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Helical vortical effects, helical waves, and anomalies of Dirac fermions

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Helicity is a classically conserved quantity which can be used, in addition to and independently of the (vector) charge and chirality, to characterize thermodynamic ensembles of massless Dirac fermions. We identify a symmetry of the Dirac Lagrangian which is responsible, via the Noether theorem, to the classical conservation of the helicity current. We demonstrate the existence of new nondissipative transport phenomena, helical vortical effects, that emerge in a helically-imbalanced rotating fermionic system. These phenomena lead to appearance of a new gapless hydrodynamic excitation, the helical vortical wave. Our results also imply that the helical symmetry suffers from a quantum anomaly. We conjecture the existence of a new type of triangle anomalies in QED which involve the helicity currents in addition to the standard vector and axial currents.

Introduction. Massless or nearly-massless fermions appear in many areas of physics, including theories of fundamental interactions, cosmological models of early Universe, ultra-hot relativistic plasmas, and superfluids, to mention a few [1]. Many relativistic phenomena are now available for experimental verification in recently discovered crystals of Dirac and Weyl semimetals, where the massless fermions appear as quasiparticle excitations [2]. The most important properties of these excitations are usually associated with their vector (gauge) and axial (chiral) symmetries that affect, in the case of semimetals, electromagnetic [3], thermal [4], and elastic [5] responses of these materials. Many unusual features of these semimetals are associated with the quantum anomaly that break the continuous axial symmetry of an underlying classical theory [6]. Similar anomalies lead to exotic transport phenomena of quarks, mediated by the topology of evolving gluon fields in expanding quark-gluon plasma of heavy-ion collisions [7].

In our paper, we remind that, in addition to the vector and axial charges, there is also a third, well-known, and, simultaneously, often-forgotten quantity that characterizes massless fermions: the helicity. The fermionic helicity is sometimes confused with the chirality, even though these quantities reflect different physical properties of fermions [8]. To highlight the importance of helicity, we demonstrate the existence of a set of new transport phenomena emerging in a gas of rotating massless fermions, the Helical Vortical Effects (HVE), that differ substantially from their chiral counterparts, the Chiral Vortical Effects [9–14].

The HVE may see its applications in noncentral ultrarelativistic heavy-ion collisions that create a nearly perfect fluid of quark-gluon plasma, the most vortical fluid ever known [15]. Hydrodynamics of relativistic plasmas with nonzero vorticity has attracted significant attention recently [16–20]. We uncover a new gapless hydrodynamic excitation, the Helical Vortical Wave (HVW) which is analogous to the Chiral Vortical Wave

(CVW) [16]. In the existing low-density quark-gluon plasmas, the HVW propagates faster than CVW, being able to proliferate even in a globally-neutral plasma.

Vector-axial-helical triad for massless Dirac fermions. We consider one species of free massless Dirac fermions in a flat (3+1) dimensional Minkowski spacetime, described by the Lagrangian

$$\mathcal{L} = \frac{i}{2} (\bar{\psi} \gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi), \quad (1)$$

where $\bar{\psi} = \psi^\dagger \gamma^0$ is the Dirac adjoint of the 4-component spinor ψ and the units $\hbar = c = 1$ are used. The 4×4 gamma matrices γ^μ are taken, for definiteness, in the Dirac representation, with $\mu = 0, \dots, 3$.

The classical Dirac Lagrangian (1) is invariant under the action of the (global) vector symmetry group:

$$U(1)_V : \quad \psi \rightarrow e^{i\alpha_V} \psi, \quad \bar{\psi} \rightarrow e^{-i\alpha_V} \bar{\psi}. \quad (2)$$

According to the Noether theorem, the symmetry (2) implies the existence of the classically conserved vector current, $J_V^\mu = \bar{\psi} \gamma^\mu \psi$, with $\partial_\mu J_V^\mu = 0$. If the fermions would carry an electric charge e and were coupled to an electromagnetic field A_μ , then the global symmetry (2) would become a local (gauge) symmetry, $\alpha_V = \alpha_V(x)$. The coupling to electromagnetism is achieved by adding a source term with the electric current eJ_V^μ to the Lagrangian (1): $\mathcal{L} \rightarrow \mathcal{L} + eJ_V^\mu A_\mu$. The symmetry (2) is unbroken at the quantum level, thus reflecting the fundamental property of the electric charge conservation in quantum theory.

The Lagrangian (1) possesses also the axial symmetry:

$$U(1)_A : \quad \psi \rightarrow e^{i\alpha_A \gamma^5} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{i\alpha_A \gamma^5}, \quad (3)$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is the fifth gamma matrix. The Noether theorem gives us the classically conserved axial (sometimes called “chiral”) current $J_A^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi$. The axial symmetry (3) is broken at the quantum level, leading to nonconservation of the axial current in the presence

of the electric \mathbf{E} and magnetic \mathbf{B} background fields [6]:

$$\partial_\mu J_A^\mu = \frac{e^2}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \equiv -\frac{e^2}{4\pi^2} \mathbf{E} \cdot \mathbf{B}, \quad (4)$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and we use the convention $\varepsilon^{0123} = \sqrt{-g}$ for the Levi-Civita tensor.

The axial charge (chirality) χ is identified with respect to the eigenstates of the γ^5 matrix, $\gamma^5 \psi = \chi \psi$. Due to the property $(\gamma^5)^2 = 1$, one distinguishes the right-handed (R) and left-handed (L) chiral eigenstates, respectively:

$$\gamma^5 \psi_R = +\psi_R, \quad \gamma^5 \psi_L = -\psi_L. \quad (5)$$

The axial current $J_A^\mu = J_R^\mu - J_L^\mu$ represents the difference in the currents of the right-chiral, $J_R^\mu = \bar{\psi}_R \gamma^\mu \psi_R$, and left-chiral, $J_L^\mu = \bar{\psi}_L \gamma^\mu \psi_L$, Dirac fermions.

The chirality χ of a fermion state is intimately related to the helicity λ of the same state. Classically, the helicity is given by the projection of the spin \mathbf{s} on the direction of motion of the fermion given by its momentum \mathbf{p} . At the quantum level, the helicity λ is an eigenvalue, $h\psi = \lambda\psi$, of the helicity operator:

$$h = \frac{\mathbf{s} \cdot \mathbf{p}}{p} \equiv \frac{1}{2} \gamma^5 \gamma^0 \frac{\boldsymbol{\gamma} \cdot \mathbf{p}}{p}, \quad (6)$$

where $p = |\mathbf{p}|$ is the absolute value of the momentum operator $\mathbf{p} = -i\boldsymbol{\partial}$, and $s_i = \frac{1}{2} \varepsilon_{0ijk} \Sigma^{jk}$ is the spin operator which is a part of the covariant tensor $\Sigma^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]$.

Since the fermion is a spin 1/2 particle, the helicity takes two values, $\lambda = \pm 1/2$. One distinguishes the right-handed (\uparrow) and the left-handed (\downarrow) helicity eigenstates [21]:

$$2h\psi_\uparrow = +\psi_\uparrow, \quad 2h\psi_\downarrow = -\psi_\downarrow. \quad (7)$$

The chirality and helicity are different quantities. The chirality of a particle is equal to its helicity (for example, a right-chiral particle has a right-handed helicity) while the chirality of an antiparticle is opposite to its helicity (for instance, a right-chiral antiparticle has a left-handed helicity). For a single fermion the vector (particle/antiparticle) and the axial (right-/left-chiral) charges determine rigidly the helicity of this single fermion. We will see later that for an ensemble of particles a similar relation would not work: the total helicity of the ensemble is not determined by its total vector and axial charges.

The chirality operator γ^5 commutes with the helicity operator h given in Eq. (6), while both of them commute with the Dirac Hamiltonian corresponding to the Lagrangian of massless fermions (1):

$$\hat{H} = \boldsymbol{\alpha} \cdot \mathbf{p}, \quad (8)$$

where $\boldsymbol{\alpha} \equiv \gamma^0 \boldsymbol{\gamma}$ in the original Dirac notations.

These commutation relations,

$$[\gamma^5, h] = 0, \quad [\hat{H}, \gamma^5] = 0, \quad [\hat{H}, h] = 0, \quad (9)$$

indicate that all three operators possess the same eigen-system. The last relation implies that the helicity, similar to the chirality, is a classically conserved number.

What is the symmetry of the Dirac Lagrangian (1) that would lead – via the Noether theorem – to a classical conservation of the helical charge that, in turn, is suggested by the last commutation relation of Eq. (9)? The Lagrangian (1) is invariant under a ‘‘helical’’ symmetry:

$$U(1)_H : \quad \psi \rightarrow e^{2i\alpha_H h} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{-2i\alpha_H h}, \quad (10)$$

where the helicity operator h is given in Eq. (6). One may readily check that the helical symmetry (10) leads to the classically conserved helicity current: $J_H^\mu = 2\bar{\psi} \gamma^\mu h \psi$. We introduced here the normalization factor 2 for the helicity current in order to enforce the integer eigenvalue $2\lambda = \pm 1$ for the helicity (7), similarly to the chirality (5).

Thus, the vector-axial-helical densities

$$\begin{pmatrix} Q_V \\ Q_A \\ Q_H \end{pmatrix} = \int d^3x \bar{\psi} \gamma^0 \begin{pmatrix} 1 \\ \gamma^5 \\ 2h \end{pmatrix} \psi. \quad (11)$$

form a ‘‘triad’’ of the classically conserved – via the Noether theorem – $U(1)$ quantities of the massless Dirac fermions described by the Lagrangian (1).[22] The behavior of the charges Q_ℓ and the associated currents \mathbf{J}_ℓ of all three quantities $\ell = V, A, H$ under the C , P , and T inversions are shown in Table I.

