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# THE $p$ -RANK $\varepsilon$ -CONJECTURE ON CLASS GROUPS IS TRUE FOR $p$ -EXTENSIONS

GEORGES GRAS

ABSTRACT. We prove the  $p$ -rank  $\varepsilon$ -conjecture on the class groups  $\mathcal{C}_F$  for the  $p$ -extensions  $F/\kappa$  of degree  $p^e$  and, more generally, for a tower of degree  $p$  cyclic fields:  $\#(\mathcal{C}_F \otimes \mathbb{F}_p) \ll_{\kappa, p^e, \varepsilon} (\sqrt{|D_F|})^\varepsilon$ , where  $D_F$  is the discriminant and  $\kappa$  any base field (Theorem 3.5). This Note generalizes the case  $e = 1$  and  $\kappa = \mathbb{Q}$  (*Genus theory and  $\varepsilon$ -conjectures on  $p$ -class groups*, J. Number Theory **207** (2020), 423–459), whose techniques appear to be “universal” for all relative  $p$ -cyclic-extensions. We prove the  $p$ -rank  $\varepsilon$ -conjecture for the  $F/\kappa$  and the cohomology groups  $H^2(\mathcal{G}_F, \mathbb{Z}_p)$  of Galois  $p$ -ramification theory (Theorem 4.3).

## 1. INTRODUCTION

For any prime number  $p$  and any number field  $F$ , we denote by  $\mathcal{C}_F$  the  $p$ -class group of  $F$  in the restricted sense; to avoid any ambiguity, we shall write  $\mathcal{C}_F \otimes \mathbb{F}_p$  for the “ $p$ -torsion group”, also denoted  $\mathcal{C}_F[p]$  in the literature, giving the  $p$ -rank  $\text{rk}_p(\mathcal{C}_F) := \dim_{\mathbb{F}_p}(\mathcal{C}_F/\mathcal{C}_F^p)$  of  $\mathcal{C}_F$ .

We refer to our paper [3] for an introduction with some history about the notion of  $\varepsilon$ -conjecture, initiated by Ellenberg–Venkatesh [2], the related density results of Koymans–Pagano [9], and also for the large bibliography that we shall not reproduce in this Note. We prove, unconditionally:

**Main results.** *Let  $\mathcal{C}_F$  denotes the  $p$ -class group of a number field  $F$  and  $D_F$  its discriminant. The general  $p$ -rank  $\varepsilon$ -conjecture is true for the family of  $p$ -extensions  $F/\kappa$  of degree  $p^e$  over a fixed number field  $\kappa$ : for all  $\varepsilon > 0$ , there exists a constant  $C_{\kappa, p^e, \varepsilon}$ , such that  $\#(\mathcal{C}_F \otimes \mathbb{F}_p) \leq C_{\kappa, p^e, \varepsilon} \cdot (\sqrt{|D_F|})^\varepsilon$ , for all such  $p$ -extensions of  $\kappa$ . The result is true for any (non-necessary Galois)  $p$ -tower  $F/\kappa$ ,  $\kappa = F_0 \subset \cdots \subset F_{i-1} \subset F_i \subset \cdots \subset F_e = F$  for which  $F_i/F_{i-1}$  is Galois for all  $i \in [1, e]$ , which is equivalent to consider towers (called “ $p$ -cyclic-towers”) of successive degree  $p$  cyclic extensions. In Section 4 we obtain a similar statement for the cohomology groups  $H^2(\mathcal{G}_F, \mathbb{Z}_p)$  of Galois  $p$ -ramification theory.*

**Note.** After the writing of this article, we have been informed of papers by Wang [13] and Klüners–Wang [10] containing similar results on class groups with other information (e.g., Cornell paper [1] using genus theory in elementary  $p$ -extensions with some restriction on the ramification, then generalized in [10]). We thank Jiuya Wang for these communications. It is interesting to see that the two methods in [3, 10] use classical analytical results on prime numbers and refer to [12]. Our result is unconditional contrary to [10] where the “ $\varepsilon$ -inequality” is valid for  $D_F \gg 0$ .

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## 2. PRINCIPLE OF THE METHOD

We will perform an induction in the successive degree  $p$  cyclic extensions, the general method for such  $p$ -cyclic steps having been given in [3]. The method involves using genus theory and minimal relative discriminants. This is of course the trivial part of the proof of the general  $\varepsilon$ -conjecture in degree  $d$  number fields, but  $p \mid d$  may give large  $p$ -class groups as explained in [3, §2].

