alpha)} \\ \downarrow & & \downarrow \\ h\omega & \xrightarrow{u} & X^{(\alpha+1)}, \end{array}$$

and $\iota^{(\alpha+1)}$ to be the induced map $X^{(\alpha+1)} \rightarrow MvX$. For $x \in X^{(\alpha+1)} - X^{(\alpha)}$, set $\deg x := \alpha + 1$.

We claim that $\iota^{(\alpha+1)}$ is a monomorphism. Towards a contradiction, assume that it is not. For $\nu \subseteq \omega$ the two-nodes subtree corresponding to the inner face f , write $x \in X_{h\nu}^{(\alpha+1)}$ the cell selected by the arrow

$$h\nu \hookrightarrow h\omega \xrightarrow{u} X^{(\alpha+1)}$$

Then there exist $y \in X_{h\nu}^{(\alpha)}$ such that $\iota_{(\alpha+1)}(x) = \iota_{(\alpha+1)}(y)$.

- (1) If l has degree ≤ 2 , then the lifting problem l was not unsolved in $X^{(\alpha)}$, a contradiction with condition **(L1)**.
- (2) If the degree of l is ≥ 3 , then the cell y gives a factorization of l as

$$\begin{array}{ccc} \Lambda^f h\omega & \xrightarrow{l} & X^{(\alpha)} \\ \downarrow & \nearrow l' & \\ \partial h\omega, & & \end{array}$$

with l' unsolved in $X^{(\alpha)}$. But this contradicts condition **(L2)**. Indeed, the step of the induction that created y but not $u(h\omega)$ only considered a lifting problem of degree 2, whereas l could have been considered instead.

Therefore, $\iota^{(\alpha+1)}$ is a monomorphism.

If κ is the cardinal of the set of horn lifting problems of MvX . Then the sequence X_α stabilizes after $\kappa + 1$, as all lifting problems have been exhausted. Clearly,

$$MvX \cong \operatorname{colim}_{\alpha < \kappa+1} X^{(\alpha)},$$

and by construction, η_X is an inner horn complex. \square

Corollary 4.4.4 (Generation lemma). *Take $X \in \mathcal{Psh}(\Lambda)$, $A \in \mathcal{Alg}$, and $m : X \hookrightarrow MA$ be a monomorphism such that its transpose $\bar{m} : vX \rightarrow A$ is an isomorphism. Then m is an inner horn complex, and in particular, an inner anodyne extension.*

Proof. The condition states that up to isomorphism, m is the unit map $\eta_X : X \rightarrow MvX$. We can apply lemma 4.4.3 to conclude. \square

Proposition 4.4.5. *The pair $(\mathfrak{J}, \mathcal{An}_{\mathfrak{J}})$ satisfies condition **(An1)** of definition 2.1.8.*

Proof. By [Cis06, proposition 1.1.16], it is enough to check the claim when m is a boundary inclusion, say $m = \mathbf{b}_{h\omega} : X = \partial h\omega \rightarrow Y = h\omega$, where $\omega \in \mathbb{O}_{n+1}$.

- (1) Assume $\deg h\omega = 0$, i.e. ω is degenerate, say $\omega = \mathbf{l}_\phi$. Then $\partial h\omega = \emptyset$, and since \mathfrak{J} preserves colimits, $\mathfrak{J}\partial h\omega = \emptyset$ as well. Thus g is the inclusion $h\omega^{(e)} = \mathbf{h}\mathbf{l}_\phi \rightarrow \mathfrak{J}h\omega = M\mathfrak{J}_\phi$, i.e. an endpoint inclusion of the Rezk interval \mathfrak{J}_ϕ , which by definition is \mathfrak{J} -anodyne
- (2) Assume $\deg h\omega \geq 1$. We only treat the case $e = 0$, the other one being similar. Let $[p] \in \omega^\bullet$, and consider the n -cell $a = \mathbf{s}_{[p]}\omega$ of $h\omega$. By equation (3.5.2), in $\mathfrak{J}h\omega$, we have

$$a^{(1)} = \mathbf{j}_t a \circ_{\square} a^{(0)} \bigcirc_{[q_i]}^{-1} j_{\mathbf{s}_{[q_i]} a}$$

where $[q_i]$ ranges over $(\mathbf{s}_{[p]}\omega)^\bullet$. Therefore, $\mathfrak{J}h\omega$ is freely generated by $\mathfrak{J}\partial h\omega \cup h\omega^{(0)}$. By corollary 4.4.4, g is an inner anodyne extension. \square

Proposition 4.4.6. *The pair $(\mathfrak{J}, \mathcal{An}_{\mathfrak{J}})$ satisfies condition **(An2)** of definition 2.1.8.*

Proof. By [Cis06, proposition 1.1.16], it is enough to check the claim when m is an inner horn inclusion or an endpoint inclusion of a Rezk interval.

- (1) Assume m is a inner horn inclusion, say $m = h_{h\omega}^f : X = \Lambda^f h\omega \longrightarrow Y = h\omega$, where $\omega \in \mathbb{O}_{n+1}$. Recall that by proposition 4.2.12, $v\Lambda^f h\omega = h\omega$. Thus applying v to the diagram of definition 2.1.7 (**An2**) yeilds

$$\begin{array}{ccc}
 h\omega + h\omega & \xlongequal{\quad} & h\omega + h\omega \\
 \downarrow & \lrcorner & \downarrow \\
 \mathfrak{J}h\omega & \xlongequal{\quad} & \mathfrak{J}h\omega \\
 & \searrow & \dashrightarrow^{vg} \\
 & & \mathfrak{J}h\omega.
 \end{array}$$

Thus $g : \mathfrak{J}\Lambda^f h\omega \cup (h\omega + h\omega) \longrightarrow \mathfrak{J}h\omega = M\mathfrak{J}h\omega$ is such that its transpose

$$\mathfrak{J}h\omega \xrightarrow{vg} vM\mathfrak{J}h\omega \cong \mathfrak{J}h\omega$$

is an isomorphism. By corollary 4.4.4, g is an inner anodyne extension.

- (2) Assume m is an endpoint inclusion of a Rezk interval, say $m : X = Mh\phi \hookrightarrow Y = M\mathfrak{J}\phi$, for a $\phi \in \mathbb{O}_{n-1}$. We thus have $\mathfrak{J}X \cup (Y + Y) = \mathfrak{J}Mh\phi \cup (M\mathfrak{J}\phi + M\mathfrak{J}\phi) = M\mathfrak{J}\phi \cup (M\mathfrak{J}\phi + M\mathfrak{J}\phi)$.

We have $\mathfrak{J}Y = \mathfrak{J}(M\mathfrak{J}\phi) = M(\mathfrak{J}v(M\mathfrak{J}\phi)) \cong M(\mathfrak{J}\mathfrak{J}\phi)$. The algebra $\mathfrak{J}\mathfrak{J}\phi$ contains four $(n-1)$ -cells 00_ϕ , 01_ϕ , 10_ϕ , and 11_ϕ of shape ϕ , and is generated by the n -cells $j_\phi^{(0*)} : 00_\phi \longrightarrow 01_\phi$, $j_\phi^{(1*)} : 10_\phi \longrightarrow 11_\phi$, $j_\phi^{(*0)} : 00_\phi \longrightarrow 10_\phi$, $j_\phi^{(*1)} : 10_\phi \longrightarrow 11_\phi$ and their inverses. Further, the equality on the right holds, which can be depicted as a commutative square of invertible arrows on the right:

$$\begin{array}{ccc}
 j_\phi^{(*1)} \circ j_\phi^{(0*)} & = & j_\phi^{(1*)} \circ j_\phi^{(*0)}, \\
 \begin{array}{ccc}
 00_\phi & \xrightarrow{j_\phi^{(0*)}} & 01_\phi \\
 j_\phi^{(*0)} \downarrow & & \downarrow j_\phi^{(*1)} \\
 10_\phi & \xrightarrow{j_\phi^{(1*)}} & 11_\phi.
 \end{array}
 \end{array}$$

On the other hand, the pushout $M\mathfrak{J}\phi \cup (M\mathfrak{J}\phi + M\mathfrak{J}\phi)$ contains $j_\phi^{(0*)}$, $j_\phi^{(*0)}$, $j_\phi^{(*1)}$, and their inverses. Thus it generates $\mathfrak{J}\mathfrak{J}\phi$, i.e. the cocartesian gap map g satisfies the conditions of corollary 4.4.4. Consequently, it is an inner anodyne extension. \square

Theorem 4.4.7. *The category \mathbb{A} endowed with the functorial cylinder $\mathfrak{J} : \mathcal{Psh}(\mathbb{A}) \longrightarrow \mathcal{Psh}(\mathbb{A})$ and the class $\mathbf{An}_{\mathfrak{J}}$ of \mathfrak{J} -anodyne extension forms a homotopical structure (definition 2.1.9)*

Using theorem 2.1.11, we obtain the following model structure for ∞ -algebras $\mathcal{Psh}(\mathbb{A})_\infty$ on $\mathcal{Psh}(\mathbb{A})$:

- (1) a morphism f is a naive fibration if $\mathbf{An}_{\mathfrak{J}} \dashv f$ (definition 4.4.1); a presheaf $X \in \mathcal{Psh}(\mathbb{A})$ is fibrant if the terminal morphism $X \longrightarrow 1$ is a naive fibration;
- (2) a morphism $f : X \longrightarrow Y$ is a weak equivalence if for all fibrant object $P \in \mathcal{Psh}(\mathbb{A})$, the induced map $f^* : \mathcal{H}\mathcal{O}\mathcal{Psh}(\mathbb{A})(Y, P) \longrightarrow \mathcal{H}\mathcal{O}\mathcal{Psh}(\mathbb{A})(X, P)$ is a bijection, where \simeq is the \mathfrak{J} -homotopy relation of definitions 2.1.6 and 4.3.2;
- (3) a morphism f is a cofibrations if it is a monomorphisms, it is a acyclic cofibrations if it is a cofibration and a weak equivalence;
- (4) a morphism f is a fibration if it has the right lifting property with respect to acyclic cofibrations, it is an acyclic fibration if it has the right lifting property with respect to all cofibrations.