	Q_V	Q_A	Q_H	\mathbf{J}_V	\mathbf{J}_A	\mathbf{J}_H	$\boldsymbol{\Omega}$
C	–	+	–	–	+	–	+
P	+	–	–	–	+	+	+
T	+	+	+	–	–	–	–

TABLE I. Parities of the vector (V), axial (A), and helical (H) charges (Q) and currents (\mathbf{J}) of a massless Dirac fermion, as well as the vorticity $\boldsymbol{\Omega}$, under the C -, P -, and T -inversions. The signs $+/-$ denote the even/odd parities.

A complete set of mode solutions of the Dirac equation, $i\gamma^\mu \partial_\mu \psi = 0$, includes the particle modes U_j and the corresponding antiparticle modes $V_j = i\gamma^2 U_j^*$, where j is a cumulative label that indexes the eigenmodes. The commutation relations (9) imply that these modes are eigenfunctions of the Hamiltonian, chirality and the helicity operators, simultaneously:

$$\hat{H}U_j = E_j U_j, \quad \hat{H}V_j = -E_j V_j, \quad (12a)$$

$$\gamma^5 U_j = \chi_j U_j, \quad \gamma^5 V_j = -\chi_j V_j, \quad (12b)$$

$$hU_j = \lambda_j U_j, \quad hV_j = \lambda_j V_j, \quad (12c)$$

where $\lambda_j = \pm 1/2$ and $\chi_j = \pm 1$, while E_j is the mode energy, satisfying $|E_j| = p_j$ (for future convenience, we allow E_j to take negative values). Using Eq. (6) and the Dirac equation for massless particles, the relationship between λ_j and χ_j can be readily established:

$$2hU_j = \gamma^5 \frac{1}{p} HU_j \quad \Rightarrow \quad \lambda_j = \frac{\chi_j E_j}{2|E_j|}. \quad (13)$$

The field operator is

$$\hat{\psi}(x) = \sum_j [U_j(x)\hat{b}_j + V_j(x)\hat{d}_j^\dagger], \quad (14)$$

where canonical anticommutation rules for particle \hat{b}_j and antiparticle \hat{d}_j operators are assumed.

The operators of all three conserved charges (11) are:

$$: \hat{Q}_V := \sum_j (\hat{b}_j^\dagger \hat{b}_j - \hat{d}_j^\dagger \hat{d}_j), \quad (15a)$$

$$: \hat{Q}_A := \sum_j \chi_j (\hat{b}_j^\dagger \hat{b}_j + \hat{d}_j^\dagger \hat{d}_j), \quad (15b)$$

$$: \hat{Q}_H := \sum_j 2\lambda_j (\hat{b}_j^\dagger \hat{b}_j - \hat{d}_j^\dagger \hat{d}_j), \quad (15c)$$

where the colons denote Wick (normal) ordering. The particle and antiparticle states contribute differently to the axial (15b) and helicity (15c) charges. Equations (12) and (15) also imply that the helicities and chiralities are indeed equal (opposite) to each other for particle (antiparticle) modes.

Helical chemical potential. Similarly to the vector (15a) and axial (15b) charges, the existence of the third classically conserved number, helicity (15c), requires the introduction of the appropriate thermodynamically conjugated, “helical”, chemical potential μ_H . Let us recall that a right-chirality particle (antiparticle) has a right-handed (left-handed) helicity with spins parallel (anti-parallel) to its direction of motion. An ensemble of an equal number of right-helicity particles (that all have a right-handed chirality) and right-helicity antiparticles (characterized by a left-handed chirality) has a zero global vector (electric) charge density, $\langle Q_V \rangle = 0$, and a zero total axial (chiral) charge, $\langle Q_A \rangle = 0$, while the global helicity charge of this ensemble is nonzero, $\langle Q_H \rangle \neq 0$. To describe such helicity-imbalanced, but otherwise neutral systems of Dirac fermions, we introduce the helical chemical potential μ_H , as the use of vector, μ_V , and axial, μ_A , chemical potentials is not enough.

Formally, the helicity operator (6) is an ambiguous, Lorentz-frame-dependent quantity. However, we consider the Dirac systems at finite temperature $T \neq 0$ and/or in the presence of a finite helicity with $\mu_H \neq 0$ that explicitly break the Lorentz invariance. Therefore, our physical environment sets a natural Lorentz frame to define the helicity operator and removes the mentioned ambiguity.

It is customary to label the massless Dirac eigenmodes by their chiralities. One usually distinguishes the right-handed and left-handed “chiral Weyl cones” with the appropriate chemical potentials μ_R and μ_L , respectively, shown in Fig. 1(a). The difference in the occupation numbers between the modes is controlled by the axial (chiral) chemical potential, $\mu_A = (\mu_R - \mu_L)/2$.

Since the helicity operator shares the same basis with the chirality, we can also use the helicity to characterize the energy branches in an independent, and non-equivalent, manner, Fig. 1(b). Now, the occupation numbers of the “helical Weyl cones” are labelled by the right-helical (μ_\uparrow) and the left-helical (μ_\downarrow) chemical potentials.

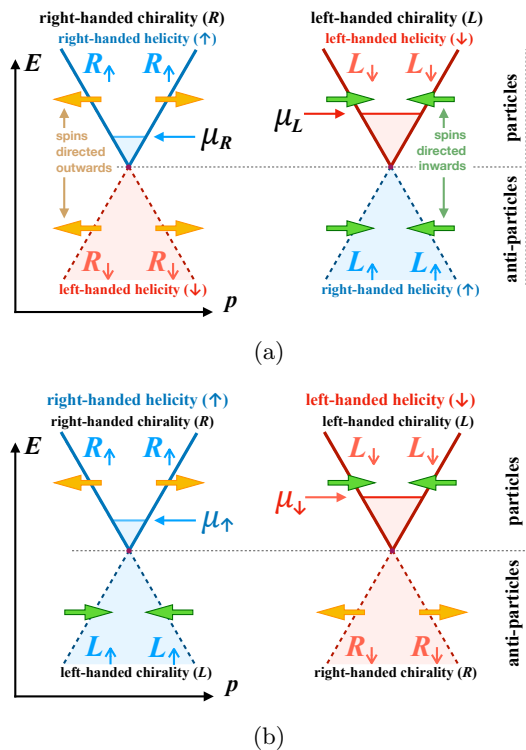


FIG. 1. The dispersion relations $E_p = \pm|\mathbf{p}|$ for Dirac fermions in (a) chiral and (b) helical basis. The chemical potentials $\mu_{R/L}$ determine the occupation numbers for the right-/left-handed chiralities, while the chemical potentials $\mu_{\uparrow/\downarrow}$ dictate the densities of the right-/left-handed helicities. The spin orientations \mathbf{s} with respect to the fermion’s momentum \mathbf{p} are shown by the horizontal arrows for each energy branch.

The difference between these potentials gives the helical chemical potential, $\mu_H = (\mu_\uparrow - \mu_\downarrow)/2$, which determines the helical imbalance of the system.

Rigidly-rotating thermal states. To reveal a role of the helical potential, we consider a gas of Dirac fermions at a finite temperature, uniformly rotating about the axis z .

It is convenient to introduce a particular basis of kinematic vectors for the rigid motion with the four-velocity

$$u = \Gamma(\partial_t + \Omega\partial_\varphi), \quad \Gamma = \frac{1}{\sqrt{1 - \rho^2\Omega^2}}, \quad (16)$$

where Ω is the angular velocity and Γ is the Lorentz factor. We use cylindrical coordinates (ρ, φ, z) .

The acceleration and vorticity four-vectors are:

$$a = \nabla_u u = -\rho\Omega^2\Gamma^2\partial_\rho, \quad \omega = \bar{w} \cdot u = \Omega\Gamma^2\partial_z, \quad (17)$$

where $(\nabla_u u)^\mu = u^\nu \nabla_\nu u^\mu$, $\bar{w}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}\nabla_\lambda u_\sigma$ is the vorticity tensor. The fourth vector, which is orthogonal to u , a and ω , is $\tau^\mu = \varepsilon^{\mu\nu\lambda\sigma}\omega_\nu a_\lambda u_\sigma$, or

$$\tau = a \cdot \bar{w} \cdot \Delta = -\rho\Omega^3\Gamma^5(\rho\Omega\partial_t + \rho^{-1}\partial_\varphi), \quad (18)$$

where $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$.

The expectation value of an operator \hat{A} is given by the thermal average over the Fock space [23, 24]:

$$\langle \hat{A} \rangle = Z^{-1} \text{Tr}(\hat{\rho} \hat{A}), \quad (19)$$

where $Z = \text{Tr}(\hat{\rho})$ is the partition function,

$$\hat{\rho} = \exp \left[-\beta_0 \left(\hat{H} - \Omega \hat{M}^z - \sum_{\ell=V,A,H} \mu_{\ell;0} \hat{Q}_\ell \right) \right], \quad (20)$$

is the density operator [25], \hat{H} is the Hamiltonian (8), \hat{M}^z is the z component of the total angular momentum operator, $\beta_0 \equiv 1/T_0$ and $\mu_{\ell;0}$ are, respectively, the values of the inverse temperature T_0 and the chemical potentials at the rotation axis $\rho = 0$. Using the fact that $[\hat{H}, \hat{M}^z] = 0$, the modes (12) are, simultaneously, the eigenvectors of \hat{M}^z . These mode solutions have been previously derived in Ref. [26] and are reproduced, for convenience, in Eq. (A5) of the Supplementary Material (SM). The SM describes also the analytical approaches used to compute the thermal expectation values below.

The rigidly rotating gas of Dirac fermions generates the vector, axial, and helical 4-currents ($\ell = V, A, H$),

$$J_\ell^\mu \equiv \langle : \hat{J}_\ell^\mu : \rangle = Q_\ell u^\mu + \sigma_\ell^\omega \omega^\mu + \sigma_\ell^\tau \tau^\mu, \quad (21)$$

along the 4-velocity u^μ , the 4-vorticity ω^μ , and the 4-circumference τ^μ . The radial components along the 4-acceleration vector a^μ are absent for all currents (21). Henceforth, we work in the β (thermometer) frame, by fixing the four-velocity u^μ to be equal to the one given in Eq. (16) [27–29].

