Hydrodynamic Limits and Clausius inequality for Isothermal Non-linear Elastodynamics with Boundary Tension

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Abstract

We consider a chain of particles connected by an-harmonic springs, with a boundary force (tension) acting on the last particle, while the first particle is kept pinned at a point. The particles are in contact with stochastic heat baths, whose action on the dynamics conserves the volume and the momentum, while energy is exchanged with the heat baths in such way that, in equilibrium, the system is at a given temperature T. We study the space empirical profiles of volume stretch and momentum under hyperbolic rescaling of space and time as the size of the system growth to be infinite, with the boundary tension changing slowly in the macroscopic time scale. We prove that the probability distributions of these profiles concentrate on L^2-valued weak solutions of the isothermal Euler equations (i.e. the non-linear wave equation, also called p-system), satisfying the boundary conditions imposed by the microscopic dynamics. Furthermore, the weak solutions obtained satisfy the Clausius inequality between the work done by the boundary force and the change of the total free energy in the system. This result includes the shock regime of the system.

1 Introduction

Boundary conditions in hyperbolic systems of conservation laws introduce challenging mathematical problems, in particular for weak solutions that are not of bounded variations. The solution may depend on the particular approximation used, and reflects

different microscopic origins of the equation. Recently (cf. [6]) we have considered L^2-valued weak solution to the isothermal Euler equation in Lagrangian coordinates on $[0,1]$ (also called in the literature non-linear wave equation or p-system):

$$\begin{cases} \partial_t r - \partial_x p = 0 \\ \partial_t p - \partial_x \tau(r) = 0 \end{cases}, \tag{1.1}$$

with the following boundary conditions: $p(t,0) = 0$ (the material is attached to a fixed point on the left side), $\tau(r(t,1)) = \bar{\tau}(t)$ (a time dependent force $\bar{\tau}(t)$ is acting on the right side). The precise sense an L^2-valued solution satisfies the boundary condition is given in Definition 2.2. In [6] we consider viscous approximations of (1.1), that requires two extra boundary conditions. We choose these extra boundary conditions to be of Neumann type (i.e. *conservative*). Adapting the L^2-version of the compensated-compactness argument of Shearer [9] and [8], we prove in [6] the existence of vanishing viscosity solutions to the p-system. Furthermore, these solutions satisfy the usual Lax-entropy production characterisation and the thermodynamic Clausius inequality, which relates the change of the total free energy to the work done by the boundary force (see Section 5). We call such weak solutions *thermodynamic entropy solutions*.

In the present article we study the microscopic *statistical mechanics* origin of (1.1). We want to understand how equation (1.1) emerges in a *hydrodynamic limit*, i.e. a hyperbolic space-time rescaling of a microscopic dynamics. We consider a chain of $N+1$ particles connected by N anharmonic springs (see Figure 1). The first particle on the left is fixed at a point, while on the rightmost particle is acting a time-dependent force (tension). The Hamiltonian dynamics of this system is perturbed by the action of stochastic heat baths at temperature T. Each heat bath is acting, independently from the others, between two springs connected by a particle, randomly exchanging momenta and volume stretch. The energy of the particles is not conserved but exchanged with the heath baths in such a way that, in equilibrium, the system is at temperature T. The intensity of the action of the heat baths is such that it does not affect the macroscopic equation directly, but sufficiently strong to provide the required regularity at certain microscopic scales and establish an *isothermal* macroscopic evolution. In this sense these heat baths act like a *stochastic viscosity*, vanishing after the space-time hyperbolic rescaling. The conservative nature of these stochastic heat baths also provides at the boundaries the analogue of the extra Neumann conditions as used in [6].

We rescale space and time using N as parameter in such a way that the time-dependent external force is changing on the macroscopic scale (i.e. very slowly on the microscopic time scale). We prove that the probability distributions of the random profiles of volume stretch and momenta concentrate on the L^2-valued weak solutions of (1.1) (in the sense of Definition 2.2), that satisfy the Clausius inequality. Proving uniqueness would complete the convergence theorem. Unfortunately, uniqueness for such weak solutions is a well known and challenging open problem.

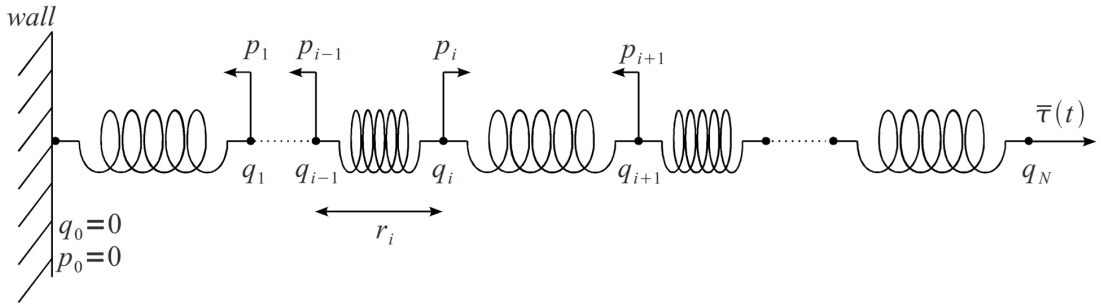

Figure 1: Microscopic model

The proof of the convergence to the weak solutions is adapted from the stochastic version of compensated compactness developed by Fritz in [3] for the same dynamics but without boundaries (see also Fritz and Toth [4] for a different two component dynamics). In a previous work [7], we considered the same problem as here, but we proved that (1.1) were satisfied only in the bulk by the limit profiles, without giving any information of the boundary conditions, nor on the *entropic* properties of these solutions.

The main new contributions of the present article are the followings:

- the limit profiles obtained are L^2-valued weak solutions that satisfy the boundary conditions, in the sense of Definition 2.2;
- the work done by the boundary force is larger than the change in the total free energy (Clausius inequality).

The proof of the Clausius inequality is the content of Section 5. It uses the variational characterisation of the (microscopic) relative entropy in order to connect it to the macroscopic free energy and estimate its time derivative. In other words, Clausius inequality follows from the microscopic entropy production.

2 The Model and the Main Theorem

We study a one-dimensional Hamiltonian system of $N+1$ particles of unitary mass. The position of the i-th particle ($i = 0, 1, \dots, N$) is denoted by $q_i \in \mathbb{R}$ and its momentum by $p_i \in \mathbb{R}$. We assume that particle 0 is kept fixed, i.e. $(q_0, p_0) \equiv (0, 0)$, while on particle N is applied a time-dependent force, $\bar{\tau}(t)$.

Denote by $\mathbf{q} = (q_0, \dots, q_N)$ and $\mathbf{p} = (p_0, \dots, p_N)$. The interaction between particles i and $i-1$ is described by the potential energy $V(q_i - q_{i-1})$ of an anharmonic spring, where V is a uniformly convex function that grows quadratically at infinity: there exist constants c_1 and c_2 such that for any $r \in \mathbb{R}$:

$$0 < c_1 \leq V''(r) \leq c_2. \tag{2.1}$$

Moreover, there are some positive constants V''_+, V''_-, α and R such that

$$\left|V''(r) - V''_+\right| \leq e^{-\alpha r}, \quad r > R \tag{2.2}$$
$$\left|V''(r) - V''_-\right| \leq e^{\alpha r}, \quad r < -R. \tag{2.3}$$

For $\tau \in \mathbb{R}$ and $\beta > 0$ we define the canonical Gibbs function as

$$G(\tau) := \log \int_{-\infty}^{+\infty} e^{-\beta V(r) + \beta \tau r}\, dr. \tag{2.4}$$

For $\ell \in \mathbb{R}$, the free energy is given by the Legendre transform of G:

$$F(\ell) := \sup_{\tau \in \mathbb{R}} \left\{ \tau \ell - \beta^{-1} G(\tau) \right\}, \tag{2.5}$$

so that its inverse is

$$G(\tau) = \beta \sup_{\ell \in \mathbb{R}} \left\{ \tau \ell - F(\ell) \right\}. \tag{2.6}$$

Note that we neglect to write the dependence of F and G on β, as it shall be fixed throughout the paper.

We denote by $\ell(\tau)$ and $\tau(\ell)$ the corresponding convex conjugate variables, that depend parametrically on β and satisfy

$$\ell(\tau) = \beta^{-1} G'(\tau), \quad \tau(\ell) = F'(\ell). \tag{2.7}$$

We identify $\tau(\ell)$ with the equilibrium tension of the system of length ℓ, and we assume that the potential V, besides satisfying the assumptions above, is such that τ is strictly convex (i.e. $\tau''(\ell) > 0$ for all $\ell \in \mathbb{R}$).

Remark. At the present time, we do not know a general condition on V that yield a strictly convex tension, but as example (cf [7], Proposition A.7) one may take V to be a mollification of the function

$$r \longmapsto \frac{1}{2}(1-\kappa) r^2 + \frac{1}{2} \kappa r |r|_+, \tag{2.8}$$

where $|r|_+ = \max\{r, 0\}$ and $\kappa \in (0, 1/3)$.

Define the Hamiltonian:

$$\mathcal{H}_N(\mathbf{q}, \mathbf{p}, t) := \sum_{i=0}^{N} \left(\frac{p_i^2}{2} + V(q_i - q_{i-1}) \right) - q_N \bar{\tau}(t), \tag{2.9}$$

where $\bar{\tau}(t)$ is the external tension. Since the interaction depends only on the distance between particles, we define

$$r_i := q_i - q_{i-1}, \quad i = 1, \ldots, N. \tag{2.10}$$

Consequently, recalling that $p_0 \equiv 0$, the configuration of the system is given by $(\mathbf{r}, \mathbf{p}) := (r_1, \dots, r_N, p_1, \dots, p_N)$ and the phase space is $\mathbb{R}^{2N}$. Thus, the Hamiltonian reads

$$\mathcal{H}_N(\mathbf{r}, \mathbf{p}, t) = \sum_{i=1}^{N} \left(\frac{p_i^2}{2} + V(r_i) - \bar{\tau}(t) r_i \right). \tag{2.11}$$

We add to the Hamiltonian dynamics physical and artificial noise. Thus, the full dynamics of the system is determined by the generator

$$\mathcal{G}_N^{\bar{\tau}(t)} := N L_N^{\bar{\tau}(t)} + N\sigma \left(S_N + \tilde{S}_N^{\bar{\tau}(t)} \right). \tag{2.12}$$

$\sigma = \sigma(N)$ is a positive number that tunes the strength of the noise. We take it such that

$$\lim_{N\to\infty} \frac{\sigma}{N} = \lim_{N\to\infty} \frac{N}{\sigma^2} = 0. \tag{2.13}$$

The Liouville operator $L_N^{\bar{\tau}(t)}$ is given by

$$L_N^{\bar{\tau}(t)} = \sum_{i=1}^{N} (p_i - p_{i-1}) \frac{\partial}{\partial r_i} + \sum_{i=1}^{N-1} \left(V'(r_{i+1}) - V'(r_i) \right) \frac{\partial}{\partial p_i} + (\bar{\tau}(t) - V'(r_N)) \frac{\partial}{\partial p_N}, \tag{2.14}$$

together with $p_0 \equiv 0$. Note that the time scale in the tension is chosen such that it changes smoothly on the macroscopic scale.

The operators S_N and $\tilde{S}_N^{\bar{\tau}(t)}$ generate the stochastic part of the dynamics, modelling the interaction with a heat bath at constant temperature β^{-1}, and are defined by

$$S_N := -\sum_{i=0}^{N-1} D_i^* D_i, \quad \tilde{S}_N^{\bar{\tau}(t)} := -\sum_{i=1}^{N} \tilde{D}_i^* \tilde{D}_i, \tag{2.15}$$

where, for $i = 1, \dots, N-1$,

$$D_i := \frac{\partial}{\partial p_{i+1}} - \frac{\partial}{\partial p_i}, \qquad D_i^* := p_{i+1} - p_i - \beta^{-1} D_i \tag{2.16}$$

$$\tilde{D}_i := \frac{\partial}{\partial r_{i+1}} - \frac{\partial}{\partial r_i}, \qquad \tilde{D}_i^* := V'(r_{i+1}) - V'(r_i) - \beta^{-1} \tilde{D}_i. \tag{2.17}$$

The extra boundary operators were first considered in [5] and are given by

$$D_0 := \frac{\partial}{\partial p_1}, \qquad D_0^* := p_1 - \beta^{-1} D_0, \tag{2.18}$$

$$\tilde{D}_N := -\frac{\partial}{\partial r_N}, \qquad \tilde{D}_N^* := \bar{\tau}(t) - V'(r_N) - \beta^{-1} \tilde{D}_N. \tag{2.19}$$

On the one-particle state space $\mathbb{R}^2$ we define a family of probability measures

$$\lambda_{\beta,\bar{p},\tau}(dr,dp) := \frac{1}{\sqrt{2\pi\beta^{-1}}} e^{-\frac{\beta}{2}(p-\bar{p})^2 - \beta V(r) + \beta\tau r - G(\beta,\tau)}\, dr\, dp. \tag{2.20}$$

The mean elongation and momentum are

$$\int r\ d\lambda_{\beta,\bar{p},\tau} = \ell(\tau), \quad \int p\ d\lambda_{\beta,\bar{p},\tau} = \bar{p}. \tag{2.21}$$

We also have the relations

$$\int V'(r)\ d\lambda_{\beta,\bar{p},\tau} = \tau, \quad \int p^2\ d\lambda_{\beta,\bar{p},\tau} - \bar{p}^2 = \beta^{-1}, \tag{2.22}$$

that identify τ as the tension and β^{-1} as the temperature.

Define the family of product measures $\lambda_t^N = \lambda^N_{\beta,0,\bar{\tau}(t)}$ where

$$\lambda^N_{\beta,\bar{p},\tau}(d\mathbf{r},d\mathbf{p}) = \prod_{i=1}^N \lambda_{\beta,\bar{p},\tau}(dr_i,dp_i). \tag{2.23}$$

Notice that $L_N^{\bar{\tau}(t)}$ is antisymmetric with respect to λ_t^N, while S_N and $\tilde{S}_N^{\bar{\tau}(t)}$ are symmetric. It follows that, in the case $\bar{\tau}$ is constant in time, $\lambda^N_{\beta,0,\bar{\tau}}$ is the unique stationary measure for the dynamics. This is the canonical Gibbs measure at a temperature β^{-1}, pressure $\bar{\tau}$ and velocity 0.

Define the discrete gradients and Laplacian by

$$\nabla a_i := a_{i+1} - a_i, \qquad \nabla^* a_i := a_{i-1} - a_i, \tag{2.24}$$

$$\Delta a_i := -\nabla\nabla^* a_i = -\nabla^*\nabla a_i = a_{i+1} + a_{i-1} - 2a_i. \tag{2.25}$$

The time evolution of the system is described by the following system of stochastic differential equations

$$\begin{cases} dr_1 = Np_1 dt + \sigma N \nabla V'(r_1)dt - \sqrt{2\beta^{-1}\sigma N}\, d\widetilde{w}_1 \\ dr_i = -N\nabla^* p_i - dt + \sigma N \Delta V'(r_i)dt + \sqrt{2\beta^{-1}\sigma N}\, \nabla^* d\widetilde{w}_i, & 2 \le i \le N-1 \\ dr_N = -N\nabla^* p_N dt + \sigma N\left(\bar{\tau}(t) + V'(r_{N-1}) - 2V'(r_N)\right) + \sqrt{2\beta^{-1}\sigma N}\, \nabla^* d\widetilde{w}_N, \\ dp_1 = N\nabla V'(r_1)dt + \sigma N\left(p_2 - 2p_1\right)dt + \sqrt{2\beta^{-1}\sigma N}\, \nabla^* dw_1, \\ dp_j = N\nabla V'(r_j)dt + \sigma N \Delta p_j dt + \sqrt{2\beta^{-1}\sigma N}\, \nabla^* dw_j, & 2 \le j \le N-1 \\ dp_N = N(\bar{\tau}(t) - V'(r_N))dt - \sigma N \nabla p_{N-1} dt + \sqrt{2\beta^{-1}\sigma N}\, dw_{N-1} \end{cases} \tag{2.26}$$

Here $\{w_i\}_{i=0}^\infty$, $\{\widetilde{w}_i\}_{i=1}^\infty$ are independent families of independent Brownian motions on a common probability space $(\Omega,\mathbb{P})$. The expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$.

Notice that the noise introduced by the heat bath respects the *boundary conditions* $V'(r_{N+1}(t)) = \bar{\tau}(t)$ and $p_0(t) = 0$ already present in the Hamiltonian part of the dynamics, while it introduces the *Neumann type boundary conditions* $r_0(t) = r_1(t)$ and $p_{N+1}(t) = p_N(t)$. In this sense these boundary conditions are the microscopic analogous of those taken in the viscous approximation used in reference [6].

Thanks to the assumptions we made on the interaction V, it is possible to show (cf [3] or Appendix A of [7]) the following

Proposition 2.1. *For any fixed $\beta > 0$, the application $\tau : \mathbb{R} \to \mathbb{R}$ is smooth and has the following properties:*

i) $c_2^{-1} \leq \tau(\ell) \leq c_1^{-1}$ *for all* $\ell \in \mathbb{R}$*;*

ii) $\tau'', \tau''' \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Furthermore, we assume $\tau''(\ell) > 0$ for all $\ell \in \mathbb{R}$.