In particular, $\mathcal{Psh}(\mathbb{A})_\infty$ is of Cisinski type (definition 2.1.10), cellular, and proper. Fibrant objects in $\mathcal{Psh}(\mathbb{A})_\infty$ are called ∞ -algebras (or inner Kan complexes).

Proof. By definition, $\mathbf{An}_{\mathfrak{J}}$ is the class of cell complexes over a set of monomorphisms, thus it satisfies axiom (**An0**). Axioms (**An1**) and (**An2**) are checked by propositions 4.4.5 and 4.4.6 respectively. \square

This construction is a direct generalization of the model structure $\mathcal{Psh}(\mathbb{A})_{\text{Joyal}}$ for quasi-categories [JT07, theorem 1.9], and of $\mathcal{Psh}(\mathbb{O})_{\text{CM}}$ for planar ∞ -operads [CM11, theorem 2.4].

Lemma 4.4.8. *For all $X \in \mathcal{Psh}(\mathbb{A})$ we have $\mathbf{E}_{\mathfrak{J}} \dashv X$. In particular, if $\mathbf{H}_{\text{inner}} \dashv X$ if and only X is an ∞ -algebra.*

Proof. Let $\phi \in \mathbb{O}_{n-1}$, and consider a lifting problem $f : h\phi \rightarrow X$. In particular, f exhibits a cell $x \in X_{h\phi}$, but also a cell $\text{id}_x \in X_{hY_\phi}$. Then a lift \bar{f} of f as in

$$\begin{array}{ccc} h\phi & \xrightarrow{f} & X \\ i_\phi \downarrow & \nearrow \bar{f} & \\ \mathcal{J}_\phi & & \end{array}$$

can be obtained by mapping 0_ϕ and 1_ϕ to x , and j_ϕ and j_ϕ^{-1} to id_x (see definition 3.4.9 for notations). \square

The following results comes as a ‘‘sanity-check’’ for the model structure of theorem 4.4.7:

Proposition 4.4.9. *For $A \in \text{Alg}$, its nerve MA is an ∞ -algebra.*

Proof. By theorem 3.1.5, $S \perp MA$, and thus by proposition 4.2.12, $H_{\text{inner}} \perp MA$. We apply lemma 4.4.8 to conclude. \square

Proposition 4.4.10. *We have a Quillen adjunction $v : \mathcal{Psh}(\mathbb{A})_\infty \xleftrightarrow{\quad} \text{Alg}_{\text{folk}} : M$.*

Proof. Trivially, if $f : X \rightarrow Y$ is a monomorphism in $\mathcal{Psh}(\mathbb{A})$, then vf is injective on $(n-1)$ -cells. Therefore, v preserves cofibrations. To conclude, it suffices to show that M preserves fibrations.

First, note that v maps H_{inner} to isomorphisms, and since the adjunction is reflective, it maps $E_{\mathcal{J}} = ME_{\mathcal{J}}$ (see definition 4.3.1) to $E_{\mathcal{J}}$ up to isomorphism. Therefore, v maps $\text{An}_{\mathcal{J}}$ to acyclic cofibrations. Let now f be a fibration in Alg_{folk} . Then f has the right lifting property against all acyclic cofibrations, and in particular, $v\text{An}_{\mathcal{J}} \pitchfork f$. By adjointness, $\text{An}_{\mathcal{J}} \pitchfork Mf$, and thus Mf is a naive fibration. By proposition 4.4.9, the codomain of Mf is an ∞ -algebra, and we apply lemma 2.1.12 to conclude that Mf is a fibration. \square

Definition 4.4.11 (Quillen model structure). Let \mathbb{H} be the set of all horn inclusions of \mathbb{A} , not only the inner ones. The *Quillen model structure* $\mathcal{Psh}(\mathbb{A})_{\text{Quillen}}$ on $\mathcal{Psh}(\mathbb{A})$ is the Bousfield localization $\mathbb{H}^{-1}\mathcal{Psh}(\mathbb{A})_\infty$, which exists by [Hir03, theorem 4.1.1].

Note that $\mathcal{Psh}(\mathbb{A})_{\text{Quillen}}$ is still of Cisinski type, cellular, and proper. In this structure, fibrant objects are called *Kan complexes*, and fibrations are *Kan fibrations*. For instance, in the case $(k, n) = (1, 1)$, we recover the classical Quillen structure on simplicial sets.

4.5. Underlying categories. This section is devoted to prove the technical lemma 4.5.7. If $n = 1$, i.e. $\mathbb{A}_{1,1} = \Delta$, then the result is stated in [JT07, proposition 1.13] and [Joy08, 2.13]. The bulk of the work is to study the *underlying category* adjunction

$$C_{\phi!} : \mathcal{Psh}(\Delta) \xleftrightarrow{\quad} \mathcal{Psh}(\mathbb{A}) : C_\phi^*, \quad (4.5.1)$$

where $\phi \in \mathbb{O}_{n-1}$, to reduce a similar statements for an arbitrary $n \geq 1$ to $n = 1$.

In this section, we assume $n \geq 1$, and for once, do not omit if from various notations, e.g. $\mathbb{A}_{1,n}$, $\text{Alg}_{1,n}$, etc. Recall that $\mathbb{A}_{1,1} = \Delta$ and $\text{Alg}_{1,1} = \text{Cat}$. In this case, the adjunction $v_{1,n} : \mathcal{Psh}(\mathbb{A}_{1,n}) \xleftrightarrow{\quad} \text{Alg}_{1,n} : M_{1,n}$ is more commonly denoted by $\tau : \mathcal{Psh}(\Delta) \xleftrightarrow{\quad} \text{Cat} : N$.

Pick $\phi \in \mathbb{O}_{n-1}$, and define an functor $C_\phi : \Delta \rightarrow \mathbb{A}_{1,n}$ as

$$[0] \mapsto h\phi, \quad [1] \mapsto hY_\phi, \quad [i] \mapsto h \left(Y_\phi \circ_{\square} Y_\phi \circ_{\square} \cdots \circ_{\square} Y_\phi \right),$$

where on the right, there are i instances of Y_ϕ . In other words, C_ϕ maps $[i]$ to $h\omega$, where $\omega \in \mathbb{O}_{n+1}$ is the linear tree with i nodes, all decorated by $Y_\phi \in \mathbb{O}_n$. Note that C_ϕ is an embedding that exhibits Δ as an *sieve* of $\mathbb{A}_{1,n}$ in that of for all morphism $\lambda \rightarrow \lambda'$ in $\mathbb{A}_{1,n}$, if $\lambda' \in \text{im } C_\phi$, then so is λ .

This functor induces an adjunction

$$C_{\phi!} : \mathcal{Psh}(\Delta) \xleftrightarrow{\quad} \mathcal{Psh}(\mathbb{A}_{1,n}) : C_\phi^*, \quad (4.5.2)$$

where $C_{\phi!}$ is the left Kan extension of $\Delta \xrightarrow{C_\phi} \mathbb{A}_{1,n} \rightarrow \mathcal{Psh}(\mathbb{A}_{1,n})$ along the Yoneda embedding, and C_ϕ^* is the precomposition by C_ϕ . Since C_ϕ is an embedding, so is $C_{\phi!}$, and furthermore, it exhibits $\mathcal{Psh}(\Delta)$ as an sieve of $\mathcal{Psh}(\mathbb{A}_{1,n})$. The following result follows directly from this observation:

Lemma 4.5.3. *Let $k \in \mathbb{N}$.*

(1) We have $C_{\phi!}S[k] \cong S[C_{\phi}[k]]$.

(2) If $\Lambda^l[k]$ is an inner horn of $[k]$, then $C_{\phi!}\Lambda^l[k]$ is an inner horn of $C_{\phi}[k]$, and all inner horns of $C_{\phi}[k]$ are obtained in this way.

