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Paul Raynaud de Fitte

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# Almost periodicity and periodicity for nonautonomous random dynamical systems

Paul RAYNAUD DE FITTE \*

Keywords : metric dynamical system; random dynamical system; crude cocycle; nonautonomous dynamical system; stochastic partial differential equation; almost periodic

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## Abstract

We present a notion of almost periodicity which can be applied to random dynamical systems as well as almost periodic stochastic differential equations in Hilbert spaces (abstract stochastic partial differential equations). This concept allows for improvements of known results of almost periodicity in distribution, for general random processes and for solutions to stochastic differential equations.

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# 1 Introduction

Since the introduction of almost periodicity by H. Bohr in the 1920s [10, 11, 12], many new definitions and variants of almost periodicity appeared for functions of a real variable (almost periodicity in the sense of Stepanov, or Weyl, or Besicovich, almost automorphy...), with applications in various fields of mathematics, especially ordinary differential equations and dynamical systems. See [1] for an overview.

In the case of random processes, each of these notions forks into several possible notions, mainly: in distribution (in various senses), in probability (or in  $p$ -mean), or in path distribution. Surveys on such notions in the case of Bohr almost periodicity can be found in [6, 35].

The study of almost periodicity for SDEs seems to start with the Romanian school, which studied Bohr almost periodicity in one-dimensional distribution (APOD) of solutions to almost periodic SDEs: first Halanay [23], then mostly Constantin Tudor and his collaborators in many papers, among them [4, 30, 33, 34, 36]. It was Tudor [35] who proposed the notions of almost periodicity in finite-dimensional distribution (APFD) and almost periodicity in (path) distribution, that we call here APPD. With G. Da Prato in [16], Tudor proved APPD for solutions to semilinear evolution equations in Hilbert spaces. Many papers continue to appear on almost periodicity of solutions to various almost periodic SDEs. In most of these papers, it is the weaker APOD property which is proved, under the name “almost periodicity in distribution”.

In a series of papers which started in 2007, some authors claimed the existence of nontrivial square-mean almost periodic solutions to general semilinear SDEs in Hilbert spaces. Unfortunately, these claims proved to be wrong, even in most elementary examples such as Ornstein-Uhlenbeck stationary process [5, 29]. The error shared by all these papers was an impossible change of variable in the Itô integral. This error is one of the motivations of the present work. Indeed, this change of variable problem leads naturally to the Wiener shift and metric dynamical systems.

Recently, W. Zhang and Z. H. Zheng [39] proposed a notion of Bohr almost periodicity for orbits of random dynamical systems which is the same as ours, in the line of the notion of periodicity proposed by Zhao and his collaborators [18, 19, 20, 21, 40]. Our setting is however less restrictive, since cocycles are only optional, and the random dynamical systems we consider are nonautonomous and they are not necessarily perfect cocycles.

The paper is organized as follows: we present in Section 2 our general setting of a probability space endowed with a group of measure preserving

transformations (a metric dynamical system), and we present the notion of nonautonomous random dynamical system that we use in some places: it is simply a mixture of nonautonomous dynamical system and very crude cocycle over a metric dynamical system. In Section 3, we study a general notion of almost periodicity, called  $\theta$ -almost periodicity, for random processes with values in a Polish space, in connection with the underlying shift  $\theta$  on the probability space. This notion encompasses almost periodicity in probability and, under a uniform integrability condition, almost periodicity in  $p$ -mean. In Section 4, we examine the relation between  $\theta$ -almost periodicity and different notions of almost periodicity in distribution. The strongest one is almost periodicity in path distribution (APPD). This notion is not implied by  $\theta$ -almost periodicity (we exhibit a counterexample), but we provide a sufficient condition, in the case of continuous processes, for both  $\theta$ -almost periodicity and APPD. We conclude in Section 5 with an application to stochastic differential equations in Hilbert spaces, where we improve and simplify a general result on existence and uniqueness of almost periodic solutions, thanks to the metric dynamical system point of view. Our method allows us to tackle periodicity as a particular case. We show that the unique bounded mild solution to some almost periodic semilinear stochastic differential equation in Hilbert spaces is  $\theta$ -almost periodic and almost periodic in path distribution.

## 2 General setting

In all the sequel,  $\mathbb{X}$  is a Polish space, that is, a separable topological space, whose topology is induced by a metric  $\mathfrak{d}$  such that  $(\mathbb{X}, \mathfrak{d})$  is complete.

Unless specifically stated, we identify random variables which are equal  $P$ -almost everywhere, and we denote by  $L^0(\Omega; \mathbb{X})$  the space of equivalence classes, for almost everywhere equality, of measurable mappings from  $\Omega$  to  $\mathbb{X}$ . This space is endowed with the distance

$$\mathfrak{d}_{L^0}(X, Y) = E(\mathfrak{d}(X, Y) \wedge 1),$$

where  $E$  denotes the expectation with respect to  $P$ . The distance  $\mathfrak{d}_{L^0}$  is complete and compatible with the topology of convergence in  $P$ -probability. The law of an element  $X$  of  $L^0(\Omega; \mathbb{X})$  is denoted by  $\text{law}(X)$ . The set of Borel probability measures on  $\mathbb{X}$  is denoted by  $\mathcal{M}^{+,1}(\mathbb{X})$ . We endow it with the topology of narrow (or weak) convergence, that is, the coarsest topology for which the function

$$\begin{cases} \mathcal{M}^{+,1}(\mathbb{X}) & \rightarrow \mathbb{R} \\ \mu & \mapsto \mu(f) := \int_{\mathbb{X}} f d\mu \end{cases}$$

is continuous for every bounded continuous function  $f : \mathbb{X} \rightarrow \mathbb{R}$ . The space  $\mathcal{M}^{+,1}(\mathbb{X})$  is Polish, see, e.g., [31]. A distance which is complete and compatible with the topology of  $\mathcal{M}^{+,1}(\mathbb{X})$  is the Wasserstein distance  $\text{Wass}_0$  associated with the truncated metric  $\mathfrak{d} \wedge 1$ , defined by

$$\text{Wass}_0(\mu, \nu) = \inf_{\text{law}(X)=\mu, \text{law}(Y)=\nu} \mathfrak{d}_{L^0}(X, Y).$$

**Nonautonomous random dynamical systems** In the sequel, we are given a *metric dynamical system*  $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ , that is,  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space, and the *shift transformation*

$$\theta : \begin{cases} \mathbb{R} \times \Omega & \rightarrow \Omega \\ (t, \omega) & \mapsto \theta_t \omega \end{cases}$$

is  $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$ -measurable, where  $\mathcal{B}(\mathbb{R})$  is the Borel  $\sigma$ -algebra of  $\mathbb{R}$ , such that  $\theta_0 = \text{Id}_\Omega$ ,  $\theta_{t+s} = \theta_t \circ \theta_s$  for all  $s, t \in \mathbb{R}$ , and  $\mathbb{P}$  is invariant under  $\theta_t$  for all  $t \in \mathbb{R}$  (we shall express this by saying, for short, that  $\mathbb{P}$  is  $\theta$ -invariant). Let  $\vartheta$  denote the shift transformation on  $\mathbb{R}$  defined by  $\vartheta_t s = s + t$  for all  $s, t \in \mathbb{R}$ . Then

$$\Theta : \begin{cases} \mathbb{R} \times \mathbb{R} \times \Omega & \rightarrow \mathbb{R} \times \Omega \\ (t, s, \omega) & \mapsto \Theta_t(s, \omega) = (\vartheta_t s, \theta_t \omega) \end{cases}$$

is a flow on  $(\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F})$  which preserves the measure  $\lambda \otimes \mathbb{P}$ , where  $\lambda$  is the Lebesgue measure on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ .

We shall sometimes assume the existence of a *very crude cocycle*  $\varphi$  over  $(\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \lambda \otimes \mathbb{P}, \Theta)$ , more precisely, a measurable mapping

$$\varphi : \begin{cases} \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathbb{X} & \rightarrow \mathbb{X} \\ (t, \tau, \omega, x) & \mapsto \varphi(t, \tau, \omega, x) =: \varphi(t, \tau, \omega)x \end{cases}$$

such that  $\varphi(0, \tau, \omega) = \text{Id}_{\mathbb{X}}$  for every  $\tau \in \mathbb{R}$  and  $\mathbb{P}$ -almost every  $\omega \in \Omega$ , and satisfying the *crude cocycle property*

$$\varphi(r+s, \tau, \omega) = \varphi(r, \Theta_s(\tau, \omega)) \circ \varphi(s, \tau, \omega) = \varphi(r, \tau+s, \theta_s \omega) \circ \varphi(s, \tau, \omega) \quad (2.1)$$

(with a slight abuse of notations) for all  $(r, s, \tau) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$  and for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ . Note that the almost sure set may depend on  $(t, s, \tau)$ .

**Remark 2.1.** 1. The cocycle  $\varphi$  is called a *measurable random dynamical system (measurable RDS)* [2] if it is independent of the second variable, that is, if  $\varphi$  has the form  $\varphi(t, \tau, \omega) = \varphi(t, \omega)$  for all  $(t, \tau, \omega) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega$ . On the other hand, if  $\varphi$  is deterministic (that is, independent of  $\omega$ ), it is called a *nonautonomous dynamical system* (see [13, 26]). Thus nonautonomous random dynamical systems are a combination of these two notions. A similar definition is given in [38].

2. When the almost sure set in (2.1) is independent of  $(t, s, \tau)$ , the cocycle  $\varphi$  is said to be *perfect*. Perfection theorems allow to construct perfect modifications of crude cocycles, see [3] or [2, Theorem 1.3.2]. Perfection of the cocycle provides powerful tools such as the multiplicative ergodic theorem (see [2]). However, we are interested here in applications to stochastic differential equations, possibly in infinite dimensions, see Section 5. For such systems, even with global Lipschitz and growth conditions, a stochastic flow does not always exist, see [17, Section 9.1.2] or [22], in that case they are not perfectible.

In this paper, the cocycles we consider satisfy only a mild continuity assumption:

**Definition 2.2.** We say that the cocycle  $\varphi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathbb{X} \rightarrow \mathbb{X}$  is *continuous in probability*, or  $L^0(\Omega; \mathbb{X})$ -*continuous*, if the mapping

$$\begin{cases} \mathbb{R} \times \mathbb{R} \times \mathbb{X} & \rightarrow L^0(\Omega; \mathbb{X}) \\ (t, \tau, x) & \mapsto \varphi(t, \tau, \cdot)x. \end{cases}$$

is continuous.

*In the sequel,  $\varphi$  always denotes a very crude  $L^0(\Omega; \mathbb{X})$ -continuous cocycle over  $(\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \lambda \otimes P, \Theta)$ .*

**Definition 2.3.** A (*complete*) *orbit* of  $\varphi$  is a measurable mapping  $X : \mathbb{R} \times \Omega \rightarrow \mathbb{X}$  such that, for all  $t, s \in \mathbb{R}$ ,

$$X(t + s, \cdot) = \varphi(t, s, \theta_s \cdot)X(s, \cdot), \quad (2.2)$$

where the equality (2.2) holds in  $L^0(\Omega; \mathbb{X})$ , that is,  $P$ -almost everywhere, and the almost sure set may depend on  $t$  and  $s$ .

**Proposition 2.4.** *Every orbit of  $\varphi$  is continuous in probability.*

*Proof.* Let  $X$  be an orbit of  $\varphi$ . Continuity in probability of  $X$  follows from

$$X(t, \cdot) = \varphi(t - t_0, t_0, \theta_{t_0} \cdot)X(t_0, \cdot)$$

which shows that  $X(t, \cdot)$  is continuous on  $[t_0, +\infty[$  for any  $t_0 \in \mathbb{R}$ .  $\square$

### 3 $\theta$ -almost periodicity and $\theta$ -periodicity for random processes

#### 3.1 Almost periodicity in metric spaces

**Definition 3.1.** (a) A set  $A \subset \mathbb{R}$  is said to be *relatively dense* if, for every  $\varepsilon > 0$ , there exists  $l > 0$  such that every interval of length  $l$  has a nonempty intersection with  $A$ .

(b) Let  $x : \mathbb{R} \rightarrow \mathbb{X}$  be a continuous function. Let  $\tau > 0$ . We say that  $\tau$  is an  $\varepsilon$ -almost period of  $x$  if, for every  $t \in \mathbb{R}$ ,  $\mathfrak{d}(x(t), x(t + \tau)) \leq \varepsilon$ .

(c) A continuous function  $x : \mathbb{R} \rightarrow \mathbb{X}$  is said to be *almost periodic* (in Bohr's sense) if, for each  $\varepsilon > 0$ , the set of its  $\varepsilon$ -almost periods is relatively dense.

(d) Let  $\mathbb{Y}$  be a topological space and  $\mathcal{K}$  be the set of compact subsets of  $\mathbb{Y}$ . For each  $K \in \mathcal{K}$ , let  $C_u(K; \mathbb{X})$  denote the space of continuous functions from  $K$  to  $\mathbb{X}$  endowed with the topology of uniform convergence. A continuous function  $x : \mathbb{R} \times \mathbb{Y} \rightarrow \mathbb{X}$  is said to be *almost periodic uniformly with respect to compact subsets of  $\mathbb{Y}$*  if, for each  $K \in \mathcal{K}$ , the mapping

$$\begin{cases} \mathbb{R} & \rightarrow C_u(K; \mathbb{X}) \\ t & \mapsto x(t, \cdot) \end{cases}$$

is almost periodic.

The proof of the following fundamental theorem can be found in classical textbooks, see, e.g., [27, 15].

**Theorem 3.2** (Bochner's criteria). *Let  $x : \mathbb{R} \rightarrow \mathbb{X}$  be a continuous function. The following statements are equivalent:*

- (i)  $x$  is almost periodic.
- (ii) The family of translated mappings  $t \mapsto x(t + \cdot)$ , where  $t$  runs over  $\mathbb{R}$ , is relatively compact in the space  $C_u(\mathbb{R}; \mathbb{X})$  of continuous functions from  $\mathbb{R}$  to  $\mathbb{X}$  endowed with the topology of uniform convergence.
- (iii) For every pair of sequences  $(\alpha'_n)$  and  $(\beta'_n)$  in  $\mathbb{R}$ , there are subsequences  $(\alpha_n)$  of  $(\alpha'_n)$  and  $(\beta_n)$  of  $(\beta'_n)$  respectively, with same indices, such that, for every  $t \in \mathbb{R}$ , the limits

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} x(t + \alpha_n + \beta_m) \text{ and } \lim_{n \rightarrow \infty} x(t + \alpha_n + \beta_n)$$

exist and are equal.

**Remark 3.3.** Characterization (iii) shows that almost periodicity depends only on the topology of  $\mathbb{X}$ , not on its metric nor on any uniform structure on  $\mathbb{X}$ .