At the rotation axis ($\rho = 0$), the 4-velocity $u^\mu = (1, \mathbf{0})$ points along the time coordinate, the 4-vorticity $\omega^\mu = (0, \Omega \mathbf{e}_z)$ is directed along the axis of rotation (17), while the 4-acceleration obviously vanishes, $a^\mu \equiv 0$. In a close vicinity of the axis, the vector $\tau^\mu = (0, \boldsymbol{\tau})$ is aligned along the circular angular coordinate (18), $\boldsymbol{\tau} \simeq \rho \Omega^3 \mathbf{e}_\varphi$.

We calculate the 4-currents (21) in a high-temperature expansion. The charge densities Q_ℓ are as follows:

$$Q_V = \frac{\mu_V T^2}{3} + \frac{4\mu_A \mu_H T}{\pi^2} \ln 2 + \frac{\mu_V^3 + 3\mu_V(\mu_A^2 + \mu_H^2)}{3\pi^2} + \frac{\boldsymbol{\omega}^2 + \mathbf{a}^2}{4\pi^2} \left(\mu_V + \frac{\mu_A \mu_H}{2T} \right) + \dots, \quad (22a)$$

$$Q_A = \frac{\mu_A T^2}{3} + \frac{4\mu_H \mu_V T}{\pi^2} \ln 2 + \frac{\mu_A^3 + 3\mu_A(\mu_H^2 + \mu_V^2)}{3\pi^2} + \frac{\boldsymbol{\omega}^2 + \mathbf{a}^2}{4\pi^2} \left(\mu_A + \frac{\mu_H \mu_V}{2T} \right) + \dots, \quad (22b)$$

$$Q_H = \frac{\mu_H T^2}{3} + \frac{4\mu_V \mu_A T}{\pi^2} \ln 2 + \frac{\mu_H^3 + 3\mu_H(\mu_V^2 + \mu_A^2)}{3\pi^2} + \frac{\boldsymbol{\omega}^2 + \mathbf{a}^2}{4\pi^2} \left(\mu_H + \frac{\mu_V \mu_A}{2T} \right) + \dots, \quad (22c)$$

where we ignore terms of order $O(T^{-1}, \Omega^2/T^2, \Omega^4)$. In the limit of vanishing μ_H , our results for Q_V and Q_A coincide with those found in Ref. [30].

The rotation generates the vector, axial, and helical ($\ell = V, A, H$) spatial currents (21), both along the axis of rotation $\mathbf{e}_z \parallel \boldsymbol{\Omega}$ and along the circumference $\mathbf{e}_\varphi \parallel \boldsymbol{\Omega} \times \boldsymbol{\rho}$:

$$\mathbf{J}_\ell = \sigma_\ell^z \boldsymbol{\Omega} + \sigma_\ell^\varphi \Omega^2 \boldsymbol{\Omega} \times \boldsymbol{\rho} + \dots \quad (\rho \rightarrow 0), \quad (23)$$

where the ellipsis denotes higher-order terms in the radial distance ρ with respect to the leading terms in Eq. (23).

The vortical conductivities along the rotation axis are:

$$\begin{aligned} \sigma_V^z &= \frac{2\mu_H T}{\pi^2} \ln 2 + \frac{\mu_V \mu_A}{\pi^2} + \frac{\mu_H^3 + 3\mu_H(\mu_V^2 + \mu_A^2)}{12\pi^2 T} \\ &\quad + \frac{\mu_H(\boldsymbol{\omega}^2 + 3\mathbf{a}^2)}{48\pi^2 T} + \dots, \\ \sigma_A^z &= \frac{T^2}{6} + \frac{\mu_V^2 + \mu_A^2 + \mu_H^2}{2\pi^2} + \frac{\mu_V \mu_A \mu_H}{2\pi^2 T} + \dots, \\ \sigma_H^z &= \frac{2\mu_V T}{\pi^2} \ln 2 + \frac{\mu_H \mu_A}{\pi^2} + \frac{\mu_V^3 + 3\mu_V(\mu_H^2 + \mu_A^2)}{12\pi^2 T} \\ &\quad + \frac{\mu_V(\boldsymbol{\omega}^2 + 3\mathbf{a}^2)}{48\pi^2 T} + \dots, \end{aligned} \quad (24)$$

where we ignored terms of order $O(T^{-3}, \Omega^4)$.

The circular conductivities in Eq. (23) are given by:

$$\sigma_V^\varphi = \frac{\mu_V}{6\pi^2} + \frac{\mu_A \mu_H}{12\pi^2 T} + \dots, \quad (25a)$$

$$\sigma_A^\varphi = \frac{\mu_A}{6\pi^2} + \frac{\mu_H \mu_V}{12\pi^2 T} + \dots, \quad (25b)$$

$$\sigma_H^\varphi = \frac{\mu_H}{6\pi^2} + \frac{\mu_V \mu_A}{12\pi^2 T} + \dots, \quad (25c)$$

where terms of order $O(T^{-3}, \Omega^2)$ were ignored. On the rotation axis ($\rho = 0$), the circumferential currents vanish, and the current (23) points exactly along the vorticity $\boldsymbol{\Omega}$.

The vortical transport effects (23) are consistent with the C -, P -, and T -symmetries of the vector, axial, and helical currents and charges, as shown in Table I. All vortical effects are dissipationless phenomena, because the laws (23) and (24) are even under the T -inversion.

In addition to the usual chiral vortical effects (CVE's), the currents (23) exhibit a plethora of new helical vortical effects (HVE's). For example, the rotating dense (charged) Dirac matter generates the helical current \mathbf{J}_H that is linearly proportional to the vector chemical potential μ_V and temperature T (24). On the other hand, the neutral Dirac matter with nonzero helicity ($\mu_H \neq 0$) generates the vector (charged) current \mathbf{J}_V . Remarkably, the mentioned helical terms, linearly proportional to a chemical potential and temperature, are allowed for HVE's and, at the same time, are forbidden for the CVE's by virtue of the C -, P -, and T -symmetries.

Helical anomalies in QED. It is known that the vector σ_V^z and axial σ_A^z vortical conductivities (24), at a vanishing helical chemical potential, $\mu_H = 0$, are determined by the axial quantum anomalies (for a review, see [31]).

For example, the $\mu_V \mu_A$ term in the vector vortical conductivity σ_V^z is generated by the axial-vector-vector (AVV) vertex of the axial anomaly (4), which is also responsible for the μ_V^2 term in the axial vortical conductivity σ_A^z . Both these terms share similar prefactors with

the axial anomaly (4). The axial conductivity σ_A^z contains also the μ_A^2 term due to the axial-axial-axial (AAA) triangular anomaly, as well as the T^2 term which originates from the axial-graviton-graviton (ATT) vertex of the mixed axial-gravitational anomaly [32].

The presence of the helical component in the vortical conductivities (24) strongly suggests the existence of new types of triangle anomalies which involve the helicity vertex (6). For example, the leading term $\mu_H T$ ($\mu_V T$) in the vector (helical) conductivity σ_V^z (σ_H^z) could have its origin in the new triangle VHT anomaly involving vector (γ^μ), helical ($\gamma^\mu h$), and graviton ($\hat{T}^{\mu\nu}$) vertices.

The new helical anomalies must reveal themselves in the background of a “helical vector field” A_μ^H which couples with the Dirac fermions via the source term $A_\mu^H J_H^\mu$ added to the Lagrangian (1). For instance, the quadratic μ_H dependence (24) of the axial vortical conductivity σ_A^z implies the existence of a particular form of the mixed axial-helical anomaly responsible for the nonconservation of the axial current in the background of the A_μ^H field. The new AHH vertex, $\partial_\mu J_A^\mu = \frac{1}{16\pi^2} F_{\mu\nu}^H \tilde{F}^{H,\mu\nu}$, shares similarity with the standard AVV vertex of the axial anomaly (4). A detailed structure of triangular anomalies with helical operators will be explored elsewhere [33].

Helical vortical waves. The emergence of the new degree of freedom, the helicity, allows us to uncover new hydrodynamic excitations in the helical sector, that are similar to the chiral magnetic [34] and chiral vortical [16] waves. To illustrate this fact, we take a globally neutral plasma ($\bar{n}_\ell = 0$, $\ell = V, A, H$) at finite temperature ($T \neq 0$), and consider the simplest gapless excitation that propagates along the vorticity vector at the rotational axis, where the mean fluid velocity vanishes, $\bar{\mathbf{v}} = 0$. The bar over a symbol means a local thermodynamic average.

We consider linear modes in hydrodynamic fluctuations. We notice that in a neutral plasma, the local densities, $n_\ell = \bar{n}_\ell + \delta n_\ell = \delta n_\ell$, and the local fluid velocity, $\mathbf{v} = \bar{\mathbf{v}} + \delta \mathbf{v} = \delta \mathbf{v}$, are linear in fluctuations. The “non-anomalous” component of the currents, $n_\ell \mathbf{v} = \delta n_\ell \delta \mathbf{v}$, vanishes at the first order, and all the currents are thus equal to their “anomalous” contributions (21).

The form of the densities (22) implies that in a neutral plasma, the temperature fluctuations δT decouple from hydrodynamics of charge density waves δn_ℓ at the linear order (we set $T = \bar{T}$). Moreover, the axial current (24) becomes quadratic in the fluctuations, so that the chiral vortical wave – that involves the vector and axial sectors – does not propagate in the neutral plasma [35–38]. On the contrary, the vector and helical densities (22) and their vortical conductivities (24) are linearly cross-coupled and therefore the vector-helical wave does propagate.