Proof. The spine and inner horns of $C_{\phi}[k]$ are obtained as colimits of presheaves over $C_{\phi}[k]$. Since $C_{\phi!}$ exhibits $\mathcal{Psh}(\Delta)$ as an sieve of $\mathcal{Psh}(\Lambda_{1,n})$, those colimits can be computed in $\mathcal{Psh}(\Delta)$. \square

Lemma 4.5.4. *The adjunction equation (4.5.2) restricts and corestricts as an adjunction $C_{\phi!} : \mathcal{Cat} \rightleftarrows \mathcal{Alg}_{1,n} : C_{\phi}^*$. We thus have three commutative squares, where the last one is obtained by adjunction from the second one:*

$$\begin{array}{ccc} \mathcal{Psh}(\Delta) \xrightarrow{C_{\phi!}} \mathcal{Psh}(\Lambda_{1,n}) & \mathcal{Psh}(\Delta) \xleftarrow{C_{\phi}^*} \mathcal{Psh}(\Lambda_{1,n}) & \mathcal{Psh}(\Delta) \xrightarrow{C_{\phi!}} \mathcal{Psh}(\Lambda_{1,n}) \\ N \uparrow & N \uparrow & \tau \downarrow \\ \mathcal{Cat} \xrightarrow{C_{\phi!}} \mathcal{Alg}_{1,n} & \mathcal{Cat} \xleftarrow{C_{\phi}^*} \mathcal{Alg}_{1,n} & \mathcal{Cat} \xrightarrow{C_{\phi!}} \mathcal{Alg}_{1,n} \\ & & \downarrow v_{1,n} \end{array}$$

Proof. (1) Take $K \in \mathcal{Psh}(\Delta)$. Note that if $\phi' \in \mathbb{O}_{n-1}$, $\phi' \neq \phi$, then $(C_{\phi!}K)_{\phi'} = \emptyset$, i.e. $C_{\phi!}K$ does not have cells of shape $\phi' \neq \phi$. If $\omega \in \mathbb{O}_{\geq n-1}$, $\omega \notin \text{im } C_{\phi}$, then ω has a $(n-1)$ -dimensional face different from ϕ , whence $\mathcal{Psh}(\Lambda_{1,n})(S[h\omega], C_{\phi!}K) = \emptyset$. By lemma 4.5.3 and the fact that $C_{\phi!}$ is fully faithful, if $S_{1,1} \perp K$, then $S_{1,n} \perp d_{\phi!}K$, and $C_{\phi!}$ restricts and corestricts as a functor $\mathcal{Cat} \rightarrow \mathcal{Alg}_{1,n}$.

(2) Take $A \in \mathcal{Alg}_{1,n}$, i.e. a presheaf $A \in \mathcal{Psh}(\Lambda_{1,n})$ such that $S_{1,n} \perp A$. By lemma 4.5.3, $C_{\phi!}S_{1,1} \subseteq S_{1,n}$, so in particular, $C_{\phi!}S_{1,1} \perp A$, whence $S_{1,1} \perp C_{\phi}^*A$, i.e. $C_{\phi}^*A \in \mathcal{Cat}$. \square

Lemma 4.5.5. (1) For $A \in \mathcal{Cat}$, we have $C_{\phi!}\mathfrak{J}A \cong \mathfrak{J}C_{\phi!}A$.

(2) For $K \in \mathcal{Psh}(\Delta)$, we have $C_{\phi!}\mathfrak{J}K \cong \mathfrak{J}C_{\phi!}K$.

(3) Let $f, g : K \rightarrow L$ be two parallel maps in $\mathcal{Psh}(\Delta)$. If $f \simeq g$ (definition 2.1.6), then $C_{\phi!}f \simeq C_{\phi!}g$.

Proof. Point (1) is by definition, and (3) follows from (2). To prove (2), consider

$$\begin{aligned} C_{\phi!}\mathfrak{J}K &= C_{\phi!}N\mathfrak{J}\tau K && \text{definition 4.3.2} \\ &\cong M_{1,n}C_{\phi!}\mathfrak{J}\tau K && \text{lemma 4.5.4} \\ &\cong M_{1,n}\mathfrak{J}C_{\phi!}\tau K && \text{lemma 4.5.4} \\ &\cong M_{1,n}\mathfrak{J}v_{1,n}C_{\phi!}K && \text{lemma 4.5.4} \\ &= \mathfrak{J}C_{\phi!}K && \text{definition 4.3.2} \end{aligned}$$

\square

Proposition 4.5.6. *We have a Quillen adjunction $C_{\phi!} : \mathcal{Psh}(\Delta)_{\infty} \rightleftarrows \mathcal{Psh}(\Lambda_{1,n})_{\infty} : C_{\phi}^*$.*

Proof. Clearly, $C_{\phi!}$ preserves monomorphisms, and by lemma 4.5.5, it preserves the homotopy relation, thus the weak equivalences. \square

Lemma 4.5.7 (Generalization of [JT07, proposition 1.13] and [Joy08, 2.13]). *Let $X, Y \in \mathcal{Psh}(\Lambda)$ be ∞ -algebras, and $f : X \rightarrow Y$ be an inner fibration (definition 4.2.6). If vf is a fibration in $\mathcal{Alg}_{\text{folk}}$ (i.e. $E_{\mathfrak{J}} \dashv v_{1,n}f$), then $E_{\mathfrak{J}} \dashv f$.*

Proof. If $n = 1$, i.e. $\Lambda = \Delta$, then the result holds by [JT07, proposition 1.13]. Assume $n > 1$, and take $\phi \in \mathbb{O}_{n-1}$. By proposition 4.5.6, C_{ϕ}^* is a right Quillen functor, thus C_{ϕ}^*X and C_{ϕ}^*Y are quasi-categories. By lemma 4.5.3, C_{ϕ}^*f is an inner fibration, since f is. By assumption, $E_{\mathfrak{J}} \dashv v_{1,n}f$, and in particular, $C_{\phi!}j_{\bullet} = j_{\phi} \dashv v_{1,n}f$. By adjunction and lemma 4.5.4, $j_{\bullet} \dashv C_{\phi}^*v_{1,n}f = \tau C_{\phi}^*f$, i.e. C_{ϕ}^*f is a fibration in $\mathcal{Cat}_{\text{folk}}$. Finally, C_{ϕ}^*f satisfies the conditions of [JT07, proposition 1.13], hence $i_{\bullet} \dashv C_{\phi}^*f$. By adjunction, $i_{\phi} \dashv f$. \square

4.6. Simplicial tensor and cotensor.

Definition 4.6.1. For $k \in \mathbb{N}$ and $\lambda \in \Lambda$, let $\Delta[k] \otimes \lambda$ be the nerve

$$\Delta[k] \otimes \lambda := M \left(\mathfrak{J}\lambda \coprod_{\lambda} \mathfrak{J}\lambda \coprod_{\lambda} \mathfrak{J}\lambda \coprod_{\lambda} \cdots \coprod_{\lambda} \mathfrak{J}\lambda \right),$$

where there is k instances of $\mathfrak{J}\lambda$. In other words, it is the nerve of k instances of the cylinder $\mathfrak{J}\lambda$ ‘‘glued end-to-end’’. Extending in both variables by colimits yields a tensor product $- \otimes - : \mathcal{Psh}(\Delta) \times \mathcal{Psh}(\Lambda) \rightarrow \mathcal{Psh}(\Lambda)$.

Let us unfold the definition a little bit. Take $X \in \mathcal{Psh}(\mathbb{A})$ and $K \in \mathcal{Psh}(\mathbb{A})$. Then for each $x \in X_{h\phi}$ with $\phi \in \mathbb{O}_{n-1}$, and $k \in K_0$, there is a cell $k \otimes x \in (K \otimes X)_{h\phi}$. For all edge $e \in K_1$, there is an isomorphism $e \otimes x \in (K \otimes X)_{h\phi}$ with source $d_1 e \otimes x$ and target $d_0 e \otimes x$. More generally, for every m -cell $k \in K_m$, writing $k_0, \dots, k_m \in K_0$ its vertices, and $k_{i,j}$ its edge from k_i to k_j , where $0 \leq i < j \leq m$, we have a cell

$$k \otimes x \in (K \otimes X)_{h\omega}, \quad \omega := C_\phi[m] = Y_\phi \circlearrowleft Y_\phi \circlearrowleft \dots \circlearrowleft Y_\phi,$$

such that $s_{[*i]}(k \otimes x) = k_{i,i+1} \otimes x$.

With this description, it is clear that for $\mathcal{J}_\bullet \in \mathcal{Psh}(\mathbb{A})$ the nerve of the groupoid generated by one isomorphism (definition 4.3.1), we have $\mathcal{J}X \cong \mathcal{J}_\bullet \otimes X$ for all $X \in \mathcal{Psh}(\mathbb{A})$.

Definition 4.6.2. A mapping space and cotensor can be constructed from the tensor product \otimes of definition 4.6.1 so as to make $\mathcal{Psh}(\mathbb{A})$ tensored and cotensored over $\mathcal{Psh}(\mathbb{A})$:

$$\text{Map}(X, Y)_k := \mathcal{Psh}(\mathbb{A})(\Delta[k] \otimes X, Y), \quad (Y^K)_\lambda := \mathcal{Psh}(\mathbb{A})(K \otimes \lambda, Y),$$

where $X, Y \in \mathcal{Psh}(\mathbb{A})$, $K \in \mathcal{Psh}(\mathbb{A})$, and $k \in \mathbb{N}$.