Using Characterization (iii), one gets immediately the following useful result:

**Corollary 3.4** (almost periodicity in product spaces). *Let  $\mathbb{Y}$ ,  $\mathbb{Z}$  be metric spaces, and let  $x$ ,  $y$ ,  $z$  be continuous functions from  $\mathbb{R}$  to  $\mathbb{X}$ ,  $\mathbb{Y}$  and  $\mathbb{Z}$  respectively. The following statements are equivalent:*

- (i) *The functions  $x$ ,  $y$  and  $z$  are almost periodic.*
- (ii) *The function  $t \mapsto (x(t), y(t), z(t))$  with values in  $\mathbb{X} \times \mathbb{Y} \times \mathbb{Z}$  is almost periodic.*

The definition of almost periodicity can be extended without change to semimetric spaces, and, by passing to a quotient space, one sees that Theorem 3.2 remains true if  $\mathfrak{d}$  is only a semidistance. Furthermore, if the topology of  $\mathbb{X}$  is defined by a family  $(\mathfrak{d}_i)_{i \in I}$  of semidistances, we can define almost periodicity using these semidistances. The following result (see, e.g., [6, Lemma 4.4]) will also be useful in the sequel.

**Proposition 3.5** (almost periodicity for a family of semidistances). *Assume that the topology of  $\mathbb{X}$  is defined by a family  $(\mathfrak{d}_i)_{i \in I}$  of semidistances. For each  $i \in I$ , let us denote by  $(\mathbb{X}, \mathfrak{d}_i)$  the space  $\mathbb{X}$  endowed with the (non separated) topology associated with  $\mathfrak{d}_i$ . Let  $x : \mathbb{R} \rightarrow \mathbb{X}$  be a continuous function. The following statements are equivalent:*

- (i)  *$x$  is almost periodic.*
- (ii) *For each  $i \in I$ , the function  $x : \mathbb{R} \rightarrow (\mathbb{X}, \mathfrak{d}_i)$  is almost periodic.*

### 3.2 $\theta$ -almost periodicity and $\theta$ -periodicity

The following notion of almost periodicity appeared in [39], in the context of continuous random dynamical systems. It is the natural generalization of the notion of periodicity investigated by Zhao and his collaborators [18, 19, 20, 21, 40], see also Cherubini et al [14] for periodicity in the nonautonomous case. Similarly, the notion of stationarity below (in the autonomous case) can be found in [28].

**Definition 3.6** ( $\theta$ -almost periodicity and  $\theta$ -periodicity). Let  $X : \mathbb{R} \mapsto L^0(\Omega; \mathbb{X})$  be a random process.



- (a) Let  $\varepsilon > 0$ . We say that a number  $\tau \in \mathbb{R}$  is a  $\theta$ - $\varepsilon$ -almost period of  $X$  in probability (or simply a  $\theta$ - $\varepsilon$ -almost period of  $X$ ) if

$$\sup_{t \in \mathbb{R}} \mathfrak{D}_{L^0}(X(t + \tau, \theta_{-\tau} \cdot), X(t, \cdot)) \leq \varepsilon. \quad (3.1)$$

- (b) We say that  $X$  is  $\theta$ -almost periodic in probability (or simply  $\theta$ -almost periodic) if Conditions (i) and (ii) below are satisfied:

- (i) the mapping

$$\begin{cases} \mathbb{R} \times \mathbb{R} & \rightarrow L^0(\Omega; \mathbb{X}) \\ (t, s) & \mapsto X(t + s, \theta_{-s} \cdot) \end{cases}$$

is continuous in probability,

- (ii) for any  $\varepsilon > 0$ , the set of  $\theta$ - $\varepsilon$ -almost periods of  $X$  is relatively dense.

In the case when  $\theta_t = \text{Id}_\Omega$  for all  $t$ , we say that  $X$  is almost periodic in probability.

- (c) Let  $\tau \in \mathbb{R}$ . We say that  $X$  is  $\theta$ - $\tau$ -periodic if, for every  $t \in \mathbb{R}$ ,

$$X(t + \tau, \theta_{-\tau} \cdot) = X(t, \cdot).$$

- (d) We say that  $X$  is  $\theta$ -stationary if  $X$  is  $\theta$ - $\tau$ -periodic for every  $\tau \in \mathbb{R}$ .

**Remark 3.7.**

1. The notions of  $\theta$ - $\varepsilon$ -period,  $\theta$ -almost periodicity,  $\theta$ -periodicity and  $\theta$ -stationarity do not depend on a cocycle  $\varphi$ , they depend only on the underlying shift  $\theta$ . However, the association of  $\theta$ -almost periodicity with Property (2.2) will prove useful, in particular in applications to stochastic differential equations, see Section 5.
2. We can generalize Definition 3.6 by replacing  $\theta_{-\tau}$  by  $\theta_{\ell(\tau)}$  in (3.1), for some fixed linear mapping  $\ell : \mathbb{R} \rightarrow \mathbb{R}$ . The reader can check that all results of this section and of Section 4 remain valid for any other choice of  $\ell$  than  $\ell(\tau) = -\tau$ . Actually, each choice of  $\ell$  amounts to a change of the metric dynamical system  $(\omega, \mathcal{F}, \mathbb{P}, \theta)$  by replacing  $\theta$  with  $\theta'$  given by  $\theta'_t = \theta_{-\ell(t)}$ . Another way to see this equivalence is to notice that  $\theta$ -almost periodicity of  $X$  amounts to almost periodicity in probability of  $\tilde{X} : t \mapsto (X(t, \theta_t \cdot))$  (or more generally  $t \mapsto (X(t, \theta_{\ell(t)} \cdot))$ ). This shows also that  $\theta$ -almost periodicity of  $X$  can be interpreted as ordinary almost periodicity in the sense of Definition 3.1 of some function  $\tilde{X}$  with values in the metric space  $L^0(\Omega; \mathbb{X})$  (see also Proposition 3.17).

However, for orbits of a given cocycle  $\varphi$ , since  $\ell$  is not taken into account in (2.1), each choice of  $\ell$  gives rise to a different class of almost periodic, periodic or stationary orbits.

For applications to stochastic differential equations, the choice  $\ell(\tau) = -\tau$  appears to be more relevant, see Section 5. The choice of  $\ell = 0$  (almost periodicity in probability) led to wrongful claims in many papers, see [5, 29] for details.

3. It is obvious that every  $\theta$ -stationary random process is strictly stationary. Conversely, for every strictly stationary random process, its canonical process is a  $\theta$ -stationary process. More precisely, let  $X : \mathbb{R} \mapsto L^0(\Omega; \mathbb{X})$  be strictly stationary. Let  $\tilde{\Omega} = \mathbb{X}^{\mathbb{R}}$ , endowed with the  $\sigma$ -algebra  $\tilde{\mathcal{F}}$  generated by cylinder sets, and let  $\tilde{\mathbb{P}}$  be the law of  $X$  on  $(\tilde{\Omega}, \tilde{\mathcal{F}})$ . Set  $\tilde{X}(t, \tilde{\omega}) = \tilde{\omega}(t)$  for  $\tilde{\omega} \in \tilde{\Omega}$  and  $t \in \mathbb{R}$ . The mapping  $\tilde{X}$  is a version of a random process  $\mathbb{R} \mapsto L^0(\tilde{\Omega}; \mathbb{X})$  which is  $\theta$ -stationary for the shift transformation  $\theta$  on  $\tilde{\Omega}$  defined by

$$\theta_t(\tilde{\omega}) = \tilde{\omega}(t + \cdot), \quad \tilde{\omega} \in \tilde{\Omega}, \quad t \in \mathbb{R},$$

see [32, Chapter IV] for more details.

**Proposition 3.8** (Closure property). *Let  $(X_n)$  be a sequence of  $\theta$ -almost periodic random processes. Assume further that there exists a random process  $X$  such that*

$$\lim_{n \rightarrow \infty} \sup_{t \in \mathbb{R}} \mathfrak{d}_{L^0}(X_n(t, \cdot), X(t, \cdot)) = 0.$$

*Then  $X$  is  $\theta$ -almost periodic.*

*Proof.* Let  $\varepsilon > 0$ , and let  $N$  such that

$$\sup_{t \in \mathbb{R}} \mathfrak{d}_{L^0}(X_N(t, \cdot), X(t, \cdot)) \leq \frac{\varepsilon}{3}. \quad (3.2)$$

Let  $\tau$  be an  $\varepsilon/3$ -period of  $X_N$ . We have, for every  $t \in \mathbb{R}$ ,

$$\begin{aligned} \mathfrak{d}_{L^0}(X(t + \tau, \theta_{-\tau} \cdot), X(t, \cdot)) &\leq \mathfrak{d}_{L^0}(X(t + \tau, \theta_{-\tau} \cdot), X_N(t + \tau, \theta_{-\tau} \cdot)) \\ &\quad + \mathfrak{d}_{L^0}(X_N(t + \tau, \theta_{-\tau} \cdot), X_N(t, \cdot)) \\ &\quad + \mathfrak{d}_{L^0}(X_N(t, \cdot), X(t, \cdot)) \\ &\leq \varepsilon. \end{aligned}$$

We deduce that the set of  $\varepsilon$ -almost periods of  $X$  is relatively dense.

To prove continuity in probability of  $(t, s) \mapsto X(t+s, \theta_{-s} \cdot)$ , let  $t_0, s_0 \in \mathbb{R}$ , and choose again  $N$  satisfying (3.2). Let  $\eta > 0$  such that

$$\max\{|t - t_0|, |s - s_0|\} < \eta \Rightarrow \mathfrak{d}_{L^0}(X_N(t+s, \theta_{-s}), X_N(t_0+s_0, \theta_{-s_0})) \leq \frac{\varepsilon}{3}.$$

We have, using the invariance of  $\mathbb{P}$  by  $\theta$ ,

$$\begin{aligned} \mathfrak{d}_{L^0}(X(t+s, \theta_{-s}), X(t_0+s_0, \theta_{-s_0})) &\leq \mathfrak{d}_{L^0}(X(t+s, \theta_{-s}), X_N(t+s, \theta_{-s})) \\ &\quad + \mathfrak{d}_{L^0}(X_N(t+s, \theta_{-s}), X_N(t_0+s_0, \theta_{-s_0})) \\ &\quad + \mathfrak{d}_{L^0}(X_N(t_0+s_0, \theta_{-s_0}), X(t_0+s_0, \theta_{-s_0})) \\ &\leq \varepsilon. \end{aligned}$$

□

We focus now on continuity and compactness properties. If  $X$  is an orbit of  $\varphi$ , Condition (b)-(i) of Definition 3.6 can be decomposed into simpler conditions.

**Proposition 3.9** (Joint continuity in probability for orbits of  $\varphi$ ). *Let  $X$  be an orbit of a very crude  $L^0(\Omega; \mathbb{X})$ -continuous cocycle  $\varphi$ . Assume that*

- (i) for every  $\varepsilon > 0$ , the set of  $\theta$ - $\varepsilon$ -almost periods of  $X$  is relatively dense,
- (ii) the mapping

$$\begin{cases} \mathbb{R} & \rightarrow L^0(\Omega; \mathbb{X}) \\ s & \mapsto X(s, \theta_{-s} \cdot) \end{cases}$$

is continuous in probability.

Then  $X$  satisfies Property (b)-(i) of Definition 3.6.

*Proof.* Note that, by (2.2), Hypothesis (ii) implies that, for every  $t \geq 0$ , the mapping  $s \mapsto X(t+s, \theta_{-s} \cdot)$  is continuous in probability. For  $t < 0$ , we arrive at the same conclusion with the help of (i): let  $\varepsilon > 0$ , and let  $\tau$  be an  $\varepsilon$ -almost period of  $X$  such that  $t + \tau > 0$ . We have, for  $s_0, s \in \mathbb{R}$

$$\begin{aligned} \mathfrak{d}_{L^0}(X(t+s, \theta_{-s}), X(t+s, \theta_{-s})) &\leq \mathfrak{d}_{L^0}(X(t+s, \theta_{-s}), X(t+s+\tau, \theta_{-s-\tau})) \\ &\quad + \mathfrak{d}_{L^0}(X(t+s+\tau, \theta_{-s-\tau}), X(t+s_0+\tau, \theta_{-s_0-\tau})) \\ &\quad + \mathfrak{d}_{L^0}(X(t+s_0+\tau, \theta_{-s_0-\tau}), X(t+s, \theta_{-s})) \end{aligned}$$

$$\leq \mathfrak{d}_{L^0}\left(X(t+s+\tau, \theta_{-s-\tau}), X(t+s_0+\tau, \theta_{-s_0-\tau})\right) + 2\varepsilon.$$

The claim follows from the continuity of  $s \mapsto X(t+\tau+s, \theta_{-s-\tau} \cdot)$ .

Let us now prove that the mapping  $t \mapsto X(t+s, \theta_{-s} \cdot)$  is continuous in probability, uniformly with respect to  $s$  in compact intervals. Let  $t_0 \in \mathbb{R}$ , let  $J$  be a compact interval, and let  $\varepsilon > 0$ . By continuity of  $s \mapsto X(t_0+s, \theta_{-s} \cdot)$ , the family of random elements of the form  $X(t_0+s, \theta_{-s} \cdot)$  with  $s \in J$  is uniformly tight, thus there exists a compact subset  $K$  of  $\mathbb{X}$  such that, for each  $s \in J$ ,  $\mathbb{P}\{X(t_0+s, \theta_{-s} \cdot) \notin K\} \leq 1 - \varepsilon/2$ .

On the other hand, by uniform continuity in probability of  $(r, s, x) \mapsto \varphi(t, t_0+s, \theta_{t_0} \cdot)x$  on the compact set  $[0, 1] \times (J+t_0+[-1, 1]) \times K$ , and since  $\mathfrak{d}_{L^0}(\varphi(0, t_0+s, \theta_{t_0} \cdot)x, x) = 0$ , we can find  $\delta > 0$  such that

$$0 \leq r \leq \delta \Rightarrow \mathfrak{d}_{L^0}(\varphi(r, t+s, \cdot)x, x) \leq \varepsilon/2 \quad (t-t_0 \in [-1, 1], s \in J, x \in K). \quad (3.3)$$

From (3.3) and the definition of  $\mathfrak{d}_{L^0}$ , we deduce that, for  $s \in J$  and  $0 \leq t-t_0 \leq \delta$ , we have

$$\begin{aligned} & \mathfrak{d}_{L^0}(X(t+s, \theta_{-s} \cdot), X(t_0+s, \theta_{-s} \cdot)) \\ &= \mathfrak{d}_{L^0}(\varphi(t-t_0, t_0+s, \theta_{t_0} \cdot)X(t_0+s, \theta_{-s} \cdot), X(t_0+s, \theta_{-s} \cdot)) \leq \varepsilon. \end{aligned}$$

If  $t \leq t_0$ , using again (3.3) and the definition of  $\mathfrak{d}_{L^0}$ , we get, since  $\theta$  is measure preserving,

$$\begin{aligned} & \mathfrak{d}_{L^0}(X(t+s, \theta_{-s} \cdot), X(t_0+s, \theta_{-s} \cdot)) \\ &= \mathfrak{d}_{L^0}(X(t+s, \theta_{-(t+s)} \cdot), X(t_0+s, \theta_{-(t+s)} \cdot)) \\ &= \mathfrak{d}_{L^0}(X(t+s, \theta_{-(t+s)} \cdot), \varphi(t_0-t, t+s, \theta_t \cdot)X(t+s, \theta_{-(t+s)} \cdot)) \leq \varepsilon. \end{aligned}$$

So, we have proved our second claim.