Remark. The condition $\tau' \geq c_2^{-1} > 0$ is a condition of strict hyperbolicity. On the other hand, $\tau'' > 0$ is a condition of genuine nonlinearity, and it is easy to see that it rules out symmetric interactions ($V(-r) = V(r)$). Nevertheless, such a condition may be relaxed as in [8] and we can allow τ'' to vanish at most at one point, which is compatible with having a symmetric interaction.

Denote by μ_t^N the probability measure of the system a time t. Then, the density f_t^N of μ_t^N with respect to λ_t^N solves the Fokker-Plank equation

$$\frac{\partial}{\partial t}\left(f_t^N \lambda_t^N\right) = \left(\mathcal{G}_N^{\bar{\tau}(t),*} f_t^N\right) \lambda_t^N. \tag{2.27}$$

Here

$$\mathcal{G}_N^{\bar{\tau}(t),*} = -N L_N^{\bar{\tau}(t)} + N\sigma\left(S_N + S_N^{\bar{\tau}(t)}\right) \tag{2.28}$$

is the adjoint of $\mathcal{G}_N^{\bar{\tau}(t)}$ with respect to λ_t^N.

We define the relative entropy

$$H_N(t) := \int f_t^N \log f_t^N d\lambda_t^N, \tag{2.29}$$

and require that the initial distribution f_0^N is such that $H_N(0) \leq CN$ for some C independent of N.

We are interested in the macroscopic behaviour of the volume stretch and momentum of the particles, at time t, as $N \to \infty$. Note that t is already the macroscopic time, as we have already multiplied by N in the generator. We shall use Lagrangian coordinates, that is our space variables will belong to the lattice $\{1/N, \ldots, (N-1)/N, 1\}$.

Consequently, we set $\mathbf{u}_i := (r_i, p_i)$. For a fixed macroscopic time T, we introduce the empirical measures on $[0,T] \times [0,1]$ representing the space-time distributions on the interval $[0,1]$ of volume stretch and momentum:

$$\boldsymbol{\zeta}_N(dx,dt) := \frac{1}{N}\sum_{i=1}^{N} \delta\left(x - \frac{i}{N}\right) \mathbf{u}_i(t)dx\ dt. \tag{2.30}$$

We expect that the measures $\boldsymbol{\zeta}_N(dx,dt)$ converge, as $N \to \infty$ to an absolutely continuous measure with density $(r(t,x), p(t,x))$, satisfying the following system of conservation laws:

$$\begin{cases} \partial_t r(t,x) - \partial_x p(t,x) = 0 \\ \partial_t p(t,x) - \partial_x \tau(r(t,x)) = 0 \end{cases}, \qquad \begin{matrix} p(t,0) = 0, & \tau(r(t,1)) = \bar{\tau}(t) \\ r(0,x) = r_0(x) & p(0,x) = p_0(x) \end{matrix}. \tag{2.31}$$

Since (2.31) is a hyperbolic system of nonlinear partial differential equations, its solutions may develop shocks in a finite time, even if smooth initial data are given. Therefore, we shall look for *weak* solutions, which are defined even if discontinuities appear.

Definition 2.2. *Fix $T > 0$ and let $Q_T = [0,T] \times [0,1]$. We say that $(r,p) \in L^2(Q_T)$ is a* weak solution *of* (2.31) *provided*

$$\int_0^1 \varphi(0,x) r_0(x) dx + \int_0^T \int_0^1 \left(r\partial_t\varphi - p\partial_x\varphi\right) dxdt = 0 \tag{2.32}$$

$$\int_0^1 \psi(0,x) p_0(x) dx + \int_0^T \int_0^1 \left(p\partial_t\psi - \tau(r)\partial_x\psi\right) dxdt + \int_0^T \psi(t,1)\bar{\tau}(t)dt = 0 \tag{2.33}$$

for all functions $\varphi, \psi \in C^2(Q_T)$ such that $\varphi(t,1) = \psi(t,0) = 0$ for all $t \in [0,T]$.

Denote by $\mathfrak{Q}_N$ the probability distribution of $\boldsymbol{\zeta}_N$ on $\mathcal{M}(Q_T)^2$. Observe that $\boldsymbol{\zeta}_N \in C([0,T], \mathcal{M}([0,1])^2)$, where $\mathcal{M}([0,1])$ is the space of signed measures on $[0,1]$, endowed by the weak topology. Our aim is to show the convergence

$$\boldsymbol{\zeta}_N(J) \to \left(\int_0^T \int_0^1 J(t,x) r(t,x) dxdt, \int_0^T \int_0^1 J(t,x) p(t,x) dxdt\right), \tag{2.34}$$

where $r(t,x)$ and $p(t,x)$ satisfy (2.32)-(2.33). Since we do not have uniqueness for the solution of these equations, we need a more precise statement.

Theorem 2.3 (Main theorem)**.** *Assume that the initial distribution satisfies the entropy condition $H_N(0) \leq CN$. Then sequence $\mathfrak{Q}_N$ is compact and any limit point of $\mathfrak{Q}_N$ has support on absolutely continuous measures with densities $r(t,x)$ and $p(t,x)$ solutions of* (2.32)-(2.33) *and belonging to $L^\infty(0,T;L^2(0,1))$.*

Moreover, if the system at time $t = 0$ is at a local equilibrium, namely if

$$f_0^N(\mathbf{r},\mathbf{p}) := \prod_{i=1}^N e^{\beta\left[\left(\tau\left(r_0\left(\frac{i}{N}\right)\right)-\bar\tau(0)\right)r_i + p_0\left(\frac{i}{N}\right)p_i\right] - G\left(\tau\left(r_0\left(\frac{i}{N}\right)\right)\right)+G(\bar\tau(0))-\frac{\beta}{2}p_0\left(\frac{i}{N}\right)^2}, \tag{2.35}$$

then we have the following Clausius inequality

$$\mathbb{E}^{\mathfrak{Q}}\left(\int_0^T\int_0^1 [\mathcal{F}(t,y)-\mathcal{F}(0,y)]dy\right) \le \mathbb{E}^{\mathfrak{Q}}\left(\int_0^T W(t)dt\right), \tag{2.36}$$

where

$$\mathcal{F}(t,x) := \frac{p(t,x)^2}{2} + F(r(t,x)) \tag{2.37}$$

is the free energy and

$$W(t) := -\int_0^t \bar\tau'(s)\int_0^1 r(s,y)dy + \bar\tau(t)\int_0^1 r(t,y)dy - \bar\tau(0)\int_0^1 r_0(y)dy \tag{2.38}$$

is the work done by the tension $\bar\tau$.

Notice that in the case the total length $L(t) = \int_0^1 r(t,y)dy$ is time differentiable, the definition of work coincide with the usual one: $W(t) := \int_0^t \bar\tau(s)L'(s)ds$.

3 Some bounds from Relative entropy and Dirichlet forms

Define the Dirichlet forms

$$\begin{aligned}
\mathcal{D}_N(t) &:= \int f_t^N(-S_N\log f_t^N)d\lambda_t^N = \int \frac{1}{f_t^N}\left[\left(\frac{\partial f_t^N}{\partial p_1}\right)^2 + \sum_{i=1}^{N-1}\left(\frac{\partial f_t^N}{\partial p_{i+1}} - \frac{\partial f_t^N}{\partial p_i}\right)^2\right]d\lambda_t^N,\\
\widetilde{\mathcal{D}}_N(t) &:= \int f_t^N(-\tilde S_N^{\bar\tau(t)}\log f_t^N)d\lambda_t^N = \int \frac{1}{f_t^N}\left[\sum_{i=1}^{N-1}\left(\frac{\partial f_t^N}{\partial r_{i+1}} - \frac{\partial f_t^N}{\partial r_i}\right)^2 + \left(\frac{\partial f_t^N}{\partial r_N}\right)^2\right]d\lambda_t^N.
\end{aligned} \tag{3.1}$$

Proposition 3.1. *The following inequality holds for any $t \ge 0$:*

$$H_N(t) - H_N(0) \le -\beta\int_0^t \bar\tau'(s)\int\sum_{i=1}^N r_i(s)f_s^N d\lambda_s^N + N\beta\int_0^t \bar\tau'(s)\ell(\bar\tau(s))ds. \tag{3.2}$$

Moreover, there is $C(t)$ independent of N such that

$$H_N(t) + N\sigma\beta^{-1}\int_0^t \left(\mathcal{D}_N(s) + \tilde{\mathcal{D}}_N(s)\right)ds \le C(t)N. \tag{3.3}$$

Proof.

$$\frac{dH_N(t)}{dt} = \int \partial_t(f_t^N \lambda_t^N) \log f_t^N d\mathbf{r}d\mathbf{p} + \int \lambda_t^N \partial_t f_t^N d\mathbf{r}d\mathbf{p} \tag{3.4}$$

$$= \int \left(\mathcal{G}_N^{\bar\tau(t),*} f_t^N\right) \log f_t^N d\lambda_t^N - \int f_t^N \partial_t \lambda_t^N d\mathbf{r}d\mathbf{p} + \partial_t \int f_t d\lambda_t^N \tag{3.5}$$

$$= \int f_t^N \mathcal{G}_N^{\bar\tau(t)} \log f_t^N d\lambda_t^N - \beta\bar\tau'(t) \int \left[q_N - N\ell(\bar\tau(t))\right] f_t^N d\lambda_t^N \tag{3.6}$$

$$= - N\sigma\beta^{-1} \left(\mathcal{D}_N(t) + \tilde{\mathcal{D}}_N(t)\right) - \beta\bar\tau'(t) \int \left[q_N - N\ell(\bar\tau(t))\right] f_t^N d\lambda_t^N, \tag{3.7}$$

since $\int f_t^N L_N^{\bar\tau(t)} \log f_t^N d\lambda_t^N = 0$, by the antisymmetry of $L_N^{\bar\tau(t)}$. Thus, (3.2) follows after an integration in time and recalling that $q_N = \sum_{i=1}^N r_i$ and that the Dirichlet forms are non-negative.

By the entropy inequality and the strict convexity of $G(\tau)$ we have, for any $\alpha > 0$,

$$\begin{aligned} -\beta\bar\tau'(t) \int \left[q_N - N\ell(\bar\tau(t))\right] f_t^N d\lambda_t^N &\le \alpha^{-1} H_N(t) + \alpha^{-1} \log \int e^{-\alpha\beta\bar\tau'(t)[q_N - N\ell(\bar\tau(t))]} d\lambda_t^N \\ &= \alpha^{-1} H_N(t) + \alpha^{-1} N \left[G(\bar\tau(t) - \alpha\bar\tau'(t)) - G(\bar\tau(t)) + \alpha\bar\tau'(t)\ell(\bar\tau'(t))\right] \\ &\le \alpha^{-1} H_N(t) + \alpha\bar\tau'(t)^2 N C_G. \end{aligned} \tag{3.8}$$

By choosing $\alpha = |\bar\tau'(t)|^{-1}$ we obtain the bound

$$\frac{d}{dt} H_N(t) + N\sigma\beta^{-1} \left(\mathcal{D}_N(t) + \tilde{\mathcal{D}}_N(t)\right) \le |\bar\tau'(t)| \left(H_N(t) + NC_G\right). \tag{3.9}$$

By Gronwall's inequality we get

$$\begin{aligned} H_N(t) + N\sigma\beta^{-1} \int_0^t \left(\mathcal{D}_N(s) + \tilde{\mathcal{D}}_N(s)\right) ds &\le H_N(0) e^{\int_0^t |\tau'(s)| ds} + NC_G \int_0^t |\tau'(s)| e^{\int_s^t |\tau'(u)| du} ds \\ &\le C(t)N. \end{aligned} \tag{3.10}$$

□

Observe that $C(t)$ in this proposition is equal to C_0 if $\bar\tau'(t) = 0$, and that can be chosen independent of t if $\bar\tau'(t) = 0$ for $t > t_0$ for some t_0.

The energy bound is a standard consequence of the bound on the relative entropy.

Proposition 3.2 (Energy estimate)**.** *For any $t \ge 0$*

$$\int \left[\sum_{i=1}^N \left(\frac{p_i^2}{2} + V(r_i)\right)\right] f_t^N d\lambda_t^N \le C(t)N, \tag{3.11}$$

and the constant $C(t)$ can be chosen independent of t in the case $\bar\tau'(t) = 0$ for $t \ge t_0$.

Proof. By the entropy inequality and for $0 < \alpha < \beta$,

$$\int \left[\sum_{i=1}^{N}\left(\frac{p_i^2}{2}+V(r_i)\right)\right] f_t d\lambda_t \le \frac{1}{\alpha}H_N(t)+\frac{1}{\alpha}\log\int e^{\alpha\sum_{i=1}^{N}\left(\frac{p_i^2}{2}+V(r_i)\right)}d\lambda_t^N \tag{3.12}$$
$$= \frac{1}{\alpha}H_N(t)+\frac{N}{\alpha}\log\int e^{(\alpha-\beta)\left(\frac{p_1^2}{2}+V(r_1)\right)+\beta\bar\tau(t)r_1-G(\bar\tau(t))}dr_1\frac{dp_1}{\sqrt{2\pi\beta^{-1}}}.$$

Note that thanks to our choice of α the last integral is convergent and bounded with respect to t. Thus, the conclusion follows as a consequence of Proposition 3.1. □

4 The hydrodynamic limit

4.1 Microscopic solutions

Let $(r_i(t),p_i(t))_{i=1}^N$ be solutions of (2.26). Let $\varphi,\psi\in C^2(Q_T)$ be such that $\varphi(t,1)=\psi(t,0)=0$ for all $t\in[0,T]$, and let

$$\varphi_i(t):=\varphi\left(t,\frac{i}{N}\right),\quad \psi_i(t):=\psi\left(t,\frac{i}{N}\right),\qquad i=1,\dots,N. \tag{4.1}$$

We set $V_i':=V'(r_i)$ and evaluate

$$\frac{1}{N}\sum_{i=1}^{N}\varphi_i(T)r_i(T)-\frac{1}{N}\sum_{i=1}^{N}\varphi_i(0)r_i(0)=\int_0^T\frac{1}{N}\sum_{i=1}^{N}\dot\varphi_i(t)r_i(t)dt+ \tag{4.2}$$
$$+\int_0^T\left(p_1\varphi_1-\sum_{i=2}^{N-1}\nabla^*p_i\varphi_i+\nabla^*p_N\varphi_N\right)dt+$$
$$+\sigma\int_0^T\left(\varphi_1\nabla V_1'+\sum_{i=2}^{N-1}\varphi_i\Delta V_i'+\varphi_N(\bar\tau(t)+V_{N-1}'-2V_N')\right)dt+$$
$$+\sqrt{2\beta^{-1}\frac{\sigma}{N}}\int_0^T\left(-\varphi_1d\tilde w_1+\sum_{i=2}^{N-1}\varphi_i\nabla^*d\tilde w_i+\varphi_N\nabla^*d\tilde w_N\right).$$

We use the summation by parts formula

$$\sum_{i=2}^{N-1}\varphi_i\nabla^*a_i=\sum_{i=1}^{N-1}a_i\nabla\varphi_i+\varphi_1a_1-\varphi_Na_{N-1} \tag{4.3}$$

with $a_i = p_i$, $a_i = V'_{i+1} - V'_i$ and $a_i = d\tilde{w}_i$ in order to obtain

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\varphi_i(T)r_i(T) - \frac{1}{N}\sum_{i=1}^{N}\varphi_i(0)r_i(0) \\
&= \int_0^T \frac{1}{N}\sum_{i=1}^{N}\dot{\varphi}_i(t)r_i(t)dt - \int_0^T \sum_{i=1}^{N-1}\nabla p_i\varphi_i dt + \int_0^T \varphi_N p_N dt + \qquad (4.4)\\
&- \sigma\int_0^T \sum_{i=1}^{N-1}\nabla\varphi_i\nabla V'_i dt + \sigma\int_0^T \varphi_N(\bar{\tau}(t) - V'_N)dt + \\
&+ \sqrt{2\beta^{-1}\frac{\sigma}{N}}\int_0^T \sum_{i=1}^{N-1}\nabla\varphi_i d\tilde{w}_i \\
&= \int_0^T \frac{1}{N}\sum_{i=1}^{N}\dot{\varphi}_i(t)r_i(t)dt - \int_0^T \sum_{i=1}^{N-1}\nabla\varphi_i p_i dt - \sigma\int_0^T \sum_{i=1}^{N-1}\nabla\varphi_i\nabla V'_i dt + \qquad (4.5)\\
&+ \sqrt{2\beta^{-1}\frac{\sigma}{N}}\int_0^T \sum_{i=1}^{N-1}\nabla\varphi_i d\tilde{w}_i,
\end{aligned}$$

as $\varphi_N(t) = \varphi(t,1) = 0$. After a second summation by parts we obtain

$$-\sum_{i=1}^{N-1}\nabla\varphi_i\nabla V'_i = \sum_{i=1}^{N-1} V'_i\Delta\varphi_i + V'_1\nabla\varphi_0 - V'_N\nabla\varphi_{N-1} \qquad (4.6)$$

so that we can write

$$\frac{1}{N}\sum_{i=1}^{N}\varphi_i(T)r_i(T) - \frac{1}{N}\sum_{i=1}^{N}\varphi_i(0)r_i(0) = \int_0^T \frac{1}{N}\sum_{i=1}^{N}\dot{\varphi}_i(t)r_i(t)dt - \int_0^T \sum_{i=1}^{N-1} p_i\nabla\varphi_i dt + \tilde{R}_N, \qquad (4.7)$$

with

$$\begin{aligned}
\tilde{R}_N =& \sigma\int_0^T \sum_{i=1}^{N-1} V'_i(t)\Delta\varphi_i dt + \sqrt{2\beta^{-1}\frac{\sigma}{N}}\int_0^T \sum_{i=1}^{N-1}\nabla\varphi_i(t)d\tilde{w}_i(t) \qquad (4.8)\\
&+ \sigma\int_0^T V'_1(t)\nabla\varphi_0 dt - \sigma\int_0^T V'_N(t)\nabla\varphi_{N-1}dt.
\end{aligned}$$

Lemma 4.1. $\mathbb{E}[|\tilde{R}_N|^2] \to 0$.