**2.1. Tower of degree  $p$  cyclic fields.** Let  $\kappa$  be any fixed number field and let:

$$F_0 = \kappa \subset F_1 \cdots \subset F_e = F, \quad e \geq 1,$$

be a  $p$ -cyclic-tower of fields with  $\text{Gal}(F_i/F_{i-1}) \simeq \mathbb{Z}/p\mathbb{Z}$ , for  $1 \leq i \leq e$  (by abuse, we use the word of tower even if the fields  $F_i$  are not necessarily Galois over  $\kappa$ ).

Put  $k = F_{i-1}$ ,  $K = F_i$ ,  $G = \text{Gal}(K/k) =: \langle \sigma \rangle$ . We shall use the obvious general exact sequence:

$$1 \rightarrow \mathcal{C}_K^* \rightarrow \mathcal{C}_K \xrightarrow{\nu} \mathcal{C}_K^\nu,$$

where  $\nu$  is the algebraic norm  $1 + \cdots + \sigma^{p-1}$  and  $\mathcal{C}_K^* = \text{Ker}(\nu)$ .

Recall that  $\nu = \text{J} \circ \text{N}$ , where  $\text{N}$  is the arithmetic norm (or the restriction  $\text{Gal}(H_K/K) \rightarrow \text{Gal}(H_k/k)$  of automorphisms in the corresponding Hilbert's class fields  $H_K, H_k$ ), which yields  $\text{N}(\mathcal{C}_K) \subseteq \mathcal{C}_k$  and where  $\text{J} : \mathcal{C}_k \rightarrow \mathcal{C}_K$  comes from the extension of ideals of  $k$  to  $K$  (or is the transfer map  $\text{Gal}(H_k/k) \rightarrow \text{Gal}(H_K/K)$ ); so  $\mathcal{C}_K^\nu \simeq \mathcal{C}_k$  if and only if  $\text{N}$  is surjective (equivalent to  $H_k \cap K = k$ ) and  $\text{J}$  injective (no capitulation).

Whence the obvious inequalities for the  $p$ -ranks:

$$(1) \quad \text{rk}_p(\mathcal{C}_K) \leq \text{rk}_p(\mathcal{C}_k) + \text{rk}_p(\mathcal{C}_K^*).$$

So, the main problem is to estimate  $\text{rk}_p(\mathcal{C}_K^*)$  by giving an explicit upper bound only depending on  $\mathcal{C}_k$ ,  $p$  and the number of ramified prime numbers.

**2.2.  $\mathbb{Z}_p[G]$ -modules annihilated by  $\nu$  [6].** Let  $G = \langle \sigma \rangle \simeq \mathbb{Z}/p\mathbb{Z}$ . We consider a finite  $\mathbb{Z}_p[G]$ -module  $M^*$  annihilated by  $\nu = 1 + \cdots + \sigma^{p-1}$  (which will be later of the form  $\mathcal{C}_{F_i}^*$ ) for which we define the following filtration, for  $h \geq 0$  and  $M_0^* = 1$ :

$$M_{h+1}^*/M_h^* := (M^*/M_h^*)^G.$$

For all  $h \geq 0$ ,  $M_h^* = \{x \in M^*, x^{(1-\sigma)^h} = 1\}^1$ , the  $p$ -groups  $M_{h+1}^*/M_h^*$  are elementary and the maps  $M_{h+1}^*/M_h^* \xrightarrow{1-\sigma} M_h^*/M_{h-1}^*$  are injective, giving a decreasing sequence for the orders  $\#(M_{h+1}^*/M_h^*)$  as  $h$  grows; whence:

$$\#(M_{h+1}^*/M_h^*) \leq \#M_1^*, \quad \text{for all } h.$$

Since  $M^*$  is a  $\mathbb{Z}_p[G]/(\nu)$ -module and since  $\mathbb{Z}_p[G]/(\nu) \simeq \mathbb{Z}_p[\zeta]$ , where  $\zeta$  is a primitive  $p$ th root of unity, we may write for  $\mathfrak{p} = (1 - \zeta)\mathbb{Z}_p[\zeta]$  and  $s \geq 0$ :

$$M^* \simeq \bigoplus_{j=1}^s \mathbb{Z}_p[\zeta]/\mathfrak{p}^{n_j}, \quad n_1 \leq n_2 \leq \cdots \leq n_s,$$

and the sub-modules  $M_h^*$  are, in this  $\mathbb{Z}_p[\zeta]$ -structure, the following ones:

$$M_h^* \simeq \bigoplus_{j, n_j \leq h} \mathbb{Z}_p[\zeta]/\mathfrak{p}^{n_j}, \quad \text{for all } h \geq 1.$$

<sup>1</sup>Note that if  $M$  is a  $\mathbb{Z}_p[G]$ -module and  $M^* = \text{Ker}(\nu)$ , we have, in an obvious meaning,  $(M^*)_h = (M_h)^*$ , whence the writing  $M_h^*$ .