We consider the wave excitation along the vorticity vector, $J_\ell^z = J_\ell^{(0),z} e^{-i\omega t + ik_z z}$. We impose the conservation of the vector ($\ell = V$) and helical ($\ell = H$) charges, $\partial_t Q_\ell + \partial_z J_\ell^z = 0$, which is valid in the absence of background fields. Using Eqs. (21), (22), and (24), we arrive to the wave equation $(\partial_t^2 - v_{\text{HVW}}^2 \partial_z^2) Q_\ell = 0$, which

describes the propagation of the helical vortical wave (HVW). The HVW is a gapless (massless) hydrodynamic excitation with the dispersion relation

$$\omega = v_{\text{HVW}} |k_z|, \quad v_{\text{HVW}} = \frac{6 \ln 2}{\pi^2} \frac{|\Omega|}{T}, \quad (26)$$

where v_{HVW} is the HVW velocity. The spectrum of the helical waves will be studied in more detail elsewhere [33].

To estimate the velocity of the HVW in ultra-relativistic heavy-ion collisions, we take the temperature $T \simeq 150$ MeV [39] matching the pseudocritical QCD value [40], and the angular frequency $\Omega \simeq 6.6$ MeV $\simeq 10^{22} s^{-1}$ revealed in a RHIC experiment [15, 42]. We find that at these parameters, the Helical Vortical Wave propagates with the velocity $v_{\text{HVW}} \simeq 2 \times 10^{-2} c$.

It is instructive to compare the velocity of the helical vortical wave with its chiral analog, $v_{\text{CVW}} = \frac{3\mu_V \Omega}{\pi^2 T^2}$ [16, 35]. To this end, we set for the chemical potential $\mu_V = \mu_q \simeq 30$ MeV [41], and obtain $v_{\text{CVW}} \simeq 3 \times 10^{-3} c$, which falls into a range of the original estimation of Ref. [16]. The helical wave propagates much faster than the chiral wave since $\frac{v_{\text{CVW}}}{v_{\text{HVW}}} = \frac{1}{2 \ln 2} \frac{\mu_V}{T} \approx 0.7 \mu_V / T$ while $\mu_V \ll T$ in the quark-gluon plasma being created in the ongoing heavy-ion collision experiments.

Conclusions. We stress that the fermionic helicity is an independent characteristic of a massless Dirac fermion similar to its vector and axial degrees of freedom. We construct a classically conserved helicity current and find a variety of new nondissipative helical vortical effects (HVE) in rotating media of Dirac fermions. These effects include, for example, the generation of the vector (helical) current in the presence of the helicity (vector charge) density at finite temperature. The helical vortical effects lead to the appearance of a new gapless hydrodynamic excitation, the helical vortical wave (HVW), that involves coherent oscillation of vector and helical densities. The HVW propagates in a neutral plasma contrary to the chiral vortical wave that requires the presence of matter. In low-density quark-gluon plasmas created in ultrarelativistic heavy-ion collisions, the helical wave propagates much faster than its chiral cousin. We point out that the helical vortical effects have an anomalous origin, and conjecture the existence of a set of new “helical” quantum anomalies in QED that appear at the same footing as the well-known axial and axial-gravitational anomalies.

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Appendix A: Supplementary material

The Supplementary Material presented herein supplements the main text with various technical details. In Subsec. A 1, we present an analysis of the properties of the helicity charge operator : Q_H : under the CPT transformations, which are summarized in Table I. Then, the details regarding the computation of thermal expectation values are given in Subsec. A 2. The details regarding the computation of the t.e.v.s of the vector and helicity charge currents are given in Subsec. A 3. Finally, the t.e.v.s. of the components of the axial charge current (ACC) and stress-energy tensor (SET) are derived in Subsections A 4 and A 5.

1. Discrete symmetries of the helicity charge operator

Taking into account that $P\psi(\mathbf{x})P = \gamma^0\psi(-\mathbf{x})$, it can be seen that:

$$\begin{aligned} Q_H &\xrightarrow{P} \int d^3x [\bar{\psi}(-\mathbf{x})\gamma^0(\gamma^0\gamma^5\gamma^0\frac{-i\boldsymbol{\gamma}\cdot\nabla_{\mathbf{x}}}{p})\gamma^0\psi(-\mathbf{x}) \\ &= \int d^3x [\bar{\psi}(-\mathbf{x})(\gamma^0\gamma^5\gamma^0\frac{i\boldsymbol{\gamma}\cdot\nabla_{-\mathbf{x}}}{p})\psi(-\mathbf{x}) \\ &= -Q_H, \end{aligned} \quad (\text{A1})$$

where $\nabla_{\mathbf{x}} = -\nabla_{-\mathbf{x}}$ was employed.

For charge conjugation, $C\psi C = i\gamma^2\psi^*$ and $C\bar{\psi}C = -i\bar{\psi}^*\gamma^2$. However, the expression needs to be presented in normal order, such that

$$\begin{aligned} : \hat{Q}_H : &\xrightarrow{C} \int d^3x : [\hat{\bar{\psi}}^*(-i\gamma^2)(\gamma^5\gamma^0\frac{-i\boldsymbol{\gamma}\cdot\nabla}{p})i\gamma^2\hat{\psi}^*] : \\ &= - \int d^3x : [\hat{\bar{\psi}}\gamma^0 2h\hat{\psi}]^* : \\ &= \int d^3x : \text{tr}[\gamma^0 2h\hat{\bar{\psi}}\hat{\psi}]^* : , \end{aligned} \quad (\text{A2})$$

where the minus sign on the last line is due to changing the operator order of $\hat{\psi}$ and $\hat{\bar{\psi}}$.

Finally, for time reversal $T\psi(t)T = -\gamma^1\gamma^3\psi(-t)$, $T\bar{\psi}(t)T = \bar{\psi}(-t)\gamma^1\gamma^3$ and $T\gamma^\mu T = (\gamma^\mu)^*$, we have:

$$\begin{aligned} Q_H &\xrightarrow{T} \int d^3x T[\bar{\psi}\gamma^5\gamma^0\frac{-i\boldsymbol{\gamma}\cdot\nabla}{p}\psi]T \\ &= \int d^3x [\bar{\psi}(-t)\gamma^1\gamma^3(\gamma^0 2h)^*(-\gamma^1\gamma^3)\psi(-t)] \\ &= \int d^3x [\bar{\psi}(-t)\gamma^0\gamma^5\gamma^0[\gamma^1\gamma^3\boldsymbol{\gamma}^*\cdot i\nabla/p](-\gamma^1\gamma^3)]\psi(-t) \\ &= \int d^3x [\bar{\psi}(-t)\gamma^0\gamma^5\gamma^0[\frac{-i\boldsymbol{\gamma}\cdot\nabla}{p}]\psi(-t)] \\ &= Q_H. \end{aligned} \quad (\text{A3})$$

2. Modes and thermal expectation values

In this subsection, we discuss the construction of thermal expectation values at finite vector, axial and helicity chemical potentials for massless fermions under rigid rotation. The basis of this construction is Eq. (19), which requires the computation of the trace over Fock space. This trace can be computed by first considering a full set of mode solutions of the Dirac equation, comprised of the simultaneous eigenfunctions of the system formed by the Hamiltonian \hat{H} , helicity \hat{h} , and angular momentum \hat{M}^z operators, which appear in the expression of the density operator $\hat{\rho}$, introduced in Eq. (19). Specifically, the particle and anti-particle mode solutions, U_j and $V_j = i\gamma^2 U_j^*$, must satisfy Eq. (12), together with:

$$\begin{aligned} (-i\partial_\varphi + S^z)U_j &= +m_j U_j, & -i\partial_z U_j &= +k_j U_j, \\ (-i\partial_\varphi + S^z)V_j &= -m_j V_j, & -i\partial_z V_j &= -k_j V_j. \end{aligned} \quad (\text{A4})$$

In the above, the eigenvalue of the total angular momentum operator, $m_j = \pm\frac{1}{2}, \pm\frac{3}{2}, \dots$, takes only half-odd-integer values. The eigenvalue of the linear momentum operator along the z axis is k_j . The particle mode solutions $U_j \equiv U_{E,k,m}^\lambda$ satisfying the eigenvalue equations (12) and (A4) can be written as [26]:

$$\begin{aligned} U_{E,k,m}^\lambda(x) &= \frac{e^{-iEt+ikz}}{4\pi} \begin{pmatrix} 1 \\ 2\lambda E/|E| \end{pmatrix} \\ &\otimes \begin{pmatrix} \sqrt{1 + \frac{2\lambda k}{p}} e^{i(m-\frac{1}{2})\varphi} J_{m-\frac{1}{2}}(q\rho) \\ 2i\lambda \sqrt{1 - \frac{2\lambda k}{p}} e^{i(m+\frac{1}{2})\varphi} J_{m+\frac{1}{2}}(q\rho) \end{pmatrix}, \end{aligned} \quad (\text{A5})$$

where $q = \sqrt{p^2 - k^2}$.

For the construction of rigidly-rotating thermal states, Iyer [45] argued that the modes with positive co-rotating energy $\tilde{E}_j = E_j - \Omega m_j > 0$ should be interpreted as particle modes. Following this prescription, the field operator can be expanded with respect to the particle modes U_j and the corresponding anti-particle modes $V_j = i\gamma^2 U_j^*$ as follows:

$$\begin{aligned} \hat{\psi}(x) &= \sum_{\lambda=\pm 1/2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dE |E| \theta(\tilde{E}) \int_{-p}^p dk \\ &\times [U_{E,k,m}^\lambda(x) \hat{b}_{E;k;m}^{\lambda;\Omega} + V_{E,k,m}^\lambda(x) \hat{d}_{E;k;m}^{\lambda;\Omega;\dagger}], \\ &\equiv \sum_j [U_j \hat{b}_j^\Omega + V_j \hat{d}_j^{\Omega;\dagger}], \end{aligned} \quad (\text{A6})$$

where the integral over E runs over the full real axis, while the step function $\theta(\tilde{E})$ ensures that only positive co-rotating energy $\tilde{E}_j = E_j - \Omega m_j > 0$ particle states are taken into account. After second quantization, the mode decomposition in Eq. (A6) naturally introduces the rotating vacuum, $|0_\Omega\rangle$, which is annihilated by the operators \hat{b}_j^Ω and $\hat{d}_j^{\Omega;\dagger}$ for which $\tilde{E}_j > 0$.

