



**HAL**  
open science

# Dynamics of a Fleming-Viot type particle system on the cycle graph

Josué M. Corujo

► **To cite this version:**

Josué M. Corujo. Dynamics of a Fleming-Viot type particle system on the cycle graph. 2020. hal-02447747v1

**HAL Id: hal-02447747**

**<https://hal.science/hal-02447747v1>**

Preprint submitted on 21 Jan 2020 (v1), last revised 9 Apr 2021 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# DYNAMICS OF A FLEMING–VIOT TYPE PARTICLE SYSTEM ON THE CYCLE GRAPH

JOSUÉ M. CORUJO

ABSTRACT. This work is devoted to the study of interacting asymmetric continuous time random walks on the cycle graph, with uniform killing. The process is of Fleming–Viot or Moran type and allows to approximate the quasi-stationary distribution of a related process. We show that this model has a remarkable exact solvability, despite the fact that it is non-reversible with non-explicit invariant distribution. Our main results include quantitative propagation of chaos and exponential ergodicity with explicit constants, as well as formulas for covariances at equilibrium in terms of the Chebyshev polynomials. In particular we obtain a uniform bound in time for the convergence of the proportion of particles in each state when the number of particles goes to infinity. This work extends previous works by Cloez and Thai around the complete graph. It can be seen as a further step towards the study of such processes on graphs with general geometry.

## 1. INTRODUCTION

This paper deals with a time-continuous Markov process describing the position of  $N$  particles moving around on the cycle graph. This type of model is usually known as Fleming–Viot process, or Moran type process, [18, 17, 10]. Consider a continuous time Markov process on  $E \cup \{\partial\}$ , where  $E$  is finite and  $\partial$  is an absorbing state. Briefly, the Fleming–Viot process consists in  $N$  particles moving in  $E$  as independent copies of the original process, until one of the particles get absorbed. When this happens, the absorbed particle jumps instantaneously and uniformly to one of the positions of the others particles. The Fleming–Viot processes were originally and independently introduced by Del Moral, Guionnet, Miclo [15, 16] and Burdzy, Holyst, March [5] to approximate the law of a Markov process conditioned on non-absorption, and its *Quasi-Stationary Distribution* (QSD), which is the limit of this conditional law when  $t \rightarrow \infty$ . See e.g. [24], [11] and [26], excellent references for an introduction to the theory related to QSD. For recent and quite general results about the convergence of Markov processes conditioned to non-absorption to a QSD, see [6], [7] and [4].

The convergence of the empirical distribution induced by Fleming–Viot process defined on discrete state spaces when the size of the population and the time increase have been assured under some assumptions. For example, [18] and [1] study the convergence of the empirical distribution induced by the Fleming–Viot process to the unique QSD in countable and finite discrete space settings, respectively. With the aim to study the convergence of the particle process under the stationary distribution to the QSD, [20] prove a Central Limit Theorem for the finite state case. Additionally, [28] and [2] study the convergence to the minimal QSD in a Galton-Watson type model and in a birth and death process, respectively. Similarly, [3] and [22] address the study of the  $N$ -particle system associated to a random walk on  $\mathbb{N}$  with a drift towards the origin, which is an absorbing state. In these scenarios there exist infinitely many QSD for each model, so it is important to assure the ergodicity of the  $N$ -particle system and to determine to which QSD it converges. Additionally, [8] study the convergence of the Fleming–Viot process to the minimal QSD under general conditions, providing also some specific examples.

In addition, some works have been devoted to the study of the speed of convergence when the number of particles and time tend to infinity. In particular, Cloez and Thai [10] study the  $N$ -particle system in a discrete state space settings. They study the convergence of the empirical measure induced by the Fleming–Viot process when both  $t \rightarrow \infty$  (ergodicity) and  $N \rightarrow \infty$  (propagation of chaos), providing explicit bounds for the speed of convergence. Following the results in this paper Cloez and Thai [9] study two examples in details: the random walk on the complete graph with uniform killing and the random walk on the two-site graph. The simple geometries of the graphs of these models simplify the study of the  $N$ -particle dynamic and allows them to give explicit expressions for the stationary distributions of the  $N$ -particle processes and explicit bounds for its convergence to the QSD.

---

*Date:* January 2020.

*2000 Mathematics Subject Classification.* 60K35, 60B10, 37A25.

*Key words and phrases.* Quasi-stationary distribution; Fleming–Viot type particle system; Moran type model; propagation of chaos; ergodicity.

Consider the quantity  $\lambda$  defined in [10] as

$$\lambda = \inf_{x,y} \left( Q_{x,y} + Q_{y,x} + \sum_{s \neq x,y} Q_{x,s} \wedge Q_{y,s} \right), \quad (1.1)$$

where  $Q = (Q_{x,y})_{x,y}$  is the infinitesimal generator matrix of the process until absorption. When  $\lambda = 0$  some of the results of [10] do not hold and most of the bounds given become not interesting or too rough. Note that  $\lambda > 0$  for the two examples studied in [9], but  $\lambda$  is equal to zero for those models where there exist two vertices such that the distance between them is greater than two. The quantity  $\lambda$  is somehow related to the geometry of the graph associated to the Markov process. So, it becomes interesting to find explicit bounds for the speed of convergence of Fleming–Viot processes with more complex geometry.

In this article we focus on the random walk on the cycle graph  $\mathbb{Z}/K\mathbb{Z}$  for  $K \geq 3$ . Note that for this graph it holds  $\lambda = 0$  when  $K \geq 6$ . For simplicity, we assume that the  $N$  particles jump to the absorbing state with the same rate, i.e. we consider a process with *uniform killing*, [24]. Even if in this case the distribution of the conditional process is trivial, the study of the Fleming–Viot process becomes more complicated due to its non reversibility and the geometry of the cyclic graph. We focus on providing bounds for the speed of the convergence of the empirical distribution induced by the particle system to the unique QSD when  $t$  and  $N$  tend to infinity. This example can be seen as a further step towards the study of the speed of convergence of Fleming–Viot process with more general geometry.

**1.1. Model and notations.** Consider a Markov process  $(Z_t)_{t \geq 0}$  with state space  $\mathbb{Z}/K\mathbb{Z} \cup \{\partial\}$ , where  $K \geq 3$  and  $\partial$  is an absorbing state. Specifically, the infinitesimal generator of the process is given by

$$\mathcal{G}f(x) = f(x+1) - f(x) + \theta[f(x-1) - f(x)] + p[f(\partial) - f(x)],$$

where  $x \in \mathbb{Z}/K\mathbb{Z}$  and  $\mathcal{G}f(\partial) = 0$ , where  $\theta, p \in \mathbb{R}_+^*$  and  $f$  is a real function defined on  $\mathbb{Z}/K\mathbb{Z} \cup \{\partial\}$ . In words,  $(Z_t)_{t \geq 0}$  is an asymmetric random walk on the  $K$ -cycle graph which jumps with rates 1 and  $\theta$  in the clockwise and the anti-clockwise directions, respectively. Also, with uniform rate  $p$  the process jumps to the absorbing state  $\partial$ , i.e. it is killed. Note that  $\mathbb{Z}/K\mathbb{Z}$  is an irreducible class.

The process generated by  $\mathcal{G}$  is a particular case of the processes with *uniform killing in a finite state space* considered by Méléard and Villemonais [24, Sec. 2.3].

Let  $(X_t)_{t \geq 0}$  be the analogous asymmetric random walk on the cycle graph  $\mathbb{Z}/K\mathbb{Z}$  without killing. The generator of this process, denoted by  $\mathcal{H}$ , is given by

$$\mathcal{H}f(x) = f(x+1) - f(x) + \theta[f(x-1) - f(x)] \quad \text{for all } x \in \mathbb{Z}/K\mathbb{Z}.$$

Note that, because of the uniform killing, the process  $(Z_t)_{t \geq 0}$  could also be defined in the following way

$$Z_t = \begin{cases} X_t & \text{if } t < \tau_p \\ \partial & \text{if } t \geq \tau_p, \end{cases}$$

where  $\tau_p$  is an exponential random variable with mean  $1/p$  and independent of the random walk  $(X_t)_{t \geq 0}$ . This means that the law of the process  $(Z_t)_{t \geq 0}$  conditioned on non absorption is given by

$$\mathbb{P}_\mu[Z_t = k \mid t < \tau_p] = \mathbb{P}_\mu[X_t = k],$$

for  $k \in \mathbb{Z}/K\mathbb{Z}$  and every initial distribution  $\mu$  on  $\mathbb{Z}/K\mathbb{Z}$ . As a consequence, the QSD of  $(Z_t)_{t \geq 0}$ , denoted by  $\nu_{qs}$ , is the stationary distribution of  $(X_t)_{t \geq 0}$ , which is the uniform distribution on  $\mathbb{Z}/K\mathbb{Z}$ , as we will prove in Theorem 1.1.

Recall that the total variation distance between two probability measures  $\mu_1$  and  $\mu_2$  defined on a discrete probability space  $E$  is defined, see for instance [21, Sec. 4], by  $d_{TV}(\mu_1, \mu_2) = \frac{1}{2} \|\mu_1 - \mu_2\|_1$  where  $\|\mu_1 - \mu_2\|_p = (\sum_{x \in E} |\mu_1(x) - \mu_2(x)|^p)^{1/p}$  is the vector  $l_p$  norm. If  $(f_N)$  and  $(g_N)$  are two real sequences,  $f_N \underset{N \rightarrow \infty}{\sim} g_N$  means  $f_N - g_N = o(g_N)$ .

Now, assume we have  $N$  particles with independent behaviour driven by the generator  $\mathcal{G}$ , until one of them jumps to the absorbing state. When this happens, the particle instantaneously and uniformly jumps to one of the positions of the other  $N - 1$  particles. We denote by  $(\eta_t^{(N)})_{t \geq 0}$  the Markov process accounted for the positions of the  $N$  particles in the  $K$ -cycle graph at time  $t$ . Consider the state space  $\mathcal{E}_{K,N}$  of this process, which is given by

$$\mathcal{E}_{K,N} = \left\{ \eta : \mathbb{Z}/K\mathbb{Z} \rightarrow \mathbb{N}, \sum_{k=0}^{K-1} \eta(k) = N \right\}. \quad (1.2)$$

At time  $t$  the system is in state  $\eta_t = (\eta_t(0), \eta_t(1), \dots, \eta_t(K-1))$  if there are  $\eta_t(k)$  particles on site  $k$ , for  $k = 0, 1, \dots, K-1$ . Note that the cardinality of  $\mathcal{E}_{K,N}$  is equal to that of the set of non-negative

solutions of the integer equation  $x_1 + x_2 + \dots + x_K = N$ , which is  $\text{card}(\mathcal{E}_{K,N}) = \binom{K+N-1}{N}$ , see e.g. [12, Th. D, Sec. 1.7].

The generator of the  $N$ -particle process  $(\eta_t^{(N)})_{t \geq 0}$ , denoted by  $\mathcal{L}_{K,N}$ , applied to a function  $f$  on  $\mathcal{E}_{K,N}$  reads

$$\mathcal{L}_{K,N}f(\eta) = \sum_{i,j \in \mathbb{Z}/K\mathbb{Z}} \eta(i)[f(T_{i \rightarrow j}\eta) - f(\eta)] \left( \mathbb{1}_{\{j=i+1\}} + \theta \mathbb{1}_{\{j=i-1\}} + p \frac{\eta(j)}{N-1} \right), \quad (1.3)$$

where  $\theta, p > 0$  and for every  $\eta \in \mathcal{E}_{K,N}$  satisfying  $\eta(i) > 0$ , the configuration  $T_{i \rightarrow j}\eta$  is defined as  $T_{i \rightarrow j}\eta = \eta - e_i + e_j$  and  $e_i$  is the  $i$ -th canonical vector of  $\mathbb{R}^K$ . Under this dynamics each of the  $N$  particles, no matter where it is, can jump to every site  $j \in \mathbb{Z}/K\mathbb{Z}$  such that  $\eta(j) > 0$ . Note that the process  $(\eta_t^{(N)})_{t \geq 0}$  is irreducible. Consequently, it has a unique stationary distribution denoted  $\nu_N$ .

For every  $\eta \in \mathcal{E}_{K,N}$  the empirical distribution  $m(\eta)$  associated to the configuration  $\eta$  is defined by

$$m(\eta) = \frac{1}{N} \sum_{k=0}^{N-1} \eta(k) \delta_{\{k\}},$$

where  $\delta_{\{k\}}$  is the Dirac distribution at  $k \in \mathbb{Z}/K\mathbb{Z}$ .

The (random) empirical distribution  $m(\eta_t^{(N)})$  approximates the QSD of the process  $(Z_t)_{t \geq 0}$ , see [18, 1, 27], which due to Theorem 1.1 is the uniform distribution. We are interested in studying how fast  $m(\eta_t^{(N)})$  converges (in some sense) to the uniform distribution on  $\mathbb{Z}/K\mathbb{Z}$  when both  $t$  and  $N$  tend to infinity. Consider  $\eta_\infty^{(N)}$  a random variable with distribution  $\nu_N$ , the stationary distribution of the process  $(\eta_t^{(N)})_{t \geq 0}$ . In this work we develop a similar analysis to that of the complete graph dynamics in [9]. We focus on the convergences when both  $N$  and  $t$  tend to infinity, as shown in the following diagram

$$\begin{array}{ccc} m(\eta_t^{(N)}) & \xrightarrow{t \rightarrow \infty} & m(\eta_\infty^{(N)}) \\ N \downarrow & & \downarrow N \\ \mathcal{L}(Z_t \mid t < \tau_p) & \xrightarrow{t \rightarrow \infty} & \nu_{qs} \end{array}$$

where the limits are in distribution. Theorem 1.1 provides lower and upper exponential bounds for the speed of convergence of  $\mathcal{L}(Z_t \mid t < \tau_p)$  to  $\nu_{qs}$  in the 2-norm, when  $t \rightarrow \infty$ . Likewise, Corollary 1.7 and Theorem 1.9 give bounds for the speed of convergence of  $m(\eta_t^{(N)})$  to  $\mathcal{L}(Z_t \mid t < \tau_p)$  and  $m(\eta_\infty^{(N)})$  to  $\nu_{qs}$ , when  $N \rightarrow \infty$ .

The quantitative long time behaviour of the  $N$ -particle system in countable state spaces is studied in [10]. Using a coupling technique, an exponential bound is provided for the convergence of  $m(\eta_t^{(N)})$  to  $m(\eta_\infty^{(N)})$  in the sense of a Wasserstein distance under certain conditions. In particular, the parameter  $\lambda$  defined by (1.1) needs to be positive. As we said, this is not the case of the asymmetric random walk on the  $K$ -cycle graph with uniform killing, when  $K \geq 6$ . A potential solution of this problem for our model could be the study of the spectrum of the generator of the  $N$ -particle process,  $\mathcal{L}_{K,N}$ . In addition, upper bounds for the speed of convergence could also be obtained using the results in the recent paper of Villemonais, [29], for a suitable distance. These are possible directions for future research.

**1.2. Main results.** We first prove that the uniform distribution on  $\mathbb{Z}/K\mathbb{Z}$  is the QSD of  $(Z_t)_{t \geq 0}$ . We also establish an exponential bound in the 2-distance between the distribution of this process at time  $t$  and its QSD.

Let us denote by  $\mathcal{L}_\nu(Z_t \mid t < \tau_p)$  the law at time  $t$  of the asymmetric random walk on the cycle graph,  $(Z_t)_{t \geq 0}$ , with initial distribution  $\nu$  on  $\mathbb{Z}/K\mathbb{Z}$  and conditioned to non absorption up to time  $t$ . Since  $\mathbb{Z}/K\mathbb{Z}$  is finite, we know that the convergence of  $\mathcal{L}_\nu(Z_t \mid t < \tau_p)$  to the QSD of  $(X_t)_{t \geq 0}$  is exponential, [13]. The following theorem gives exponential lower and upper bounds for this convergence in the 2-norm.

**Theorem 1.1** (Convergence in  $L^2$  distance). *The QSD of the process  $(Z_t)_{t \geq 0}$ ,  $\nu_{qs}$ , is the uniform distribution on  $\mathbb{Z}/K\mathbb{Z}$ . Also, we have*

$$e^{-\alpha_K t} \|\nu - \mu\|_2 \leq \|\mathcal{L}_\nu(Z_t \mid t < \tau_p) - \mathcal{L}_\mu(Z_t \mid t < \tau_p)\|_2 \leq e^{-\rho_K t} \|\nu - \mu\|_2, \quad (1.4)$$

for every  $\nu$  and  $\mu$  initial distributions on  $\mathbb{Z}/K\mathbb{Z}$ , where

$$\alpha_K = \begin{cases} 2(1 + \theta) & \text{if } K \text{ is even,} \\ 2(1 + \theta) \cos^2\left(\frac{\pi}{2K}\right) & \text{if } K \text{ is odd,} \end{cases} \quad (1.5)$$

$$\rho_K = 2(1 + \theta) \sin^2\left(\frac{\pi}{K}\right). \quad (1.6)$$

As a consequence, the convergence of  $\mathcal{L}_\nu(Z_t | t < \tau_p)$  to  $\nu_{qs}$  is exponentially fast:

$$e^{-\alpha_K t} \|\nu - \nu_{qs}\|_2 \leq \|\mathcal{L}_\nu(Z_t | t < \tau_p) - \nu_{qs}\|_2 \leq e^{-\rho_K t} \|\nu - \nu_{qs}\|_2, \quad (1.7)$$

for every initial distribution  $\nu$ .

