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# ON IDENTITIES FOR ZETA VALUES IN TATE ALGEBRAS

HUY HUNG LE AND TUAN NGO DAC

ABSTRACT. Zeta values in Tate algebras were introduced by Pellarin in 2012. They are generalizations of Carlitz zeta values and play an increasingly important role in function field arithmetic. In this paper we prove a conjecture of Pellarin on identities for these zeta values. The proof is based on arithmetic properties of Carlitz zeta values and an explicit formula for Bernoulli-type polynomials attached to Pellarin's zeta values.

## CONTENTS

Introduction	1
1. Identities for zeta values in Tate algebras	3
2. The several variable Bernoulli-type polynomial	6
3. Coefficients of the Bernoulli-type polynomial	16
4. Proof of the main result	28
5. Final remarks	29
References	29

## INTRODUCTION

A classical topic in number theory is the study of the Riemann zeta function  $\zeta(\cdot)$  and its special values  $\zeta(n)$  for  $n \in \mathbb{N}$  and  $n \geq 2$ . By a well-known analogy between the arithmetic of number fields and global function fields, Carlitz suggested to transport the classical results to the function field setting in positive characteristic. In [10] he considered the rational function field equipped with the infinity place and introduced the Carlitz zeta values  $\zeta_A(n)$  which are considered as the analogues of  $\zeta(n)$ . Let  $\mathbb{F}_q$  be a finite field having  $q$  elements,  $q$  being a power of a prime number  $p$ , and  $\theta$  an indeterminate over  $\mathbb{F}_q$ . Let  $A = \mathbb{F}_q[\theta]$ , and let  $K = \mathbb{F}_q(\theta)$  equipped with the rational place  $\infty$ . Let  $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$  be the completion of  $K$  at  $\infty$ , and let  $\mathbb{C}_\infty$  be the completion of a fixed algebraic closure of  $K_\infty$  at  $\infty$ . For  $d \in \mathbb{N}$ ,  $A_{+,d}$  denotes the set of monic elements in  $A$  of degree  $d$ . For  $n \in \mathbb{Z}$ , the value at  $n$  of

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the Carlitz-Goss zeta function is given by

$$\zeta_A(n) := \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{1}{a^n} \in K_\infty.$$

One can show that  $\zeta_A(n) \in A$  if  $n \leq 0$  and even  $\zeta_A(n) = 0$  if  $n < 0$  and  $n \equiv 0 \pmod{q-1}$  (see [19, Chapter 8]).

We now move to the context of Tate algebras. Let  $s \geq 1$  be an integer, and let  $t_1, \dots, t_s$  be  $s$  variables over  $K$  and we write  $\underline{t}_s$  for the family of variables  $(t_1, \dots, t_s)$ . Let  $\mathbb{T}_s$  be the Tate algebra in the variables  $\underline{t}_s$  with coefficients in  $\mathbb{C}_\infty$  (see Section 1.2). In 2012 Pellarin [24] introduced the following element in  $\mathbb{T}_s^\times$  called *the zeta value in the variables  $\underline{t}_s$*

$$\zeta_A(1, \underline{t}_s) := \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t_1) \cdots a(t_s)}{a} \in \mathbb{T}_s^\times.$$

We also call it the zeta value in Tate algebras, and also Pellarin's zeta value in the variables  $\underline{t}_s$ .

For  $s = 1$ , he proved the remarkable identity (see [24, Theorem 1]):

$$\frac{\zeta_A(1, t_1) \omega(t_1)}{\tilde{\pi}} = \frac{1}{\theta - t_1}$$

where  $\tilde{\pi}$  is the Carlitz fundamental period (see [19, 30]), and  $\omega(t_1)$  is the special function introduced by Anderson and Thakur in [1] and given by

$$\omega(t_1) = (-\theta)^{\frac{1}{q-1}} \prod_{j \geq 0} \left(1 - \frac{t_1}{\theta^{q^j}}\right)^{-1}$$

for a fixed choice of the  $(q-1)$ th root of  $(-\theta)$  in  $\mathbb{C}_\infty$ .

Since their introduction various works have revealed the importance of these zeta values for both their proper interest and their applications to values of the Goss  $L$ -functions, characteristic  $p$  multiple zeta values, Anderson's log-algebraicity identities, Taelman's units, and Drinfeld modular forms in Tate algebras (see for example [4, 6, 7, 8, 9, 11, 12, 14, 15, 26, 27, 31]). We should mention that generalizations of Pellarin's zeta values to various settings have been also conducted (see for example [2, 3, 21, 22, 23]).

In a recent work [25] Pellarin investigated a more general setting for Drinfeld modular forms and formulated several conjectural identities for zeta values in Tate algebras. Roughly speaking, he predicted that the multi-variable zeta value  $\zeta_A(1, \underline{t}_s)$  belongs to the  $\mathbb{F}_p$ -algebra generated by negative or positive twists of zeta values in one variable  $\zeta_A(1, t_i)$  for  $1 \leq i \leq s$ , and that an elegant explicit expression is even provided *for  $q$  large enough*. These conjectures combined with other results could lead to new identities for Eisenstein series as explained in [25, Section 9]. We refer the reader to Conjectures 1.1, 1.3 and 2.4 for the precise statements.

The main result of the present paper is to give an affirmative answer to the above Conjectures 1.3 and 2.4 with an explicit bound for  $q$  (see Theorem 1.4). To prove this result we study the several variable Bernoulli-type polynomial  $\mathbb{B}_s$  which encodes the deep connection between Pellarin's zeta values and Taelman's class formula thanks to the work of Anglès, Pellarin and Tavares Ribeiro in [5, 7]. We introduce a notion of weight for polynomials and show that if the weight of the polynomial

$\mathbb{B}_s$  is greater than 1, then Theorem 1.4 holds. Therefore, to prove this theorem it suffices to prove the above estimation for the weight of  $\mathbb{B}_s$  when  $q$  is large enough. We succeed in doing so by computing explicitly  $\mathbb{B}_s$  (for  $q$  large enough) based on combinatorial arguments. We should mention that combinatorial properties of  $\mathbb{B}_s$  have already had important applications in function field arithmetic (see [4, 20, 26] for more details).

Our paper is organized as follows. In Section 1 we review the aforementioned conjectures of Pellarin and state our main result in Theorem 1.4. In Section 2 we study the several variable Bernoulli-type polynomial  $\mathbb{B}_s$ . We introduce a notion of weight for polynomials and explain how to deduce Pellarin's conjectures from a lower bound on the weight of  $\mathbb{B}_s$  (see Theorem 2.9). Section 3 is devoted to prove a key result, Theorem 3.1, which gives an explicit expression of  $\mathbb{B}_s$  in terms of symmetric polynomials. Putting altogether, we prove Theorem 1.4 in Section 4. At the end we discuss some interesting questions in Section 5, which we will investigate in a future project.

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## 1. IDENTITIES FOR ZETA VALUES IN TATE ALGEBRAS

In this section we first set up the necessary preliminaries. We then review Pellarin's conjectures given in [25] and state the main result of the present paper in Section 1.3.

### 1.1. Notation.

In this paper we adopt the following notation.

- $\mathbb{N}$ : the set of non-negative integers.
- $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ : the set of positive integers.
- $\mathbb{Z}$ : the set of integers.
- $\mathbb{F}_q$ : a finite field having  $q$  elements.
- $p$ : the characteristic of  $\mathbb{F}_q$ .
- $\mathbb{F}_p$ : a finite field having  $p$  elements.
- $\theta$ : an indeterminate over  $\mathbb{F}_q$ .
- $A$ : the polynomial ring  $\mathbb{F}_q[\theta]$ .
- $A_+$ : the set of monic elements in  $A$ .
- For  $d \in \mathbb{N}$ ,  $A_{+,d}$  denotes the set of monic elements in  $A$  of degree  $d$ .
- $K = \mathbb{F}_q(\theta)$ : the fraction field of  $A$ .
- $\infty$ : the unique place of  $K$  which is a pole of  $\theta$ .
- $v_\infty$ : the discrete valuation on  $K$  corresponding to the place  $\infty$  normalized such that  $v_\infty(\theta) = -1$ .
- $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$ : the completion of  $K$  at  $\infty$ .

- $\mathbb{C}_\infty$ : the completion of a fixed algebraic closure of  $K_\infty$ . The unique valuation of  $\mathbb{C}_\infty$  which extends  $v_\infty$  will still be denoted by  $v_\infty$ .
- $(-\theta)^{\frac{1}{q-1}}$ : a fixed  $(q-1)$ th root of  $(-\theta)$  in  $\mathbb{C}_\infty$ .
- For  $s \in \mathbb{N}$ ,  $\{t_1, t_2, \dots, t_s\}$  denotes a family of  $s$  variables, and we will also denote this family by  $\underline{t}_s$ .

We will work with the set of all (finite) sequences of integers  $\underline{\ell}$ . When we consider a sequence  $\underline{\ell} = (\ell_1, \dots, \ell_d)$  of integers, the reader should keep in mind that  $d$  depends on the sequence  $\underline{\ell}$ , and that  $\ell_i$  may be 0.

### 1.2. Tate algebras.

Let  $L$  be an extension of  $K_\infty$  in  $\mathbb{C}_\infty$  such that  $L$  is complete with respect to  $v_\infty$ . Let  $s \in \mathbb{N}$  be a non-negative integer. The polynomial ring  $L[\underline{t}_s] = L[t_1, \dots, t_s]$  is equipped with the Gauss valuation: for a polynomial  $f \in L[\underline{t}_s]$ , if we write

$$f = \sum_{i_1, \dots, i_s \in \mathbb{N}} a_{i_1, \dots, i_s} t_1^{i_1} \dots t_s^{i_s}, \quad a_{i_1, \dots, i_s} \in L,$$

then the Gauss valuation of  $f$  is defined by

$$v_\infty(f) := \inf\{v_\infty(a_{i_1, \dots, i_s}), i_1, \dots, i_s \in \mathbb{N}\}.$$

By definition, the Tate algebra  $\mathbb{T}_s(L)$  in the variables  $\underline{t}_s$  with coefficients in  $L$  is the completion of  $L[\underline{t}_s]$  with respect to the Gauss valuation. Explicitly,  $\mathbb{T}_s(L)$  is the set of formal series

$$f = \sum_{i_1, \dots, i_s \in \mathbb{N}} a_{i_1, \dots, i_s} t_1^{i_1} \dots t_s^{i_s}, \quad a_{i_1, \dots, i_s} \in L,$$

such that

$$\lim_{i_1 + \dots + i_s \rightarrow +\infty} v_\infty(a_{i_1, \dots, i_s}) = +\infty.$$

When  $L = \mathbb{C}_\infty$ , we will write  $\mathbb{T}_s$  instead of  $\mathbb{T}_s(\mathbb{C}_\infty)$ . Let  $\tau : \mathbb{T}_s \rightarrow \mathbb{T}_s$  be the continuous homomorphism of  $\mathbb{F}_q[\underline{t}_s]$ -algebras such that

$$\tau(c) = c^q, \quad c \in \mathbb{C}_\infty.$$

Explicitly, for a formal series  $f \in \mathbb{T}_s$ , if we write

$$f = \sum_{i_1, \dots, i_s \in \mathbb{N}} a_{i_1, \dots, i_s} t_1^{i_1} \dots t_s^{i_s}, \quad a_{i_1, \dots, i_s} \in \mathbb{C}_\infty,$$

then

$$\tau(f) = \sum_{i_1, \dots, i_s \in \mathbb{N}} a_{i_1, \dots, i_s}^q t_1^{i_1} \dots t_s^{i_s}.$$

### 1.3. Conjectures of Pellarin and the main result.

For the rest of this paper we will always suppose that  $s \equiv 1 \pmod{q-1}$ . We set

$$(1.1) \quad \Sigma := \{1, \dots, s\},$$

and

$$(1.2) \quad m := \frac{s-1}{q-1} \in \mathbb{N}.$$

Recall that the Carlitz fundamental period  $\tilde{\pi}$  is given by (see [19, 30])

$$\tilde{\pi} = \theta(-\theta)^{\frac{1}{q-1}} \prod_{j \geq 1} \left(1 - \frac{\theta}{\theta^{q^j}}\right)^{-1} \in \mathbb{C}_\infty^\times.$$

Following Anderson and Thakur [1], for  $i \in \Sigma$ , we introduce the following function

$$\omega(t_i) = (-\theta)^{\frac{1}{q-1}} \prod_{j \geq 0} \left(1 - \frac{t_i}{\theta^{q^j}}\right)^{-1} \in \mathbb{T}_s^\times.$$

In 2012 Pellarin [24] introduced the following function in  $\mathbb{T}_s^\times$  called *the zeta value in the variables  $t_s$*

$$\zeta_A(1, t_s) := \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t_1) \cdots a(t_s)}{a} \in \mathbb{T}_s^\times.$$

We also call it the zeta value in Tate algebras, and also Pellarin's zeta value in the variables  $t_s$ . We observe that it can be written as a Euler product

$$\zeta_A(1, t_s) = \prod_{\substack{P \in A_+ \\ P \text{ irreducible}}} \left(1 - \frac{P(t_1) \cdots P(t_s)}{P}\right)^{-1}.$$

In particular, when we have one variable  $t_i$ , we get the zeta value in one variable

$$\zeta_A(1, t_i) := \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t_i)}{a}.$$

In [25] Pellarin revisited the theory of Drinfeld modular forms which were initially developed by Goss in [16, 17, 18] and Gekeler in [13]. In his investigation he proposed several conjectures for the zeta value  $\zeta_A(1, t_s)$  which would lead to new identities for Eisenstein series. We refer the reader to [25, Section 9] for more details.