**Proposition 2.1.** *Under the condition  $M^{*\nu} = 1$ , the  $p$ -rank of  $M^*$  fulfills the inequality  $\text{rk}_p(M^*) \leq (p-1) \cdot \text{rk}_p(M_1^*)$ .*

*Proof.* Let  $M^*[p] := \{x \in M^*, x^p = 1\}$ ; since  $p\mathbb{Z}_p[\zeta] = \mathfrak{p}^{p-1}$ , we obtain that  $M^*[p] = M_{p-1}^*$ ; then  $\#M_{p-1}^* = \prod_{h=0}^{p-2} \#(M_{h+1}^*/M_h^*) \leq (\#M_1^*)^{p-1}$ . Since  $M_{p-1}^*$  and  $M_1^*$  are elementary  $p$ -groups, the inequality on the  $p$ -ranks follows.  $\square$

**Corollary 2.2.** *Let  $K/k$  be a degree  $p$  cyclic extension of number fields and let  $G = \text{Gal}(K/k)$ . Then  $\text{rk}_p(\mathcal{C}_K^*) \leq (p-1) \text{rk}_p(\mathcal{C}_K^{G*}) = (p-1) \text{rk}_p(\mathcal{C}_K^G)$ .*

**2.3. Majoration of the  $\text{rk}_p(\mathcal{C}_{F_i}^G)$ .** As above, put  $k = F_{i-1}$ ,  $K = F_i$ ,  $1 \leq i \leq e$ , and  $G = \text{Gal}(K/k) =: \langle \sigma \rangle$ . Consider  $M = \mathcal{C}_K$ . Then Corollary 2.2 gives  $\text{rk}_p(\mathcal{C}_K^*) \leq (p-1) \text{rk}_p(\mathcal{C}_K^G)$  and we obtain the following upper bound:

**Proposition 2.3.** *We have  $\text{rk}_p(\mathcal{C}_K^G) \leq \text{rk}_p(\mathcal{C}_k) + t_k + \text{rk}_p(E_k/E_k^p)$ , where  $t_k$  is the number of prime ideals of  $k$  ramified in  $K/k$  and  $E_k$  the group of (totally positive) units of  $k$ .*

*Proof.* We have the classical exact sequence:

$$1 \rightarrow \mathcal{C}_K(I_K^G) \longrightarrow \mathcal{C}_K^G \xrightarrow{\theta} E_k \cap N_{K/k}(K^\times)/N_{K/k}(E_K) \rightarrow 1,$$

where  $I_K$  is the  $\mathbb{Z}[G]$ -module of ideals of  $K$  and where  $\theta$  associates with  $\mathcal{C}_K(\mathfrak{A})$ , such that  $\mathfrak{A}^{1-\sigma} = (\alpha)$ ,  $\alpha \in K^\times$ , the class of the unit  $N_{K/k}(\alpha)$  of  $k$ , modulo  $N_{K/k}(E_K)$ . The surjectivity and the kernel are immediate.

The  $\mathbb{Z}[G]$ -module  $I_K^G$  is generated by the extension in  $K$  of the ideals of  $k$  and by the  $t_k$  prime ideals of  $K$  ramified in  $K/k$ . Thus, using the obvious inequality  $\text{rk}_p(E_k \cap N_{K/k}(K^\times)/N_{K/k}(E_K)) \leq \text{rk}_p(E_k/E_k^p)$ , we obtain:

$$\text{rk}_p(\mathcal{C}_K^G) \leq \text{rk}_p(\mathcal{C}_K(I_K^G)) + \text{rk}_p(E_k/E_k^p) \leq \text{rk}_p(\mathcal{C}_k) + t_k + \text{rk}_p(E_k/E_k^p),$$

which proves the claim.  $\square$

Denote by  $\{\ell_1, \dots, \ell_N\}$ ,  $N \geq 0$ , the set of tame primes ramified in  $K/k$  (for such an  $\ell$ , there exist prime ideals  $\mathfrak{l}_u \mid \ell$  in  $k$ ,  $u \in [1, t_{k,\ell}]$ ,  $t_{k,\ell} \geq 1$ , ramified in  $K/k$ ).

