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THE p -RANK ε -CONJECTURE ON CLASS GROUPS IS TRUE FOR TOWERS OF p -EXTENSIONS

GEORGES GRAS

ABSTRACT. Let $p \geq 2$ be a given prime number. We prove, for any number field \mathcal{K} and any integer $e \geq 1$, the p -rank ε -conjecture, on the p -class groups \mathcal{C}_F , for the family $\mathcal{F}_\kappa^{p^e}$ of towers F/\mathcal{K} built as successive degree p cyclic extensions (without any other Galois conditions) such that F/\mathcal{K} be of degree p^e , namely: $\#(\mathcal{C}_F \otimes \mathbb{F}_p) \ll_{\kappa, p^e, \varepsilon} (\sqrt{D_F})^\varepsilon$ for all $F \in \mathcal{F}_\kappa^{p^e}$, where D_F is the absolute value of the discriminant (Theorem 3.6), and more generally $\#(\mathcal{C}_F \otimes \mathbb{Z}/p^r\mathbb{Z}) \ll_{\kappa, p^e, \varepsilon} (\sqrt{D_F})^\varepsilon$ ($r \geq 1$ fixed). This Note generalizes the case of the family $\mathcal{F}_\mathbb{Q}^p$ (*Genus theory and ε -conjectures on p -class groups*, J. Number Theory 207, 423–459 (2020)), whose techniques appear to be “universal” for all relative degree p cyclic extensions and use the Montgomery–Vaughan result on prime numbers. Then we prove, for $\mathcal{F}_\kappa^{p^e}$, the p -rank ε -conjecture on the cohomology groups $H^2(\mathcal{G}_F, \mathbb{Z}_p)$ of Galois p -ramification theory over F (Theorem 4.3) and for some other classical finite p -invariants of F , as the Hilbert kernels and the logarithmic class groups.

1. INTRODUCTION

For any prime number $p \geq 2$ and any number field F , we denote by \mathcal{C}_F the p -class group of F (in the restricted sense for $p = 2$); the precise sense does not matter, the case of ordinary sense being a consequence.

To avoid any ambiguity, we shall write $\mathcal{C}_F \otimes \mathbb{F}_p$, isomorphic to the “ p -torsion group” also denoted $\mathcal{C}_F[p]$ in the literature, only giving the p -rank of \mathcal{C}_F :

$$\mathrm{rk}_p(\mathcal{C}_F) := \dim_{\mathbb{F}_p}(\mathcal{C}_F/\mathcal{C}_F^p).$$

We refer to our paper [14] for an introduction with some history about the notion of ε -conjecture, initiated by Ellenberg–Venkatesh [11] by means of reflection theorems [20] and Arakelov class groups [37], then developed by many authors as Frei–Pierce–Turnage–Butterbaugh–Widmer–Wood [10, 11, 12, 35, 36, 41, 42], . . . , then the related density results of Koymans–Pagano [28], all these questions being in relation with the classical heuristics/conjectures on class groups [1, 4, 5, 9, 13, 31].

We prove, unconditionally:

Main results. *Let \mathcal{C}_F denotes the p -class group of a number field F and D_F the absolute value of its discriminant. For κ and e fixed, the general p -rank ε -conjecture is true for the family $\mathcal{F}_\kappa^{p^e}$ of p -towers F/κ of degree p^e ($\kappa = F_0 \subset \cdots \subset F_{i-1} \subset F_i \subset \cdots \subset F_e = F$) with F_i/F_{i-1} cyclic of degree p for all*

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$i \in [1, e]$ (the F/κ will be called “ p -cyclic-towers”): for all $\varepsilon > 0$, there exists a constant $C_{\kappa, p^e, \varepsilon}$, such that (Theorem 3.6):

$$\#(\mathcal{C}_F \otimes \mathbb{F}_p) \leq C_{\kappa, p^e, \varepsilon} \cdot (\sqrt{D_F})^\varepsilon, \quad \text{for all } F \in \mathcal{F}_\kappa^{p^e}.$$

Moreover, the parameter κ only intervenes by its degree $d_\kappa \leq d$ and the p -rank $\text{rk}_p(\mathcal{C}_\kappa) \leq \rho$ of its class group (for d and ρ given), so that, in some sense, $C_{\kappa, p^e, \varepsilon} = C_{d, \rho, p^e, \varepsilon}$.

We obtain the same result for the cohomology groups $H^2(\mathcal{G}_F, \mathbb{Z}_p)$ of p -ramification theory, where \mathcal{G}_F is the Galois group of the maximal p -ramified pro- p -extension of F , then, for example, for p -adic regulators, Hilbert’s and regular kernels, Jaulent’s logarithmic class groups (Theorem 4.3 and Remark 4.4).

Note. During the writing of this article, we have been informed of papers by Wang [41, Theorem 1.1] (dealing with the non-trivial case $\ell \neq p$ and mentioning, without proofs, the easier case $\ell = p$, for $\text{Gal}(F/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^e$) and Klüners–Wang [29] giving a proof for $\ell = p$ and extensions F/κ contained in (Galois) p -extensions; their results are based on other information (Cornell paper [6] using genus theory in elementary p -extensions, then a generalization of Cornell approach in [29]). We thank Jiuya Wang for these communications, in particular the text [29] (in order to be published) from a lecture by the authors.

Our result, using another point of view, is unconditional with computable constants and without any Galois conditions, contrary to [29] where the “ ε -inequality” is valid for $D_F \gg 0$ and where the Galois closure of F/κ is assumed to be a p -extension. In fact, if our result does not give the “strong ε -conjecture”

$$\#(\mathcal{C}_F \otimes \mathbb{Z}_p) \ll_{\kappa, p^e, \varepsilon} \cdot (\sqrt{D_F})^\varepsilon,$$

it gives, for any fixed $r \geq 1$, the ε -inequality (see Corollary 3.8):

$$\#(\mathcal{C}_F \otimes \mathbb{Z}/p^r\mathbb{Z}) \ll_{\kappa, p^e, \varepsilon} \cdot (\sqrt{D_F})^\varepsilon, \quad \text{for all } F \in \mathcal{F}_\kappa^{p^e}.$$

However, this suggests well, considering also the density results of [28], that the strong ε -conjecture is true “for almost all” elements of $\mathcal{F}_\kappa^{p^e}$.

2. PRINCIPLE OF THE METHOD

We will perform an induction on the successive degree p cyclic extensions of a tower $F \in \mathcal{F}_\kappa^{p^e}$, the principles for such p -cyclic steps coming from [14] dealing with the family $\mathcal{F}_\mathbb{Q}^p$. The method involves using fixed points exact sequences, for the invariants considered, and the definition of “minimal relative discriminants” built by means of the Montgomery–Vaughan result on prime numbers.

2.1. Tower of degree p cyclic fields and relative class groups. Let κ be any fixed number field and let:

$$F_0 = \kappa \subset F_1 \cdots F_{i-1} \subset F_i \cdots \subset F_e = F, \quad e \geq 1,$$

be a p -cyclic-tower of fields with $\text{Gal}(F_i/F_{i-1}) \simeq \mathbb{Z}/p\mathbb{Z}$, for $1 \leq i \leq e$ (by abuse, we use the word of tower even if the fields F_i are not necessarily Galois over κ).