The decompositions of the charge operators in Eq. (15), together with equivalent ones for \widehat{H} and \widehat{M}^z given below,

$$\begin{aligned} : \widehat{H} : &:= \sum_j E_j (b_j^\dagger b_j + d_j^\dagger d_j), \\ : \widehat{M}^z : &:= \sum_j m_j (b_j^\dagger b_j + d_j^\dagger d_j), \end{aligned} \quad (\text{A7})$$

allow the following relations to be derived:

$$\begin{aligned} \widehat{\varrho} b_j^{\Omega;\dagger} \widehat{\varrho}^{-1} &= e^{\beta_0(\widetilde{E}_j t - \mu_{\lambda_j, \chi_j, 0})} b_j^{\Omega;\dagger}, \\ \widehat{\varrho} d_j^{\Omega;\dagger} \widehat{\varrho}^{-1} &= e^{\beta_0(\widetilde{E}_j t + \mu_{\lambda_j, -\chi_j, 0})} d_j^{\Omega;\dagger}, \end{aligned} \quad (\text{A8})$$

where

$$\mu_{\lambda_j; \chi_j; 0} = \mu_{V; 0} + 2\lambda_j \mu_{H; 0} + \chi_j \mu_{A; 0}, \quad (\text{A9})$$

is the total chemical potential. The chirality χ_j and helicity λ_j are related through Eq. (13). Starting from

Eq. (A8), the following t.e.v.s can be derived [25]:

$$\begin{aligned} \langle \widehat{b}_j^{\Omega;\dagger} \widehat{b}_{j'}^\Omega \rangle &= n_{\beta_0} (\widetilde{E}_j - \mu_{\lambda_j, \chi_j, 0}) \delta(j, j'), \\ \langle \widehat{d}_j^{\Omega;\dagger} \widehat{d}_{j'}^\Omega \rangle &= n_{\beta_0} (\widetilde{E}_j + \mu_{\lambda_j, -\chi_j, 0}) \delta(j, j'), \end{aligned} \quad (\text{A10})$$

where n_{β_0} is the Fermi-Dirac factor:

$$n_{\beta_0}(a) = \frac{1}{e^{\beta_0 a} + 1}. \quad (\text{A11})$$

An extensive discussion on how to compute the t.e.v.s of various operators using the formalism employed in this paper can be found in Refs. [43, 44]. Briefly, we consider operators \widehat{F} for which the t.e.v.s can be put in the form:

$$\langle : \widehat{F} : \Omega \rangle = \sum_j \left[\mathcal{F}(U_j, U_j) n_{\beta_0} (\widetilde{E}_j - \mu_{\lambda_j, \chi_j, 0}) - \mathcal{F}(V_j, V_j) n_{\beta_0} (\widetilde{E}_j + \mu_{\lambda_j, -\chi_j, 0}) \right], \quad (\text{A12})$$

where $\mathcal{F}(\psi, \chi)$ is a bilinear form with respect to $\bar{\psi}$ and χ . The normal ordering is taken with respect to the rotating vacuum $|0_\Omega\rangle$, i.e.

$$: \widehat{F} : \Omega = \widehat{F} - \langle 0_\Omega | \widehat{F} | 0_\Omega \rangle. \quad (\text{A13})$$

In computing the t.e.v. in Eq. (A12), the integration over E in Eq. (A6) can be manipulated such that the range of E spans only the positive real semiaxis, as follows:

$$\begin{aligned} \langle : \widehat{F} : \Omega \rangle &= \frac{1}{2} \sum_{\lambda=\pm\frac{1}{2}} \sum_{m=-\infty}^{\infty} \int_0^\infty dp p \int_{-p}^p dk \left\{ [\mathcal{F}(U_j, U_j) + \mathcal{F}(V_j, V_j)] \left[n_{\beta_0} (|\widetilde{E}| - \mu_{\lambda; \chi; 0}) + -n_{\beta_0} (|\widetilde{E}| + \mu_{\lambda; -\chi; 0}) \right] \right. \\ &\quad \left. + \text{sgn}(\widetilde{E}) [\mathcal{F}(U_j, U_j) - \mathcal{F}(V_j, V_j)] \left[n_{\beta_0} (|\widetilde{E}| - \mu_{\lambda; \chi; 0}) + n_{\beta_0} (|\widetilde{E}| + \mu_{\lambda; -\chi; 0}) \right] \right\}, \end{aligned} \quad (\text{A14})$$

where $\chi = 2\lambda \text{sgn}(\widetilde{E})$. As discussed in Ref. [44], the t.e.v. of the operator \widehat{F} can be expressed with respect to the Minkowski (non-rotating) vacuum $|0_M\rangle$, in which case it is given by:

$$\langle : \widehat{F} :_M \rangle = \langle : \widehat{F} : \Omega \rangle - \langle 0_M | : \widehat{F} : \Omega | 0_M \rangle, \quad (\text{A15})$$

where the subscript M indicates that the normal ordering is taken with respect to the Minkowski vacuum, $|0_M\rangle$. The one-particle operators b_j^M and d_j^M that annihilate $|0_M\rangle$ can be introduced by considering the following mode expansion:

$$\begin{aligned} \widehat{\psi}(x) &= \sum_{\lambda=\pm 1/2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dE |E| \theta(E) \int_{-p}^p dk \\ &\quad \times [U_{E,k,m}^\lambda(x) \widehat{b}_{E;k;m}^{\lambda;M} + V_{E,k,m}^\lambda(x) \widehat{d}_{E;k;m}^{\lambda;M;\dagger}], \end{aligned} \quad (\text{A16})$$

In Eq. (A15), $\langle 0_M | : \widehat{F} : \Omega | 0_M \rangle$ represents the expectation value of the operator \widehat{F} expressed in normal order with respect to the co-rotating vacuum, taken in the Minkowski (stationary) vacuum state. It receives contributions only from the modes for which $\widetilde{E}E < 0$ [44]:

$$\begin{aligned} \langle 0_M | : \widehat{F} : \Omega | 0_M \rangle &= - \sum_{\lambda=\pm\frac{1}{2}} \sum_{m=-\infty}^{\infty} \int_0^\infty dp p \int_{-p}^p dk \\ &\quad \times \theta(-\widetilde{p}) [\mathcal{F}(U_j, U_j) - \mathcal{F}(V_j, V_j)]. \end{aligned} \quad (\text{A17})$$

Inserting the above into Eq. (A15) gives:

$$\begin{aligned} \langle : \widehat{F} :_M \rangle = & \frac{1}{2} \sum_{\lambda=\pm\frac{1}{2}} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dp p \int_{-p}^p dk \left\{ [\mathcal{F}(U_j, U_j) + \mathcal{F}(V_j, V_j)] [n_{\beta_0}(\tilde{p} - \mu_{\lambda; 2\lambda; 0}) - n_{\beta_0}(\tilde{p} + \mu_{\lambda; -2\lambda; 0})] \right. \\ & \left. + [\mathcal{F}(U_j, U_j) - \mathcal{F}(V_j, V_j)] [n_{\beta_0}(\tilde{p} - \mu_{\lambda; 2\lambda; 0}) + n_{\beta_0}(\tilde{p} + \mu_{\lambda; -2\lambda; 0})] \right\}. \quad (\text{A18}) \end{aligned}$$

The form (A18) is more convenient for analytical manipulations.

In what follows, the computation relies on the following steps. First, the Fermi-Dirac factors are expanded with respect to the rotation parameter Ω , as follows:

$$n_{\beta_0}(\tilde{p} \mp \mu) = \sum_{n=0}^{\infty} \frac{(-\Omega m)^n}{n!} \frac{d^n}{dp^n} n_{\beta_0}(p \mp \mu). \quad (\text{A19})$$

The sum over m appearing in Eq. (A18) is then performed using the following formulas:

$$\begin{aligned} \sum_{m=-\infty}^{\infty} m^{2n} J_m^+(z) &= \sum_{j=0}^n \frac{2\Gamma(j + \frac{1}{2})}{j! \sqrt{\pi}} s_{n,j}^+ z^{2j}, \\ \sum_{m=-\infty}^{\infty} m^{2n+1} J_m^-(z) &= \sum_{j=0}^n \frac{2\Gamma(j + \frac{3}{2})}{j! \sqrt{\pi}} s_{n,j}^+ z^{2j}, \\ \sum_{m=-\infty}^{\infty} m^{2n+1} J_m^\times(z) &= \sum_{j=0}^n \frac{2\Gamma(j + \frac{3}{2})}{(j+1)! \sqrt{\pi}} s_{n,j}^+ z^{2j+1}, \quad (\text{A20}) \end{aligned}$$

where, for future convenience, the following notation was introduced:

$$\begin{aligned} J_m^+(q\rho) &= J_{m-\frac{1}{2}}^2(q\rho) \pm J_{m+\frac{1}{2}}^2(q\rho), \\ J_m^\times(q\rho) &= 2J_{m-\frac{1}{2}}(q\rho)J_{m+\frac{1}{2}}(q\rho). \quad (\text{A21}) \end{aligned}$$

The coefficients $s_{n,j}^+$ can be obtained from:

$$s_{n,j}^+ = \frac{1}{(2j+1)!} \lim_{\alpha \rightarrow 0} \frac{d^{2n+1}}{d\alpha^{2n+1}} \left(2 \sinh \frac{\alpha}{2} \right)^{2j+1}. \quad (\text{A22})$$