Thus,  $t_k = t_{k,p} + \sum_{j=1}^N t_{k,\ell_j}$ . We shall replace  $\text{rk}_p(E_k/E_k^p)$  then  $t_{k,p}$ ,  $t_{k,\ell_j}$  by the rough upper bound  $d_k := [k : \mathbb{Q}]$ , which gives, using inequality (1), Corollary 2.2 and Proposition 2.3:

**Corollary 2.4.** *Put  $d_k = [k : \mathbb{Q}]$ ; then we have the inequalities:*

$$\begin{aligned} \text{rk}_p(\mathcal{C}_K^*) &\leq (p-1) \cdot [\text{rk}_p(\mathcal{C}_k) + (N+2)d_k], \\ \text{rk}_p(\mathcal{C}_K) &\leq p \cdot \text{rk}_p(\mathcal{C}_k) + (p-1)(N+2)d_k. \end{aligned}$$

### 3. ABOUT THE DISCRIMINANTS IN THE TOWER $(F_i)_{1 \leq i \leq e}$

Now, the purpose is to give lower bounds for discriminants, contrariwise for the previous case of  $p$ -ranks. Give some essential explanations:

**Remark 3.1.** We have given, in the previous section, an upper bound for the difference of  $p$ -ranks in  $K/k$ , where the number of ramified primes is crucial because of genus theory, so that any *real*  $p$ -ranks at the step  $K/k$  (whatever the  $p$ -cyclic tower  $F/\kappa$  having the same parameter  $N$ ) will be *smaller*.

In the present section, we shall give a definition of the “minimal tame relative discriminant  $D_N^* \in \mathbb{N}$ ” with  $N$  tame primes  $\ell_j$ , possibly ramified in  $K/k$  (so that any *real* relative discriminant  $D_{K/k} \in \mathbb{N}$  (whatever  $F/\kappa$ ), eventually with a non-trivial  $p$ -part, will be *larger*). Then we shall apply:  $D_K = D_k^p D_{K/k} \geq D_k^p D_N^*$ . Thus, under this facts, if an “ $\varepsilon$ -inequality” does exist between such fictitious  $p$ -ranks and discriminants, a fortiori, any real situation will fulfill the  $p$ -rank  $\varepsilon$ -inequality at this step.

**3.1. The tame relative discriminant.** As above, let  $k = F_{i-1}$ ,  $K = F_i$ , for  $1 \leq i \leq e$ , and let  $\tilde{D}_k$  and  $\tilde{D}_K$  be the absolute values of *the tame parts* of the discriminants of  $k$  and  $K$ , respectively; then let  $\tilde{D}_{K/k} \in \mathbb{N}$  be *the tame part* of the relative discriminant of  $K/k$ . If  $N$  is the number of tame prime numbers  $\ell$  ramified in  $K/k$ , let  $\mathfrak{l}_1, \dots, \mathfrak{l}_{t_{k,\ell}}$  be the prime ideals of  $k$  above  $\ell$ , ramified in  $K/k$ . Taking the norm of the different of  $K/k$ , the tame part of the relative discriminant

is  $\tilde{D}_{K/k} = \prod_{j=1}^N \prod_{u=1}^{t_{k,\ell_j}} \ell_{j,u}^{p-1}$ , and, taking its norm, the “tame discriminant formula” [11, Ch. IV, Prop. 4, Ch. III, Prop. 8] is ( $f_{k,\mathfrak{l}_{j,u}}$  is the residue degree of  $\mathfrak{l}_{j,u}$  in  $k/\mathbb{Q}$ ):

$$(2) \quad \tilde{D}_K = \tilde{D}_k^p \cdot N_{k/\mathbb{Q}}(\tilde{D}_{K/k}) = \tilde{D}_k^p \cdot \prod_{j=1}^N \prod_{u=1}^{t_{k,\ell_j}} \ell_j^{(p-1)f_{k,\mathfrak{l}_{j,u}}}.$$

We put  $\tilde{D}_{K/k} := \prod_{j=1}^N \prod_{u=1}^{t_{k,\ell_j}} \ell_j^{(p-1)f_{k,\mathfrak{l}_{j,u}}} = \prod_{j=1}^N \ell_j^{(p-1) \cdot \sum_{u=1}^{t_{k,\ell_j}} f_{k,\mathfrak{l}_{j,u}}}$  and call it the tame relative discriminant.