Remark 2.1. For $p = 2$, all the 2-cyclic-towers are obtained inductively by means of Kummer extensions $F_i = F_{i-1}(\sqrt{a_{i-1}})$, $a_{i-1} \in F_{i-1}^\times \setminus F_{i-1}^{\times 2}$; for $p \neq 2$, one uses the classical Kummer process, considering $F'_{i-1} := F_{i-1}(\mu_p)$ and then taking $a'_{i-1} = b'_{i-1} e_\omega$ in F'_{i-1} , where e_ω is the idempotent of $\mathbb{Z}_p[\text{Gal}(F'_{i-1}/F_{i-1})]$ corresponding to the Teichmüller character ω , so that $F'_i := F'_{i-1}(\sqrt{a'_{i-1}})$ is decomposed over F_{i-1} into F_i/F_{i-1} cyclic of degree p .

We check easily that, for any p , the (non- p -cyclic) towers of degree p^e , of the form $F_i = F_{i-1}(\sqrt[e]{a_{i-1}})$, fulfill the p -rank ε -conjecture.

This gives many more extensions F/κ of degree p^e which are not necessarily (Galois) p -extensions.

Put $k = F_{i-1}$, $K = F_i$ and $G = \text{Gal}(K/k) =: \langle \sigma \rangle$. We shall use the obvious general exact sequence:

$$1 \rightarrow \mathcal{C}_K^* \rightarrow \mathcal{C}_K \xrightarrow{\nu} \mathcal{C}_K^\nu,$$

where $\nu := 1 + \sigma + \cdots + \sigma^{p-1}$ is the algebraic norm and $\mathcal{C}_K^* = \text{Ker}(\nu)$.

Recall that $\nu = \text{J} \circ \text{N}$, where N is the arithmetic norm (or the restriction of automorphisms $\text{Gal}(H_K/K) \rightarrow \text{Gal}(H_k/k)$ in the corresponding Hilbert's class fields H_K, H_k), which yields $\text{N}(\mathcal{C}_K) \subseteq \mathcal{C}_k$, and where $\text{J} : \mathcal{C}_k \rightarrow \mathcal{C}_K$ comes from extension of ideals (or is the transfer map $\text{Gal}(H_k/k) \rightarrow \text{Gal}(H_K/K)$); so $\mathcal{C}_K^\nu \simeq \mathcal{C}_k$ if and only if N is surjective (equivalent to $H_k \cap K = k$) and J injective (no capitulation). Whence the following inequality for the p -ranks:

$$(1) \quad \text{rk}_p(\mathcal{C}_K) \leq \text{rk}_p(\mathcal{C}_k) + \text{rk}_p(\mathcal{C}_K^*).$$

So, the main problem is to give an explicit upper bound of $\text{rk}_p(\mathcal{C}_K^*)$ only depending on $\text{rk}_p(\mathcal{C}_k)$ and the number of ramified prime numbers in K/k . For this, we shall recall some elementary properties of the finite $\mathbb{Z}_p[G]$ -modules annihilated by ν [17].

Let $G = \langle \sigma \rangle \simeq \mathbb{Z}/p\mathbb{Z}$. We consider a $\mathbb{Z}_p[G]$ -module M^* (of finite type) annihilated by $\nu = 1 + \cdots + \sigma^{p-1}$ (which will be \mathcal{C}_K^*) for which we define the following filtration, for all $h \geq 0$ and $M_0^* = 1$:

$$(2) \quad M_{h+1}^*/M_h^* := (M^*/M_h^*)^G.$$

For all $h \geq 0$, $M_h^* = \{x \in M^*, x^{(1-\sigma)^h} = 1\}^1$, the p -groups M_{h+1}^*/M_h^* are elementary and the maps $M_{h+1}^*/M_h^* \xrightarrow{1-\sigma} M_h^*/M_{h-1}^*$ are injective, giving a decreasing sequence for the orders $\#(M_{h+1}^*/M_h^*)$ as h grows; whence:

$$(3) \quad \#(M_{h+1}^*/M_h^*) \leq \#M_1^*, \text{ for all } h \geq 0.$$

Since M^* is a $\mathbb{Z}_p[G]/(\nu)$ -module and since $\mathbb{Z}_p[G]/(\nu) \simeq \mathbb{Z}_p[\zeta]$, where ζ is a primitive p th root of unity, we may write for $\mathfrak{p} = (1 - \zeta)\mathbb{Z}_p[\zeta]$ and $s \geq 0$:

$$M^* \simeq \bigoplus_{j=1}^s \mathbb{Z}_p[\zeta]/\mathfrak{p}^{n_j}, \quad n_1 \leq n_2 \leq \cdots \leq n_s,$$

¹Note that if M is a $\mathbb{Z}_p[G]$ -module of finite type and $M^* = \text{Ker}(\nu)$, we have, in an obvious meaning, $(M^*)_h = (M_h)^*$, whence the writing M_h^* .

and the sub-modules M_h^* are, in this $\mathbb{Z}_p[\zeta]$ -structure, the following ones:

$$M_h^* \simeq \bigoplus_{j, n_j \leq h} \mathbb{Z}_p[\zeta]/\mathfrak{p}^{n_j}, \text{ for all } h \geq 0.$$

Proposition 2.2. *Under the condition $M^{*\nu} = 1$, the p -rank of M^* fulfills the inequality $\text{rk}_p(M^*) \leq (p-1) \cdot \text{rk}_p(M_1^*)$.*

Proof. Let $M^*[p] := \{x \in M^*, x^p = 1\}$; since $p\mathbb{Z}_p[\zeta] = \mathfrak{p}^{p-1}$, we obtain that $M^*[p] = M_{p-1}^*$; then (from (3)):

$$\#M_{p-1}^* = \prod_{h=0}^{p-2} \#(M_{h+1}^*/M_h^*) \leq (\#M_1^*)^{p-1}.$$

Since M_{p-1}^* and M_1^* are elementary, the inequality on the p -ranks follows. \square

Considering $M = \mathcal{C}_K$ and M^* in K/k yields:

Corollary 2.3. *Let K/k be a degree p cyclic extension of number fields and let $G = \text{Gal}(K/k)$. Then $\text{rk}_p(\mathcal{C}_K^G) \leq (p-1) \text{rk}_p(\mathcal{C}_K^{G*}) = (p-1) \text{rk}_p(\mathcal{C}_K^G)$.*

2.2. Majoration of $\text{rk}_p(\mathcal{C}_K^G)$ and $\text{rk}_p(\mathcal{C}_K)$.

Proposition 2.4. *We have $\text{rk}_p(\mathcal{C}_K^G) \leq \text{rk}_p(\mathcal{C}_k) + t_k + \text{rk}_p(E_k/E_k^p)$, where t_k is the number of prime ideals of k ramified in K/k and where E_k is the group of units of k .*

Proof. We have the classical exact sequence:

$$(4) \quad 1 \rightarrow \mathcal{C}_K(I_K^G) \rightarrow \mathcal{C}_K^G \xrightarrow{\theta} E_k \cap N_{K/k}(K^\times)/N_{K/k}(E_K) \rightarrow 1,$$

where I_K is the $\mathbb{Z}[G]$ -module of ideals of K and where θ associates with $\mathcal{C}_K(\mathfrak{A})$, such that $\mathfrak{A}^{1-\sigma} = (\alpha)$, $\alpha \in K^\times$, the class of the unit $N_{K/k}(\alpha)$ of k , modulo $N_{K/k}(E_K)$. The surjectivity and the kernel are immediate.