Proof. Since $\varphi \in C^2([0,1])$, we can estimate $|\Delta\varphi_i| \leq \|\partial^2_{xx}\varphi\|_\infty N^{-2}$. Moreover, since $c_1 \leq V''(r) \leq c_2$ we have $|V'(r)|^2 \leq C\,(1+r^2)$, and

$$\mathbb{E}\left[\left|\int_0^T \sigma \sum_{i=1}^{N-1} V_i' \Delta\varphi_i dt\right|^2\right] \leq \sigma^2 T \left(\sum_{i=1}^{N-1} (\Delta\varphi_i)^2\right) \int_0^T \mathbb{E}\left[\sum_{i=1}^{N-1} (V_i'(t))^2\right] dt \tag{4.9}$$
$$\leq C\frac{\sigma^2 T}{N^3} \int_0^T \mathbb{E}\left[\sum_{i=1}^{N-1} \left(1 + r_i^2(t)\right)\right] dt \leq C(T)\frac{\sigma^2}{N^2}.$$

Since the Brownian motions $d\tilde{w}_i$ are independent, we evaluate

$$\mathbb{E}\left[\left|\sqrt{\frac{2\sigma}{\beta N}} \int_0^T \sum_{i=1}^{N-1} \nabla\varphi_i d\tilde{w}_i\right|^2\right] \leq \frac{2\sigma}{\beta N} \int_0^T \sum_{i=1}^{N-1} (\nabla\varphi_i)^2 dt \leq \frac{CT\sigma}{N^2}$$

In order to evaluate the boundary terms in (4.8), we estimate, for any $i = 1$ and $i = N$,

$$\int (V_i')^2 f_t^N d\lambda_t^N = \int \bar{\tau}(t) V_i' f_t^N d\lambda_t^N + \int (V_i' - \bar{\tau}(t)) V_i' f_t^N d\lambda_t^N \tag{4.10}$$
$$= \int \bar{\tau}(t) V_i' f_t^N d\lambda_t^N + \int V''(r_i) f_t^N d\lambda_t^N + \int V_i' \frac{\partial f_t^N}{\partial r_i} d\lambda_t^N$$
$$\leq C + \alpha \int (V_i')^2 f_t^N d\lambda_t^N + \int \left(\frac{\partial f_t^N}{\partial r_i}\right)^2 \frac{1}{f_t^N} d\lambda_t^N,$$

where we have used Cauchy-Schwartz twice and used the boundedness of $\bar{\tau}(t)$ and V''. Here we can choose $C > 0$ and $0 < \alpha < 1$ such that do not depend on t or N, so that we have

$$\int (V_i')^2 f_t^N d\lambda_t^N \leq C' \left(1 + \int \left(\frac{\partial f_t^N}{\partial r_i}\right)^2 \frac{1}{f_t^N} d\lambda_t^N\right)$$

For $i = N$ this gives directly $(1-\alpha)\int (V_i')^2 f_t^N d\lambda_t^N \leq C'\left(1 + \tilde{\mathcal{D}}_N(t)\right)$. For $i = 1$, by writing

$$\left(\frac{\partial f_t^N}{\partial r_1}\right)^2 = \left(-\sum_{j=1}^{N-1} \left(\frac{\partial f_t^N}{\partial r_{j+1}} - \frac{\partial f_t^N}{\partial r_j}\right) + \frac{\partial f_t^N}{\partial r_N}\right)^2 \tag{4.11}$$
$$\leq N \left(\sum_{j=1}^{N-1} \left(\frac{\partial f_t^N}{\partial r_{j+1}} - \frac{\partial f_t^N}{\partial r_j}\right)^2 + \left(\frac{\partial f_t^N}{\partial r_N}\right)^2\right)$$

we obtain

$$(1-\alpha) \int (V_1')^2 f_t^N d\lambda_t^N \leq C + N\tilde{\mathcal{D}}_N(t), \tag{4.12}$$

and, in turn,

$$\int_0^T \int (V_1')^2 f_t^N d\lambda_t^N \leq C(T)\left(1+\frac{N}{\sigma}\right) \leq C'(T)\frac{N}{\sigma}. \tag{4.13}$$

This allows us to estimate

$$\mathbb{E}\left[\left(\sigma\int_0^T V_1'(t)\nabla\varphi_0 dt\right)^2\right] \leq T\frac{\sigma^2}{N^2}\mathbb{E}\left[\int_0^T (V_1'(t))^2 dt\right] \leq C_T\frac{\sigma}{N}, \tag{4.14}$$

which vanishes as $N\to\infty$. In a similar way we estimate the boundary term involving V_N'.

□

Thus, we have obtained the following

Proposition 4.2. *Let $\varphi\in C^2(Q_T)$ such that $\varphi(t,1)=0$ for all $t\in[0,T]$. Then*

$$\frac{1}{N}\sum_{i=1}^N \varphi_i(T)r_i(T) - \frac{1}{N}\sum_{i=1}^N \varphi_i(0)r_i(0) - \int_0^T \frac{1}{N}\sum_{i=1}^N \dot\varphi_i(t)r_i(t)dt + \int_0^T \sum_{i=1}^{N-1} p_i\nabla\varphi_i dt \to 0 \tag{4.15}$$

in probability as $N\to\infty$

From similar calculations and recalling that $\psi_0=0$, we evaluate

$$\begin{aligned}&\frac{1}{N}\sum_{i=1}^N \psi_i(T)p_i(T) - \frac{1}{N}\sum_{i=1}^N \psi_i(0)p_i(0) \\ &= \int_0^T \frac{1}{N}\sum_{i=1}^N \dot\psi_i(t)p_i(t)dt - \sum_{i=1}^{N-1} V_i'\nabla\psi_i dt + \int_0^T \psi_N\bar\tau(t)dt + R_N,\end{aligned} \tag{4.16}$$

where

$$\begin{aligned}R_N =& \sigma\int_0^T \sum_{i=1}^{N-1} p_i\Delta\psi_i dt + \sqrt{2\beta^{-1}\frac{\sigma}{N}}\int_0^T \sum_{i=1}^{N-1}\nabla\psi_i dw_i + \\ &- \sigma\int_0^T p_N\nabla\psi_{N-1}dt + \sqrt{2\beta^{-1}\frac{\sigma}{N}}\int_0^T \varphi_1 dw_0,\end{aligned} \tag{4.17}$$

Similarly to Lemma 4.1, we prove the following

Lemma 4.3. $\mathbb{E}[|R_N|^2]\to 0$.

Thus, we have proved

Proposition 4.4. *Let $\psi \in C^2(Q_T)$ such that $\psi(t,0) = 0$ for all $t \in [0,T]$. Then*

$$\frac{1}{N}\sum_{i=1}^{N}\psi_i(T)p_i(T) - \frac{1}{N}\sum_{i=1}^{N}\psi_i(0)p_i(0) - \int_0^T \frac{1}{N}\sum_{i=1}^{N}\dot{\psi}_i(t)p_i(t)dt + \tag{4.18}$$
$$+ \int_0^T \sum_{i=1}^{N-1} V_i'\nabla\psi_i dt + \int_0^T \psi(t,1)\bar{\tau}(t)dt \to 0$$

in probability as $N \to \infty$.

4.2 Mesoscopic solutions

For any sequence $(a_i)_{i\in\mathbb{N}}$ and any $l \in \mathbb{N}$, smoother block averages are defined as

$$\hat{a}_{l,i} := \frac{1}{l}\sum_{|j|<l}\frac{l-|j|}{l}a_{i-j}, \quad i \geq l, \tag{4.19}$$

where we will choose

$$l = l(N) := \left[N^{\frac{1}{4}}\sigma(N)^{\frac{1}{2}}\right]. \tag{4.20}$$

Remark. Since σ/N and N/σ^2 vanish as $N \to \infty$, we may choose $\sigma = N^{\frac{1}{2}+\alpha}$, with $\alpha \in (0, 1/2)$. This means l is of order $N^{\frac{1}{2}+\frac{\alpha}{2}}$.

Let $1_{N,i}(x)$ be the indicator function of the interval $\left[\frac{i}{N} - \frac{1}{2N}, \frac{i}{N} + \frac{1}{2N}\right]$, then we define functions on $[0,1]$ from the sequence $(a_i)_{i\in\mathbb{N}}$ as

$$\hat{a}_N(x) := \sum_{i=l+1}^{N-l} 1_{N,i}(x)\hat{a}_{l,i}, \qquad x \in [0,1]. \tag{4.21}$$

Notice that for any function f on $\mathbb{R}$ and $\varphi(x) \in L^1([0,1])$ we have

$$\int_0^1 f(\hat{a}_N(x))\varphi(x)dx = \frac{1}{N}\sum_{i=l+1}^{N-l} f(\hat{a}_{l,i})\bar{\varphi}_i + \tag{4.22}$$
$$+ f(0)\left(\int_0^{l/N+1/(2N)} \varphi(x)dx + \int_{1-l/N-1/(2N)}^1 \varphi(x)dx\right)$$

where

$$\bar{\varphi}_i = N\int_0^1 \varphi(x)1_{N,i}(x)dx = N\int_{i/N-1/(2N)}^{i/N+1/(2N)} \varphi(x)dx, \qquad i = 1,\dots,N-1. \tag{4.23}$$

We have, by Cauchy-Schwarz inequality,

$$\int_0^1 \hat{a}_N(x)^2 dx \leq \frac{1}{N}\sum_{k=2}^{N-1} a_k^2$$

Recalling that $\mathbf{u}_i = (r_i, p_i)$, we define

$$\hat{\mathbf{u}}_N(t,x) = \sum_{i=l+1}^{N-l} 1_{N,i}(x)\hat{\mathbf{u}}_{l,i}(t), \qquad (t,x) \in Q_T. \tag{4.24}$$

As a consequence of (4.24) and the energy estimate given by Proposition 3.2, we have that

$$\mathbb{E}\left(\int_{Q_T} |\hat{\mathbf{u}}_N(t,x)|^2\right) \leq C_T, \tag{4.25}$$

i.e. almost surely $\hat{\mathbf{u}}_N(t,x)$ is uniformly bounded in $L^2(Q_T)$ and is therefore weakly convergent, up to a subsequence. The following proposition ensures us that $\boldsymbol{\zeta}_N(dx,dt)$ has the same weak limit points as $\hat{\mathbf{u}}_N(t,x)$.

Proposition 4.5. *For any function $\varphi \in C^1([0,1])$ we have that*

$$\lim_{N\to\infty} \mathbb{E}\left(\left|\int_0^1 \hat{\mathbf{u}}_N(t,x)\varphi(x)dx - \frac{1}{N}\sum_{i=1}^{N} \mathbf{u}_i(t)\varphi\left(\frac{i}{N}\right)\right|\right) = 0. \tag{4.26}$$

Proof. By (4.22) with $f(\hat{\mathbf{u}}_N) = \hat{\mathbf{u}}_N$ we have

$$\int_0^1 \hat{\mathbf{u}}_N(t,x)\varphi(x)dx = \frac{1}{N}\sum_{i=l+1}^{N-l} \hat{\mathbf{u}}_{l,i}\bar{\varphi}_i. \tag{4.27}$$

Next, we note that we can neglect the first and the last l points at the boundaries. Namely, we have

$$\left|\frac{1}{N}\sum_{i=1}^{l} \mathbf{u}_i(t)\varphi\left(\frac{i}{N}\right)\right| \leq \left(\frac{1}{l}\sum_{i=1}^{l}\varphi\left(\frac{i}{N}\right)^2\right)^{1/2}\left(\frac{1}{l}\sum_{i=1}^{l}\frac{l^2}{N^2}|\mathbf{u}_i(t)|^2\right)^{1/2} \tag{4.28}$$

$$\leq C\sqrt{\frac{l}{N}}\left(\frac{1}{N}\sum_{i=1}^{N}|\mathbf{u}_i(t)|^2\right)^{1/2}, \tag{4.29}$$

so that

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{l} \mathbf{u}_i(t)\varphi\left(\frac{i}{N}\right)\right|\right] \to 0. \tag{4.30}$$

Similarly, we have

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=N-l+1}^{N}\mathbf{u}_i(t)\varphi\left(\frac{i}{N}\right)\right|\right]\to 0. \tag{4.31}$$

Therefore, we evaluate

$$\begin{aligned}\frac{1}{N}\sum_{i=l+1}^{N-l}\hat{\mathbf{u}}_{l,i}\bar{\varphi}_i-\frac{1}{N}\sum_{i=l+1}^{N-l}\mathbf{u}_i\varphi\left(\frac{i}{N}\right)&=\frac{1}{N}\sum_{i=l+1}^{N-l}(\hat{\mathbf{u}}_{l,i}-\mathbf{u}_i)\varphi\left(\frac{i}{N}\right)+\\&+\frac{1}{N}\sum_{i=l+1}^{N-l}\hat{\mathbf{u}}_{l,i}\left(\bar{\varphi}_i-\varphi\left(\frac{i}{N}\right)\right).\end{aligned} \tag{4.32}$$

The last summation is estimated by noting that there is a point $\xi_i\in\left[\frac{i}{N}-\frac{1}{2N},\frac{i}{N}+\frac{1}{2N}\right]$ such that $\bar{\varphi}_i=\varphi(\xi_i)$. Thus,

$$\left|\bar{\varphi}_i-\varphi\left(\frac{i}{N}\right)\right|\le\|\varphi'\|_\infty\left|\xi_i-\frac{i}{N}\right|\le\frac{\|\varphi'\|_\infty}{2N} \tag{4.33}$$

and therefore

$$\begin{aligned}\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=l+1}^{N-l}\hat{\mathbf{u}}_{l,i}\left(\bar{\varphi}_i-\varphi\left(\frac{i}{N}\right)\right)\right|\right]&\le\frac{C}{N}\left(\mathbb{E}\left[\frac{1}{N}\sum_{i=l+1}^{N-l}|\hat{\mathbf{u}}_{l,i}|^2\right]\right)^{1/2}\\&\le\frac{C}{N}\left(\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}|\mathbf{u}_i|^2\right]\right)^{1/2}\le\frac{C}{N}.\end{aligned} \tag{4.34}$$

Finally, defining $c_j=\dfrac{l-|j|}{l^2}$ and recalling that $\sum_{|j|<l}c_j=1$, we write perform a change of variables and write

$$\begin{aligned}\frac{1}{N}\sum_{i=l+1}^{N-l}(\hat{\mathbf{u}}_{l,i}-\mathbf{u}_i)\varphi\left(\frac{i}{N}\right)&=\sum_{|j|<l}c_j\left[\frac{1}{N}\sum_{i=l+1-j}^{N-l-j}\mathbf{u}_i\varphi\left(\frac{i+j}{N}\right)-\frac{1}{N}\sum_{i=l+1}^{N-l}\mathbf{u}_i\varphi\left(\frac{i}{N}\right)\right]\\&=\sum_{|j|<l}c_j\frac{1}{N}\sum_{i=l+1}^{N-l}\mathbf{u}_i\left[\varphi\left(\frac{i+j}{N}\right)-\varphi\left(\frac{i}{N}\right)\right]+\mathcal{O}\left(\sqrt{\frac{l}{N}}\right).\end{aligned} \tag{4.35}$$

The conclusion then follows similarly to (4.34). □

The proposition allows us to replace each $\mathbf{u}_i$ with their average $\hat{\mathbf{u}}_{l,i}$. In the same way we can replace $V'(r_i)$ by the average

$$\hat{V}'_{l,i} := \frac{1}{l}\sum_{|j|<l} \frac{l-|j|}{l} V'(r_{i-j}) \tag{4.36}$$

and then replace $\hat{V}'_{l,i}$ by $\tau(\hat{r}_{l,i})$ via the following proposition, which we shall prove in Section A.1.