**3.2. Minimal tame relative discriminant.** We fix  $N \geq 0$  and intend to determine an absolute lower bound  $D_N^*$  of  $\tilde{D}_{K/k}$ , only depending on  $N$ , so that for every concrete ramification in  $K/k$ , with  $N$  tame primes  $\ell_j$  and possibly that of  $p$ , the effective relative discriminant  $D_{K/k} \geq \tilde{D}_{K/k}$  will be necessarily larger than  $D_N^*$  as explained in Remark 3.1. So we can state:

**Lemma 3.2.** *An universal lower bound of the tame relative discriminant  $\tilde{D}_{K/k}$  is obtained when  $t_{k,\ell_j} = f_{k,\mathfrak{l}_j} = 1$  for all  $j = 1, \dots, N$ ; thus the “fictive” minimum of  $\tilde{D}_{K/k}$  is then  $D_N^* := \prod_{j=1}^N \ell_j^{p-1}$ , taking the  $N$  successive prime numbers  $\neq p$ .*

By abuse, we call  $D_N^* = \prod_{j=1}^N \ell_j^{p-1}$  the minimal tame relative discriminant (for  $p = 2$ ,  $D_N^* = 3 \cdot 5 \cdot 7 \cdot 11 \cdots \ell_N$ ); whence, an analogous framework as for the case of degree  $p$  cyclic number fields  $K/\mathbb{Q}$  given in [3], which allows an induction process in the  $p$ -cyclic-tower  $F/\kappa$ . Note that, when  $\kappa \neq \mathbb{Q}$  the primes  $\ell$  ramified in  $K/k$  are not necessarily  $\equiv 1 \pmod{p}$ ; this shall not modify the nature of the results.

**3.3. Induction.** The case  $i = 0$  ( $k = \kappa$ ) is obvious since to get  $p^{\text{rk}_p(\mathcal{O}_\kappa)} \leq C_{\kappa,p,\varepsilon} \cdot (\sqrt{D_\kappa})^\varepsilon$  it suffices to take  $C_{\kappa,p,\varepsilon} = \frac{p^{\text{rk}_p(\mathcal{O}_\kappa)}}{\sqrt{D_\kappa}}$ , a constant independent of  $\varepsilon$ , subject to state the  $\varepsilon$ -conjecture with  $0 < \varepsilon < 1$  which is natural.

By induction, we assume (for  $k := F_{i-1}$ ) that for all  $\varepsilon > 0$  there exists a constant  $C_{k,p,\varepsilon}$  such that  $p^{\text{rk}_p(\mathcal{O}_k)} \leq C_{k,p,\varepsilon} \cdot (\sqrt{D_k})^\varepsilon$  independently of the number  $N_i$  of tame ramified primes  $\ell_j$  in  $F_i/F_{i-1}$ ; then we shall prove the property at the level  $i$ .

From Corollary 2.4 we obtain the following inequality ( $k = F_{i-1}$ ,  $K = F_i$ ,  $N = N_i$ ,  $d_k := [k : \mathbb{Q}]$ ):

$$(3) \quad p^{\text{rk}_p(\mathcal{A}_K)} \leq p^{p \cdot \text{rk}_p(\mathcal{A}_k) + (p-1)(N+2)d_k} \leq C_{k,p,\varepsilon}^p \cdot \left(\sqrt{D_k^p}\right)^\varepsilon \cdot p^{(p-1)(N+2)d_k}.$$

Then, as explained in Remark 3.1, we will compare  $p^{(p-1)(N+2)d_k}$  and  $(\sqrt{D_N^*})^\varepsilon$ ,

where (Lemma 3.2)  $D_N^* = \prod_{j=1}^N \ell_j^{p-1}$ , the  $\ell_j$  being the  $N$  consecutive primes  $\neq p$ .

But we have the required computations in [3, § 2.3, Formulas (4, 5), p. 10] that we give again with some obvious modifications.

**Lemma 3.3.** *There exists  $c_{k,p,\varepsilon}$  such that  $p^{(p-1)(N+2)d_k} \leq c_{k,p,\varepsilon} \cdot (\sqrt{D_N^*})^\varepsilon$ .*

*Proof.* For  $N = 0$ ,  $D_N^* = 1$ , so that the result (independent of  $\varepsilon$ ) is true since the constant  $c_{k,p,\varepsilon}$  will be much larger than  $p^{2(p-1)d_k}$ ; we assume  $N > 0$  in what follows. The existence of  $c_{k,p,\varepsilon}$  is equivalent to the fact that, a fortiori:

$$p^{(p-1)(N+2)d_k} \cdot (\sqrt{D_N^*})^{-\varepsilon}$$

is bounded over  $N$ , whence  $(p-1)(N+2)d_k \log(p) - \varepsilon \cdot \frac{p-1}{2} \sum_{j=1}^N \log(\ell_j) < \infty$ . We have, replacing the consecutive primes  $\ell_j \neq p$  by the lower bounds:

$$\ell'_j = \frac{1}{2} j \log\left(\frac{\ell_j}{2}\right), \quad j \geq 2, \quad \ell'_1 = 2,$$