Now, let  $t_0, s_0 \in \mathbb{R}$ , and let  $\varepsilon > 0$ . By our hypothesis, there exists  $\delta_1 > 0$  such that  $|s-s_0| \leq \delta_1$  implies

$$\mathfrak{d}_{L^0}(X(t_0+s, \theta_{-s} \cdot), X(t_0+s_0, \theta_{-s_0} \cdot)) \leq \varepsilon/2. \quad (3.4)$$

But we have proved that there exists  $\delta_2 > 0$  such that, for  $|t-t_0| \leq \delta_2$ , and for all  $s \in [s_0 - \delta_1, s_0 + \delta_1]$ ,

$$\mathfrak{d}_{L^0}(X(t+s, \theta_{-s} \cdot), X(t_0+s, \theta_{-s} \cdot)) \leq \varepsilon/2. \quad (3.5)$$

The result follows immediately from (3.4) and (3.5).  $\square$

**Proposition 3.10** (Compactness). *Let  $X : \mathbb{R} \rightarrow L^0(\Omega; \mathbb{X})$  be an almost periodic random process, and let  $J$  be a compact interval of  $\mathbb{R}$ . Then*

- (i) *The set  $\mathcal{L}_J = \{X(s+t, \theta_{-t} \cdot); s \in J, t \in \mathbb{R}\}$  is relatively compact in  $L^0(\Omega; \mathbb{X})$ ,*
- (ii) *the set  $\mathcal{K} = \{\text{law}(X(t, \cdot)); t \in \mathbb{R}\}$  is uniformly tight, that is, for each  $\varepsilon > 0$ , there exists a compact subset  $K$  of  $\mathbb{X}$  such that*

$$\sup_{t \in \mathbb{R}} \mathbb{P} \{\omega \in \Omega; X(t, \omega) \notin K\} \leq \varepsilon.$$

*Proof.* Let  $\varepsilon > 0$ , and let  $l > 0$  such that any interval of length  $l$  contains an  $\varepsilon$ -almost period of  $X$ . Let  $I = [-l/2, l/2]$ . By continuity in probability of  $(s, t) \mapsto X(s+t, \theta_{-t} \cdot)$ , the set of random variables  $\mathcal{J} = \{X(s+t, \theta_{-t} \cdot); s \in J, t \in I\}$  is a compact subset of  $L^0(\Omega; \mathbb{X})$ . Now, let  $s \in J, t \in \mathbb{R}$ , and let  $\tau \in [-t - l/2, -t + l/2]$  be an  $\varepsilon$ -almost period of  $X$ . We have  $t + \tau \in I$  and

$$\mathfrak{d}_{L^0}(X(s+t, \theta_{-t} \cdot), X(s+t+\tau, \theta_{-t-\tau} \cdot)) \leq \varepsilon,$$

with  $X(s+t+\tau, \theta_{-t-\tau} \cdot) \in \mathcal{J}$ . Thus  $\mathcal{J}$  is an  $\varepsilon$ -net of  $\mathcal{L}_J$  for the distance  $\mathfrak{d}_{L^0}$ , which proves (i).

Relative compactness of  $\mathcal{K}$  follows from (i) with  $J = \{0\}$  by continuity of the mapping  $Y \mapsto \text{law}(Y)$  from  $L^0(\Omega; \mathbb{X})$  to  $\mathcal{M}^{+,1}(\mathbb{X})$ , because, since  $\theta_{-t}$  is measure preserving, we have

$$\mathcal{K} = \{\text{law}(X(0+t, \theta_{-t} \cdot)); t \in \mathbb{R}\} = \{\text{law}(Y); Y \in \mathcal{L}_0\}.$$

Since  $\mathbb{X}$  is Polish,  $\mathcal{K}$  is uniformly tight, by Prokhorov's well-known theorem (see, e.g., [9]), which proves (ii).  $\square$

**Theorem 3.11** (Equicontinuity and uniform continuity in probability). *Let  $X : \mathbb{R} \rightarrow L^0(\Omega; \mathbb{X})$  be a  $\theta$ -almost periodic random process. Then,*

- (a) *The mapping  $t \mapsto X(t+s, \theta_{-s} \cdot)$  is continuous in probability, uniformly with respect to  $s \in \mathbb{R}$ .*
- (b) *The mapping  $s \mapsto X(t+s, \theta_{-s} \cdot)$  is uniformly continuous in probability, uniformly with respect to  $t \in \mathbb{R}$ .*

*Proof.* (a) Joint continuity in probability of  $(t, s) \mapsto X(t+s, \theta_{-s} \cdot)$  is provided by Condition (b)-(i) of Definition 3.6. For the uniformity with respect to  $s \in \mathbb{R}$ , let  $\varepsilon > 0$ , and let  $l > 0$  such that any interval of length  $l$  contains an  $\varepsilon/3$ -almost period of  $X$ . For each relative integer  $k$ , set  $I_k = [-l/2 +$

$kl, l/2 + kl]$ , and let  $J = [-l, l]$ . Let  $t_0 \in \mathbb{R}$ . By uniform continuity of  $(t, s) \mapsto X(t + s, \theta_{-s} \cdot)$  on  $[t_0 - 1, t_0 + 1] \times J$ , there exists  $\delta \in ]0, 1]$  such that, for all  $s \in J$  and for  $|t - t_0| \leq \delta$ , we have

$$\mathfrak{d}_{L^0}\left(X(t + s, \theta_{-s} \cdot), X(t_0 + s, \theta_{-s} \cdot)\right) \leq \varepsilon/3.$$

Let  $s \in \mathbb{R}$ , let  $k$  such that  $s \in I_k$ , and let  $\tau_{-k} \in I_{-k}$  be an  $\varepsilon/3$ -almost period of  $X$ . We have  $s + \tau_k \in J$ , thus

$$\begin{aligned} & \mathfrak{d}_{L^0}\left(X(t + s, \theta_{-s} \cdot), X(t_0 + s, \theta_{-s} \cdot)\right) \\ & \leq \mathfrak{d}_{L^0}\left(X(t + s, \theta_{-s} \cdot), X(t + s + \tau_k, \theta_{-s-\tau_k} \cdot)\right) \\ & \quad + \mathfrak{d}_{L^0}\left(X(t + s + \tau_k, \theta_{-s-\tau_k} \cdot), X(t_0 + s + \tau_k, \theta_{-s-\tau_k} \cdot)\right) \\ & \quad + \mathfrak{d}_{L^0}\left(X(t_0 + s + \tau_k, \theta_{-s-\tau_k} \cdot), X(t_0 + s, \theta_{-s} \cdot)\right) \\ & \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

(b) Let  $t_0 \in \mathbb{R}$ . Let  $\varepsilon > 0$ , and let  $l > 0$  such that any interval of length  $l$  contains an  $\varepsilon/3$ -almost period of  $X$ . Let  $J = [-l, l]$ . The mapping  $s \mapsto X(t_0 + s, \theta_{-s} \cdot)$  is uniformly continuous in probability on  $J$ , thus we can find  $\delta > 0$  such that  $\mathfrak{d}_{L^0}(X(t_0 + s, \theta_{-s} \cdot), X(t_0 + r, \theta_{-r} \cdot)) < \varepsilon/3$  for all  $r, s \in J$  such that  $|s - r| < \delta$ . We can choose  $\delta < l$ . Let  $r, s \in \mathbb{R}$  with  $|s - r| < \delta$ , and let  $m = (r + s)/2$ , so that  $r, s \in [m - l/2, m + l/2]$ . Let  $\tau \in [-m - l/2, -m + l/2]$  be an  $\varepsilon/3$ -almost period of  $X$ . We have

$$\begin{aligned} & \mathfrak{d}_{L^0}(X(t_0 + r, \theta_{-r} \cdot), X(t_0 + s, \theta_{-s} \cdot)) \\ & \leq \mathfrak{d}_{L^0}(X(t_0 + r, \theta_{-r} \cdot), X(t_0 + r + \tau, \theta_{-r-\tau} \cdot)) \\ & \quad + \mathfrak{d}_{L^0}(X(t_0 + r + \tau, \theta_{-r-\tau} \cdot), X(t_0 + s + \tau, \theta_{-s-\tau} \cdot)) \\ & \quad + \mathfrak{d}_{L^0}(X(t_0 + s + \tau, \theta_{-s-\tau} \cdot), X(t_0 + s, \theta_{-s} \cdot)) \\ & \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Then, for any  $t \in \mathbb{R}$ , and for  $r, s \in \mathbb{R}$  such that  $|s - r| < \delta$ , we get, since  $\theta_{t-t_0}$  is measure preserving,

$$\begin{aligned} & \mathfrak{d}_{L^0}(X(t + r, \theta_{-r} \cdot), X(t + s, \theta_{-s} \cdot)) \\ & = \mathfrak{d}_{L^0}(X(t_0 + (r + t - t_0), \theta_{-r-t+t_0} \cdot), X(t_0 + (s + t - t_0), \theta_{-s-t+t_0} \cdot)) \\ & \leq \varepsilon. \end{aligned}$$

□

**Theorem 3.12** (Bochner's criteria for  $\theta$ -almost periodicity). *Let  $X : \mathbb{R} \mapsto L^0(\Omega; \mathbb{X})$  be a random process satisfying Property (b)-(i) of Definition 3.6. The following statements are equivalent:*

(i)  $X$  is  $\theta$ -almost periodic.

(ii) (Bochner's criterion [7]) *The family of mappings*

$$\mathfrak{T}_s X : \begin{cases} \mathbb{R} & \rightarrow L^0(\Omega; \mathbb{X}) \\ t & \mapsto X(t + s, \theta_{-s} \cdot), \end{cases}$$

*where  $s$  runs over  $\mathbb{R}$ , is relatively compact in the space  $C_u(\mathbb{R}; L^0(\Omega; \mathbb{X}))$  of continuous functions from  $\mathbb{R}$  to  $L^0(\Omega; \mathbb{X})$  endowed with the topology of uniform convergence, that is, for every sequence  $(\gamma'_n)$  of real numbers, there exists a subsequence  $(\gamma_n)$  of  $(\gamma'_n)$  and a random process  $Y : \mathbb{R} \rightarrow L^0(\Omega; \mathbb{X})$ , continuous in probability, such that*

$$\limsup_{n \rightarrow \infty} \sup_{t \in \mathbb{R}} \mathfrak{d}_{L^0}(\mathfrak{T}_{\gamma_n} X(t, \cdot), Y(t, \cdot)) = 0.$$

(iii) (Bochner's double sequence criterion [8]) *For every pair of sequences  $(\alpha'_n)$  and  $(\beta'_n)$  in  $\mathbb{R}$ , there are subsequences  $(\alpha_n)$  of  $(\alpha'_n)$  and  $(\beta_n)$  of  $(\beta'_n)$  respectively, with same indices, such that, for every  $t \in \mathbb{R}$ , the limits in probability*

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} X(t + \alpha_n + \beta_m, \theta_{-\alpha_n - \beta_m} \cdot) \text{ and } \lim_{n \rightarrow \infty} X(t + \alpha_n + \beta_n, \theta_{-\alpha_n - \beta_n} \cdot) \quad (3.6)$$

*exist and are equal.*

*Proof.* (i) $\Rightarrow$ (iii): Let  $(\alpha'_n)$  and  $(\beta'_n)$  be two sequences in  $\mathbb{R}$ , and let  $t \in \mathbb{R}$ . Let  $\varepsilon > 0$ , let  $l$  be such that each interval of length  $l$  contains an  $\varepsilon/2$ -almost period of  $X$ , and let  $J = [-l/2, l/2]$ . For each  $s \in \mathbb{R}$ , let  $\tau_s \in [s - l/2, s + l/2]$  be an  $\varepsilon/2$ -almost period of  $X$ , and let  $s^\varepsilon = s - \tau_s \in J$ . We can extract from the sequences  $(\alpha'_n)$  and  $(\beta'_n)$  two subsequences  $(\alpha_n)$  and  $(\beta_n)$  respectively, with same indices, which converge in  $J$  to some limits  $\alpha^\varepsilon$  and  $\beta^\varepsilon$  respectively. Then, by the continuity property (b)-(i) of Definition 3.6, the following limits in probability exist for any  $t \in \mathbb{R}$ :

$$\begin{aligned} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} X(t + \alpha_n^\varepsilon + \beta_m^\varepsilon, \theta_{-\alpha_n^\varepsilon - \beta_m^\varepsilon} \cdot) &= X(t + \alpha^\varepsilon + \beta^\varepsilon, \theta_{-\alpha^\varepsilon - \beta^\varepsilon} \cdot) \\ &= \lim_{n \rightarrow \infty} X(t + \alpha_n^\varepsilon + \beta_n^\varepsilon, \theta_{-\alpha_n^\varepsilon - \beta_n^\varepsilon} \cdot), \end{aligned} \quad (3.7)$$

whith, for all integers  $n, m$

$$\begin{aligned}
& \mathfrak{d}_{L^0}\left(X(t + \alpha_n + \beta_m, \theta_{-\alpha_n - \beta_m} \cdot), X(t + \alpha_n^\varepsilon + \beta_m^\varepsilon, \theta_{-\alpha_n^\varepsilon - \beta_m^\varepsilon} \cdot)\right) \\
& \leq \mathfrak{d}_{L^0}\left(X(t + \alpha_n + \beta_m, \theta_{-\alpha_n - \beta_m} \cdot), X(t + \alpha_n^\varepsilon + \beta_m, \theta_{-\alpha_n^\varepsilon - \beta_m} \cdot)\right) \\
& \quad + \mathfrak{d}_{L^0}\left(X(t + \alpha_n^\varepsilon + \beta_m, \theta_{-\alpha_n^\varepsilon - \beta_m} \cdot), X(t + \alpha_n^\varepsilon + \beta_m^\varepsilon, \theta_{-\alpha_n^\varepsilon - \beta_m^\varepsilon} \cdot)\right) \\
& \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad (3.8)
\end{aligned}$$

Let us repeat this procedure for  $\varepsilon = 1/k$ , where  $k \geq 1$  is an integer, in such a way that, for each  $k$ , the sequences  $(\alpha_n^{1/(k+1)})$  and  $(\beta_n^{1/(k+1)})$  are subsequences of  $(\alpha_n^{1/k})$  and  $(\beta_n^{1/k})$  respectively. Let  $(\alpha_n)$  and  $(\beta_n)$  be the subsequences of  $(\alpha_n')$  and  $(\beta_n')$  respectively corresponding to  $(\alpha_n^{1/n})$  and  $(\beta_n^{1/n})$ . By (3.7), for any integer  $N \geq 1$ , the following limits in probability exist:

$$\begin{aligned}
& \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} X(t + \alpha_n^{1/N} + \beta_m^{1/N}, \theta_{-\alpha_n^{1/N} - \beta_m^{1/N}} \cdot) \\
& = \lim_{n \rightarrow \infty} X(t + \alpha_n^{1/N} + \beta_n^{1/N}, \theta_{-\alpha_n^{1/N} - \beta_n^{1/N}} \cdot). \quad (3.9)
\end{aligned}$$

On the other hand, we deduce from (3.8) that, for  $n, m \geq N$ ,

$$\mathfrak{d}_{L^0}\left(X(t + \alpha_n + \beta_m, \theta_{-\alpha_n - \beta_m} \cdot), X(t + \alpha_n^{1/n} + \beta_m^{1/n}, \theta_{-\alpha_n^{1/n} - \beta_m^{1/n}} \cdot)\right) \leq \frac{1}{N}. \quad (3.10)$$

The result follows from (3.9) and (3.10), since  $N$  is arbitrary.

