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# ON IDENTITIES FOR ZETA VALUES IN TATE ALGEBRAS

HUY HUNG LE AND TUAN NGO DAC

ABSTRACT. Zeta values in Tate algebras were introduced by Pellarin in 2012. They are generalizations of Carlitz's zeta values and play an increasingly important role in function field arithmetic. In this paper, we prove a conjecture of Pellarin on identities for these zeta values. The proof is based on arithmetic properties of Carlitz's zeta values and an explicit formula for Bernoulli-type polynomials attached to Pellarin's zeta values.

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## INTRODUCTION

A classical topic in number theory is the study of the Riemann zeta function  $\zeta(\cdot)$  and its special values  $\zeta(n)$  for  $n \in \mathbb{N}$  and  $n \geq 2$ . By a well-known analogy between the arithmetic of number fields and global function fields, Carlitz suggested to transport the classical results to the function field setting in positive characteristic. In [10], he considered the rational function field equipped with the infinity place and introduced the Carlitz zeta values  $\zeta_A(n)$  which are considered as the analogues of  $\zeta(n)$ . Let  $\mathbb{F}_q$  be a finite field having  $q$  elements,  $q$  being a power of a prime number  $p$ . Let  $A = \mathbb{F}_q[\theta]$  with  $\theta$  an indeterminate over  $\mathbb{F}_q$ ,  $K = \mathbb{F}_q(\theta)$ ,  $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$  and let  $\mathbb{C}_\infty$  be the completion of a fixed algebraic closure of  $K_\infty$ . For  $d \in \mathbb{N}$ ,  $A_{+,d}$  denotes the set of monic elements in  $A$  of degree  $d$ . For  $n \in \mathbb{Z}$ , the value at  $n$  of the Carlitz-Goss zeta function is defined as follows:

$$\zeta_A(n) := \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{1}{a^n} \in K_\infty.$$

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One can show that  $\zeta_A(n) \in A$  if  $n \leq 0$  and even  $\zeta_A(n) = 0$  if  $n < 0$  and  $n \equiv 0 \pmod{q-1}$  (see [19], Chapter 8).

We now move to the context of Tate algebras. Let  $s \geq 1$  be an integer and let  $\mathbb{T}_s$  be the Tate algebra in  $s$  variables  $t_1, \dots, t_s$  with coefficients in  $\mathbb{C}_\infty$  (see Section 1.2). In 2012, Pellarin [24] introduced the following elements in  $\mathbb{T}_s^\times$  called the *several variable L-series*

$$\zeta_A(1, \underline{t}_s) := L_s(\underline{t}_s) = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t_1) \cdots a(t_s)}{a} \in \mathbb{T}_s^\times.$$

For  $s = 1$ , he proved the following remarkable identity (see [24], Theorem 1):

$$\frac{L_1(t_1)\omega(t_1)}{\tilde{\pi}} = \frac{1}{\theta - t_1}$$

where  $\omega(t_1)$  is the special function introduced by Anderson and Thakur in [1] and given by

$$\omega(t_1) = \lambda_\theta \prod_{j \geq 0} \left(1 - \frac{t_1}{\theta^{q^j}}\right)^{-1} \in \mathbb{T}_1^\times$$

where  $\lambda_\theta$  is a fixed  $(q-1)$ -th root of  $-\theta$  in  $\mathbb{C}_\infty$ .

Since their introduction, various works have revealed the importance of these zeta values for both their proper interest or their applications to values of the Goss  $L$ -functions, values of multizetas, Anderson's log-algebraicity identities, Taelman's units and Drinfeld modular forms in Tate algebras (see [4, 6, 7, 8, 9, 11, 12, 14, 15, 26, 27, 30]). We should mention that generalizations of Pellarin's zeta values to various settings have been also conducted (see for example [2, 3, 21, 22, 23]).

In a recent work [25], Pellarin investigated a more general setting for Drinfeld modular forms and formulated several conjectural identities for zeta values in Tate algebras. Roughly speaking, he predicts that the several variable zeta values belong to the  $\mathbb{F}_p$ -algebra generated by negative or positive powers of one-variable zeta values and that an elegant explicit expression is even provided *for  $q$  large*. These conjectures combined with other results could lead to identities for Eisenstein series as explained in [25], Section 9. We refer the reader to Conjectures 1.1, 1.2 and 2.3 for the precise statements.