Lemma 4.6.3. For $K \in \mathcal{Psh}(\mathbb{A})$ and $X, Y \in \mathcal{Psh}(\mathbb{A})$, consider the natural hom-set isomorphism

$$\Phi : \mathcal{Psh}(\mathbb{A})(K \otimes X, Y) \longrightarrow \mathcal{Psh}(\mathbb{A})(X, Y^K)$$

of the adjunction $K \otimes - : \mathcal{Psh}(\mathbb{A}) \rightleftarrows \mathcal{Psh}(\mathbb{A}) : (-)^K$. The map Φ preserves and reflects the \mathcal{J} -homotopy relation (definitions 2.1.6 and 4.3.2), i.e. it induces an isomorphism

$$\Phi : \mathcal{Psh}(\mathbb{A})(K \otimes X, Y)/\simeq \longrightarrow \mathcal{Psh}(\mathbb{A})(X, Y^K)/\simeq.$$

Proof. It is enough to show that Φ preserves and reflects the elementary \mathcal{J} -homotopy relation. Let $f, g : K \otimes X \rightarrow Y$ be elementary homotopic maps, i.e. such that there exist a homotopy $H : \mathcal{J}(K \otimes X) \rightarrow Y$ making the following triangle commute:

$$\begin{array}{ccc} (K \otimes X) + (K \otimes X) & & \\ (i_0, i_1) \downarrow & \searrow^{f+g} & \\ \mathcal{J}(K \otimes X) & \xrightarrow{H} & Y. \end{array}$$

Note that $(K \otimes X) + (K \otimes X) \cong K \otimes (X + X)$, and $\mathcal{J}(K \otimes X) \cong \mathcal{J}_\bullet \otimes K \otimes X \cong (\mathcal{J}_\bullet \times K) \otimes X \cong (K \times \mathcal{J}_\bullet) \otimes X \cong K \otimes \mathcal{J}X$. Under the adjunction, the triangle above transposes as

$$\begin{array}{ccc} X + X & & \\ (i_0, i_1) \downarrow & \searrow^{\Phi f + \Phi g} & \\ \mathcal{J}X & \xrightarrow{\Phi H} & Y^K, \end{array}$$

exhibiting a homotopy from Φf to Φg . Reflection of homotopies is done similarly. \square

Lemma 4.6.4. Let $K \in \mathcal{Psh}(\mathbb{A})$.

- (1) For $\omega \in \mathbb{O}_{\geq n-1}$, and e an inner face of $h\omega$, the map $K \otimes h_{h\omega}^e : K \otimes \Lambda^e h\omega \rightarrow K \otimes h\omega$ is an anodyne extension.
- (2) For $\phi \in \mathbb{O}_{n-1}$, the map $K \otimes i_\phi : K \otimes h\phi \rightarrow K \otimes \mathcal{J}_\phi$ is an anodyne extension.

Proof. In both cases, it is enough to check the claim where K is representable, say $K = \Delta[m]$ for some $m \in \mathbb{N}$.

- (1) Clearly, the inclusion $\Delta[m] \otimes h_{h\omega}^e : \Delta[m] \otimes \Lambda^e h\omega \rightarrow \Delta[m] \otimes h\omega$ satisfies the condition of corollary 4.4.4, i.e. $\Delta[m] \otimes \Lambda^e h\omega$ generates $v(\Delta[m] \otimes h\omega)$.
- (2) Write $*$ the unique element of $h\phi_{h\phi}$ (corresponding to the identity map $h\phi \rightarrow h\phi$ in \mathbb{A}), and $0_m, \dots, m_m$ the vertices of $\Delta[m]$. In particular, $(\Delta[m] \otimes h\phi)_{h\phi} = \{0_m \otimes *, \dots, m_m \otimes *\}$.

Let $T \in \mathcal{Psh}(\mathbb{A})$ be the sum of $m+1$ copies of $h\phi$, and write $T_{h\phi} = \{*_0, \dots, *_m\}$. Clearly, $\mathcal{J}T$ is the sum of $m+1$ copies of \mathcal{J}_ϕ , and the inclusion $i_0 : T \rightarrow \mathcal{J}T$ is the sum of $m+1$ copies of the endpoint inclusion

$i_\phi : h\phi \rightarrow \mathfrak{J}_\phi$. There is an obvious inclusion $t : T \rightarrow \Delta[m] \otimes h\phi$ mapping the $*_i$ to $i_m \otimes *$. Let $X \in \mathcal{Psh}(\Lambda)$ be defined by the following pushout:

$$\begin{array}{ccc} T & \xrightarrow{t} & \Delta[m] \otimes h\phi \\ i_0 \downarrow & & \downarrow u \\ \mathfrak{J}T & \xrightarrow{\quad} & X. \end{array}$$

In other words, X is $\Delta[m] \otimes h\phi$ where an instance of \mathfrak{J}_ϕ has been glued to each cell $i_m \otimes *$. The map $\Delta[m] \otimes i_\phi$ factors as

$$\Delta[m] \otimes h\phi \xrightarrow{u} X \xrightarrow{v} \Delta[m] \otimes \mathfrak{J}_\phi,$$

and by construction, u is an anodyne extension. On the other hand, v satisfies the conditions of corollary 4.4.4, as the cells not in its image can be obtained as composites of cells in X . Therefore, v is an anodyne extension, and so is $\Delta[m] \otimes i_\phi$. \square

Corollary 4.6.5. *For $K \in \mathcal{Psh}(\Delta)$, the functor $(-)^K$ preserves naive fibration (definition 2.1.10), and in particular, ∞ -algebras.*

Proof. Let $L := H_{\text{inner}} \cup E_{\mathfrak{J}}$, so that $\text{An}_{\mathfrak{J}} = \mathfrak{h}(L^{\mathfrak{h}})$. Let p be a naive fibration, i.e. a map such that $\text{An}_{\mathfrak{J}} \mathfrak{h} p$, or equivalently, such that $L \mathfrak{h} p$. By lemma 4.6.4, we have $K \otimes L \mathfrak{h} p$, and by adjunction, $L \mathfrak{h} p^K$. Consequently, p^K is a naive fibration. \square

Lemma 4.6.6. (1) *For $K \in \mathcal{Psh}(\Delta)$, the tensor $K \otimes - : \mathcal{Psh}(\Lambda)_\infty \rightarrow \mathcal{Psh}(\Lambda)_\infty$ preserves cofibrations and weak equivalences.*

(2) *Let $X \in \mathcal{Psh}(\Lambda)$ be an ∞ -algebra. Then $- \otimes X : \mathcal{Psh}(\Delta)_{\text{Quillen}} \rightarrow \mathcal{Psh}(\Lambda)_\infty$ preserves cofibrations, and weak equivalences between Kan complexes.*

Proof. (1) Clearly, $K \otimes -$ preserves monomorphisms. Let $u : X \rightarrow Y$ be a weak equivalence, and $P \in \mathcal{Psh}(\Lambda)$ be an ∞ -algebra. Recall from lemma 4.6.3 that we have a natural isomorphism

$$\Phi : \mathcal{Psh}(\Lambda)(K \otimes X, Y)/\simeq \rightarrow \mathcal{Psh}(\Lambda)(X, Y^K)/\simeq,$$

where Φ is the natural hom-set isomorphism of the adjunction $K \otimes - \dashv (-)^K$. Therefore, we have the following naturality square:

$$\begin{array}{ccc} \mathcal{Psh}(\Lambda)(K \otimes X, P)/\simeq & \xrightarrow{(K \otimes u)^*} & \mathcal{Psh}(\Lambda)(K \otimes Y, P)/\simeq \\ \Phi \downarrow & & \downarrow \Phi \\ \mathcal{Psh}(\Lambda)(X, P^K)/\simeq & \xrightarrow{u^*} & \mathcal{Psh}(\Lambda)(Y, P^K)/\simeq. \end{array}$$

The vertical maps are bijections. By corollary 4.6.5, P^K is an ∞ -algebra, thus u^* is a bijection as well. Therefore, $(K \otimes u)^*$ is a bijection for all ∞ -algebra P . By definition, $K \otimes u$ is a weak equivalence.