(iii) $\Rightarrow$ (ii): Taking  $\beta_n' = 0$ , we see that, for each sequence  $(\alpha_n')$  in  $\mathbb{R}$ , there exists a subsequence  $(\alpha_n)$  and a random process  $X : \mathbb{R} \rightarrow L^0(\Omega; \mathbb{X})$  such that, for every  $t \in \mathbb{R}$ ,  $X(t + \alpha_n, \theta_{-\alpha_n} \cdot)$  converges in probability to  $Y(t, \cdot)$ . So, we only need to prove that this convergence is uniform with respect to  $t$ . Assuming this is not the case, we get the existence of  $\varepsilon > 0$  and a sequence  $(\beta_n)$  in  $\mathbb{R}$  such that, for every  $n$ ,

$$\mathfrak{d}_{L^0}\left(X(\beta_n + \alpha_n, \theta_{-\alpha_n} \cdot), Y(\beta_n, \cdot)\right) \geq \varepsilon,$$

that is,

$$\mathfrak{d}_{L^0}\left(X(\beta_n + \alpha_n, \theta_{-\alpha_n - \beta_n} \cdot), Y(\beta_n, \theta_{-\beta_n} \cdot)\right) \geq \varepsilon, \quad (3.11)$$

By (iii), extracting subsequences, we may assume that we have the following limits in probability:

$$\begin{aligned}
\lim_{m \rightarrow \infty} Y(\beta_m, \theta_{-\beta_m} \cdot) & = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} X(\alpha_n + \beta_m, \theta_{-\alpha_n - \beta_m} \cdot) \\
& = \lim_{n \rightarrow \infty} X(\alpha_n + \beta_n, \theta_{-\alpha_n - \beta_n} \cdot) \\
& = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} X(\alpha_n + \beta_m, \theta_{-\alpha_n - \beta_m} \cdot),
\end{aligned}$$



which contradicts (3.11).

(ii) $\Rightarrow$ (i): Let  $\varepsilon > 0$ . By total boundedness of the family  $\{\mathfrak{T}_s X; s \in \mathbb{R}\}$ , we can find a finite sequence  $\gamma_1, \dots, \gamma_n$  such that, for each  $s \in \mathbb{R}$ , there exists  $k \in \{1, \dots, n\}$  such that

$$\sup_{t \in \mathbb{R}} \mathfrak{d}_{L^0}(\mathfrak{T}_s X(t), \mathfrak{T}_{\gamma_k} X(t)) \leq \varepsilon,$$

that is,

$$\sup_{t \in \mathbb{R}} \mathfrak{d}_{L^0}(X(t, \cdot), X(t + s - \gamma_k, \theta_{-s + \gamma_k}), X(t, \cdot)) \leq \varepsilon,$$

which shows that  $s - \gamma_k$  is an  $\varepsilon$ -almost period of  $X$ . Let  $l = \max\{\gamma_1, \dots, \gamma_n\}$ . Then  $s - \gamma_k \in [s - l, s + l]$ , thus each interval of length  $2l$  contains an  $\varepsilon$ -almost period of  $X$ .  $\square$

**Remark 3.13.** Bochner's double sequence criterion and the proof of Theorem 3.12 show that  $\theta$ -almost periodicity is a property which remains unchanged if we replace  $\mathfrak{d}$  by any other distance compatible with the topology of  $\mathbb{X}$ , or if we replace  $\mathfrak{d}_{L^0}$  by any other distance compatible with the topology of convergence in probability.

### 3.3 $\theta$ -almost periodicity in $p$ -mean

We present here a stronger notion of almost periodicity, which depends on the distance  $\mathfrak{d}$  on  $\mathbb{X}$ .

Let  $p \geq 1$ . Let us denote by  $L^p(\Omega; \mathbb{X})$  the set of elements  $X$  of  $L^0(\Omega; \mathbb{X})$  such that, for some (equivalently, for any)  $x_0 \in \mathbb{X}$ ,

$$E(\mathfrak{d}(X, x_0))^p < \infty.$$

We endow  $L^p(\Omega; \mathbb{X})$  with the distance

$$\mathfrak{d}_{L^p}(X, Y) = \left( E(\mathfrak{d}(X, Y))^p \right)^{1/p}.$$

Similarly to definition 3.6, replacing  $\mathfrak{d}_{L^0}$  by  $\mathfrak{d}_{L^p}$ , we set:

**Definition 3.14.** Let  $X : \mathbb{R} \mapsto L^0(\Omega; \mathbb{X})$  be a random process such that  $X(t, \cdot) \in L^p(\Omega; \mathbb{X})$  for each  $t \in \mathbb{R}$ .

(a) We say that a number  $\tau \in \mathbb{R}$  is a  $\theta$ - $\varepsilon$ -almost period of  $X$  in  $p$ -mean if

$$\sup_{t \in \mathbb{R}} \mathfrak{d}_{L^p}(X(t + \tau, \theta_{-\tau} \cdot), X(t, \cdot)) \leq \varepsilon.$$

- (b) We say that  $X$  is  $\theta$ -almost periodic in  $p$ -mean if the mapping  $(t, s) \mapsto X(t + s, \theta_{-s} \cdot)$  is continuous for the distance  $\mathfrak{d}_{L^p}$  and if, for each  $\varepsilon > 0$ , the set of  $\theta$ - $\varepsilon$ -almost periods in  $p$ -mean of  $X$  is relatively dense.

In the case when  $\theta_t = \text{Id}_\Omega$  for all  $t$ , we say that  $X$  is *almost periodic in  $p$ -mean*.

It is not difficult to check that all preceding results remain true if we replace  $\mathfrak{d}_{L^0}$  by  $\mathfrak{d}_{L^p}$ . Furthermore, using Vitali's theorem and Bochner's double sequence criterion in Theorem 3.12-(iii), we have immediately:

**Proposition 3.15.** *Let  $p \geq 1$ , and let  $X : \mathbb{R} \mapsto L^0(\Omega; \mathbb{X})$  be a random process such that  $X(t, \cdot) \in L^p(\Omega; \mathbb{X})$  for each  $t \in \mathbb{R}$ . The following statements are equivalent:*

- (i)  $X$  is  $\theta$ -almost periodic in  $p$ -mean.
- (ii)  $X$  is  $\theta$ -almost periodic and, for some (equivalently, for any)  $x_0 \in \mathbb{X}$ , the random variables  $(\mathfrak{d}(X(t, \cdot), x_0))^p$ ,  $t \in \mathbb{R}$ , are uniformly integrable.

### 3.4 Almost periodicity in $C_u(\mathbb{R}; L^0(\Omega; \mathbb{X}))$ of the translation map

**Definition 3.16.** Let  $X : \mathbb{R} \mapsto L^0(\Omega; \mathbb{X})$  be a random process. Assume that  $X$  is continuous in probability.

- (a) The *translation operator* of  $X$  is the map

$$\mathfrak{T} : \begin{cases} \mathbb{R} & \rightarrow C(\mathbb{R}; L^0(\Omega; \mathbb{X})) \\ t & \mapsto \mathfrak{T}_t X(\cdot, \cdot) = X(t + \cdot, \theta_{-t} \cdot). \end{cases}$$

- (b) The process  $\mathfrak{T}_t X$  is called the  $t$ -translate of  $X$ .

We denote by  $C_u(\mathbb{R}; L^0(\Omega; \mathbb{X}))$  the space  $C_u(\mathbb{R}; L^0(\Omega; \mathbb{X}))$  of continuous functions from  $\mathbb{R}$  to  $L^0(\Omega; \mathbb{X})$  endowed with the topology of uniform convergence associated with  $\mathfrak{d}_{L^0}$ . We denote by  $\mathfrak{d}_{L^0, \infty}$  the distance on  $C_u(\mathbb{R}; L^0(\Omega; \mathbb{X}))$  defined by

$$\mathfrak{d}_{L^0, \infty}(X, Y) = \sup_{t \in \mathbb{R}} \mathfrak{d}_{L^0}(X(t), Y(t)).$$

The next result shows that we can see  $\theta$ -almost periodicity of a random process  $X$  as ordinary almost periodicity of a function  $Y$  with values in a metric space.

**Proposition 3.17.** *Let  $X : \mathbb{R} \rightarrow L^0(\Omega; \mathbb{X})$  be a random process. Assume that  $X$  is continuous in probability. Let  $Y$  be the translation mapping*

$$Y : \begin{cases} \mathbb{R} & \rightarrow C_u(\mathbb{R}; L^0(\Omega; \mathbb{X})) \\ t & \mapsto Y(t) = \mathfrak{T}_t X \end{cases}$$

*Then, for each  $\varepsilon > 0$ ,  $X$  and  $Y$  have the same  $\varepsilon$ -almost periods. Furthermore,  $X$  is  $\theta$ -almost periodic in the sense of Definition 3.6 if, and only if,  $Y$  is almost periodic in the sense of Definition 3.1.*

*Proof.* Let  $\tau \in \mathbb{R}$ . We have, for any  $t \in \mathbb{R}$ ,

$$\begin{aligned} \mathfrak{d}_{L^0, \infty}(Y(t + \tau), Y(t)) &= \sup_{s \in \mathbb{R}} \mathfrak{d}_{L^0}(X(s + t + \tau, \theta_{-t-\tau} \cdot), X(s + t, \theta_{-t} \cdot)) \\ &= \sup_{s \in \mathbb{R}} \mathfrak{d}_{L^0}(X(s + \tau, \theta_{-\tau} \cdot), X(s, \cdot)). \end{aligned}$$

Since the last term is independent of  $t$ , this yields

$$\sup_{t \in \mathbb{R}} \mathfrak{d}_{L^0, \infty}(Y(t + \tau), Y(t)) = \sup_{s \in \mathbb{R}} \mathfrak{d}_{L^0}(X(s + \tau, \theta_{-\tau} \cdot), X(s, \cdot)). \quad (3.12)$$

Thus, for every  $\varepsilon > 0$ ,  $\tau$  is a  $\theta$ - $\varepsilon$ -almost period of  $X$  if, and only if, it is an  $\varepsilon$ -almost period of  $Y$ .

If  $X$  is  $\theta$ -almost periodic, then (3.12) and Part (b) of Theorem 3.11 show that  $Y$  is continuous, thus  $Y$  is almost periodic in the sense of Definition 3.1.

Conversely, if  $Y$  is continuous, then (3.12) and the reasoning of the second part of the proof of Proposition 3.9 show that  $X$  satisfies Condition (b)-(i) of Definition 3.6, thus it is  $\theta$ -almost periodic.  $\square$

**Remark 3.18** (Almost periodicity of the translate function does not depend on any uniform structure). To define the state space of  $Y$  in Proposition 3.17, we have used the topology of uniform convergence on  $C(\mathbb{R}; L^0(\Omega; \mathbb{X}))$ , which is related to the distance  $\mathfrak{d}_{L^0}$ . So, it might seem that the property of almost periodicity of  $Y$  depends on the distance  $\mathfrak{d}_{L^0}$ . But Remark 3.13 combined with Proposition 3.17 show that this is not the case and that any other distance than  $\mathfrak{d}_{L^0}$  compatible with the topology of convergence in probability lets the almost periodicity property of  $Y$  unchanged.

## 4 Almost periodicity in distribution

### 4.1 Different notions of almost periodicity in distribution

The following definitions are inspired by Tudor [35], see also [6].

In the sequel, we denote by  $C_k(\mathbb{R}; \mathbb{X})$  the space of continuous functions from  $\mathbb{R}$  to  $\mathbb{X}$  endowed with the compact-open topology, that is, the topology of uniform convergence on compact subsets (equivalently, on compact intervals) of  $\mathbb{R}$ . It is well known that  $C_k(\mathbb{R}; \mathbb{X})$  is Polish.

**Definition 4.1.** (a) A random process  $X : \mathbb{R} \rightarrow L^0(\Omega; \mathbb{X})$  is *almost periodic in one-dimensional distribution*, or *APOD* (respectively, *periodic in one-dimensional distribution*, or *POD*) if the mapping

$$\begin{cases} \mathbb{R} & \rightarrow \mathcal{M}^{+,1}(\mathbb{X}) \\ t & \mapsto \text{law}(X(t, \cdot)) \end{cases}$$

is almost periodic (resp. periodic).

(b) The process  $X$  is *almost periodic in finite-dimensional distribution*, or *APFD* (respectively, *periodic in finite-dimensional distribution*, or *PFD*), if, for every finite sequence  $(t_1, \dots, t_n)$ , the mapping

$$\begin{cases} \mathbb{R} & \rightarrow \mathcal{M}^{+,1}(\mathbb{X}^n) \\ t & \mapsto \text{law}(\mathfrak{T}_t X(t_1, \cdot), \dots, \mathfrak{T}_t X(t_n, \cdot)) \end{cases}$$

is almost periodic (resp.  $\tau$ -periodic, for some  $\tau > 0$  which does not depend on  $(t_1, \dots, t_n)$ ).

(c) If the process  $X$  has a version with continuous trajectories (for simplicity, let us denote by  $X$  this version), we say that  $X$  is *almost periodic in path distribution*, or *APPD*, if the mapping

$$\begin{cases} \mathbb{R} & \rightarrow \mathcal{M}^{+,1}(C_k(\mathbb{R}; \mathbb{X})) \\ t & \mapsto \text{law}(\mathfrak{T}_t X) \end{cases}$$

is almost periodic.