In this paper, we succeed in giving an affirmative answer to the above Conjectures 1.2 and 2.3 (see Theorem 1.3). In Section 2, we first recall the Bernoulli-type polynomials  $\mathbb{B}_s$  which encodes the deep connection between Pellarin's zeta values and Taelman's class formula thanks to the work of Anglès, Pellarin and Tavares Ribeiro [5, 7]. Next we show that *for  $q$  large*, this Conjecture follows from an explicit expression for  $\mathbb{B}_s$  (see Theorem 2.7). The arguments are based on a notion of weights and arithmetic properties of Carlitz's zeta values. Then we prove this explicit formula for  $\mathbb{B}_s$  for  $q$  large by a combinatorial proof, see Section 3. We should mention that combinatorial properties of  $\mathbb{B}_s$  have already had important applications in function field arithmetic (see [4, 20] for more details). Finally, we conjecture that the explicit formula for  $\mathbb{B}_s$  holds for all  $q$  and provide some hints to support it.

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## 1. IDENTITIES FOR ZETA VALUES

## 1.1. Notation.

In this paper, we will use the following notation.

- $\mathbb{N}$ : the set of non-negative integers.
- $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ : the set of positive integers.
- $\mathbb{Z}$ : the set of integers.
- $\mathbb{F}_q$ : a finite field having  $q$  elements.
- $p$ : the characteristic of  $\mathbb{F}_q$ .
- $\theta$ : an indeterminate over  $\mathbb{F}_q$ .
- $A$ : the polynomial ring  $\mathbb{F}_q[\theta]$ .
- $A_+$ : the set of monic elements in  $A$ .
- For  $d \in \mathbb{N}$ ,  $A_{+,d}$  denotes the set of monic elements in  $A$  of degree  $d$ .
- $K = \mathbb{F}_q(\theta)$ : the fraction field of  $A$ .
- $\infty$ : the unique place of  $K$  which is a pole of  $\theta$ .
- $v_\infty$ : the discrete valuation on  $K$  corresponding to the place  $\infty$  normalized such that  $v_\infty(\theta) = -1$ .
- $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$ : the completion of  $K$  at  $\infty$ .
- $\mathbb{C}_\infty$ : the completion of a fixed algebraic closure of  $K_\infty$ . The unique valuation of  $\mathbb{C}_\infty$  which extends  $v_\infty$  will still be denoted by  $v_\infty$ .
- $\lambda_\theta$ : a fixed  $(q-1)$ th-root of  $-\theta$  in  $\mathbb{C}_\infty$ .
- For  $s \in \mathbb{N}$ ,  $\{t_1, t_2, \dots, t_s\}$  denotes a set of  $s$  variables and we will also denote this set by  $\underline{t}_s$ .

We will work with the set of all (finite) sequences of non-negative integers  $\underline{m}$ . When we consider a sequence  $\underline{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$  of non-negative integers, the reader should keep in mind that  $d$  depends on the sequence  $\underline{m}$  and that  $m_i$  may be 0.

## 1.2. Tate algebras.

Let  $L$  be an extension of  $K_\infty$  in  $\mathbb{C}_\infty$  such that  $L$  is complete with respect to  $v_\infty$ . The absolute value of  $L$  is defined by

$$|a| = q^{-v_\infty(a)} \quad \text{for } a \in L.$$

Let  $s \in \mathbb{N}$  be a non-negative integer. The polynomial ring  $L[\underline{t}_s] = L[t_1, \dots, t_s]$  is equipped with the Gauss valuation: for a polynomial  $f \in L[\underline{t}_s]$ , if we write

$$f = \sum_{i_1, \dots, i_s \in \mathbb{N}} a_{i_1, \dots, i_s} t_1^{i_1} \cdots t_s^{i_s}, \quad a_{i_1, \dots, i_s} \in L,$$

then the Gauss valuation of  $f$  is defined by

$$v_\infty(f) := \inf\{v_\infty(a_{i_1, \dots, i_s}), i_1, \dots, i_s \in \mathbb{N}\}.$$