(2) Clearly, $- \otimes X$ preserves monomorphisms. Let $w : K \rightarrow L$ be a weak equivalence between Kan complexes. By [Hov99, theorem 1.2.10], it is a homotopy equivalence, meaning that it admits an inverse $w^{-1} : L \rightarrow K$ up to homotopy, and $w^{-1} \otimes X$ is an inverse of $w \otimes X$ up to homotopy. \square

Corollary 4.6.7. *Let $u : K \rightarrow L$ be a cofibration between Kan complexes, $v : X \rightarrow Y$ be a cofibration in $\mathcal{Psh}(\Lambda)$, and consider the Leibniz tensor $u \hat{\otimes} v$ (definition 2.2.4). Then $u \hat{\otimes} v$ is a cofibration. If either u or v is an acyclic cofibration, then so is $u \hat{\otimes} v$.*

Proof. Surely, since u and v are monomorphisms, $u \hat{\otimes} v$ is too. Assume that u is an acyclic cofibration. By lemma 4.6.6, $u \otimes X$ and $u \otimes Y$ are too, and so is the pushout u' of $u \otimes X$ along $K \otimes v$. By 3-for-2, $u \hat{\otimes} v$ is a weak equivalence. The case where v is an acyclic cofibration instead of u is done similarly. \square

Proposition 4.6.8. *Let $X \in \mathcal{Psh}(\Lambda)$ be an ∞ -algebra, and $v : K \rightarrow L$ be a cofibration (resp. acyclic cofibration) between Kan complexes. Then $X^v : X^L \rightarrow X^K$ is a fibration (resp. acyclic fibration).*

Proof. Assume that v is an cofibration (resp. acyclic cofibration). In order to show that X^v is an fibration (resp. acyclic fibration), we must show that $u \pitchfork X^v$ for all acyclic cofibration (resp. cofibration) u in $\mathcal{Psh}(\mathbb{A})_\infty$. This is equivalent to $u \hat{\otimes} v \pitchfork X$. By corollary 4.6.7, $u \hat{\otimes} v$ is an acyclic cofibration, and since X is an ∞ -algebra, the result holds. \square

Remark 4.6.9. Unfortunately, the model structure $\mathcal{Psh}(\mathbb{A})_\infty$, together with the tensor and cotensor of definitions 4.6.1 and 4.6.2 cannot be promoted into a simplicial model category. In fact, this already fails if $n = 1$ [JT07, section 6].

5. ∞ -ALGEBRAS VS. COMPLETE SEGAL SPACES

In [HTLS20a], we introduced the Segal and Rezk model structure on $\mathcal{Sp}(\mathbb{A})$ as Bousfield localization of the projective structure. Here, we take a slightly different approach by starting with the *vertical* (or *Reedy*) structure. This corresponds to the theory we generalize [JT07, CM13], while still being equivalent to the notions of [HTLS20a] as there is a Quillen equivalence $\text{id} : \mathcal{Sp}(\mathbb{A})_{\text{proj}} \xleftrightarrow{\sim} \mathcal{Sp}(\mathbb{A})_v : \text{id}$.

5.1. Segal spaces.

Definition 5.1.1. Recall from definition 2.3.5 that $\mathcal{Sp}(\mathbb{A})_v$ is the Reedy structure on $\mathcal{Sp}(\mathbb{A}) = \mathcal{Psh}(\mathbb{A})^{\Delta^{\text{op}}}$ induced by $\mathcal{Psh}(\mathbb{A})_{\text{Quillen}}$. Let $\mathcal{Sp}(\mathbb{A})_{\text{Segal}}$, the *Segal model structure* on $\mathcal{Sp}(\mathbb{A})$, be the left Bousfield localization of $\mathcal{Sp}(\mathbb{A})_v$ at the set \mathcal{S} of spine inclusions (definition 3.1.3), which exists by [Hir03, theorem 4.1.1].

Fibrant objects (resp. weak equivalences) in $\mathcal{Sp}(\mathbb{A})_{\text{Segal}}$ are called *Segal spaces* (resp. *Segal weak equivalences*). Explicitly, a Segal space $X \in \mathcal{Psh}(\mathbb{A})$ is a vertically fibrant space such that for all $\omega \in \mathbb{O}_{n-1}$, the map $\text{sh}_\omega \backslash X : h\omega \backslash X = X_{h\omega} \rightarrow S[h\omega] \backslash X$ is a weak equivalence.

Lemma 5.1.2. *Let \mathcal{K} be a saturated class of monomorphism of $\mathcal{Psh}(\mathbb{A})$ having the right cancellation property, i.e. such that for all composable pair or morphisms $f, g \in \mathcal{Psh}(\mathbb{A})$, if $fg, g \in \mathcal{K}$, then $f \in \mathcal{K}$.*

- (1) (Generalization of [JT07, lemma 3.5] and [CM13, proposition 2.5]) *If $\mathcal{S} \subseteq \mathcal{K}$, then $\text{An}_{\text{inner}} \subseteq \mathcal{K}$.*
- (2) (Generalization of [JT07, lemma 3.7]) *If \mathcal{K} contains all elementary face embeddings, then $\text{An} \subseteq \mathcal{K}$.*

Proof. (1) Since \mathcal{K} is saturated, we have $\text{Cell}(\mathcal{S}) \subseteq \mathcal{K}$, so by lemma 4.2.10, $\text{H}_{\text{inner}} \subseteq \mathcal{K}$. By saturation again, $\text{An}_{\text{inner}} \subseteq \mathcal{K}$.

- (2) For $\omega \in \mathbb{O}_{n+1}$, let $F(\omega)$ be the set of elementary faces of $h\omega$. It suffices to show that for all $\omega \in \mathbb{O}_{n+1}$ and non empty set $I \subsetneq F(\omega)$, the inclusion $h_{h\omega}^I : \Lambda^I h\omega \rightarrow$ (cf. section 4.2 for notations) is in \mathcal{K} .

Take a set $I \subsetneq F(\omega)$ and $f \in F(\omega) - I$, say $f : h\omega' \rightarrow h\omega$. If $I = F(\omega) - \{f\}$, then clearly, $\Lambda^I h\omega = \text{im } f \cong h\omega'$, and $h_{h\omega}^I \cong f \in \mathcal{K}$.

Otherwise, let $I \subsetneq J \subsetneq F(\omega)$ be a set of elementary faces of $h\omega$ not containing f . Then f factors as

$$h\omega' \xrightarrow{u} \Lambda^J h\omega \xrightarrow{v} \Lambda^I h\omega \xrightarrow{h_{h\omega}^I} h\omega.$$

Since $f \in \mathcal{K}$ be assumption, and since \mathcal{K} has the right cancellation property, in order to prove that $h_{h\omega}^I$, it suffices to show that $u, v \in \mathcal{K}$.

- (a) Assuming $v \in \mathcal{K}$ for all $J \supsetneq I$ not containing f , it is enough to show that $u \in \mathcal{K}$ in the case where $J = F(\omega) - \{f\}$. But in this case, as before, $\Lambda^J h\omega = \text{im } f \cong h\omega'$, and u is an isomorphism.
- (b) It is enough to consider the case $J = I + \{g\}$, for some elementary face $g \in F(\omega)$ different from f . Since $\Lambda^I h\omega = \text{im } f \cup \Lambda^J h\omega$, we have the following pushout square:

$$\begin{array}{ccc} \text{im } f \cap \Lambda^J h\omega & \hookrightarrow & \Lambda^J h\omega \\ w \downarrow & \lrcorner & \downarrow v \\ \text{im } f & \hookrightarrow & \Lambda^I h\omega. \end{array}$$

Recall that $\text{im } f \cong h\omega'$, and it is easy to see that the inclusion $w : \text{im } f \cap \Lambda^J h\omega \hookrightarrow \text{im } f$ is isomorphic to an elementary face of $h\omega'$. Consequently, v is the pushout of w , which is in \mathcal{K} by assumption, and thus $v \in \mathcal{K}$. \square

Proposition 5.1.3 (Generalization of [JT07, proposition 3.4]). *Let $X \in \text{Sp}(\mathbb{A})$ be vertically fibrant. The following are equivalent:*

- (1) X is a Segal space;
- (2) the map $\mathbf{h}_{h\omega}^f \backslash X$ is an acyclic Kan fibration, for all $\omega \in \mathbb{O}_{n+1}$ and all inner horn inclusion $\mathbf{h}_{h\omega}^f : \mathbb{A}^f h\omega \rightarrow h\omega$;
- (3) the map $u \backslash X$ is an acyclic Kan fibration, for all inner anodyne extension $u \in \mathcal{Psh}(\mathbb{A})$;
- (4) the map X/\mathbf{b}_n is an inner fibration (definition 4.2.6), for all $n \in \mathbb{N}$;
- (5) the map X/v is an inner fibration, for all monomorphism $v \in \mathcal{Psh}(\mathbb{A})$.

Proof. All equivalences are straightforward, except for (1) \iff (3).