**Remark 4.2.**

1. Clearly, APPD  $\Rightarrow$  APFD  $\Rightarrow$  APOD and PFD  $\Rightarrow$  POD. The notion of periodicity in path distribution (PPD) is not relevant since it is equivalent to PFD. Indeed, assume that  $X$  has a continuous version and that all finite distributions of  $X$  are  $\tau$ -periodic. Let  $J = [t_0, t_0 + T]$  be a fixed interval. Let  $X_J$  be the random variable with values in  $C_u(J, \mathbb{X})$  defined by

$$X_J(\omega)(t) = X(t, \omega) \quad (t \in J).$$

We define similarly the random variable  $X_{J+\tau}$ . To prove that  $X_J$  and  $X_{J+\tau}$  have the same distribution, we embed isometrically  $\mathbb{X}$  in some separable Banach space  $\mathbb{B}$ , see, e.g., [24] on such embeddings. For all integers  $n \geq 1$  and  $k = 0, \dots, n$ , set  $t_k^n = t_0 + kT/n$ . Let  $X_J^n$  be the random variable with piecewise linear values in  $C_u(J, \mathbb{B})$  which coincides with  $X_J$  at  $t_0^n, \dots, t_n^n$ . We have

$$\lim_{n \rightarrow \infty} \text{law}(X_J^n) = \text{law}(X_J) \quad \text{and} \quad \lim_{n \rightarrow \infty} \text{law}(X_{J+\tau}^n) = \text{law}(X_{J+\tau}),$$

since  $X_J^n$  and  $X_{J+\tau}^n$  converge respectively to  $X_J$  and  $X_{J+\tau}$  a.e. in  $C_u(J, \mathbb{B})$ . Let  $f : C_u(J, \mathbb{B}) \rightarrow \mathbb{R}$  be a bounded continuous function, and let  $\varepsilon > 0$ . For  $n$  large enough, we have

$$|\mathbb{E}(f(X_J^n) - f(X_J))| \leq \varepsilon \quad \text{and} \quad |\mathbb{E}(f(X_{J+\tau}^n) - f(X_{J+\tau}))| \leq \varepsilon.$$

But, by the periodicity assumption, we have  $\mathbb{E}(f(X_{J+\tau}^n) - f(X_J^n)) = 0$ , thus

$$|\mathbb{E}(f(X_{J+\tau}) - f(X_J))| \leq 2\varepsilon.$$

The result follows since  $\varepsilon$  is arbitrary.

2. The topology of  $C_k(\mathbb{R}; \mathbb{X})$  is defined in a natural way by a countable family of semidistances, e.g., for  $x, y \in C_k(\mathbb{R}; \mathbb{X})$ ,

$$\mathfrak{d}_k(x, y) = \sup_{t \in [-k, k]} \mathfrak{d}(x(t), y(t)), \quad (k \geq 1).$$

It is also metrized by, e.g., the distance  $\mathfrak{d}_{C_k}$  defined by

$$\mathfrak{d}_{C_k}(x, y) = \sum_{k \geq 1} \mathfrak{d}_k(x, y).$$

A distance on  $\mathcal{M}^{+,1}(C_k(\mathbb{R}; \mathbb{X}))$  associated with  $\mathfrak{d}_{C_k}$  is

$$\text{Wass}_{C_k} = \inf_{\text{law}(X)=\mu, \text{law}(Y)=\nu} \mathbb{E}(\mathfrak{d}_{C_k}(X, Y) \wedge 1).$$

Thus a random process  $X$  is APDD if, for each  $\varepsilon > 0$ , the set of  $\varepsilon$ -periods of  $t \mapsto \text{law}(\mathfrak{T}_t X)$  for  $\text{Wass}_{C_k}$  is relatively dense.

Another equivalent approach, thanks to Proposition 3.5, consists in defining the APPD property using semidistances on  $\mathcal{M}^{+,1}(C_k(\mathbb{R}; \mathbb{X}))$  associated with  $\mathfrak{d}_k$ ,  $k \geq 1$ :  $X$  is APPD if, and only if, for each integer  $k \geq 1$ , the map  $t \mapsto \text{law}(\mathfrak{T}_t X)$  is almost periodic for the semidistance

$$\text{Wass}_{[-k, k]}(\mu, \nu) = \inf_{\text{law}(X)=\mu, \text{law}(Y)=\nu} \mathbb{E}(\mathfrak{d}_k(X, Y) \wedge 1).$$

3. Definition (c) can be easily generalized to other spaces of trajectories as follows:

(c') If  $X$  has a version (again denoted by  $X$ ) whose trajectories lie in a metrizable topological space of trajectories  $\mathbb{Y} \subset \mathbb{X}^{\mathbb{R}}$  which is stable by translations, that is, such that  $x \in \mathbb{Y} \Rightarrow x(t + \cdot) \in \mathbb{Y}$  for all  $t \in \mathbb{R}$ , we say that  $X$  is *almost periodic in  $\mathbb{Y}$ -path distribution* (let us say,  $\mathbb{Y}$ -APPD) if the mapping

$$\begin{cases} \mathbb{R} & \rightarrow \mathcal{M}^{+,1}(\mathbb{Y}) \\ t & \mapsto \text{law}(\mathfrak{T}_t X) \end{cases}$$

is almost periodic.

4. Since  $\theta$  is measure preserving, Definition (4.1) as well as Definition (c') above remain unchanged if we replace the operator  $\mathfrak{T}_t$  by the simpler operator  $\mathfrak{T}_t^0$  defined by  $\mathfrak{T}_t^0 X(s, \cdot) = X(t + s, \cdot)$  for all  $s \in \mathbb{R}$ .

The following criterion from [6, Theorem 2.3] is based on the Arzelà-Ascoli theorem:

**Proposition 4.3** ([6]). *Let  $X : \mathbb{R} \mapsto L^0(\Omega; \mathbb{X})$  be a random process. Assume that  $X$  is APFD, and that  $X$  has a continuous modification. Then  $X$  is APPD if, and only if, it satisfies, for every compact interval  $J$ ,*

$$\lim_{\delta \rightarrow 0} \sup_{t \in \mathbb{R}} \mathbb{E} \left( \sup_{r, s \in J, |r-s| < \delta} \mathfrak{d}(X(t+r, \cdot), X(t+s, \cdot)) \wedge 1 \right) = 0. \quad (4.1)$$

## 4.2 Comparison with $\theta$ -almost periodicity

For applications to stochastic differential equations, we will use the following sufficient criterion for APPD property for  $\theta$ -almost periodic processes. For any  $J \subset I$  be denote by  $C_u(J; \mathbb{X})$  the space of continuous functions from  $J$  to  $\mathbb{X}$  endowed with the topology of uniform convergence.

**Theorem 4.4** ( $\theta$ -almost periodicity vs almost periodicity in distribution). *If  $X$  is  $\theta$ -almost periodic (respectively  $\theta$ -periodic), it is APFD (respectively PFD).*

*If furthermore  $X$  has a continuous modification (that we denote by  $X$  for simplicity), a sufficient condition for  $X$  to be APPD is Condition (C) below:*

(C) For every compact interval  $J$ , the mapping  $Z_J : \mathbb{R} \rightarrow L^0(C_u(J; \mathbb{X}))$  defined by

$$Z_J(t)(\omega)(s) = \mathfrak{T}_t X(s, \omega) = X(t + s, \theta_{-t} \omega) \quad (t \in \mathbb{R}, \omega \in \Omega, s \in J) \quad (4.2)$$

is almost periodic.

**Remark 4.5.** 1. By Proposition 3.17, with the notation of (C),  $\theta$ -almost periodicity amounts to almost periodicity of  $Z_{\{0\}}$ . Thus Condition (C) implies at the same time  $\theta$ -almost periodicity and APPD property.

2. By Proposition 3.5 (see also Part 2 of Remark 4.2), (C) is equivalent to:

(C)' The mapping  $Z : \mathbb{R} \rightarrow L^0(C_k(\mathbb{R}; \mathbb{X}))$  defined by

$$Z(t)(\omega)(s) = \mathfrak{T}_t X(s, \omega) = X(t + s, \theta_{-t} \omega) \quad (t \in \mathbb{R}, \omega \in \Omega, s \in \mathbb{R})$$

is almost periodic.

*Proof of Theorem 4.4.* Let  $X$  be  $\theta$ -almost periodic. Let  $(t_1, \dots, t_n)$  be a finite sequence in  $\mathbb{R}$ . Let us endow  $\mathbb{X}^n$  with the distance

$$\mathfrak{d}_n((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max_{1 \leq i \leq n} \mathfrak{d}(x_i, y_i).$$

Let  $\varepsilon > 0$ , and let  $\tau$  be an  $\varepsilon$ -period of  $X$ . For every  $t \in \mathbb{R}$ , we have

$$\begin{aligned} & \text{Wass}_0 \left( \text{law}(\mathfrak{T}_{t_1} X(t, \cdot), \dots, \mathfrak{T}_{t_n} X(t, \cdot)), \text{law}(\mathfrak{T}_{t_1 + \tau} X(t, \cdot), \dots, \mathfrak{T}_{t_n + \tau} X(t, \cdot)) \right) \\ & \leq \mathbb{E} \left( \max_{1 \leq i \leq n} \mathfrak{d}(\mathfrak{T}_{t_i} X(t, \cdot), \mathfrak{T}_{t_i + \tau} X(t, \cdot)) \wedge 1 \right) \\ & \leq n \max_{1 \leq i \leq n} \mathbb{E} \left( \mathfrak{d}(\mathfrak{T}_{t_i} X(t, \cdot), \mathfrak{T}_{t_i + \tau} X(t, \cdot)) \wedge 1 \right) \leq n\varepsilon, \end{aligned}$$

which shows that  $X$  is APPD.

If  $X$  is  $\theta$ - $\tau$ -periodic, the same reasoning with  $\varepsilon = 0$  yields the PFD property.

Assume now that  $X$  has a continuous modification, that we also denote by  $X$  for simplicity. Assume (C), and let  $J \subset \mathbb{R}$  be a compact interval. Let  $\varepsilon > 0$ . Let  $l$  such that each interval of length  $l$  contains an  $\varepsilon/3$ -almost period

of  $Z_J$ . We can choose  $l$  large enough that  $J \subset [-l/2, l/2]$ . Let  $I = [-l, l]$ . By uniform continuity on  $I$  of the trajectories of  $X$ , we can find an  $\mathcal{F}$ -measurable random variable  $\eta(\cdot) > 0$  such that, for every  $\omega \in \Omega$ ,

$$(r, s \in I \text{ and } |r - s| \leq \eta(\omega)) \Rightarrow \mathfrak{d}(X(r, \omega), X(s, \omega)) \leq \frac{\varepsilon}{6}.$$

By tightness of  $\eta(\cdot)$ , we can find a number  $\delta > 0$  such that

$$\mathbb{P}(\eta(\cdot) \geq \delta) \geq 1 - \frac{\varepsilon}{6}.$$

Let  $A = \{\eta(\cdot) \geq \delta\}$  and  $A^c = \Omega \setminus A$ . We have thus

$$\begin{aligned} \mathbb{E} \left( \sup_{r, s \in I, |r-s| \leq \delta} \mathfrak{d}(X(r, \cdot), X(s, \cdot)) \wedge 1 \right) \\ \leq \mathbb{E} \left( \sup_{r, s \in I, |r-s| \leq \delta} \mathfrak{d}(X(r, \cdot), X(s, \cdot)) \mathbf{1}_A \wedge 1 \right) \\ + \mathbb{E} \left( \sup_{r, s \in I, |r-s| \leq \delta} \mathfrak{d}(X(r, \cdot), X(s, \cdot)) \mathbf{1}_{A^c} \wedge 1 \right) \\ \leq \frac{\varepsilon}{6} + \mathbb{P}(A^c) \leq \frac{\varepsilon}{3}. \end{aligned}$$

Let  $t \in \mathbb{R}$ , and let  $\tau \in [-t - l/2, -t + l/2]$  be an  $\varepsilon/3$ -almost period of  $Z_J$ , so that  $J + t \subset I$ . We have

$$\mathbb{E} \left( \sup_{s \in J} \mathfrak{d}(\mathfrak{T}_t(X(s, \cdot)), \mathfrak{T}_{t+\tau}(X(s, \cdot))) \right) \leq \frac{\varepsilon}{3}.$$

On the other hand, we can find measurable random variables  $r(\cdot)$  and  $s(\cdot)$  with values in  $J$  such that  $|r(\omega) - s(\omega)| \leq \delta$  for each  $\omega \in \Omega$  and

$$\begin{aligned} \mathbb{E} \left( \sup_{r, s \in J, |r-s| < \delta} \mathfrak{d}(X(t+r, \cdot), X(t+s, \cdot)) \wedge 1 \right) \\ = \mathbb{E}(\mathfrak{d}(X(t+r(\cdot), \cdot), X(t+s(\cdot), \cdot)) \wedge 1). \end{aligned}$$

Using that  $\mathbb{P}$  is  $\theta$ -invariant, we get thus

$$\mathbb{E} \left( \sup_{r, s \in J, |r-s| \leq \delta} \mathfrak{d}(X(t+r, \cdot), X(t+s, \cdot)) \wedge 1 \right)$$



$$\begin{aligned}
&= \mathbb{E} \left( \mathfrak{d}(X(t+r(\cdot), \theta_{-t}\cdot), X(t+s(\cdot), \theta_{-t}\cdot)) \wedge 1 \right) \\
&\leq \mathbb{E} \left( \mathfrak{d}(X(t+r(\cdot), \theta_{-t}\cdot), X(t+\tau+r(\cdot), \theta_{-t-\tau}\cdot)) \wedge 1 \right) \\
&\quad + \mathbb{E} \left( \mathfrak{d}(X(t+\tau+r(\cdot), \theta_{-t-\tau}\cdot), X(t+\tau+s(\cdot), \theta_{-t-\tau}\cdot)) \wedge 1 \right) \\
&\quad + \mathbb{E} \left( \mathfrak{d}(X(t+\tau+s(\cdot), \theta_{-t-\tau}\cdot), X(t+s(\cdot), \theta_{-t}\cdot)) \wedge 1 \right) \\
&\leq \mathbb{E} \left( \mathfrak{d}(X(t+r(\cdot), \theta_{-t}\cdot), X(t+\tau+r(\cdot), \theta_{-t-\tau}\cdot)) \wedge 1 \right) \\
&\quad + \mathbb{E} \left( \sup_{r,s \in I, |r-s| \leq \delta} \mathfrak{d}(X(r, \cdot), X(s, \cdot)) \wedge 1 \right) \\
&\quad + \mathbb{E} \left( \mathfrak{d}(X(t+\tau+s(\cdot), \theta_{-t-\tau}\cdot), X(t+s(\cdot), \theta_{-t}\cdot)) \wedge 1 \right) \\
&\leq \varepsilon.
\end{aligned}$$

Since this estimation is independent of  $t$ , this proves Condition (4.1) of Proposition 4.3.  $\square$

The following counterexample is inspired from Ursell [37]. It shows that  $\theta$ -almost periodicity does not imply the APPD property, in particular this property is strictly stronger than the APFD property.