**Conjecture 1.1** ([25], Conjecture 9.1). *We have*

$$\zeta_A(1, t_s) \in \mathbb{F}_p \left[ \tau^k(\zeta_A(1, t_i)) : 1 \leq i \leq s, k \in \mathbb{Z} \right].$$

As Pellarin mentioned in his paper (see the discussion just before [25, Conjecture 9.1]), the central point of this Conjecture is that negative twists are allowed, and that the coefficients belong to  $\mathbb{F}_p$ . Further, Pellarin suggested an explicit formula for  $\zeta_A(1, t_s)$  when  $q$  is large enough. More precisely, letting  $k \in \mathbb{Z}$  and  $U$  be a subset of  $\Sigma$ , we set

$$(1.3) \quad \mathcal{L}_U^{(k)} := \tau^k \left( \prod_{i \in U} \zeta_A(1, t_i) \right).$$

**Definition 1.2.** An *ordered set partition* of  $\Sigma$  is a set partition  $U_1 \sqcup \dots \sqcup U_d$  of  $\Sigma$  equipped with a total order on its blocks  $U_1 \prec \dots \prec U_d$ . Here we require that  $U_d \neq \emptyset$  but the other blocks may be empty.

We will denote this ordered set partition of  $\Sigma$  by

$$\underline{U} = (U_1 \mid U_2 \mid \dots \mid U_d).$$

**Conjecture 1.3** ([25], Conjecture 9.4). *Let  $\Sigma$  and  $m$  be defined as in (1.1) and (1.2), respectively. Suppose that  $q$  is large enough, depending on  $m$ . Then the following formula holds*

$$\zeta_A(1, \underline{t}_s) = \sum \mathcal{L}_{U_1}^{(-1)} \cdots \mathcal{L}_{U_d}^{(-d)}$$

where the sum runs through the set of ordered set partitions  $\underline{U} = (U_1 | \cdots | U_d)$  of  $\Sigma$  satisfying

$$\frac{|U_1|}{q} + \cdots + \frac{|U_d|}{q^d} = 1.$$

The aim of the present paper is to give an affirmative answer to this conjecture with an explicit bound for  $q$ .

**Theorem 1.4.** *Conjecture 1.3 holds for  $q > m$ .*

A proof of Theorem 1.4 will be given in Section 4. Let us outline the main ideas of the proof.

- (1) First, using the link between Pellarin's zeta values and Taelman's class formula due to Anglès, Pellarin and Tavares Ribeiro in [5, 7], we state an equivalent statement of Conjecture 1.3 (see Conjecture 2.4). Instead of identities on Pellarin's zeta values, it gives conjectural expressions on a certain several variable Bernoulli-type polynomial  $\mathbb{B}_s \in \mathbb{F}_q[t_s, \theta]$ .
- (2) Next, using the polynomial  $\mathbb{B}_s$  we give an expression of the zeta value  $\zeta_A(1, \underline{t}_s)$  in terms of products of twists of zeta values  $\zeta_A(1, t_i)$  for  $i \in \Sigma$  with coefficients in  $\mathbb{C}_\infty$  (see Proposition 2.7). Using some specialization arguments we are able to compute explicitly some coefficients of this expression (see Lemma 2.8). Furthermore, we introduce a notion of weight for polynomials in Definition 2.6 and show that if the weight of  $\mathbb{B}_s$  is bounded below by 1, then the other coefficients vanish which implies Conjecture 1.3 (see Theorem 2.9).
- (3) Finally, we succeed in proving the previous bound for  $q$  large enough (see Section 3). In order to do so we express  $\mathbb{B}_s$  as a linear combination of symmetric polynomials in  $\underline{t}_s$ . For  $q$  large enough we then compute explicitly this expression of  $\mathbb{B}_s$  (see Theorem 3.1), which implies immediately the desired estimation of its weight (see Section 4). We mention that the proof of Theorem 3.1 is of combinatorial nature.

## 2. THE SEVERAL VARIABLE BERNOULLI-TYPE POLYNOMIAL

In this section we study the several variable Bernoulli-type polynomial  $\mathbb{B}_s$ . In Section 2.1 we recall its definition, its basic properties, and its connection with zeta values in Tate algebras. In Section 2.2 we use this polynomial to formulate a conjecture equivalent to Conjecture 1.3 (see Conjecture 2.4). Section 2.3 is devoted to express the zeta value  $\zeta_A(1, \underline{t}_s)$  in terms of products of twists of zeta values in one variable  $\zeta_A(1, t_i)$  for  $i \in \Sigma$  (see Proposition 2.7). The key result states that under some mild condition on  $\mathbb{B}_s$ , Conjecture 1.3 holds (see Theorem 2.9).

### 2.1. The several variable polynomial $\mathbb{B}_s$ .

We briefly recall the deep connection between the zeta value  $\zeta_A(1, \underline{t}_s)$  and some several variable Bernoulli-type polynomial  $\mathbb{B}_s$  as explained in [5, 7].

Recall that for  $s = 1$ , Pellarin proved the following identity (see [24, Theorem 1]):

$$(2.1) \quad \frac{\zeta_A(1, t_1) \omega(t_1)}{\tilde{\pi}} = \frac{1}{\theta - t_1}.$$

For  $s \geq 2$  and  $s \equiv 1 \pmod{q-1}$ , we define

$$(2.2) \quad \mathbb{B}_s := (-1)^m \frac{\zeta_A(1, \underline{t}_s) \omega(t_1) \cdots \omega(t_s)}{\tilde{\pi}} \in \mathbb{T}_s$$

where  $m$  is given by (1.2). Then by [7, Lemma 7.6] (see also [6, Corollary 21]), we have

**Proposition 2.1.** *The element  $\mathbb{B}_s$  is a polynomial in  $\mathbb{F}_q[\underline{t}_s, \theta]$ . Moreover, it is a monic polynomial in the variable  $\theta$  of degree  $m-1$  and a symmetric polynomial in the variables  $\underline{t}_s$ .*

Inspired by Taelman's theory in [28, 29], Anglès, Pellarin, and Tavares Ribeiro showed that the polynomial  $\mathbb{B}_s$  is closely connected to the class module  $H_\phi$  of a certain Drinfeld  $A[\underline{t}_s]$ -module  $\phi$  of rank one as follows (see [7, Section 7] for more details). Let  $\phi : A[\underline{t}_s] \rightarrow \mathbb{T}_s[\underline{t}_s]\{\tau\}$  be the Drinfeld  $A[\underline{t}_s]$ -module over  $\mathbb{T}_s$  given by a homomorphism of  $\mathbb{F}_q[\underline{t}_s]$ -algebras such that

$$\phi_\theta = \theta + (t_1 - \theta) \cdots (t_s - \theta)\tau.$$

There exists a unique formal series  $\exp_\phi \in \mathbb{T}_s\{\{\tau\}\}$  called the exponential series attached to  $\phi$  such that

$$\exp_\phi \equiv 1 \pmod{\tau},$$

and

$$\phi_a \exp_\phi = \exp_\phi a, \quad a \in A[\underline{t}_s].$$

One can show that the exponential series induces a natural  $\mathbb{F}_q[\underline{t}_s]$ -linear map

$$\exp_\phi : \mathbb{T}_s \rightarrow \mathbb{T}_s.$$

Following Taelman [28, 29], we define the class module  $H_\phi$  by

$$H_\phi := \frac{\phi(\mathbb{T}_s(K_\infty))}{\exp_\phi(\mathbb{T}_s(K_\infty)) + \phi(A[\underline{t}_s])}$$

where  $\phi(A[\underline{t}_s])$  is the  $\mathbb{F}_q[\underline{t}_s]$ -module  $A[\underline{t}_s]$  equipped with the  $A[\underline{t}_s]$ -module structure induced by  $\phi$ . By [7, Proposition 7.2] the class module  $H_\phi$  is a finitely generated  $\mathbb{F}_q[\underline{t}_s]$ -module of rank  $m-1$ . The importance of the polynomials  $\mathbb{B}_s$  is explained in the following theorem.

**Theorem 2.2** ([7], Theorem 7.7). *We denote by  $\text{Fitt}_{A[\underline{t}_s]}(H_\phi)$  the Fitting ideal of the torsion  $A[\underline{t}_s]$ -module of finite type  $H_\phi$ . Then*

$$\text{Fitt}_{A[\underline{t}_s]}(H_\phi) = \mathbb{B}_s A[\underline{t}_s].$$

*In particular,*

$$\mathbb{B}_s = \det_{\mathbb{F}_q[\underline{t}_s][Z]} \left( Z \cdot \text{Id} - \phi_\theta |_{H_\phi \otimes_{\mathbb{F}_q[\underline{t}_s]} \mathbb{F}_q[\underline{t}_s][Z]} \right) |_{Z=\theta}.$$



A few explicit examples of the polynomials  $\mathbb{B}_s$  are given in [4, 7] (see also [26]). We need to introduce some more notation.

**Definition 2.3.** For any sequence  $\underline{\ell} = (\ell_1, \dots, \ell_d) \in \mathbb{N}^d$ , we set

$$\ell_0 := s - (\ell_1 + \dots + \ell_d) \in \mathbb{Z},$$

and define

$$\sigma_s(\underline{\ell}) := \sigma_s(\ell_1, \dots, \ell_d) = \sum \prod_{k=1}^d \prod_{i \in U_k} t_i^k,$$

where the sum runs through the set of ordered set partitions  $\underline{U} = (U_0 \mid \dots \mid U_d)$  of  $\Sigma$  such that  $|U_k| = \ell_k$  for  $0 \leq k \leq d$ .

In particular,  $\sigma_s(\underline{\ell}) = 0$  if  $\ell_1 + \dots + \ell_d > s$ , which is equivalent to the condition  $\ell_0 < 0$ . The reader should keep in mind that  $\ell_i$  may be 0. For example,

$$\sigma_s(0, 0, 1) = \sum_{i=1}^s t_i^3.$$

Here are some more explicit examples that will appear in the explicit formulas of  $\mathbb{B}_s$  for small values of  $s$ :

$$\begin{aligned} \sigma_{2q-1}(q) &= \sum_{1 \leq i_1 < \dots < i_q \leq 2q-1} t_{i_1} \cdots t_{i_q}, \\ \sigma_{3q-2}(q) &= \sum_{1 \leq i_1 < \dots < i_q \leq 3q-2} \prod_{j=1}^q t_{i_j}, \\ \sigma_{3q-2}(2q-1) &= \sum_{1 \leq i_1 < \dots < i_{2q-1} \leq 3q-2} \prod_{j=1}^{2q-1} t_{i_j}, \\ \sigma_{3q-2}(2q) &= \sum_{1 \leq i_1 < \dots < i_{2q} \leq 3q-2} \prod_{j=1}^{2q} t_{i_j}, \\ \sigma_{3q-2}(q-1, q) &= \sum_{1 \leq i_1 < \dots < i_{q-1} \leq 3q-2} \sum_{\substack{1 \leq k_1 < \dots < k_q \leq 3q-2 \\ k_\ell \neq i_j}} \prod_{j=1}^{q-1} t_{i_j} \prod_{\ell=1}^q t_{k_\ell}^2. \end{aligned}$$

By [4, Lemma 3.4] we have

$$(2.3) \quad \mathbb{B}_q = 1,$$

$$(2.4) \quad \mathbb{B}_{2q-1} = \theta - \sigma_{2q-1}(q),$$

$$(2.5) \quad \mathbb{B}_{3q-2} = \theta^2 - \theta[\sigma_{3q-2}(q) + \sigma_{3q-2}(2q-1)] + [\sigma_{3q-2}(q-1, q) + \sigma_{3q-2}(2q)].$$

## 2.2. A conjecture equivalent to Conjecture 1.3.

In this section we use the several variable polynomial  $\mathbb{B}_s$  to formulate a conjecture equivalent to Conjecture 1.3 (see Conjecture 2.4).