(cf. [12, Notes on Chapitre I, x 4.6]), to verify that:

$$X(N) := (p-1)(N+2)d_k \log(p) - \varepsilon \cdot \frac{p-1}{2} \left( -\log(p) + \log_2 + \sum_{j=2}^N \left[ \log\left(\frac{1}{2} \cdot j\right) + \log_2\left(\frac{\ell_j}{2}\right) \right] \right) < \infty.$$

We check that  $\log(2) + \sum_{j=2}^N \log\left(\frac{1}{2}\right) + \log_2\left(\frac{\ell_j}{2}\right)$  can be neglected subject to adding  $-1$  to the global sum, and we consider, instead:

$$X(N) = (p-1)(N+2)d_k \log(p) - \varepsilon \cdot \frac{p-1}{2} \left( -1 - \log(p) + \sum_{j=2}^N \log(j) \right).$$

We have, for  $N \geq 1$ ,  $\log(N!) = N \log(N) - N + \frac{1}{2} \log(N) + O(1)$ , giving:

$$X(N) = (p-1)(N+2)d_k \log(p) - \varepsilon \cdot \frac{p-1}{2} \left[ N \log(N) - N + \frac{1}{2} \log(N) + O(1) \right],$$

whence:

$$X(N) = -\varepsilon \frac{p-1}{2} N \log(N) + N \left[ (p-1)d_k \log(p) - \varepsilon O(1) + o(1) \right].$$

The dominant term (for  $\varepsilon$  fixed):

$$-\varepsilon \frac{p-1}{2} N \log(N)$$

ensures the existence of a bound  $c_{k,p,\varepsilon}$  and completes the proof.  $\square$

**Remark 3.4.** It is easy to verify that  $X(N)$ , as function of  $N \geq 0$ , admits, for an  $N_0(\varepsilon) \gg 0$ , a maximum, roughly speaking of the form  $e^{O(1) \cdot \varepsilon^{-1}}$ , giving enormous constants  $c_{k,p,\varepsilon}$ . This is due in part to the method using certain extreme bounds which are not achieved in practice; for example the use of the systematic upper bounds  $d_k = [k : \mathbb{Q}]$  at each step of the tower. It would perhaps be interesting to give heuristics for more reasonable constants.

Finally, we may write in the degree  $p$  cyclic extension  $K/k$  (Formula (3) and Lemma 3.3 giving  $p^{(p-1)(N+2)d_k} \leq c_{k,p,\varepsilon} \cdot (\sqrt{D_N^*})^\varepsilon \leq c_{k,p,\varepsilon} \cdot (\sqrt{D_{K/k}})^\varepsilon$ ):

$$\begin{aligned} p^{\text{rk}_p(\mathcal{A}_K)} &\leq C_{k,p,\varepsilon}^p \left(\sqrt{D_k^p}\right)^\varepsilon \cdot p^{(p-1)(N+2)d_k} \\ &\leq C_{k,p,\varepsilon}^p \left(\sqrt{D_k^p}\right)^\varepsilon \cdot c_{k,p,\varepsilon} \left(\sqrt{D_{K/k}}\right)^\varepsilon \leq C_{k,p,\varepsilon}^p \cdot c_{k,p,\varepsilon} \left(\sqrt{D_k^p \cdot D_{K/k}}\right)^\varepsilon \\ &\leq C_{K,p,\varepsilon} \cdot \left(\sqrt{D_K}\right)^\varepsilon, \end{aligned}$$

with  $C_{K,p,\varepsilon} := C_{k,p,\varepsilon}^p \cdot c_{k,p,\varepsilon}$ . The degree  $[F : \kappa]$  being fixed, the above induction leads to (denoting  $C_{F,p,\varepsilon} := C_{\kappa,p^e,\varepsilon}$ ):

**Theorem 3.5.** *Let  $p \geq 2$  be prime. The  $p$ -rank  $\varepsilon$ -conjecture for  $p$ -cyclic-towers  $F/\kappa$  of given degree  $p^e$ , on the existence, for all  $\varepsilon > 0$ , of a constant  $C_{\kappa,p^e,\varepsilon}$  such that  $\#(\mathcal{A}_F \otimes \mathbb{F}_p) \leq C_{\kappa,p^e,\varepsilon} \cdot (\sqrt{D_F})^\varepsilon$ , is fulfilled unconditionally.*