In spite of its simplicity, we did not find this result in the literature. Therefore, for the sake of completeness, we provide a proof of this theorem in Section 2.

**Remark 1.1** (Total variation). *Using the equivalence of the norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  in  $\mathbb{R}^K$  we obtain a similar result to that of the Theorem 1.1 in the total variation distance:*

$$\frac{1}{\sqrt{K}} d_{\text{TV}}(\nu, \mu) \leq d_{\text{TV}}(\mathcal{L}_\nu(Z_t | t < \tau_p), \mathcal{L}_\mu(Z_t | t < \tau_p)) \leq \sqrt{K} d_{\text{TV}}(\nu, \mu).$$

Consider the function  $\phi : \mathcal{E}_{K,N} \rightarrow \mathcal{E}_{K,N}$  defined by

$$\phi(\eta_0, \eta_1, \dots, \eta_{K-1}) = (\eta_1, \eta_2, \dots, \eta_{K-1}, \eta_0) \quad (1.8)$$

and its  $l$ -composed  $\phi^{(l)} = \underbrace{\phi \circ \phi \circ \dots \circ \phi}_{l \text{ times}}$  which acts on the cycle graph by rotating it  $l$  sites clockwise,

for  $l \in \{1, 2, \dots, K-1\}$ .

Even if the dynamics induced by  $\mathcal{G}$  has some symmetry (in fact, it is symmetric when  $\theta = 1$ ), we prove that  $(\eta_t^{(N)})_{t \geq 0}$  is not reversible when  $K \geq 4$  or when  $K = 3$  and  $\theta \neq 1$ . However, we show that the stationary distribution of the  $N$ -particle process is rotation invariant. Using this invariance, we calculate the mean of the proportion of particles in each state under the stationary distribution.

**Theorem 1.2** (Non-reversibility and rotation invariance). *The  $N$ -particle system with generator given by (1.3) has the following properties*

- a) *It is not reversible, except when  $K = 3$  and  $\theta = 1$ .*
- b) *Its stationary distribution, denoted by  $\nu_N$ , is invariant by rotations, i.e.*

$$\nu_N = \nu_N \circ \phi^{(l)}, \quad l \in \{1, 2, \dots, K-1\}.$$

- c) *Under the stationary dynamics the empirical distribution of the  $N$ -particle system is an unbiased estimator of the QSD of  $(Z_t)_{t \geq 0}$ , i.e.*

$$\mathbb{E}_{\nu_N} \left[ \frac{\eta(k)}{N} \right] = \frac{1}{K}, \quad k \in \mathbb{Z}/K\mathbb{Z}.$$

Theorem 1.2 is proved in Section 3. Using parts b) and c) of Theorem 1.2 the following result is immediate.

**Corollary 1.3** (Cyclic symmetry). *For every  $K \geq 3$  we have*

$$\text{Cov}_{\nu_N} \left( \frac{\eta(0)}{N}, \frac{\eta(k)}{N} \right) = \text{Cov}_{\nu_N} \left( \frac{\eta(0)}{N}, \frac{\eta(K-k)}{N} \right), \quad k \in \mathbb{Z}/K\mathbb{Z}. \quad (1.9)$$

Let  $T_n$  and  $U_n$  be the  $n$ -th degree Chebyshev polynomials of first and second kind, respectively, for  $n \geq 1$ . We recall that polynomials  $(T_n)_{n \geq 0}$  and  $(U_n)_{n \geq 0}$  satisfy both the recurrence relation

$$p_{n+1}(x) = 2x p_n(x) - p_{n-1}(x), \quad \text{for all } n \geq 1, \quad (1.10)$$

with initial conditions  $T_0(x) = U_0(x) = 1$ ,  $T_1(x) = x$  and  $U_1(x) = 2x$ . See [23] for a reference about Chebyshev polynomials. We also extend the definition of the Chebyshev polynomials of second kind for  $n = -1$ , by putting  $U_{-1}(x) = 0$ .

The following theorem provides explicit expressions for  $\text{Cov}_{\nu_N} [\eta(0)/N, \eta(k)/N]$  in terms of the Chebyshev polynomials of first and second kind, for  $k \in \{0, 1, \dots, K-1\}$  and the constant  $\beta_N$ , defined by

$$\beta_N = 2 \left( 1 + \frac{p}{(N-1)(1+\theta)} \right). \quad (1.11)$$

**Theorem 1.4** (Explicit expressions for the covariances). *We have*

- *If  $K = 2K_2$ ,  $K_2 \geq 2$ ,*

$$\text{Var}_{\nu_N} \left[ \frac{\eta(0)}{N} \right] = \frac{N-1}{KN} \frac{2}{\beta_N + 2} \frac{T_{K_2}(\beta_N/2)}{U_{K_2-1}(\beta_N/2)} + \frac{1}{KN} - \frac{1}{K^2}, \quad (1.12)$$

$$\text{Cov}_{\nu_N} \left[ \frac{\eta(0)}{N}, \frac{\eta(k)}{N} \right] = \frac{N-1}{KN} \frac{2}{\beta_N + 2} \frac{T_{K_2-k}(\beta_N/2)}{U_{K_2-1}(\beta_N/2)} - \frac{1}{K^2}, \quad (1.13)$$

for all  $1 \leq k \leq K_2 - 1$ .

- If  $K = 2K_2 + 1$ ,  $K_2 \geq 1$ ,

$$\text{Var}_{\nu_N} \left[ \frac{\eta(0)}{N} \right] = \frac{N-1}{KN} \frac{U_{K_2}(\beta_N/2) - U_{K_2-1}(\beta_N/2)}{U_{K_2}(\beta_N/2) + U_{K_2-1}(\beta_N/2)} + \frac{1}{KN} - \frac{1}{K^2}, \quad (1.14)$$

$$\text{Cov}_{\nu_N} \left[ \frac{\eta(0)}{N}, \frac{\eta(k)}{N} \right] = \frac{N-1}{KN} \frac{U_{K_2-k}(\beta_N/2) - U_{K_2-k-1}(\beta_N/2)}{U_{K_2}(\beta_N/2) + U_{K_2-1}(\beta_N/2)} - \frac{1}{K^2}, \quad (1.15)$$

for all  $1 \leq k \leq K_2$ .

Theorem 1.4 is proved in Section 3.2. Using Theorem 1.4 it is possible to show that the covariance between the proportions of particles under the stationary distribution in two different states decreases as a function of the graph distance between the states.

**Corollary 1.5** (Geometry of the cycle graph and covariances). *The covariance between two states under the stationary measure,  $\nu_N$ , is decreasing as a function of the graph distance between these states, i.e. for all  $k = 0, 1, \dots, \lfloor \frac{K}{2} \rfloor - 1$  we have*

$$\text{Cov}_{\nu_N} \left[ \frac{\eta(0)}{N}, \frac{\eta(k)}{N} \right] \geq \text{Cov}_{\nu_N} \left[ \frac{\eta(0)}{N}, \frac{\eta(k+1)}{N} \right].$$

With the aim of proving the convergence of the proportion of particles in each state to  $1/K$ , we study the behaviour of  $\text{Var}_{\nu_N}[\eta(0)/N]$  as a function of  $1/N$  when  $N$  tends to infinity. In [1, Thm. 2] it was proved that these variances vanishes when  $N$  goes to infinity. Our interest is to study the speed of convergence. For this purpose, we find the asymptotic development of second order for  $\text{Cov}_{\nu_N}[\eta(0)/N, \eta(k)/N]$  as a function of  $1/N$  when  $N$  tends to infinity, for  $k \in \mathbb{Z}/K\mathbb{Z}$ .

**Theorem 1.6** (Asymptotic development of two-particle covariances). *The asymptotic series expansion of order 2 when  $N \rightarrow +\infty$  of  $\text{Cov}_{\nu_N} \left[ \frac{\eta(0)}{N}, \frac{\eta(k)}{N} \right]$ , for  $k \in \mathbb{Z}/K\mathbb{Z}$ , is given by*

$$\begin{aligned} \text{Cov}_{\nu_N} \left[ \frac{\eta(0)}{N}, \frac{\eta(k)}{N} \right] &= \frac{1}{N} \left( \frac{1}{K} \mathbb{1}_{\{k=0\}} - \frac{1}{K^2} + \frac{p(6k(k-K) + K^2 - 1)}{6K^2(\theta + 1)} \right) \\ &+ \frac{p^2}{N^2} \frac{30k(K-k)[k(K-k) + 2] - (K-1)(K+1)(K^2 + 11)}{180K^2(\theta + 1)^2} \\ &+ o\left(\frac{1}{N^2}\right). \end{aligned} \quad (1.16)$$

The following result provides a bound for the speed of convergence of the empirical distribution induced by the  $N$ -particle system to the QSD when  $N \rightarrow \infty$ .

**Corollary 1.7** (Convergence to the QSD). *We have*

$$\mathbb{E}_{\nu_N} [\|m(\eta) - \nu_{qs}\|_2] \leq \sqrt{\frac{K-1}{N}} \sqrt{1 + \frac{p(K+1)}{6(1+\theta)}} + o\left(\frac{1}{\sqrt{N}}\right). \quad (1.17)$$

Theorem 1.6 and Corollary 1.7 are proved in Section 3.3. In particular, Corollary 1.7 implies the convergence at rate  $1/\sqrt{N}$  under the stationary distribution of  $m(\eta)$  towards the uniform distributions, when  $N \rightarrow \infty$ . [9, Cor. 2.10] provided the same rate of convergence for the Fleming–Viot process in the  $K$ -complete graph. As Cloez and Thai [10] remarked and to the best of our knowledge, there are no general results on Fleming–Viot process in discrete spaces assuring the rate of convergence  $1/\sqrt{N}$ , under the stationary distribution, of the empirical distribution to the QSD.

Finally, in Section 4 we study the convergence of the empirical distribution,  $m(\eta_t)$ , to the quasi-stationary distribution of  $(Z_t)_{t \geq 0}$  when  $t$  tends to infinity.

Let us denote by  $\overline{m}(\eta_t^{(N)})$  the empirical mean measure induced by the  $N$ -particle process at time  $t$ , defined by  $\overline{m}(\eta_t^{(N)})(k) = \mathbb{E}[m(\eta_t^{(N)})(k)] = \mathbb{E}[\eta_t^{(N)}(k)/N]$ . Using (3.4) we can prove the following two theorems.

**Theorem 1.8** (Mean empirical distribution). *Consider  $\eta_0 \in \mathcal{E}_{K,N}$  and  $(\eta_t^{(N)})_{t \geq 0}$  the  $N$ -particle process with initial distribution concentrated at  $\eta_0$ . We have*

$$\overline{m}(\eta_t^{(N)}) = \mathcal{L}_{m(\eta_0)}(Z_t \mid t < \tau_p).$$

Furthermore, for every probability measure  $\nu$  on  $\mathbb{Z}/K\mathbb{Z}$  we obtain

$$e^{-\alpha_K t} \|m(\eta_0) - \nu\|_2 \leq \left\| \overline{m}(\eta_t^{(N)}) - \mathcal{L}_{\nu}(Z_t \mid t < \tau_p) \right\|_2 \leq e^{-\rho_K t} \|m(\eta_0) - \nu\|_2, \quad (1.18)$$

where  $\alpha_K$  and  $\rho_K$  are defined by (1.5) and (1.6), respectively.

Thus, the proportion of particles in each state is an unbiased estimator of the distribution of the conditioned process for all  $t \geq 0$ . Using [18, Thm. 1.2] we know that the variance of the proportion of particles in each state at time  $t \geq 0$  vanishes when  $N$  goes to infinity, for every  $t \geq 0$ . The following result provides a bound for this convergence.

**Theorem 1.9** (Convergence to the Conditioned Process). *We have the following uniform upper bound for the variance of the proportion of particles in each state*

$$\max_{\substack{\eta \in \mathcal{E}_{K,N} \\ k \in \mathbb{Z}/K\mathbb{Z}}} \left| \text{Var}_\eta \left[ \frac{\eta_t^{(N)}(k)}{N} \right] - \text{Var}_{\nu_N} \left[ \frac{\eta_t^{(N)}(k)}{N} \right] \right| \leq C_{K,N} \frac{e^{-p_N t} - e^{-\rho_K t}}{\rho_K - p_N} + e^{-p_N t} \text{Var}_{\nu_N} \left[ \frac{\eta(0)}{N} \right] \quad (1.19)$$

where  $\rho_K$  is given by (1.6) and

$$p_N = \frac{2p}{N-1}, \quad (1.20)$$

$$C_{K,N} = \frac{2}{N} \left( 1 + \theta + \frac{p}{N-1} + \frac{pN(K+1)\sqrt{K-1}}{K\sqrt{K}(N-1)} \right). \quad (1.21)$$

Furthermore,

$$\begin{aligned} e^{-\alpha_K t} \|m(\eta) - \mu\|_2 &\leq \mathbb{E}_\eta \left[ \|m(\eta_t) - \mathcal{L}_\mu(X_t | t \leq \tau_p)\|_2 \right] \\ &\leq \sqrt{\frac{K}{N}} \left( D_K \frac{1 - e^{-\rho_K t}}{\rho_K} + E_K \right)^{1/2} + e^{-\rho_K t} \|m(\eta) - \mu\|_2 + o\left(\frac{1}{\sqrt{N}}\right), \end{aligned} \quad (1.22)$$

where  $\alpha_K$  and  $\rho_K$  are given by (1.5) and (1.6), respectively, and

$$D_K = 2 \left( 1 + \theta + \frac{p(K+1)\sqrt{K-1}}{K\sqrt{K}} \right), \quad E_K = \frac{K-1}{K^2} + \frac{K^2-1}{6K^2(1+\theta)}. \quad (1.23)$$

Theorems 1.8 and 1.9 is proved in Section 4. Similar results are proved in [9] for the Fleming–Viot process on the complete graph and for the two-point process.

**Remark 1.2** (Uniform bound). *Note that the bound given by (1.19) tends exponentially towards zero when  $t \rightarrow \infty$ . In particular, the right side of (1.19) is bounded in  $t$  and can be used to obtain a uniform bound for the variance of the proportion of particles in each state of order  $1/\sqrt{N}$ . Namely, using (1.19) and the inequality  $(e^{-p_N t} - e^{-\rho_K t})/(\rho_K - p_N) \leq 1/\max(\rho_K, p_N)$ , we obtain*

$$\sup_{t \geq 0} \max_{\substack{\eta \in \mathcal{E}_{K,N} \\ k \in \mathbb{Z}/K\mathbb{Z}}} \text{Var}_\eta \left[ \frac{\eta_t^{(N)}(k)}{N} \right] \leq \frac{C_{K,N}}{\max(\rho_K, p_N)} + 2 \text{Var}_{\nu_N} \left[ \frac{\eta(0)}{N} \right],$$

where  $\rho_K$ ,  $p_N$  and  $C_{K,N}$  is given by (1.6), (1.20) and (1.21), respectively.

Similar bounds have been obtained for the convergence to the conditional distribution for Fleming–Viot process in discrete state spaces, see e.g. [16, Thm. 1.1] and [27, Thm. 2.2]. However, these results are not uniform in  $t \geq 0$ . Corollary 1.5 in [10] does provide a uniform bound of order  $1/N^\gamma$ , with  $\gamma < 1/2$ , under certain conditions, but this result does not hold for the Fleming–Viot process on the  $K$ -cycle graph, for  $K \geq 6$ , since the parameter  $\lambda$  given by (1.1) is null.

The rest of this paper is organized as follows. Section 2 gives the proof of Theorem 1.1. In Section 3 we study the covariances of the proportions of particles in each state under the stationary distribution. In this section we prove Theorems 1.2, 1.4 and 1.6. Finally, Section 4 is devoted to the proof of Theorems 1.8 and 1.9 related to the variance of the proportion of particles in each site at a given time  $t \geq 0$ .

## 2. THE ASYMMETRIC RANDOM WALK ON THE CYCLE GRAPH

We first prove that the QSD of  $(Z_t)_{t \geq 0}$ , denoted by  $\nu_{qs}$ , which is the stationary distribution of  $(X_t)_{t \geq 0}$ , is the uniform distribution on  $\mathbb{Z}/K\mathbb{Z}$ . We also give an exponential bound for the speed of convergence in the 2-distance of  $\mathcal{L}_\nu(Z_t | t < \tau_p)$  to  $\nu_{qs}$ .

Recall that a square matrix  $C$  is called circulant if it takes the form

$$C = \begin{pmatrix} c_0 & c_1 & \dots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & \ddots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ c_2 & c_3 & \ddots & c_0 & c_1 \\ c_1 & c_2 & \dots & c_{n-1} & c_0 \end{pmatrix}. \quad (2.1)$$

It is evident that a circulant matrix is completely determined by its first row, therefore we will denote a circulant matrix with the form given by (2.1) by  $C = \text{circ}(c_0, c_1, \dots, c_{n-1})$ .