Let  $k \in \mathbb{N}^*$ . Since  $\tau\omega(t_1) = (t_1 - \theta)\omega(t_1)$ , we get

$$\tau^{-k} \left( \frac{1}{(t_1 - \theta)\omega(t_1)} \right) = \frac{(t_1 - \theta^{\frac{1}{q^{k-1}}}) \cdots (t_1 - \theta^{\frac{1}{q}})}{\omega(t_1)}.$$

By Equation (2.1), we know that

$$\zeta_A(1, t_1) = \frac{\tilde{\pi}}{(\theta - t_1)\omega(t_1)} = -\frac{\tilde{\pi}}{(t_1 - \theta)\omega(t_1)}.$$

It follows that

$$\tau^{-k}(\zeta_A(1, t_1)) = \tau^{-k}\left(-\frac{\tilde{\pi}}{(t_1 - \theta)\omega(t_1)}\right) = -\frac{\tilde{\pi}^{\frac{1}{q^k}}(t_1 - \theta^{\frac{1}{q^{k-1}}}) \cdots (t_1 - \theta^{\frac{1}{q}})}{\omega(t_1)}.$$

Similarly, for  $1 \leq i \leq s$ , we obtain

$$(2.6) \quad \tau^{-k}(\zeta_A(1, t_i)) = -\frac{\tilde{\pi}^{\frac{1}{q^k}}(t_i - \theta^{\frac{1}{q^{k-1}}}) \cdots (t_i - \theta^{\frac{1}{q}})}{\omega(t_i)} = -\frac{\tilde{\pi}^{\frac{1}{q^k}} b_k^*(t_i)}{\omega(t_i)}$$

where we set

$$(2.7) \quad b_k^*(t_i) := (t_i - \theta^{\frac{1}{q^{k-1}}}) \cdots (t_i - \theta^{\frac{1}{q}}).$$

For a subset  $U$  de  $\Sigma$ , we define

$$B_k^*(t_U) := \prod_{i \in U} b_k^*(t_i).$$

By the previous discussion we deduce that Conjecture 1.3 is equivalent to the following conjecture.

**Conjecture 2.4** ([25], Conjecture 9.7). *Suppose that  $q$  is large enough, depending on  $m$ . Then the following formula holds*

$$(2.8) \quad \mathbb{B}_s = (-1)^{m-1} \sum B_1^*(t_{U_1}) \cdots \sum B_d^*(t_{U_d})$$

where the sum runs through the set of ordered set partitions  $\underline{U} = (U_1 | \cdots | U_d)$  of  $\Sigma$  satisfying

$$\frac{|U_1|}{q} + \cdots + \frac{|U_d|}{q^d} = 1.$$

We now present the cases  $m = 1$  and  $m = 2$  to illustrate combinatorial computations which we may encounter. We follow the presentation of Pellarin given in [25, Section 9.1.1] and see that by direct calculations Conjecture 2.4 holds in these cases.

- (1) For  $m = 1$ , Conjecture 2.4 holds since both sides of (2.8) are equal to 1 (see (2.3) for the left-hand side).
- (2) For  $m = 2$ , by (2.4) the left-hand side of (2.8) equals

$$\mathbb{B}_{2q-1} = \theta - \sigma_{2q-1}(q).$$

Since  $|\Sigma| = s = 2q - 1$ , we see that the only ordered set partitions appearing on the right-hand side of (2.8) are  $(U_1 | U_2)$  with  $|U_1| = q - 1$  and  $|U_2| = q$ . It follows that the right-hand side of (2.8) is equal to

$$-\sum_{U_2 \subset \Sigma, |U_2|=q} \prod_{i \in U_2} (t_i - \theta^{1/q}).$$

We claim that this expression is equal to  $\theta - \sigma_{2q-1}(q)$ , which confirms Conjecture 2.4 for  $m = 2$ . In fact, it is easy to see that all the terms defined over  $\mathbb{F}_q[\theta^{1/q}]$  but not over  $\mathbb{F}_q[\theta]$  cancel. Further, the terms over  $\mathbb{F}_q[\theta]$  give exactly the polynomial  $\theta - \sigma_{2q-1}(q)$  as desired.

- (3) More generally, our strategy follows that given in the case  $m = 2$ . On the one hand, we show that on the right-hand side of (2.8) all the terms not defined over  $\mathbb{F}_q[\theta]$  cancel, which is exactly explained in the rest of this Section. On the other hand, we compute the terms over  $\mathbb{F}_q[\theta]$  and prove that they give exactly  $\mathbb{B}_s$ , which will be done in Section 3.

**Remark 2.5.** Let  $\underline{U} = (U_1 \mid \cdots \mid U_d)$  be an ordered set partition of  $\Sigma$  satisfying

$$\frac{|U_1|}{q} + \cdots + \frac{|U_d|}{q^d} = 1.$$

We set  $\ell_j = |U_j|$  for  $1 \leq j \leq d$ . Then the sequence  $\underline{\ell} = (\ell_1, \dots, \ell_d) \in \mathbb{N}^d$  with  $\ell_d \geq 1$  is a solution of the system

$$(2.9) \quad \begin{cases} \ell_1 + \cdots + \ell_d = s \\ \frac{\ell_1}{q} + \cdots + \frac{\ell_d}{q^d} = 1 \end{cases}$$

We assume further that  $m < q$  where  $m$  is defined as in (1.2). Then one can show easily that if  $\underline{\ell} = (\ell_1, \dots, \ell_d) \in \mathbb{N}^d$  with  $\ell_d \geq 1$  is a solution of the above system, then we can write  $\ell_1 = q - s_1, \ell_2 = s_1q - s_2, \dots, \ell_d = s_{d-1}q$  for a sequence  $(s_1, \dots, s_{d-1}) \in (\mathbb{N}^*)^{d-1}$  with  $s_1 + \cdots + s_{d-1} = m - 1$ . In fact, the map

$$(\ell_1, \dots, \ell_d) \mapsto (s_1, \dots, s_{d-1})$$

gives rise to a bijection between the set  $\mathcal{U}_m$  of solutions  $\underline{\ell} = (\ell_1, \dots, \ell_d) \in \mathbb{N}^d$  with  $\ell_d \geq 1$  of the system (2.9) and that of sequences  $(s_1, \dots, s_{d-1}) \in (\mathbb{N}^*)^{d-1}$  with  $s_1 + \cdots + s_{d-1} = m - 1$ . In particular, for  $m \geq 2$ , the cardinal of the set  $\mathcal{U}_m$  equals  $2^{m-2}$ .

We will give, for  $m = 1, 2, 3, 4$  and  $m < q$ , the explicit list of the elements of  $\mathcal{U}_m$ .

- $m = 1$ :  $\mathcal{U}_1 = \{(q)\}$ ;
- $m = 2$ :  $\mathcal{U}_2 = \{(q-1, q)\}$ ;
- $m = 3$ :  $\mathcal{U}_3 = \{(q-1, q-1, q), (q-2, 2q)\}$ ;
- $m = 4$ :  $\mathcal{U}_4 = \{(q-1, q-1, q-1, q), (q-1, q-2, 2q), (q-2, 2q-1, q), (q-3, 3q)\}$ .

### 2.3. Twists of zeta values in one variable.

In this section we will first give an expression for the zeta value  $\zeta_A(1, \underline{t}_s)$  in terms of products of twists of zeta values in one variable  $\zeta_A(1, t_i)$  for  $i \in \Sigma$  with coefficients in  $\mathbb{C}_\infty$  (see Proposition 2.7). Next, using specialization properties we determine some coefficients of this expression (see Lemma 2.8). Finally, under some mild condition on  $\mathbb{B}_s$ , we deduce Conjecture 1.3 (and its equivalent form, Conjecture 2.4) from the previous calculations (see Theorem 2.9).

We start introducing a notions of weight for polynomials.

**Definition 2.6.** 1) Let  $\underline{\ell} = (\ell_1, \dots, \ell_s) \in \mathbb{N}^s$  be an  $s$ -tuple of integers. We consider the monomial  $\underline{t}_s^\ell := \prod_{i=1}^s t_i^{\ell_i}$  and define its weight by

$$\mathbf{w}(\underline{t}_s^\ell) := \sum_{i=1}^s \frac{1}{q^{\ell_i+1}}.$$

2) Let  $P(t_s) \in \mathbb{C}_\infty[t_s]$  be a non-zero polynomial. If we express

$$P(t_s) = \sum_{\underline{\ell}} a_{\underline{\ell}} t_s^{\underline{\ell}}, \quad a_{\underline{\ell}} \in \mathbb{C}_\infty,$$

where the sum runs through the set of  $s$ -tuples  $\underline{\ell} = (\ell_1, \dots, \ell_s) \in \mathbb{N}^s$ , then we define its weight by

$$\mathbf{w}(P) := \min\{\mathbf{w}(t_s^{\underline{\ell}}) : a_{\underline{\ell}} \neq 0\}.$$

**Proposition 2.7.** *With the notation as above, we can express*

$$(2.10) \quad \zeta_A(1, t_s) = (-1)^{m-1} \sum_{\underline{\ell} \in (\mathbb{N}^*)^s} \tilde{\pi}^{1-\sum_{i=1}^s \frac{1}{q^{\ell_i}}} \gamma_{\underline{\ell}} \prod_{i=1}^s \tau^{-\ell_i} (\zeta_A(1, t_i)), \quad \gamma_{\underline{\ell}} \in \mathbb{C}_\infty,$$

where the sum runs through a finite set of  $s$ -tuples  $\underline{\ell} = (\ell_1, \dots, \ell_s) \in (\mathbb{N}^*)^s$  such that if  $\gamma_{\underline{\ell}} \neq 0$ , then

$$\mathbf{w}(\mathbb{B}_s) \leq \sum_{i=1}^s \frac{1}{q^{\ell_i}}.$$

*Proof.* The proof is divided into several steps.

**Step 1.** First, we will express the polynomial  $\mathbb{B}_s \in \mathbb{F}_q[t_s, \theta]$  as a sum of products of  $b_k^*(t_i)$  defined as in (2.7).

Recall that for  $k \in \mathbb{N}^*$  and for  $1 \leq i \leq s$ , we have set in (2.7)

$$b_k^*(t_i) = (t_i - \theta^{\frac{1}{q^{k-1}}}) \cdots (t_i - \theta^{\frac{1}{q}}) \in \mathbb{C}_\infty[t_i]$$

which is a polynomial in the variable  $t_i$  of degree  $k-1$ . It follows that for  $n \in \mathbb{N}$ , we can write

$$(2.11) \quad t_i^n = \sum_{k=1}^{n+1} a_{k,n} b_k^*(t_i), \quad a_{k,n} \in \mathbb{C}_\infty.$$

We note that the coefficients  $a_{k,n}$  do not depend on  $i \in \Sigma$ .

For the polynomial  $\mathbb{B}_s \in \mathbb{F}_q[t_s, \theta]$ , we write

$$\mathbb{B}_s = \sum_{\underline{j}} \beta_{\underline{j}} t_s^{\underline{j}} = \sum_{\underline{j}} \beta_{\underline{j}} t_1^{j_1} \cdots t_s^{j_s}, \quad \text{with } \beta_{\underline{j}} \in \mathbb{F}_q[\theta],$$

where the sum runs through a finite set of  $s$ -tuples  $\underline{j} = (j_1, \dots, j_s) \in \mathbb{N}^s$ .

For any  $s$ -tuple of positive integers  $\underline{\ell} = (\ell_1, \dots, \ell_s) \in (\mathbb{N}^*)^s$ , we set

$$(2.12) \quad \gamma_{\underline{\ell}} := \sum_{\underline{j}} \beta_{\underline{j}} \prod_{i=1}^s a_{\ell_i, j_i} \in \mathbb{C}_\infty,$$

where the coefficients  $a_{\ell_i, j_i}$  are defined as in (2.11), and the sum runs through the set of  $s$ -tuples  $\underline{j} = (j_1, \dots, j_s) \in \mathbb{N}^s$  such that  $j_i + 1 \geq \ell_i$  for  $1 \leq i \leq s$ .

By (2.11), we get

$$\begin{aligned}
(2.13) \quad \mathbb{B}_s &= \sum_{\underline{j}} \beta_{\underline{j}} t_1^{j_1} \cdots t_s^{j_s} \\
&= \sum_{\underline{j}} \beta_{\underline{j}} \prod_{i=1}^s \sum_{\ell_i=1}^{j_i+1} a_{\ell_i, j_i} b_{\ell_i}^*(t_i) \\
&= \sum_{\underline{\ell}} \gamma_{\underline{\ell}} b_{\ell_1}^*(t_1) \cdots b_{\ell_s}^*(t_s).
\end{aligned}$$

Here

- the first and second sums run through a finite set of  $s$ -tuples  $\underline{j} = (j_1, \dots, j_s) \in \mathbb{N}^s$ .
- the third sum runs through a finite set of  $s$ -tuples  $\underline{\ell} = (\ell_1, \dots, \ell_s) \in (\mathbb{N}^*)^s$ .