**Remark 3.6.** If  $F/\kappa$ , of degree  $p^e$ , is inside the  $p$ -Hilbert tower of a number field  $\kappa$ , we have  $D_F = D_\kappa^{p^e}$ , whence  $\#(\mathcal{A}_F \otimes \mathbb{F}_p) \leq C_{\kappa,p^e,\varepsilon} \cdot (\sqrt{D_\kappa})^\varepsilon p^e$ ; renormalizing  $\varepsilon$  since  $p^e$  is a constant, this may be written: for all  $\varepsilon > 0$ , there exists a constant  $C'_{\kappa,p^e,\varepsilon}$  such that, for all such unramified  $p$ -towers:

$$\#(\mathcal{A}_F \otimes \mathbb{F}_p) \leq C'_{\kappa,p^e,\varepsilon} \cdot (\sqrt{D_\kappa})^\varepsilon;$$

this kind of results (with huge constants) is probably necessary because of some results giving large ranks as the degree grows, as it is done by Hajir–Maire in [8].

#### 4. THE $p$ -RANK $\varepsilon$ -CONJECTURE IN $p$ -RAMIFICATION THEORY

More precisely, we shall replace, in the  $p$ -cyclic-towers  $F/\kappa$  of given degree  $p^e$ , the  $p$ -class group  $\mathcal{A}_F$  by the pro- $p$ -group  $\mathcal{A}_F$  which is the Galois group of the maximal  $p$ -ramified abelian pro- $p$ -extension  $\mathcal{H}_F^{\text{ab}}$  of  $F$ . As we know, this pro- $p$ -group is related to the  $p$ -class group  $\mathcal{A}_F$ , the normalized  $p$ -adic regulator  $\mathcal{R}_F$  and its  $\mathbb{Z}_p$ -rank depends on Leopoldt’s conjecture (see, e.g., [4, IV, §§1,2,3], [7] and the almost exhaustive bibliography of [5] for the story of abelian  $p$ -ramification theory, especially the items [3,16,17,18,19,26,40,50,57,58,59,63,65,67,70,72]).

**4.1. Recalls about  $\mathcal{A}_F$ .** We assume the Leopoldt conjecture for  $p$  in all the fields considered. Let  $\mathcal{G}_F$  be the Galois group of the maximal pro- $p$ -extension  $\mathcal{H}_F$  of  $F$ ,  $p$ -ramified in the ordinary sense (i.e., unramified outside  $p$  and non-complexified (= totally split) at the real infinite places of  $F$  when  $p = 2$ ). We know that  $\mathcal{A}_F = \text{Gal}(\mathcal{H}_F^{\text{ab}}/F)$  is a  $\mathbb{Z}_p$ -module of finite type for which  $\mathcal{T}_F := \text{tor}_{\mathbb{Z}_p}(\mathcal{A}_F)$  is isomorphic to the dual of the cohomology group  $H^2(\mathcal{G}_F, \mathbb{Z}_p)$ . Under Leopoldt’s conjecture, we have

$$(4) \quad \mathcal{A}_F \simeq \mathbb{Z}_p^{r_2(F)+1} \oplus \mathcal{T}_F,$$

where  $[F : \mathbb{Q}] =: r_1(F) + 2r_2(F)$  and where  $\mathcal{T}_F$  fixes the compositum  $\widetilde{F}$  of the  $\mathbb{Z}_p$ -extensions of  $F$ . Let  $U_F := \bigoplus_{\mathfrak{p}|p} U_{\mathfrak{p}}$ , be the product of the principal local units of the completions  $F_{\mathfrak{p}}$  of  $F$ , let  $\overline{E}_F$  be the closure of the image of  $E_F$  in  $U_F$  and let  $\mathcal{W}_F = \text{tor}_{\mathbb{Z}_p}(U_F)/\mu_p(F)$ . Then  $U_F/\overline{E}_F$  fixes the  $p$ -Hilbert class field  $H_F$  and  $\mathcal{W}_F$  fixes the Bertrandias–Payan field  $\mathcal{H}_F^{\text{bp}}$ :

$$\begin{array}{ccccc}
& & \mathcal{T}_F \simeq \mathrm{H}^2(\mathcal{G}_F, \mathbb{Z}_p)^* & & \\
& \text{---} & \text{---} & \text{---} & \text{---} \\
\tilde{F} & \text{---} & \tilde{F}H_F & \xrightarrow[\simeq \mathcal{R}_F]{\mathcal{H}_F^{\mathrm{bp}}} & \mathcal{H}_F^{\mathrm{ab}} \\
| & & | & & | \\
\tilde{F} \cap H_F & \text{---} & H_F & \xrightarrow[\simeq \mathcal{A}_F]{\simeq U_F/\overline{E}_F} & \mathcal{H}_F^{\mathrm{ab}} \\
| & & | & & | \\
F & & & & 
\end{array}$$