**Counterexample 4.6.** Let  $\Omega = [0, 1]$ , endowed with Lebesgue measure. let  $(\varepsilon_n)_{n \geq 1}$  be a sequence in  $]0, 1]$  such that  $\sum_n \varepsilon_n < \infty$ . For each positive integer  $n$  and each integer  $k$  (positive or nonpositive), let  $x_{n,k} = (2k+1)n$ , and define

$$\begin{aligned}
f_n(t) &= \sum_{-\infty < k < \infty} \left( \frac{1}{\varepsilon_n} - \left| \frac{t}{\varepsilon_n} - x_{n,k} \right| \right) \mathbf{1}_{\{|t-x_{n,k}| \leq \varepsilon_n\}}, \\
f(t) &= \sum_{n \geq 1} f_n(t).
\end{aligned}$$

Each  $f_n$  is periodic and continuous, and  $f$  is continuous, but  $f$  is not uniformly continuous, nor bounded, thus it is not almost periodic in Bohr's sense. However,  $f$  is almost periodic in Stepanov's sense, that is,  $f$  is locally integrable and the mapping

$$\begin{cases} \mathbb{R} & \mapsto L^1([0, 1]) \\ t & \mapsto f(t + \cdot) \end{cases}$$

is almost periodic.

Set  $X(t, \omega) = f(t + \omega)$  for  $t \in \mathbb{R}$  and  $\omega \in \Omega$ . Then  $X$  is  $\theta$ -almost periodic for the shift transformation

$$\theta_t(\omega) = t + \omega \pmod{1}.$$

Let  $J = [0, 2]$ . Set, for  $n \geq 1$ ,  $\omega \in \Omega$  and  $\delta \in ]0, 1]$ ,

$$t_n = n(2n + 1) - 1, \quad r_n(\omega) = 1 - \omega, \quad s_n(\omega) = 1 - \delta - \omega.$$

We have

$$\begin{aligned} & \sup_{t \in \mathbb{R}} \mathbb{E} \left( \sup_{r, s \in J, |r-s| < \delta} \mathfrak{d}(X(t+r, \cdot), X(t+s, \cdot)) \wedge 1 \right) \\ &= \sup_{t \in \mathbb{R}} \int_0^1 \left( \sup_{r, s \in J, |r-s| < \delta} |f(t+r+\omega) - f(t+s+\omega)| \wedge 1 \right) d\omega \\ &\geq \sup_{n \geq 1} \int_0^1 (|f(t_n + r_n(\omega) + \omega) - f(t_n + s_n(\omega) + \omega)| \wedge 1) d\omega \\ &\geq \sup_{n \geq 1} \frac{1}{\varepsilon_n} = +\infty, \end{aligned}$$

which shows by Proposition 4.3 that  $X$  is not APPD.

## 5 Application to stochastic differential equations

We apply here the results of the previous sections. The novelty is that we can reduce the proof of almost periodicity to a fixed point problem, unlike in usual proofs of almost periodicity in distribution. Recall that the “naive” almost periodicity in  $p$ -mean (that corresponds to  $\theta_t = \text{Id}_\Omega$ , or to  $\ell(t) = 0$  in Remark 3.7-(2)) does not apply to stochastic differential equations [5, 29].

In this section,  $\mathbb{U}$  and  $\mathbb{H}$  are separable Hilbert spaces,  $\Omega = \mathbb{C}(\mathbb{R}; \mathbb{U})$  is endowed with the compact-open topology (the topology of uniform convergence on compact subsets of  $\mathbb{R}$ ),  $\mathcal{F}$  is the Borel  $\sigma$ -algebra of  $\Omega$ , and  $\mathbb{P}$  is the Wiener measure on  $\Omega$  with trace class<sup>1</sup>covariance operator  $Q$ , that is, the process  $W$  with values in  $\mathbb{U}$  defined by

$$W(t, \omega) = \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R},$$

is a Brownian motion with covariance operator  $Q$ . Our group  $\theta$  of measure preserving transformations on  $(\Omega, \mathcal{F}, \mathbb{P})$  is the *Wiener shift*, defined by

$$\theta_\tau(\omega)(t) = \omega(t + \tau) - \omega(\tau) = W(t + \tau, \omega) - W(\tau, \omega) \quad (5.1)$$

---

<sup>1</sup>It is possible to consider a more general, nonnecessarily nuclear, nonnegative symmetric operator  $Q$ . If  $\text{tr}(Q) = +\infty$ ,  $W$  is only a cylindrical Brownian motion on  $\mathbb{U}$ . However, it is possible to embed  $\mathbb{U}$  in a “larger” Hilbert space  $\mathbb{U}_1$  such that  $W$  has trace class covariance operator on  $\mathbb{U}_1$ . In that case, we have to take  $\Omega = \mathbb{C}(\mathbb{R}; \mathbb{U}_1)$  and  $\mathbb{P}$  is the distribution of  $W$  on  $\mathbb{U}_1$ , see [17, Chapter 4].

for all  $\tau, t \in \mathbb{R}$  and  $\omega \in \Omega$ . This yields

$$W(t+\tau, \theta_{-\tau} \omega) = (\theta_{-\tau} \omega)(t+\tau) = \omega(t+\tau-\tau) - \omega(-\tau) = W(t, \omega) - W(-\tau, \omega), \quad (5.2)$$

so that the translation operator of Definition 3.16 leaves invariant the increments of  $W$ : Indeed, we have, for all  $t, s, \tau \in \mathbb{R}$ ,

$$\begin{aligned} \mathfrak{T}_\tau(W(t+s, \omega) - W(t, \omega)) &= (W(t+s, \omega) - W(-\tau, \omega)) - (W(t, \omega) - W(-\tau, \omega)) \\ &= W(t+s, \omega) - W(t, \omega). \end{aligned}$$

We refer to Da Prato and Zabczyk's treatise [17] for stochastic integration and stochastic differential equations in Hilbert spaces. For the needs of stochastic integration with respect to  $W$ , we endow  $(\Omega, \mathcal{F}, \mathbb{P})$  with the augmented natural filtration  $(\mathcal{F}_t)$  of  $W$ . We denote by  $\mathbb{U}_0$  the Hilbert space  $Q^{1/2}\mathbb{U}$  with norm  $\|u\|_{\mathbb{U}_0} = \|Q^{-1/2}u\|_{\mathbb{U}}$ . Let  $L_2^0$  be the space of Hilbert-Schmidt operators from  $\mathbb{U}_0$  to  $\mathbb{H}$ . We consider the semilinear stochastic differential equation, for  $t \in \mathbb{R}$ ,

$$dX(t, \cdot) = (AX(t, \cdot) + F(t, X(t, \cdot))) dt + G(t, X(t, \cdot)) dW(t, \cdot) \quad (5.3)$$

where the unknown process  $X$  takes its values in  $\mathbb{H}$ ,  $A : \text{dom}(A) \subset \mathbb{H} \rightarrow \mathbb{H}$  is a linear operator which may be unbounded, and  $F : \mathbb{R} \times \mathbb{H} \rightarrow \mathbb{H}$ , and  $G : \mathbb{R} \times \mathbb{H} \rightarrow L_2^0$  are continuous functions. We assume that

(H1)  $A$  is the infinitesimal generator of a  $C_0$ -semigroup  $(S(t))_{t \geq 0}$  satisfying, for some constant numbers  $\mathfrak{d} > 0$ , and  $M > 0$ , and for all  $t \geq 0$ ,

$$\|S(t)\|_{L(\mathbb{H})} \leq M e^{-\mathfrak{d}t}, \quad (5.4)$$

where  $L(\mathbb{H})$  is the space of continuous linear mappings from  $\mathbb{H}$  to itself,

(H2)  $F$  and  $G$  satisfy the usual Lipschitz and growth conditions, that is, for some constant numbers  $\mathfrak{g}, \mathfrak{h} > 0$  and for all  $x, y \in \mathbb{H}$ ,

$$\|F(t, x) - F(t, y)\| + \|G(t, x) - G(t, y)\| \leq \mathfrak{h} \|x - y\|, \quad (5.5)$$

$$\|F(t, x)\| + \|G(t, x)\| \leq \mathfrak{g}(1 + \|x\|), \quad (5.6)$$

(H3)  $F$  and  $G$  are almost periodic uniformly with respect to compact subsets of  $\mathbb{H}$  (see Definition 3.1).

A *mild solution* to (5.3) is a random process  $X$  such that, for all  $t, s \in \mathbb{R}$  such that  $s \leq t$ , we have  $E \int_s^t \|X(u, \cdot)\|^2 du < \infty$ , and

$$\begin{aligned} X(t, \cdot) &= S(t-s)X(s, \cdot) + \int_s^t S(t-u)F(u, X(u, \cdot)) du \\ &\quad + \int_s^t S(t-u)G(u, X(u, \cdot)) dW(u, \cdot). \end{aligned} \quad (5.7)$$

By [17, Theorem 7.2], for any  $t_1 \in \mathbb{R}$  and any  $\xi \in L^2(\Omega, \mathcal{F}_t; \mathbb{H})$ , there exists a mild solution to Equation (5.3) which is defined on  $[t_1, \infty[$  and starts from  $\xi$  at  $t_1$ , and each mild solution has a continuous version. Furthermore, Equation (5.3) defines an ‘‘almost flow’’ (see the discussion in [17, Section 9.1.2]), that is, a mapping

$$\Pi : \begin{cases} \mathbb{R} \times \mathbb{R} \times \Omega \times \mathbb{H} & \rightarrow \mathbb{H} \\ (t_2, t_1, \omega, x) & \mapsto \Pi(t_2, t_1, \omega)x \end{cases}$$

defined for  $t_2 \geq t_1$ , and satisfying  $\Pi(t_1, t_1, \omega) = \text{Id}_{\mathbb{H}}$  and

$$\Pi(t_3, t_1, \omega) = \Pi(t_3, t_2, \omega) \circ \Pi(t_2, t_1, \omega) \quad (5.8)$$

for  $t_1 \leq t_2 \leq t_3$  and P-almost all  $\omega \in \Omega$ . Denote  $\tau = t_1$ ,  $s = t_2 - t_1$ ,  $r = t_3 - t_2$ , and

$$\varphi(s, \tau, \theta_t \omega) = \Pi(\tau + s, \tau, \omega),$$

where  $\theta$  is defined as in (5.1), then Equation (5.8) becomes the crude cocycle relation (2.1). Any mild solution which is defined on  $\mathbb{R}$  is an orbit of  $\varphi$ .

We seek solutions that are defined on  $\mathbb{R}$  and  $\theta$ -almost periodic in quadratic mean. A solution  $X$  of this type is uniformly bounded in  $L^2(\Omega; \mathbb{H})$ . By (5.4), it satisfies

$$X(t, \cdot) = \int_{-\infty}^t S(t-u)F(u, X(u, \cdot)) du + \int_{-\infty}^t S(t-u)G(u, X(u, \cdot)) dW(u, \cdot). \quad (5.9)$$

In the sequel, we denote

- $\text{CUB}(\Omega; \mathbb{H})$  the space of continuous uniformly bounded functions from  $\mathbb{R}$  to  $L^2(\Omega; \mathbb{H})$ , endowed with the norm  $\|X\| = \sup_{t \in \mathbb{R}} \|X(t)\|_{L^2(\Omega; \mathbb{H})}$ ,
- $\text{AP} \subset \text{CUB}(\Omega; \mathbb{H})$  the space of  $\theta$ -almost periodic in square mean  $(\mathcal{F}_t)$ -predictable random processes,

- $\text{Per}(\tau) \subset \text{AP}$  the space of  $\theta$ - $\tau$ -periodic and square integrable  $(\mathcal{F}_t)$ -predictable random processes.

Any constant (in  $t$  and  $\omega$ ) random process belongs to each of these spaces, thus none of them is empty. We define an operator  $\Gamma : \text{AP} \rightarrow \text{CUB}(\Omega; \mathbb{H})$  by

$$\Gamma X(t) = \int_{-\infty}^t S(t-s)F(s, X(s))ds + \int_{-\infty}^t S(t-s)G(s, X(s))dW(s)$$

for all  $t \in \mathbb{R}$ . By [17, Proposition 7.3], for any  $X \in \text{AP}$ , the random process  $\Gamma X$  has a continuous modification. Furthermore, for any  $u \in \mathbb{R}$ , the family  $(X(u+s, \theta_{-s} \cdot))_{s \in \mathbb{R}}$  is uniformly square integrable, since it is relatively compact in  $L^2(\Omega, \mathbb{P}; \mathbb{H})$ . Since  $\theta$  is measure preserving, this entails that the family  $(X(s, \cdot))_{s \in \mathbb{R}}$  is uniformly square integrable. Using Vitali's theorem, we deduce that  $\Gamma X$  is continuous in square mean.

Combining  $\Gamma$  with the translation operator of Definition 3.16, we get

**Proposition 5.1.** *The operator  $\Gamma$  maps AP into itself.*

*Proof.* Let  $X \in \text{AP}$  be a  $\theta$ -almost periodic  $(\mathcal{F}_t)$ -predictable random process. Let  $\varepsilon_0 > 0$ , and let us show that the set of  $\theta$ - $\varepsilon_0$ -almost periods of  $\Gamma X$  is relatively dense.

As we already noticed, the family  $(X(s, \cdot))_{s \in \mathbb{R}}$  is uniformly square integrable. From the growth condition (5.6), we deduce that the families  $(F(r, X(s, \cdot)))_{r, s \in \mathbb{R}}$  and  $(G(r, X(s, \cdot)))_{r, s \in \mathbb{R}}$  are also uniformly square integrable. Let  $\alpha > 0$ . There exists  $\eta > 0$ , with  $\eta < \min(\alpha, 1)$ , such that, for any  $A \in \mathcal{F}$ , and for all  $u, s \in \mathbb{R}$ ,

$$P(A) < \eta \Rightarrow \begin{cases} \mathbb{E} \left( \|X(u+s, \theta_{-s} \cdot)\|^2 \mathbf{1}_A \right) < \alpha, \\ \mathbb{E} \left( \|F(r, X(u+s, \theta_{-s} \cdot))\|^2 \mathbf{1}_A \right) < \alpha, \\ \mathbb{E} \left( \|G(r, X(u+s, \theta_{-s} \cdot))\|^2 \mathbf{1}_A \right) < \alpha. \end{cases} \quad (5.10)$$

Furthermore, by Proposition 3.10, the family  $(X(s, \cdot))_{s \in \mathbb{R}}$  is uniformly tight. We can find thus a compact subset  $K_\alpha$  of  $\mathbb{H}$  such that, for all  $u, s \in \mathbb{R}$ ,

$$P \{ X(u+s, \theta_{-s} \cdot) \in K_\alpha \} \geq 1 - \eta. \quad (5.11)$$

On the other hand, we deduce from Corollary 3.4 and Proposition 3.17, that the mapping

$$\begin{cases} \mathbb{R} & \rightarrow & \mathbb{H} \times L_2^0 \times L^2(\Omega; \mathbb{X}) \\ t & \mapsto & (F(t, x), G(t, x), X(t, \cdot)) \end{cases}$$

is  $\theta$ -almost periodic uniformly with respect to  $x$  in compact subsets of  $\mathbb{H}$ . Let  $\varepsilon > 0$ , and let  $T_\varepsilon$  be the relatively dense set of common  $\varepsilon$ -almost periods of  $X$ ,  $F(\cdot, x)$  and  $G(\cdot, x)$  for all  $x \in K_\alpha$ . We are going to show that, for an appropriate choice of  $\varepsilon$  and  $\alpha$ , the set  $T_\varepsilon$  is contained in the set of  $\varepsilon_0$ -almost periods in square mean of  $\Gamma X$ .