It can be seen that $s_{n,j}^+$ vanishes when $j > n$. For small values of $n - j \geq 0$, the first few coefficients are given by [26]:

$$\begin{aligned} s_{j,j}^+ &= 1, \quad s_{j+1,j} = \frac{1}{24} (2j+1)(2j+2)(2j+3), \\ s_{j+2,j}^+ &= \frac{1}{5760} (2j+1)(2j+2)(2j+3) \\ & \quad \times (2j+4)(2j+5)(10j+3). \quad (\text{A23}) \end{aligned}$$

For general values of $n > j$, the following recurrences can be established:

$$\begin{aligned} s_{n+1,j}^+ &= s_{n,j-1}^+ + \left(j + \frac{1}{2} \right)^2 s_{n,j}^+, \\ s_{n,j+1}^+ &= \frac{1}{(j+1)(2j+3)} \sum_{k=1}^{n-j} \binom{2n+1}{2k} s_{n-k,j}^+. \quad (\text{A24}) \end{aligned}$$

The last step involves the integration with respect to k , which can be performed using:

$$\int_0^p dk q^n = \frac{\Gamma[(n+2)/2] \sqrt{\pi}}{2\Gamma[(n+3)/2]} p^{n+1}. \quad (\text{A25})$$

3. Vector and helicity charge currents

The bilinear forms $\mathcal{F}(\psi, \chi)$ for the vector charge current (VCC) and helicity charge current (HCC) are:

$$\mathcal{J}_V^\mu(\psi, \chi) = \bar{\psi} \gamma^\mu \chi, \quad \mathcal{J}_H^\mu(\psi, \chi) = 2\bar{\psi} \gamma^\mu h \chi. \quad (\text{A26})$$

By noting that $V_j = i\gamma^2 U_j^*$, it is not difficult to see that

$$\mathcal{J}_{V/H}^\mu(V_j, V_j) = [\mathcal{J}_{V/H}^\mu(U_j, U_j)]^* = \mathcal{J}_{V/H}^\mu(U_j, U_j). \quad (\text{A27})$$

The above equality implies that

$$\langle : \widehat{\mathcal{J}}_{V/H}^\mu :_\Omega \rangle = \langle : \widehat{\mathcal{J}}_{V/H}^\mu :_M \rangle, \quad (\text{A28})$$

since $\langle 0_M | : \widehat{\mathcal{J}}^\mu :_\Omega | 0_M \rangle = 0$ by virtue of Eq. (A17). Furthermore, it can be seen that $\mathcal{J}_H^\mu(U_j, U_j) = 2\lambda_j \mathcal{J}_V^\mu(U_j, U_j)$, such that it is convenient to discuss the t.e.v.s of the VCC and HCC collectively using the notation:

$$\widehat{\mathcal{J}}_\pm^\mu = \widehat{\mathcal{J}}_V^\mu \pm \widehat{\mathcal{J}}_H^\mu. \quad (\text{A29})$$

Using the explicit expressions for the modes, given in Eq. (A5), the following expressions can be obtained [44]:

$$\begin{aligned} \left(\begin{array}{l} \langle : \widehat{\mathcal{J}}_\pm^\mu :_\Omega \rangle \\ \rho \langle : \widehat{\mathcal{J}}_\pm^\nu :_\Omega \rangle \\ \langle : \widehat{\mathcal{J}}_\pm^\mu :_\Omega \rangle \end{array} \right) &= \frac{1}{2\pi^2} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dp \\ & \quad \times [n_{\beta_0}(\tilde{p} - \mu_{V;0} \mp \mu_{H;0} \mp \mu_{A;0}) \\ & \quad - n_{\beta_0}(\tilde{p} + \mu_{V;0} \pm \mu_{H;0} \pm \mu_{A;0})] \\ & \quad \times \int_0^p dk \left(\begin{array}{l} p J_m^+(q\rho) \\ q J_m^\times(q\rho) \\ \pm p J_m^-(q\rho) \end{array} \right). \quad (\text{A30}) \end{aligned}$$

We illustrate the computational procedure for the temporal component. Expanding the Fermi-Dirac factors using

Eq. (A19), we obtain:

$$\begin{aligned} \langle : \widehat{J}_{\pm}^t : \Omega \rangle &= \frac{1}{2\pi^2} \sum_{n=0}^{\infty} \frac{\Omega^{2n}}{(2n)!} \int_0^{\infty} dp p \\ &\times \frac{d^{2n}}{dp^{2n}} [n_{\beta_0} (p - \mu_{V;0} \mp \mu_{H;0} \mp \mu_{A;0}) \\ &- n_{\beta_0} (p + \mu_{V;0} \pm \mu_{H;0} \mp \mu_{A;0})] \\ &\times \int_0^p dk \sum_{m=-\infty}^{\infty} m^{2n} J_m^+(q\rho). \quad (\text{A31}) \end{aligned}$$

The sum over m and integration over k can be carried out using Eqs. (A20) and (A25):

$$\begin{aligned} \int_0^p dk \sum_{m=-\infty}^{\infty} m^{2n} J_m^+(q\rho) &= \sum_{j=0}^n \frac{2\Gamma(j + \frac{1}{2})}{j! \sqrt{\pi}} s_{n,j}^+ \rho^{2j} \int_0^p dk q^{2j} \\ &= \sum_{j=0}^n \frac{2\rho^{2j} p^{2j+1}}{2j+1} s_{n,j}^+. \quad (\text{A32}) \end{aligned}$$

Upon substitution of the above result into Eq. (A31), the summation over n and j can be reversed, in which case the range of j is from 0 to ∞ , while n runs from j to ∞ . Shifting downwards the summation over n via $n \rightarrow n+j$, the following result is obtained:

$$\begin{aligned} \langle : \widehat{J}_{\pm}^t : \Omega \rangle &= \frac{1}{2\pi^2} \sum_{j=0}^{\infty} \frac{(\rho\Omega)^{2j}}{2j+1} \sum_{n=0}^{\infty} \frac{\Omega^{2n} (2j+2)!}{(2n+2j)!} s_{n+j,j}^+ \\ &\times \int_0^{\infty} dp p^2 \frac{d^{2n}}{dp^{2n}} [n_{\beta_0} (p - \mu_{V;0} \mp \mu_{H;0} \mp \mu_{A;0}) \\ &- n_{\beta_0} (p + \mu_{V;0} \pm \mu_{H;0} \mp \mu_{A;0})], \quad (\text{A33}) \end{aligned}$$

where integration by parts was performed $2j$ times in the integral over p . In the above expression, the series over n contributes corrections of order Ω^{2n} to the t.e.v. Retaining only terms up to $O(\Omega^2)$, the series over n can be truncated at $n=1$. The sum over j can be performed at each value of n using the following results:

$$\sum_{j=0}^{\infty} \frac{(j+n)!}{j!} (\rho\Omega)^{2j} = n! \Gamma^{2n+2}, \quad (\text{A34})$$

where $\Gamma = (1 - \rho^2\Omega^2)^{-1/2}$ is the Lorentz factor corresponding to rigid rotation. Changing the integration variable to $x = \beta_0 p$ and using

$$\int_0^{\infty} \frac{x^n dx}{e^{x-a} + 1} = -n! \text{Li}_{n+1}(-e^a), \quad (\text{A35})$$

where $\text{Li}_n(z)$ is the polylogarithm function, and $n = 0, 1, 2, \dots$ is a natural number, Eq. (A33) can be put

in the form:

$$\begin{aligned} \langle : \widehat{J}_{\pm}^t : \Omega \rangle &= \frac{2\Gamma T^3}{\pi^2} \left\{ \text{Li}_3[-e^{-(\mu_V \pm \mu_H \mp \mu_A)/T}] \right. \\ &\quad \left. - \text{Li}_3[-e^{(\mu_V \pm \mu_H \pm \mu_A)/T}] \right\} \\ &+ \frac{\Gamma^3 \Omega^2 T}{12\pi^2} (4\Gamma^2 - 1) \ln \frac{1 + e^{(\mu_V \pm \mu_H \pm \mu_A)/T}}{1 + e^{-(\mu_V \pm \mu_H \mp \mu_A)/T}} + O(\Omega^4). \quad (\text{A36}) \end{aligned}$$

The high temperature limit can be extracted by noting that

$$\begin{aligned} \text{Li}_3(-e^a) &= -\frac{3\zeta(3)}{4} - \frac{\pi^2 a}{12} - \frac{a^2}{2} \ln 2 - \frac{a^3}{12} + O(a^4), \\ \ln(1 + e^a) &= \ln 2 + \frac{a}{2} + \frac{a^2}{8} + O(a^4), \quad (\text{A37}) \end{aligned}$$

where $\zeta(z)$ is the Riemann zeta function. The following result is obtained:

$$\begin{aligned} \langle : \widehat{J}_{\pm}^t : \Omega \rangle &= \Gamma(\mu_V \pm \mu_H) \left\{ \frac{T^2}{3} \pm \frac{4T\mu_A}{\pi^2} \ln 2 \right. \\ &\quad \left. + \frac{(\mu_V \pm \mu_H)^2 + 3\mu_A^2}{3\pi^2} + O(T^{-1}) \right. \\ &\quad \left. + \frac{3\omega^2 + \mathbf{a}^2}{12\pi^2} \left[1 \pm \frac{\mu_A}{2T} + O(T^{-2}) \right] + O(\Omega^4) \right\}, \quad (\text{A38}) \end{aligned}$$

where $\omega^2 = \Omega^2 \Gamma^4$ and $\mathbf{a}^2 = \rho^2 \Omega^4 \Gamma^4 = \Omega^2 \Gamma^2 (\Gamma^2 - 1)$. Performing the exact same steps for the φ component yields:

$$\begin{aligned} \langle : \widehat{J}_{\pm}^{\varphi} : \Omega \rangle &= \Omega \Gamma(\mu_V \pm \mu_H) \left\{ \frac{T^2}{3} \pm \frac{4T\mu_A}{\pi^2} \ln 2 \right. \\ &\quad \left. + \frac{(\mu_V \pm \mu_H)^2 + 3\mu_A^2}{3\pi^2} + O(T^{-1}) \right. \\ &\quad \left. + \frac{\omega^2 + 3\mathbf{a}^2}{12\pi^2} \left[1 \pm \frac{\mu_A}{2T} + O(T^{-2}) \right] + O(\Omega^4) \right\}. \quad (\text{A39}) \end{aligned}$$