Let  $Q$  be the infinitesimal generator matrix of the process  $(X_t)_{t \geq 0}$ . Then,  $Q$  is circulant and it satisfies

$$Q = \text{circ}(-(1 + \theta), 1, 0, \dots, 0, \theta). \quad (2.2)$$

Let us also denote by  $i$  the complex root of  $-1$ . Since the matrix  $Q$  is circulant, its spectrum is explicitly known, as follows in the next lemma.

**Lemma 2.1** (Spectrum of  $Q$ ). *The matrix  $Q$  satisfies  $Q = F_K \Lambda F_K^*$ , where*

- $F_K$  is the  $K$ -dimensional Fourier matrix, i.e. the unitary matrix defined by

$$[F_K]_{r,c} = \frac{1}{\sqrt{K}} (\omega_K)^{-rc}, \quad (2.3)$$

for each  $r, c \in \{0, 1, \dots, K-1\}$ , where  $\omega_K = e^{i\frac{2\pi}{K}}$ ,

- $F_K^*$  is the conjugate of  $F_K$  (and also its inverse because  $F_K$  is unitary and symmetric),
- $\Lambda$  is the  $K \times K$  diagonal matrix with  $[\Lambda]_{k,k} = \lambda_k$ , for all  $0 \leq k \leq K-1$ , where

$$\lambda_k = -(1 + \theta) \sin^2 \left( \frac{\pi k}{K} \right) + i(1 - \theta) \sin \left( \frac{2\pi k}{K} \right),$$

for  $k = 0, 1, \dots, K-1$ .

*Proof of Lemma 2.1.* Let us define the polynomial  $p_Q : s \mapsto -(1 + \theta) + s + \theta s^{K-1}$ . Since  $Q$  is a circulant matrix, we can use [14, Thm. 3.2.2] to diagonalize  $Q$  in the following way

$$Q = F_K \text{Diag}(\lambda_0, \lambda_1, \dots, \lambda_{K-1}) F_K^*,$$

where  $F_K$  is the Fourier matrix defined by (2.3) and

$$\begin{aligned} \lambda_k &= p_Q(e^{i\frac{2k\pi}{K}}) = -(1 + \theta) + e^{i\frac{2k\pi}{K}} + \theta \left( e^{i\frac{2k\pi}{K}} \right)^{K-1} \\ &= -(1 + \theta) \left[ 1 - \cos \left( \frac{2\pi k}{K} \right) \right] + i(1 - \theta) \sin \left( \frac{2\pi k}{K} \right) \\ &= -2(1 + \theta) \sin^2 \left( \frac{\pi k}{K} \right) + i(1 - \theta) \sin \left( \frac{2\pi k}{K} \right), \end{aligned}$$

for  $k = 0, 1, \dots, K-1$ . □

**Remark 2.1** (Eigenvalues of  $Q$ ). *Note that  $\frac{[\Re(\lambda_k) + (1 + \theta)]^2}{(1 + \theta)^2} + \frac{[\Im(\lambda_k)]^2}{(1 - \theta)^2} = 1$ , for all  $\theta \neq 1$ , where  $\Re(\lambda_k)$  and  $\Im(\lambda_k)$  are the real and the imaginary parts of  $\lambda_k$ , respectively, for  $k = 0, 1, \dots, K-1$ . Thus, all the eigenvalues  $\lambda_k$  are on the ellipse with center  $(0, -(1 + \theta))$  and equation*

$$\frac{(x + 1 + \theta)^2}{(1 + \theta)^2} + \frac{y^2}{(1 - \theta)^2} = 1.$$

*Of course, for  $\theta = 1$ , since the matrix  $Q$  is symmetric, all the eigenvalues are real.*

*Also, the spectral gap of  $Q$ , denoted by  $\rho_K$ , is given by (1.6) and it is reached for  $-\Re(\lambda_1)$  and  $-\Re(\lambda_{K-1})$ . The minimum of  $\Re(\lambda_k)$  is reached for  $\Re(\lambda_{K/2})$  if  $K$  is even and for  $\Re(\lambda_{(K-1)/2})$  and  $\Re(\lambda_{(K+1)/2})$  if  $K$  is odd. Note that  $\alpha_k$  defined by (1.5) satisfies  $\alpha_k = -\min_k \Re(\lambda_k)$ .*

## 2.1. Proof of Theorem 1.1.

*Proof of Theorem 1.1.* We know that  $Q = F_K \Lambda F_K^*$ . Therefore  $e^{tQ} = F_K e^{t\Lambda} F_K^*$ , and it follows that

$$e^{tQ} = \sum_{k=0}^{K-1} e^{\lambda_k t} F_K U_k F_K^* = \sum_{k=0}^{K-1} e^{\lambda_k t} \Omega_k,$$

where  $U_k$ ,  $0 \leq k \leq K-1$ , is the  $K \times K$  matrix with  $[U_k]_{k,k} = 1$  and 0 elsewhere, and  $\Omega_k$  is defined as  $\Omega_k = F_K U_k F_K^*$ . In fact,  $\Omega_k$  is the symmetric circulant matrix satisfying  $[\Omega_k]_{r,c} = \frac{1}{K} \omega^{k(r-c)}$ , for all  $0 \leq r, c \leq K-1$  and for every  $k \in \{0, 1, \dots, K-1\}$ . In particular  $[\Omega_0]_{r,c} = \frac{1}{K}$  for all  $0 \leq r, c \leq K-1$ , and  $\Omega_k \Omega_l = \mathbf{0}$ , for all  $k \neq l$ .

For two probability measures  $\mu$  and  $\nu$  on  $\{0, 1, \dots, K-1\}$  we have

$$(\mu - \nu) \Omega_0 = \mathbf{0} \quad (2.4)$$



and therefore

$$(\mu - \nu)e^{tQ} = \sum_{k=1}^{K-1} e^{\lambda_k t} (\mu - \nu)\Omega_k. \quad (2.5)$$

Let us denote by  $\langle \cdot, \cdot \rangle$  the usual inner product in  $\mathbb{C}$  and for a matrix  $A$  let us denote by  $A^T$  its transpose. Note that for every  $K$ -dimensional vector  $\mathbf{x}$  and  $k \neq l$  we have  $\langle \mathbf{x}\Omega_k, \mathbf{x}\Omega_l \rangle = \mathbf{x}\Omega_k [(\Omega_l^T)^*] (\mathbf{x}^*)^T = \mathbf{x}\Omega_k \Omega_l (\mathbf{x}^*)^T = 0$ . Thus, the set of vectors  $(\mathbf{x}\Omega_k)_{k=1}^{K-1}$  are orthogonal in  $(\mathbb{C}, \langle \cdot, \cdot \rangle)$ . Now, using (2.5) and Pythagoras' theorem we have

$$\begin{aligned} \|(\mu - \nu)e^{tQ}\|_2^2 &= \sum_{k=1}^{K-1} \|e^{\lambda_k t} (\mu - \nu)\Omega_k\|_2^2 \\ &= \sum_{k=1}^{K-1} e^{2\Re(\lambda_k)t} \|(\mu - \nu)\Omega_k\|_2^2. \end{aligned} \quad (2.6)$$

Since  $\rho_K = -\max_{k=1, \dots, K-1} \Re(\lambda_k)$  we obtain

$$\begin{aligned} \|(\mu - \nu)e^{tQ}\|_2^2 &\leq e^{-2\rho_K t} \sum_{k=1}^{K-1} \|(\mu - \nu)\Omega_k\|_2^2 \\ &= e^{-2\rho_K t} \left\| \sum_{k=1}^{K-1} (\mu - \nu)\Omega_k \right\|_2^2 \\ &= e^{-2\rho_K t} \left\| \sum_{k=0}^{K-1} (\mu - \nu)\Omega_k \right\|_2^2 \\ &= e^{-2\rho_K t} \|\mu - \nu\|_2^2. \end{aligned} \quad (2.7)$$

Note that the first equality holds due the Pythagoras' theorem, the second one uses (2.4) and the last one uses the fact that

$$\sum_{k=0}^{K-1} (\mu - \nu)\Omega_k = \mu - \nu.$$

Similarly, we obtain

$$\|(\mu - \nu)e^{tQ}\|_2^2 \geq e^{-2\alpha_K t} \|\mu - \nu\|_2^2. \quad (2.8)$$

Using (2.7) and (2.8) we prove (1.4). Expression (1.7) is obtained taking  $\nu = \nu_{qs}$ .  $\square$

### 3. COVARIANCES OF THE PROPORTIONS OF PARTICLES UNDER THE STATIONARY DISTRIBUTION

The following lemma gives us informations about the invariance of the generator  $\mathcal{L}_K^{(N)}$ , defined in (1.3), by the rotation function  $\phi$  defined in (1.8).

**Lemma 3.1** (Rotation invariance of the generator). *The generator  $\mathcal{L}_{K,N}$  of  $(\eta_t^{(N)})_{t \geq 0}$  satisfies*

$$\mathcal{L}_K^{(N)} \mathbb{1}_\eta(\eta') = \mathcal{L}_K^{(N)} \mathbb{1}_{\phi(\eta)}(\phi(\eta')), \quad (3.1)$$

for all  $\eta, \eta' \in \mathcal{E}_{K,N}$ .

*Proof.* Note that

$$\mathcal{L}_K^{(N)} \mathbb{1}_\eta(\eta') = \eta'(i) \left( \mathbb{1}_{\{j=i+1\}} + \theta \mathbb{1}_{\{j=i-1\}} + p \frac{\eta'(j)}{N-1} \right), \quad (3.2)$$

if  $\eta = T_{i \rightarrow j} \eta'$ , for some  $i, j \in \mathbb{Z}/K\mathbb{Z}$ , and it is null otherwise. Now, if  $\eta = T_{i \rightarrow j} \eta'$ , then we have  $\phi(\eta) = T_{(i+1) \rightarrow (j+1)} \phi(\eta')$ . Thus,

$$\mathcal{L}_K^{(N)} \mathbb{1}_{\phi(\eta)}(\phi(\eta')) = \phi(\eta')(i+1) \left( \mathbb{1}_{\{j=i+1\}} + \theta \mathbb{1}_{\{j=i-1\}} + p \frac{\phi(\eta')(j+1)}{N-1} \right). \quad (3.3)$$

Using (3.2) and (3.3) we can see that (3.1) holds, since  $\eta'(i) = \phi(\eta')(i+1)$  and  $\eta(j) = \phi(\eta)(j+1)$ .  $\square$

**3.1. Proof of Theorem 1.2.** We will now prove Theorem 1.2, which describes some properties of  $\nu_N$ , the stationary distribution of the  $N$ -particle process  $(\eta_t^{(N)})_{t \geq 0}$ .

*Proof of Theorem 1.2.*

a) *The process  $(\eta_t^{(N)})_{t \geq 0}$  is not reversible, except when  $K = 3$  and  $\theta = 1$ .*

For  $K = 3$  and  $N \geq 2$ , let us consider the three states in  $\mathcal{E}_{3,N}$ ,

$$\eta_1 = [N, 0, 0], \quad \eta_2 = [N - 1, 1, 0], \quad \eta_3 = [N - 1, 0, 1].$$

It is straightforward to verify that

$$\begin{aligned} \mathcal{L}_{K,N} \mathbb{1}_{\eta_2}(\eta_1) &= N, & \mathcal{L}_{K,N} \mathbb{1}_{\eta_3}(\eta_1) &= N\theta, & \mathcal{L}_{K,N} \mathbb{1}_{\eta_1}(\eta_2) &= p + \theta, \\ \mathcal{L}_{K,N} \mathbb{1}_{\eta_3}(\eta_2) &= 1, & \mathcal{L}_{K,N} \mathbb{1}_{\eta_1}(\eta_3) &= p + 1, & \mathcal{L}_{K,N} \mathbb{1}_{\eta_2}(\eta_3) &= \theta. \end{aligned}$$

Because

$$\begin{aligned} \mathcal{L}_{K,N} \mathbb{1}_{\eta_1}(\eta_3) \mathcal{L}_{K,N} \mathbb{1}_{\eta_2}(\eta_1) \mathcal{L}_{K,N} \mathbb{1}_{\eta_3}(\eta_2) &= (p + 1)N, \\ \mathcal{L}_{K,N} \mathbb{1}_{\eta_3}(\eta_1) \mathbb{1}_{\eta_2}(\eta_3) \mathcal{L}_{K,N} \mathbb{1}_{\eta_1}(\eta_2) &= N\theta^2(p + \theta), \end{aligned}$$

the Kolmogorov cycle reversibility criterion, see [19, Thm. 1.8], is not satisfied unless  $\theta = 1$ . Indeed, note that a necessary condition to have reversibility is that the polynomial

$$\alpha(\theta) = \theta^3 + p(N - 1)\theta^2 - p(N - 1) - 1 = (\theta - 1)[\theta^2 + (\theta + 1)(p + 1)]$$

is equal to zero. Now, since  $\theta^2 + (\theta + 1)(p + 1) > 0$  for all  $\theta \geq 0$ , the polynomial  $\alpha(\theta)$  only has one positive root, which is  $\theta = 1$ . So the Kolmogorov cycle reversibility criterion is not satisfied unless  $\theta = 1$ .

For  $K \geq 4$ ,  $N \geq 2$  and  $p > 0$ , let us consider the two states in  $\mathcal{E}_{K,N}$ :  $\eta_1 = [N, 0, \dots, 0]$  and  $\eta_2 = [N - 1, 0, 1, \dots, 0]$ . Because  $\mathcal{L}_{K,N} \mathbb{1}_{\eta_2}(\eta_1) = 0$  and  $\mathcal{L}_{K,N} \mathbb{1}_{\eta_1}(\eta_2) = p \neq 0$ , the detailed balanced property for a reversible process, see [19, Thm. 1.3],  $\nu_N(\eta_1) \mathcal{L}_{K,N} \mathbb{1}_{\eta_2}(\eta_1) = \nu_N(\eta_2) \mathcal{L}_{K,N} \mathbb{1}_{\eta_1}(\eta_2)$ , is not satisfied.

Therefore, a) is proved except in the special case  $K = 3$ ,  $N \geq 2$  and  $\theta = 1$ . Note that in this case the model is a complete graph model, which was proved to be reversible in [9, Thm. 2.4].

b) *The stationary distribution  $\nu_N$  is invariant by rotation.*

Since  $\nu_N$  is the unique stationary distribution of  $(\eta_t^{(N)})_{t \geq 0}$ , we know that  $\nu_N \mathcal{L}f = 0$  for every function  $f$  on  $\mathcal{E}_{K,N}$ . Thus, in order to prove that  $\nu_N$  is invariant by rotation, it is sufficient to prove that  $\nu_N \circ \phi$  also satisfies  $(\nu_N \circ \phi) \mathcal{L}f = 0$  for every function  $f$  on  $\mathcal{E}_{K,N}$ . Since  $\mathcal{E}_{K,N}$  is finite, it is enough to consider the indicator functions  $\mathbb{1}_\eta$ , for every  $\eta \in \mathcal{E}_{K,N}$ . Using Lemma 3.1, we have

$$\begin{aligned} (\nu_N \circ \phi) \mathcal{L}_{K,N} \mathbb{1}_\eta &= \sum_{\eta' \in \mathcal{E}_{K,N}} \nu_N \circ \phi(\eta') \mathcal{L} \mathbb{1}_\eta(\eta') \\ &= \sum_{\eta'' \in \mathcal{E}_{K,N}} \nu_N(\eta'') \mathcal{L} \mathbb{1}_\eta(\phi^{-1}(\eta'')) \\ &= \sum_{\eta'' \in \mathcal{E}_{K,N}} \nu_N(\eta'') \mathcal{L} \mathbb{1}_{\phi(\eta)}(\eta'') \\ &= \nu_N \mathcal{L}_{K,N} \mathbb{1}_{\phi(\eta)} = 0, \end{aligned}$$

for all  $\eta \in \mathcal{E}_{K,N}$ . Consequently, by the uniqueness of the stationary distribution, we have  $\nu_N = \nu_N \circ \phi$ . The result trivially holds for any rotation  $\phi^{(l)}$ ,  $l \geq 1$ .

c) *Mean of the proportion of particles in each state.*

Using part b) we have  $\mathbb{E}_{\nu_N}[\eta(0)] = \mathbb{E}_{\nu_N}[\phi^{(k)}(\eta)(0)] = \mathbb{E}_{\nu_N}[\eta(k)]$ , for all  $k = 0, 1, \dots, K - 1$ . Also, we know that  $\eta(0) + \eta(1) + \dots + \eta(K - 1) = N$ . Thus,  $\mathbb{E}_{\nu_N}[\eta(k)] = \frac{N}{K}$ , for all  $k = 0, 1, \dots, K - 1$ .  $\square$

Let us define the functions  $f_k$  and  $f_{k,l}$  on  $\mathcal{E}_{K,N}$  as  $f_k : \eta \mapsto \eta(k)$  and  $f_{k,l} : \eta \mapsto \eta(k)\eta(l)$ , for all  $k, l \in \{0, 1, \dots, K - 1\}$ . The following lemma provides explicit expressions for the evaluation of the generator of the  $N$ -particle process on these functions.