Let  $\tau \in T_\varepsilon$ . Assume first that  $\tau > 0$ . We have, using (5.2),

$$\begin{aligned}
\mathfrak{T}_\tau \Gamma X(t, \cdot) &= \int_{-\infty}^{t+\tau} S(t-s)F(s, X(s, \theta_{-\tau} \cdot)) ds \\
&\quad + \int_{-\infty}^{t+\tau} S(t-s)G(s, X(s, \theta_{-\tau} \cdot)) dW(s, \theta_{-\tau} \cdot) \\
&= \int_{-\infty}^t S(t-u)F(u+\tau, X(u+\tau, \theta_{-\tau} \cdot)) du \\
&\quad + \int_{-\infty}^t S(t-u)G(u+\tau, X(u+\tau, \theta_{-\tau} \cdot)) dW(u+\tau, \theta_{-\tau} \cdot) \\
&= \int_{-\infty}^t S(t-u)F(u+\tau, X(u+\tau, \theta_{-\tau} \cdot)) du \\
&\quad + \int_{-\infty}^t S(t-u)G(u+\tau, X(u+\tau, \theta_{-\tau} \cdot)) dW(u, \cdot).
\end{aligned} \tag{5.12}$$

We deduce

$$\begin{aligned}
&\mathbb{E} \|\mathfrak{T}_\tau \Gamma X(t, \cdot) - \Gamma X(t, \cdot)\|^2 \\
&= \mathbb{E} \left\| \int_{-\infty}^t S(t-u) \left( F(u+\tau, X(u+\tau, \theta_{-\tau} \cdot)) - F(u, X(u, \cdot)) \right) du \right. \\
&\quad \left. + \int_{-\infty}^t S(t-u) \left( G(u+\tau, X(u+\tau, \theta_{-\tau} \cdot)) - G(u, X(u, \cdot)) \right) dW(u, \cdot) \right\|^2 \\
&\leq 4 \mathbb{E} \left\| \int_{-\infty}^t S(t-u) \left( F(u+\tau, X(u+\tau, \theta_{-\tau} \cdot)) - F(u, X(u+\tau, \theta_{-\tau} \cdot)) \right) ds \right\|^2 \\
&\quad + 4 \mathbb{E} \left\| \int_{-\infty}^t S(t-u) \left( F(u, X(u+\tau, \theta_{-\tau} \cdot)) - F(u, X(u, \cdot)) \right) du \right\|^2 \\
&\quad + 4 \mathbb{E} \left\| \int_{-\infty}^t S(t-u) \left( G(u+\tau, X(u+\tau, \theta_{-\tau} \cdot)) \right. \right. \\
&\quad \quad \left. \left. - G(u, X(u+\tau, \theta_{-\tau} \cdot)) \right) dW(u, \cdot) \right\|^2 \\
&\quad + 4 \mathbb{E} \left\| \int_{-\infty}^t S(t-u) \left( G(u, X(u+\tau, \theta_{-\tau} \cdot)) - G(u, X(u, \cdot)) \right) dW(u, \cdot) \right\|^2
\end{aligned}$$

$$=: 4I_1 + 4I_2 + 4I_3 + 4I_4.$$

Let us denote, for  $u \in \mathbb{R}$ ,

$$\begin{aligned} F^\tau(u, \cdot) &= F(u + \tau, X(u + \tau, \theta_{-\tau} \cdot)) - F(u, X(u + \tau, \theta_{-\tau} \cdot)), \\ G^\tau(u, \cdot) &= G(u + \tau, X(u + \tau, \theta_{-\tau} \cdot)) - G(u, X(u + \tau, \theta_{-\tau} \cdot)), \\ A_\alpha(u) &= \{\omega \in \Omega; X(u + \tau, \theta_{-\tau} \omega) \in K_\alpha\}. \end{aligned}$$

Since  $\tau$  is an  $\varepsilon$ -period of  $F(\cdot, x)$  and  $G(\cdot, x)$ , uniformly with respect to  $x \in K_\alpha$ , we have

$$\|F^\tau(u, \cdot)\| \mathbf{1}_{A_\alpha(u)} \leq \varepsilon \text{ and } \|G^\tau(u, \cdot)\| \mathbf{1}_{A_\alpha(u)} \leq \varepsilon. \quad (5.13)$$

On the other hand, from (5.11), we get

$$\mathbb{P}\{A_\alpha(u)\} \geq 1 - \eta.$$

We deduce, by (5.10),

$$\begin{aligned} \mathbb{E}\left(\|F^\tau(u)\|^2 \mathbf{1}_{\Omega \setminus A_\alpha(u)}\right) &\leq 2 \mathbb{E}\left(\|F(u + \tau, X(u + \tau, \theta_{-\tau} \cdot))\|^2 \mathbf{1}_{\Omega \setminus A_\alpha(u)}\right) \\ &\quad + 2 \mathbb{E}\left(\|F(u, X(u + \tau, \theta_{-\tau} \cdot))\|^2 \mathbf{1}_{\Omega \setminus A_\alpha(u)}\right) \\ &\leq 4\alpha, \\ \text{and } \mathbb{E}\left(\|G^\tau(u)\|^2 \mathbf{1}_{\Omega \setminus A_\alpha(u)}\right) &\leq 4\alpha. \end{aligned} \quad (5.14)$$

We deduce from (5.13) and (5.14):

$$\begin{aligned} I_1 &= \mathbb{E}\left\|\int_{-\infty}^t S(t-u)F^\tau(u) du\right\|^2 \leq M^2 \left(\int_{-\infty}^t e^{-\vartheta(t-u)} \mathbb{E}\|F^\tau(u)\| du\right)^2 e^{-\vartheta(t-u)} \\ &\leq M^2 \int_{-\infty}^t e^{-\vartheta(t-u)} du \int_{-\infty}^t e^{-\vartheta(t-u)} \mathbb{E}\|F^\tau(u)\|^2 du \\ &\leq \frac{M^2}{\vartheta^2} (\varepsilon^2 + 4\alpha). \end{aligned}$$

Similarly, using Itô's isometry, we get

$$\begin{aligned} I_3 &= \mathbb{E}\left\|\int_{-\infty}^t S(t-u)G^\tau(u) dW(u, \cdot)\right\|^2 \leq M^2 \mathbb{E} \int_{-\infty}^t e^{-2\vartheta(t-u)} \|G^\tau(u)\|^2 du \\ &\leq \frac{M^2}{2\vartheta} (\varepsilon^2 + 4\alpha). \end{aligned}$$

To estimate  $I_2$  and  $I_4$ , we use the Lipschitz condition (5.5):

$$\begin{aligned}
I_2 &\leq M^2 \left( \int_{-\infty}^t e^{-\mathfrak{d}(t-u)} du \right)^2 \sup_{u \in \mathbb{R}} \mathbb{E} \left\| F(u, X(u + \tau, \theta_{-\tau} \cdot)) - F(u, X(u, \cdot)) \right\|^2 \\
&= \frac{M^2}{\mathfrak{d}^2} \sup_{u \in \mathbb{R}} \mathbb{E} \left\| F(u, X(u + \tau, \theta_{-\tau} \cdot)) - F(u, X(u, \cdot)) \right\|^2 \\
&\leq \frac{M^2 \mathfrak{h}^2 \varepsilon^2}{\mathfrak{d}^2}, \\
I_4 &\leq M^2 \mathbb{E} \int_{-\infty}^t e^{-2\mathfrak{d}(t-u)} \left\| G(u, X(u + \tau, \theta_{-\tau} \cdot)) - G(u, X(u, \cdot)) \right\|^2 du \\
&\leq \frac{M^2 \mathfrak{h}^2 \varepsilon^2}{2\mathfrak{d}}.
\end{aligned}$$

Gathering the estimations for  $I_1$ – $I_4$ , we get

$$\mathbb{E} \|\mathfrak{I}_\tau \Gamma X(t, \cdot) - \Gamma X(t, \cdot)\|^2 \leq 4M^2 \left( \frac{1}{\mathfrak{d}^2} + \frac{1}{2\mathfrak{d}} \right) (\varepsilon^2(1 + \mathfrak{h}^2) + 4\alpha). \quad (5.15)$$

If  $\tau < 0$ , we make the change of variables  $\tau' = -\tau$ ,  $t' = t + \tau$ . Then

$$\begin{aligned}
\mathbb{E} \|\mathfrak{I}_\tau \Gamma X(t, \cdot) - \Gamma X(t, \cdot)\|^2 &= \mathbb{E} \|(\Gamma X)(t + \tau, \theta_{-\tau} \cdot) - \Gamma X(t, \cdot)\|^2 \\
&= \mathbb{E} \|(\Gamma X)(t', \theta_{\tau'} \cdot) - \Gamma X(t' + \tau', \theta_{-\tau'} \theta_{\tau'} \cdot)\|^2 \\
&= \mathbb{E} \|(\Gamma X)(t', \cdot) - \Gamma X(t' + \tau', \theta_{-\tau'} \cdot)\|^2 \\
&= \mathbb{E} \|\Gamma X(t', \cdot) - \mathfrak{I}_{\tau'} \Gamma X(t', \cdot)\|^2,
\end{aligned}$$

so that the preceding calculation leads again to (5.15). Finally, choosing  $\varepsilon$  and  $\alpha$  such that

$$4M^2 \left( \frac{1}{\mathfrak{d}^2} + \frac{1}{2\mathfrak{d}} \right) (\varepsilon^2(1 + \mathfrak{h}^2) + 4\alpha) \leq \varepsilon_0^2,$$

we have that  $\tau$  is an  $\varepsilon_0$ -almost period in square mean of  $\Gamma X$ . Thus the set of  $\theta$ - $\varepsilon_0$ -almost periods in square mean of  $\Gamma X$  is contained in  $T_\varepsilon$ , thus it is relatively dense.

Finally, there remains to prove the continuity condition (b)-(i) of definition 3.6. By Proposition 3.9, we only need to prove continuity at any  $s_0 \in \mathbb{R}$  of  $s \mapsto \Gamma X(s, \theta_{-s} \cdot) = \mathfrak{I}_s \Gamma X(0, \cdot)$ . The calculation follows similar lines as above. Let  $\alpha > 0$ , and let  $\eta > 0$  and  $K_\alpha$  as in (5.10) and (5.11). Using the calculation in and Set

$$A_\alpha(u) = \{\omega \in \Omega; X(u + s_0, \theta_{-s_0} \omega) \in K_\alpha\}.$$



By the almost periodicity hypotheses (H3),  $F$  and  $G$  are uniformly continuous on  $\mathbb{R} \times K_\alpha$  (see, e.g., the proof of [15, Theorem 2.3]). We can thus choose  $\eta$  such that

$$|s - s_0| < \eta \Rightarrow \begin{cases} \sup_{x \in K_\alpha} \|F(s, x) - F(s_0, x)\|^2 < \alpha \\ \sup_{x \in K_\alpha} \|G(s, x) - G(s_0, x)\|^2 < \alpha \\ \sup_{u \in \mathbb{R}} \mathbb{E} \|X(u + s, \theta_{-s} \cdot) - X(u + s_0, \theta_{-s_0} \cdot)\|^2 < \alpha. \end{cases}$$

For  $s_0 > 0$ , using (5.12), we have that, for any  $s > 0$ ,

$$\begin{aligned} & \mathbb{E} \|\Gamma X(s, \theta_{-s} \cdot) - \Gamma X(s_0, \theta_{-s_0} \cdot)\|^2 \\ & \leq 4M^2 \left( \frac{1}{\mathfrak{d}} \int_{-\infty}^0 e^{\mathfrak{d}u} \mathbb{E} \left\| F(u + s, X(u + s, \theta_{-s} \cdot)) - F(u + s, X(u + s_0, \theta_{-s_0} \cdot)) \right\|^2 du \right. \\ & \quad + \frac{1}{\mathfrak{d}} \int_{-\infty}^0 e^{\mathfrak{d}u} \mathbb{E} \left\| F(u + s, X(u + s_0, \theta_{-s_0} \cdot)) - F(u + s_0, X(u + s_0, \theta_{-s_0} \cdot)) \right\|^2 du \\ & \quad + \int_{-\infty}^0 e^{2\mathfrak{d}u} \mathbb{E} \left\| G(u + s, X(u + s, \theta_{-s} \cdot)) - G(u + s, X(u + s_0, \theta_{-s_0} \cdot)) \right\|^2 du \\ & \quad \left. + \int_{-\infty}^0 e^{2\mathfrak{d}u} \mathbb{E} \left\| G(u + s, X(u + s_0, \theta_{-s_0} \cdot)) \right. \right. \\ & \quad \quad \quad \left. \left. - G(u + s_0, X(u + s_0, \theta_{-s_0} \cdot)) \right\|^2 du \right) \\ & \leq \frac{4M^2 \mathfrak{h}^2}{\mathfrak{d}} \int_{-\infty}^0 e^{\mathfrak{d}u} \mathbb{E} \left\| X(u + s, \theta_{-s} \cdot) - X(u + s_0, \theta_{-s_0} \cdot) \right\|^2 du \\ & \quad + \frac{4M^2}{\mathfrak{d}} \int_{-\infty}^0 e^{\mathfrak{d}u} \mathbb{E} \left\| F(u + s, X(u + s_0, \theta_{-s_0} \cdot)) \right. \\ & \quad \quad \quad \left. - F(u + s_0, X(u + s_0, \theta_{-s_0} \cdot)) \right\|^2 \mathbf{1}_{A_\alpha(u)} du \\ & \quad + \frac{4M^2}{\mathfrak{d}} \int_{-\infty}^0 e^{\mathfrak{d}u} \mathbb{E} \left\| F(u + s, X(u + s_0, \theta_{-s_0} \cdot)) \right. \\ & \quad \quad \quad \left. - F(u + s_0, X(u + s_0, \theta_{-s_0} \cdot)) \right\|^2 \mathbf{1}_{\Omega \setminus A_\alpha(u)} du \\ & \quad + 4M^2 \mathfrak{h}^2 \int_{-\infty}^0 e^{2\mathfrak{d}u} \mathbb{E} \left\| X(u + s, \theta_{-s} \cdot) - X(u + s_0, \theta_{-s_0} \cdot) \right\|^2 du \\ & \quad + 4M^2 \int_{-\infty}^0 e^{2\mathfrak{d}u} \mathbb{E} \left\| G(u + s, X(u + s_0, \theta_{-s_0} \cdot)) \right. \\ & \quad \quad \quad \left. - G(u + s_0, X(u + s_0, \theta_{-s_0} \cdot)) \right\|^2 \mathbf{1}_{A_\alpha(u)} du \end{aligned}$$

$$\begin{aligned}
& + 4M^2 \int_{-\infty}^0 e^{2\vartheta u} \mathbb{E} \left\| G(u+s, X(u+s_0, \theta_{-s_0} \cdot)) \right. \\
& \qquad \qquad \qquad \left. - G(u+s_0, X(u+s_0, \theta_{-s_0} \cdot)) \right\|^2 \mathbf{1}_{\Omega \setminus A_\alpha(u)} du. \\
& \leq 4M^2 \alpha \left( \frac{1}{\vartheta^2} + \frac{1}{2\vartheta} \right) (\mathfrak{h}^2 + 5).
\end{aligned}$$

The result follows since  $\alpha$  is arbitrary.