For $\langle : \widehat{J}_{\pm}^z : \Omega \rangle$, the following expression is obtained:

$$\begin{aligned} \langle : \widehat{J}_{\pm}^z : \Omega \rangle &= \pm \frac{\Gamma^2 \Omega T^2}{\pi^2} \left\{ \text{Li}_2[-e^{-(\mu_V \pm \mu_H \mp \mu_A)/T}] \right. \\ &\quad \left. - \text{Li}_2[-e^{(\mu_V \pm \mu_H \pm \mu_A)/T}] \right\} \\ &\pm \frac{\Gamma^4 \Omega^3}{24\pi^2} \frac{(4\Gamma^2 - 3) \sinh[(\mu_V \pm \mu_H)/T]}{\cosh[(\mu_V \pm \mu_H)/T] + \cosh(\mu_A/T)} + O(\Omega^4). \quad (\text{A40}) \end{aligned}$$

Using the following expansion for the polylogarithm:

$$\text{Li}_2(-e^a) = -\frac{\pi^2}{12} - a \ln 2 - \frac{a^2}{4} - \frac{a^3}{24} + O(a^5), \quad (\text{A41})$$

the following expansion is obtained:

$$\langle : \widehat{J}_{\pm}^z : \Omega \rangle = \pm \frac{\Omega \Gamma^2}{\pi^2} (\mu_V \pm \mu_H) \left\{ \begin{aligned} & 2T \ln 2 \pm \mu_A + \frac{(\mu_V \pm \mu_H)^2 + 3\mu_A^2}{12T} + O(T^{-2}) \\ & + \frac{\omega^2 + 3\mathbf{a}^2}{48T} [1 + O(T^{-2})] + O(\Omega^4) \end{aligned} \right\}. \quad (\text{A42})$$

The charge density Q_{\pm} and azimuthal conductivity σ_{\pm}^{τ} can be obtained by inverting the expansion (21). Noting that $Q_{\pm} = u \cdot \langle : J_{\pm}^t : \Omega \rangle = \Gamma(\langle : J_{\pm}^t : \Omega \rangle - \rho^2 \Omega \langle : J_{\pm}^{\varphi} : \Omega \rangle)$, the following expression is obtained:

$$Q_{\pm} = (\mu_V \pm \mu_H) \left\{ \begin{aligned} & \frac{T^2}{3} \pm \frac{4T\mu_A}{\pi^2} \ln 2 \\ & + \frac{\mu_V^2 + \mu_H^2 + 3\mu_A^2}{3\pi^2} \pm \frac{\mu_V \mu_H}{\pi^2} + O(T^{-1}) \\ & + \frac{\omega^2 + \mathbf{a}^2}{4\pi^2} \left[1 \pm \frac{\mu_A}{2T} + O(T^{-2}) \right] + O(\Omega^4) \end{aligned} \right\}. \quad (\text{A43})$$

The charge conductivity can be obtained using

$$\sigma_{\pm}^{\tau} = \frac{\tau_{\mu} \langle : J_{\pm}^{\mu} : \Omega \rangle}{\tau^2} = \frac{\Omega \langle : J_{\pm}^t : \Omega \rangle - \langle : J_{\pm}^{\varphi} : \Omega \rangle}{\Omega^3 \Gamma^3} \\ = \frac{\mu_V \pm \mu_H}{6\pi^2} \left[1 \pm \frac{\mu_A}{2T} + O(T^{-2}) \right] + O(\Omega^2). \quad (\text{A44})$$

Finally, the vortical conductivity σ_{\pm}^{ω} can be obtained as follows:

$$\sigma_{\pm}^{\omega} = \frac{\omega_{\mu} \langle : J_{\pm}^{\omega} : \Omega \rangle}{\omega^2} = \frac{\langle : J_{\pm}^z : \Omega \rangle}{\Omega \Gamma^2} \\ = \pm \frac{2\mu_{\pm} T}{\pi^2} \ln 2 + \frac{\mu_A \mu_{\pm}}{\pi^2} \pm \frac{\mu_{\pm}(\mu_{\pm}^2 + 3\mu_A^2)}{12\pi^2 T} + O(T^{-2}) \\ \pm \frac{\mu_{\pm}(\omega^2 + 3\mathbf{a}^2)}{48\pi^2 T} [1 + O(T^{-2})] + O(\Omega^4). \quad (\text{A45})$$

The vector and helical quantities can be reconstructed from the above results using:

$$\langle : J_{V/H}^{\mu} : \Omega \rangle = \frac{1}{2} (\langle : J_{+}^{\mu} : \Omega \rangle \pm \langle : J_{-}^{\mu} : \Omega \rangle). \quad (\text{A46})$$

4. Axial charge current

The bilinear form $\mathcal{F}(\psi, \chi)$ for the axial charge current (ACC) is:

$$\mathcal{J}_A^{\mu}(\psi, \chi) = \bar{\psi} \gamma^{\mu} \gamma^5 \chi. \quad (\text{A47})$$

Taking into account the charge conjugation symmetry, $V_j = i\gamma^2 U_j^*$, it can be shown that

$$\mathcal{J}_A^{\mu}(V_j, V_j) = -[\mathcal{J}_A^{\mu}(U_j, U_j)]^* = \mathcal{J}_A^{\mu}(U_j, U_j). \quad (\text{A48})$$

The above property implies that the term on the first line in Eq. (A12) makes a vanishing contribution to the t.e.v.s of the components of the ACC. Using the explicit expression for the modes given in Eq. (A5), the following results can be obtained [44]:

$$\mathcal{J}_A^t(U_j, U_j) = \frac{p_j}{8\pi^2 |E_j|} \left[2\lambda_j J_{m_j}^+(q_j \rho) + \frac{k_j}{p_j} J_{m_j}^-(q_j \rho) \right], \\ \mathcal{J}_A^{\varphi}(U_j, U_j) = \frac{\lambda_j q_j}{4\pi^2 p_j \rho} J_{m_j}^{\times}(q_j \rho), \\ \mathcal{J}_A^z(U_j, U_j) = \frac{1}{8\pi^2} \left[J_{m_j}^-(q_j \rho) + \frac{2\lambda_j k_j}{p_j} J_{m_j}^+(q_j \rho) \right]. \quad (\text{A49})$$

The results above show that $\mathcal{J}_A^t(U_j, U_j)$ and $\mathcal{J}_A^{\varphi}(U_j, U_j)$ are odd under the simultaneous flip $(k_j, \lambda_j) \rightarrow (-k_j, -\lambda_j)$. Because of this property, Eq. (A17) shows that

$$\langle 0_M | : \widehat{J}_A^t : \Omega | 0_M \rangle = \langle 0_M | : \widehat{J}_A^{\varphi} : \Omega | 0_M \rangle = 0. \quad (\text{A50})$$

Thus, the t.e.v.s of the t and φ components of the ACC with respect to the rotating vacuum are the same as when expressed with respect to the Minkowski vacuum. This conclusion does not hold true for the z component, for which there will be non-vanishing vacuum terms in the t.e.v. computed with respect to the Minkowski vacuum.

Using Eq. (A49), the following expressions can be obtained [44]:

$$\left(\begin{array}{c} \langle : \widehat{J}_A^t : \Omega \rangle \\ \rho \langle : \widehat{J}_A^{\varphi} : \Omega \rangle \\ \langle : \widehat{J}_A^z : \Omega \rangle \end{array} \right) = \frac{1}{4\pi^2} \sum_{\lambda=\pm\frac{1}{2}} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dp \\ \times [n_{\beta_0}(\tilde{p} - \mu_{\lambda; 2\lambda; 0}) + n_{\beta_0}(\tilde{p} + \mu_{\lambda; -2\lambda; 0})] \\ \times \int_0^p dk \left(\begin{array}{c} 2\lambda p J_m^+(q\rho) \\ 2\lambda q J_m^{\times}(q\rho) \\ p J_m^-(q\rho) \end{array} \right). \quad (\text{A51})$$

Employing the same steps as in Subsec. A 3, we obtain:

$$\langle : \widehat{J}_A^t : \Omega \rangle = \frac{1}{4\pi^2} \sum_{\lambda=\pm\frac{1}{2}} 2\lambda \sum_{j=0}^{\infty} \frac{(\rho\Omega)^{2j}}{2j+1} \sum_{n=0}^{\infty} \frac{\Omega^{2n} (2j+2)!}{(2n+2j)!} \\ \times s_{n+j}^+ \int_0^{\infty} dp p^2 \frac{d^{2n}}{dp^{2n}} [n_{\beta_0}(p - \mu_{\lambda; 2\lambda; 0}) \\ + n_{\beta_0}(p + \mu_{\lambda; -2\lambda; 0})]. \quad (\text{A52})$$