**Step 2.** Next, letting  $\underline{\ell} = (\ell_1, \dots, \ell_s) \in (\mathbb{N}^*)^s$  be an  $s$ -tuple of positive integers, we claim that if  $\gamma_{\underline{\ell}} \neq 0$ , then

$$\mathbf{w}(\mathbb{B}_s) \leq \sum_{i=1}^s \frac{1}{q^{\ell_i}}.$$

In fact, if  $\gamma_{\underline{\ell}} \neq 0$ , then by (2.12), there exists an  $s$ -tuple  $\underline{j} = (j_1, \dots, j_s) \in \mathbb{N}^s$  such that  $\ell_i \leq j_i + 1$  for  $1 \leq i \leq s$  and

$$\beta_{\underline{j}} \prod_{i=1}^s a_{\ell_i, j_i} \neq 0.$$

It implies that  $\beta_{\underline{j}} \neq 0$ . Thus we obtain

$$\mathbf{w}(\mathbb{B}_s) \leq \mathbf{w}(t_1^{j_1} \cdots t_s^{j_s}) = \sum_{i=1}^s \frac{1}{q^{j_i+1}} \leq \sum_{i=1}^s \frac{1}{q^{\ell_i}}.$$

Here the first inequality and the second equality follow from the fact that  $\beta_{\underline{j}} \neq 0$  and Definition 2.6. The last inequality comes from the fact that  $\ell_i \leq j_i + 1$  for all  $1 \leq i \leq s$ .

**Step 3.** We now switch to zeta values in Tate algebras. We have

$$\begin{aligned}
\zeta_A(1, \underline{t}_s) &= \frac{(-1)^m \tilde{\pi} \mathbb{B}_s}{\omega(t_1) \cdots \omega(t_s)} \quad \text{by (2.2)} \\
&= \frac{(-1)^m \tilde{\pi} \sum_{\underline{\ell} \in (\mathbb{N}^*)^s} \gamma_{\underline{\ell}} b_{\ell_1}^*(t_1) \cdots b_{\ell_s}^*(t_s)}{\omega(t_1) \cdots \omega(t_s)} \quad \text{by (2.13)} \\
&= (-1)^m \tilde{\pi} \sum_{\underline{\ell} \in (\mathbb{N}^*)^s} \gamma_{\underline{\ell}} \prod_{i=1}^s \frac{(-1) \tau^{-\ell_i} (\zeta_A(1, t_i))}{\tilde{\pi} q^{\frac{1}{\ell_i}}} \quad \text{by (2.6)} \\
&= (-1)^{m-1} \sum_{\underline{\ell} \in (\mathbb{N}^*)^s} \tilde{\pi}^{1-\sum_{i=1}^s \frac{1}{\ell_i}} \gamma_{\underline{\ell}} \prod_{i=1}^s \tau^{-\ell_i} (\zeta_A(1, t_i)) \quad \text{since } (-1)^s = -1
\end{aligned}$$

where the sum runs through a finite set of  $s$ -tuples of positive integers  $\underline{\ell} \in (\mathbb{N}^*)^s$ . The proof of Proposition 2.7 is finished.  $\square$

We now calculate some coefficients of the expression (2.10) using specialization arguments. Let  $\underline{k} = (k_1, \dots, k_s) \in \mathbb{N}^s$  be an  $s$ -tuple of non-negative integers. We study the following specialization of  $(t_1, \dots, t_s)$ :

$$t_i = \theta^{q^{-k_i}} = \theta^{\frac{1}{q^{k_i}}}, \quad i = 1, \dots, s.$$

Let  $i \in \Sigma$ . For an  $s$ -tuple  $\underline{\ell} = (\ell_1, \dots, \ell_s) \in (\mathbb{N}^*)^s$ , we have

$$\begin{aligned} \tau^{-\ell_i} (\zeta_A(1, t_i)) \Big|_{t_i = \theta^{q^{-k_i}}} &= \left( \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t_i)}{a^{q^{-\ell_i}}} \right) \Big|_{t_i = \theta^{q^{-k_i}}} \\ &= \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(\theta^{q^{-k_i}})}{a^{q^{-\ell_i}}} \\ &= \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a^{q^{-k_i}}}{a^{q^{-\ell_i}}} \\ &= \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{1}{a^{\left(\frac{1}{q^{\ell_i}} - \frac{1}{q^{k_i}}\right)}}. \end{aligned}$$

Recall that  $\zeta_A(n) = 0$  if  $n < 0$  and  $n \equiv 0 \pmod{q-1}$ ,  $\zeta_A(0) = 1$  and  $\zeta_A(n) \neq 0$  if  $n > 0$  (see for example [19, Chapter 8]). It follows that

$$(2.14) \quad \tau^{-\ell_i} (\zeta_A(1, t_i)) \Big|_{t_i = \theta^{q^{-k_i}}} = \begin{cases} 0 & \text{if } \ell_i > k_i, \\ 1 & \text{if } \ell_i = k_i, \\ \neq 0 & \text{otherwise.} \end{cases}$$

We now analyze the term  $\zeta_A(1, \underline{t}_s) \Big|_{t_i = \theta^{q^{-k_i}}$ . We write

$$\begin{aligned} \zeta_A(1, \underline{t}_s) \Big|_{t_i = \theta^{q^{-k_i}}} &= \left( \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t_1) \dots a(t_s)}{a} \right) \Big|_{t_i = \theta^{q^{-k_i}}} \\ &= \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(\theta^{q^{-k_1}}) \dots a(\theta^{q^{-k_s}})}{a} \\ &= \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{1}{a^{\left(1 - \sum_{i=1}^s \frac{1}{q^{k_i}}\right)}}. \end{aligned}$$

Since  $s \equiv 1 \pmod{q-1}$ , we can write

$$1 - \sum_{i=1}^s \frac{1}{q^{k_i}} = \frac{u}{q^k}$$

with  $u \equiv 0 \pmod{q-1}$  and  $k \in \mathbb{N}^*$ . Again, since  $\zeta_A(n) = 0$  if  $n < 0$  and  $n \equiv 0 \pmod{q-1}$ ,  $\zeta_A(0) = 1$  and  $\zeta_A(n) \neq 0$  if  $n > 0$ , we deduce

$$(2.15) \quad \zeta_A(1, \underline{t}_s) \Big|_{t_i = \theta^{q^{-k_i}}} = \begin{cases} 0 & \text{if } \sum_{i=1}^s \frac{1}{q^{k_i}} > 1, \\ 1 & \text{if } \sum_{i=1}^s \frac{1}{q^{k_i}} = 1, \\ \neq 0 & \text{otherwise.} \end{cases}$$

**Lemma 2.8.** *We continue with the notation of Proposition 2.7. Then for any  $s$ -tuple  $\underline{k} = (k_1, \dots, k_s) \in (\mathbb{N}^*)^s$ , we have*

$$\gamma_{\underline{k}} = \begin{cases} 0 & \text{if } \sum_{i=1}^s \frac{1}{q^{k_i}} > 1, \\ (-1)^{m-1} & \text{if } \sum_{i=1}^s \frac{1}{q^{k_i}} = 1. \end{cases}$$

*Proof.* We divide the proof into two steps.

**Step 1.** Recall that the coefficients  $\gamma_{\underline{\ell}}$  are defined as in Proposition 2.7. We consider the set  $\mathfrak{B}$  of  $s$ -tuples  $\underline{\ell} = (\ell_1, \dots, \ell_s) \in (\mathbb{N}^*)^s$  such that  $\gamma_{\underline{\ell}} \neq 0$ . We choose one  $s$ -tuple  $\underline{k} = (k_1, \dots, k_s) \in \mathfrak{B}$  such that the sum  $\sum_{i=1}^s \frac{1}{q^{k_i}}$  is maximal. Thus  $\gamma_{\underline{k}} \neq 0$ .

We claim that  $\sum_{i=1}^s \frac{1}{q^{k_i}} \leq 1$ . In fact, suppose that  $\sum_{i=1}^s \frac{1}{q^{k_i}} > 1$ . We consider (2.10) and study the specialization of  $(t_1, \dots, t_s)$  given as above:

$$t_i = \theta^{q^{-k_i}} = \theta^{\frac{1}{q^{k_i}}}, \quad i = 1, \dots, s.$$

Since  $\sum_{i=1}^s \frac{1}{q^{k_i}} > 1$ , Equation (2.15) implies

$$(2.16) \quad \zeta_A(1, \underline{t}_s) \Big|_{t_i = \theta^{q^{-k_i}}} = 0.$$

Thus the specialization value of the left-hand side of (2.10) equals 0.

We now analyze the right-hand side of (2.10). First, we consider the term corresponding to the  $s$ -tuple  $\underline{k} = (k_1, \dots, k_s) \in (\mathbb{N}^*)^s$ . By Equation (2.14), we get

$$(-1)^{m-1} \left( \tilde{\pi}^{1-\sum_{i=1}^s \frac{1}{q^{k_i}}} \gamma_{\underline{k}} \prod_{i=1}^s \tau^{-k_i} (\zeta_A(1, t_i)) \right) \Big|_{t_i = \theta^{q^{-k_i}}} = (-1)^{m-1} \tilde{\pi}^{1-\sum_{i=1}^s \frac{1}{q^{k_i}}} \gamma_{\underline{k}}.$$

Next, for other  $s$ -tuple  $\underline{\ell} \in \mathfrak{B}$ , that means  $\gamma_{\underline{\ell}} \neq 0$ , we claim that there exists  $1 \leq i \leq s$  such that  $\ell_i > k_i$ . Suppose that  $\ell_i \leq k_i$  for  $1 \leq i \leq s$  and  $\ell_j < k_j$  for some  $1 \leq j \leq s$ . Thus we get  $\sum_{i=1}^s \frac{1}{q^{k_i}} < \sum_{i=1}^s \frac{1}{q^{\ell_i}}$ , which contradicts with the fact that  $\sum_{i=1}^s \frac{1}{q^{k_i}}$  is maximal.

Since  $\ell_i > k_i$  for some  $1 \leq i \leq s$ , by (2.14) we have

$$\tau^{-\ell_i} (\zeta_A(1, t_i)) \Big|_{t_i = \theta^{q^{-k_i}}} = 0.$$

Thus we obtain

$$(-1)^{m-1} \left( \tilde{\pi}^{1-\sum_{i=1}^s \frac{1}{q^{\ell_i}}} \gamma_{\underline{\ell}} \prod_{i=1}^s \tau^{-\ell_i} (\zeta_A(1, t_i)) \right) \Big|_{t_i = \theta^{q^{-k_i}}} = 0.$$

Putting altogether, the specialization value of the right-hand side of (2.10) equals

$$(2.17) \quad (-1)^{m-1} \tilde{\pi}^{1-\sum_{i=1}^s \frac{1}{q^{k_i}}} \gamma_{\underline{k}}.$$

By (2.16) and (2.17) we conclude that  $(-1)^{m-1} \tilde{\pi}^{1-\sum_{i=1}^s \frac{1}{q^{k_i}}} \gamma_{\underline{k}} = 0$ . Thus  $\gamma_{\underline{k}} = 0$ , which is a contradiction.

To summarize we have proved that for any  $s$ -tuple  $\underline{\ell} = (\ell_1, \dots, \ell_s) \in (\mathbb{N}^*)^s$ , if  $\sum_{i=1}^s \frac{1}{q^{\ell_i}} > 1$ , then  $\gamma_{\underline{\ell}} = 0$ .

**Step 2.** We consider an  $s$ -tuple  $\underline{k} = (k_1, \dots, k_s) \in (\mathbb{N}^*)^s$  such that the sum  $\sum_{i=1}^s \frac{1}{q^{k_i}} = 1$ .

We claim that  $\gamma_{\underline{k}} = (-1)^{m-1}$ . As before, we consider (2.10) and study the specialization of  $(t_1, \dots, t_s)$  given as above:

$$t_i = \theta^q^{-k_i} = \theta^{\frac{1}{q^{k_i}}}, \quad i = 1, \dots, s.$$

Since  $\sum_{i=1}^s \frac{1}{q^{k_i}} = 1$ , Equation (2.15) implies that

$$(2.18) \quad \zeta_A(1, \underline{t}_s) \Big|_{t_i = \theta^q^{-k_i}} = 1.$$

Thus the specialization value of the left-hand side of (2.10) equals 1.