Proposition 4.6 (One-block estimate)**.**

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\sum_{i=l+1}^{N-l}\int_0^T \left(\hat{V}'_{l,i} - \tau(\hat{r}_{l,i})\right)^2 dt\right] = 0. \tag{4.37}$$

Therefore, combining Propositions 4.2, 4.4, 4.5 and 4.6, we obtain the following

Proposition 4.7. *Let $\varphi, \psi \in C^2(Q_T)$ such that $\varphi(t,1) = \psi(t,0) = 0$ for all $t \in [0,T]$ and let $\mathbf{u}_N(t,x) = (r_N(t,x), p_N(t,x))$. Then, the following convergences happen in probability as $N \to \infty$*

$$\begin{aligned}\int_0^1 (\varphi(T,x)\hat{r}_N(T,x) - \varphi(0,x)\hat{r}_N(0,x))\,dx + \\ - \int_{Q_T} (\hat{r}_N(t,x)\partial_t\varphi(t,x) + \hat{p}_N(t,x)\partial_x\varphi(t,x))\,dxdt \to 0\end{aligned} \tag{4.38}$$

and

$$\begin{aligned}\int_0^1 (\psi(T,x)\hat{p}_N(T,x) - \psi(0,x)\hat{p}_N(0,x))\,dx + \\ - \int_{Q_T} (\hat{p}_N(t,x)\partial_t\psi(t,x) + \tau(\hat{r}_N(t,x))\partial_x\psi(t,x))\,dxdt + \int_0^T \psi(t,1)\bar{\tau}(t)dt \to 0\end{aligned} \tag{4.39}$$

4.3 Random Young Measures and Weak Convergence

The purpose of this section is to prove that any weakly convergent subsequence will converge strongly. We use the compensated compactness argument of Fritz [3], inspired from the work of Di Perna, Serre and Shearer, properly adapted to the presence of boundaries [6].

Denote by $\hat{\nu}^N_{t,x} = \delta_{\hat{\mathbf{u}}_N(t,x)}$ the random Young measure on $\mathbb{R}^2$ associated to the empirical process $\hat{\mathbf{u}}_N(t,x)$:

$$\int_{\mathbb{R}^2} f(\mathbf{y}) d\hat{\nu}^N_{t,x}(\mathbf{y}) = f(\hat{\mathbf{u}}_N(t,x)) \tag{4.40}$$

for any $f : \mathbb{R}^2 \to \mathbb{R}$. Since $\hat{\mathbf{u}}_N \in L^2(\Omega \times Q_T)$, we say that $\hat{\nu}^N_{t,x}$ is a L^2-random Dirac mass. The following

$$\mathbb{E}\left[\int_{Q_T}\int_{\mathbb{R}^2} |\mathbf{y}|^2 d\hat{\nu}^N_{t,x}(\mathbf{y})dxdt\right] = \mathbb{E}\left[\int_{Q_T} |\hat{\mathbf{u}}_N(t,x)|^2 dxdt\right] \leq C_T \tag{4.41}$$

with C_T independent of N, implies that there exists a subsequence of random Young measures $(\hat{\nu}^{N_n}_{t,x})$ and a subsequence of real random variables $(\|\hat{\mathbf{u}}_{N_n}\|_{L^2(Q_T)})$ that converges in law.

We can now apply the Skorohod's representation theorem to the laws of $(\hat{\nu}^{N_n}_{t,x}, \|\hat{\mathbf{u}}_{N_n}\|_{L^2(Q_T)})$ and find a common probability space such that the convergence happens almost surely. This proves the following proposition:

Proposition 4.8. *There exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, random Young measures $\tilde{\nu}^n_{t,x}, \tilde{\nu}_{t,x}$ and real random variables a_n, a such that $\tilde{\nu}^n_{t,x}$ has the same law of $\hat{\nu}^{N_n}_{t,x}$, a_n has the same law of $\|\hat{\mathbf{u}}_{N_n}\|_{L^2(Q_T)}$ and $\tilde{\nu}^n_{t,x} \overset{*}{\rightharpoonup} \tilde{\nu}_{t,x}$, $a_n \to a$, $\tilde{\mathbb{P}}$-almost surely.*

Remark. Since $\hat{\nu}^{N_n}_{t,x}$ is a random Dirac mass and $\tilde{\nu}^n_{t,x}$ and $\hat{\nu}^{N_n}_{t,x}$ have the same law, $\tilde{\nu}^n_{t,x}$ is a L^2-random Dirac mass, too: $\tilde{\nu}^n_{t,x} = \delta_{\tilde{\mathbf{u}}_n(t,x)}$ for some $\tilde{\mathbf{u}}_n \in L^2(\tilde{\Omega} \times Q_T)$. $\tilde{\mathbf{u}}_n$ and $\hat{\mathbf{u}}_{N_n}$ have the same law. Since $a_n \to a$ almost surely, we have that (a_n) is bounded and so $\tilde{\mathbf{u}}_n$ is uniformly bounded in $L^2(Q_T)$ with $\tilde{\mathbb{P}}$-probability 1.

Since from a uniformly bounded sequence in L^p we can extract a weakly convergent subsequence, we obtain the following proposition:

Proposition 4.9. *There exist $L^2(Q_T)$-valued random variables $(\tilde{\mathbf{u}}_n), \tilde{\mathbf{u}}$ such that $\tilde{\mathbf{u}}_n$ and $\hat{\mathbf{u}}_{N_n}$ have the same law and, $\tilde{\mathbb{P}}$-almost surely and up to a subsequence, $\tilde{\mathbf{u}}_n \rightharpoonup \tilde{\mathbf{u}}$ in $L^2(Q_T)$.*

The condition $\tilde{\nu}^n_{t,x} \overset{*}{\rightharpoonup} \tilde{\nu}_{t,x}$ in Proposition 4.8 reads

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} f(\mathbf{y})d\tilde{\nu}^n_{t,x}(\mathbf{y})dxdt = \int_{\mathbb{R}^2} f(\mathbf{y})d\tilde{\nu}_{t,x}(\mathbf{y})dxdt \tag{4.42}$$

for all continuous and *bounded* $f : \mathbb{R}^2 \to \mathbb{R}$. By a simple adaptation of Proposition 4.2 of [1], (4.42) can be extended to a functiion $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(\mathbf{y})/|\mathbf{y}|^2 \to 0$ as $|\mathbf{y}| \to +\infty$.

Because of Proposition 4.7 we are interested in the weak limit of $\tau(\hat{r}_N(t,x))$. Since τ is linearly bounded, the main Theorem 2.3 is proved once we show that $\tilde{\nu}_{t,x} = \delta_{\tilde{\mathbf{u}}(t,x)}$, almost surely and for almost all $(t,x) \in Q_T$.

We shall now prove that the support of $\tilde{\nu}_{t,x}$ is almost surely and almost everywhere a point. The result will then follow from the lemma:

Lemma 4.10. *$\tilde{\nu}_{t,x} = \delta_{\tilde{\mathbf{u}}(t,x)}$ almost surely and for almost all $(t,x) \in Q_T$ if and only if the support of $\tilde{\nu}_{t,x}$ is a point for almost all $(t,x) \in Q_T$. In this case, $\tilde{\mathbf{u}}_n \to \tilde{\mathbf{u}}$ in $L^p(Q_T)^2$-strong for all $1 \leq p < 2$.*

Proof. Suppose there is a measurable function $\mathbf{u}^* : Q_T \to \mathbb{R}^2$ such that $\tilde{\nu}_{t,x} = \delta_{\mathbf{u}^*(t,x)}$ for almost all $(t,x) \in Q_T$. For any test function $\mathbf{J} : Q_T \to \mathbb{R}^2$ consider the quantity

$$\int_{Q_T} \mathbf{J}(t,x) \cdot \tilde{\mathbf{u}}_n(t,x) dx dt = \int_{Q_T} \int_{\mathbb{R}^2} \mathbf{J}(t,x) \cdot \mathbf{y} d\tilde{\nu}^n_{t,x}(\mathbf{y}) dx dt. \tag{4.43}$$

By taking the limit for $n \to \infty$ in the sense of L^2-weak first and in the sense of (4.42) then, we obtain

$$\int_{Q_T} \int_{\mathbb{R}^2} \mathbf{J}(t,x) \cdot \tilde{\mathbf{u}}(t,x) dx dt = \int_{Q_T} \int_{\mathbb{R}^2} \mathbf{J}(t,x) \cdot \mathbf{y} d\tilde{\nu}_{t,x}(\mathbf{y}) dx dt = \int_{Q_T} \int_{\mathbb{R}^2} \mathbf{J}(t,x) \cdot \mathbf{u}^*(t,x) dx dt \tag{4.44}$$

almost surely. Then $\tilde{\mathbf{u}}(t,x) = \mathbf{u}^*(t,x)$ for almost all $(t,x) \in Q_T$ follows from the fact that $\mathbf{J}$ was arbitrary.

Next, fix $1 < p < 2$. Taking $f(\mathbf{y}) = |\mathbf{y}|^p$ in (4.42) gives $\|\tilde{\mathbf{u}}_n\|_{L^p(Q_T)} \to \|\tilde{\mathbf{u}}\|_{L^p(Q_T)}$, which, together with $\tilde{\mathbf{u}}_n \rightharpoonup \tilde{\mathbf{u}}$ in $L^p(Q_T)$ and the fact that $L^p(Q_T)$ is uniformly convex for $1 < p < \infty$ implies strong convergence.

The case $p = 1$ follows from the result for $p > 1$ and Hölder's inequality. □

4.4 Reduction of the Limit Young Measure

In this section we prove that the support of $\tilde{\nu}_{t,x}$ is almost surely and almost everywhere a point.

We recall that Lax entropy-entropy flux pair for the system

$$\begin{cases} \partial_t r(t,x) - \partial_x p(t,x) = 0 \\ \partial_t p(t,x) - \partial_x \tau(p(t,x)) = 0 \end{cases} \tag{4.45}$$

is a pair of functions $\eta, q : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\partial_t \eta(\mathbf{u}(t,x)) + \partial_x q(\mathbf{u}(t,x)) = 0 \tag{4.46}$$

for any smooth solution $\mathbf{u}(t,x) = (r(t,x), p(t,x))$ of (4.45). This is equivalent to the following:

$$\begin{cases} \partial_r \eta(r,p) + \partial_p q(r,p) = 0 \\ \tau'(r)\partial_p \eta(r,p) + \partial_r q(r,p) = 0 \end{cases}. \tag{4.47}$$

Under appropriate conditions on τ, Shearer ([9]) constructs a family of entropy-entropy flux pairs (η, q) such that η, q, their first and their second derivatives are bounded (cf also [6]). As shown in [7] and [3], our choice of the potential V ensures that the tension τ has the required properties, so the result of Shearer applies to our case.

In particular, following Section 5 of [9], we have that the support $\tilde{\nu}_{t,x}$ is almost surely and almost everywhere a point provided Tartar's commutation relation

$$\langle \eta_1 q_2 - \eta_2 q_1, \tilde{\nu}_{t,x} \rangle = \langle \eta_1, \tilde{\nu}_{t,x} \rangle \langle q_2, \tilde{\nu}_{t,x} \rangle - \langle \eta_2, \tilde{\nu}_{t,x} \rangle \langle q_1, \tilde{\nu}_{t,x} \rangle \tag{4.48}$$

holds almost surely and almost everywhere for any bounded pairs (η_1, q_1), (η_2, q_2) with bounded first and second derivatives.

Obtaining (4.48) in a deterministic setting is standard and relies on the div-curl and Murat-Tartar lemma. Both of these lemmas have a stochastic extension (cf Appendix A of [2]) and what we ultimately need to prove in order to obtain (4.48) is that the hypotheses for the stochastic Murat-Tartar lemma are satisfied (cf [1] , Proposition 5.6).

This is ensured next theorem, for which we will give a preliminary definition. Let $(\eta, q) \in C^2(\mathbb{R}^2)$ be a Lax entropy-entropy flux pair with bounded derivatives. We assume, without loss of generality, $\eta(0,0) = q(0,0) = 0$.

For $\varphi \in H_0^1(Q_T) \cap L^\infty(Q_T)$ define the corresponding entropy production functional as

$$X_N(\varphi, \eta) = -\int_{Q_T} (\eta(\hat{\mathbf{u}}_N)\partial_t \varphi + q(\hat{\mathbf{u}}_N)\partial_x \varphi)\, dxdt. \tag{4.49}$$

Theorem 4.11. *The entropy production X_N decomposes as $X_N = Y_N + Z_N$, such that (Y_N) is compact in $H^{-1}(Q_T)$ and (Z_N) is uniformly bounded as a signed measure. Namely,*

$$\mathbb{E}\left[|Y_N(\varphi, \eta)|\right] \leq a_N \|\varphi\|_{H^1(Q_T)} \quad \textit{with} \quad \lim_{N\to\infty} a_N = 0, \tag{4.50}$$

$$\mathbb{E}\left[|Z_N(\varphi, \eta)|\right] \leq b_N \|\varphi\|_{L^\infty(Q_T)} \quad \textit{with} \quad \limsup_{N\to\infty} b_N < \infty, \tag{4.51}$$

where $a_N, b_N > 0$ are independent of φ.

Definition 4.12. *We say that the random variables $Y_N(\varphi, \eta)$ are of* type Y *provided* (4.50) *holds for some a_N independent of φ. We further say that the random variables $Z_N(\varphi, \eta)$ are of* type Z *provided* (4.51) *holds for some b_N independent of φ.*

By recalling that φ vanishes on ∂Q_T, a direct calculation involving Ito formula we can integrate by parts in time and obtain

$$X_N = X_{a,N} + X_{s,N} + \tilde{X}_{s,N} + \mathcal{M}_N + \tilde{\mathcal{M}}_N + Q_N, \tag{4.52}$$

where

$$\begin{aligned} X_{a,N}(\varphi, \eta) = &\int_0^T \sum_{i=l+1}^{N-l} \bar{\varphi}_i \left[\partial_p \eta(\hat{\mathbf{u}}_{l,i}) \nabla \hat{V}'_{l,i} - \partial_r \eta(\hat{\mathbf{u}}_{l,i}) \nabla^* \hat{p}_{l,i}\right] dt + \\ &+ \int_0^T \sum_{i=l+1}^{N-l} \bar{\varphi}_i \left[\partial_r q(\hat{\mathbf{u}}_{l,i}) \nabla \hat{r}_{l,i} - \partial_p q(\hat{\mathbf{u}}_{l,i}) \nabla^* \hat{p}_{l,i}\right] dt, \end{aligned} \tag{4.53}$$

$$X_{s,N}(\varphi,\eta)=\sigma\int_0^T\sum_{i=l+1}^{N-l}\bar{\varphi}_i\partial_p\eta(\hat{\mathbf{u}}_{l,i})\Delta\hat{p}_{l,i}dt+\frac{2\sigma}{l^3}\int_0^T\sum_{i=l+1}^{N-l}\bar{\varphi}_i\partial^2_{pp}\eta(\hat{\mathbf{u}}_{l,i})dt, \tag{4.54}$$

$$\tilde{X}_{s,N}(\varphi,\eta)=\sigma\int_0^T\sum_{i=l+1}^{N-l}\bar{\varphi}_i\partial_r\eta(\hat{\mathbf{u}}_{l,i})\Delta\hat{V}'_{l,i}dt+\frac{2\sigma}{l^3}\int_0^T\sum_{i=l+1}^{N-l}\bar{\varphi}_i\partial^2_{rr}\eta(\hat{\mathbf{u}}_{l,i})dt, \tag{4.55}$$

$$M_N(\varphi,\eta)=-\sqrt{2\frac{\sigma}{N}}\int_0^T\sum_{i=l+1}^{N-l}\bar{\varphi}_i\partial_p\eta(\hat{\mathbf{u}}_{l,i})d\nabla^*\hat{w}_{l,i}, \tag{4.56}$$

$$\tilde{M}_N(\varphi,\eta)=-\sqrt{2\frac{\sigma}{N}}\int_0^T\sum_{i=l+1}^{N-l}\bar{\varphi}_i\partial_r\eta(\hat{\mathbf{u}}_{l,i})d\nabla^*\hat{\tilde{w}}_{l,i} \tag{4.57}$$

$$\begin{aligned}Q_N(\varphi,\eta)=&-\int_0^T\int_0^1\partial_x\varphi(t,x)q(\hat{\mathbf{u}}_N(t,x))dxdt \\ &-\int_0^T\sum_{i=l+1}^{N-l}\bar{\varphi}_i\left[\partial_r q(\hat{\mathbf{u}}_{l,i})\nabla\hat{r}_{l,i}-\partial_p q(\hat{\mathbf{u}}_{l,i})\nabla^*\hat{p}_{l,i}\right]dt.\end{aligned} \tag{4.58}$$

We shall prove Theorem 4.11 via a series of lemmas. We start with two preliminary ones.