The $\mathbb{Z}[G]$ -module I_K^G is generated by $J(I_k)$, extension in K of the ideals of k , and by the t_k prime ideals of K ramified in K/k . Thus, using the obvious inequality $\text{rk}_p(E_k \cap N_{K/k}(K^\times)/N_{K/k}(E_K)) \leq \text{rk}_p(E_k/E_k^p)$, we obtain:

$$\begin{aligned} \text{rk}_p(\mathcal{C}_K^G) &\leq \text{rk}_p(\mathcal{C}_K(I_K^G)) + \text{rk}_p(E_k \cap N_{K/k}(K^\times)/N_{K/k}(E_K)) \\ &\leq \text{rk}_p(\mathcal{C}_k) + t_k + \text{rk}_p(E_k/E_k^p), \end{aligned}$$

which proves the claim. \square

Remark 2.5. The Chevalley formula [3] gives, in K/k , the order:

$$\#(\mathcal{C}_K^G) = \#\mathcal{C}_k \cdot \frac{p^{t_k-1}}{(E_k : E_k \cap N_{K/k}(K^\times))};$$

one sees some similarities between this formula and the inequality given by Proposition 2.4, but we can not deduce a more efficient inequality on the ranks because we only have $\text{rk}_p(\mathcal{C}_K^G) \leq \frac{\log(\#\mathcal{C}_k)}{\log(p)} + t_k - 1$, and the order of \mathcal{C}_k may be huge even if its p -rank may be evaluated.

The Chevalley formula gives a more precise inequality when $\#\mathcal{C}_k$ is small (e.g., $\mathcal{C}_k = 1$, giving $\text{rk}_p(\mathcal{C}_K^G) \leq t_k - 1$); on the contrary, our inductive method controls more the p -ranks than the orders.

Denote by $\{\ell_1, \dots, \ell_N\}$, $N = N_{i-1} \geq 0$, the set of tame prime numbers ramified in $K/k = F_i/F_{i-1}$ (for such a ℓ , there exist prime ideals $\mathfrak{l}_u \mid \ell$ in k , $u \in [1, t_{k,\ell}]$, $t_{k,\ell} \geq 1$, ramified in K/k and similarly for p). Thus, with the above notations:

$$t_k = t_{k,p} + \sum_{j=1}^N t_{k,\ell_j}.$$

We shall replace, in Proposition 2.4, $\text{rk}_p(E_k/E_k^p)$, then $t_{k,p}$ and t_{k,ℓ_j} , by the rough upper bound $d_k := [k : \mathbb{Q}] = p^{i-1} [\kappa : \mathbb{Q}]$, which gives, using inequality (1), Corollary 2.3 and Proposition 2.4:

Proposition 2.6. *Put $d_k = [k : \mathbb{Q}]$ and let N be the number of tame prime numbers ramified in K/k (cyclic of degree p); we have the inequalities:*

$$\begin{aligned} \text{rk}_p(\mathcal{C}_K^*) &\leq (p-1) \cdot [\text{rk}_p(\mathcal{C}_k) + (N+2)d_k], \\ \text{rk}_p(\mathcal{C}_K) &\leq p \cdot \text{rk}_p(\mathcal{C}_k) + (p-1)(N+2)d_k. \end{aligned}$$

3. ABOUT THE DISCRIMINANTS IN THE TOWER $(F_i)_{0 \leq i \leq e}$

Now, the goal is to give lower bounds for discriminants, unlike for the case of p -ranks. Give some essential explanations:

Remark 3.1. We have given, in the previous section, an upper bound for the p -rank of \mathcal{C}_K in $K/k = F_i/F_{i-1}$, where the number of ramified primes is crucial because of “genus theory” aspects, so that *any true p -rank* at the step K/k (whatever $F \in \mathcal{F}_k^{p^e}$) will be *smaller*; note that each integer $N = N_{i-1}$ is relative to the step $K/k = F_i/F_{i-1}$, which will be also the case of the relative discriminants $D_{K/k}$.

In the present section, we shall give a definition of the “minimal tame relative discriminant” $D_N^{\text{ta}} \in \mathbb{N}$ only depending on N , then a minor modification giving $D_N \geq D_N^{\text{ta}}$ (so that any *true* relative discriminant $D_{K/k} = D_{K/k}^{\text{ta}} \times (p\text{-power}) \in \mathbb{N}$ (from ramification of tame ℓ_j and possibly that of p), will be *larger* than D_N). Then we shall apply the relation $D_K = D_k^p D_{K/k} \geq D_k^p D_N$.

Thus, under this facts, if an “ ε -inequality” $\#(\mathcal{C}_K \otimes \mathbb{F}_p) \ll (\sqrt{D_k^p D_N})^\varepsilon$ does exist between such fictitious p -ranks and discriminants, a fortiori, any effective situation will fulfill the p -rank ε -inequality at this step.

3.1. Tame relative discriminant – Minimal relative discriminant. As above, let $k = F_{i-1}$, $K = F_i$, $N = N_{i-1}$ and let D_k^{ta} and D_K^{ta} be the absolute values of *the tame parts* of the discriminants of k and K , respectively. For the N tame prime numbers ℓ , ramified in K/k , let $\mathfrak{l}_1, \dots, \mathfrak{l}_{t_{k,\ell}}$ be the prime ideals of k above ℓ , ramified in K/k .

The relative norm of the different of K/k gives the tame part of the relative

ideal discriminant of K/k , namely $\mathcal{D}_{K/k}^{\text{ta}} = \prod_{j=1}^N \prod_{u=1}^{t_{k,\ell_j}} \mathfrak{l}_{j,u}^{p-1}$, and its absolute norm is $D_{K/k}^{\text{ta}}$ (tame relative discriminant); the tame “discriminant formula” [38, Propositions IV.4 and III.8] yields ($f_{k,\mathfrak{l}_{j,u}}$ is the residue degree of $\mathfrak{l}_{j,u}$ in k/\mathbb{Q}):

$$D_K^{\text{ta}} = (D_k^{\text{ta}})^p \cdot N_{k/\mathbb{Q}}(\mathcal{D}_{K/k}^{\text{ta}}) = (D_k^{\text{ta}})^p \cdot D_{K/k}^{\text{ta}} \leq D_k^p \cdot D_{K/k}^{\text{ta}},$$

where $D_{K/k}^{\text{ta}} = \prod_{j=1}^N \prod_{u=1}^{t_{k,\ell_j}} \ell_j^{(p-1)f_{k,t_j,u}} = \prod_{j=1}^N \ell_j^{(p-1)\cdot \sum_{u=1}^{t_{k,\ell_j}} f_{k,t_j,u}}$.