We now analyze the right-hand side of (2.10). For any  $s$ -tuple  $\underline{\ell} \in (\mathbb{N}^*)^s$  such that  $\gamma_{\underline{\ell}} \neq 0$ , we know that

$$\sum_{i=1}^s \frac{1}{q^{\ell_i}} \leq 1 = \sum_{i=1}^s \frac{1}{q^{k_i}}.$$

Thus the arguments given in Step 1 can apply so that the specialization value of the right-hand side of (2.10) equals

$$(2.19) \quad (-1)^{m-1} \tilde{\pi}^{1 - \sum_{i=1}^s \frac{1}{q^{k_i}}} \gamma_{\underline{k}} = (-1)^{m-1} \gamma_{\underline{k}}.$$

Here the equality comes from the fact that  $\sum_{i=1}^s \frac{1}{q^{k_i}} = 1$ .

By (2.18) and (2.19) we get  $(-1)^{m-1} \gamma_{\underline{k}} = 1$ . Thus  $\gamma_{\underline{k}} = (-1)^{m-1}$  as required.

The proof of Lemma 2.8 is complete.  $\square$

As a consequence of Proposition 2.7 and Lemma 2.8, we prove the key result of this section.

**Theorem 2.9.** *Suppose that  $\mathbf{w}(\mathbb{B}_s) \geq 1$ . Then*

- 1) *We have  $\mathbf{w}(\mathbb{B}_s) = 1$ .*
- 2) *Conjecture 1.3 holds, that means we have*

$$\zeta_A(1, \underline{t}_s) = \sum \mathcal{L}_{U_1}^{(-1)} \cdots \mathcal{L}_{U_d}^{(-d)}$$

where the sum runs through the set of ordered set partitions  $\underline{U} = (U_1 \mid \cdots \mid U_d)$  of  $\Sigma$  such that

$$\frac{|U_1|}{q} + \cdots + \frac{|U_d|}{q^d} = 1.$$

*Proof.* Proposition 2.7 states that we can write

$$\zeta_A(1, \underline{t}_s) = (-1)^{m-1} \sum_{\underline{\ell} \in (\mathbb{N}^*)^s} \tilde{\pi}^{1 - \sum_{i=1}^s \frac{1}{q^{\ell_i}}} \gamma_{\underline{\ell}} \prod_{i=1}^s \tau^{-\ell_i} (\zeta_A(1, t_i)), \quad \gamma_{\underline{\ell}} \in \mathbb{C}_{\infty},$$

where the sum runs through a finite set of  $s$ -tuples  $\underline{\ell} = (\ell_1, \dots, \ell_s) \in (\mathbb{N}^*)^s$  such that if  $\gamma_{\underline{\ell}} \neq 0$ , then

$$\mathbf{w}(\mathbb{B}_s) \leq \sum_{i=1}^s \frac{1}{q^{\ell_i}}.$$



Thus the hypothesis  $\mathbf{w}(\mathbb{B}_s) \geq 1$  implies that if  $\gamma_{\underline{\ell}} \neq 0$ , then  $\sum_{i=1}^s \frac{1}{q^{\ell_i}} \geq \mathbf{w}(\mathbb{B}_s) \geq 1$ . Combining this property with Lemma 2.8, we conclude that  $\mathbf{w}(\mathbb{B}_s) = 1$ , and that

$$\zeta_A(1, \underline{t}_s) = \sum_{\underline{\ell} \in (\mathbb{N}^*)^s} \prod_{i=1}^s \tau^{-\ell_i} (\zeta_A(1, t_i))$$

where the sum runs through the set of  $s$ -tuples  $\underline{\ell} = (\ell_1, \dots, \ell_s) \in (\mathbb{N}^*)^s$  satisfying

$$\sum_{i=1}^s \frac{1}{q^{\ell_i}} = 1.$$

We wish to re-index the above sum by ordered set partitions of  $\Sigma$ . For an  $s$ -tuple  $\underline{\ell} = (\ell_1, \dots, \ell_s) \in (\mathbb{N}^*)^s$ , we can associate an ordered set partition  $\underline{U} = (U_1 \mid \dots \mid U_d)$  of  $\Sigma$  as follows. We put  $d = \max\{\ell_i : i \in \Sigma\}$  and for  $1 \leq j \leq d$ ,

$$U_j = \{i \in \Sigma : \ell_i = j\}.$$

In fact, we see that this association gives rise to a bijection between the set of  $s$ -tuple  $\underline{\ell} \in (\mathbb{N}^*)^s$  and the set of ordered set partitions of  $\Sigma$ . Furthermore, it is clear that

$$\sum_{j=1}^d \frac{|U_j|}{q^j} = \sum_{i=1}^s \frac{1}{q^{\ell_i}}.$$

Using this bijection we conclude that

$$\zeta_A(1, \underline{t}_s) = \sum \mathcal{L}_{U_1}^{(-1)} \cdots \mathcal{L}_{U_d}^{(-d)}$$

where  $\mathcal{L}_U^{(k)}$  are defined as in (1.3), and the sum runs through the set of ordered set partitions  $\underline{U} = (U_1 \mid \dots \mid U_d)$  of  $\Sigma$  satisfying

$$\frac{|U_1|}{q} + \dots + \frac{|U_d|}{q^d} = 1.$$

The proof is complete.  $\square$

### 3. COEFFICIENTS OF THE BERNOULLI-TYPE POLYNOMIAL

In this (long) section we study the expression of  $\mathbb{B}_s$  as a linear combination of symmetric polynomials in  $\underline{t}_s$ . We will give explicit formulas for some coefficients of this expression (see Theorem 3.1). To do so we need to write down similar expressions of zeta values (see Propositions 3.10 and 3.12) and Anderson-Thakur's special functions. We then deduce such an expression for  $\mathbb{B}_s$  (see Sections 3.3 and 3.4). For the desired coefficients we are able to compute them by combinatorial tools (see Section 3.4).

#### 3.1. The key result.

We recall that for any sequence  $\underline{\ell} = (\ell_1, \dots, \ell_d) \in \mathbb{N}^d$ , we have defined the symmetric polynomial  $\sigma_s(\underline{\ell})$  as in Definition 2.3.

In what follows, we define

$$(3.1) \quad \mathfrak{A}^+ := \{\underline{a} = (a_0, \dots, a_d) \in (\mathbb{N}^*)^{d+1} : \sum_{j=0}^d a_j = m\},$$

$$\mathfrak{A} := \{\underline{a} = (a_0, \dots, a_d) \in (\mathbb{N}^*)^d \times \mathbb{N} : \sum_{j=0}^d a_j = m\},$$

$$\mathfrak{N}_\ell^+ := \{\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d : n_1 \leq \dots \leq n_d, \sum_{j=1}^d n_j = \ell\}, \quad \ell \in \mathbb{N}^*,$$

$$\mathfrak{N}_\ell := \{\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d : \sum_{j=1}^d n_j = \ell\}, \quad \ell \in \mathbb{N}^*.$$

It is clear that  $\mathfrak{A}^+ \subset \mathfrak{A}$  and  $\mathfrak{N}_\ell^+ \subset \mathfrak{N}_\ell$  for all  $\ell \in \mathbb{N}^*$ .

We now state the main result of this section whose proof will be given in Section 3.4.

**Theorem 3.1.** *Recall that (see Proposition 2.1)*

$$\mathbb{B}_s = \theta^{m-1} + B_1 \theta^{m-2} + \dots + B_{m-1}, \quad B_\ell \in \mathbb{F}_q[t_s].$$

Let  $\ell \in \mathbb{N}^*$  such that  $1 \leq \ell \leq q-2$ , and let  $\mathfrak{N}_\ell$  and  $\mathfrak{A}^+$  be defined as in (3.1). Then we have

$$B_\ell = \sum_{\underline{n} \in \mathfrak{N}_\ell} \sum_{\underline{a} \in \mathfrak{A}^+} B(\underline{n}, \underline{a}) \sigma_s(a_1(q-1)+n_1-n_2, \dots, a_{d-1}(q-1)+n_{d-1}-n_d, a_d(q-1)+n_d)$$

where

- the first sum runs through the set of sequences  $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d$  satisfying  $\sum_{j=1}^d n_j = \ell$ , that means  $\underline{n} \in \mathfrak{N}_\ell$ ,
- the second sum runs through the set of sequences  $\underline{a} = (a_0, \dots, a_d) \in (\mathbb{N}^*)^{d+1}$  satisfying  $\sum_{j=0}^d a_j = m$ , that means  $\underline{a} \in \mathfrak{A}^+$ ,
- the coefficient  $B(\underline{n}, \underline{a}) \in \mathbb{F}_p$  is given by

$$B(\underline{n}, \underline{a}) = (-1)^\ell \prod_{j=1}^d \binom{a_j - 1}{n_j - 1}.$$

**Remark 3.2.** 1) We note that if the coefficient  $B(\underline{n}, \underline{a}) \neq 0$ , then  $a_j \geq n_j$  for  $1 \leq j \leq d$ .

2) The reader may compare the above expression with formulas given in (2.3), (2.4) and (2.5). We leave the reader to write down explicitly the polynomial  $\mathbb{B}_{4q-3}$  for  $q > 4$ .

**Remark 3.3.** We now present a heuristic explanation for the formulas given in Theorem 3.1. We assume that  $m < q$  (see the discussion after Conjecture 2.4 for  $m = 1, 2$ ). By Conjecture 2.4 we write

$$\mathbb{B}_s = (-1)^{m-1} \sum B_1^*(t_{U_1}) \cdots \sum B_d^*(t_{U_d})$$

where the sum runs through the set of ordered set partitions  $\underline{U} = (U_1 \mid \cdots \mid U_d)$  of  $\Sigma$  satisfying

$$\frac{|U_1|}{q} + \cdots + \frac{|U_d|}{q^d} = 1.$$

By Remark 2.5 we have an explicit description of the set of such partitions when  $m < q$ . Using this description we can write down all the terms defined over  $\mathbb{F}_q[\theta]$  of the right-hand side. By this way we obtain a nice formula for  $\mathbb{B}_s$  as given in Theorem 3.1.

### 3.2. Preparatory lemmas.

We first collect several combinatorial lemmas which will be necessary in the sequel.

**Lemma 3.4.** *Let  $n \in \mathbb{N}^*$  with  $n < q$ , and let  $a \in \mathbb{N}^*$ . Then we have*

$$\binom{a(q-1) + n - 1}{n-1} = (-1)^{n-1} \binom{a-1}{n-1} \pmod{p}.$$

*Proof.* This lemma is an application of Lucas's theorem. We write down completely the proof for the convenience of the reader.

We always work in  $\mathbb{F}_p$ . Since  $1 \leq n < q$ , by Lucas's theorem we can assume that  $1 \leq a \leq q$ . By Lucas's theorem and the fact  $1 \leq n < q$  again, we get

$$\begin{aligned} \binom{a(q-1) + n - 1}{n-1} &= \binom{(a-1)q + q - a + n - 1}{n-1} \\ &= \binom{q - a + n - 1}{n-1} \\ &= \frac{(q - a + n - 1) \cdots (q - a + 1)}{(n-1)!} \\ &= \frac{(-a + n - 1) \cdots (-a + 1)}{(n-1)!} \\ &= (-1)^{n-1} \frac{(a - n + 1) \cdots (a - 1)}{(n-1)!} \\ &= (-1)^{n-1} \binom{a-1}{n-1} \end{aligned}$$

as required.  $\square$

**Lemma 3.5.** *For any integer  $j \geq 0$  and any sequence  $(\ell_1, \dots, \ell_d) \in \mathbb{N}^d$ , we have*

$$\begin{aligned} \sigma_s(j) \sigma_s(\ell_1, \dots, \ell_d) &= \sum_{\underline{j}=(j_1, \dots, j_{d+1}) \in \mathbb{N}^{d+1}} \binom{\ell_1 + j_1 - j_2}{j_1} \cdots \binom{\ell_d + j_d - j_{d+1}}{j_d} \times \\ &\quad \times \sigma_s(\ell_1 + j_1 - j_2, \dots, \ell_d + j_d - j_{d+1}, j_{d+1}) \end{aligned}$$

where the sum runs through the set of sequences  $\underline{j} = (j_1, \dots, j_{d+1}) \in \mathbb{N}^{d+1}$  such that  $j_1 + \dots + j_{d+1} = j$ .

*Proof.* Left to the reader.  $\square$

In what follows, we fix  $x$  to be an indeterminate over  $K$ . We recall that for any  $k \in \mathbb{N}$ , the binomial polynomial

$$\binom{x}{k} := \frac{x(x-1)\dots(x-k+1)}{k!} \in \mathbb{Q}[x]$$

represents a polynomial in the variable  $x$  with rational coefficients. Note that its value at  $\ell \in \mathbb{N}$  is equal to the binomial coefficient  $\binom{\ell}{k}$ .