Lemma 4.13. *Let $(A_i)_{i\in\mathbb{N}}$ and $(B_i)_{i\in\mathbb{N}}$ be families of $L^2(\mathbb{R})$-valued random variables such that*

$$\limsup_{N\to\infty}\left(\mathbb{E}\left[\sum_{i=1}^N\int_0^T A_i(s)^2ds\right]\mathbb{E}\left[\sum_{i=1}^N\int_0^T B_i(s)^2ds\right]\right)<\infty. \tag{4.59}$$

Let $\varphi\in L^\infty(Q_T)$ and let

$$\bar{\varphi}_i(t):=N\int_0^1 1_{N,i}(x)\varphi(t,x)dx=N\int_{\frac{i}{N}-\frac{1}{2N}}^{\frac{i}{N}+\frac{1}{2N}}\varphi(t,x)dx \tag{4.60}$$

Then

$$\mathbb{E}\left[\left|\sum_{i=1}^N\int_0^T\bar{\varphi}_iA_iB_idt\right|\right]\le b_N\|\varphi\|_{L^\infty(Q_T)}, \tag{4.61}$$

where b_N is independent of φ such that

$$\limsup_{N\to\infty}b_N<\infty. \tag{4.62}$$

Proof.

$$|\bar{\varphi}_i| = \left| N \int_0^1 \varphi(t,x) 1_{N,i}(x) dx \right| \leq \|\varphi\|_{L^\infty(Q_T)}. \tag{4.63}$$

Then, by the Cauchy-Schwarz inequality, we have

$$\mathbb{E}\left[\left| \sum_{i=1}^N \int_0^T \bar{\varphi}_i A_i B_i dt \right| \right] \leq \|\varphi\|_\infty \left(\mathbb{E}\left[\sum_{i=1}^N \int_0^T A_i^2 dt \right] \right)^{1/2} \left(\mathbb{E}\left[\sum_{i=1}^N \int_0^T B_i^2 dt \right] \right)^{1/2}. \tag{4.64}$$

□

Lemma 4.14. *Let $(A_i)_{i\in\mathbb{N}}$ be a family of $L^2(\mathbb{R})$-valued random variables such that*

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N} \sum_{i=1}^N \int_0^T A_i(s)^2 ds \right] = 0. \tag{4.65}$$

Let $\varphi \in H^1(Q_T)$ and $\bar{\varphi}_i$ as in (4.60). Then

$$\mathbb{E}\left[\sum_{i=1}^{N-1} \int_0^T |A_i (\bar{\varphi}_{i+1} - \bar{\varphi}_i)| \, dt \right] \leq a_N \|\varphi\|_{H^1(Q_T)}, \tag{4.66}$$

where a_N is independent of φ and

$$\lim_{N\to\infty} a_N = 0. \tag{4.67}$$

Proof. By Cauchy-Schwarz we have

$$\begin{aligned} \mathbb{E}\left[\sum_{i=1}^{N-1} \int_0^T |A_i (\bar{\varphi}_{i+1} - \bar{\varphi}_i)| \, dt \right] \leq & \left(\frac{1}{N} \sum_{i=1}^{N-1} \int_0^T N^2 (\bar{\varphi}_{i+1} - \bar{\varphi}_i)^2 \, dt \right)^{1/2} \times \\ & \times \left(\mathbb{E}\left[\frac{1}{N} \sum_{i=1}^{N-1} \int_0^T A_i^2 dt \right] \right)^{1/2}. \end{aligned} \tag{4.68}$$

We write

$$\begin{aligned} \bar{\varphi}_{i+1} - \bar{\varphi}_i &= N \int_0^1 1_{N,i+1}(x) \varphi(t,x) dx - N \int_0^1 1_{N,i}(x) \varphi(t,x) dx \\ &= N \int_0^1 1_{N,i}(x) \left(\varphi\left(t, x + \frac{1}{N}\right) - \varphi(t,x) \right) dx \\ &= N \int_0^1 1_{N,i}(x) \int_x^{x+\frac{1}{N}} \partial_y \varphi(t,y) dy dx \\ &= N \int_0^1 1_{N,i}(x) \int_0^1 1_{\left[x, x+\frac{1}{N}\right]}(y) \partial_y \varphi(t,y) dy dx \end{aligned} \tag{4.69}$$

Thus, Cauchy-Schwarz inequality implies

$$\begin{aligned}|\bar\varphi_{i+1}-\bar\varphi_i| &\le N\int_0^1 1_{N,i}(x)\left|\int_0^1 1_{\left[x,x+\frac{1}{N}\right]}(y)\partial_y\varphi(t,y)dy\right|dx \\ &\le N\int_0^1 1_{N,i}(x)\left(\int_0^1 1_{\left[x,x+\frac{1}{N}\right]}(y)dy\right)^{1/2}\left(\int_0^1|\partial_y\varphi(t,y)|^2dy\right)^{1/2}dx \\ &= \frac{1}{\sqrt{N}}\left(\int_0^1|\partial_y\varphi(t,y)|^2dy\right)^{1/2}\end{aligned} \tag{4.70}$$

and so

$$\begin{aligned}\int_0^T\frac{1}{N}\sum_{i=1}^{N-1}N^2\left(\bar\varphi_{i+1}-\bar\varphi_i\right)^2dt &\le \int_0^T\int_0^1|\partial_x\varphi(t,x)|^2dxdt \\ &= \|\partial_x\varphi\|^2_{L^2(Q_T)}\le\|\varphi\|^2_{H^1(Q_T)}.\end{aligned} \tag{4.71}$$

□

Remark. The same result applies if we replace $\nabla\bar\varphi_i$ by $\nabla^*\bar\varphi_i$.

In the following we shall diffusely use the following formulae, which hold for any two sequences $(a_i)_{i\in\mathbb{N}}$, $(b_i)_{i\in\mathbb{N}}$.

$$\nabla(a_ib_i) = b_{i+1}\nabla a_i + a_i\nabla b_i \tag{4.72}$$

$$\nabla^*(a_ib_i) = b_{i-1}\nabla^* a_i + a_i\nabla^* b_i \tag{4.73}$$

$$\sum_{i=l+1}^{N-l}a_i\nabla b_i = \sum_{i=l+1}^{N-l}b_i\nabla^*a_i + a_{N-l}b_{N-l+1} - a_lb_{l+1} \tag{4.74}$$

Lemma 4.15. *$X_{a,N}$ be defined in* (4.53)*. Then it decomposes as $X_{a,N}=Y_{a,N}+Z_{a,N}$ such that $Y_{a,N}$ is of type Y and $Z_{a,N}$ is of type Z in the sense of Definition* 4.12*. Moreover, $b_N\to 0$ as $N\to\infty$.*

Proof. Since (η,q) is a Lax entropy-entropy flux pair, we have

$$\begin{cases}\partial_r\eta+\partial_pq=0\\ \tau'(r)\partial_p\eta+\partial_rq=0\end{cases}, \tag{4.75}$$

so that we can write

$$X_{a,N} = \int_0^T\sum_{i=l+1}^{N-l}\bar\varphi_i\partial_p\eta(\hat u_{l,i})\left[\nabla\hat V'_{l,i}-\tau'(\hat r_{l,i})\nabla\hat r_{l,i}\right]dt \tag{4.76}$$

$$= X^1_{a,N}+Z^1_{a,N}, \tag{4.77}$$

where

$$X^1_{a,N} = \int_0^T \sum_{i=l+1}^{N-l} \bar{\varphi}_i \partial_p \eta(\hat{u}_{l,i}) \nabla \left(\hat{V}'_{l,i} - \tau(\hat{r}_{l,i}) \right) dt \tag{4.78}$$

$$Z^1_{a,N} = \int_0^T \sum_{i=l+1}^{N-l} \bar{\varphi}_i \partial_p \eta(\hat{u}_{l,i}) \left[\nabla \tau(\hat{r}_{l,i}) - \tau'(\hat{r}_{l,i}) \nabla \hat{r}_{l,i} \right] dt \tag{4.79}$$

The term $Z^1_{a,N}$ is of type Z with $b_N \to 0$. This follows from Lemma 4.13, the fact that $\partial_p \eta$ and τ'' are bounded Corollary $A.7$, and the following estimate,

$$\begin{aligned} |\nabla \tau(\hat{r}_{l,i}) - \tau'(\hat{r}_{l,i}) \nabla \hat{r}_{l,i}| &\leq |[\tau'(\alpha \hat{r}_{l,i} + (1-\alpha)\hat{r}_{l,i+1}) - \tau'(\hat{r}_{l,i})] \nabla \hat{r}_{l,i}| \\ &\leq \|\tau''\|_\infty (1-\alpha) (\nabla r_{l,i})^2, \end{aligned} \tag{4.80}$$

which holds for some $\alpha \in (0,1)$.

After a summation by parts, we write

$$X^1_{a,N} = Y_{a,N} + Z^2_{a,N} + Z^b_{a,N}, \tag{4.81}$$

where

$$Y_{a,N} = \int_0^T \sum_{i=l+1}^{N-l} (\nabla^* \bar{\varphi}_i) \partial_p \eta(\hat{u}_{l,i-1}) (\hat{V}'_{l,i} - \tau(\hat{r}_{l,i})) dt \tag{4.82}$$

$$Z^2_{a,N} = \int_0^T \sum_{i=l+1}^{N-l} \bar{\varphi}_i (\nabla^* \partial_p \eta(\hat{u}_{l,i})) (\hat{V}'_{l,i} - \tau(\hat{r}_{l,i})) dt \tag{4.83}$$

$$\begin{aligned} Z^b_{a,N} = &\int_0^T \bar{\varphi}_{N-l} \partial_p \eta(\hat{u}_{l,N-l}) (\hat{V}'_{l,N-l+1} - \tau(\hat{r}_{l,N-l+1}) dt + \\ &- \int_0^T \bar{\varphi}_l \partial_p \eta(\hat{u}_{l,l}) (\hat{V}'_{l,l+1} - \tau(\hat{r}_{l,l+1})) dt. \end{aligned} \tag{4.84}$$

$Y_{a,N}$ is of type Y. This follows from Lemma 4.14, the fact that $\partial_p \eta$ is bounded and Corollary $A.11$. $Z^2_{a,N}$ is of type Z with $b_N \to 0$. This follows from Lemma 4.13, Corollary $A.11$ and Corollary $A.7$ after writing, for some intermediate $\tilde{u}_{l,i}$,

$$\nabla^* \partial_p \eta(\hat{u}_{l,i}) = \partial^2_{rp} \eta(\tilde{u}_{l,i}) \nabla \hat{r}_{l,i} + \partial^2_{pp} \eta(\tilde{u}_{l,i}) \nabla \hat{p}_{l,i} \tag{4.85}$$

and using the fact that the second derivatives of η are bounded. Finally, $Z^b_{a,N}$ is of type Z with $b_N \to 0$.This follows from

$$|Z^b_{a,N}| \leq \|\varphi\|_{L^\infty(Q_T)} \|\partial_p \eta\|_\infty \int_0^T \left(|\hat{V}'_{l,N-l+1} - \tau(\hat{r}_{l,N-l+1})| + |\hat{V}'_{l,l+1} - \tau(\hat{r}_{l,l+1})| \right) dt \tag{4.86}$$

and Corollary $A.11$. □

Lemma 4.16. *Let $X_{s,N}$ be defined in* (4.54)*. Then it decomposes as $X_{s,N} = Y_{s,N} + Z_{s,N}$ such that $Y_{s,N}$ is of type Y and $Z_{s,N}$ is of type Z in the sense of Definition* 4.12*.*

Proof. Write

$$X_{s,N} = X^1_{s,N} + Z^1_{s,N} \tag{4.87}$$

where

$$X^1_{s,N} = \sigma \int_0^T \sum_{i=l+1}^{N-l} \bar{\varphi}_i \partial_p \eta(\hat{\mathbf{u}}_{l,i}) \Delta \hat{p}_{l,i} dt \tag{4.88}$$

$$Z^1_{s,N} = \frac{2\sigma}{l^3} \int_0^T \sum_{i=l+1}^{N-l} \bar{\varphi}_i \partial^2_{pp} \eta(\hat{\mathbf{u}}_{l,i}) dt \tag{4.89}$$

Since $\partial^2_{pp}\eta$ is bounded, we have that $Z^1_{s,N}$ is of type Z with $b_N \to 0$. In order to evaluate $X^1_{s,N}$ we sum by parts and write

$$X^1_{s,N} = Y^1_{s,N} + Z^2_{s,N} + Y^b_{s,N}, \tag{4.90}$$

where

$$Y^1_{s,N} = -\sigma \int_0^T \sum_{i=l+1}^{N-l} (\nabla \bar{\varphi}_i) \partial_p \eta(\hat{\mathbf{u}}_{l,i+1}) \nabla \hat{p}_{l,i} dt \tag{4.91}$$

$$Z^2_{s,N} = -\sigma \int_0^T \sum_{i=l+1}^{N-l} \bar{\varphi}_i (\nabla \partial_p \eta(\hat{\mathbf{u}}_{l,i})) \nabla \hat{p}_{l,i} dt \tag{4.92}$$

$$Y^b_{s,N} = \sigma \int_0^T \bar{\varphi}_{N-l} \partial_p \eta(\hat{u}_{l,N-l}) \nabla \hat{p}_{l,N-l+1} dt - \sigma \int_0^T \bar{\varphi}_l \partial_p \eta(\hat{u}_{l,i}) \nabla \hat{p}_{l,l+1} dt. \tag{4.93}$$

Thanks to Lemma 4.14, Lemma 4.13 and Corollary *A*.7, $Y^1_{s,N}$ is of type Y, and $Z^2_{s,N}$ is of type Z. However, we obtain $|Z^2_{s,N}| \le b_N \|\varphi\|_\infty$ with b_N only bounded, since

$$\mathbb{E}\left[\sum_{i=l+1}^{N-l} \int_0^T (\sigma(\nabla \hat{p}_{l,i})^2 + \sigma(\nabla \hat{r}_{l,i})^2) dt \right] \tag{4.94}$$

is bounded but does not necessarily vanish as $N \to \infty$. $Y^b_{s,N}$ is of type Y. We focus only on the boundary term in l, as the boundary term in $N-l$ is analogous. Since $\varphi(t,0) = 0$ we write

$$\begin{aligned} \bar{\varphi}_l = N \int_0^1 1_{N,l}(x) \varphi(x) dx &= N \int_0^1 1_{N,l}(x) \int_0^x \partial_y \varphi(t,y) dy \\ &= N \int_0^1 1_{N,l}(x) \int_0^1 1_{[0,x]}(y) \partial_y \varphi(t,y) dy. \end{aligned} \tag{4.95}$$

By Cauchy-Schwarz, we estimate

$$\left|\int_0^1 1_{[0,x]}(y)\partial_y\varphi(t,y)dy\right| \leq \sqrt{x}\left(\int_0^1 |\partial_y\varphi(t,y)|^2 dy\right)^{1/2}, \tag{4.96}$$

so that

$$\begin{aligned}|\bar\varphi_l| &\leq \left(\int_0^1 |\partial_y\varphi(t,y)|^2 dy\right)^{1/2} N\int_{\frac{l}{N}-\frac{1}{2N}}^{\frac{l}{N}+\frac{1}{2N}} \sqrt{x}dx \\ &= \left(\int_0^1 |\partial_y\varphi(t,y)|^2 dy\right)^{1/2} \sqrt{\frac{l}{N}}\left(1+\mathcal{O}\left(\frac{l}{N}\right)\right).\end{aligned} \tag{4.97}$$

Thus, we obtain

$$\begin{aligned}\mathbb{E}\left[\sigma\int_0^T |\bar\varphi_l\partial_p\eta(\hat u_{l,i})\nabla\hat p_{l,l+1}|dt\right] &\leq \|\partial_p\eta\|_\infty\left(\int_0^T|\bar\varphi_l|^2dt\right)^{1/2}\left(\mathbb{E}\left[\int_0^T\sigma^2(\nabla\hat p_{l,i})^2dt\right]\right)^{1/2} \\ &\leq C\left(\int_0^T\int_0^1|\partial_y\varphi(t,y)|^2dy\right)^{1/2}\left(\frac{l}{N}\frac{\sigma^2}{l\sigma}\right)^{1/2} \\ &\leq C\|\varphi\|_{H^1(Q_T)}\left(\frac{\sigma}{N}\right)^{1/2}\end{aligned} \tag{4.98}$$

□

Similarly, we have the following

Lemma 4.17. *Let $\tilde X_{s,N}$ be defined in* (4.55)*. Then it decomposes as $\tilde X_{s,N} = \tilde Y_{s,N} + \tilde Z_{s,N}$ such that $\tilde Y_{s,N}$ is of type Y and $\tilde Z_{s,N}$ is of type Z in the sense of Definition* 4.12*.*

Lemma 4.18. *Let M_N be defined in* (4.56)*. Then it decomposes as $M_N = Y_{m,N} + Z_{m,N}$ such that $Y_{m,N}$ is of type Y and $Z_{m,N}$ is of type Z in the sense of Definition* 4.12*. Moreover, $b_N \to 0$ as $N\to\infty$.*

Proof. By a summation by parts we obtain

$$M_N = Y_{m,N} + Z_{m,N} + Z^b_{m,N}, \tag{4.99}$$

with

$$Y_{m,N} = -\sqrt{2\frac{\sigma}{N}}\int_0^T\sum_{i=l+1}^{N-l}(\nabla\bar\varphi_i)\partial_p\eta(\hat u_{l,i+1})d\hat w_{l,i} \tag{4.100}$$

$$Z_{m,N} = -\sqrt{2\frac{\sigma}{N}}\int_0^T\sum_{i=l+1}^{N-l}\bar\varphi_i(\nabla\partial_p\eta(\hat u_{l,i}))d\hat w_{l,i} \tag{4.101}$$

$$Z^b_{m,N} = \sqrt{2\frac{\sigma}{N}}\int_0^T\left(\bar\varphi_{N-l+1}\partial_p\eta(\hat u_{N-l+1})d\hat w_{l,N-l} - \bar\varphi_{l+1}\partial_p\eta(\hat u_{l,l+1})d\hat w_{l,l}\right). \tag{4.102}$$