We intend now to determine a lower bound D_N of $D_{K/k}$, only depending on N , so that for every concrete ramification in K/k , with N tame primes ℓ_j and possibly that of p , the effective relative discriminant $D_{K/k}$ will be necessarily larger than D_N as explained in Remark 3.1. This needs two obvious lemmas.

Lemma 3.2. *A lower bound of the tame relative discriminant $D_{K/k}^{\text{ta}}$ is obtained when $t_{k,\ell_j} = f_{k,t_j} = 1$ for all $j = 1, \dots, N$; thus the “fictive” minimum of $D_{K/k}^{\text{ta}}$ is then $D_N^{\text{ta}} := \prod_{j=1}^N q_j^{p-1}$, taking the N successive prime numbers $q_j \neq p$.*

We may call D_N^{ta} the *minimal tame relative discriminant* (for instance, for $p = 2$, $D_N^{\text{ta}} = 3 \cdot 5 \cdot 7 \cdots q_N$); whence, an analogous framework as for the case of degree p cyclic number fields given in [14]. Note that, when $\kappa \neq \mathbb{Q}$ the primes ℓ , ramified in K/k , are not necessarily such that $\ell \equiv 1 \pmod{p}$ (e.g., $p = 3$, $\kappa = \mathbb{Q}(\mu_3)$ and $F = \kappa(\sqrt[3]{2})$).

The second lemma will simplify the forthcoming computations:

Lemma 3.3. *One can replace $D_N^{\text{ta}} = \prod_{j=1, q_j \neq p}^N q_j^{p-1}$ by $D_N = \prod_{j=1}^N q_j^{p-1}$.*

Proof. The claim is obvious if $p > q_N$. Otherwise, there are two cases for a true relative discriminant $D_{K/k}$:

- (i) $p \mid D_{K/k}$. In this case the p -part of $D_{K/k}$ is a p -power larger than p and the required inequality between $D_{K/k}$ and D_N holds;
- (ii) $p \nmid D_{K/k}$. In this case, since $p \mid D_N$ and since $D_{K/k}$ has N tame ramified primes, there exists at least an $\ell \mid D_{K/k}$ (“replacing” p), $\ell \nmid D_N$; so this prime is larger than $q_N \geq p$, thus $D_{K/k} > D_N$. \square

3.2. Induction. Let $F \in \mathcal{F}_\kappa^{p^\varepsilon}$. The case $i = 0$ ($k = \kappa$) is obvious since to get, for all $\varepsilon > 0$:

$$\#(\mathcal{C}_\kappa \otimes \mathbb{F}_p) = p^{\text{rk}_p(\mathcal{A}_\kappa)} \leq C_{\kappa,p,\varepsilon} \cdot (\sqrt{D_\kappa})^\varepsilon,$$

it suffices to take $C_{\kappa,p,\varepsilon} = C_{\kappa,p} := p^{\text{rk}_p(\mathcal{A}_\kappa)}$ since $(\sqrt{D_\kappa})^\varepsilon > 1$; in other words one may replace $\mathcal{F}_\kappa^{p^\varepsilon}$ by the family $\mathcal{F}_{d,\rho}^{p^\varepsilon}$ where κ , of degree $d_\kappa \leq d$, varies under the condition $\text{rk}_p(\mathcal{C}_\kappa) \leq \rho$, d, ρ given (e.g., one may consider all the p -principal base fields κ of degree $d_\kappa \leq d$ with $\rho = 0$).

By induction, we assume (for $k := F_{i-1}$, $K = F_i$) that for all $\varepsilon > 0$ there exists a constant $C_{k,p,\varepsilon}$ such that $p^{\text{rk}_p(\mathcal{A}_k)} \leq C_{k,p,\varepsilon} \cdot (\sqrt{D_k})^\varepsilon$ independently of the number N of tame ramified primes ℓ_j in K/k ; then we shall prove the property for K .

Proposition 2.6 implies the following inequality (where $d_k := [k : \mathbb{Q}]$):

$$(5) \quad p^{\text{rk}_p(\mathcal{A}_K)} \leq p^{p \cdot \text{rk}_p(\mathcal{A}_k) + (p-1)(N+2)d_k} \leq C_{k,p,\varepsilon}^p \cdot (\sqrt{D_k}^p)^\varepsilon \cdot p^{(p-1)(N+2)d_k}.$$

Then, as explained in Remark 3.1, we will compare $p^{(p-1)(N+2)d_k}$ and $(\sqrt{D_N})^\varepsilon$,

where (Lemmas 3.2, 3.3) $D_N = \prod_{j=1}^N q_j^{p-1}$, the q_j being the N consecutive prime numbers whatever p , then using the inequality $D_K \geq D_k^p D_N$.

But we have the required computations in [14, §2.3, Formulas (4, 5), p.10] that we improve with some obvious modifications.

Proposition 3.4. *There exists $c_{k,p,\varepsilon}$ such that $p^{(p-1)(N+2)d_k} \leq c_{k,p,\varepsilon} \cdot (\sqrt{D_N})^\varepsilon$.*

Proof. For $N = 0$, $D_N = 1$ (K/k is at most p -ramified), so that the result (independent of ε) is true since the constant $c_{k,p,\varepsilon}$, resulting of the computation of a maximum taken over $N \in \mathbb{N}$ (see Remark 3.5), will be much larger than $p^{2(p-1)d_k}$ as we shall verify; we assume $N \geq 1$ in what follows.

The existence of $c_{k,p,\varepsilon}$ is equivalent to the fact that $p^{(p-1)(N+2)d_k} (\sqrt{D_N})^{-\varepsilon}$ is bounded over N , whence $(p-1)(N+2)d_k \log(p) - \varepsilon \cdot \frac{p-1}{2} \sum_{j=1}^N \log(q_j) < \infty$, in which case, $\log(c_{k,p,\varepsilon})$ is given by the upper bound.

The main purpose is to estimate the sum $\sum_{j=1}^N \log(q_j)$, knowing that we can replace this sum by any convenient lower bound but noting that this will increase the constant $c_{k,p,\varepsilon}$.