**Lemma 3.2** (Dynamics of the  $N$ -particle process). *We have that*

$$\begin{aligned} \mathcal{L}_{K,N} f_k &= f_{k-1} - (1 + \theta)f_k + \theta f_{k+1}, \\ \mathcal{L}_{K,N} f_{k,k} &= 2 \left[ f_{k-1,k} - \left( 1 + \theta + \frac{p}{N-1} \right) f_{k,k} + \theta f_{k,k+1} \right] \end{aligned} \tag{3.4}$$

$$+f_{k-1} + \left(1 + \theta + \frac{2pN}{N-1}\right) f_k + \theta f_{k+1}, \quad (3.5)$$

$$\begin{aligned} \mathcal{L}_{K,N} f_{k,k+1} &= -2 \left(1 + \theta + \frac{p}{N-1}\right) f_{k,k+1} + f_{k-1,k+1} + \theta f_{k+1,k+1} + f_{k,k} + \theta f_{k,k+2} \\ &\quad - f_k - \theta f_{k+1}, \end{aligned} \quad (3.6)$$

$$\mathcal{L}_{K,N} f_{k,l} = -2 \left(1 + \theta + \frac{p}{N-1}\right) f_{k,l} + f_{k-1,l} + \theta f_{k+1,l} + f_{k,l-1} + \theta f_{k,l+1}, \quad (3.7)$$

for all  $k, l \in \mathbb{Z}/K\mathbb{Z}$  such that  $|k-l| > 2$ .

Lemma 3.2 is proved in Appendix A. The expression (3.4) given by this lemma is used to study the behaviour of the mean of the proportion of particles in each state. Also, (3.5), (3.6) and (3.7) are used to study the covariances of the number of particles when  $t$  and  $N$  tend to infinity.

Let us denote

$$s_k = \mathbb{E}_{\nu_N} \left[ \frac{f_{l,l+k}(\eta)}{N^2} \right] = \mathbb{E}_{\nu_N} \left[ \frac{f_{0,k}(\eta)}{N^2} \right] = \mathbb{E}_{\nu_N} \left[ \frac{\eta(0)}{N} \frac{\eta(k)}{N} \right], \quad (3.8)$$

for all  $k, l \in \mathbb{Z}/K\mathbb{Z}$ . Note that the second equality comes from part b) of Theorem 1.2. The following two lemmas will be useful for obtaining explicit expressions for the quantities  $s_k$ , for  $k = 0, 1, \dots, K-1$ .

Let us consider the constant

$$\gamma_N = -2 \left(1 + \frac{Np}{(N-1)(1+\theta)}\right). \quad (3.9)$$

**Lemma 3.3.** *Then, for  $K \geq 3$ , the values  $s_k$ , for  $0 \leq k \leq K-2$ , satisfy the following linear system:*

$$-s_{K-1} + \beta_N s_0 - s_1 = -\frac{\gamma_N}{KN}, \quad (3.10)$$

$$-s_0 + \beta_N s_1 - s_2 = -\frac{1}{KN}, \quad (3.11)$$

and when  $K \geq 4$ :

$$-s_{l-1} + \beta_N s_l - s_{l+1} = 0, \quad (3.12)$$

for  $2 \leq l \leq K-2$ , where  $\beta_N$  and  $\gamma_N$  are defined by (1.11) and (3.9), respectively.

*Proof of Lemma 3.3.* Using (3.5) we have

$$\begin{aligned} \mathbb{E}_{\nu_N} [\mathcal{L}_{K,N} f_{k,k}(\eta)] &= 2 \left[ \mathbb{E}_{\nu_N} [f_{k-1,k}(\eta)] - \left(1 + \theta + \frac{p}{N-1}\right) \mathbb{E}_{\nu_N} [f_{k,k}(\eta)] + \theta \mathbb{E}_{\nu_N} [f_{k,k+1}(\eta)] \right] + \\ &\quad \mathbb{E}_{\nu_N} [f_{k-1}(\eta)] + \left(1 + \theta + \frac{2pN}{N-1}\right) \mathbb{E}_{\nu_N} [f_k(\eta)] + \theta \mathbb{E}_{\nu_N} [f_{k+1}(\eta)]. \end{aligned}$$

Since  $\nu_N$  is the stationary distribution, we know that  $\mathbb{E}_{\nu_N} [\mathcal{L}_{K,N} f(\eta)] = 0$ , for all  $f$  on  $\mathcal{E}_{K,N}$ . Thus, using parts a) and b) of Theorem 1.2 and dividing by  $N^2$  we have the equality

$$2(1+\theta)s_1 - 2 \left(1 + \theta + \frac{p}{N-1}\right) s_0 = -\frac{2}{KN} \left(1 + \theta + \frac{pN}{N-1}\right).$$

Dividing by  $2(1+\theta)$ , this last equality is equivalent to

$$\beta_N s_0 - 2s_1 = -\frac{\gamma_N}{KN}. \quad (3.13)$$

Note that  $s_1 = s_{K-1}$  due to Corollary 1.3. Using this fact we deduce that (3.13) is equivalent to (3.10).

Furthermore, using (3.6) we get

$$\begin{aligned} \mathbb{E}_{\nu_N} [\mathcal{L}_{K,N} f_{k,k+1}(\eta)] &= -2 \left(1 + \theta + \frac{p}{N-1}\right) \mathbb{E}_{\nu_N} [f_{k,k+1}(\eta)] + \mathbb{E}_{\nu_N} [f_{k-1,k+1}(\eta)] + \\ &\quad \theta \mathbb{E}_{\nu_N} [f_{k+1,k+1}(\eta)] + \mathbb{E}_{\nu_N} [f_{k,k}(\eta)] + \theta \mathbb{E}_{\nu_N} [f_{k,k+2}(\eta)] - \\ &\quad \mathbb{E}_{\nu_N} [f_k(\eta)] - \theta \mathbb{E}_{\nu_N} [f_{k+1}(\eta)]. \end{aligned}$$

In a similar way to the previous case we obtain the equation

$$-s_0 + 2\beta_N s_1 - s_2 = -\frac{1}{KN}, \quad (3.14)$$

which is equivalent to (3.11).

Similarly, using (3.7), the equality (3.12) is proved for all  $2 \leq l \leq K-2$ .  $\square$

Using Corollary 1.3 and Equation (3.11) we can obtain the following relation

$$-s_{K-2} + \beta_N s_{K-1} - s_0 = -\frac{1}{KN}. \quad (3.15)$$

Let us define the  $K \times K$  circulant matrix  $A_K(\beta_N)$  and the  $K$ -vector  $\mathbf{b}_K(\gamma_N)$  by

$$\begin{aligned} A_K(\beta_N) &= \text{circ}(\beta_N, -1, 0, \dots, \dots, 0, -1), \\ \mathbf{b}_K(\gamma_N) &= (\gamma_N, 1, 0, 0, \dots, 0, 1)^T, \end{aligned}$$

for  $K \geq 3$ , where  $\beta_N$  and  $\gamma_N$  are defined by (1.11) and (3.9), respectively.

Using Equations (3.10), (3.11), (3.12) and (3.15), the quantities  $s_k$ ,  $0 \leq k \leq K-1$ , defined in (3.8) are proved to verify the linear system of equations

$$A_K(\beta_N) \mathbf{s}_K = -\frac{1}{KN} B_K(\gamma_N), \quad (3.16)$$

where  $\mathbf{s}_K = (s_0, s_1, \dots, s_{K-1})^T$  and  $\beta_N$  and  $\gamma_N$  are defined by (1.11).

Note that the vector  $\mathbf{b} = -\frac{1}{KN} B_K(\gamma_N)$  is *almost symmetric*, in the sense that  $b_k = b_{K-k}$ ,  $1 \leq k \leq K-1$ , where  $b_k$ ,  $0 \leq k \leq K-1$ , are the  $K$  components of  $\mathbf{b}$ . Note that a vector  $\mathbf{b}$  is almost symmetric if and only if the equality  $J\mathbf{b} = \mathbf{b}$  holds, where

$$J = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

In addition, any symmetric circulant matrix of size  $n$  can be expressed in the following way

$$A = a_0 I + a_1 \Pi + a_2 \Pi^2 + \dots + a_{n-1} \Pi^{n-1},$$

where  $(a_0, a_1, \dots, a_{n-1})$  is an almost symmetric vector and  $\Pi = \text{circ}(0, 1, 0, \dots, 0)$ .

The following result gives us informations about the solution of a symmetric circulant system when the vector of constant terms is almost symmetric.

**Proposition 3.4** (Circulant matrices). *Let  $A$  be a  $n$ -dimensional invertible circulant symmetric matrix and let  $\mathbf{b}$  be an almost symmetric vector of dimension  $n$ , then  $\mathbf{x} = A^{-1}\mathbf{b}$ , the solution of the linear system  $A\mathbf{x} = \mathbf{b}$ , is an almost symmetric vector.*

*Proof.* Since  $A$  is a invertible matrix, we know that  $\mathbf{x}$  is the unique vector of dimension  $n$  satisfying  $A\mathbf{x} = \mathbf{b}$  and this vector  $\mathbf{x}$  is almost symmetric if and only if  $\mathbf{x} = J\mathbf{x}$ . So, it is sufficient to prove that  $J\mathbf{x}$  is also solution of the linear system, i.e.  $A(J\mathbf{x}) = \mathbf{b}$ . Since  $\mathbf{b}$  is almost symmetric, the equation  $A(J\mathbf{x}) = \mathbf{b}$  becomes equivalent to

$$JA(J\mathbf{x}) = \mathbf{b}. \quad (3.17)$$

It is sufficient to prove that  $JAJ = A$ . Note that the matrix  $J$  is an involutory matrix, i.e.  $J^{-1} = J$ , and

$$\begin{aligned} JAJ &= J(a_0 I + a_1 \Pi + a_2 \Pi^2 + \dots + a_{n-1} \Pi^{n-1})J \\ &= a_0 I + a_1 J\Pi J + a_2 J\Pi^2 J + \dots + a_{n-1} J\Pi^{n-1} J. \end{aligned} \quad (3.18)$$

The matrix  $\Pi$  is orthogonal, satisfying  $\Pi^{-1} = \Pi^T$ . Moreover,

$$J\Pi J = J(\Pi J) = J \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \Pi^T,$$

which implies

$$J\Pi^n J = J\Pi J^2 \Pi^{n-1} J = \Pi^T J \Pi^{n-1} J = \dots = (\Pi^T)^n. \quad (3.19)$$

Thus, using (3.18) and (3.19), we get

$$\begin{aligned} JAJ &= a_0 I + a_1 \Pi^T + a_2 (\Pi^T)^2 + \dots + a_{n-1} (\Pi^T)^{n-1} \\ &= (a_0 I + a_1 \Pi + a_2 (\Pi)^2 + \dots + a_{n-1} (\Pi)^{n-1})^T \end{aligned}$$

$$= A^T = A.$$

Thus, (3.17) holds and hence  $J\mathbf{x}$  is solution of the equation  $A\mathbf{x} = \mathbf{b}$ . By uniqueness of the solution we get  $\mathbf{x} = J\mathbf{x}$ , proving that  $\mathbf{x}$  is almost symmetric.  $\square$

Because the  $K \times K$  matrix  $A_K(\beta_N)$  is a symmetric circulant matrix, it is possible to obtain explicit formulas for all its eigenvalues and eigenvectors using [14, Thm. 3.2.2]. Since all its eigenvalues are non-null, we conclude that the matrix  $A_K(\beta_N)$  is invertible. Thus, using Proposition 3.4, the linear system (3.16) has as its unique solution the vector  $\mathbf{s}_K$ , which is almost symmetric. In addition to its almost symmetry, the vector  $\mathbf{b}_K$  satisfies  $b_1 = b_{K-1}$ ,  $b_k = 0$ ,  $2 \leq k \leq K-2$ . This simple structure of  $\mathbf{b}_K$  allows us to deduce explicit expressions for  $s_k$ ,  $0 \leq k \leq K-1$ , given in Theorem 1.4.

**3.2. Proof of Theorem 1.4.** Consider the four families of orthogonal polynomials  $N_{\text{even},n}(x)$ ,  $D_{\text{even},n}(x)$ ,  $N_{\text{odd},n}(x)$ ,  $D_{\text{odd},n}(x)$ ,  $n \geq 0$ , defined by

$$\begin{aligned} N_{\text{even},0}(x) &= 2, & D_{\text{even},0}(x) &= 0, & N_{\text{odd},0}(x) &= 1, & D_{\text{odd},0}(x) &= 1 \\ N_{\text{even},1}(x) &= x, & D_{\text{even},1}(x) &= x+2, & N_{\text{odd},1}(x) &= x-1, & D_{\text{odd},1}(x) &= x+1, \end{aligned}$$

satisfying all of them the recurrence relation

$$p_{n+1}(x) = x p_n(x) - p_{n-1}(x), \quad (3.20)$$

for all  $n \geq 1$ .

The next proposition will prove useful in the sequel.

**Lemma 3.5.** *The following relations hold, for all  $n \geq 0$ :*

$$2N_{\text{even},n}(x) - xN_{\text{even},n+1}(x) + (x-2)D_{\text{even},n+1}(x) = 0, \quad (3.21)$$

$$2N_{\text{odd},n}(x) - xN_{\text{odd},n+1}(x) + (x-2)D_{\text{odd},n+1}(x) = 0. \quad (3.22)$$

Furthermore, we have the following identities involving the Chebyshev polynomials of first and second kind, for all  $n \geq 0$ :

$$N_{\text{even},n}(x) = 2T_n(x/2), \quad (3.23)$$

$$D_{\text{even},n}(x) = (x+2)U_{n-1}(x/2), \quad (3.24)$$

$$N_{\text{odd},n}(x) = U_n(x/2) - U_{n-1}(x/2), \quad (3.25)$$

$$D_{\text{odd},n}(x) = U_n(x/2) + U_{n-1}(x/2). \quad (3.26)$$

*Proof of Lemma 3.5.* Setting  $P_n(x) = 2N_{\text{even},n}(x) - xN_{\text{even},n+1}(x) + (x-2)D_{\text{even},n+1}(x)$ , for all  $n \geq 0$ , it follows from the definitions of  $N_{\text{even},n}(x)$  and  $D_{\text{even},n}(x)$  that  $P_0(x) = 0$ ,  $P_1(x) = 0$  and  $P_n(x)$  satisfies the recurrence relation (3.20). Therefore  $P_n(x) = 0$  for every  $n \geq 0$  and (3.21) is proved. The proof of (3.22) is similar.

Now, note that the sequence of polynomials  $(2T_n(x/2))_{n \geq 0}$  satisfy the recurrence relation (1.10). Furthermore,  $2T_0(x) = 2 = N_{\text{even},0}(x)$  and  $2T_1(x) = 2x = N_{\text{even},1}(x)$ . Consequently, identity (3.23) is proved. Analogously, identities (3.24), (3.25) and (3.26) are proved.  $\square$

We now prove Lemma 3.6, which provides explicit expressions for  $s_k$ ,  $k \in \{0, 1, \dots, K-1\}$ , in terms of the polynomials  $N_{\text{even},n}(x)$ ,  $D_{\text{even},n}(x)$ ,  $N_{\text{odd},n}(x)$  and  $D_{\text{odd},n}(x)$ .

**Lemma 3.6** (Explicit formulas for  $s_k$ ). *The values of  $s_k$ ,  $0 \leq k \leq K-1$ , are given by*

a) *If  $K = 2K_2$ ,  $K_2 \geq 2$ ,*

$$s_0 = \frac{N-1}{KN} \frac{N_{\text{even},K_2}(\beta_N)}{D_{\text{even},K_2}(\beta_N)} + \frac{1}{KN}, \quad (3.27)$$

$$s_k = \frac{N-1}{KN} \frac{N_{\text{even},K_2-k}(\beta_N)}{D_{\text{even},K_2}(\beta_N)}, \quad 1 \leq k \leq K_2, \quad (3.28)$$

$$s_{K-k} = s_k, \quad 1 \leq k \leq K_2 - 1,$$

b) *If  $K = 2K_2 + 1$ ,  $K_2 \geq 1$ ,*

$$s_0 = \frac{N-1}{KN} \frac{N_{\text{odd},K_2}(\beta_N)}{D_{\text{odd},K_2}(\beta_N)} + \frac{1}{KN},$$

$$s_k = \frac{N-1}{KN} \frac{N_{\text{odd},K_2-k}(\beta_N)}{D_{\text{odd},K_2}(\beta_N)}, \quad 1 \leq k \leq K_2,$$

$$s_{K-k} = s_k, \quad 1 \leq k \leq K_2,$$

where  $\beta_N$  is defined by (1.11).

*Proof.* We separate the proof into two cases: when  $K$  is even and when  $K$  is odd.