By definition, the Tate algebra  $\mathbb{T}_s(L)$  in the variables  $t_1, \dots, t_s$  with coefficients in  $L$  is the completion of  $L[\underline{t}_s]$  with respect to the Gauss valuation. Explicitly,  $\mathbb{T}_s(L)$  is the set of formal series

$$f = \sum_{i_1, \dots, i_s \in \mathbb{N}} a_{i_1, \dots, i_s} t_1^{i_1} \cdots t_s^{i_s}, \quad a_{i_1, \dots, i_s} \in L,$$

such that

$$\lim_{i_1 + \dots + i_s \rightarrow +\infty} v_\infty(a_{i_1, \dots, i_s}) = +\infty.$$

When  $L = \mathbb{C}_\infty$ , we will write  $\mathbb{T}_s$  instead of  $\mathbb{T}_s(\mathbb{C}_\infty)$ . Let  $\tau : \mathbb{T}_s \rightarrow \mathbb{T}_s$  be the continuous homomorphism of  $\mathbb{F}_q[\underline{t}_s]$ -algebras such that

$$\tau(c) = c^q, \quad \text{for } c \in \mathbb{C}_\infty.$$

Explicitly, for a formal series  $f \in \mathbb{T}_s$ , we write

$$f = \sum_{i_1, \dots, i_s \in \mathbb{N}} a_{i_1, \dots, i_s} t_1^{i_1} \dots t_s^{i_s}, \quad a_{i_1, \dots, i_s} \in \mathbb{C}_\infty,$$

then

$$\tau(f) = \sum_{i_1, \dots, i_s \in \mathbb{N}} a_{i_1, \dots, i_s}^q t_1^{i_1} \dots t_s^{i_s}.$$

### 1.3. Conjectures of Pellarin and the main results.

For the rest of this paper, we will always suppose that  $s \equiv 1 \pmod{q-1}$ . We set

$$\Sigma := \{t_1, \dots, t_s\}$$

and

$$m = \frac{s-1}{q-1} \in \mathbb{N}.$$

Recall that  $\lambda_\theta$  is a fixed  $(q-1)$ -th root of  $-\theta$  in  $\mathbb{C}_\infty$ . We have set

$$\tilde{\pi} = \lambda_\theta \theta \prod_{j \geq 1} \left(1 - \theta^{1-q^j}\right)^{-1} \in \mathbb{C}_\infty^\times,$$

and for  $i = 1, \dots, s$ , we set

$$\omega(t_i) = \lambda_\theta \prod_{j \geq 0} \left(1 - \frac{t_i}{\theta^{q^j}}\right)^{-1} \in \mathbb{T}_s^\times.$$

In 2012, F. Pellarin [24] introduced the following elements in  $\mathbb{T}_s^\times$  called the *several variables L-series or several variables zeta values*

$$\zeta_A(1, \underline{t}_s) := L_s(\underline{t}_s) = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t_1) \dots a(t_s)}{a} \in \mathbb{T}_s^\times.$$

We observe that it can be written as a Euler product

$$\zeta_A(1, \underline{t}_s) = \prod_{\substack{P \in A_+ \\ P \text{ irreducible}}} \left(1 - \frac{P(t_1) \dots P(t_s)}{P}\right)^{-1}.$$

In [25], Pellarin revisited the theory of Drinfeld modular forms which were initially developed by Goss [16, 17, 18] and Gekeler [13]. In his investigation, he proposed several conjectures for these zeta values  $\zeta_A(1, \underline{t}_s)$  which would lead to new identities for Eisenstein series. We refer the reader to [25], Section 9 for more details.

**Conjecture 1.1** ([25], Conjecture 9.1). *We have*

$$\zeta_A(1, \underline{t}_s) \in \mathbb{F}_p[\tau^m(\zeta_A(1, t_i)) : 1 \leq i \leq s, m \in \mathbb{Z}].$$

As Pellarin mentioned in his paper (see the discussion just before [25], Conjecture 9.1), the central point of this Conjecture is that negative powers are allowed and that the coefficients belong to  $\mathbb{F}_p$ . Further, Pellarin suggested an explicit formula for  $\zeta_A(1, \underline{t}_s)$  when  $q$  is large enough. Precisely, let  $m \in \mathbb{Z}$  and  $U$  be a subset of  $\Sigma$ . We set

$$\mathcal{L}_U^{(m)} := \tau^m \left( \prod_{i \in U} \zeta_A(1, t_i) \right).$$

Recall that an ordered set partition of  $\Sigma$  is a set partition of  $\Sigma$  equipped with a total order on its blocks. We will write

$$U_1 \sqcup \cdots \sqcup U_m = \Sigma$$

for the partition  $U_1 \cup \cdots \cup U_m = \Sigma$  of  $\Sigma$  equipped with the total order  $U_1 \prec \cdots \prec U_m$ .