We replace the consecutive primes q_j (except for $q_1 = 2$) by the lower bounds:

$$q'_1 = 2, \quad q'_j = \frac{1}{2} j \log\left(\frac{q_j}{2}\right), \quad j \geq 2$$

(cf. [39, Notes on Ch. I, §4.6] about the Montgomery–Vaughan result giving, for all prime numbers, this lower bound without error term). Thus a sufficient condition to have $p^{(p-1)(N+2)d_k} \cdot (\sqrt{D_N})^{-\varepsilon}$ bounded is that:

$$X(N) := (p-1)(N+2)d_k \log(p) - \varepsilon \cdot \frac{p-1}{2} \left(\log(2) + \sum_{j=2}^N \left[\log\left(\frac{1}{2}\right) + \log(j) + \log_2\left(\frac{q_j}{2}\right) \right] \right) < \infty.$$

We check that $e := \log(2) + \sum_{j=2}^N \left[\log\left(\frac{1}{2}\right) + \log_2\left(\frac{q_j}{2}\right) \right]$ is positive, except few values of N , but that this does not contradict the whole lower bound, as shown by the following PARI/GP [34] program computing $E = S - t$ and $e = s - t$, with:

$$S = \sum_{j=1}^N \log(q_j), \quad s = \log(2) + \sum_{j=2}^N \left[\log\left(\frac{1}{2}\right) + \log(j) + \log_2\left(\frac{q_j}{2}\right) \right], \quad t = \sum_{j=2}^N \log(j):$$

thus E measures the approximation regarding the lower bound t of S .

```
{for(N=1,100,S=log(2);s=log(2);t=0;for(j=2,N,e1=prime(j);
S=S+log(e1);s=s+log(j/2)+log(log(e1/2));t=t+log(j));E=S-t;e=s-t;
print("N=",N," E=S-t=",precision(E,1)," e=s-t=",precision(e,1));
print("S=",precision(S,1)," s=",precision(s,1)," t=",precision(t,1)))}
```

```
N=1  E=S-t=0.693147180559945309  e=s-t=0.6931471805599453095
      S=0.693147180559945309  s=0.693147180559945309  t=0
N=2  E=S-t=1.098612288668109691  e=s-t=-0.902720455717879982
      S=1.791759469228055000  s=-0.20957327515793467  t=0.693147180559945309
N=3  E=S-t=1.609437912434100374  e=s-t=-1.683289208068580387
      S=3.401197381662155376  s=0.108470261159474613  t=1.791759469228055000
N=4  E=S-t=2.169053700369523061  e=s-t=-2.151084901802564193
      S=5.347107530717468681  s=1.026968928545381426  t=3.178053830347945620
N=5  E=S-t=2.957511060733793231  e=s-t=-2.310814729030344016
      S=7.745002803515839225  s=2.476677013751701978  t=4.787491742782045994
```


N=6	E=S-t=3.730700948967274966	e=s-t=-2.377060211850912773
	S=10.30995216097737596	s=4.20219100015918822
	t=6.579251212010100996	
N=7	E=S-t=4.618004143968177741	e=s-t=-2.309370646333412833
	S=13.143165505033592041	s=6.21579071473200146
	t=8.525161361065414300	
N=8	E=S-t=5.483001581454782273	e=s-t=-2.191013642714161849
	S=16.087604484200032501	s=8.41358926003108838
	t=10.60460290274525022	
N=9	E=S-t=6.421271220047712581	e=s-t=-1.9912013465566233694
	S=19.223098700129182192	s=10.8106261335248462
	t=12.80182748008146961	
N=10	E=S-t=7.485981957040140924	e=s-t=-1.7007174593212892239
	S=22.590394530115656220	s=13.4036951137542260
	t=15.10441257307551529	
N=11	E=S-t=8.522073888726916626	e=s-t=-1.3856001883918199308
	S=26.024381734600802466	s=16.1167076574820659
	t=17.50230784587388584	
N=12	E=S-t=9.64808515158314076	e=s-t=-1.0079274921626889861
	S=29.63529964724502690	s=18.9792870034991971
	t=19.98721449566188615	
N=13	E=S-t=10.7967078608259118	e=s-t=-0.5956771604555961438
	S=33.34887171394933471	s=21.9564866926678267
	t=22.55216385312342288	
N=14	E=S-t=11.9188506469042156	e=s-t=-0.16778120370448471046
	S=37.11007182964289714	s=25.0234399790341967
	t=25.19122118273868150	
N=15	E=S-t=13.0609480475120641	e=s-t=0.2886939587134154542
	S=40.96021943135295572	s=28.1879653425543070
	t=27.89927138384089156	
N=16	E=S-t=14.2586512388244047	e=s-t=0.7825193132031076079
	S=44.93051134490507756	s=31.4543794192837804
	t=30.67186010608067280	
N=17	E=S-t=15.5029753386739081	e=s-t=1.3085458937854073287
	S=49.00804878881079701	s=34.8136193439222962
	t=33.50507345013688888	
N=18	E=S-t=16.7234774449510546	e=s-t=1.8443743308869917078
	S=53.11892265298410826	s=38.2398195389200452
	t=36.39544520803305358	
N=19	E=S-t=17.9837310851755802	e=s-t=2.407283386866249599
	S=57.32361527237507432	s=41.7471675740657436
	t=39.33988418719949404	
N=20	E=S-t=19.250678688662904708	e=s-t=2.986570896185892937
	S=61.58629514941638974	s=45.3221873569393779
	t=42.33561646075348503	
N=21	E=S-t=20.496615692087872840	e=s-t=3.573610687966718782
	S=65.87675459056478087	s=48.9537495864436268
	t=45.38013889847690803	
N=22	E=S-t=21.775021091196578482	e=s-t=4.182370501783588325
	S=70.24620244303180236	s=52.6535518536188122
	t=48.47118135183522388	
N=23	E=S-t=23.058367483064026714	e=s-t=4.804476310693340535
	S=74.66504305082840029	s=56.4111518784577141
	t=51.60667556776437357	
N=24	E=S-t=24.368950022448220934	e=s-t=5.445142436276319277
	S=79.15367942056054013	s=60.2298718343886384
	t=54.78472939811231919	
	(...)	
N=99	E=S-t=140.388608477587341	e=s-t=72.45660863104452031
	S=499.5228138471627406	s=431.5908140006199191
	t=359.1342053695753988	
N=100	E=S-t=142.07685757044573	e=s-t=73.48627663602501092
	S=505.8162331260092221	s=437.2256521915885011
	t=363.7393755555634902	
	(...)	
N=10000	E=S-t=22283.274179035430378	e=s-t=15748.9389769817289
	S=104392.20201584978383	s=97857.86681379608238
	t=82108.9278368143	
N=10001	E=S-t=22285.623003678229650	e=s-t=15750.6314792930841
	S=104403.76128085955962	s=97868.76975647441412
	t=82118.1382771813	

Thus, since $E = S - t$ is always positive, we may consider, instead:

$$X(N) = (p-1)(N+2) d_k \log(p) - \varepsilon \cdot \frac{p-1}{2} \sum_{j=1}^N \log(j)$$

for which it suffices to prove that $X(N) < \infty$. We have, for all $N \geq 1$, $\log(N!) = N \log(N) - N + \frac{\log(N)}{2} + 1 - o(1)$, giving:

$$X(N) = (p-1)(N+2) d_k \log(p) - \varepsilon \cdot \frac{p-1}{2} \left[N \log(N) - N + \frac{\log(N)}{2} + 1 - o(1) \right],$$

whence:

$$X(N) = -\varepsilon \frac{p-1}{2} N \log(N) + N \left[(p-1) d_k \log(p) + \varepsilon \frac{p-1}{2} - \varepsilon \frac{p-1}{2} o(1) \right],$$

where the new $o(1)$ is of the form $\frac{\log(N)}{2N} + \frac{1-o(1)}{N} > 0$. The dominant term:

$$-\varepsilon \frac{p-1}{2} N \log(N), \quad N \geq 1,$$

ensures the existence of a bound $c_{k,p,\varepsilon}$ over $N \geq 1, N \rightarrow \infty$. \square