- (1) \implies (3). Let \mathbf{K} be the class of morphisms $u \in \mathcal{Psh}(\mathbb{A})$ such that $u \backslash X$ is an acyclic Kan fibration. Since X is a segal space, $\mathbf{S} \subseteq \mathbf{K}$, and clearly, \mathbf{K} has the right cancellation property. We now show that it is saturated. By definition, $u \in \mathbf{K}$ if and only if $u \backslash X$ is an acyclic Kan fibration, i.e. $\mathbf{b}_n \mathbin{\lrcorner} u \backslash X$ for all $n \in \mathbb{N}$. But this is equivalent to $u \mathbin{\lrcorner} X/\mathbf{b}_n$, and thus $\mathbf{K} = \mathbin{\lrcorner} \{X/\mathbf{b}_n \mid n \in \mathbb{N}\}$, and in particular, \mathbf{K} is saturated. Finally, by lemma 5.1.2, \mathbf{K} contains all inner anodyne extensions.
- (3) \implies (1). Recall that spine inclusions are inner anodyne extensions by proposition 4.2.11. \square

Corollary 5.1.4 (Generalization of [JT07, corollary 3.6]). *Take $X \in \text{Sp}(\mathbb{A})$ be a Segal space, and $K \in \mathcal{Psh}(\mathbb{A})$. Then X/K is an inner Kan complex. In particular, $X_{-,n} = X/\Delta[n]$ is an inner Kan complex for all $n \in \mathbb{N}$.*

Proof. For $k : \emptyset \hookrightarrow K$ the initial map, the map $X/k : X/K \rightarrow X/\emptyset = 1$ is an inner fibration by proposition 5.1.3. \square

Lemma 5.1.5 (Generalization of [JT07, lemma 3.8]). *Let $f : X \rightarrow Y$ be an inner fibration between ∞ -algebras. It is an acyclic fibration if and only if it is a weak equivalence surjective on $(n-1)$ -cells.*

- Proof.*
- (1) (\implies) Surely, if f is an acyclic fibration, it is a weak equivalence. Moreover, f has the right lifting property against all monomorphisms (recall theorem 4.4.7), and in particular, against inclusions of the form $\emptyset \hookrightarrow h\phi$, for $\phi \in \mathbb{O}_{n-1}$. Therefore, f is surjective on $(n-1)$ -cells.
 - (2) (\impliedby) By theorem 4.4.7 and lemma 2.1.12 f is a fibration if and only if it is an inner fibration, and $\mathbf{E}_\gamma \mathbin{\lrcorner} f$. By lemma 4.5.7, it suffices to show that $\mathbf{E}_\gamma \mathbin{\lrcorner} vf$. Note that f is a weak equivalence between cofibrant objects. By Ken Brown's lemma [Hov99, lemma 1.1.12], vf is a weak equivalence. Besides, it is also surjective on $(n-1)$ -cells, so by theorem 3.4.8, it is an acyclic fibration. In particular, it is a fibration, and we apply lemma 4.5.7 to conclude. \square

Proposition 5.1.6 (Generalization of [JT07, proposition 3.9]). *A presheaf $X \in \text{Sp}(\mathbb{A})$ is a Segal space if and only if the following conditions are satisfied:*

- (1) for all $n \in \mathbb{N}$, the map X/\mathbf{b}_n is an inner fibration;
- (2) $X_{h\phi}$ is a Kan complex for all $\phi \in \mathbb{O}_{n-1}$;
- (3) X is homotopically constant (see definition 2.3.8).

Proof. • (\implies) Assume that $X \in \text{Sp}(\mathbb{A})$ is a Segal space. In particular, it is vertically fibrant.

- (1) This is proposition 5.1.3 (4).
- (2) The terminal map $! : X \rightarrow 1$ is a vertical fibration, so by definition 2.3.5, for $\phi \in \mathbb{O}_{n-1}$, the map $\langle \mathbf{b}_{h\phi} \backslash ! \rangle : X_{h\phi} \rightarrow 1$ is a Kan fibration.
- (3) This is proposition 2.3.10.
- (\impliedby) Let $v : K \rightarrow L$ be an anodyne extension in $\mathcal{Psh}(\mathbb{A})$ such that X/v is a weak equivalence in $\mathcal{Psh}(\mathbb{A})_\infty$. We claim that X/v is an acyclic fibration. By lemma 5.1.5, it suffices to show that $X/v : X/L \rightarrow X/K$ is surjective on $(n-1)$ -cells. Note that for $\phi \in \mathbb{O}_{n-1}$,

$$(X/L)_{h\phi} = \mathcal{Psh}(\mathbb{A})(h\phi, X/L) \cong \text{Sp}(\mathbb{A})(h\phi \boxtimes L, X) \cong \mathcal{Psh}(\mathbb{A})(L, X_{h\phi}),$$

and likewise, $(X/K)_{h\phi} \cong \mathcal{Psh}(\Delta)(K, X_{h\phi})$. By assumption, $X_{h\phi}$ is a Kan complex, and since v is an anodyne extension, the precomposition map on top is surjective

$$\begin{array}{ccc} \mathcal{Psh}(\Delta)(L, X_{h\phi}) & \xrightarrow{v^*} & \mathcal{Psh}(\Delta)(K, X_{h\phi}) \\ \cong \downarrow & & \downarrow \cong \\ (X/L)_{h\phi} & \xrightarrow{(X/v)_{h\phi}} & (X/K)_{h\phi}. \end{array}$$

Therefore, X/v is surjective on $(n-1)$ -cells, and thus an acyclic fibration as claimed.

We now show that X is vertically fibrant. By proposition 2.3.7, this is equivalent to X/v being an acyclic fibration for all anodyne extension $v \in \mathcal{Psh}(\Delta)$. Let \mathbf{K} be the class of anodyne extensions v such that X/v is an acyclic fibration. Note that by the first claim, this is equivalent to X/v being a weak equivalence. Using lemma 5.1.2, it suffices to show that \mathbf{K} has the right cancellation property, is saturated, and that it contains all simplicial face maps.

- (1) Right cancellation follows from 3-for-2.
- (2) For saturation, note that $v \in \mathbf{K}$ if and only if $\mathbf{b}_{h\omega} \dashv X/v$ for all $\omega \in \mathbb{O}_{\geq n-1}$ by definition. This is equivalent to $v \dashv \mathbf{b}_{h\omega} \setminus X$, thus $\mathbf{K} = \{ \mathbf{b}_{h\omega} \setminus X \mid \omega \in \mathbb{O}_{\geq n-1} \}$ is saturated.
- (3) Lastly, let us show that the simplicial face maps $d^i : \Delta[n-1] \rightarrow \Delta[n]$ belong to \mathbf{K} . Since X is homotopically constant, the simplicial map $X_{-,n} \rightarrow X_{-,0}$ is a weak equivalence for all $n \in \mathbb{N}$, so by 3-for-2, X/d^i is too, as displayed by the following triangle

$$\begin{array}{ccc} X_{-,n} & \xrightarrow{X/d^i} & X_{-,n-1} \\ & \searrow \sim & \swarrow \sim \\ & X_{-,0} & \end{array}$$

Therefore, by proposition 2.3.7, \mathbf{K} contains all anodyne extensions, and by proposition 2.3.7, X is vertically fibrant.

Lastly, we show that X satisfies the Segal condition definition 5.1.1. It suffices to show that for all $\omega \in \mathbb{O}_{\geq n-1}$, the map $\mathbf{s}_{h\omega} \setminus X$ is an acyclic fibration, i.e. that for all $n \in \mathbb{N}$, we have $\mathbf{b}_n \dashv \mathbf{s}_{h\omega} \setminus X$. This is equivalent to $\mathbf{s}_{h\omega} \dashv X/\mathbf{b}_n$, which holds since $\mathbf{s}_{h\omega}$ is an inner anodyne extension by proposition 4.2.11. \square

Proposition 5.1.7 (Generalization of [JT07, proposition 3.10]). *Let $f : X \rightarrow Y$ be a vertical fibration between two Segal spaces.*

- (1) *If $u : A \rightarrow B$ is an inner anodyne extension in $\mathcal{Psh}(\Delta)$ (definition 4.2.6), then $\langle u \setminus f \rangle : B \setminus X \rightarrow B \setminus Y \prod_{A \setminus Y} A \setminus X$ is an acyclic fibration.*
- (2) *If $v : K \rightarrow L$ is a monomorphism in $\mathcal{Psh}(\Delta)$, then $\langle f \setminus v \rangle : X/L \rightarrow Y/L \prod_{Y/K} X/K$ is an inner fibration between ∞ -algebras.*

Proof. (1) By proposition 2.3.7, the map $\langle v \setminus f \rangle$ is a Kan fibration. It remains to show that it is a weak equivalence. Consider the following diagram:

$$\begin{array}{ccc} B \setminus X & \xrightarrow{u \setminus X} & A \setminus X \\ \langle u \setminus f \rangle \searrow & & \downarrow A \setminus f \\ B \setminus Y & \xrightarrow{u \setminus Y} & A \setminus Y \\ \downarrow B \setminus f & \lrcorner & \downarrow p_1 \\ \cdot & \xrightarrow{p_2} & A \setminus X \end{array}$$

By proposition 5.1.3, $u \setminus X$ and $u \setminus Y$ are trivial fibrations, and so is the pullback map p_2 . By 3-for-2, $\langle u \setminus f \rangle$ is a weak equivalence.

(2) By proposition 5.1.3, X/S and X/L are ∞ -algebras. In the pullback square

$$\begin{array}{ccc} Y/L \amalg_{Y/K} X/K & \xrightarrow{p_2} & X/K \\ \downarrow & \lrcorner & \downarrow \\ Y/L & \xrightarrow{Y/v} & Y/K, \end{array}$$

the bottom map Y/v is an inner fibration by proposition 5.1.3, and thus p_2 is too. Since X/K is an ∞ -algebra, so is $Y/L \amalg_{Y/K} X/K$.