**Lemma 3.6.** *For  $M, N \in \mathbb{N}$ , we have the following equality in  $\mathbb{Q}[x]$ :*

$$\sum_{k=0}^M (-1)^k \binom{x+N-k}{M-k} \binom{x}{k} = \binom{N}{M}.$$

*Proof.* For  $M, N \in \mathbb{N}$ , we define

$$P_{M,N}(x) := \sum_{k=0}^M (-1)^k \binom{x+N-k}{M-k} \binom{x}{k} \in \mathbb{Q}[x].$$

We claim that

$$P_{M,N}(x) = \binom{N}{M}.$$

The proof is by induction on  $M \in \mathbb{N}$ . For  $M = 0$ , the assertion is clear. Suppose that we have proved the claim for  $M - 1$  with  $M \in \mathbb{N}^*$ , i.e. for all  $N \in \mathbb{N}$ ,

$$P_{M-1,N}(x) = \binom{N}{M-1}.$$

We now show that the claim is true for  $M$ . In other words, we have to show that for all  $N \in \mathbb{N}$ , the following equality holds

$$P_{M,N}(x) = \binom{N}{M}.$$

For  $N \in \mathbb{N}$ , we have the following equality in  $\mathbb{Q}[x]$ :

$$P_{M,N+1}(x) = P_{M,N}(x) + P_{M-1,N}(x),$$

which implies

$$P_{M,N+1}(x) - \binom{N+1}{M} = \left( P_{M,N}(x) - \binom{N}{M} \right) + \left( P_{M-1,N}(x) - \binom{N}{M-1} \right).$$

By the induction hypothesis, we know that the second term in the above sum vanishes. Thus

$$(3.2) \quad P_{M,N+1}(x) - \binom{N+1}{M} = P_{M,N}(x) - \binom{N}{M}.$$

Since (3.2) holds for all  $N \in \mathbb{N}$ , we deduce

$$P_{M,N}(x) - \binom{N}{M} = P_{M,N-1}(x) - \binom{N-1}{M} = \dots = P_{M,0}(x).$$

To conclude, it suffices to prove that  $P_{M,0}(x) = 0$ . In fact, we have

$$P_{M,0}(x) = \sum_{k=0}^M (-1)^k \binom{x-k}{M-k} \binom{x}{k} = \binom{x}{M} \sum_{k=0}^M (-1)^k \binom{M}{k} = 0.$$

The proof is finished.  $\square$

**Lemma 3.7.** For  $M, N \in \mathbb{N}$  with  $M \leq N$ , we have the following equality in  $\mathbb{Q}[x]$ :

$$\sum_{k=0}^M (-1)^k \binom{x-N+k-1}{k} \binom{x}{M-k} = \binom{N}{M}.$$

*Proof.* We consider the polynomial in  $\mathbb{Q}[x]$  defined by

$$P(x) := \sum_{k=0}^M (-1)^k \binom{x-N+k-1}{k} \binom{x}{M-k}.$$

We write

$$\begin{aligned} P(x) &:= \sum_{k=0}^M (-1)^k \binom{x-N+k-1}{k} \binom{x}{M-k} \\ &= \sum_{k=0}^M (-1)^k \frac{(x-N) \dots (x-N+k-1)}{k!} \times \frac{x \dots (x-(M-k)+1)}{(M-k)!} \\ &= \sum_{k=0}^M \frac{(N-x) \dots (N-x-k+1)}{k!} \times \frac{x \dots (x-(M-k)+1)}{(M-k)!}. \end{aligned}$$

Since  $M \leq N$ , we have  $\deg P \leq M \leq N$ . We know that for all integer  $\ell$  with  $0 \leq \ell \leq N$ , we have the equality

$$P(x = \ell) = \sum_{k=0}^M \binom{N-\ell}{k} \binom{\ell}{M-k} = \binom{N}{M}.$$

It follows that  $P(x)$  is the constant polynomial  $\binom{N}{M}$ . The proof is finished.  $\square$

### 3.3. An expression of the multi-variable zeta value.

The main goal of this section is to express the zeta value  $\zeta_A(1, \underline{t}_s)$  as a series in  $\theta^{-1}$  with coefficients as symmetric polynomials in  $\underline{t}_s$ . We make use of the notion of basic sums introduced in [4, Section 5.2] to obtain such an expression (see Proposition 3.10).

Following [4, Section 5.2], we recall some facts of basic sums. For a sequence  $\underline{k} = (k_0, \dots, k_{d-1}) \in \mathbb{N}^d$ , we set

$$\begin{aligned} w(\underline{k}) &:= dk_0 + (d-1)k_1 + \dots + k_{d-1}, \\ |\underline{k}| &:= k_0 + \dots + k_{d-1}, \\ C_{\underline{k}} &:= (-1)^{|\underline{k}|} \frac{|\underline{k}|!}{k_0! \dots k_{d-1}!} \in \mathbb{F}_p. \end{aligned}$$

Letting  $a \in A_{+,d}$ , we write  $a = a_0 + a_1\theta + \dots + a_{d-1}\theta^{d-1} + \theta^d$ . Thus we get

$$\frac{1}{a} = \frac{1}{\theta^d} \sum_{\underline{k}=(k_0, \dots, k_{d-1}) \in \mathbb{N}^d} C_{\underline{k}} a^{\underline{k}} \frac{1}{\theta^{w(\underline{k})}}$$

where we put  $a^{\underline{k}} = \prod_{j=0}^{d-1} a_j^{k_j}$ .

It follows that

$$\begin{aligned}
& \sum_{a \in A_{+,d}} \frac{a(t_1) \dots a(t_s)}{a} \\
&= \frac{1}{\theta^d} \sum_{\underline{k}=(k_0, \dots, k_{d-1}) \in \mathbb{N}^d} C_{\underline{k}} \frac{1}{\theta^{w(\underline{k})}} \sum_{a \in A_{+,d}} a^{\underline{k}} a(t_1) \dots a(t_s) \\
&= \frac{1}{\theta^d} \sum_{\underline{k}=(k_0, \dots, k_{d-1}) \in \mathbb{N}^d} C_{\underline{k}} \frac{1}{\theta^{w(\underline{k})}} \sum_{a \in A_{+,d}} \sum_{\substack{\underline{\ell}=(\ell_0, \dots, \ell_d) \in \mathbb{N}^{d+1}, \\ |\underline{\ell}|=s}} a^{\underline{k}} a^{\underline{\ell}} \sigma_s(\ell_1, \dots, \ell_d) \\
&= \frac{1}{\theta^d} \sum_{\underline{k}=(k_0, \dots, k_{d-1}) \in \mathbb{N}^d} C_{\underline{k}} \frac{1}{\theta^{w(\underline{k})}} \sum_{\substack{\underline{\ell}=(\ell_0, \dots, \ell_d) \in \mathbb{N}^{d+1}, \\ |\underline{\ell}|=s}} \sigma_s(\ell_1, \dots, \ell_d) \sum_{a \in A_{+,d}} a^{\underline{k}+\underline{\ell}}.
\end{aligned}$$

Here we put  $a^{\underline{\ell}} = \prod_{j=0}^{d-1} a_j^{\ell_j}$  and  $a^{\underline{k}+\underline{\ell}} = \prod_{j=0}^{d-1} a_j^{k_j+\ell_j}$ .

Letting  $\underline{k} = (k_0, \dots, k_{d-1}) \in \mathbb{N}^d$  and  $\underline{\ell} = (\ell_0, \dots, \ell_d) \in \mathbb{N}^{d+1}$  two sequences of integers, we say that  $\underline{\ell}$  is  $\underline{k}$ -admissible if  $(k_0 + \ell_0, \dots, k_{d-1} + \ell_{d-1}) \in ((q-1)\mathbb{N}^*)^d$ . We see that if  $\underline{\ell}$  is  $\underline{k}$ -admissible, then the sum  $\sum_{a \in A_{+,d}} a^{\underline{k}+\underline{\ell}}$  is equal to  $(-1)^d$ . Otherwise, this sum is equal to 0.

Given a sequence  $\underline{k} = (k_0, \dots, k_{d-1}) \in \mathbb{N}^d$  as above, we define another sequence  $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d$  by

$$\begin{aligned}
n_1 &:= k_0 + 1, \\
n_2 &:= k_0 + k_1 + 1, \\
&\dots \\
n_d &:= k_0 + \dots + k_{d-1} + 1.
\end{aligned}$$

This sequence satisfies

$$\begin{aligned}
i) \quad & n_1 \leq \dots \leq n_d, \\
ii) \quad & \sum_{j=1}^d n_j = w(\underline{k}) + d.
\end{aligned}$$

We observe that the sequence  $\underline{k}$  is completely determined by the associated sequence  $\underline{n}$ . In fact, we have  $k_0 = n_1 - 1$  and  $k_j = n_{j+1} - n_j$  for  $1 \leq j \leq d-1$ .

Let  $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d$  satisfying  $n_1 \leq \dots \leq n_d$  as above, and let  $\underline{k} = (k_0, \dots, k_{d-1}) \in \mathbb{N}^d$  be the associated sequence. We set

$$(3.3) \quad \mathfrak{L}_{\underline{n},s} := \{\underline{\ell} = (\ell_0, \dots, \ell_d) \in \mathbb{N}^{d+1} : \underline{\ell} \text{ is } \underline{k}\text{-admissible and } |\underline{\ell}| = s\}.$$

Then

$$\sum_{a \in A_{+,d}} \frac{a(t_1) \dots a(t_s)}{a} = \frac{(-1)^d}{\theta^d} \sum_{\underline{k} \in \mathbb{N}^d} C_{\underline{k}} \frac{1}{\theta^{w(\underline{k})}} \sum_{\underline{\ell} \in \mathfrak{L}_{\underline{n},s}} \sigma_s(\ell_1, \dots, \ell_d).$$

Let  $\underline{\ell} = (\ell_0, \dots, \ell_d)$  be a sequence in  $\mathfrak{L}_{\underline{n}, s}$  defined as above. Then there exist  $a_0, \dots, a_{d-1} \in \mathbb{N}^*$  such that

$$\begin{aligned}\ell_0 &= a_0(q-1) - n_1 + 1, \\ \ell_1 &= a_1(q-1) + n_1 - n_2, \\ &\dots \\ \ell_{d-1} &= a_{d-1}(q-1) + n_{d-1} - n_d.\end{aligned}$$

Thus we get

$$\ell_d = s - \sum_{j=0}^{d-1} \ell_j = a_d(q-1) + n_d$$

where we put  $a_d := m - \sum_{j=0}^{d-1} a_j$  and recall that  $m$  is defined as in (1.2). Since  $\underline{\ell} \in \mathbb{N}^{d+1}$ , we deduce the following lemma.

**Lemma 3.8.** *The set  $\mathfrak{L}_{\underline{n}, s}$  consists of the elements  $\underline{\ell} = (\ell_0, \dots, \ell_d) \in \mathbb{N}^{d+1}$  of the form*

$$\begin{aligned}\ell_0 &= a_0(q-1) - n_1 + 1, \\ \ell_1 &= a_1(q-1) + n_1 - n_2, \\ &\dots \\ \ell_{d-1} &= a_{d-1}(q-1) + n_{d-1} - n_d, \\ \ell_d &= a_d(q-1) + n_d,\end{aligned}$$

where  $a_0, \dots, a_d$  are integers such that

- $a_0 > 0, \dots, a_{d-1} > 0,$
- $\sum_{j=0}^d a_j = m.$

**Remark 3.9.** We note that  $a_d$  may be negative.

To summarize we have proved the following proposition.

**Proposition 3.10.** *We have*

$$\zeta_A(1, \underline{t}_s) = \sum_{\ell \geq 0} \alpha_{\ell, s} \theta^{-\ell}$$

with

$$\alpha_{\ell, s} = \sum_{\underline{n} \in \mathfrak{N}_{\ell}^+} \tilde{C}(\underline{n}) \sum_{\underline{\ell} \in \mathfrak{L}_{\underline{n}, s}} \sigma_s(\underline{\ell})$$

where

- the first sum runs through the set of sequences  $\underline{n} = (n_1, \dots, n_d) \in \mathfrak{N}_{\ell}^+$  defined as in (3.1),
- the second sum runs through the set of sequences  $\underline{\ell} \in \mathfrak{L}_{\underline{n}, s}$  defined as in (3.3),
- the coefficient  $\tilde{C}(\underline{n}) \in \mathbb{F}_p$  equals

$$(-1)^{d+n_d-1} \frac{(n_d-1)!}{(n_1-1)!(n_2-n_1)! \dots (n_d-n_{d-1})!}.$$

### 3.4. Proof of Theorem 3.1.

This section is devoted to prove Theorem 3.1 which compute the first  $q - 2$  coefficients of the polynomial  $\mathbb{B}_s$ . We start proving intermediate results and give a proof of Theorem 3.1 at the end of this section.