$Y_{m,N}$ is of type Y. In fact, we have

$$\begin{aligned}\mathbb{E}\left[Y_{m,N}^2\right] &= 2\frac{\sigma}{N}\mathbb{E}\left[\left(\frac{1}{l}\sum_{|j|<l}\int_0^T\sum_{i=l+1}^{N-l}(\nabla\bar{\varphi}_i)\partial_p\eta(\hat{u}_{l,i+1})dw_{i-j}\right)^2\right] \\ &\leq \frac{2\sigma}{N}\frac{1}{l}\sum_{|j|<l}\frac{l-|j|}{l}\sum_{i=l+1}^{N-l}\mathbb{E}\left[\int_0^T\left((\nabla\bar{\varphi}_i)\partial_p\eta(\hat{u}_{l,i+1})\right)^2\right]dt \\ &\leq \frac{2\sigma}{N}\|\partial_p\eta\|_\infty^2\frac{1}{N^2}\int_0^T\sum_{i=l+1}^{N-l}N^2(\nabla\bar{\varphi}_i)^2dt \\ &\leq \|\varphi\|_{H^1(Q_T)}^2\frac{2\sigma}{N^2}\|\partial_p\eta\|_\infty^2.\end{aligned} \tag{4.103}$$

Thanks to the coefficient $\sqrt{\sigma/N}$, the boundary term $Z_{m,N}^b$ is of type Z with $b_N \to 0$. Finally, we show that $Z_{m,N}$ is of type Z as well, by estimating

$$\begin{aligned}\mathbb{E}\left[Z_{m,N}^2\right] &\leq C\|\varphi\|_{L^\infty(Q_T)}^2\frac{\sigma}{N}\int_0^T\sum_{i=l+1}^{N-l}\mathbb{E}\left[(\nabla\hat{r}_{l,i})^2+(\nabla\hat{p}_{l,i})^2\right]dt \\ &\leq \|\varphi\|_{L^\infty(Q_T)}^2\frac{C}{N}\end{aligned}$$

□

Similarly, we prove the following

Lemma 4.19. *Let $\tilde{M}_N$ be defined in* (4.57)*. Then it decomposes as $M_N = \tilde{Y}_{m,N} + \tilde{Z}_{m,N}$ such that $\tilde{Y}_{m,N}$ is of type Y and $\tilde{Z}_{m,N}$ is of type Z in the sense of Definition* 4.12*. Moreover, $b_N \to 0$ as $N \to \infty$.*

Lemma 4.20. *Let Q_N be defined in* (4.58)*. Then it decomposes as $Q_N = Y_{q,N} + Z_{q,N}$ such that $Y_{q,N}$ is of type Y and $Z_{q,N}$ is of type Z in the sense of Definition* 4.12*. Moreover, $b_N \to 0$ as $N \to \infty$.*

Proof. Let $\tilde{\varphi}_i(t) = \varphi\left(t, \frac{i}{N} + \frac{1}{2N}\right)$. Then

$$
\begin{aligned}
-\int_0^T \int_0^1 \partial_x \varphi(t,x) q(\hat{u}_N(t,x)) dx dt &= -\int_0^T \sum_{i=l+1}^{N-l} \left(\int_0^1 \partial_x \varphi(t,x) 1_{N,i}(x) dx \right) q(\hat{u}_{l,i}) dt \\
&= \int_0^T \sum_{i=l+1}^{N-l} (\nabla^* \tilde{\varphi}_i) q(\hat{u}_{l,i}) dt \qquad (4.104) \\
&= \int_0^T \sum_{i=l+1}^{N-l} \tilde{\varphi}_i \nabla q(\hat{u}_{l,i}) dt + \int_0^T \big(\tilde{\varphi}_l q(\hat{u}_{l,l+1}) + \\
&\quad - \tilde{\varphi}_{N-l} q(\hat{u}_{N-l+1}) \big) dt.
\end{aligned}
$$

Thus, since we may write, for some $\alpha \in (0,1)$,

$$
\nabla q(\hat{u}_{l,i}) = \partial_r \eta(\hat{u}^\alpha_{l,i}) \nabla \hat{r}_{l,i} + \partial_p \eta(\hat{u}^\alpha_{l,i}) \nabla \hat{p}_{l,i}, \qquad (4.105)
$$

where $\hat{u}^\alpha_{l,i} = \alpha \hat{u}_{l,i} + (1-\alpha)\hat{u}_{l,i+1}$, we obtain

$$
Q_N = Q^r_{q,N} + Q^p_{q,N} + Z^b_{q,N}, \qquad (4.106)
$$

where

$$
Q^r_{q,N} = \int_0^T \sum_{i=l+1}^{N-l} \left(\tilde{\varphi}_i \partial_r q(\hat{u}^\alpha_{l,i}) \nabla \hat{r}_{l,i} - \bar{\varphi}_i \partial_r q(\hat{u}_{l,i}) \nabla \hat{r}_{l,i} \right) dt \qquad (4.107)
$$

$$
Q^{*p}_{q,N} = \int_0^T \sum_{i=l+1}^{N-l} \left(\tilde{\varphi}_i \partial_p q(\hat{u}^\alpha_{l,i}) \nabla \hat{p}_{l,i} + \bar{\varphi}_i \partial_p q(\hat{u}_{l,i}) \nabla^* p_{l,i} \right) dt \qquad (4.108)
$$

$$
Z^b_{q,N} = \int_0^T \left(\tilde{\varphi}_l q(\hat{u}_{l,l+1}) - \tilde{\varphi}_{N-l} q(\hat{u}_{N-l+1}) \right) dt. \qquad (4.109)
$$

Since q is bounded, $Z^b_{q,N}$ is at glance of type Z. Next, we write

$$
Q^r_{q,N} = Y^r_{q,N} + Z^r_{q,N}, \qquad (4.110)
$$

where

$$
Y^r_{q,N} = \int_0^T \sum_{i=l+1}^{N-l} (\tilde{\varphi}_i - \bar{\varphi}_i) \partial_r q(\hat{u}^\alpha_{l,i}) \nabla \hat{r}_{l,i} dt \qquad (4.111)
$$

$$
Z^r_{q,N} = \int_0^T \sum_{i=l+1}^{N-l} \bar{\varphi}_i \left(\partial_r q(\hat{u}^\alpha_{l,i}) - \partial_r q(\hat{u}_{l,i}) \right) \nabla \hat{r}_{l,i} dt. \qquad (4.112)
$$

In order to estimate $Y^r_{q,N}$ we estimate

$$
\begin{aligned}
|\tilde{\varphi}_i - \bar{\varphi}_i| &= N \left| \int_0^1 1_{N,i}(x) \left(\varphi\left(\frac{i}{N} + \frac{1}{2N} \right) - \varphi(x) \right) dx \right| \\
&\leq N \int_0^1 1_{N,i}(x) \left| \int_0^1 1_{[x, \frac{i}{N} + \frac{1}{2N}]}(y) \partial_y \varphi(t,y) dy \right| dx \\
&\leq N \int_0^1 1_{N,i}(x) \left(\frac{i}{N} + \frac{1}{2N} - x \right)^{1/2} \left(\int_0^1 |\partial_y \varphi(t,y)|^2 dy \right)^{1/2} dx \\
&= \frac{2}{3\sqrt{N}} \left(\int_0^1 |\partial_y \varphi(t,y)|^2 dy \right)^{1/2}.
\end{aligned} \tag{4.113}
$$

Thus, by Cauchy-Schwarz,

$$
\begin{aligned}
|Y^r_{q,N}| &\leq C \left(\int_0^T \sum_{i=l+1}^{N-l} (\tilde{\varphi}_i - \bar{\varphi}_i)^2 dt \right)^{1/2} \left(\int_0^T \sum_{i=l+1}^{N-l} (\nabla \hat{r}_{l,i})^2 dt \right)^{1/2} \\
&\leq C \left(\int_0^T \int_0^1 |\partial_y \varphi(t,y)|^2 dy dt \right)^{1/2} \left(\int_0^T \sum_{i=l+1}^{N-l} (\nabla \hat{r}_{l,i})^2 dt \right)^{1/2} \\
&\leq C \|\varphi\|_{H^1(Q_T)} \left(\int_0^T \sum_{i=l+1}^{N-l} (\nabla \hat{r}_{l,i})^2 dt \right)^{1/2}
\end{aligned} \tag{4.114}
$$

and so $Y^r_{q,N}$ is of type Y. $Z^r_{q,N}$ is easily seen to be of type Z with $b_N \to 0$ since we can estimate, for some intermediate value $\tilde{u}_{l,i}$,

$$
\begin{aligned}
\left| \partial_r q(\hat{u}^\alpha_{l,i}) - \partial_r q(\hat{u}_{l,i}) \right| &= |\partial_r q(\hat{\alpha} u_{l,i} + (1-\alpha)\hat{u}_{l,i+1}) - \partial_r q(\hat{u}_{l,i})| \\
&= |\partial^2_{rr} q(\tilde{u}_{l,i})(1-\alpha)\nabla \hat{r}_{l,i} + \partial^2_{rp} q(\tilde{u}_{l,i})(1-\alpha)\nabla \hat{p}_{l,i}| \\
&\leq \|q''\|_\infty (1-\alpha) \left(|\nabla \hat{r}_{l,i}| + |\nabla \hat{p}_{l,i}| \right).
\end{aligned} \tag{4.115}
$$

In order to evaluate $Q^{*p}_{q,N}$ we write

$$
Q^{*p}_{q,N} = Q^p_{q,N} + X_{q,N}, \tag{4.116}
$$

where

$$
Q^p_{q,N} = \int_0^T \sum_{i=l+1}^{N-l} \left(\tilde{\varphi}_i \partial_p q(\hat{u}^\alpha_{l,i}) \nabla \hat{p}_{l,i} - \bar{\varphi}_i \partial_p q(\hat{u}_{l,i}) \nabla p_{l,i} \right) dt \tag{4.117}
$$

$$
X_{q,N} = \int_0^T \sum_{i=p+1}^{N-l} \bar{\varphi}_i \partial_p q(\hat{u}_{l,i}) \left(\nabla^* \hat{p}_{l,i} + \nabla \hat{p}_{l,i} \right) dt. \tag{4.118}
$$

$Q^p_{p,N}$ is evaluated exactly as $Q^r_{p,N}$ and so it writes as a sum of a type Y term and of a type Z term with $b_N \to \infty$. Finally, since

$$\nabla^* \hat{p}_{l,i} + \nabla \hat{p}_{l,i} = \Delta \hat{p}_{l,i}, \tag{4.119}$$

the term $X_{q,N}$ is entirely similar to the term $X^1_{s,N}$ of Lemma 4.16, except the latter has an extra factor σ. Thus, $X_{q,N}$ writes as a sum of a type Y term and a type Z term with $b_N \to 0$. □

5 Clausius Inequality

This section is devoted to proving the second law of Thermodynamics in the form of the Clausius inequality. We recall here the variational formula for the relative entropy

$$H_N(t) = \sup_{\phi} \left\{ \int \phi d\mu^N_t - \log \int e^{\phi} d\lambda^N_t \right\}, \tag{5.1}$$

where the supremum is carried over all measurable function ϕ such that $\int e^{\phi} d\lambda^N_t < +\infty$.

Lemma 5.1. *Any solution* $\tilde{\mathbf{u}}$ *belongs almost surely to* $L^\infty(0,T;L^2(0,1))$.

Proof. Fix $1 \le p < 2$. By Lemma 4.10 there exists a set A of probability 1 such that $\tilde{\mathbf{u}}_n \to \tilde{\mathbf{u}}$ in L^p-strong for for any $\omega \in A$. For any such ω we can find a subsequence $\{n^\omega_k\}$ such that $\tilde{\mathbf{u}}_{n^\omega_k}(t,x) \to \tilde{\mathbf{u}}(t,x)$ for almost all t and x. In particular, for almost all t, the sequence $\tilde{\mathbf{u}}_{n^\omega_k}(t,x)$ converges for almost all x. Thus, by Fatou lemma and the remark following Proposition 4.8 we have

$$\int_0^1 |\tilde{\mathbf{u}}(t,x)|^2 dx \le \liminf_{k\to\infty} \int_0^1 |\tilde{\mathbf{u}}_{n^\omega_k}(t,x)|^2 dx \le C \tag{5.2}$$

for almost all t. □

Lemma 5.2. *For any limit point* $\mathfrak{Q}$ *of* $\mathfrak{Q}_N$

$$\int_0^T \liminf_{N\to\infty} \frac{H_N(t)}{N} dt \;\ge\; \beta \mathbb{E}^{\mathfrak{Q}} \left(\int_0^T \int_0^1 \left[\mathcal{F}(t,y) - \bar{\tau}(t) r(t,y) \right] dy dt \right) + \int_0^T G(\bar{\tau}(t)) dt, \tag{5.3}$$

where

$$\mathcal{F}(t,y) := \frac{p(t,y)^2}{2} + F(r(t,y)). \tag{5.4}$$

Proof. To get the lower bound for any fixed $t \geq 0$ choose two Lipschitz functions $\tilde{\tau}(t,\cdot)$ and $\tilde{p}(t,\cdot)$ on $[0,1]$ and define $\widetilde{\tau_i(t)} := \tilde{\tau}\left(t, \frac{i}{N}\right)$ and $\widetilde{p_i(t)} := \tilde{p}\left(t, \frac{i}{N}\right)$. Then take ϕ in (5.1) as

$$\phi(\mathbf{r},\mathbf{p}) = \sum_{i=1}^{N} \left\{ \beta \left[(\widetilde{\tau_i(t)} - \bar{\tau}(t)) r_i + \widetilde{p_i(t)} p_i \right] - G(\widetilde{\tau_i(t)}) + G(\bar{\tau}(t)) - \beta \frac{\widetilde{p_i(t)}^2}{2} \right\}. \tag{5.5}$$

Observe that ϕ is such that $\int e^{\phi} d\lambda_t^N = 1$. Then using the results on the hydrodynamic limit, along a sub-sequence, we have, for almost all t,

$$\begin{aligned} \liminf_{N\to\infty} \frac{H_N(t)}{N} \;\geq\; & \sup_{\tilde{\tau}(t,\cdot),\tilde{p}(t,\cdot)\in \mathrm{Lip}([0,1])} \mathbb{E}^{\mathfrak{Q}} \int_0^1 dy \Big\{ \left[\beta \left(\tilde{\tau}(t,y) - \bar{\tau}(t)\right) r(t,y) + \tilde{p}(t,y) p(t,y)\right] \\ & - G(\tilde{\tau}(t,y)) + G(\bar{\tau}(t)) - \frac{\beta}{2} \tilde{p}(t,y)^2 \Big\} \\ = \mathbb{E}^{\mathfrak{Q}} \int_0^1 dy \Big\{ & \sup_{\tau\in\mathbb{R}} \left[\beta\tau r(t,y) - G(\tau)\right] - \beta \bar{\tau}(t) r(t,y) + \sup_{p'\in\mathbb{R}} \left[p' p(t,y) - \frac{\beta}{2} p'^2 \right] \Big\} + G(\bar{\tau}(t)) \\ = & \; \beta \mathbb{E}^{\mathfrak{Q}} \int_0^1 dy \left[\mathcal{F}(t,y) - \beta \bar{\tau}(t) r(t,y)\right] + G(\bar{\tau}(t)). \end{aligned} \tag{5.6}$$

The conclusion then follows after a time integration. □

Theorem 5.3 (Clausius inequality)**.**

$$\mathbb{E}^{\mathfrak{Q}} \left(\int_0^T \int_0^1 [\mathcal{F}(t,y) - \mathcal{F}(0,y)] dy \right) \leq \mathbb{E}^{\mathfrak{Q}} \left(\int_0^T W(t) dt \right). \tag{5.7}$$

where the macroscopic work is given by

$$W(t) := -\int_0^t \bar{\tau}'(s) \int_0^1 r(s,y) dy + \bar{\tau}(t) \int_0^1 r(t,y) dy - \bar{\tau}(0) \int_0^1 r_0(y) dy. \tag{5.8}$$

Proof. By our assumptions on the initial conditions we have

$$\lim_{N\to\infty} \frac{H_N(0)}{N} \;=\; \beta \int_0^1 \left[\mathcal{F}(0,y) - \bar{\tau}(0) r_0(y)\right] dy + G(\bar{\tau}(0)). \tag{5.9}$$

This, together with the previous lemma, yields for almost all t,

$$\begin{aligned} \beta \mathbb{E}^{\mathfrak{Q}} \left(\int_0^1 \mathcal{F}(t,y) dy \right) - \beta \int_0^1 \mathcal{F}(0,y) dy \leq & \liminf_{N\to\infty} \frac{H_N(t) - H_N(0)}{N} \\ + \beta \mathbb{E}^{\mathfrak{Q}} & \left(\int \left[\bar{\tau}(t) r(t,y) - \bar{\tau}(0) r_0(y)\right] dy \right) - G(\bar{\tau}(t)) + G(\bar{\tau}(0)). \end{aligned} \tag{5.10}$$

Finally, from Proposition 3.1 we have

$$\liminf_{N\to\infty}\frac{H_N(t)-H_N(0)}{N} \le -\beta\int_0^t ds\ \mathbb{E}^{\mathfrak{Q}}\left(\bar\tau'(s)\int_0^1 r(s,y)dy\right)+\beta\int_0^t ds\ \bar\tau'(s)\ell(\bar\tau(s)) \tag{5.11}$$

$$\begin{aligned}&= \beta\mathbb{E}^{\mathfrak{Q}}\left(W(t)\right)-\beta\mathbb{E}^{\mathfrak{Q}}\left(\int_0^1\left[\bar\tau(t)r(t,y)-\bar\tau(0)r_0(y)\right]dy\right)+\beta\int_0^t ds\ \bar\tau'(s)\ell(\bar\tau(s))\\ &= \beta\mathbb{E}^{\mathfrak{Q}}\left(W(t)\right)-\beta\mathbb{E}^{\mathfrak{Q}}\left(\int_0^1\left[\bar\tau(t)r(t,y)-\bar\tau(0)r_0(y)\right]dy\right)+G(\bar\tau(t))-G(\bar\tau(0)),\end{aligned}$$

which, together with (5.10) and after an integration in time gives the conclusion. □