(a) When  $K$  is even, say  $K = 2K_2$ , Equation (3.16) is equivalent to the following linear system for  $s_k$ ,  $0 \leq k \leq K_2$ ,

$$\beta_N s_0 - 2s_1 = -\frac{1}{KN}\gamma_N, \quad (3.29)$$

$$-s_0 + \beta_N s_1 - s_2 = -\frac{1}{KN}, \quad (3.30)$$

$$-s_{k-1} + \beta_N s_k - s_{k+1} = 0, \quad (3.31)$$

for  $2 \leq k \leq K_2 - 1$  and

$$\beta_N s_{K_2} - 2s_{K_2-1} = 0. \quad (3.32)$$

Note that (3.32) follows from the equality  $s_{K_2-1} = s_{K_2+1}$ .

Consider  $A \in \mathbb{R}$  such that  $s_{K_2} = 2A = N_{\text{even},0}(\beta_N)A$ . Equation (3.32) implies

$$s_{K_2-1} = A\beta_N = AN_{\text{even},1}(\beta_N).$$

Equation (3.31) may be written as

$$s_{k-1} = \beta_N s_k - s_{k+1},$$

for  $2 \leq k \leq K_2 - 1$ . This proves that  $s_k$ , for  $k$  decreasing from  $K_2$  to 1, may be written

$$s_k = AN_{\text{even},K_2-k}(\beta_N).$$

From Equation (3.30), we get

$$\begin{aligned} s_0 &= \beta_N s_1 - s_2 + \frac{1}{KN} \\ &= A[\beta_N N_{\text{even},K_2-1}(\beta_N) - N_{\text{even},K_2-2}(\beta_N)] + \frac{1}{KN} \\ &= AN_{\text{even},K_2}(\beta_N) + \frac{1}{KN}. \end{aligned} \quad (3.33)$$

Plugging (3.33) into Equation (3.29), we get

$$\begin{aligned} A[\beta_N N_{\text{even},K_2}(\beta_N) - 2N_{\text{even},K_2-1}(\beta_N)] &= -\frac{1}{KN}(\beta_N + \gamma_N) \\ &= \frac{1}{KN} \frac{2p}{1+\theta}. \end{aligned} \quad (3.34)$$

Using Equation (3.21) we get

$$\begin{aligned} A[\beta_N N_{\text{even},K_2}(\beta_N) - 2N_{\text{even},K_2-1}(\beta_N)] &= A(\beta_N - 2)D_{\text{even},K_2}(\beta_N) \\ &= A \frac{2p}{(N-1)(1+\theta)} D_{\text{even},K_2}(\beta_N). \end{aligned} \quad (3.35)$$

Thus, using (3.34) and (3.35), we obtain  $A = \frac{N-1}{KN D_{\text{even},K_2}(\beta_N)}$ , that achieves the proof of (3.28) for an even value of  $K$ .

(b) The proof when  $K$  is odd is similar. For  $K = 2K_2 + 1$ , the linear system for  $s_k$ ,  $0 \leq k \leq K_2$  is

$$\beta_N s_0 - 2s_1 = -\frac{1}{KN}\gamma_N, \quad (3.36)$$

$$-s_0 + \beta_N s_1 - s_2 = -\frac{1}{KN}, \quad (3.37)$$

$$-s_{k-1} + \beta_N s_k - s_{k+1} = 0, \quad (3.38)$$

for  $2 \leq k \leq K_2 - 1$  and

$$-s_{K_2} + \beta_N s_{K_2} - s_{K_2-1} = 0. \quad (3.39)$$

Equation (3.39) may be written as

$$(\beta_N - 1)s_{K_2} = s_{K_2-1},$$

and so

$$s_{K_2} = B = BN_{\text{odd},0}(\beta_N), \quad s_{K_2-1} = B(\beta_N - 1) = BN_{\text{odd},1}(\beta_N).$$

From Equations (3.22) and (3.38), it follows that

$$s_k = BN_{\text{odd},K_2-k}(\beta_N), \quad 1 \leq k \leq K_2.$$

Then, from Equation (3.37), we get

$$s_0 = \beta_N s_1 - s_2 + \frac{1}{KN}$$

$$\begin{aligned}
&= B [\beta_N N_{\text{odd}, K_2-1}(\beta_N) - N_{\text{odd}, K_2-2}(\beta_N)] + \frac{1}{NK} \\
&= AN_{\text{odd}, K_2}(\beta_N) + \frac{1}{KN}.
\end{aligned}$$

From Equation (3.36), it follows, using Equation (3.22), that  $B = \frac{N-1}{KND_{\text{odd}, K_2}(\beta_N)}$ .

The proof of Lemma 3.6 is therefore complete.  $\square$

We are now able to prove Theorem 1.4, which provides explicit expressions for the covariances of the proportions of particles in two states under the stationary distribution, in terms of the orthogonal Chebyshev polynomials of first and second kind.

*Proof of Theorem 1.4.* Using expressions (3.23), (3.24), (3.25) and (3.26), and Lemma 3.6 we obtain explicit expressions for  $s_k$  in terms of the Chebyshev polynomials of first and second kind, for  $0 \leq k \leq K-1$ . Since  $\text{Cov}_{\nu_N}[\eta(0)/N, \eta(k)/N] = s_k - 1/K^2$ , for all  $0 \leq k \leq K-1$ , we deduce that (1.12), (1.13), (1.14) and (1.15) hold.  $\square$

Now, using Theorem 1.4 we are able to study the monotony of the covariance of the proportions of particles in two sites as a function of the graph distances between these two sites.

*Proof of Corollary 1.5.* Note that  $\text{Cov}_{\nu_N} \left[ \frac{\eta(0)}{N}, \frac{\eta(k)}{N} \right] \geq \text{Cov}_{\nu_N} \left[ \frac{\eta(0)}{N}, \frac{\eta(k+1)}{N} \right]$  holds if and only if  $s_k \geq s_{k+1}$ , for all  $k = 0, 1, \dots, \lfloor \frac{K}{2} \rfloor$ . So, for  $K$  even, using (1.12) and (1.13), it is sufficient to prove that  $T_{k+1}(\beta_N/2) \leq T_k(\beta_N/2)$ . Let us prove it by induction. We know that  $T_1(\beta_N/2) = \beta_N/2 \geq 1 = T_0(\beta_N/2)$ . Assume that  $T_k(\beta_N/2) \geq T_{k-1}(\beta_N/2)$ . Since  $(T_n(x))_{n \geq 0}$  satisfies the recurrence relation (1.10) we have

$$T_{k+1}(\beta_N/2) - T_k(\beta_N/2) = (\beta_N - 1)T_k(\beta_N/2) - T_{k-1}(\beta_N/2) \geq T_k(\beta_N/2) - T_{k-1}(\beta_N/2) \geq 0,$$

where the first inequality is true due to the inequality  $\beta_N \geq 2$  and the second one because, by assumption,  $T_k(\beta_N/2) \geq T_{k-1}(\beta_N/2)$ . Then,  $T_{k+1}(\beta_N/2) \geq T_k(\beta_N/2)$ , for all  $k \geq 0$ .

Analogously, for  $K$  odd the inequality  $\text{Cov}_{\nu_N} \left[ \frac{\eta(0)}{N}, \frac{\eta(k)}{N} \right] \geq \text{Cov}_{\nu_N} \left[ \frac{\eta(0)}{N}, \frac{\eta(k+1)}{N} \right]$  holds for all  $k = 0, 1, \dots, \lfloor \frac{K}{2} \rfloor$  if

$$U_{k+1}(\beta_N/2) - U_k(\beta_N/2) \geq U_k(\beta_N/2) - U_{k-1}(\beta_N/2), \quad (3.40)$$

for all  $k \geq 1$ . For  $k = 1$  we have that (3.40) is equivalent to

$$\beta_N - 1 \leq 4 \left( \frac{\beta_N}{2} \right)^2 - 1 - \beta_N,$$

which is trivially true since  $\beta_N \geq 2$ . Assume that (3.40) holds and let us prove the inequality for  $k+1$ . Indeed, using that  $(U_n)_{n \geq 0}$  satisfies the recurrence relation (1.10), we have

$$U_{k+2}(\beta_N/2) - U_{k+1}(\beta_N/2) = (\beta_N - 1)U_{k+1}(\beta_N/2) - U_k(\beta_N/2) \geq U_{k+1}(\beta_N/2) - U_k(\beta_N/2).$$

Thus, (3.40) holds for all  $k = 0, 1, \dots, K_2$ .  $\square$

**3.3. Proof of Theorem 1.6.** Theorem 1.4 allows us to get a Taylor series expansion for  $s_k$ ,  $0 \leq k \leq K-1$  as a function of  $\frac{1}{N}$ , as soon as we are able to obtain such a series expansion for  $\beta_N$ , as a function of  $1/N$ , as well as for the polynomials  $N_{\text{odd}, n}(x)$ ,  $N_{\text{even}, n}(x)$ ,  $D_{\text{odd}, n}(x)$ ,  $D_{\text{even}, n}(x)$ ,  $n \geq 0$  around  $x = 2$ , using their definitions by recurrence given in (3.20).

**Lemma 3.7.** *The polynomials  $N_{\text{odd}, n}(x)$ ,  $N_{\text{even}, n}(x)$ ,  $D_{\text{odd}, n}(x)$ ,  $D_{\text{even}, n}(x)$ , for  $n \geq 0$ , satisfy the following Taylor series expansion of order 2 around  $x = 2$ :*

$$N_{\text{even}, n}(x) = 2 + n^2(x-2) + \frac{n^4 - n^2}{12}(x-2)^2 + o(x-2)^2, \quad (3.41)$$

$$D_{\text{even}, n}(x) = 4n + \frac{2n^3 + n}{3}(x-2) + \frac{n^5 - n}{30}(x-2)^2 + o(x-2)^2, \quad (3.42)$$

$$N_{\text{odd}, n}(x) = 1 + \frac{n^2 + n}{2}(x-2) + \frac{n^4 + 2n^3 - n^2 - 2n}{24}(x-2)^2 + o(x-2)^2, \quad (3.43)$$

$$\begin{aligned}
D_{\text{odd}, n}(x) &= 2n + 1 + \frac{2n^3 + 3n^2 + n}{6}(x-2) + \frac{2n^5 + 5n^4 - 5n^2 - 2n}{120}(x-2)^2 + \\
&\quad + o(x-2)^2.
\end{aligned} \quad (3.44)$$

*Proof.* Assume  $N_{\text{even},n}(x) = a_0^{(n)} + a_1^{(n)}(x-2) + a_2^{(n)}(x-2)^2 + o(x-2)^2$ , for all  $n \geq 0$ . Note that the polynomials  $N_{\text{even},n}(x)$  can also be defined as

$$\begin{aligned} N_{\text{even},0}(x) &= 2, \\ N_{\text{even},1}(x) &= (x-2) + 2, \\ N_{\text{even},n}(x) &= (x-2)N_{\text{even},n-1}(x) + 2N_{\text{even},n-1}(x) - N_{\text{even},n-2}(x), \quad n \geq 2. \end{aligned} \quad (3.45)$$

Thus, the coefficients  $(a_0^{(n)})_{n \geq 0}$  satisfy the recurrence relation  $a_0^{(0)} = a_0^{(1)} = 2$  and  $a_0^{(n)} = 2a_0^{(n-1)} - a_0^{(n-2)}$ , for every  $n \geq 2$ , which yields  $a_0^{(n)} = 2$ , for all  $n \geq 1$ .

Also, using (3.45), the coefficients  $(a_1^{(n)})_{n \geq 0}$  satisfy  $a_1^{(0)} = 0$ ,  $a_1^{(1)} = 1$  and

$$a_1^{(n)} = 2a_1^{(n-1)} - a_1^{(n-2)} + a_0^{(n-1)} = 2a_1^{(n-1)} - a_1^{(n-2)} + 2,$$

for all  $n \geq 2$ . Solving this recurrence gives  $a_1^{(n)} = n^2$ , for all  $n \geq 2$ .

Similarly, the coefficients  $(a_2^{(n)})_{n \geq 0}$  satisfy  $a_2^{(0)} = a_2^{(1)} = 0$  and

$$a_2^{(n)} = 2a_2^{(n-1)} - a_2^{(n-2)} + a_1^{(n-1)} = 2a_2^{(n-1)} - a_2^{(n-2)} + (n-1)^2,$$

for all  $n \geq 2$ . which yields  $a_2^{(n)} = \frac{n^4 - n^2}{12}$ , for all  $n \geq 2$ , proving (3.41).

The proofs of (3.42), (3.43) and (3.44) are similar.  $\square$

We now prove Theorem 1.6, which provides a second order Taylor series expansion of the variance of the proportion of particles in each state, as a function of  $1/N$ , when  $N$  tends to infinity.

*Proof of Theorem 1.6.* Suppose  $K$  is even, say  $K = 2K_2$ . Using Lemma 3.6, we have

$$s_k = \frac{1}{K} \left(1 - \frac{1}{N}\right) \frac{N_{\text{even},K_2-k}(\beta_N)}{D_{\text{even},K_2}(\beta_N)},$$

for all  $k = 1, 2, \dots, K_2$ . Note that  $\beta_N$ , defined by (1.11), tends to 2 when  $N$  tends to infinity, specifically

$$\beta_N - 2 = \frac{2p}{(N-1)(1+\theta)} = \frac{2p}{1+\theta} \left(\frac{1}{N} + \frac{1}{N^2}\right) + o\left(\frac{1}{N^2}\right).$$

Using (3.41) and (3.42), we have

$$\begin{aligned} \frac{N_{\text{even},K_2-k}(\beta_N)}{D_{\text{even},K_2}(\beta_N)} &= \frac{2 + (K_2 - k)^2(\beta_N - 2) + \frac{(K_2 - k)^4 - (K_2 - k)^2}{12}(\beta_N - 2)^2 + o((\beta_N - 2)^2)}{4K_2 + \frac{2K_2^3 + K_2}{3}(\beta_N - 2) + \frac{K_2^5 - K_2}{30}(\beta_N - 2)^2 + o((\beta_N - 2)^2)} \\ &= \frac{1}{K} + \frac{(6k(k-K) + K^2 - 1)}{12K}(\beta_N - 2) \\ &\quad + \frac{30k(K-k)[k(K-k) + 2] - (K^2 - 1)(K^2 + 10)}{720K}(\beta_N - 2)^2 \\ &\quad + o((\beta_N - 2)^2), \end{aligned} \quad (3.46)$$

where  $K = 2K_2$ . Finally,

$$\begin{aligned} s_k &= \frac{1}{K} \left(1 - \frac{1}{N}\right) \frac{N_{\text{even},K_2-k}(\beta_N)}{D_{\text{even},K_2}(\beta_N)} \\ &= \frac{1}{K} \left(1 - \frac{1}{N}\right) \left[ \frac{1}{K} + \frac{(6k(k-K) + K^2 - 1)}{12K} \frac{2p}{1+\theta} \left(\frac{1}{N} + \frac{1}{N^2}\right) \right. \\ &\quad \left. + \frac{(30k(K-k)[k(K-k) + 2] - K^2(K^2 + 10) + 11)}{720K} \frac{2p}{1+\theta} \left(\frac{1}{N}\right)^2 \right] + o\left(\frac{1}{N^2}\right) \\ &= \frac{1}{K^2} + \left(-\frac{1}{K} + \frac{p(6k(k-K) + K^2 - 1)}{6(1+\theta)K}\right) \frac{1}{KN} \\ &\quad + \frac{p^2(30k(K-k)[k(K-k) + 2] - (K^2 - 1)(K^2 + 11))}{180(1+\theta)^2 K^2} \frac{1}{N^2} + o\left(\frac{1}{N^2}\right). \end{aligned}$$

Using (3.27), we get the following expression for  $s_0$ ,

$$s_0 = \frac{1}{K^2} + \left(-\frac{1}{K} + \frac{p(K^2 - 1)}{6(1+\theta)K}\right) \frac{1}{KN} + \frac{p^2(K^2 - 1)(K^2 + 11)}{180(1+\theta)^2 K^2} \frac{1}{N^2} + \frac{1}{KN} + o\left(\frac{1}{N^2}\right)$$



$$= \frac{1}{K^2} + \left(1 - \frac{1}{K} + \frac{p(K^2 - 1)}{6K(1 + \theta)}\right) \frac{1}{KN} + \frac{p^2(K^2 - 1)(K^2 + 11)}{180(1 + \theta)^2 K^2} \frac{1}{N^2} + o\left(\frac{1}{N^2}\right).$$

Now, the expression (1.16) for  $\text{Cov}_{\nu_N}[\eta(0)/N, \eta(k)/N]$  with  $K$  even follows by noting that  $\mathbb{E}_{\nu_N}\left[\frac{\eta(k)}{N}\right] = \frac{1}{K}$ , for all  $k = 0, 1, 2, \dots, K - 1$ .