Remark 3.5. To obtain an explicit upper bound of the constant $c_{k,p,\varepsilon}$, one may replace the function $X(N)$ by $Y(N) > X(N)$ defined by:

$$Y(N) := -\varepsilon \frac{p-1}{2} N \log(N) + N \left[(p-1) d_k \log(p) + \varepsilon \frac{p-1}{2} \right].$$

One obtains that $Y(N)$ admits, for an $N_0(\varepsilon) > 0$, a maximum given by

$$\log(N_0(\varepsilon)) = 2 d_k \log(p) \cdot \varepsilon^{-1},$$

which yields a maximum for $\log(c_{k,p,\varepsilon})$ less than:

$$\frac{p-1}{2} \varepsilon \cdot e^{2 d_k \log(p) \cdot \varepsilon^{-1}},$$

giving an important constant. This is due in part to the method using certain extreme bounds which are not achieved in practice (for example the systematic use of the upper bound $d_k = [k : \mathbb{Q}]$ to get Proposition 2.6).

Finally, from formula (5) and Proposition 3.4 giving:

$$p^{(p-1)(N+2) d_k} \leq c_{k,p,\varepsilon} \cdot (\sqrt{D_N})^\varepsilon \leq c_{k,p,\varepsilon} \cdot (\sqrt{D_{K/k}})^\varepsilon,$$

we may write in K/k :

$$\begin{aligned} p^{\text{rk}_p(\mathcal{A}_K)} &\leq C_{k,p,\varepsilon}^p \cdot (\sqrt{D_k^p})^\varepsilon \cdot p^{(p-1)(N+2) d_k} \\ &\leq C_{k,p,\varepsilon}^p \cdot (\sqrt{D_k^p})^\varepsilon \cdot c_{k,p,\varepsilon} \cdot (\sqrt{D_{K/k}})^\varepsilon = C_{k,p,\varepsilon}^p \cdot c_{k,p,\varepsilon} \cdot (\sqrt{D_k^p \cdot D_{K/k}})^\varepsilon \\ &= C_{K,p,\varepsilon} \cdot (\sqrt{D_K})^\varepsilon, \end{aligned}$$

with $C_{K,p,\varepsilon} := C_{k,p,\varepsilon}^p \cdot c_{k,p,\varepsilon}$. The degree $[F : \kappa] = p^e$ being fixed, the above induction leads to (denoting $C_{F,p,\varepsilon} := C_{\kappa,p^e,\varepsilon}$):

Theorem 3.6. *Let $p \geq 2$ be prime. The p -rank ε -conjecture for the family $\mathcal{F}_\kappa^{p^e}$ of p -cyclic-towers F/κ of degree p^e , on the existence, for all $\varepsilon > 0$, of a constant $C_{\kappa,p^e,\varepsilon}$ such that $\#(\mathcal{C}_F \otimes \mathbb{F}_p) \leq C_{\kappa,p^e,\varepsilon} (\sqrt{D_F})^\varepsilon$, is fulfilled unconditionally for all $F \in \mathcal{F}_\kappa^{p^e}$.*

Remark 3.7. Let $r \geq 1$. It is obvious that, using the filtration $(M_h^*)_{h \geq 0}$, analogous computations with $M^*[p^r] := \{x \in M^*, x^{p^r} = 1\}$ give (from (3)):

$$\#M^*[p^r] = M_{r(p-1)}^* \leq (\#M_1^*)^{r(p-1)},$$

then, for $M^* = \mathcal{C}_K^*$ and $\text{rk}_p(\mathcal{C}_K^{*G}) \leq \text{rk}_p(\mathcal{C}_k) + (N+2) d_k$ (Proposition 2.4), written under the form $(\#\mathcal{C}_K^{*G})^{r(p-1)} \leq p^{r(p-1) \text{rk}_p(\mathcal{C}_k) + r(p-1)(N+2) d_k}$, we get:

$$\#(\mathcal{C}_K^* \otimes \mathbb{Z}/p^r \mathbb{Z}) \leq p^{r(p-1) \text{rk}_p(\mathcal{C}_k) + r(p-1)(N+2) d_k}.$$

Whence, using $1 \rightarrow \mathcal{C}_K^* \otimes \mathbb{Z}/p^r\mathbb{Z} \rightarrow \mathcal{C}_K \otimes \mathbb{Z}/p^r\mathbb{Z} \rightarrow \mathcal{C}_k \otimes \mathbb{Z}/p^r\mathbb{Z}$, and since $\mathcal{C}_k \otimes \mathbb{Z}/p^r\mathbb{Z} \leq p^{r \operatorname{rk}_p(\mathcal{C}_k)}$, we obtain:

$$\#(\mathcal{C}_K \otimes \mathbb{Z}/p^r\mathbb{Z}) \leq p^{r \operatorname{rk}_p(\mathcal{C}_k) + r(p-1)(N+2)d_k}.$$

Finally, we may consider the following statement as a consequence of the above computations for $r = 1$:

Corollary 3.8. *Let $r \geq 1$ be a fixed integer. Then, for all $\varepsilon > 0$, there exists a constant $C_{\kappa, p^e, \varepsilon}^{(r)}$ such that $\#(\mathcal{C}_F \otimes \mathbb{Z}/p^r\mathbb{Z}) \leq C_{\kappa, p^e, \varepsilon}^{(r)} (\sqrt{D_F})^\varepsilon$.*

Proof. The introduction of r changes:

$$X(N) = (p-1)(N+2)d_k \log(p) - \varepsilon \cdot \frac{p-1}{2} \sum_{j=1}^N \log(j)$$

into the expression:

$$X^{(r)}(N) = r(p-1)(N+2)d_k \log(p) - \varepsilon \cdot \frac{p-1}{2} \sum_{j=1}^N \log(j),$$

which does not modify the dominant term $-\varepsilon \frac{p-1}{2} N \log(N)$ coming from the right term. Then the induction is similar with the exponent r which is a constant and does not modify the reasoning (we omit the details). \square

Remark 3.9. If $F \in \mathcal{F}_\kappa^{p^e}$ is contained in the p -Hilbert tower of κ (which defines a sub-family $\mathcal{F}_\kappa'^{p^e}$ of p -cyclic towers when F varies, especially when the p -Hilbert tower is infinite), we have $D_F = D_\kappa^{p^e}$, whence:

$$\#(\mathcal{C}_F \otimes \mathbb{F}_p) \leq C_{\kappa, p^e, \varepsilon} \cdot (\sqrt{D_\kappa})^{\varepsilon p^e};$$

renormalizing ε , since p^e is a constant, we may write: for all $\varepsilon > 0$, there exists a constant $C'_{\kappa, p^e, \varepsilon}$ such that, for all such unramified p -towers $F \in \mathcal{F}_\kappa'^{p^e}$:

$$\#(\mathcal{C}_F \otimes \mathbb{F}_p) \leq C'_{\kappa, p^e, \varepsilon} \cdot (\sqrt{D_\kappa})^\varepsilon.$$

For other approaches about p -ranks in towers as the degree grows, see for instance Hajir [23] and Hajir–Maire [24].