Remark. Assume that the external tension varies smoothly from τ_0 at $t=0$ to τ_1 as $t\to\infty$. Assume also that the system is at equilibrium both at time zero and as $t\to\infty$:

$$p_0(x)=0,\quad \tau(r_0(x))=\tau_0,\qquad \forall x\in[0,1] \tag{5.12}$$

$$\lim_{t\to\infty}p(t,x)=0,\quad \lim_{t\to\infty}\tau(r(t,x))=\tau_1\qquad \text{a.e. } x\in[0,1]. \tag{5.13}$$

Then, the following version of the Clausius inequality holds

$$F(\tau_1)-F(\tau_0)\le E^{\mathfrak{Q}}(W), \tag{5.14}$$

where the total work W is given by

$$W:=-\int_0^\infty\bar\tau'(s)\int_0^1 r(s,y)dy+\tau_1\ell(\tau_1)-\tau_0\ell(\tau_0). \tag{5.15}$$

A Appendix

A.1 Microscopic estimates

In the following we will denote, for any sequence $(a_i)_{i\in\mathbb{N}}$ and any $l\in\mathbb{N}$ the usual block averages

$$\bar a_{l,i}:=\frac{1}{l}\sum_{j=1}^{l}a_{i-j+1},\quad i\ge l. \tag{A.1}$$

For $1\le m\le i\le N$, denote by $d\mu_{m,i,t}\in\mathcal{M}\left(\mathbb{R}^{2m}\right)$ the projection of the probability measure μ_t^N on $\{r_{i-m+1},\dots,r_i,p_{i-m+1},\dots,p_i\}$. This decompose in $d\mu_{m,i,t}(\cdot)=$

$d\mu_{m,i,t}(\cdot|\bar r_{m,i},\bar p_{m,i})d\mu_{i,t}(\bar r_{m,i},\bar p_{m,i})$, where $\mu_{m,i,t}(\cdot|\bar r_{m,i},\bar p_{m,i})$ is the measure $\mu_{m,i,t}$ conditioned to $\bar r_{m,i},\bar p_{m,i}$, while $d\mu_{i,t}(\bar r_{m,i},\bar p_{m,i})$ is the marginal distribution of $(\bar r_{m,i},\bar p_{m,i})$ under μ_t^N.

Correspondingly, from the measure λ_t^N, we define $d\lambda_{m,i,t}(\cdot)=d\lambda_{m,i,t}(\cdot|\bar r_{m,i},\bar p_{m,i})d\lambda_{i,t}(\bar r_{m,i},\bar p_{m,i})$.

For every value of $\ell,\bar p$, we can choose the corresponding regular conditional probabilities $\bar\mu^{\ell,\bar p}_{m,i,t}=\mu_{m,i,t}(\cdot|\bar r_{m,i}=\ell,\bar p_{m,i}=\bar p)$ and $\bar\lambda^{\ell,\bar p}_{m,i,t}=\lambda_{m,i,t}(\cdot|\bar r_{m,i}=\ell,\bar p_{m,i}=\bar p)$. These are probability measures on $\mathbb{R}^{2m}$ supported on the $2(m-1)$ hyperplane $\Sigma_m(\ell,\bar p)$ defined by $\frac{1}{m}\sum_{j=0}^{m-1} r_j=\ell$, $\frac{1}{m}\sum_{j=0}^{m-1} p_j=\bar p$. Observe that $\bar\lambda^{\ell,\bar p}_{m,i,t}$ does not depend on $\bar\tau(t)$ nor on i.

Since the potential V is uniformly convex, the Bakry-Emery criterion applies and we have the following logarithmic Sobolev inequality (LSI)

$$\int_{\Sigma_m(\ell,\bar p)} g^2\log g^2 d\bar\lambda^{\ell,\bar p}_m \le C_{lsi}m^2\sum_{j=1}^{m-1}\int_{\Sigma_m(\ell,\bar p)}\left[(D_jg)^2+(\tilde D_jg)^2\right]d\bar\lambda^{\ell,\bar p}_m \tag{A.2}$$

for any smooth g on $\mathbb{R}^{2m}$ such that $\int_{\Sigma_m(\ell,\bar p)} g^2 d\bar\lambda^{\ell,\bar p}_m=1$. Here C_{lsi} is a universal constant depending on the interaction V only. In particular (A.2) holds for $g^2=\bar f^{\ell,\bar p}_{m,i,t}=d\bar\mu^{\ell,\bar p}_{m,i,t}/d\bar\lambda^{\ell,\bar p}_m$. Denote $d\mu_{m,i,t}(\ell,\bar p)$ the marginal distribution of $\bar r_{m,i},\bar p_{m,i}$ of μ_t.

Lemma A.1. *Let $m<i<N$. Then there exists a constant C such that, for any $\ell,\bar p\in\mathbb{R}$,*

$$\int_{\mathbb{R}^2} d\mu_{m,i,t}(\bar r_{m,i},\bar p_{m,i})\int_{\Sigma_m(\bar r_{m,i},\bar p_{m,i})} \bar f^{\bar r_{m,i},\bar p_{m,i}}_{m,i,t}\log \bar f^{\bar r_{m,i},\bar p_{m,i}}_{m,i,t} d\bar\lambda^{\bar r_{m,i},\bar p_{m,i}}_m \tag{A.3}$$
$$\le C_{lsi}m^2\sum_{j=1}^{m-1}\int_{\mathbb{R}^{2N}}\frac{\left(D_{i-j}f_t^N\right)^2+\left(\tilde D_{i-j}f_t^N\right)^2}{f_t^N}d\lambda_t^N$$

Proof. Let $f_{m,i,t}=\dfrac{d\mu_{m,i,t}}{d\lambda_{m,i,t}}$. It decomposes as

$$f_{m,i,t}(r_{i-m},p_{i-m},\dots,r_i,p_i)=\bar f^{\bar r_{m,i},\bar p_{m,i}}_{m,i,t}(r_{i-m},p_{i-m},\dots,r_i,p_i)\bar f_{i,t}(\bar r_{m,i},\bar p_{m,i}), \tag{A.4}$$

where $\bar f_{i,t}=\dfrac{d\mu_{i,t}}{d\lambda_{i,t}}$.

By (A.2), the left hand side of (A.3) is less or equal to

$$C_{lsi}m^2\int_{\mathbb{R}^2}d\mu_{i,t}(\bar r_{m,i},\bar p_{m,i})\sum_{j=1}^{m-1}\int\frac{(D_{i-j}\bar f^{\bar r_{m,i},\bar p_{m,i}}_{m,i,t})^2+(\tilde D_{i-j}\bar f^{\bar r_{m,i},\bar p_{m,i}}_{m,i,t})^2}{\bar f^{\bar r_{m,i},\bar p_{m,i}}_{m,i,t}}d\bar\lambda^{\bar r_{i,m},\bar p_{i,m}}_{m,i,t} \tag{A.5}$$

$$=C_{lsi}m^2\int_{\mathbb{R}^2}f_{i,t}(\bar r_{m,i},\bar p_{m,i})d\lambda_{i,t}(\bar r_{m,i},\bar p_{m,i})\sum_{j=1}^{m-1}\int\frac{(D_{i-j}\bar f^{\bar r_{m,i},\bar p_{m,i}}_{m,i,t})^2+(\tilde D_{i-j}\bar f^{\bar r_{m,i},\bar p_{m,i}}_{m,i,t})^2}{\bar f^{\bar r_{m,i},\bar p_{m,i}}_{m,i,t}}d\bar\lambda^{\bar r_{i,m},\bar p_{i,m}}_{m,i,t}$$

$$=C_{lsi}m^2\int_{\mathbb{R}^2}d\lambda_{i,t}(\bar r_{m,i},\bar p_{m,i})\sum_{j=1}^{m-1}\int\frac{(D_{i-j}f_{m,i,t})^2+(\tilde D_{i-j}f_{m,i,t})^2}{f_{m,i,t}}d\bar\lambda^{\bar r_{i,m},\bar p_{i,m}}_{m,i,t}$$

$$=C_{lsi}m^2\sum_{j=1}^{m-1}\int\frac{(D_{i-j}f_{m,i,t})^2+(\tilde D_{i-j}f_{m,i,t})^2}{f_{m,i,t}}d\lambda_{m,i,t},$$

and (A.3) follows by Jensen's inequality. □

Proposition A.2 (One-block estimate - interior)**.** *There exists $l_0\in\mathbb{N}$ such that, for $l_0<l\le N$, we have*

$$\sum_{i=l+1}^{N-1}\int_0^t\int\left(\bar V'_{l,i}-\tau\left(\bar r_{l,i}\right)\right)^2d\mu^N_s ds\le C\left(\frac{N}{l}t+\frac{l^2}{\sigma}\right)\le C(T)\frac{l^2}{\sigma}. \tag{A.6}$$

Proof. Fix $\alpha>0$. By the entropy inequality and Lemma $A.1$:

$$\sum_{i=l+1}^{N-1}\alpha l\int_0^t ds\int\left(\bar V'_{l,i}-\tau\left(\bar r_{l,i}\right)\right)^2d\mu^N_s \tag{A.7}$$

$$=\sum_{i=l+1}^{N-1}\alpha l\int_0^t ds\int d\mu_{i,s}(\bar r_{l,i},\bar p_{l,i})\int\left(\bar V'_{l,i}-\tau\left(\bar r_{l,i}\right)\right)^2d\bar\mu^{\bar r_{l,i},\bar p_{l,i}}_{l,i,s}$$

$$\le l^3C_{lsi}\int_0^t(\mathcal{D}_N(s)+\widetilde{\mathcal{D}}_N(s))ds+\sum_{i=l+1}^{N-1}\int_0^t ds\int d\mu_{i,s}(\bar r_{l,i},\bar p_{l,i})\log\left(\int e^{\alpha l\left(\bar V'_{l,i}-\tau(\bar r_{l,i})\right)^2}d\bar\lambda^{\bar r_{l,i},\bar p_{l,i}}_{l,i,t}\right)$$

$$\le C\frac{l^3}{\sigma}+\sum_{i=l+1}^{N-1}\int_0^t ds\int d\mu_{i,s}(\bar r_{l,i},\bar p_{l,i})\log\left(\int e^{\alpha l\left(\bar V'_{l,i}-\tau(\bar r_{l,i})\right)^2}d\bar\lambda^{\bar r_{l,i},\bar p_{l,i}}_{l}\right)$$

where we have used the bound on the time integral of the Dirichlet form .

We prove now that for $\alpha<(4c_0)^{-1}$ we have

$$\sup_{\ell,\bar p\in\mathbb{R}}\int e^{\alpha l\left(\bar V'_{l,i}-\tau(\ell)\right)^2}d\bar\lambda^{\ell,\bar p}_l\le C, \tag{A.8}$$

and (A.15) will follow.

We take $l > l_0$ so that

$$\int e^{\alpha l \left(\bar{V}'_{l,i} - \tau(\ell)\right)^2} d\bar{\lambda}_l^{\ell,\bar{p}} \leq C \int e^{\alpha l \left(\bar{V}'_{l,i} - \tau(\ell)\right)^2} d\lambda^l_{\beta,\bar{p},\tau(\ell)}. \tag{A.9}$$

Let us introduce a normally distributed random variable $\xi \sim \mathcal{N}(0,1)$ so that we can use the identity

$$e^{\alpha l \left(\bar{V}'_{l,i} - \tau(\ell)\right)^2} = \mathbb{E}_\xi \left[e^{\xi \sqrt{2\alpha l} \left(\bar{V}'_{l,i} - \tau(\ell)\right)} \right] \tag{A.10}$$

in order to write

$$\int e^{\alpha l \left(\bar{V}'_{l,i} - \tau(\ell)\right)^2} d\lambda^l_{\beta,\bar{p},\tau(\ell)} = \mathbb{E}_\xi \left[\int e^{\xi \sqrt{2\alpha l} \left(\bar{V}'_{l,i} - \tau(\ell)\right)} d\lambda^l_{\beta,\bar{p},\tau(\ell)} \right] \tag{A.11}$$

$$= \mathbb{E}_\xi \left[e^{-\tau(\ell)\xi\sqrt{2\alpha l}} \left(\int e^{\frac{\xi\sqrt{2\alpha l}}{l} V'(r)} d\lambda_{\beta,\bar{p},\tau(\ell)} \right)^l \right], \tag{A.12}$$

It is easy to show (cf Appendix A of [7]) that,

$$\int e^{\xi \sqrt{2\alpha l^{-1}} V'(r_1)} d\lambda^N_{\beta,\bar{p},\tau(\ell)} \leq e^{\frac{c_2 \alpha}{\beta} \xi^2 + \frac{\tau(\ell)\sqrt{2\alpha l}}{l} \xi}. \tag{A.13}$$

Hence, we obtain

$$\int e^{\alpha \left(\bar{V}'_{l,i} - \tau(\ell)\right)^2} d\lambda^l_{\beta,\bar{p},\tau(\ell)} \leq \mathbb{E}_\xi \left[e^{\frac{c_2 \alpha}{\beta} \xi^2} \right], \tag{A.14}$$

and the right hand side is independent of ℓ and $\bar{p}$. Taking $\alpha = \beta/(4c_2)$ the expectation with respect to ξ is finite. □

If we do not perform the summation over i in (A.7) and we bound the right-hand side of (A.3) by the Dirichlet forms we obtain the following

Corollary A.3. *There exists $l_0 \in \mathbb{N}$ such that, for $l_0 < l < i < N$, we have*

$$\int_0^t \int \left(\bar{V}'_{l,i} - \tau\left(\bar{r}_{l,i}\right)\right)^2 d\mu^N_s ds \leq C \left(\frac{1}{l} t + \frac{l}{\sigma} \right) \leq C(T) \frac{l}{\sigma}. \tag{A.15}$$

Proposition A.4 (One-and-a-half-block estimate)**.** *Let $a_i \in \{p_i, V'(r_i)\}$. Then, for any fixed $l \leq k \leq N - l$, we have*

$$\int_0^t \int (\bar{a}_{l,k+l} - \bar{a}_{l,k})^2 d\mu^N_s ds \leq C \left(\frac{1}{l} t + \frac{l}{\sigma} \right) \leq C(T) \frac{l}{\sigma}. \tag{A.16}$$

Proof. We consider $a_i = V'(r_i)$, as the case $a_i = p_i$ is analogous. Thanks to the identity

$$V'(r_i) - V'(r_j) = -\beta^{-1}\left(\frac{\partial \lambda_s^N}{\partial r_i} - \frac{\partial \lambda_s^N}{\partial r_j}\right) \tag{A.17}$$

we compute

$$\begin{aligned}\int (\bar V'_{l,k+l} - \bar V'_{l,k})^2 f_s^N d\lambda_s^N =& \frac{1}{l}\sum_{i=k-l+1}^{k}\int (V'(r_{i+l}) - V'(r_i))(\bar V'_{l,k+l} - \bar V'_{l,k}) f_s^N d\lambda_s^N \qquad (A.18)\\ =& \frac{\beta^{-1}}{l^2}\sum_{i=k-l+1}^{k}\int (V''(r_{i+l}) + V''(r_i)) d\mu_s^N + \\ &+ \int (\bar V'_{l,k+l} - \bar V'_{l,k})\left(\frac{\beta^{-1}}{l}\sum_{i=k-l+1}^{k}\left(\frac{\partial f_s^N}{\partial r_{i+l}} - \frac{\partial f_s^N}{\partial r_i}\right)\right) d\lambda_s^N.\end{aligned}$$

By using Cauchy-Schwartz inequality on the last term and the fact that V'' is bounded, we obtain

$$\begin{aligned}\int (\bar V'_{l,k+l} - \bar V'_{l,k})^2 f_s^N d\lambda_s^N &\le C\left(\frac{1}{l} + \frac{1}{l}\int \frac{1}{f_s^N}\sum_{i=k-l+1}^{k}\left(\frac{\partial f_s^N}{\partial r_{i+l}} - \frac{\partial f_s^N}{\partial r_i}\right)^2 d\lambda_s^N\right)\\ &= C\left(\frac{1}{l} + \frac{1}{l}\int \frac{1}{f_s^N}\sum_{i=k-l+1}^{k}\left(\sum_{j=i}^{i+l-1}\left(\frac{\partial f_s^N}{\partial r_{j+1}} - \frac{\partial f_s^N}{\partial r_j}\right)\right)^2 d\lambda_s^N\right)\\ &\le C\left(\frac{1}{l} + \int \frac{1}{f_s^N}\sum_{i=k-l+1}^{k}\sum_{j=i}^{i+l-1}\left(\frac{\partial f_s^N}{\partial r_{j+1}} - \frac{\partial f_s^N}{\partial r_j}\right)^2 d\lambda_s^N\right) \qquad (A.19)\\ &\le C\left(\frac{1}{l} + \int \frac{1}{f_s^N}\sum_{i=k-l+1}^{k}\sum_{j=1}^{N-1}\left(\frac{\partial f_s^N}{\partial r_{j+1}} - \frac{\partial f_s^N}{\partial r_j}\right)^2 d\lambda_s^N\right)\\ &\le C\left(\frac{1}{l} + l\tilde{\mathcal{D}}_N(s)\right).\end{aligned}$$