Considering  $K$  odd, specifically  $K = 2K_2 + 1$ , and using (3.43) and (3.44), we have

$$\begin{aligned} \frac{N_{\text{odd}, K_2 - k}(\beta_N)}{D_{\text{odd}, K_2}(\beta_N)} &= \frac{1}{K} + \frac{(6k^2 - 6kK + K^2 - 1)}{12K}(\beta_N - 2) \\ &\quad + \frac{30k^4 - 60k^3K + 30k^2(K^2 - 2) + 60kK - (K^2 - 1)(K^2 + 11)}{720K}(\beta_N - 2)^2 \\ &\quad + o((\beta_N - 2)^2), \end{aligned}$$

which is the same expression we get for  $\frac{N_{\text{even}, K_2 - k}(\beta_N)}{D_{\text{even}, K_2}(\beta_N)}$  in (3.46). So, the general result is proved.  $\square$

*Proof of Corollary 1.7.* Using Jensen's inequality, we have

$$\begin{aligned} \mathbb{E}_{\nu_N}[\|m(\eta) - \nu_{qs}\|_2] &\leq \left(\mathbb{E}_{\nu_N} \|m(\eta) - \nu_{qs}\|_2^2\right)^{1/2} \\ &= \left(\sum_{k=0}^{K-1} \text{Var}_{\nu_N}\left[\frac{\eta(k)}{N}\right]\right)^{1/2} \\ &= \sqrt{K} \left(\text{Var}_{\nu_N}\left[\frac{\eta(0)}{N}\right]\right)^{1/2}. \end{aligned} \quad (3.47)$$

Finally, (1.17) is proved using (3.47) and Theorem 1.6.  $\square$

#### 4. COVARIANCES OF THE PROPORTIONS OF PARTICLES AT A GIVEN TIME

##### 4.1. Proof of Theorem 1.8.

*Proof of Theorem 1.8.* Consider  $\eta_0 \in \mathcal{E}_{K, N}$  and the function  $f_k : \eta \mapsto \eta(k)$ , for  $k \in \{0, 1, \dots, K - 1\}$ . Using the expression of  $\mathcal{L}_{K, N} f_k$ , for  $k = 0, 1, \dots, K - 1$ , given by (3.4), we get

$$\begin{aligned} \mathbb{E}_{\eta_0} \left[ \frac{\mathcal{L}_{K, N} f_k(\eta_t^{(N)})}{N} \right] &= \frac{d}{dt} \mathbb{E}_{\eta_0} \left[ \frac{f_k(\eta_t^{(N)})}{N} \right] \\ &= \mathbb{E}_{\eta_0} \left[ \frac{f_{k-1}(\eta_t^{(N)})}{N} \right] - (1 + \theta) \mathbb{E}_{\eta_0} \left[ \frac{f_k(\eta_t^{(N)})}{N} \right] \\ &\quad + \theta \mathbb{E}_{\eta_0} \left[ \frac{f_{k+1}(\eta_t^{(N)})}{N} \right], \end{aligned} \quad (4.1)$$

for  $k = 0, 1, \dots, K - 1$ .

Let us define  $s_t(k) = \mathbb{E}_{\eta_0} \left[ \frac{f_k(\eta_t^{(N)})}{N} \right] = \mathbb{E}_{\eta_0} \left[ \frac{\eta_t^{(N)}(k)}{N} \right] = \overline{m}(\eta_t^{(N)})(k)$ , for  $k = 0, 1, \dots, K - 1$ , and the vector  $\mathbf{s}_t = (s_t(0), s_t(1), \dots, s_t(K - 1))^T$ . Using (4.1), we get that  $\mathbf{s}_t$  satisfies the differential equation

$$\frac{d\mathbf{s}_t}{dt} = \mathbf{s}_t Q,$$

where  $Q$  is the circulant matrix defined in (2.2), with initial condition  $\mathbf{s}_0 = \eta_0/N$ .

Note that the solution of this differential equation is given by

$$\mathbf{s}_t = \frac{\eta_0}{N} e^{tQ}.$$

Thus,  $\overline{m}(\eta_t^{(N)})$  is actually equal to the distribution of the asymmetric random walk on the cycle graph  $\mathbb{Z}/K\mathbb{Z}$  with infinitesimal generator matrix  $Q$  and initial distribution  $m(\eta_0)$  at time  $t \geq 0$ , which is  $\mathcal{L}_{m(\eta_0)}(Z_t \mid t < \tau_p)$ . So, the proof of Equation (1.18) follows from (1.4) in Theorem 1.1.  $\square$

**4.2. Proof of Theorem 1.9.** In order to study the convergence of the empirical distribution  $m(\eta_t^{(N)})$  induced by the  $N$ -particle system, we will analyse the behaviour of the covariance functions in time. Let  $\eta_0 \in \mathcal{E}_{K,N}$  be fixed and let us define the functions  $s_t^{(2)}(k, r)$  as  $s_t^{(2)}(k, r) = \mathbb{E}_{\eta_0} \left[ \frac{f(k,r)}{N^2} \right] = \mathbb{E}_{\eta_0} [\eta(k)\eta(r)/N^2]$ , for all  $k, r \in \mathbb{Z}/K\mathbb{Z}$ . Using (3.5), (3.6) and (3.7), we have

$$\begin{aligned} \frac{d s_t^{(2)}(k, k)}{d t} &= 2 \left[ s_t^{(2)}(k, k-1) - \left( 1 + \theta + \frac{p}{N-1} \right) s_t^{(2)}(k, k) + \theta s_t^{(2)}(k, k+1) \right] \\ &\quad + \frac{1}{N} \left[ s_t(k-1) + \left( 1 + \theta + 2 \frac{p}{N-1} \right) s_t(k) + \theta s_t(k+1) \right], \\ \frac{d s_t^{(2)}(k, k+1)}{d t} &= -2 \left( 1 + \theta + \frac{p}{N-1} \right) s_t^{(2)}(k, k+1) + s_t^{(2)}(k-1, k+1) + \theta s_t^{(2)}(k+1, k+1) \\ &\quad + s_t^{(2)}(k, k) + \theta s_t^{(2)}(k, k+2) - \frac{1}{N} [s_t(k) + \theta s_t(k+1)] \\ \frac{d s_t^{(2)}(k, k+l)}{d t} &= -2 \left( 1 + \theta + \frac{p}{N-1} \right) s_t^{(2)}(k, k+l) + s_t^{(2)}(k-1, k+l) + \theta s_t^{(2)}(k+1, k+l) \\ &\quad + s_t^{(2)}(k, k+l-1) + \theta s_t^{(2)}(k, k+l+1). \end{aligned}$$

Consider the functions  $g_t(k, r)$  defined as

$$g_t(k, r) = \text{Cov}_{\eta_0} \left( \frac{\eta_t(k)}{N}, \frac{\eta_t(r)}{N} \right) = s_t^{(2)}(k, r) - s_t(k)s_t(r), \quad (4.2)$$

for all  $k, r \in \mathbb{Z}/K\mathbb{Z}$ .

Then, we obtain the following system of differential equations

$$\begin{aligned} \frac{d g_t(k, k)}{d t} &= \frac{d s_t^{(2)}(k, k)}{d t} - 2 s_t(k) \frac{d s_t(k)}{d t} \\ &= 2 \left[ s_t^{(2)}(k, k-1) - \left( 1 + \theta + \frac{p}{N-1} \right) s_t^{(2)}(k, k) + \theta s_t^{(2)}(k, k+1) \right] \\ &\quad + \frac{1}{N} \left[ s_t(k-1) + \left( 1 + \theta + 2 \frac{p}{N-1} \right) s_t(k) + \theta s_t(k+1) \right] \\ &\quad - 2 s_t(k) [s_t(k-1) - (1 + \theta) s_t(k) + \theta s_t(k+1)] \\ &= 2 \left[ g_t(k, k-1) - \left( 1 + \theta + \frac{p}{N-1} \right) g_t(k, k) + \theta g_t(k, k+1) \right] \\ &\quad + \frac{1}{N} \left[ s_t(k-1) + \left( 1 + \theta + 2 \frac{p}{N-1} \right) s_t(k) + \theta s_t(k+1) \right] - \frac{2p}{N-1} s_t(k)^2, \\ \frac{d g_t(k, k+1)}{d t} &= \frac{d s_t^{(2)}(k, k+1)}{d t} - s_t(k) \frac{d s_t(k+1)}{d t} - s_t(k+1) \frac{d s_t(k)}{d t} \\ &= -2 \left( 1 + \theta + \frac{p}{N-1} \right) s_t^{(2)}(k, k+1) + s_t^{(2)}(k-1, k+1) + \theta s_t^{(2)}(k+1, k+1) \\ &\quad + s_t^{(2)}(k, k) + \theta s_t^{(2)}(k, k+2) - \frac{1}{N} [s_t(k) + \theta s_t(k+1)] \\ &\quad - s_t(k) [s_t(k) + (1 + \theta) s_t(k+1) + \theta s_t(k+2)] \\ &\quad - s_t(k+1) [s_t(k-1) + (1 + \theta) s_t(k) + \theta s_t(k+1)] \\ &= -2 \left( 1 + \theta + \frac{p}{N-1} \right) g_t(k, k+1) + g_t(k-1, k+1) + \theta g_t(k+1, k+1) \\ &\quad + g_t(k, k) + \theta g_t(k, k+2) - \frac{1}{N} [s_t(k) + \theta s_t(k+1)] - \frac{2p}{N-1} s_t(k)s_t(k+1), \\ \frac{d g_t(k, l)}{d t} &= \frac{d s_t^{(2)}(k, l)}{d t} - s_t(k) \frac{d s_t(l)}{d t} - s_t(l) \frac{d s_t(k)}{d t} \\ &= -2 \left( 1 + \theta + \frac{p}{N-1} \right) s_t^{(2)}(k, l) + s_t^{(2)}(k-1, l) + \theta s_t^{(2)}(k+1, l) \\ &\quad + s_t^{(2)}(k, l-1) + \theta s_t^{(2)}(k, l+1) - s_t(k) [s_t(l-1) + (1 + \theta) s_t(l) + \theta s_t(l+1)] \end{aligned}$$

$$\begin{aligned}
& -s_t(l) [s_t(k-1) + (1+\theta)s_t(k) + \theta s_t(k+1)] \\
= & -2 \left( 1 + \theta + \frac{p}{N-1} \right) g_t(k, l) + g_t(k-1, l) + \theta g_t(k+1, l) \\
& + g_t(k, l-1) + \theta g_t(k, l+1) - \frac{2p}{N-1} s_t(k)s_t(l).
\end{aligned}$$

Then, the  $K^2$ -dimensional vector  $\mathbf{g}_t = (g_t(k, r))_{k,r}$  satisfies the differential equation

$$\frac{d\mathbf{g}_t}{dt} = \mathbf{g}_t Q_p^{(2)} + \mathbf{w}_t, \quad (4.3)$$

where  $Q_p^{(2)} = Q^{(2)} - 2\frac{p}{N-1}I$ ,  $I$  is the  $K^2$ -dimensional identity matrix,  $\mathbf{w}_t = (w_t(k, r))_{k,r}$  is the  $K^2$ -vector defined by

$$w_t(k, r) = \begin{cases} \frac{1}{N} \left[ s_t(k-1) + \left( 1 + \theta + 2\frac{p}{N-1} \right) s_t(k) + \theta s_t(k+1) \right] - \frac{2p}{N-1} s_t(k)^2 & \text{if } r = k \\ -\frac{1}{N} [s_t(k \wedge r) + \theta s_t(k \vee r)] - \frac{2p}{N-1} s_t(k)s_t(r) & \text{if } |k-r| = 1 \\ -\frac{2p}{N-1} s_t(k)s_t(r) & \text{if } |k-r| > 1, \end{cases}$$

for all  $k, r \in \mathbb{Z}/K\mathbb{Z}$ , and the matrix  $Q^{(2)} \in M_{\mathbb{R}}(K^2)$  is defined as

$$Q_{(u,v),(k,r)}^{(2)} = \begin{cases} 1 & \text{if } (k = u+1 \wedge r = v) \vee (k = u \wedge r = v+1), \\ \theta & \text{if } (k = u-1 \wedge r = v) \vee (k = u \wedge r = v-1), \\ -2(1+\theta) & \text{if } (k = u) \wedge (r = v). \end{cases} \quad (4.4)$$

Note also that

$$g_0(k, r) = \text{Cov}_{\eta_0} \left[ \frac{\eta_0(k)}{N}, \frac{\eta_0(r)}{N} \right] = 0, \quad (4.5)$$

$$g_{\infty}(k, r) = \lim_{t \rightarrow \infty} g_t(k, r) = \text{Cov}_{\nu_N} \left[ \frac{\eta(k)}{N}, \frac{\eta(r)}{N} \right], \quad (4.6)$$

and

$$w_{\infty}(k, r) = \lim_{t \rightarrow \infty} w_t(k, r) = \begin{cases} \frac{2}{KN} \left( 1 + \theta + \frac{p}{N-1} \right) - \frac{2p}{K^2(N-1)} & \text{if } k = r, \\ -\frac{1}{KN} (1 + \theta) - \frac{2p}{K^2(N-1)} & \text{if } |k-r| = 1, \\ -\frac{2p}{K^2(N-1)} & \text{if } |k-r| > 1, \end{cases} \quad (4.7)$$

for all  $k, r \in \mathbb{Z}/K\mathbb{Z}$ .

Let  $A = (a_{r,c})$  and  $B = (b_{r,c})$  be two matrices of dimensions  $m \times n$  and  $w \times q$ , respectively. Recall that the Kronecker product of  $A$  and  $B$ , denoted by  $A \otimes B$ , is the  $mw \times nq$  matrix defined as

$$A \otimes B = \begin{pmatrix} a_{0,0}B & a_{0,1}B & \dots & a_{0,n-1}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-1,0}B & a_{m-1,1}B & \dots & a_{m-1,n-1}B \end{pmatrix}.$$

It is convenient to index the elements of  $A \otimes B$  with two 2-dimensional index in the following way

$$(A \otimes B)_{(r_1, r_2), (c_1, c_2)} = (A \otimes B)_{r_1 m + r_2, c_1 n + c_2} = a_{r_1, c_1} b_{r_2, c_2},$$

for all  $0 \leq r_1 \leq m-1, 0 \leq r_2 \leq w-1, 0 \leq c_1 \leq n-1, 0 \leq c_2 \leq q-1$ . Now, consider that  $m = n$  and  $w = q$ , i.e.  $A$  and  $B$  are square matrices of dimension  $n$  and  $q$ , respectively. The Kronecker sum of  $A$  and  $B$ , denoted by  $A \oplus B$ , is defined as  $A \oplus B = A \otimes I_q + I_n \otimes B$ , where  $I_q$  and  $I_n$  are the identity matrices of dimension  $q$  and  $n$ , respectively. It is well known that the exponential of matrices transforms Kronecker sums in Kronecker products, i.e.

$$e^{A \oplus B} = e^A \otimes e^B. \quad (4.8)$$

See [25] and [14] for the proofs of this result and more details about the Kronecker product and sum of matrices.

**Lemma 4.1.** *The following properties hold:*

- (1)  $Q^{(2)} = Q \oplus Q$ ,
- (2)  $Q^{(2)}$  is circulant,
- (3)  $e^{tQ^{(2)}} = e^{tQ} \otimes e^{tQ}$ .

Consequently, the matrix  $Q^{(2)}$  is the infinitesimal rate matrix of the independent coupling of two processes driven by the infinitesimal generator matrix  $Q$ .

*Proof of Lemma 4.1.* Note that using (4.4) for all  $r_1, r_2, c_1, c_2 \in \{0, 1, \dots, K-1\}$ , we have

$$Q_{(r_1, r_2), (c_1, c_2)}^{(2)} = Q_{r_1, c_1} I_{r_2, c_2} + I_{r_1, c_1} Q_{r_2, c_2} = (Q \oplus Q)_{(r_1, r_2), (c_1, c_2)},$$

where  $I$  is the  $K$ -dimensional identity matrix. Then, property (1) holds. Note that  $Q \oplus I$  and  $I \oplus Q$  are circulant. Thus,  $Q^{(2)}$  is circulant because it is the sum of two circulant matrices. Also, using (4.8) we can easily prove the property (3).

All the non-diagonal entries of matrix  $Q^{(2)}$  are positive and the sum of each row is null, thus  $Q^{(2)}$  is an infinitesimal matrix. Furthermore,

$$e^{tQ^{(2)}}_{(r_1, r_2), (c_1, c_2)} = e^{tQ}_{r_1, c_1} e^{tQ}_{r_2, c_2},$$

which means that  $Q^{(2)}$  is the infinitesimal rate matrix of the independent coupling of two processes driven by  $Q$ .  $\square$

Note also that, when  $t$  goes to infinity in (4.3), we get  $\mathbf{g}_\infty Q_p^{(2)} + \mathbf{w}_\infty = 0$ . Since  $Q^{(2)}$  is the infinitesimal matrix generator of a Markov process and  $Q_p^{(2)} = Q^{(2)} - \frac{2p}{N-1}I$ , all the eigenvalues of  $Q_p^{(2)}$  are strictly negative and thus,  $Q_p^{(2)}$  is invertible. Then,

$$\mathbf{g}_\infty = -\mathbf{w}_\infty \left( Q_p^{(2)} \right)^{-1}. \quad (4.9)$$

We will now prove Theorem 1.9, which gives us the solution of the differential Equation (4.3) and studies the convergence of the proportion of particles at time  $t$  in each state when  $t$  and  $N$  tend to infinity.