**4.2. Fixed point formula.** Let  $K/k$  be any finite Galois extension of Galois group  $G$ ; then from [4, Theorem IV.2.1] the transfer  $J : \mathcal{A}_k \rightarrow \mathcal{A}_K$  is always injective under Leopoldt's conjecture. This will be applied to the degree  $p$  cyclic sub-extensions of a tower  $F/\kappa$ . Then the analogue of the exact sequence for class groups, used in the proof of Proposition 2.3, is given by the following result:

**Theorem 4.1.** *We have the exact sequence [4, Proposition IV.3.2.1]:*

$$1 \longrightarrow J(\mathcal{A}_k) \longrightarrow \mathcal{A}_K^G \longrightarrow \bigoplus_{\substack{\mathfrak{l}_k \uparrow p}} \mathbb{Z}_p / e_{\mathfrak{l}_k} \mathbb{Z}_p \longrightarrow 1,$$

where  $e_{\mathfrak{l}_k}$  is the ramification index of the tame prime ideals of  $k$  ramified in  $K/k$ .

**Corollary 4.2.** *We have  $\mathrm{rk}_p(\mathcal{A}_K^G) \leq \mathrm{rk}_p(\mathcal{A}_k) + t'_k$ , where  $t'_k$  is the number of tame prime ideals of  $k$  ramified in  $K/k$ .*

**4.3. Induction process in a  $p$ -cyclic tower  $F/\kappa$ .** We consider the framework of the previous sections with degree  $p$  cyclic extensions  $K/k$ . Let  $\mathcal{A}_K^* = \mathrm{Ker}(\nu)$ ; we get  $\mathrm{rk}_p(\mathcal{A}_K) \leq \mathrm{rk}_p(\mathcal{A}_k) + \mathrm{rk}_p(\mathcal{A}_K^*)$ . Then considering Proposition 2.1 (valid for the  $\mathbb{Z}_p$ -modules of finite type  $M = \mathcal{A}_K$  since the  $M_{h+1}^*/M_h^*$  are elementary finite  $p$ -groups), one obtains from Corollary 4.2 in  $K/k$ :

$$\mathrm{rk}_p(\mathcal{A}_K^*) \leq (p-1) \mathrm{rk}_p(\mathcal{A}_K^{G*}) = (p-1) \mathrm{rk}_p(\mathcal{A}_K^G) \leq (p-1) (\mathrm{rk}_p(\mathcal{A}_k) + t'_k),$$

whence  $\mathrm{rk}_p(\mathcal{A}_K) \leq p \mathrm{rk}_p(\mathcal{A}_k) + (p-1) t'_k$ . Let  $N$  be the number of tame prime numbers  $\ell_j$  ramified in  $K/k$ ; using the same upper bound  $d_k := [k : \mathbb{Q}]$ , we obtain:

$$(5) \quad \mathrm{rk}_p(\mathcal{A}_K) \leq p \mathrm{rk}_p(\mathcal{A}_k) + (p-1) N d_k.$$

Assume for instance that the real infinite places of  $k$  split in  $K/k$  (which is the case for  $p \neq 2$ ), we get easily from relations (4) and (5):

$$\mathrm{rk}_p(\mathcal{T}_K) \leq p \mathrm{rk}_p(\mathcal{T}_k) + (p-1) (1 + N d_k).$$

Seeing that, whatever  $p$  and  $F/\kappa$ , the inequalities are similar to that obtained for the  $p$ -ranks of usual class groups, we can state about the Galois cohomology groups  $\mathrm{H}^2(\mathcal{G}_F, \mathbb{Z}_p)$ , since  $\mathrm{rk}_p(\mathrm{H}^2(\mathcal{G}_F, \mathbb{Z}_p)) = \mathrm{rk}_p(\mathcal{T}_F)$ :

**Theorem 4.3.** *Let  $p \geq 2$  be prime. The  $p$ -rank  $\varepsilon$ -conjecture for  $p$ -cyclic-towers  $F/\kappa$  of given degree  $p^e$ , on the existence, for all  $\varepsilon > 0$ , of a constant  $C'_{\kappa, p^e, \varepsilon}$  such that  $\#(\mathrm{H}^2(\mathcal{G}_F, \mathbb{Z}_p) \otimes \mathbb{F}_p) \leq C'_{\kappa, p^e, \varepsilon} \cdot (\sqrt{D_F})^\varepsilon$ , is fulfilled unconditionally.*



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