4. THE p -RANK ε -CONJECTURE IN p -RAMIFICATION THEORY

We shall replace, for the family $\mathcal{F}_\kappa^{p^e}$, the p -class group \mathcal{C}_F by the Galois group \mathcal{A}_F of the maximal p -ramified abelian pro- p -extension $\mathcal{H}_F^{\text{ab}}$ of F or its torsion group \mathcal{T}_F [15, III.2, IV.3]. As we know, this pro- p -group \mathcal{A}_F is a fundamental invariant related to the p -class group \mathcal{C}_F , the normalized p -adic regulator \mathcal{R}_F and the number of independent \mathbb{Z}_p -extensions of F (\mathbb{Z}_p -rank of \mathcal{A}_F depending on Leopoldt's conjecture); see, e.g., [15, IV, §§1,2,3], [18] and the very complete bibliography of [16] for the story of abelian p -ramification theory, especially the items [3, 16, 17, 18, 19, 26, 40, 50, 57, 58, 59, 63, 65, 67, 70, 72].

Give some recalls about \mathcal{A}_F , \mathcal{T}_F and the corresponding fixed point formulas. We assume the Leopoldt conjecture for p in all the fields considered.

Let \mathcal{G}_F be the Galois group of the maximal pro- p -extension \mathcal{H}_F of F , p -ramified (i.e., unramified outside p and non-complexified (= totally split) at the real infinite places of F when $p = 2$).

Its abelianized $\mathcal{A}_F = \text{Gal}(\mathcal{H}_F^{\text{ab}}/F)$ is a \mathbb{Z}_p -module of finite type for which $\mathcal{T}_F := \text{tor}_{\mathbb{Z}_p}(\mathcal{A}_F)$, isomorphic to the dual of the cohomology group $H^2(\mathcal{G}_F, \mathbb{Z}_p)$ [33], fixes the compositum \tilde{F} of the \mathbb{Z}_p -extensions of F . Then:

$$\mathcal{A}_F \simeq \mathbb{Z}_p^{r_2(F)+1} \oplus \mathcal{T}_F,$$

where $r_2(F)$ is the number of complex embeddings of F .

Let $U_F := \bigoplus_{\mathfrak{p}|p} U_{\mathfrak{p}}$, be the product of the principal local units of the completions $F_{\mathfrak{p}}$ of F at the p -places, let \overline{E}_F be the closure in U_F of the diagonal image of the group E_F of global units and let

$$\mathcal{W}_F := \text{tor}_{\mathbb{Z}_p}(U_F)/\text{tor}_{\mathbb{Z}_p}(\overline{E}_F) = \text{tor}_{\mathbb{Z}_p}(U_F)/\mu_p(F)$$

under Leopoldt's conjecture.

Then U_F/\overline{E}_F (resp. \mathcal{W}_F) fixes the p -Hilbert class field H_F (resp. the Bertrandias–Payan field $\mathcal{H}_F^{\text{bp}}$) and \mathcal{R}_F is the normalized p -adic regulator of F (classical p -adic regulator up to an obvious p -power, cf. [18, Proposition 5.2]):

$$\begin{array}{ccccc}
 & & \mathcal{T}_F \simeq H^2(\mathcal{G}_F, \mathbb{Z}_p)^* & & \\
 & \tilde{F} & \xrightarrow{\quad} & \tilde{F}H_F & \xrightarrow{\quad} & \mathcal{H}_F^{\text{bp}} & \xrightarrow{\quad} & \mathcal{H}_F^{\text{ab}} \\
 & | & & | & \simeq_{\mathcal{R}_F} & & \simeq_{\mathcal{W}_F} & \\
 & \tilde{F} \cap H_F & \xrightarrow{\quad} & H_F & \xrightarrow{\quad} & \mathcal{H}_F^{\text{ab}} & & \\
 & | & & | & \simeq_{U_F/\overline{E}_F} & & & \\
 & F & & & \simeq_{\mathcal{A}_F} & & & \\
 & & & & \simeq_{\mathcal{A}_F} & & &
 \end{array}$$

Let K/k be *any extension* of number fields; then from [15, Theorem IV.2.1] the transfer map $J : \mathcal{A}_k \rightarrow \mathcal{A}_K$ is always injective under Leopoldt's conjecture. This will be applied to the degree p cyclic sub-extensions of a tower F/k .

The analogue of the exact sequence (4) for class groups, used in the proof of Proposition 2.4, is given by the following result [15, Proposition IV.3.2.1]:

Proposition 4.1. *Let K/k be any Galois extension of number fields and let $G = \text{Gal}(K/k)$. We have, under Leopoldt's conjecture, the exact sequence:*

$$1 \rightarrow J(\mathcal{A}_k) \simeq \mathcal{A}_k \longrightarrow \mathcal{A}_K^G \longrightarrow \bigoplus_{\mathfrak{l}_k \nmid p} \mathbb{Z}_p/e_{\mathfrak{l}_k} \mathbb{Z}_p \longrightarrow 0,$$

$e_{\mathfrak{l}_k}$ being the ramification index of the prime ideals $\mathfrak{l}_k \nmid p$ of k ramified in K/k .

Corollary 4.2. *We have $\text{rk}_p(\mathcal{A}_K^G) \leq \text{rk}_p(\mathcal{A}_k) + t_k^{\text{ta}}$, where t_k^{ta} is the number of prime ideals $\mathfrak{l}_k \nmid p$ of k ramified in K/k .*

Consider the framework of degree p cyclic extensions $K/k = F_i/F_{i-1}$ related to a p -cyclic tower $F \in \mathcal{F}_k^{p^e}$. Let $\mathcal{A}_K^* = \text{Ker}(\nu)$, with $\nu = J \circ N$, where $N : \mathcal{A}_K \rightarrow \mathcal{A}_k$ is the restriction of automorphisms; we have similarly:

$$\text{rk}_p(\mathcal{A}_K) \leq \text{rk}_p(\mathcal{A}_k) + \text{rk}_p(\mathcal{A}_K^*) \text{ and } \text{rk}_p(\mathcal{T}_K) \leq \text{rk}_p(\mathcal{T}_k) + \text{rk}_p(\mathcal{T}_K^*).$$

Then considering Proposition 2.2 (valid for the \mathbb{Z}_p -modules of finite type $M = \mathcal{A}_K$ since the M_{h+1}^*/M_h^* are elementary finite p -groups for all $h \geq 0$), one obtains from Corollary 4.2 in K/k :

$$\mathrm{rk}_p(\mathcal{A}_K^*) \leq (p-1) \mathrm{rk}_p(\mathcal{A}_K^{G^*}) = (p-1) \mathrm{rk}_p(\mathcal{A}_K^G) \leq (p-1) (\mathrm{rk}_p(\mathcal{A}_k) + t_k^{\mathrm{ta}}),$$

whence:

$$\mathrm{rk}_p(\mathcal{A}_K) \leq p \mathrm{rk}_p(\mathcal{A}_k) + (p-1) t_k^{\mathrm{ta}}.$$