**Conjecture 1.2** ([25], Conjecture 9.4). *Suppose that  $q$  is large enough depending on  $m$ . Then the following formula holds*

$$\zeta_A(1, \underline{t}_s) = \sum \mathcal{L}_{U_1}^{(-1)} \cdots \mathcal{L}_{U_m}^{(-m)}$$

where the sum runs through the set of ordered set partitions of  $\Sigma$

$$U_1 \sqcup \cdots \sqcup U_m = \Sigma$$

such that

$$\frac{|U_1|}{q} + \cdots + \frac{|U_m|}{q^m} = 1.$$

In this paper, we give an affirmative answer to this Conjecture.

**Theorem 1.3.** *Conjecture 1.2 holds.*

A proof of this Theorem will be given in Sections 2 and 3.4. Let us outline the main steps of the proof.

- (1) First, using the link between Pellarin's zeta values and Taelman's class formula due to Anglès, Pellarin and Tavares Ribeiro [5, 7], we state an equivalent statement of Conjecture 1.2 (see Conjecture 2.3). Instead of identities on Pellarin's zeta values, it gives conjectural expressions on certain several variable polynomials  $\mathbb{B}_s$ .
- (2) Next, we give an expression of  $\mathbb{B}_s$  and manage to calculate explicitly some coefficients of this formula as predicted by Conjecture 2.3 (see Lemma 2.4 and Lemma 2.5). The main ingredients are a notion of weights and arithmetic properties of Carlitz's zeta values. To conclude, it suffices to prove that the other coefficients vanish.
- (3) Finally, we succeed in proving the last assertion. It follows from another explicit formula for  $\mathbb{B}_s$  for  $q$  large whose proof is of combinatorial nature (see Theorem 2.7).

## 2. THE SEVERAL VARIABLE BERNOULLI-TYPE POLYNOMIALS

We briefly recall the deep connection between the zeta values  $\zeta_A(1, \underline{t}_s)$  and some several variable Bernoulli-type polynomials  $\mathbb{B}_s$  as explained in [5, 7].

### 2.1. The several variable polynomials $\mathbb{B}_s$ .

Recall that for  $s = 1$ , Pellarin proved the following identity (see [24], Theorem 1):

$$(2.1) \quad \frac{\zeta_A(1, t_1)\omega(t_1)}{\tilde{\pi}} = \frac{1}{\theta - t_1}.$$

For  $s \geq 2$  and  $s \equiv 1 \pmod{q-1}$ , we define

$$(2.2) \quad \mathbb{B}_s = (-1)^{\frac{s-1}{q-1}} \frac{\zeta_A(1, \underline{t}_s)\omega(t_1) \cdots \omega(t_s)}{\tilde{\pi}} \in \mathbb{T}_s.$$

Then by [7], Lemma 7.6 (see also [6], Corollary 21), we know that

**Proposition 2.1.** *The element  $\mathbb{B}_s$  is a polynomial in  $\mathbb{F}_q[\underline{t}_s, \theta]$ . Moreover, it is a monic polynomial in  $\theta$  of degree  $m-1 = \frac{s-q}{q-1}$  and a symmetric polynomial in variables  $\underline{t}_s$ .*

Thanks to the work of Anglès, Pellarin and Tavares Ribeiro, we know that the polynomial  $\mathbb{B}_s$  is closely connected to the class module  $H_\phi$  of a certain Drinfeld  $A[\underline{t}_s]$ -module  $\phi$  of rank one as follows (we refer the interested reader to [7], Section 7 for more details). Let  $\phi : A[\underline{t}_s] \rightarrow \mathbb{T}_s[\underline{t}_s]\{\tau\}$  be the Drinfeld  $A[\underline{t}_s]$ -module over  $\mathbb{T}_s$  given by a homomorphism of  $\mathbb{F}_q[\underline{t}_s]$ -algebras such that

$$\phi_\theta = \theta + (t_1 - \theta) \cdots (t_s - \theta)\tau.$$

There exists a unique formal series  $\exp_\phi \in \mathbb{T}_s\{\{\tau\}\}$  called *the exponential series* attached to  $\phi$  such that