For  $s_0 \leq 0$  and  $\varepsilon > 0$ , let  $\tau$  be an  $\varepsilon/3$  almost period of  $X$  such that  $s_0 + \tau > 0$ , and let  $\eta > 0$  such that  $s_0 + \tau - \eta > 0$  and

$$|s - s_0| < \eta \Rightarrow \mathbb{E} \|\Gamma X(s + \tau, \theta_{-s-\tau} \cdot) - \Gamma X(s_0 + \tau, \theta_{-s_0-\tau} \cdot)\|^2 < \frac{\varepsilon}{3}.$$

We have then, for  $|s - s_0| < \eta$ ,

$$\begin{aligned}
& \mathbb{E} \|\Gamma X(s, \theta_{-s} \cdot) - \Gamma X(s_0, \theta_{-s_0} \cdot)\|^2 \\
& \leq \mathbb{E} \|\Gamma X(s, \theta_{-s} \cdot) - \Gamma X(s + \tau, \theta_{-s-\tau} \cdot)\|^2 \\
& \quad + \mathbb{E} \|\Gamma X(s + \tau, \theta_{-s-\tau} \cdot) - \Gamma X(s_0 + \tau, \theta_{-s_0-\tau} \cdot)\|^2 \\
& \quad + \mathbb{E} \|\Gamma X(s_0 + \tau, \theta_{-s_0-\tau} \cdot) - \Gamma X(s_0, \theta_{-s_0} \cdot)\|^2 \\
& \leq \varepsilon.
\end{aligned}$$

□

As a byproduct, we get:

**Proposition 5.2.** *Assume that  $F(\cdot, x)$  and  $G(\cdot, x)$  are  $\tau$ -periodic, for some  $\tau > 0$ . Then the operator  $\Gamma$  maps  $\text{Per}(\tau)$  into itself.*

*Proof.* It suffices to repeat the proof of Proposition 5.1 with  $\varepsilon = 0$ . The result follows since  $\alpha$  can be chosen arbitrarily small. □

**Proposition 5.3.** *For any  $X \in \text{AP}$ , the random process  $\Gamma X$  has a continuous modification which is almost periodic in path distribution (APPD).*

*Proof.* By Theorem 4.4,  $X$  is APFD. Furthermore, by [17, Proposition 7.3],  $X$  has a continuous modification. To prove that  $X$  is APPD, one can use the method of Da Prato and Tudor [16], which is based on Bochner's double sequence criterion. Let us prove instead Condition (C) of Theorem 4.4. In both methods, the key tool is the convolution inequality [17, Theorem 6.10]: for every  $T > 0$ , there exists a constant  $\mathfrak{c}_T$  such that, for

any predictable random process  $Y$  with values in  $L_2^0$  and  $t_0 \in \mathbb{R}$  such that  $\mathbb{E} \left( \int_{t_0}^{t_0+T} \|Y(u, \cdot)\|^2 du \right) > \infty$ , we have

$$\mathbb{E} \left( \sup_{t \in [t_0, t_0+T]} \left\| \int_{t_0}^t S(t-u) Y(u, \cdot) dW(u, \cdot) \right\|^2 \right) \leq c_T \mathbb{E} \left( \int_{t_0}^{t_0+T} \|Y(u, \cdot)\|^2 du \right).$$

Let  $J = [t_0, t_0 + T]$  be a fixed compact interval. Let  $\varepsilon > 0$  and let  $\alpha > 0$ . As in the proof of Proposition 5.1, we can find  $\eta > 0$  and a compact subset  $K$  of  $\mathbb{H}$  such that, for all  $t \in \mathbb{R}$  and all  $A \in \mathcal{F}$ ,

$$\begin{aligned} \mathbb{P}(\{X(t, \cdot) \notin K\}) &\leq \eta \leq \frac{\varepsilon^2}{\alpha^2}, \\ \mathbb{P}(A) \leq \eta &\Rightarrow \mathbb{E} \left( (1 + \|X(t, \cdot)\|^2) \mathbf{1}_A \right) \leq \frac{\varepsilon^2}{\alpha^2}. \end{aligned}$$

Let  $\tau$  be a common  $(\varepsilon/\alpha)$ -almost period of  $t \mapsto X(t, \theta_t \cdot)$ ,  $t \mapsto F(t, \cdot, x)$  and  $t \mapsto G(t, \cdot, x)$  for all  $x \in K$ . We have for every  $t \geq t_0$ , using the calculation of (5.12),

$$\begin{aligned} \mathfrak{T}_\tau X(t, \cdot) &= S(t-t_0) \mathfrak{T}_\tau X(t_0, \cdot) + \int_{t_0}^t S(t-u) F(u+\tau, X(u+\tau, \theta_{-\tau} \cdot)) du \\ &\quad + \int_{-\infty}^t S(t-u) G(u+\tau, X(u+\tau, \theta_{-\tau} \cdot)) dW(u, \cdot). \end{aligned}$$

We deduce

$$\begin{aligned} &\mathbb{E} \left( \sup_{t \in J} \|\mathfrak{T}_\tau X(t) - X(t)\|^2 \right) \\ &\leq 3 \mathbb{E} \left( \sup_{t \in J} \|S(t-t_0) (\mathfrak{T}_\tau X(t_0, \cdot) - X(t_0, \cdot))\|^2 \right) \\ &\quad + 3 \mathbb{E} \left( \sup_{t \in J} \left\| \int_{t_0}^t S(t-u) (F(u+\tau, X(u+\tau, \theta_{-\tau} \cdot)) - (F(u, X(u, \cdot)))) du \right\|^2 \right) \\ &\quad + 3 \mathbb{E} \left( \sup_{t \in J} \left\| \int_{t_0}^t S(t-u) (G(u+\tau, X(u+\tau, \theta_{-\tau} \cdot)) - (G(u, X(u, \cdot)))) dW(u, \cdot) \right\|^2 \right) \\ &=: 3A + 3B + 3C. \end{aligned}$$

We have

$$A \leq \mathbb{E} \left( \|\mathfrak{T}_\tau X(t_0, \cdot) - X(t_0, \cdot)\|^2 \right) \leq \frac{\varepsilon^2}{\alpha^2}.$$

Furthermore, let us denote, for  $u \in J$ ,

$$\Omega_u = \{X(u + \tau, \theta_{-\tau} \cdot) \in K, X(u, \cdot) \in K\}.$$

We have

$$\begin{aligned} B &\leq 2M^2T \mathbb{E} \left( \int_{t_0}^{t_0+T} \|F(u + \tau, X(u + \tau, \theta_{-\tau} \cdot)) - (F(u + \tau, X(u, \cdot)))\|^2 du \right) \\ &\quad + 2M^2T \mathbb{E} \left( \int_{t_0}^{t_0+T} \|F(u + \tau, X(u, \cdot)) - (F(u, X(u, \cdot)))\|^2 du \right) \\ &\leq 2M^2T \mathfrak{h}^2 \int_{t_0}^{t_0+T} \mathbb{E} \left( \|X(u + \tau, \theta_{-\tau} \cdot) - X(u, \cdot)\|^2 \right) du \\ &\quad + 2M^2T \int_{t_0}^{t_0+T} \mathbb{E} \left( \|F(u + \tau, X(u, \cdot)) - (F(u, X(u, \cdot)))\|^2 \mathbf{1}_{\Omega_u} \right) du \\ &\quad + 2M^2T \int_{t_0}^{t_0+T} \mathbb{E} \left( \|F(u + \tau, X(u, \cdot)) - (F(u, X(u, \cdot)))\|^2 \mathbf{1}_{\Omega \setminus \Omega_u} \right) du \\ &\leq 2M^2T \left( \mathfrak{h}^2 T \frac{\varepsilon^2}{\alpha^2} + T \frac{\varepsilon^2}{\alpha^2} + 2\mathfrak{g}^2 \int_{t_0}^{t_0+T} \mathbb{E} (1 + \|X(u, \cdot)\|^2) \mathbf{1}_{\Omega \setminus \Omega_u} du \right) \\ &\leq \frac{2M^2T^2\varepsilon^2}{\alpha^2} (\mathfrak{h}^2 + 1 + 2\mathfrak{g}^2T). \end{aligned}$$

Similarly, we get, using the convolution inequality,

$$\begin{aligned} C &\leq c_T M^2 \mathbb{E} \left( \int_{t_0}^{t_0+T} \|G(u + \tau, X(u + \tau, \theta_{-\tau} \cdot)) - (G(u, X(u, \cdot)))\|^2 du \right) \\ &\leq c_T \frac{2M^2T\varepsilon^2}{\alpha^2} (\mathfrak{h}^2 + 1 + 2\mathfrak{g}^2T). \end{aligned}$$

We deduce that, for some constant  $\kappa > 0$ ,

$$\mathbb{E} \left( \sup_{t \in J} \|\mathfrak{T}_\tau X(t) - X(t)\|^2 \right) \leq 3(A + B + C) \leq \kappa \frac{\varepsilon^2}{\alpha^2},$$

thus, taking  $\alpha = \sqrt{\kappa}$ , we obtain that  $\tau$  is an  $\varepsilon$ -period of the map  $Z_J$  defined in (4.2). We deduce that the set of  $\varepsilon$ -almost periods of  $Z_J$  is relatively dense. By Theorem 4.4, this shows that  $X$  is APPD.  $\square$

The following result is given in a slightly different setting in [25, Theorem 3.1]:

**Lemma 5.4.** *Assume that Hypotheses (H1) and (H2) are satisfied, with  $M = 1$ , and that*

$$2\mathfrak{h}^2 \left( \frac{1}{\mathfrak{d}^2} + \frac{1}{2\mathfrak{d}} \right) < 1. \quad (5.16)$$

*Then the operator SDE is a contraction in  $\text{CUB}(\Omega; \mathbb{H})$ .*

*Proof.* Let  $X, Y \in \text{CUB}(\Omega; \mathbb{H})$ . We have, for any  $t \in \mathbb{R}$ ,

$$\begin{aligned} & \mathbb{E} \|\Gamma X(t, \cdot) - \Gamma Y(t, \cdot)\|^2 \\ & \leq 2 \mathbb{E} \left( \int_{-\infty}^t e^{-\mathfrak{d}(t-s)} \|F(s, X(s, \cdot)) - F(s, Y(s, \cdot))\| ds \right)^2 \\ & \quad + 2 \mathbb{E} \left( \left\| \int_{-\infty}^t S(t-s) (G(s, X(s, \cdot)) - G(s, Y(s, \cdot))) dW(s, \cdot) \right\|^2 \right) \\ & \leq 2 \int_{-\infty}^t e^{-\mathfrak{d}(t-s)} ds \int_{-\infty}^t e^{-\mathfrak{d}(t-s)} \mathbb{E} \|F(s, X(s, \cdot)) - F(s, Y(s, \cdot))\|^2 ds \\ & \quad + 2 \int_{-\infty}^t e^{-2\mathfrak{d}(t-s)} \mathbb{E} \|G(s, X(s, \cdot)) - G(s, Y(s, \cdot))\|^2 ds \\ & \leq 2\mathfrak{h}^2 \left( \left( \int_{-\infty}^t e^{-\mathfrak{d}(t-s)} ds \right)^2 + \int_{-\infty}^t e^{-2\mathfrak{d}(t-s)} ds \right) \sup_{s \in \mathbb{R}} \mathbb{E} \|X(s, \cdot) - Y(s, \cdot)\|^2 \\ & = 2\mathfrak{h}^2 \left( \frac{1}{\mathfrak{d}^2} + \frac{1}{2\mathfrak{d}} \right) \sup_{s \in \mathbb{R}} \mathbb{E} \|X(s, \cdot) - Y(s, \cdot)\|^2. \end{aligned}$$

□

The following result is similar to [17, Theorems 4.1 and 4.3], with different hypotheses. It improves [25, Theorem 3.1] on almost periodic solutions, notably since  $F$  and  $G$  are assumed to be almost periodic uniformly with respect to compact sets, instead of bounded sets. The main novelty is the addition of  $\theta$ -almost periodicity, which allows for a different proof of almost periodicity in path distribution. This method provides a result on periodic solutions as a particular case.

**Theorem 5.5** (Almost periodic or periodic solution). *Assume that Hypotheses (H1)-(H2)-(H3) are satisfied, with  $M = 1$ , and that (5.16) holds true. Then (5.3) has a unique bounded mild solution  $X$  which satisfies (5.9) and has a continuous modification, and  $X$  is  $\theta$ -almost periodic and almost periodic in path distribution (APPD).*

*Furthermore, if  $F(\cdot, x)$  and  $G(\cdot, x)$  are  $\tau$ -periodic, for some  $\tau > 0$  and all  $x \in \mathbb{H}$ , then  $X$  is  $\theta$ - $\tau$ -periodic, thus it is periodic in finite dimensional*

distribution (PFD). In particular, if  $F$  and  $G$  do not depend on the first variable,  $X$  is  $\theta$ -stationary.

*Proof.* By Proposition 5.1 or Proposition 5.2, and Proposition 3.8, to prove the existence and uniqueness of a  $\theta$ -almost periodic solution (or a  $\theta$ - $\tau$ -periodic solution if  $F(\cdot, x)$  and  $G(\cdot, x)$  are  $\tau$ -periodic, for some  $\tau > 0$  and all  $x \in \mathbb{H}$ ), it is sufficient to show that the operator  $\Gamma$  has an attracting fixed point in  $\text{CUB}(\Omega; \mathbb{H})$ , but this is a consequence of Lemma 5.4. By Proposition 5.3,  $X$  is APPD. If  $F$  and  $G$  are  $\tau$ -periodic with respect to the first variable,  $X$  is PFD by Theorem 4.4.  $\square$

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