*Proof of Theorem 1.9.* The solutions of the system of differential equations (4.3) is given by

$$\begin{aligned} \mathbf{g}_t &= \left( \int_0^t \mathbf{w}_u e^{-uQ_p^{(2)}} du \right) e^{tQ_p^{(2)}} \\ &= \left( \int_0^t (\mathbf{w}_u - \mathbf{w}_\infty) e^{-uQ_p^{(2)}} du + \mathbf{w}_\infty \int_0^t e^{-uQ_p^{(2)}} du \right) e^{tQ_p^{(2)}} \\ &= \left( \int_0^t (\mathbf{w}_u - \mathbf{w}_\infty) e^{-uQ_p^{(2)}} du + \mathbf{w}_\infty \left( Q_p^{(2)} \right)^{-1} \left( I - e^{-tQ_p^{(2)}} \right) \right) e^{tQ_p^{(2)}} \\ &= \int_0^t (\mathbf{w}_u - \mathbf{w}_\infty) e^{(t-u)Q_p^{(2)}} du + \mathbf{w}_\infty \left( Q_p^{(2)} \right)^{-1} \left( e^{tQ_p^{(2)}} - I \right) \\ &= \int_0^t (\mathbf{w}_u - \mathbf{w}_\infty) e^{(t-u)Q_p^{(2)}} du + \mathbf{g}_\infty \left( I - e^{tQ_p^{(2)}} \right). \end{aligned}$$

Note that the last equality comes from (4.9). Therefore, we have

$$\begin{aligned} \left\| \mathbf{g}_t^{(2)} - \mathbf{g}_\infty^{(2)} \right\|_\infty &\leq \left\| \int_0^t (\mathbf{w}_u^{(2)} - \mathbf{w}_\infty^{(2)}) e^{(t-u)Q_p^{(2)}} du \right\|_\infty + \left\| \mathbf{g}_\infty^{(2)} \left( e^{tQ_p^{(2)}} \right) \right\|_\infty \\ &\leq \int_0^t \left\| \mathbf{w}_u^{(2)} - \mathbf{w}_\infty^{(2)} \right\|_\infty \left\| e^{(t-u)Q_p^{(2)}} \right\|_\infty du + \left\| \mathbf{g}_\infty^{(2)} \right\|_\infty \left\| e^{tQ_p^{(2)}} \right\|_\infty \end{aligned} \quad (4.10)$$

We obtain

$$\left\| e^{sQ_p^{(2)}} \right\|_\infty = e^{-pN(t-u)} \left\| e^{sQ^{(2)}} \right\|_\infty = e^{-pNs}, \quad (4.11)$$

for all  $s \geq 0$ , where the second equality comes from the fact that  $Q^{(2)}$  is a circulant infinitesimal generator matrix.

Using Lemma 1.5, or the CauchySchwarz inequality, we get

$$\left\| \mathbf{g}_\infty^{(2)} \right\|_\infty = \text{Var}_{\nu_N} \left[ \frac{\eta(0)}{N} \right]. \quad (4.12)$$

Note that inequality (1.7) implies that  $|s_t(k) - \frac{1}{K}| \leq \sqrt{\frac{K-1}{K}} e^{-\rho_K t}$ , for every  $k \in \mathbb{Z}/K\mathbb{Z}$  and all  $t \geq 0$ . Therefore,

$$|\mathbf{w}_u(k, k) - \mathbf{w}_\infty(k, k)| \leq \frac{2}{N} \left( 1 + \theta + \frac{p}{N-1} \right) e^{-\rho_K u} + \frac{2p}{N-1} \left| s_u(k)^2 - \frac{1}{K^2} \right|.$$

But

$$\left| s_u(k)^2 - \frac{1}{K^2} \right| = \left( s_u(k) + \frac{1}{K} \right) \left| s_u(k) - \frac{1}{K} \right| \leq \frac{K+1}{K} \sqrt{\frac{K-1}{K}} e^{-\rho_K u}.$$

Thus,

$$|\mathbf{w}_u(k, k) - \mathbf{w}_\infty(k, k)| \leq \frac{2}{N} \left( 1 + \theta + \frac{p}{N-1} + \frac{pN(K+1)\sqrt{K-1}}{K\sqrt{K}(N-1)} \right) e^{-\rho_K u}. \quad (4.13)$$

Similarly we get,

$$|\mathbf{w}_u(k, k+1) - \mathbf{w}_\infty(k, k+1)| \leq \frac{2}{N} \left( 1 + \theta + \frac{pN(K+1)\sqrt{K-1}}{K\sqrt{K}(N-1)} \right) e^{-\rho_K u}, \quad (4.14)$$

$$|\mathbf{w}_u(k, l) - \mathbf{w}_\infty(k, l)| \leq \frac{p(K+1)\sqrt{K-1}}{K\sqrt{K}(N-1)} e^{-\rho_K u}, \quad |k-l| \geq 2. \quad (4.15)$$

Inequalities (4.13), (4.14) and (4.15) imply that

$$\left\| \mathbf{w}_u^{(2)} - \mathbf{w}_\infty^{(2)} \right\|_\infty \leq C_{K,N} e^{-\rho_K u}, \quad (4.16)$$

where  $C_{K,N}$  is defined by (1.21). Plugging (4.11), (4.12) and (4.16) into (4.10), we obtain

$$\begin{aligned} \left\| \mathbf{g}_t^{(2)} - \mathbf{g}_\infty^{(2)} \right\|_\infty &\leq C_{K,N} \int_0^t e^{-\rho_K u} e^{-p_N(t-u)} du + e^{-p_N t} \|\mathbf{g}_\infty^{(2)}\|_\infty \\ &= C_{K,N} e^{-p_N t} \int_0^t e^{-(\rho_K - p_N)u} du + e^{-p_N t} \text{Var}_{\nu_N} \left[ \frac{\eta(0)}{N} \right] \\ &= C_{K,N} \frac{e^{-p_N t} - e^{-\rho_K t}}{\rho_K - p_N} + e^{-p_N t} \text{Var}_{\nu_N} \left[ \frac{\eta(0)}{N} \right] \end{aligned} \quad (4.17)$$

$$\begin{aligned} &= C_{K,N} \frac{1 - e^{-\rho_K t}}{\rho_K} + \text{Var}_{\nu_N} \left[ \frac{\eta(0)}{N} \right] + o\left(\frac{1}{N}\right) \\ &= \frac{1}{N} \left\{ D_K \frac{1 - e^{-\rho_K t}}{\rho_K} + E_K \right\} + o\left(\frac{1}{N}\right), \end{aligned} \quad (4.18)$$

where  $D_K$  and  $E_K$  are given by (1.23). Note that (1.19) is obtained from (4.17).

In order to prove (1.22), note that

$$\|\bar{m}(\eta_t) - \mathcal{L}_\mu(X_t | t \leq \tau_p)\|_2 \leq \mathbb{E}_\eta \left[ \|m(\eta_t) - \mathcal{L}_\mu(X_t | t \leq \tau_p)\|_2 \right] \quad (4.19)$$

$$\leq \mathbb{E}_\eta \left[ \|m(\eta_t) - \bar{m}(\eta_t)\|_2 \right] + \|\bar{m}(\eta_t) - \mathcal{L}_\mu(X_t | t \leq \tau_p)\|_2. \quad (4.20)$$

Inequality (4.19) is obtained using the convexity of the 2-norm and Jensen's inequality. Inequality (4.20) is proved using the triangular inequality. From Theorem 1.8 we know that

$$e^{-\alpha_K t} \|m(\eta) - \mu\|_2 \leq \|\bar{m}(\eta_t) - \mathcal{L}_\mu(X_t | t \leq \tau_p)\|_2 \leq e^{-\rho_K t} \|m(\eta) - \mu\|_2, \quad (4.21)$$

where  $\alpha_K$  and  $\rho_K$  are given by (1.5) and (1.6), respectively. Also,

$$\begin{aligned} \mathbb{E}_\eta \left[ \|m(\eta_t) - \bar{m}(\eta_t)\|_2^2 \right] &= \sum_{k=0}^{K-1} \text{Var}_\eta \left[ \frac{\eta(k)}{N} \right] \leq K \left\| g_t^{(2)} \right\|_\infty \\ &\leq \frac{2K}{N} \left( D_K \frac{1 - e^{-\rho_K t}}{\rho_K} + E_K \right) + o\left(\frac{1}{N}\right), \end{aligned} \quad (4.22)$$

where  $D_K$  and  $E_K$  are defined by (1.23). Finally, (1.22) is proved using (4.19), (4.20), (4.21), (4.22) and Jensen's inequality.  $\square$

#### APPENDIX A. PROOF OF LEMMA 3.2

In order to calculate  $\mathcal{L}_{K,N} f_k$ , note that

$$\mathcal{L}_{K,N} f_k(\eta) = \sum_{i,j} \eta(i) [f_k(T_{i \rightarrow j} \eta) - f_k(\eta)] \left( \mathbb{1}_{\{j=i+1\}} + \theta \mathbb{1}_{\{j=i-1\}} + \eta(j) \frac{p}{N-1} \right).$$

But  $f_k(T_{i \rightarrow j} \eta) = f_k(\eta)$  if  $i \neq k$  and  $j \neq k$ . Thus,

$$\begin{aligned} \mathcal{L}_{K,N} f_k(\eta) &= \eta(k) \sum_{j \neq k} [T_{k \rightarrow j} \eta(k) - \eta(k)] \left( \mathbb{1}_{\{j=k+1\}} + \theta \mathbb{1}_{\{j=k-1\}} + \eta(j) \frac{p}{N-1} \right) \\ &\quad + \sum_{i \neq k} \eta(i) [T_{i \rightarrow k} \eta(k) - \eta(k)] \left( \mathbb{1}_{\{k=i+1\}} + \theta \mathbb{1}_{\{k=i-1\}} + \eta(k) \frac{p}{N-1} \right) \end{aligned}$$

$$\begin{aligned}
&= -\eta(k) \left[ 1 + \theta + p \frac{N - \eta(k)}{N - 1} \right] + \eta(k - 1) + \theta \eta(k + 1) + p \eta(k) \frac{N - \eta(k)}{N - 1} \\
&= \eta(k - 1) - (1 + \theta) \eta(k) + \theta \eta(k + 1),
\end{aligned}$$

for all  $\eta \in \mathcal{E}_{K,N}$ . Thus, (3.4) is proved.

Now, for computing  $\mathcal{L}_{K,N} f_{k,l}$  for all  $1 \leq k, l \leq K$ , we separate the proof in three cases:  $l = k$ ,  $l = k + 1$  and  $l > k + 1$ , for all  $0 \leq k \leq K - 2$ .

**Case  $l = k$ :**

From (1.3) we have

$$\mathcal{L}_{K,N} f_{k,k}(\eta) = \sum_{i,j} \eta(i) [f_{k,k}(T_{i \rightarrow j} \eta) - f_{k,k}(\eta)] \left( \mathbb{1}_{\{j=i+1\}} + \theta \mathbb{1}_{\{j=i-1\}} + \eta(j) \frac{p}{N-1} \right),$$

for all  $\eta \in \mathcal{E}_{K,N}$ .

Denote

$$S_{i,j}(\eta) = \eta(i) [T_{i \rightarrow j} \eta(k)^2 - \eta(k)^2] \left( \mathbb{1}_{\{j=i+1\}} + \theta \mathbb{1}_{\{j=i-1\}} + \eta(j) \frac{p}{N-1} \right).$$

Note that if  $\{i, j\} \cap \{k\} = \emptyset$ , then we have  $S_{i,j}(\eta) = 0$ . So,

$$\mathcal{L}_{K,N} f_{k,k}(\eta) = \sum_{j \neq k} S_{k,j}(\eta) + \sum_{i \neq k} S_{i,k}(\eta).$$

Note that

$$\begin{aligned}
\sum_{j \neq k} S_{k,j}(\eta) &= \sum_{j \neq k} \eta(k) [T_{k \rightarrow j} \eta(k)^2 - \eta(k)^2] \left( \mathbb{1}_{\{j=k+1\}} + \theta \mathbb{1}_{\{j=k-1\}} + \eta(j) \frac{p}{N-1} \right) \\
&= \eta(k) [(\eta(k) - 1)^2 - \eta(k)^2] \left( 1 + \theta + \frac{p}{N-1} \sum_{j \neq k} \eta(j) \right) \\
&= [-2\eta(k) + 1] \left( \eta(k) + \theta \eta(k) + p \eta(k) \frac{N - \eta(k)}{N - 1} \right), \tag{A.1}
\end{aligned}$$

$$\begin{aligned}
\sum_{i \neq k} S_{i,k}(\eta) &= \sum_{i \neq k} \eta(i) [T_{i \rightarrow k} \eta(k)^2 - \eta(k)^2] \left( \mathbb{1}_{\{k=i+1\}} + \theta \mathbb{1}_{\{k=i-1\}} + \eta(k) \frac{p}{N-1} \right) \\
&= [(\eta(k) + 1)^2 - \eta(k)^2] \left( \eta(k - 1) + \theta \eta(k + 1) + \eta(k) \frac{p}{N-1} \sum_{i \neq k} \eta(i) \right) \\
&= [2\eta(k) + 1] \left( \eta(k - 1) + \theta \eta(k + 1) + p \eta(k) \frac{N - \eta(k)}{N - 1} \right). \tag{A.2}
\end{aligned}$$

Summing (A.1) and (A.2), we obtain

$$\begin{aligned}
\mathcal{L}_{K,N} f_{k,k}(\eta) &= \sum_{j \neq k} S_{k,j}(\eta) + \sum_{i \neq k} S_{i,k}(\eta) \\
&= 2\eta(k) [\eta(k - 1) - \eta(k) + \theta(\eta(k + 1) - \eta(k))] \\
&\quad + (\eta(k) + \eta(k - 1)) + \theta(\eta(k + 1) + \eta(k)) + 2p \eta(k) \frac{N - \eta(k)}{N - 1} \\
&= 2 \left[ \eta(k - 1) \eta(k) - \left( 1 + \theta + \frac{p}{N - 1} \right) \eta(k)^2 + \theta \eta(k) \eta(k + 1) \right] \\
&\quad + \eta(k - 1) + \left( 1 + \theta + \frac{2pN}{N - 1} \right) \eta(k) + \theta \eta(k + 1),
\end{aligned}$$

for all  $\eta \in \mathcal{E}_{K,N}$ . Thus, (3.5) holds.

**Case  $l = k + 1$ :**

From (1.3), similarly to the previous case, we have

$$\mathcal{L}_{K,N} f_{k,k+1}(\eta) = \sum_{i,j} \eta(i) [f_{k,k+1}(T_{i \rightarrow j} \eta) - f_{k,k+1}(\eta)] \left( \mathbb{1}_{\{j=i+1\}} + \theta \mathbb{1}_{\{j=i-1\}} + \eta(j) \frac{p}{N-1} \right).$$

Denote

$$R_{i,j}(\eta) = \eta(i) [T_{i \rightarrow j} \eta(k) T_{i \rightarrow j} \eta(k + 1) - \eta(k) \eta(k + 1)] \left( \mathbb{1}_{\{j=i+1\}} + \theta \mathbb{1}_{\{j=i-1\}} + \eta(j) \frac{p}{N-1} \right).$$

If  $\{i, j\} \cap \{k, k+1\} = \emptyset$ , then  $R_{i,j} = 0$ . Thus,

$$\mathcal{L}_{K,N} f_{k,k+1}(\eta) = \sum_{j \neq k} R_{k,j}(\eta) + \sum_{i \neq k, k+1} R_{i,k+1}(\eta) + \sum_{j \neq k+1} R_{k+1,j}(\eta) + \sum_{i \neq k, k+1} R_{i,k}(\eta).$$