We now show that $\langle f/v \rangle$ is an inner fibration, i.e. that $u \lrcorner \langle f/v \rangle$ for all $u \in \mathbf{An}_{\text{inner}}$. By (1), $\langle u \setminus f \rangle$ is an acyclic fibration, so $v \lrcorner \langle u \setminus f \rangle$, and by adjunction, $u \lrcorner \langle f/v \rangle$ as desired. \square

5.2. Complete Segal spaces.

Definition 5.2.1 (Rezk map). For $\phi \in \mathbb{O}_{n-1}$, recall from definition 4.3.1 the definition of the Rezk interval \mathcal{J}_ϕ , and the endpoint inclusion $i_\phi : h\phi \rightarrow \mathcal{J}_\phi$. There is a canonical morphism, called the *Rezk map* at ϕ ,

$$r_\phi : \mathcal{J}_\phi \rightarrow h\phi,$$

mapping j_ϕ and j_ϕ^{-1} to id_ϕ (see definition 3.4.9 for notations), and let $\mathbf{R} = \{r_\phi \mid \phi \in \mathbb{O}_{n-1}\}$ be the set of Rezk maps.

Definition 5.2.2. Let $\text{Sp}(\mathbb{A})_{\text{Rezk}}$, the *Rezk model structure* on $\text{Sp}(\mathbb{A})$, be the left Bousfield localization of $\text{Sp}(\mathbb{A})_{\text{Segal}}$ (definition 5.1.1) at the set of Rezk maps \mathbf{R} , which exists by [Hir03, theorem 4.1.1].

Fibrant objects (resp. weak equivalences) in $\text{Sp}(\mathbb{A})_{\text{Rezk}}$ are called *complete Segal spaces* (resp. *Rezk weak equivalences*). Explicitly, a Segal space $X \in \text{Psh}(\mathbb{A})$ is complete if for all $\phi \in \mathbb{O}_{n-1}$, the map $r_\phi \setminus X : X_{h\phi} \rightarrow \mathcal{J}_\phi \setminus X$ is a weak equivalence.

Lemma 5.2.3 (Generalization of [JT07, lemma 4.2]). *A Segal space $X \in \text{Sp}(\mathbb{A})$ is complete if and only if for all $\phi \in \mathbb{O}_{n-1}$, the map $i_\phi \setminus X$ is a trivial fibration, where $i_\phi : h\phi \rightarrow \mathcal{J}_\phi$ is the endpoint inclusion of the Rezk interval \mathcal{J}_ϕ (definition 4.3.1).*

Proof. By definition, X is complete if and only if for all $\phi \in \mathbb{O}_{n-1}$, the map $r_\phi \setminus X$ is a weak equivalence. Since $r_\phi i_\phi = \text{id}_{h\phi}$, we have $(i_\phi \setminus X)(r_\phi \setminus X) = \text{id}_{X_{h\phi}}$, and by 3-for-2, $i_\phi \setminus X$ is a weak equivalence if and only if $r_\phi \setminus X$ is. On the other hand, $i_\phi \setminus X$ is always a Kan fibration by proposition 2.3.7. Hence it is a trivial fibration if and only if it is a weak equivalence. \square

Lemma 5.2.4 (Generalization of [JT07, lemma 4.3]). *Let $f : X \rightarrow Y$ be a Rezk fibration (i.e. a fibration in $\text{Sp}(\mathbb{A})_{\text{Rezk}}$) between two complete Segal spaces, and $u : K \hookrightarrow L$ be a monomorphism in $\text{Psh}(\mathbb{A})$. Then the map $\langle f/u \rangle : X/L \rightarrow (Y/L) \amalg_{Y/K} (X/K)$ is a fibration.*

Proof. By proposition 5.1.7, $\langle f/u \rangle$ is an inner fibration between ∞ -algebras. By lemma 2.1.12, $\langle f/u \rangle$ is a fibration if and only if it is a naive fibration, so it remains to show that $\mathbf{E}_{\mathcal{J}} \lrcorner \langle f/u \rangle$.

By adjunction, for $\phi \in \mathbb{O}_{n-1}$, we have $i_\phi \lrcorner \langle f/u \rangle$ if and only if $u \lrcorner \langle i_\phi \setminus f \rangle$. Thus, we must show that $\langle i_\phi \setminus f \rangle$ is an acyclic fibration. By proposition 2.3.7, it is a fibration. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{J}_\phi \setminus X & \xrightarrow{i_\phi \setminus X} & h\phi \setminus Y \\ \downarrow \langle i_\phi \setminus f \rangle & \lrcorner & \downarrow h\phi \setminus f \\ \mathcal{J}_\phi \setminus f & \xrightarrow{p_2} & h\phi \setminus Y \\ \downarrow p_1 & \lrcorner & \downarrow h\phi \setminus f \\ \mathcal{J}_\phi \setminus Y & \xrightarrow{i_\phi \setminus Y} & h\phi \setminus Y. \end{array}$$

By lemma 5.2.3, $i_\phi \setminus X$ and $i_\phi \setminus Y$ are acyclic fibrations. Since f is a Rezk fibration, it is a vertical fibration, and its matching map $\langle (\partial h\phi \hookrightarrow h\phi) \setminus f \rangle$ is a fibration by proposition 2.3.7. Since $\phi \in \mathbb{O}_{n-1}$, $\partial h\phi = \emptyset$, thus $h\phi \setminus f = \langle (\partial h\phi \hookrightarrow h\phi) \setminus f \rangle$ is a fibration. The pullback map p_2 is an acyclic fibration, and by 3-for-2, so is $\langle i_\phi \setminus f \rangle$. \square

Proposition 5.2.5 (Generalization of [JT07, proposition 4.4]). *A presheaf $X \in \mathcal{S}p(\mathbb{A})$ is a complete Segal space if and only if the following conditions are satisfied⁵:*

- (1) X/\mathbf{b}_n is a fibration for all $n \in \mathbb{N}$, i.e. X is horizontally fibrant (definition 2.3.5);
- (2) X is homotopically constant (definition 2.3.8).

Proof. • (\implies) By proposition 5.1.6, X is homotopically constant. By lemma 5.2.4, $X/\mathbf{b}_n = \langle (X \rightarrow 1)/\mathbf{b}_n \rangle$ is a fibration as $X \rightarrow 1$ is a Rezk fibration.

- (\impliedby) We first show that X is vertically fibrant. By proposition 2.3.7, this is equivalent to X/u being an acyclic fibration for all anodyne extension $u \in \mathcal{P}sh(\Delta)$. Let \mathbf{K} be the class of monomorphisms u such that X/u is an acyclic fibration. Using lemma 5.1.2, it suffices to show that \mathbf{K} has the right cancellation property, is saturated, and that it contains all simplicial face maps.

- (1) By condition (1), X/u is a fibration for any monomorphism $u \in \mathcal{P}sh(\Delta)$. Thus, X/u is an acyclic fibration if and if it is a weak equivalence. The right cancellation property of \mathbf{K} then follows from 3-for-2.
- (2) For saturation, note that $u \in \mathbf{K}$ if and only if $\mathbf{b}_{h\omega} \dashv X/u$ for all $\omega \in \mathbb{O}_{\geq n-1}$ by definition. This is equivalent to $v \dashv \mathbf{b}_{h\omega} \setminus X$, thus $\mathbf{K} = \dashv \{ \mathbf{b}_{h\omega} \mid \omega \in \mathbb{O}_{\geq n-1} \}$ is saturated.
- (3) Lastly, let us show that the simplicial face maps $d^i : \Delta[n-1] \rightarrow \Delta[n]$ belong to \mathbf{K} . Since X is homotopically constant, the structure map $X_{-,n} \rightarrow X_{-,0}$ is a weak equivalence for all $n \in \mathbb{N}$, so by 3-for-2, X/d^i is too, as displayed by the following triangle

$$\begin{array}{ccc} X_{-,n} & \xrightarrow{X/d^i} & X_{-,n-1} \\ & \searrow \sim & \swarrow \sim \\ & & X_{-,0} \end{array}$$

Since d^i is a monomorphism, X/d^i is a fibration by condition (1). Finally, X/d^i is an acyclic fibration. Therefore, by proposition 2.3.7, \mathbf{K} contains all anodyne extensions, and by proposition 2.3.7, X is vertically fibrant.

Since the terminal map $! : X \rightarrow 1$ is a vertical fibration, by definition 2.3.5, for $\phi \in \mathbb{O}_{n-1}$, the map $\langle \mathbf{b}_{h\phi} \setminus ! \rangle : X_{h\phi} \rightarrow 1$ is a Kan fibration, and $X_{h\phi}$ is a Kan complex. By proposition 5.1.6, X is a Segal space.