The above expression can be approximated up to $O(\Omega^2)$ by taking into account the $n=0$ and $n=1$ terms. The following result is obtained:

$$\langle : \widehat{J}_A^t : \Omega \rangle = -\frac{\Gamma T^3}{\pi^2} \left[\text{Li}_3(-e^{\mu_{\frac{1}{2}, 1}/T}) - \text{Li}_3(-e^{\mu_{-\frac{1}{2}, -1}/T}) \right. \\ \left. + \text{Li}_3(-e^{-\mu_{\frac{1}{2}, -1}/T}) - \text{Li}_3(-e^{-\mu_{-\frac{1}{2}, 1}/T}) \right] \\ + \frac{\Gamma^3 \Omega^2 T}{24\pi^2} (4\Gamma^2 - 1) \ln \frac{e^{\mu_A/T} (\cosh \frac{\mu_V + \mu_H}{T} + \cosh \frac{\mu_A}{T})}{e^{-\mu_A/T} (\cosh \frac{\mu_V - \mu_H}{T} + \cosh \frac{\mu_A}{T})} \\ + O(\Omega^4). \quad (\text{A53})$$

In the high temperature limit, the following result is obtained:

$$\begin{aligned} \langle : \widehat{J}_A^i : \Omega \rangle = \Gamma \left\{ \left[\frac{\mu_A T^2}{3} + \frac{4T\mu_V\mu_H}{\pi^2} \ln 2 \right. \right. \\ \left. \left. + \frac{3\mu_A(\mu_V + \mu_H)^2 + \mu_A^3}{3\pi^2} + O(T^{-1}) \right] \right. \\ \left. + \frac{3\omega^2 + \mathbf{a}^2}{12\pi^2} \left[\mu_A + \frac{\mu_V\mu_H}{2T} + O(T^{-2}) \right] + O(\Omega^4) \right\}. \end{aligned} \quad (\text{A54})$$

Similarly, the φ component is given by:

$$\begin{aligned} \langle : \widehat{J}_A^\varphi : \Omega \rangle = \Omega \Gamma \left\{ \left[\frac{\mu_A T^2}{3} + \frac{4T\mu_V\mu_H}{\pi^2} \ln 2 \right. \right. \\ \left. \left. + \frac{3\mu_A(\mu_V + \mu_H)^2 + \mu_A^3}{3\pi^2} + O(T^{-1}) \right] \right. \\ \left. + \frac{\omega^2 + 3\mathbf{a}^2}{12\pi^2} \left[\mu_A + \frac{\mu_V\mu_H}{2T} + O(T^{-2}) \right] + O(\Omega^4) \right\}. \end{aligned} \quad (\text{A55})$$

The charge density Q_A and azimuthal conductivity σ_A^τ can be obtained using the relations (A43) and (A44). The results is summarised in Eqs. (22b) and (25b), respectively.

For $\langle : \widehat{J}_A^z : \Omega \rangle$, the following expression is obtained:

$$\begin{aligned} \langle : \widehat{J}_\pm^z : A \rangle = -\frac{\Gamma^2 \Omega T^2}{2\pi^2} \left[\text{Li}_2(-e^{-\mu_{\frac{1}{2},1}/T}) + \text{Li}_2(-e^{-\mu_{\frac{1}{2},1}/T}) \right. \\ \left. + \text{Li}_2(-e^{-\mu_{\frac{1}{2},1}/T}) + \text{Li}_2(-e^{-\mu_{\frac{1}{2},1}/T}) \right] \\ + \frac{\Gamma^4 \Omega^3}{48\pi^2} (4\Gamma^2 - 3) \left[\frac{e^{\mu_A/T} + \cosh \frac{\mu_V + \mu_H}{T}}{\cosh \frac{\mu_A}{T} + \cosh \frac{\mu_V + \mu_H}{T}} + \frac{e^{-\mu_A/T} + \cosh \frac{\mu_V - \mu_H}{T}}{\cosh \frac{\mu_A}{T} + \cosh \frac{\mu_V - \mu_H}{T}} \right] + O(\Omega^4). \end{aligned} \quad (\text{A56})$$

The large temperature expansion is easily obtained:

$$\begin{aligned} \langle : \widehat{J}_A^z : M \rangle = \Omega \Gamma^2 \left\{ \left[\frac{T^2}{6} + \frac{\mu_V^2 + \mu_H^2 + \mu_A^2}{2\pi^2} \right. \right. \\ \left. \left. + \frac{\mu_V\mu_H\mu_A}{2\pi^2 T} + O(T^{-2}) \right] \right. \\ \left. + \frac{\omega^2 + 3\mathbf{a}^2}{24\pi^2} [1 + O(T^{-3})] + O(\Omega^4) \right\}. \end{aligned} \quad (\text{A57})$$

It can be seen that the term on the last line above is independent of the thermodynamic parameters T and μ_l ($l \in \{V, H, A\}$). This term represents the vacuum contribution $\langle 0_M | : \widehat{J}_A^z : \Omega | 0_M \rangle$, which was previously found in many studies [25, 26, 44]. The t.e.v. of \widehat{J}^z computed

with respect to the rotating vacuum can be obtained by subtracting this term. The result is:

$$\begin{aligned} \langle : \widehat{J}_A^z : \Omega \rangle = \Omega \Gamma^2 \left[\frac{T^2}{6} + \frac{\mu_V^2 + \mu_H^2 + \mu_A^2}{2\pi^2} \right. \\ \left. + \frac{\mu_V\mu_H\mu_A}{2\pi^2 T} + O(T^{-2}, \Omega^4) \right]. \end{aligned} \quad (\text{A58})$$

The corresponding vortical conductivity can be obtained as shown in Eq. (A45) and the result is summarised in Eq. (24).

5. Stress-energy tensor

For completeness of the overall presentation, we also provide the result for the thermal expectation value (t.e.v.) of the stress-energy tensor (SET),

$$\widehat{T}_{\mu\nu} = \frac{i}{2} [\widehat{\Psi} \gamma_{(\mu} \nabla_{\nu)} \widehat{\Psi} - \nabla_{(\mu} \widehat{\Psi} \gamma_{\nu)} \widehat{\Psi}]. \quad (\text{A59})$$

The bilinear form introduced in Eq. (A12) which corresponds to the SET is:

$$\mathcal{T}_{\mu\nu}(\psi, \chi) = \frac{i}{2} [\bar{\psi} \gamma_{(\mu} \nabla_{\nu)} \chi - \nabla_{(\mu} \bar{\psi} \gamma_{\nu)} \chi]. \quad (\text{A60})$$

It is easy to check that

$$\mathcal{T}_{\mu\nu}(V_j, V_j) = -[\mathcal{T}_{\mu\nu}(U_j, U_j)]^* = -\mathcal{T}_{\mu\nu}(U_j, U_j). \quad (\text{A61})$$

Due to the above property, the first line in Eq. (A14) vanishes, as was the case for the t.e.v. of the ACC discussed in Sec. A 4. The explicit expressions for the bilinears $\mathcal{T}_{\mu\nu}(U_j, U_j)$ are presented in Refs. [43, 44] and for brevity, they are not reproduced here. The procedure to compute the t.e.v.s is also detailed in these references, being identical to the one employed for the ACC in Sec. A 4. The hydrodynamic content of the SET can be extracted by considering the following decomposition:

$$\langle : \widehat{T}_{\mu\nu} : \rangle = P(4u_\mu u_\nu - g_{\mu\nu}) + \Pi_{\mu\nu} + u_\mu W_\nu + u_\nu W_\mu, \quad (\text{A62})$$

which contains the isotropic pressure component,

$$\begin{aligned} P = \frac{7\pi^2 T^4}{180} + \frac{T^2}{6} (\mu_A^2 + \mu_V^2 + \mu_H^2) \\ + \frac{\mu_V^4 + \mu_H^4 + \mu_A^4 + 6(\mu_V^2 \mu_H^2 + \mu_V^2 \mu_A^2 + \mu_H^2 \mu_A^2)}{12\pi^2} \\ + \frac{3\omega^2 + \mathbf{a}^2}{72} \left[T^2 + \frac{3(\mu_V^2 + \mu_H^2 + \mu_A^2)}{\pi^2} \right] \\ + \frac{4\mu_A \mu_H \mu_V T}{\pi^2} \ln 2 + O(T^0), \end{aligned} \quad (\text{A63})$$

the anisotropic stress contribution,

$$\begin{aligned} \Pi^{\mu\nu} = \Pi_V \left(\tau^\mu \tau^\nu - \frac{\omega^2}{2} a^\mu a^\nu - \frac{\mathbf{a}^2}{2} \omega^\mu \omega^\nu \right) \\ + \Pi_H (\tau^\mu \omega^\nu + \tau^\nu \omega^\mu), \end{aligned} \quad (\text{A64})$$

and the heat flux:

The conductivities κ_τ and κ_ω are as follows:

$$\begin{aligned}\kappa^\tau &= -\frac{1}{18} \left[T^2 + \frac{3(\mu_V^2 + \mu_H^2 + \mu_A^2)}{\pi^2} \right] + O(T^{-1}), \\ \kappa^\omega &= \frac{\mu_A T^2}{3} + \frac{4\mu_H \mu_V T}{\pi^2} \ln 2 + \frac{\mu_A^3 + 3\mu_A(\mu_V^2 + \mu_H^2)}{3\pi^2} \\ &\quad + \frac{\mu_V \mu_H (3\mu_A^2 + \mu_V^2 + \mu_H^2)}{6\pi^2 T} \\ &\quad + \frac{\omega^2 + \mathbf{a}^2}{12\pi^2} \left[\mu_A + \frac{\mu_V \mu_H}{2T} \right] + O(T^{-3}, \Omega^4),\end{aligned}\quad (\text{A66})$$

while the coefficient Π_H is given by:

$$\Pi_H = -\frac{\mu_A}{3\pi^2} - \frac{\mu_H \mu_V}{6\pi^2 T} + O(T^{-2}, \Omega^2).\quad (\text{A67})$$

For massless particles, we find that $\Pi_V = 0$, while the energy density is $\mathcal{E} = 3P$.

$$W^\mu = \kappa_\tau \tau^\mu + \kappa_\omega \omega^\mu.\quad (\text{A65})$$