The conclusion then follows after an integration in time. □

Proposition A.5 (Two-block estimate)**.** *Let $a_i \in \{p_i, V'(r_i), r_i\}$ and l_0 be as in Proposition A.2. Then, for $l_0 < l \le N$, we have*

$$\sum_{i=l+1}^{N-l}\int_0^t (\bar a_{l,i+l} - \bar a_{l,i})^2 \, d\mu_s^N ds \le C\left(\frac{N}{l}t + \frac{l^2}{\sigma}\right) \le C(T)\frac{l^2}{\sigma}. \tag{A.20}$$

Proof. We prove the statement for $a_i = V'(r_i)$. From (A.19) we have

$$\begin{aligned}\sum_{i=l+1}^{N-l}\int(\bar V'_{l,i+l}-\bar V'_{l,i})^2 f_s d\lambda_s^N &\le C\left(\frac{N}{l}+\int\frac{1}{f_s^N}\sum_{i=l}^{N-l}\sum_{k=i-l+1}^{i}\sum_{j=k}^{k+l-1}\left(\frac{\partial f_s^N}{\partial r_{j+1}}-\frac{\partial f_s^N}{\partial r_j}\right)^2 d\lambda_s^N\right)\\ &\le C\left(\frac{N}{l}+l\int\frac{1}{f_s^N}\sum_{i=l}^{N-l}\sum_{j=i-l+1}^{i+l-1}\left(\frac{\partial f_s^N}{\partial r_{j+1}}-\frac{\partial f_s^N}{\partial r_j}\right)^2 d\lambda_s^N\right)\\ &\le C\left(\frac{N}{l}+l^2\int\frac{1}{f_s^N}\sum_{i=l}^{N-l}\left(\frac{\partial f_s^N}{\partial r_{i+1}}-\frac{\partial f_s^N}{\partial r_i}\right)^2 d\lambda_s^N\right) \qquad (A.21)\\ &\le C\left(\frac{N}{l}+l^2\tilde{\mathcal{D}}_N(s)\right).\end{aligned}$$

The conclusion then follows after a time integration. The proof for $a_i = p_i$ is analogous. Finally, since τ' is bounded from below by a positive constant, we have

$$\begin{aligned}(\bar r_{l,i+l}-\bar r_{l,i})^2 &\le C\left(\tau(\bar r_{l,i+l})-\tau(\bar r_{l,i})\right)^2 \qquad (A.22)\\ &\le C\left[\left(\tau(\bar r_{l,i+l})-\bar V'_{l,i+l}\right)^2+\left(\bar V'_{l,i}-\tau(\bar r_{l,i})\right)^2+\left(\bar V'_{l,i+l}-\bar V'_{l,i}\right)^2\right]\end{aligned}$$

and the statement for $a_i = r_i$ follows from the fist part of the proof and Proposition $A.2$. □

We conclude this section by showing the connection between the averages $\hat a_{l,i}$ and $\bar a_{l,i}$.

Lemma A.6. *For any sequence $(a_i)_{i\in\mathbb{N}}$, any $l\in\mathbb{N}$ and any $i\ge l$, we have*

$$\nabla\hat a_{l,i} = \frac{1}{l}(\bar a_{l,i+l}-\bar a_{l,i}) \qquad (A.23)$$

Proof. We prove the statement by induction over l. The statement for $l = 1$ is obvious, since both $\hat a_{1,i+1}-\hat a_{1,i}$ and $\bar a_{1,i+1}-\bar a_{1,i}$ are equal to $a_{i+1}-a_i$.

Assume now the statement is true for some $l\ge 1$. We prove it holds for $l+1$ as well. We have

$$\begin{aligned}\hat a_{l+1,i+1}-\hat a_{l+1,i} &= \frac{1}{l+1}\sum_{|j|<l+1}\frac{l+1-|j|}{l+1}a_{i+1-j}-\frac{1}{l+1}\sum_{|j|<l+1}\frac{l+1-|j|}{l+1}a_{i-j} \qquad (A.24)\\ &= \frac{1}{(l+1)^2}\sum_{|j|<l+1}(l+1-|j|)(a_{i+1-j}-a_{i-j})\\ &= \frac{l^2}{(l+1)^2}\frac{1}{l}\sum_{|j|<l}\frac{l-|j|}{l}(a_{i+1-j}-a_{i-j})+\frac{1}{(l+1)^2}\sum_{|j|<l+1}(a_{i+1-j}-a_{i-j}).\end{aligned}$$

For the first summation we can use the inductive hypothesis, while the second summation is telescopic. Therefore we obtain

$$\begin{aligned}\hat{a}_{l+1,i+1} - \hat{a}_{l+1,i} &= \frac{1}{(l+1)^2}\sum_{j=1}^{l}(a_{i+l-j+1} - a_{i-j+1}) + \frac{1}{(l+1)^2}(a_{i+l+1} - a_{i-l}) \\ &= \frac{1}{(l+1)^2}\sum_{j=1}^{l+1}(a_{i+l-j+1} - a_{i-j+1}) \\ &= \frac{1}{l+1}(\bar{a}_{l+1,i+l+1} - \bar{a}_{l+1,i}).\end{aligned} \tag{A.25}$$

□

Combining Proposition $A.5$ and Lemma $A.6$ we get the following

Corollary A.7. *Let $a_i \in \{p_i, V'(r_i), r_i\}$ and l_0 be as in Proposition A.2. Then, for $l_0 < l < i < N - l + 1$,*

$$\sum_{j=l+1}^{N-l}\int_0^t\int (\nabla\hat{a}_{l,j})^2\, d\mu_s^N ds \le C\left(\frac{N}{l^3}t + \frac{1}{\sigma}\right) \le C(T)\frac{1}{\sigma} \tag{A.26}$$

$$\int_0^t\int (\nabla\hat{a}_{l,i})^2\, d\mu_s^N ds \le C\left(\frac{1}{l^3}t + \frac{1}{l\sigma}\right) \le C(T)\frac{1}{l\sigma}. \tag{A.27}$$

We now show that the two averages we defined are equivalent in the limit.

Proposition A.8 (One-block comparison)**.** *Let $a_i \in \{p_i, V'(r_i)\}$. Then, for any $l \le i \le N - l$, we have*

$$\int_0^t\int (\bar{a}_{l,i} - \hat{a}_{l,i})^2\, d\mu_s^N ds \le C\left(\frac{1}{l}t + \frac{l}{\sigma}\right) \le C(T)\frac{l}{\sigma}. \tag{A.28}$$

Proof. We prove the statement for $a_i = V'(r_i)$, the proof for $a_i = p_i$ being analogous. We can write

$$\bar{a}_{l,i} - \hat{a}_{l,i} = \frac{1}{l}\sum_{j=0}^{l-1}\frac{j}{l}\left(a_{i-j} - a_{i-j+l}\right). \tag{A.29}$$

Thus,

$$\int (\bar{V}'_{l,i} - \hat{V}'_{l,i})^2 d\mu_s^N = \int \frac{1}{l} \sum_{j=0}^{l-1} \frac{j}{l} \left(V'(r_{i-j}) - V'(r_{i-j+l})\right) (\bar{V}'_{l,i} - \hat{V}'_{l,i}) f_s^N d\lambda_s^N \tag{A.30}$$

$$= \beta^{-1} \frac{1}{l^2} \int \sum_{j=0}^{l-1} \sum_{k=0}^{l-1} \frac{jk}{l^2} \left(\left(\frac{\partial}{\partial r_{i-j}} - \frac{\partial}{\partial r_{i-j+l}} \right) \left(V'(r_{i-k}) - V'(r_{i-k+l})\right) \right) f_s^N d\lambda_s^N +$$

$$+ \beta^{-1} \int (\bar{V}'_{l,i} - \hat{V}'_{l,i}) \sqrt{f_s^N} \frac{1}{l} \sum_{j=0}^{l-1} \frac{j}{l} \frac{1}{\sqrt{f_s^N}} \left(\frac{\partial f_s^N}{\partial r_{i-j}} - \frac{\partial f_s^N}{\partial r_{i-j+l}} \right) d\lambda_s^N$$

$$\leq \beta^{-1} \frac{1}{l^2} \int \sum_{j=0}^{l-1} \frac{j^2}{l^2} \left(V''(r_{i-j}) + V''(r_{i-j+l})\right) f_s^N d\lambda_s^N +$$

$$+ \frac{1}{2} \int (\bar{V}'_{l,i} - \hat{V}'_{l,i})^2 f_s^N d\lambda_s^N + \frac{\beta^{-2}}{2} \int \frac{1}{l} \sum_{j=0}^{l-1} \frac{j^2}{l^2} \frac{1}{f_s^N} \left(\frac{\partial f_s^N}{\partial r_{i-j}} - \frac{\partial f_s^N}{\partial r_{i-j+l}} \right)^2 d\lambda_s^N.$$

Thus, we obtain

$$\int (\bar{V}'_{l,i} - \hat{V}'_{l,i})^2 d\mu_s^N \leq 2\beta^{-1} \frac{\|V''\|_\infty}{l} + \beta^{-2} \int \frac{1}{l} \sum_{j=0}^{l-1} \frac{1}{f_s^N} \left(\frac{\partial f_s^N}{\partial r_{i-j}} - \frac{\partial f_s^N}{\partial r_{i-j+l}} \right)^2 d\lambda_s^N \tag{A.31}$$

and the conclusion follows as in the proof of Proposition *A*.4. □

Proposition A.9 (Block average comparison)**.** *Let $a_i \in \{p_i, V'(r_i), r_i\}$ and l_0 as in Proposition A.2. Then, for $l_0 < l \leq N$ we have*

$$\sum_{i=l+1}^{N-l} \int_0^t \int \left(\bar{a}_{l,i} - \hat{a}_{l,i}\right)^2 d\mu_s^N ds \leq C \left(\frac{N}{l} t + \frac{l^2}{\sigma} \right) \leq C(T) \frac{l^2}{\sigma}. \tag{A.32}$$

Proof. The statement for $a_i \in \{p_i, V'(r_i)\}$ is obtained by the previous proposition after summing over i as in the proof of Proposition *A*.5.

In order to prove the statement for $a_i = r_i$ we write

$$\bar{r}_{l,i} - \hat{r}_{l,i} = \frac{1}{l} \sum_{|j|<l} c_j r_{i-j}, \tag{A.33}$$

where $|c_j| < 1$ and $\sum_{|j|<l} c_j = 0$. Let $\alpha > 0$ to be chosen later and follow the proof of Proposition *A*.2 in order to obtain

$$\alpha \sum_{i=l+1}^{N-l} \int_0^t \int \left(\bar{r}_{l,i} - \hat{r}_{l,i}\right)^2 d\mu_s^N ds \leq C \frac{l^3}{\sigma} + \sum_{i=l+1}^{N-l} \int_0^t \int \log \left(\int e^{\alpha \left(\bar{r}_{l,i} - \hat{r}_{l,i}\right)^2} d\hat{\lambda}_{2l-1,i+l-1}^{\ell,\bar{p}} \right) d\mu_s^N ds. \tag{A.34}$$

We introduce a normally distributed $\xi \sim \mathcal{N}(0,1)$ and write, for large enough l,

$$\begin{aligned}\int e^{\alpha\,(\bar{r}_{l,i}-\hat{r}_{l,i})^2} d\hat{\lambda}^{\ell,\bar{p}}_{2l-1,i+l-1} &\le C\int e^{\alpha\,(\bar{r}_{l,i}-\hat{r}_{l,i})^2} d\hat{\lambda}^{N}_{\beta,\bar{p},\tau(\ell)} \\ &= C\mathbb{E}_\xi\left[\int e^{\sqrt{2\alpha}\xi\,(\bar{r}_{l,i}-\hat{r}_{l,i})} d\hat{\lambda}^{N}_{\beta,\bar{p},\tau(\ell)}\right] \\ &= C\mathbb{E}_\xi\left[\int e^{\sqrt{2\alpha}\xi\frac{1}{l}\sum_{|j|<l} c_j r_{i-j}} d\hat{\lambda}^{N}_{\beta,\bar{p},\tau(\ell)}\right] \\ &= C\mathbb{E}_\xi\left[\prod_{|j|<l} e^{G\left(\tau(\ell)+\frac{\sqrt{2\alpha}\xi}{l}c_j\right)-G(\tau(\ell))}\right] \\ &= C\mathbb{E}_\xi\left[e^{\sum_{|j|<l}\left(G'(\tau(\ell))\frac{\sqrt{2\alpha}\xi}{l}c_j+G''(\tilde{\tau})\frac{2\alpha\xi^2}{l^2}c_j^2\right)}\right]\end{aligned} \tag{A.35}$$

for some intermediate value $\tilde{\tau}$. But since $\sum_{|j|<l} c_j = 0$, $|c_j| < 1$ and G'' is bounded, we can estimate

$$\int e^{\alpha\,(\bar{r}_{l,i}-\hat{r}_{l,i})^2} d\hat{\lambda}^{\ell,\bar{p}}_{2l-1,i+l-1} \le C\mathbb{E}_\xi\left[e^{\frac{6\alpha\|G''\|_\infty}{l}\xi^2}\right] = C\frac{1}{\sqrt{1-\dfrac{12\alpha\|G''\|_\infty}{l}}}, \tag{A.36}$$

provided $\dfrac{6\alpha\|G''\|_\infty}{l} < \dfrac{1}{2}$. Note that the right-hand side of (A.36) does not depend on ℓ and $\bar{p}$. Thus, combining (A.34) and (A.36) and choosing $\alpha < \dfrac{1}{12\|G''\|_\infty} l$ leads to the conclusion. □

Corollary A.10. *There exists $l_0 \in \mathbb{N}$ such that, for $l_0 < l < i < N - l + 1$, we have*

$$\int_0^t \int (\bar{r}_{l,i} - \hat{r}_{l,i})^2 \, d\mu_s^N ds \le C\left(\frac{1}{l}t + \frac{l}{\sigma}\right) \le C(T)\frac{l}{\sigma}. \tag{A.37}$$

Corollary A.11. *Let l_0 be as in Proposition A.2. Then, for $l_0 < l < i < N - l + 1$ we have*

$$\begin{aligned}\sum_{j=l+1}^{N-l} \int_0^t \int \left(\hat{V}'_{l,j} - \tau(\hat{r}_{l,j})\right)^2 d\mu_s^N ds &\le C\left(\frac{N}{l}t + \frac{l^2}{\sigma}\right) \le C(T)\frac{l^2}{\sigma} \\ \int_0^t \int \left(\hat{V}'_{l,i} - \tau(\hat{r}_{l,i})\right)^2 d\mu_s^N ds &\le C\left(\frac{1}{l}t + \frac{l}{\sigma}\right) \le C(T)\frac{l}{\sigma}\end{aligned} \tag{A.38}$$

Proof. Follows from Proposition $A.2$, Proposition $A.9$ and the inequality

$$\begin{aligned}\left(\hat{V}'_{l,i} - \tau(\hat{r}_{l,i})\right)^2 &\leq 3\left[\left(\hat{V}'_{l,i} - \bar{V}'_{l,i}\right)^2 + \left(\bar{V}'_{l,i} - \tau(\bar{r}_{l,i})\right)^2 + (\tau(\bar{r}_{l,i}) - \tau(\hat{r}_{l,i}))^2\right] \\ &\leq C\left[\left(\hat{V}'_{l,i} - \bar{V}'_{l,i}\right)^2 + \left(\bar{V}'_{l,i} - \tau(\bar{r}_{l,i})\right)^2 + (\bar{r}_{l,i} - \hat{r}_{l,i})^2\right]\end{aligned} \tag{A.39}$$

□

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References

[1] F Berthelin and J Vovelle. Stochastic isentropic Euler equations. *to appear in Annales de l'ENS*, 2017.

[2] J Feng and D Nualart. Stochastic scalar conservation laws. *Journal of Functional Analysis*, 255(2):313–373, 2008.

[3] J Fritz. Microscopic theory of isothermal elastodynamics. *Archive for rational mechanics and analysis*, 201(1):209–249, 2011.

[4] József Fritz and Bálint Tóth. Derivation of the Leroux system as the hydrodynamic limit of a two-component lattice gas. *Communications in Mathematical Physics*, 249(1):1–27, Jul 2004.

[5] S Marchesani. Hydrodynamic limit for a diffusive system with boundary conditions. *Preprint, https://arxiv.org/abs/1903.08576*, 2019.

[6] S Marchesani and S Olla. On the existence of L^2-valued thermodynamic entropy solutions for a hyperbolic system with boundary conditions. *arXiv preprint arXiv:1811.05300v4, to appear in Comm. PDE.*, 2020.

[7] Stefano Marchesani and Stefano Olla. Hydrodynamic limit for an anharmonic chain under boundary tension. *Nonlinearity*, 31(11):4979, 2018.

[8] D Serre and J W Shearer. Convergence with physical viscosity for nonlinear elasticity. *Preprint, unpublished*, 1994.

[9] J W Shearer. Global existence and compactness in L^p for the quasi-linear wave equation. *Communications in Partial Differential Equations*, 19(11):1829–1878, 1994.

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