Note that

$$\begin{aligned} \sum_{j \neq k} R_{k,j}(\eta) &= R_{k,k+1}(\eta) + \sum_{j \neq k, k+1} R_{k,j}(\eta) \\ &= \eta(k)[(\eta(k) - 1)(\eta(k+1) + 1) - \eta(k)\eta(k+1)] \left[ 1 + p \frac{\eta(k+1)}{N-1} \right] \\ &\quad + \sum_{j \neq k, k+1} \eta(k)[(\eta(k) - 1)\eta(k+1) - \eta(k)\eta(k+1)] \\ &\quad \times \left( \mathbb{1}_{\{j=k+1\}} + \theta \mathbb{1}_{\{j=k-1\}} + \eta(j) \frac{p}{N-1} \right) \\ &= \eta(k)[\eta(k) - \eta(k+1) - 1] \left[ 1 + p \frac{\eta(k+1)}{N-1} \right] \\ &\quad - \eta(k)\eta(k+1) \left( \theta + \frac{p}{N-1} \sum_{j \neq k, k+1} \eta(j) \right) \\ &= \eta(k)[\eta(k) - 1] \left[ 1 + p \frac{\eta(k+1)}{N-1} \right] \\ &\quad - \eta(k)\eta(k+1)(1 + \theta) - p \eta(k)\eta(k+1) \frac{N - \eta(k)}{N-1}, \\ \sum_{i \neq k, k+1} R_{i,k+1}(\eta) &= \sum_{i \neq k, k+1} \eta(i)[\eta(k)(\eta(k+1) + 1) - \eta(k)\eta(k+1)] \\ &\quad \times \left( \mathbb{1}_{\{k+1=i+1\}} + \theta \mathbb{1}_{\{k+1=i-1\}} + \eta(k+1) \frac{p}{N-1} \right) \\ &= \eta(k) \left( \theta \eta(k+2) + p \eta(k+1) \frac{\sum_{i \neq k, k+1} \eta(i)}{N-1} \right) \\ &= \theta \eta(k)\eta(k+2) + p \eta(k)\eta(k+1) \frac{N - \eta(k) - \eta(k+1)}{N-1}, \\ \sum_{j \neq k+1} R_{k+1,j}(\eta) &= R_{k+1,k}(\eta) + \sum_{j \neq k, k+1} R_{k+1,j}(\eta) \\ &= \eta(k+1)[(\eta(k) + 1)(\eta(k+1) - 1) - \eta(k)\eta(k+1)] \left[ \theta + p \frac{\eta(k)}{N-1} \right] \\ &\quad + \sum_{j \neq k, k+1} \eta(k+1)[\eta(k)(\eta(k+1) - 1) - \eta(k)\eta(k+1)] \\ &\quad \times \left( \mathbb{1}_{\{j=k+2\}} + \theta \mathbb{1}_{\{j=k\}} + p \frac{\eta(j)}{N-1} \right) \\ &= \eta(k+1)[\eta(k+1) - \eta(k) - 1] \left[ \theta + p \frac{\eta(k)}{N-1} \right] \\ &\quad - \eta(k)\eta(k+1) \left( 1 + \frac{p}{N-1} \sum_{j \neq k, k+1} \eta(j) \right) \\ &= \eta(k+1)[\eta(k+1) - 1] \left[ \theta + p \frac{\eta(k)}{N-1} \right] - \eta(k)\eta(k+1)(1 + \theta) \\ &\quad - p \eta(k)\eta(k+1) \frac{N - \eta(k+1)}{N-1}, \\ \sum_{i \neq k, k+1} R_{i,k}(\eta) &= \sum_{i \neq k, k+1} \eta(i)[(\eta(k) + 1)\eta(k+1) - \eta(k)\eta(k+1)] \\ &\quad \times \left( \mathbb{1}_{\{k=i+1\}} + \theta \mathbb{1}_{\{k=i-1\}} + \eta(k) \frac{p}{N-1} \right) \end{aligned}$$

$$\begin{aligned}
&= \eta(k+1) \left( \eta(k-1) + p\eta(k) \frac{N - \eta(k) - \eta(k+1)}{N-1} \right) \\
&= \eta(k-1)\eta(k+1) + p\eta(k)\eta(k+1) \frac{N - \eta(k) - \eta(k+1)}{N-1}.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{L}_{K,N} f_{k,k+1}(\eta) &= -\eta(k)\eta(k+1) \left[ 2(1+\theta) + p \frac{2N - \eta(k) - \eta(k+1) - 2[N - \eta(k) - \eta(k+1)]}{N-1} \right] \\
&\quad + \eta(k)[\eta(k) - 1] \left( 1 + p \frac{\eta(k+1)}{N-1} \right) + \eta(k+1)[\eta(k+1) - 1] \left( \theta + p \frac{\eta(k)}{N-1} \right) \\
&\quad + \eta(k-1)\eta(k+1) + \theta\eta(k)\eta(k+2) \\
&= -\eta(k)\eta(k+1) \left[ 2(1+\theta) + p \frac{\eta(k) + \eta(k+1)}{N-1} \right] \\
&\quad + \eta(k)[\eta(k) - 1] \left( 1 + p \frac{\eta(k+1)}{N-1} \right) + \eta(k+1)[\eta(k+1) - 1] \left( \theta + p \frac{\eta(k)}{N-1} \right) \\
&\quad + \eta(k-1)\eta(k+1) + \theta\eta(k)\eta(k+2) \\
&= -2\eta(k)\eta(k+1)(1+\theta) + \eta(k)[\eta(k) - 1] + \theta\eta(k+1)[\eta(k+1) - 1] \\
&\quad - 2p \frac{\eta(k)\eta(k+1)}{N-1} + \eta(k-1)\eta(k+1) + \theta\eta(k)\eta(k+2) \\
&= -2 \left( 1 + \theta + \frac{p}{N-1} \right) \eta(k)\eta(k+1) + \eta(k-1)\eta(k+1) \\
&\quad + \theta\eta(k+1)^2 + \eta(k)^2 + \theta\eta(k)\eta(k+2) - \eta(k) - \theta\eta(k+1),
\end{aligned}$$

for all  $\eta \in \mathcal{E}_{K,N}$ , which is equivalent to (3.6).

**Case**  $l > k+1$ :

In this case we have

$$\mathcal{L}_{K,N} f_{k,l}(\eta) = \sum_{i,j \in F} \eta(i) [f_{k,l}(T_{i \rightarrow j} \eta) - f_{k,l}(\eta)] \left( \mathbb{1}_{\{j=i+1\}} + \theta \mathbb{1}_{\{j=i-1\}} + \eta(j) \frac{p}{N-1} \right).$$

Denote

$$T_{i,j}(\eta) = \eta(i) [T_{i \rightarrow j} \eta(k) T_{i \rightarrow j} \eta(l) - \eta(k)\eta(l)] \left( \mathbb{1}_{\{j=i+1\}} + \theta \mathbb{1}_{\{j=i-1\}} + \eta(j) \frac{p}{N-1} \right).$$

Obviously, if  $\{i, j\} \cap \{k, k+l\} = \emptyset$ , then

$$T_{i,j}(\eta) = \eta(i) [T_{i \rightarrow j} \eta(k) T_{i \rightarrow j} \eta(k+l) - \eta(k)\eta(k+l)] \left( \mathbb{1}_{\{j=i+1\}} + \theta \mathbb{1}_{\{j=i-1\}} + \eta(j) \frac{p}{N-1} \right) = 0.$$

Thus

$$\mathcal{L}_{K,N} f_{k,l}(\eta) = \sum_{j \neq k} T_{k,j}(\eta) + \sum_{i \neq k, k+l} T_{i, k+l}(\eta) + \sum_{j \neq k+l} T_{k+l, j}(\eta) + \sum_{i \neq k, k+l} T_{i, k}(\eta).$$

Note that

$$\begin{aligned}
\sum_{j \neq k} T_{k,j}(\eta) &= T_{k, k+l}(\eta) + \sum_{j \neq k, k+l} T_{k,j}(\eta) \\
&= \eta(k) [(\eta(k) - 1)(\eta(k+l) + 1) - \eta(k)\eta(k+l)] p \frac{\eta(k+l)}{N-1} \\
&\quad + \sum_{j \neq k, k+l} \eta(k) [(\eta(k) - 1)\eta(k+l) - \eta(k)\eta(k+l)] \\
&\quad \times \left( \mathbb{1}_{\{j=k+1\}} + \theta \mathbb{1}_{\{j=k-1\}} + p \frac{\eta(j)}{N-1} \right) \\
&= \eta(k) [\eta(k) - \eta(k+l) - 1] p \frac{\eta(k+l)}{N-1} \\
&\quad - \eta(k)\eta(k+l) \left( 1 + \theta + \frac{p}{N-1} \sum_{j \neq k, k+l} \eta(j) \right)
\end{aligned}$$



$$\begin{aligned}
&= \eta(k)\eta(k+l) \left[ \frac{p}{N-1}(\eta(k)-1) - (1+\theta) - p\frac{N-\eta(k)}{N-1} \right], \\
\sum_{i \neq k, k+l} T_{i, k+l}(\eta) &= \sum_{i \neq k, k+l} \eta(i) [\eta(k)(\eta(k+l)+1) - \eta(k)\eta(k+l)] \\
&\quad \left( \mathbb{1}_{\{k+l=i+1\}} \frac{1}{K} + \mathbb{1}_{\{k+l=i-1\}} \frac{\theta}{K} + \eta(k+l) \frac{p}{N-1} \right) \\
&= \eta(k) \left[ \eta(k+l-1) + \theta\eta(k+l+1) + p\eta(k+l) \frac{N-\eta(k)-\eta(k+l)}{N-1} \right] \\
&= \eta(k)\eta(k+l-1) + \theta\eta(k)\eta(k+l+1) + p\eta(k)\eta(k+l) \frac{N-\eta(k)-\eta(k+l)}{N-1}, \\
\sum_{j \neq k+l} T_{k+l, j}(\eta) &= T_{k+l, k}(\eta) + \sum_{j \neq k, k+l} T_{k+l, j}(\eta) \\
&= \eta(k+l) [(\eta(k)+1)(\eta(k+l)-1) - \eta(k)\eta(k+l)] p \frac{\eta(k)}{N-1} \\
&\quad + \sum_{j \neq k, k+l} \eta(k+l) [\eta(k)(\eta(k+l)-1) - \eta(k)\eta(k+l)] \\
&\quad \times \left( \mathbb{1}_{\{j=k+l+1\}} + \theta \mathbb{1}_{\{j=k+l-1\}} + p \frac{\eta(j)}{N-1} \right) \\
&= \eta(k+l) [\eta(k+l) - \eta(k) - 1] p \frac{\eta(k)}{N-1} \\
&\quad - \eta(k)\eta(k+l) \left[ 1 + \theta + \frac{p}{N-1} \sum_{j \neq k, k+l} \eta(j) \right] \\
&= \eta(k+l) [\eta(k+l) - \eta(k) - 1] p \frac{\eta(k)}{N-1} \\
&\quad - \eta(k)\eta(k+l) \left[ 1 + \theta + p \frac{N-\eta(k)-\eta(k+l)}{N-1} \right] \\
&= \eta(k)\eta(k+l) \left[ (\eta(k+l)-1) \frac{p}{N-1} - \frac{1+\theta}{N} - p \frac{N-\eta(k+l)}{N-1} \right], \\
\sum_{i \neq k, k+l} T_{i, k}(\eta) &= \sum_{i \neq k, k+l} \eta(i) [(\eta(k)+1)\eta(k+l) - \eta(k)\eta(k+l)] \\
&\quad \times \left( \mathbb{1}_{\{k=i+1\}} + \theta \mathbb{1}_{\{k=i-1\}} + p \frac{\eta(k)}{N-1} \right) \\
&= \eta(k+l) \left[ \eta(k-1) + \theta\eta(k+1) + p\eta(k) \frac{N-\eta(k)-\eta(k+l)}{N-1} \right] \\
&= \eta(k-1)\eta(k+l) + \theta\eta(k+1)\eta(k+l) + p\eta(k)\eta(k+l) \frac{N-\eta(k)-\eta(k+l)}{N-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{L}_{K, N} f_{k, l}(\eta) &= \eta(k)\eta(k+l) \left( \frac{p}{N-1} [\eta(k) + \eta(k+l) - 2] - 2(1+\theta) \right. \\
&\quad \left. - \frac{p}{N-1} [2N - \eta(k) - \eta(k+l) - 2[N - \eta(k) - \eta(k+l)]] \right) \\
&\quad + \eta(k) [\eta(k+l-1) + \theta\eta(k+l+1)] + \eta(k+l) [\eta(k-1) + \theta\eta(k+1)] \\
&= -2\eta(k)\eta(k+l) \left( 1 + \theta + \frac{p}{N-1} \right) + \eta(k) [\eta(k+l-1) + \theta\eta(k+l+1)] \\
&\quad + \eta(k+l) [\eta(k-1) + \theta\eta(k+1)],
\end{aligned}$$

for all  $\eta \in \mathcal{E}_{K, N}$ , proving (3.7).

#### ACKNOWLEDGEMENTS

I would like to thank Djalil Chafaï, Bertrand Cloez, Simona Grusea and Didier Pinchon for very useful discussions that greatly enriched this work.

## REFERENCES

- [1] A. Asselah, P. A. Ferrari, and P. Groisman. Quasistationary distributions and Fleming–Viot processes in finite spaces. *Journal of Applied Probability*, 48(2):322–332, 2011.
- [2] A. Asselah, P. A. Ferrari, P. Groisman, and M. Jonckheere. Fleming–Viot selects the minimal quasi-stationary distribution: The Galton–Watson case. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(2):647–668, 2016.
- [3] A. Asselah and M.-N. Thai. A note on the rightmost particle in a Fleming–Viot process. *arXiv e-prints*, Dec 2012.
- [4] V. Bansaye, B. Cloez, P. Gabriel, and A. Marguet. A Non-Conservative Harris’ Ergodic Theorem. Working paper or preprint, March 2019.
- [5] K. Burdzy, R. Holyst, and P. March. A Fleming–Viot Particle Representation of the Dirichlet Laplacian. *Communications in Mathematical Physics*, 214(3):679–703, Nov 2000.
- [6] N. Champagnat and D. Villemonais. Exponential convergence to quasi-stationary distribution and Q-process. *Probability Theory and Related Fields*, 164(1):243–283, Feb 2016.
- [7] N. Champagnat and D. Villemonais. General criteria for the study of quasi-stationarity. *arXiv e-prints*, Dec 2017.
- [8] N. Champagnat and D. Villemonais. Convergence of the Fleming–Viot process toward the minimal quasi-stationary distribution. *arXiv e-prints*, Oct 2018.
- [9] B. Cloez and M.-N. Thai. Fleming–Viot processes: two explicit examples. *Latin American Journal of Probability and Mathematical Statistics*, 13(1):337–356, 2016.
- [10] B. Cloez and M.-N. Thai. Quantitative results for the Fleming–Viot particle system and quasi-stationary distributions in discrete space. *Stochastic Processes and their Applications*, 126(3):680–702, May 2016.
- [11] P. Collet, S. Martínez, and J. San Martín. *Quasi-stationary distributions: Markov chains, diffusions and dynamical systems*. Springer Science & Business Media, 2012.
- [12] L. Comtet. *Advanced Combinatorics: The art of finite and infinite expansions*. Springer Science & Business Media, 2012.
- [13] J. N. Darroch and E. Seneta. On quasi-stationary distributions in absorbing continuous-time finite Markov chains. *Journal of Applied Probability*, 4(1):192–196, nov 1967.
- [14] P. J. Davis. *Circulant Matrices*. Wiley, 1979.
- [15] P. Del Moral and A. Guionnet. On the stability of measure valued processes with applications to filtering. *Comptes Rendus de l’Académie des Sciences - Series I - Mathematics*, 329(5):429–434, sep 1999.
- [16] P. Del Moral and L. Miclo. A Moran particle system approximation of Feynman–Kac formulae. *Stochastic Processes and their Applications*, 86(2):193–216, apr 2000.
- [17] A. M. Etheridge and R. C. Griffiths. A coalescent dual process in a Moran model with genic selection. *Theoretical Population Biology*, 75(4):320–330, jun 2009.
- [18] P. Ferrari and N. Maric. Quasi Stationary Distributions and Fleming–Viot Processes in Countable Spaces. *Electron. J. Probab.*, 12:684–702, 2007.
- [19] F. P. Kelly. *Reversibility and stochastic networks*. John Wiley & Sons, 1978.
- [20] T. Lelièvre, L. Pillaud-Vivien, and J. Reygner. Central Limit Theorem for stationary Fleming–Viot particle systems in finite spaces. *Latin American Journal of Probability and Mathematical Statistics*, 15(2):1163–1182, 2018.
- [21] D. A. Levin and Y. Peres. *Markov chains and mixing times*, volume 107. American Mathematical Soc., 2017.
- [22] N. Marić. Fleming–Viot Particle System Driven by a Random Walk on  $\mathbb{N}$ . *Journal of Statistical Physics*, 160(3):548–560, may 2015.
- [23] J. C. Mason and D. C. Handscomb. *Chebyshev polynomials*. Chapman and Hall/CRC, 2002.
- [24] S. Méléard and D. Villemonais. Quasi-stationary distributions and population processes. *Probability Surveys*, 9:340–410, 2012.
- [25] M. C. Pease. Chapter XIV The Direct Product and Kronecker Sum. In Marshall C. Pease, editor, *Methods of Matrix Algebra*, pages 321–334. Elsevier, 1965.
- [26] E. A. van Doorn and P. K. Pollett. Quasi-stationary distributions for discrete-state models. *European journal of operational research*, 230(1):1–14, oct 2013.
- [27] D. Villemonais. General approximation method for the distribution of Markov processes conditioned not to be killed. *ESAIM: Probability and Statistics*, 18:441–467, 2014.
- [28] D. Villemonais. Minimal quasi-stationary distribution approximation for a birth and death process. *Electron. J. Probab.*, 20:1–18, 2015.
- [29] D. Villemonais. Lower Bound for the Coarse Ricci Curvature of Continuous-Time Pure-Jump Processes. *Journal of Theoretical Probability*, pages 1–38, May 2019.

CEREMADE, UNIVERSITÉ PARIS-DAUPHINE, PSL UNIVERSITY AND INSTITUT DE MATHÉMATIQUES DE TOULOUSE AND INSTITUT NATIONAL DES SCIENCES APPLIQUES DE TOULOUSE, FRANCE

*E-mail address:* jcorujo@insa-toulouse.fr