Let N be the number of tame primes ℓ_j , ramified in K/k ; using the same upper bound $d_k := [k : \mathbb{Q}]$, we obtain:

$$(6) \quad \mathrm{rk}_p(\mathcal{A}_K) \leq p \mathrm{rk}_p(\mathcal{A}_k) + (p-1) N d_k.$$

Theorem 4.3. *Let $p \geq 2$ be a prime number. Under Leopoldt's conjecture for p , the p -rank ε -conjectures:*

$$\#(\mathcal{A}_F \otimes \mathbb{F}_p) \ll_{\kappa, p^e, \varepsilon} (\sqrt{D_F})^\varepsilon \quad \text{and} \quad \#(\mathrm{H}^2(\mathcal{G}_F, \mathbb{Z}_p) \otimes \mathbb{F}_p) \ll_{\kappa, p^e, \varepsilon} (\sqrt{D_F})^\varepsilon,$$

are fulfilled unconditionally for the family $\mathcal{F}_\kappa^{p^e}$ of p -cyclic-towers F/κ of degree p^e .

Proof. For any number field L , we have $\mathrm{rk}_p(\mathcal{A}_L) = r_2(L) + 1 + \mathrm{rk}_p(\mathcal{T}_L)$ and $r_2(L) + 1$ is the free rank, $\mathrm{rk}_{\mathbb{Z}_p}(\mathcal{A}_L)$, of \mathcal{A}_L . Thus we get, from relation (6) and $\mathrm{rk}_{\mathbb{Z}_p}(\mathcal{A}_K) = \mathrm{rk}_{\mathbb{Z}_p}(\mathcal{A}_k) + \mathrm{rk}_{\mathbb{Z}_p}(\mathcal{A}_K^*)$:

$$r_2(K) + 1 + \mathrm{rk}_p(\mathcal{T}_K) \leq p (r_2(k) + 1 + \mathrm{rk}_p(\mathcal{T}_k)) + (p-1) N d_k,$$

whence, since $r_2(K) - r_2(k) = \mathrm{rk}_{\mathbb{Z}_p}(\mathcal{A}_K^*) \geq 0$:

$$\begin{aligned} \mathrm{rk}_p(\mathcal{T}_K) &\leq p \mathrm{rk}_p(\mathcal{T}_k) + p r_2(k) + p - r_2(K) - 1 + (p-1) N d_k \\ &\leq p \mathrm{rk}_p(\mathcal{T}_k) + (p-1) r_2(k) + p - 1 + (p-1) N d_k \\ &\leq p \mathrm{rk}_p(\mathcal{T}_k) + (p-1) (N+1) d_k \end{aligned}$$

The inequalities are similar to that obtained for the p -ranks of usual class groups, whence the result since $\mathrm{rk}_p(\mathrm{H}^2(\mathcal{G}_F, \mathbb{Z}_p)) = \mathrm{rk}_p(\mathcal{T}_F)$. \square

As for the p -class groups, we have, for $r \geq 1$ fixed:

$$\#(\mathrm{H}^2(\mathcal{G}_F, \mathbb{Z}_p) \otimes \mathbb{Z}/p^r\mathbb{Z}) \ll_{\kappa, p^e, \varepsilon} (\sqrt{D_F})^\varepsilon, \quad \text{for all } F \in \mathcal{F}_\kappa^{p^e}.$$

Remark 4.4. Most arithmetic p -invariants, stemming from generalized class groups \mathcal{C}_Σ^S (regarding ramification and decomposition of given sets of places in the corresponding ray class fields), fulfill the p -rank ε -conjecture for $\mathcal{F}_\kappa^{p^e}$.

In the same way, many other p -invariants are related to the fundamental groups \mathcal{C}_F and \mathcal{T}_F by means of standard rank formulas and/or dualities (e.g., reflection theorems detailed in [20, Chapitre III]), so that all these invariants fulfill the p -rank ε -conjecture. One may cite:

- (i) The normalized p -adic regulator \mathcal{R}_F (obvious from the schema).
- (ii) The regular kernel R_F and the Hilbert kernel W_F (from results of Tate; see [21] [15, § 7.7.2, Theorem 7.7.3.1], [22]); then the corresponding study for the higher K-theory [20, § 12].
- (iii) The Jaulent logarithmic class group [2, 25, 26, 27] (finite under the conjecture of Gross and isomorphic to a quotient of \mathcal{T}_F), linked with precise formulas to the previous groups \mathcal{C}_F , \mathcal{T}_F , W_F [25, 26].

5. CONCLUSION

The main question remains the case of a *strong ε -conjecture*, for such finite invariants M , saying that:

$$\#(M_F \otimes \mathbb{Z}_p) \ll_{\kappa, p^e, \varepsilon} (\sqrt{D_F})^\varepsilon \text{ for all } F \in \mathcal{F}_\kappa^{p^e},$$

except for a subfamily of $\mathcal{F}_\kappa^{p^e}$ of zero density.

This restriction seems essential because of the existence of very rare fields giving exceptional large invariants M as shown in [7, 8, 30] for class groups (or [19] for torsion groups \mathcal{T}). This is also justified, in the framework of p -class groups, by the Koymans–Pagano density results [28] as analyzed in [14] for $\mathcal{F}_\mathbb{Q}^p$; indeed, in any relative degree p cyclic extension, the algorithm defining the filtration $(M_h)_{h \geq 0}$ is a priori unbounded, giving possibly large $\#M$ contrary to the p -ranks (or the $\#M[p^r]$ as seen in Corollary 3.7 which allows to take $r \gg 0$, but *constant* regarding the family $\mathcal{F}_\kappa^{p^e}$).

All the previous results on p -rank ε -inequalities fall within the framework of “genus theory” at the prime p for p -extensions; the case of degree d number fields, when $p \nmid d$, is highly non-trivial. For instance, the simplest case of the 3-rank ε -conjecture for quadratic fields F remains open since one only knows that $\#(\mathcal{C}_F \otimes \mathbb{F}_3) \ll_\varepsilon (\sqrt{D_F})^{\frac{2}{3} + \varepsilon}$ (we refer to the bibliographies of [11, 41], among other, for many generalizations and improvements of the exponents).

Indeed, the general case, regarding the ℓ -invariants $M \otimes \mathbb{F}_\ell$, $M \otimes \mathbb{Z}_\ell$ of M , in degree d extensions, has a fundamental difficulty since complex analytic methods consider globally $\#M$ as upper bound (like “ $\#(M \otimes \mathbb{F}_\ell) \leq \#M$ ”, in the framework of Brauer–Siegel type results [32, 36, 40, 43]) and often assume GRH, so that the “bad primes” $p \mid d$ may give large p -parts in $\#M$, thus analytic difficulties as explained in [14, §2]; this is due to the lack of direct p -adic analytic tools.

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