We first need the following consequence of Lemma 3.8.

**Lemma 3.11.** *Let  $\ell$  be an integer with  $1 \leq \ell \leq q - 2$ , and let  $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d$  satisfying  $n_1 \leq \dots \leq n_d$  and  $\sum_{j=1}^d n_j = \ell$ . Then the set  $\mathfrak{L}_{\underline{n}, s}$  defined as in (3.3) consists of the sequences  $\underline{\ell} = (\ell_0, \dots, \ell_d) \in \mathbb{N}^{d+1}$  of the form*

$$\begin{aligned} \ell_0 &= a_0(q - 1) - n_1 + 1, \\ \ell_1 &= a_1(q - 1) + n_1 - n_2, \\ &\dots \\ \ell_{d-1} &= a_{d-1}(q - 1) + n_{d-1} - n_d, \\ \ell_d &= a_d(q - 1) + n_d, \end{aligned}$$

where  $a_0, \dots, a_d$  are integers such that

- $a_0 > 0, \dots, a_{d-1} > 0$  et  $a_d \geq 0$ ,
- $\sum_{j=0}^d a_j = m$ .

*Proof.* We have to prove that  $a_d \geq 0$ . In fact, the fact that  $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d$  satisfies  $n_1 \leq \dots \leq n_d$  and  $\sum_{j=1}^d n_j = \ell$  implies  $n_d < \ell$ . Thus  $n_d < q - 2$  since  $\ell \leq q - 2$ . It follows immediately that  $a_d \geq 0$  since  $\ell_d = a_d(q - 1) + n_d$  and  $\ell_d \in \mathbb{N}$ .  $\square$

As an immediate consequence of Lemma 3.11 we obtain

**Proposition 3.12.** *Let  $\ell$  be an integer with  $1 \leq \ell \leq q - 2$ , and let  $\mathfrak{N}_\ell^+$  and  $\mathfrak{A}$  be defined as in (3.1). Then we have*

$$\alpha_{\ell, s} = \sum_{\underline{n} \in \mathfrak{N}_\ell^+} \sum_{\underline{a} \in \mathfrak{A}} \tilde{C}(\underline{n}) \sigma_s(a_1(q - 1) + n_1 - n_2, \dots, a_d(q - 1) + n_d)$$

where

- the first sum runs through the set of sequences  $\underline{n} = (n_1, \dots, n_d) \in \mathfrak{N}_\ell^+$ ,
- the second sum runs through the set of sequences  $\underline{a} = (a_0, \dots, a_d) \in \mathfrak{A}$ ,
- the coefficient  $\tilde{C}(\underline{n}) \in \mathbb{F}_p$  equals

$$\tilde{C}(\underline{n}) = (-1)^{d+n_d-1} \frac{(n_d - 1)!}{(n_1 - 1)!(n_2 - n_1)! \dots (n_d - n_{d-1})!}.$$

**Remark 3.13.** In the above formula, we could take the first sum over the bigger set of sequences  $\underline{n} = (n_1, \dots, n_d) \in \mathfrak{N}_\ell$  defined as in (3.1) since for any sequence  $\underline{n} \in \mathfrak{N}_\ell \setminus \mathfrak{N}_\ell^+$ , we have  $\tilde{C}(\underline{n}) = 0$ .

We set  $B_0 := 1$  and write

$$\mathbb{B}_s = \theta^{m-1}(B_0 + B_1\theta^{-1} + \dots + B_{m-1}\theta^{-(m-1)}), \quad B_\ell \in \mathbb{F}_q[t_s].$$

Recall that (see (2.2))

$$\mathbb{B}_s := (-1)^m \frac{\zeta_{\mathcal{A}}(1, t_s) \omega(t_1) \dots \omega(t_s)}{\tilde{\pi}}$$



To compute the first coefficients  $B_1, \dots, B_{q-2}$  of  $\mathbb{B}_s$ , it suffices to divide the above equality by  $\theta^{m-1}$  and to look at both sides modulo  $\theta^{-(q-1)}$ .

On the left-hand side, we obtain

$$B_0 + B_1\theta^{-1} + \dots + B_{q-2}\theta^{-(q-2)} \pmod{\theta^{-(q-1)}}$$

where we put  $B_k = 0$  for  $k \geq m$ .

On the right-hand side, for the zeta value  $\zeta_A(1, \underline{t}_s)$ , Proposition 3.12 gives

$$\alpha_{0,s} + \alpha_{1,s}\theta^{-1} + \dots + \alpha_{q-2,s}\theta^{-(q-2)} \pmod{\theta^{-(q-1)}}.$$

For other factors, we write

$$\prod_{j \geq 1} \left(1 - \frac{\theta}{\theta^{q^j}}\right) \equiv 1 \pmod{\theta^{-(q-1)}},$$

and

$$\begin{aligned} \prod_{i=1}^s \prod_{j \geq 0} \left(1 - \frac{t_i}{\theta^{q^j}}\right)^{-1} &\equiv \prod_{i=1}^s \left(1 - \frac{t_i}{\theta}\right)^{-1} \pmod{\theta^{-(q-1)}} \\ &\equiv \left(1 - \sigma_s(1)\theta^{-1} + \dots + (-1)^{q-2}\sigma_s(q-2)\theta^{-(q-2)}\right)^{-1} \pmod{\theta^{-(q-1)}}. \end{aligned}$$

Putting altogether, we get

$$\begin{aligned} &\left(B_0 + B_1\theta^{-1} + \dots + B_{q-2}\theta^{-(q-2)}\right) \left(1 - \sigma_s(1)\theta^{-1} + \dots + (-1)^{q-2}\sigma_s(q-2)\theta^{-(q-2)}\right) \\ &= \alpha_{0,s} + \alpha_{1,s}\theta^{-1} + \dots + \alpha_{q-2,s}\theta^{-(q-2)} \pmod{\theta^{-(q-1)}}. \end{aligned}$$

In other words, for all  $1 \leq \ell \leq q-2$ , we have

$$(3.4) \quad B_\ell - \sigma_s(1)B_{\ell-1} + \dots + (-1)^\ell \sigma_s(\ell)B_0 = \alpha_{\ell,s}.$$

Hence  $B_\ell$  is completely determined by  $B_0, \dots, B_{\ell-1}$ .

We now prove one of the key results of this section.

**Proposition 3.14.** *Let  $\ell$  be an integer with  $1 \leq \ell \leq q-2$ , and let  $\mathfrak{N}_\ell$  and  $\mathfrak{A}^+$  be defined as in (3.1). Then*

$$(3.5) \quad B_\ell = \sum_{\underline{n} \in \mathfrak{N}_\ell} \sum_{\underline{a} \in \mathfrak{A}^+} C(\underline{n}, \underline{a}) \sigma_s(a_1(q-1) + n_1 - n_2, \dots, a_d(q-1) + n_d)$$

where

- the first sum runs through the set of sequences  $\underline{n} = (n_1, \dots, n_d) \in \mathfrak{N}_\ell$ ,
- the second sum runs through the set of sequences  $\underline{a} = (a_0, \dots, a_d) \in \mathfrak{A}^+$ ,
- the coefficient  $C(\underline{n}, \underline{a}) \in \mathbb{F}_p$  is equal to

$$C(\underline{n}, \underline{a}) = (-1)^d \prod_{j=1}^d \binom{a_j(q-1) + n_j - 1}{n_j - 1}.$$

*Proof.* Let  $\ell$  be an integer with  $1 \leq \ell \leq q-2$ . It suffices to prove (3.4) where  $B_\ell$  is given by (3.5) and  $\alpha_{\ell,s}$  is given in Proposition 3.12.

If we set

$$S := B_\ell - \sigma_s(1)B_{\ell-1} + \dots + (-1)^\ell \sigma_s(\ell)B_0 = \sum_{k=0}^{\ell-1} (-1)^k \sigma_s(k)B_{\ell-k} + (-1)^\ell \sigma_s(\ell),$$

then we replace  $B_{\ell-k}$  by (3.5) for  $0 \leq k \leq \ell - 1 \leq q - 2$  to get

$$S = \sum_{k=0}^{\ell-1} (-1)^k \sigma_s(k) \sum_{\underline{\ell} \in \mathfrak{N}_{\ell-k}} \sum_{\underline{a} \in \mathfrak{A}^+} C(\underline{\ell}, \underline{a}) \sigma_s(a_1(q-1) + \ell_1 - \ell_2, \dots, a_d(q-1) + \ell_d) \\ + (-1)^\ell \sigma_s(\ell),$$

where the second sum (resp. the third sum) is over the set of sequences  $\underline{\ell} = (\ell_1, \dots, \ell_d) \in \mathfrak{N}_{\ell-k}$  (resp.  $\underline{a} = (a_0, \dots, a_d) \in \mathfrak{A}^+$ ). By Lemma 3.5 we develop the above expression to get

$$S = \sum_{k=0}^{\ell-1} (-1)^k \sum_{\underline{\ell} \in \mathfrak{N}_{\ell-k}} \sum_{\underline{a} \in \mathfrak{A}^+} C(\underline{\ell}, \underline{a}) \sigma_s(a_1(q-1) + \ell_1 - \ell_2 + j_1 - j_2, \dots, a_d(q-1) + \ell_d + j_d - j_{d+1}, j_{d+1}) \\ \times \sum_{\substack{(j_1, \dots, j_{d+1}) \in \mathbb{N}^{d+1} \\ j_1 + \dots + j_{d+1} = k}} \binom{a_1(q-1) + \ell_1 - \ell_2 + j_1 - j_2}{j_1} \dots \binom{a_d(q-1) + \ell_d + j_d - j_{d+1}}{j_d} \\ + (-1)^\ell \sigma_s(\ell).$$

For sequences  $\underline{\ell} = (\ell_1, \dots, \ell_d) \in \mathfrak{N}_{\ell-k}$  and  $(j_1, \dots, j_{d+1}) \in \mathbb{N}^{d+1}$  with  $j_1 + \dots + j_{d+1} = k$  as appeared in the above sum, we put

$$n_1 = \ell_1 + j_1, \quad \dots, \quad n_d = \ell_d + j_d, \quad n_{d+1} = j_{d+1}.$$

Then the sequence  $\underline{n} = (n_1, \dots, n_{d+1})$  belongs to  $(\mathbb{N}^*)^d \times \mathbb{N}$  and satisfies  $n_1 + \dots + n_{d+1} = \ell$ . For the last factor  $\sigma_s(\ell)$ , we put  $\underline{n} = (\ell)$ .

Using this notation and the formula for  $C(\underline{\ell}, \underline{a})$  we can re-index the sums in  $S$  to get

$$S = \sum_{\underline{n}} \sum_{\underline{a} \in \mathfrak{A}^+} \sigma_s(a_1(q-1) + n_1 - n_2, \dots, a_d(q-1) + n_d, n_{d+1}) \\ \times (-1)^{d+n_{d+1}} \sum_{\underline{\ell}} \prod_{j=1}^d (-1)^{n_j - \ell_j} \binom{a_j(q-1) + \ell_j - 1}{\ell_j - 1} \binom{a_j(q-1) + n_j - n_{j+1}}{n_j - \ell_j} \\ = \sum_{\underline{n}} \sum_{\underline{a} \in \mathfrak{A}^+} S(\underline{n}, \underline{a}) \sigma_s(a_1(q-1) + n_1 - n_2, \dots, a_d(q-1) + n_d, n_{d+1})$$

where

- the first sum runs through the set of sequences  $\underline{n} = (n_1, \dots, n_{d+1}) \in (\mathbb{N}^*)^d \times \mathbb{N}$  such that  $n_1 + \dots + n_{d+1} = \ell$ ,
- the second sum runs through the set of sequences  $\underline{a} = (a_0, \dots, a_d) \in \mathfrak{A}^+$ ,
- the third sum of the first equality runs through the set of sequences  $\underline{\ell} = (\ell_1, \dots, \ell_d) \in (\mathbb{N}^*)^d$  such that  $\ell_j \leq n_j$  for all  $1 \leq j \leq d$ ,
- the coefficients  $S(\underline{n}, \underline{a})$  are given by

$$S(\underline{n}, \underline{a}) = (-1)^{d+n_{d+1}} \sum_{\underline{\ell}} \prod_{j=1}^d (-1)^{n_j - \ell_j} \binom{a_j(q-1) + \ell_j - 1}{\ell_j - 1} \binom{a_j(q-1) + n_j - n_{j+1}}{n_j - \ell_j}.$$

where the sum runs through the set of sequences  $\underline{\ell} = (\ell_1, \dots, \ell_d) \in (\mathbb{N}^*)^d$  such that  $\ell_j \leq n_j$  for all  $1 \leq j \leq d$ .