Lastly, let us show that X is complete. By lemma 5.2.3, it suffices to show that $i_\phi \setminus X$ is an acyclic fibration, for all $\phi \in \mathbb{O}_n$. Clearly, the endpoint inclusion $i_\phi : h\phi \rightarrow \mathcal{I}_\phi$ is a weak equivalence, with the Rezklemma:saturation-rcp-rcpp to homotopy. Since i_ϕ is a monomorphism, it is an acyclic cofibration. By condition (1), $i_\phi \dashv X/\mathbf{b}_n$, for all $n \in \mathbb{N}$, so by adjunction, $\mathbf{b}_n \dashv i_\phi \setminus X$, and $i_\phi \setminus X$ is a trivial fibration. \square

Theorem 5.2.6 (Generalization of [JT07, theorem 4.5]). (1) *The model structure $\mathcal{S}p(\mathbb{A})_{\text{Rezk}}$ is a Bousfield localization of the horizontal model structure $\mathcal{S}p(\mathbb{A})_h$ (definition 2.3.5). In particular, a weak equivalence in $\mathcal{S}p(\mathbb{A})_h$ is a Rezk weak equivalence.*

- (2) *An horizontally fibrant space is a complete Segal space if and only if it is homotopically constant (definition 2.3.8).*

Proof. (1) By proposition 2.3.6, $\mathcal{S}p(\mathbb{A})_v$ and $\mathcal{S}p(\mathbb{A})_h$ are both of Cisinski type. Since $\mathcal{S}p(\mathbb{A})_{\text{Rezk}}$ is a left Bousfield localization of $\mathcal{S}p(\mathbb{A})_v$, it is also of Cisinski type. In particular, $\mathcal{S}p(\mathbb{A})_{\text{Rezk}}$ and $\mathcal{S}p(\mathbb{A})_h$ have the same cofibrations, namely the monomorphisms. Thus, in order to prove the claim, it is enough to show that the identity functor induces a Quillen adjunction $\text{id} : \mathcal{S}p(\mathbb{A})_h \xleftrightarrow{\text{id}} \mathcal{S}p(\mathbb{A})_{\text{Rezk}} : \text{id}$. By a result of Dugger [Dug01, corollary A.2] (also stated in [Hir03, proposition 8.5.4]), it suffices to show that id preserves cofibrations, and Rezk fibrations between complete Segal spaces. In both structures, cofibrations are the monomorphisms. By lemma 5.2.4, for $f : X \rightarrow Y$ a Rezk fibration between complete Segal spaces, the matching map $\langle f/\mathbf{b}_n \rangle$ is a fibration for all $n \in \mathbb{N}$.

- (2) Follows from proposition 5.2.5 \square

⁵In particular, complete Segal spaces are exactly the simplicial resolutions [Dug01, definition 4.7].

Definition 5.2.7. Recall from [JT07, proposition 1.16] the functor $J : q\text{Cat} \rightarrow \mathcal{K}\text{an}$ from quasi-categories to Kan complexes, that maps a quasi-category to its maximal sub Kan complex. It is right adjoint and preserves cofibrations, fibrations, and weak equivalences (from $\mathcal{P}\text{sh}(\Delta)_\infty$).

For $X \in \mathcal{P}\text{sh}(\Delta)$, let $\Gamma X \in \text{Sp}(\Delta)$ be given by $\Gamma X_{-,k} := X^{J(\Delta[k])}$, where the cotensor is given in definition 4.6.2.

Lemma 5.2.8 (Generalization of [JT07, lemma 4.8]). *Let \mathcal{C} be a small category, and $F : \mathcal{P}\text{sh}(\Delta)^{\text{op}} \rightarrow \mathcal{P}\text{sh}(\mathcal{C})$ be a continuous functor, i.e. mapping colimits in $\mathcal{P}\text{sh}(\Delta)$ to limits in $\mathcal{P}\text{sh}(\mathcal{C})$. Then $F \cong G/-$, where $G \in \text{Sp}(\mathcal{C})$ is the restriction of F to Δ , i.e. $G_{-,k} := F\Delta[k]$, for $k \in \mathbb{N}$.*

Proof. For $k \in \mathbb{N}$, we have $G/\Delta[k] = G_{-,k} = F\Delta[k]$, thus $G/-$ and F coincide on Δ^{op} . Since Δ^{op} freely generates $\mathcal{P}\text{sh}(\Delta)^{\text{op}}$ under small limits, we are done. \square

Proposition 5.2.9 (Generalization of [JT07, proposition 4.10]). *Let $X \in \text{Sp}(\Delta)_\infty$ be an ∞ -algebra. Then ΓX is a complete Segal space, and there is a canonical acyclic cofibration $X^{\text{disc}} \rightarrow \Gamma X$, thus exhibiting ΓX as a fibrant replacement of X^{disc} in $\text{Sp}(\Delta)_{\text{Rezk}}$.*

Proof. (1) Note that $X^{J(-)} : \mathcal{P}\text{sh}(\Delta)^{\text{op}} \rightarrow \mathcal{P}\text{sh}(\Delta)$ is a continuous functor, thus by lemma 5.2.8, there exists $G \in \text{Sp}(\Delta)$ such that $X^{J(-)} \cong G/-$. In particular, for $k \in \mathbb{N}$, we have $G_{-,k} \cong G/\Delta[k] \cong \Gamma X_{-,k}$, and therefore, $G \cong \Gamma X$, and $X^{J(-)} \cong \Gamma X/-$.

(2) As a consequence, and by proposition 4.6.8, $\Gamma X/b_k$ is a fibration, for all $k \in \mathbb{N}$. Likewise, the map $d : \Delta[0] \rightarrow \Delta[n]$ is an acyclic cofibration in $\mathcal{P}\text{sh}(\Delta)_{\text{Quillen}}$. Therefore $\Gamma X/d : \Gamma X_{-,n} \rightarrow \Gamma X_{-,0}$ is a weak equivalence, and by lemma 2.3.9, ΓX is homotopically constant. By proposition 5.2.5, it is a complete Segal space.

(3) Note that $\Gamma X_{-,0} \cong \Gamma X/\Delta[0] \cong X^{J(\Delta[0])} = X$, so we have an isomorphism $f : X \rightarrow \Gamma X_{-,0}$. We now extend f to a morphism $\gamma : X^{\text{disc}} \rightarrow \Gamma X$. First, let $\gamma_0 := f : X_{-,0}^{\text{disc}} = X \rightarrow \Gamma X_{-,0}$. Next, note that in Δ , the terminal map $s_k : [k] \rightarrow [0]$ is a retraction of any iterated coface map $d : [0] \rightarrow [k]$. Thus, in addition to being a weak equivalence (as ΓX is homotopically constant), the structure map $s_k : \Gamma X_{-,0} \rightarrow \Gamma X_{-,k}$ is a section of $d : \Gamma X_{-,k} \rightarrow \Gamma X_{-,0}$, thus injective. Letting $\gamma_k := s_k f : X_{-,k}^{\text{disc}} = X \rightarrow \Gamma X_{-,k}$ gives rise to the desired map $\gamma : X^{\text{disc}} \rightarrow \Gamma X$ which by construction is a horizontal acyclic cofibration. By theorem 5.2.6, it is a Rezk acyclic cofibration. \square

Theorem 5.2.10 (Generalization of [JT07, proposition 4.7 and theorem 4.11]). *We have a Quillen equivalence $(-)^{\text{disc}} : \mathcal{P}\text{sh}(\Delta)_\infty \xleftrightarrow{\cong} \text{Sp}(\Delta)_{\text{Rezk}} : (-)_{-,0}$ (see section 2.2).*

Proof. (1) We first shows that the adjunction is Quillen. Clearly, $(-)^{\text{disc}}$ preserves monomorphisms and maps weak equivalences to horizontal weak equivalences, which by theorem 5.2.6 are Rezk weak equivalences. Therefore, $(-)^{\text{disc}}$ is left Quillen functor.

(2) We show that for a complete Segal space $X \in \text{Sp}(\Delta)$, the map

$$(QX_{-,0})^{\text{disc}} \xrightarrow{q^{\text{disc}}} (X_{-,0})^{\text{disc}} \xrightarrow{\varepsilon} X$$

is a Rezk weak equivalence, where $q : QX_{-,0} \rightarrow X_{-,0}$ a cofibrant replacement of $X_{-,0}$. First, since all objects are cofibrant in $\mathcal{P}\text{sh}(\Delta)_\infty$, whose q to be an identity. Next, by theorem 5.2.6, it suffices to show that ε is a weak equivalence in $\text{Sp}(\Delta)_h$, i.e. that $\varepsilon_{-,n} : (X_{-,0})_{-,n}^{\text{disc}} = X_{-,0} \rightarrow X_{-,n}$ is a weak equivalence. Clearly, $\varepsilon_{-,n}$ is induced by the terminal map $[n] \rightarrow [0]$ in Δ , and thus is a weak equivalence since X is homotopically constant by proposition 5.2.5.

(3) We show that for an ∞ -algebra $X \in \mathcal{P}\text{sh}(\Delta)_\infty$, the map

$$X \xrightarrow{\cong} (X^{\text{disc}})_{-,0} \xrightarrow{r_{-,0}} (RX^{\text{disc}})_{-,0}$$

is a Rezk weak equivalence, where $r : X^{\text{disc}} \rightarrow RX^{\text{disc}}$ is a fibrant replacement of X^{disc} . Choosing it to be $\gamma : X^{\text{disc}} \rightarrow \Gamma X$ (proposition 5.2.9) concludes the proof. \square

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