In Lemma 3.15 below we compute explicitly the coefficients  $S(\underline{n}, \underline{a})$ . Combining it with Proposition 3.12, we deduce immediately Proposition 3.14.  $\square$

**Lemma 3.15.** *Let  $\ell$  be an integer with  $1 \leq \ell \leq q - 2$ . Let  $\underline{n} = (n_1, \dots, n_{d+1}) \in (\mathbb{N}^*)^d \times \mathbb{N}$  be a sequence satisfying  $n_1 + \dots + n_{d+1} = \ell$ , and let  $\underline{a} = (a_0, \dots, a_d) \in (\mathbb{N}^*)^{d+1}$  be a sequence of positive integers. We set*

$$S(\underline{n}, \underline{a}) = (-1)^{d+n_{d+1}} \sum_{\underline{\ell}} \prod_{j=1}^d (-1)^{n_j - \ell_j} \binom{a_j(q-1) + \ell_j - 1}{\ell_j - 1} \binom{a_j(q-1) + n_j - n_{j+1}}{n_j - \ell_j}.$$

where the sum runs through the set of sequences  $\underline{\ell} = (\ell_1, \dots, \ell_d) \in (\mathbb{N}^*)^d$  satisfying  $\ell_j \leq n_j$  for  $1 \leq j \leq d$ .

Then we have

(1) If  $n_{d+1} > 0$ , then

$$S(\underline{n}, \underline{a}) = \tilde{C}(n_1, \dots, n_{d+1}) = (-1)^{d+n_{d+1}} \frac{(n_{d+1} - 1)!}{(n_1 - 1)!(n_2 - n_1)! \dots (n_{d+1} - n_d)!}.$$

(2) If  $n_{d+1} = 0$ , then

$$S(\underline{n}, \underline{a}) = \tilde{C}(n_1, \dots, n_d) = (-1)^{d+n_d-1} \frac{(n_d - 1)!}{(n_1 - 1)!(n_2 - n_1)! \dots (n_d - n_{d-1})!}.$$

Here recall that the coefficients  $\tilde{C}(\underline{n})$  are defined as in Proposition 3.12.

*Proof.* In fact, we write

$$S(\underline{n}, \underline{a}) = (-1)^{d+n_{d+1}} \prod_{j=1}^d \sum_{1 \leq \ell_j \leq n_j} (-1)^{n_j - \ell_j} \binom{a_j(q-1) + \ell_j - 1}{\ell_j - 1} \binom{a_j(q-1) + n_j - n_{j+1}}{n_j - \ell_j}.$$

We consider separately each factor of the above product and distinguish three cases.

**Case 1: the  $j$ th factor for  $1 \leq j \leq d - 1$ .**

We apply Lemma 3.6 to  $x = a_j(q-1) + n_j - n_{j+1}$ ,  $M = n_j - 1$  and  $N = n_{j+1} - 1$  to obtain

$$\begin{aligned} & \sum_{1 \leq \ell_j \leq n_j} (-1)^{n_j - \ell_j} \binom{a_j(q-1) + \ell_j - 1}{\ell_j - 1} \binom{a_j(q-1) + n_j - n_{j+1}}{n_j - \ell_j} \\ &= \sum_{k=0}^{n_j-1} (-1)^k \binom{a_j(q-1) + n_j - 1 - k}{n_j - 1 - k} \binom{a_j(q-1) + n_j - n_{j+1}}{k} \quad \text{where } k = n_j - \ell_j \\ &= \binom{n_{j+1} - 1}{n_j - 1}. \end{aligned}$$

**Case 2: the  $d$ th factor with  $n_{d+1} > 0$ .**

We apply Lemma 3.6 to  $x = a_d(q-1) + n_d - n_{d+1}$ ,  $M = n_d - 1$  and  $N = n_{d+1} - 1$  to obtain

$$\begin{aligned} & \sum_{1 \leq \ell_d \leq n_d} (-1)^{n_d - \ell_d} \binom{a_d(q-1) + \ell_d - 1}{\ell_d - 1} \binom{a_d(q-1) + n_d - n_{d+1}}{n_d - \ell_d} \\ &= \sum_{k=0}^{n_d-1} (-1)^k \binom{a_d(q-1) + n_d - 1 - k}{n_d - 1 - k} \binom{a_d(q-1) + n_d - n_{d+1}}{k} \quad \text{where } k = n_d - \ell_d \\ &= \binom{n_{d+1} - 1}{n_d - 1}. \end{aligned}$$

**Case 3: the  $d$ th factor with  $n_{d+1} = 0$ .**

Note that  $n_d \geq 1$ . We apply Lemma 3.7 to  $x = a_d(q-1) + n_d$  and  $M = N = n_d - 1$  to obtain

$$\begin{aligned} & \sum_{1 \leq \ell_d \leq n_d} (-1)^{n_d - \ell_d} \binom{a_d(q-1) + \ell_d - 1}{\ell_d - 1} \binom{a_d(q-1) + n_d - n_{d+1}}{n_d - \ell_d} \\ & \sum_{1 \leq \ell_d \leq n_d} (-1)^{n_d - \ell_d} \binom{a_d(q-1) + \ell_d - 1}{\ell_d - 1} \binom{a_d(q-1) + n_d}{n_d - \ell_d} \\ &= \sum_{k=0}^{n_d-1} (-1)^{n_d-1+k} \binom{a_d(q-1) + k}{k} \binom{a_d(q-1) + n_d}{n_d - 1 - k} \quad \text{where } k = \ell_d - 1 \\ &= (-1)^{n_d-1}. \end{aligned}$$

Putting altogether, we obtain Lemma 3.15. The proof is finished.  $\square$

We are now ready to prove Theorem 3.1.

*Proof of Theorem 3.1.* By Proposition 3.14, letting  $\mathfrak{N}_\ell$  and  $\mathfrak{A}^+$  be defined as in (3.1), for  $1 \leq \ell \leq q-2$ , we have

$$B_\ell = \sum_{\underline{n}} \sum_{\underline{a}} C(\underline{n}, \underline{a}) \sigma_s(a_1(q-1) + n_1 - n_2, \dots, a_d(q-1) + n_d)$$

where

- the first sum runs through the set of sequences  $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d$  satisfying  $\sum_{j=1}^d n_j = \ell$  (e.g.  $\underline{n} \in \mathfrak{N}_\ell$ ),
- the second sum runs through the set of sequences  $\underline{a} = (a_0, \dots, a_d) \in (\mathbb{N}^*)^{d+1}$  satisfying  $\sum_{j=0}^d a_j = m$  (e.g.  $\underline{a} \in \mathfrak{A}^+$ ),
- the coefficient  $C(\underline{n}, \underline{a}) \in \mathbb{F}_p$  is equal to

$$C(\underline{n}, \underline{a}) = (-1)^d \prod_{j=1}^d \binom{a_j(q-1) + n_j - 1}{n_j - 1}.$$

To prove Theorem 3.1, it suffices to prove that for  $\underline{n} = (n_1, \dots, n_d) \in \mathfrak{N}_\ell$  and  $\underline{a} = (a_0, \dots, a_d) \in \mathfrak{A}^+$  as above, the coefficients  $C(\underline{n}, \underline{a})$  and  $B(\underline{n}, \underline{a})$  given in Theorem

3.1 are the same. In fact, by Lemma 3.4 we have

$$\begin{aligned} C(\underline{n}, \underline{a}) &= (-1)^d \prod_{j=1}^d \binom{a_j(q-1) + n_j - 1}{n_j - 1} \\ &= (-1)^d \prod_{j=1}^d (-1)^{n_j-1} \binom{a_j - 1}{n_j - 1} \\ &= (-1)^{n_1 + \dots + n_d} \prod_{j=1}^d \binom{a_j - 1}{n_j - 1}. \end{aligned}$$

Since  $\sum_{j=1}^d n_j = \ell$ , it follows that

$$C(\underline{n}, \underline{a}) = (-1)^\ell \prod_{j=1}^d \binom{a_j - 1}{n_j - 1} = B(\underline{n}, \underline{a})$$

as desired. The proof is finished.  $\square$

#### 4. PROOF OF THE MAIN RESULT

In this section we present a proof of Theorem 1.4. We have to show that if  $m < q$  where  $m$  is defined as in (1.2), then the following formula holds

$$\zeta_A(1, \underline{t}_s) = \sum \mathcal{L}_{U_1}^{(-1)} \cdots \mathcal{L}_{U_d}^{(-d)}$$

where the sum runs through the set of ordered set partitions  $\underline{U} = (U_1 \mid \cdots \mid U_d)$  of  $\Sigma$  satisfying

$$\frac{|U_1|}{q} + \cdots + \frac{|U_d|}{q^d} = 1.$$

We assume that  $m < q$ . Then the polynomial  $\mathbb{B}_s$  is completely determined by Theorem 3.1. We claim that  $\mathbf{w}(\mathbb{B}_s) \geq 1$ . In fact, by Theorem 3.1 and Definition 2.6 it suffices to prove that

$$\mathbf{w}(\sigma_s(a_1(q-1) + n_1 - n_2, \dots, a_d(q-1) + n_d)) \geq 1$$

where

- $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d$ ,
- $\underline{a} = (a_0, \dots, a_d) \in (\mathbb{N}^*)^{d+1}$  such that  $\sum_{j=0}^d a_j = m$ ,

satisfying  $a_j \geq n_j$  for  $1 \leq j \leq d$  (see Remark 3.2).

We see that

$$\begin{aligned} &\mathbf{w}(\sigma_s(a_1(q-1) + n_1 - n_2, \dots, a_d(q-1) + n_d)) \\ &= \frac{a_0 q - n_1}{q} + \frac{a_1(q-1) + n_1 - n_2}{q^2} + \cdots + \frac{a_d(q-1) + n_d}{q^{d+1}}. \end{aligned}$$

Since  $a_j \geq n_j$  for  $1 \leq j \leq d$  and  $a_0 \geq 1$ , we get

$$\mathbf{w}(\sigma_s(a_1(q-1) + n_1 - n_2, \dots, a_d(q-1) + n_d)) \geq \frac{q - n_1}{q} + \frac{n_1 q - n_2}{q^2} + \cdots + \frac{n_d q}{q^{d+1}} = 1$$

as required.

Now we know that  $\mathbf{w}(\mathbb{B}_s) \geq 1$ . Thus Theorem 1.4 follows immediately from Theorem 2.9.

## 5. FINAL REMARKS

We end this paper with some remarks. In this paper we have succeeded in proving Conjecture 1.3 and thus get a partial answer to Conjecture 1.1. We expect that Conjecture 1.1 always holds. Thus it is tempting to ask whether Theorem 3.1 holds in full generality, that means we can remove the restriction  $1 \leq \ell \leq q - 2$ .

**Conjecture 5.1.** *Recall that (see Proposition 2.1)*

$$\mathbb{B}_s = \theta^{m-1} + B_1\theta^{m-2} + \dots + B_{m-1}, \quad B_\ell \in \mathbb{F}_q[t_s].$$

Let  $\ell \in \mathbb{N}^*$  such that  $1 \leq \ell \leq m - 1$ . Then we have

$$B_\ell = \sum_{\underline{n}} \sum_{\underline{a}} B(\underline{n}, \underline{a}) \sigma_s(a_1(q-1) + n_1 - n_2, \dots, a_d(q-1) + n_d)$$

where

- the first sum runs through the set of sequences  $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d$  satisfying  $\sum_{j=1}^d n_j = \ell$ ,
- the second sum runs through the set of sequences  $\underline{a} = (a_0, \dots, a_d) \in (\mathbb{N}^*)^{d+1}$  satisfying  $\sum_{j=0}^d a_j = m$ ,
- the coefficients  $B(\underline{n}, \underline{a}) \in \mathbb{F}_p$  are given by

$$B(\underline{n}, \underline{a}) = (-1)^\ell \prod_{j=1}^d \binom{a_j - 1}{n_j - 1}.$$

By similar arguments as before we prove the following result.

**Proposition 5.2.** *Conjecture 5.1 implies Conjecture 1.1.*

*Proof.* Suppose that Conjecture 5.1 holds. From the explicit formula for  $\mathbb{B}_s$ , by similar arguments as those given in Section 4, we see that  $\mathbf{w}(\mathbb{B}_s) \geq 1$ . Combined with Theorem 2.9, it implies immediately Conjecture 1.1.  $\square$

**Remark 5.3.** 1) For  $m = 1, 2, 3$ , we have explicit formulas for  $\mathbb{B}_s$  (see Section 2.1) and see easily that Conjecture 5.1 holds for these small values. They provide the first evidence to support our conjecture.

2) We should mention that the first author is currently investigating the above Conjecture in his PhD thesis.

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