$$\exp_\phi \equiv 1 \pmod{\tau}$$

and

$$\phi_a \exp_\phi = \exp_\phi a, \quad a \in A[\underline{t}_s].$$

One can show that the exponential series induces a natural  $\mathbb{F}_q[\underline{t}_s]$ -linear map

$$\exp_\phi : \mathbb{T}_s \rightarrow \mathbb{T}_s.$$

Following Taelman [28, 29], we define the class module  $H_\phi$  by

$$H_\phi := \frac{\phi(\mathbb{T}_s(K_\infty))}{\exp_\phi(\mathbb{T}_s(K_\infty)) + \phi(A[\underline{t}_s])}$$

where  $\phi(A[\underline{t}_s])$  is the  $\mathbb{F}_q[\underline{t}_s]$ -module  $A[\underline{t}_s]$  equipped with the  $A[\underline{t}_s]$ -module structure induced by  $\phi$ . Then by [7], Proposition 7.2, the class module  $H_\phi$  is a finitely generated  $\mathbb{F}_q[\underline{t}_s]$ -module of rank  $m-1 = \frac{s-q}{q-1}$ . The importance of the polynomials  $\mathbb{B}_s$  is explained in the following theorem:

**Theorem 2.2** (Anglès-Pellarin-Tavares Ribeiro [7], Theorem 7.7).

*We denote by  $\text{Fitt}_{A[\underline{t}_s]}(H_\phi)$  the Fitting ideal of the torsion  $A[\underline{t}_s]$ -module of finite type  $H_\phi$ . Then*

$$\text{Fitt}_{A[\underline{t}_s]}(H_\phi) = \mathbb{B}_s A[\underline{t}_s].$$

*In particular,*

$$\mathbb{B}_s = \det_{\mathbb{F}_q[\underline{t}_s][Z]} \left( Z \cdot \text{Id} - \phi_\theta \mid_{H_\phi \otimes_{\mathbb{F}_q[\underline{t}_s]} \mathbb{F}_q[\underline{t}_s][Z]} \right) \Big|_{Z=\theta}.$$

A few explicit examples of the polynomials  $\mathbb{B}_s$  are given in [4, 7]. We need to introduce some more notation. Denote by  $S$  the set of all (finite) sequences of non-negative integers  $\underline{m}$ . For any sequence  $\underline{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$  of non-negative integers, we set

$$m_0 = s - (m_1 + \dots + m_d) \in \mathbb{Z}$$

and

$$\sigma_s(\underline{m}) = \sigma_s(m_1, \dots, m_d) = \sum \prod_{u=1}^d \prod_{i \in J_u} t_i^u,$$

where the sum runs through the ordered set partitions  $J_1 \sqcup \dots \sqcup J_d$  of  $\{1, \dots, s\}$  such that  $|J_u| = m_u$ , for  $1 \leq u \leq d$ . In particular,  $\sigma_s(\underline{m}) = 0$  if  $m_1 + \dots + m_d > s$ , which is equivalent to  $m_0 < 0$ . The reader should keep in mind that  $m_i$  may be 0. For example,

$$\sigma_s(0, 0, 1) = \sum_{i=1}^s t_i^3.$$

Here are some more explicit examples that will appear in the explicit formulas  $\mathbb{B}_s$  for small values of  $s$ :

$$\begin{aligned} \sigma_{2q-1}(q) &= \sum_{1 \leq i_1 < \dots < i_q \leq 2q-1} t_{i_1} \dots t_{i_q}, \\ \sigma_{3q-2}(q) &= \sum_{1 \leq i_1 < \dots < i_q \leq 3q-2} \prod_{j=1}^q t_{i_j}, \\ \sigma_{3q-2}(2q-1) &= \sum_{1 \leq i_1 < \dots < i_{2q-1} \leq 3q-2} \prod_{j=1}^{2q-1} t_{i_j}, \\ \sigma_{3q-2}(2q) &= \sum_{1 \leq i_1 < \dots < i_{2q} \leq 3q-2} \prod_{j=1}^{2q} t_{i_j}, \\ \sigma_{3q-2}(q-1, q) &= \sum_{1 \leq i_1 < \dots < i_{q-1} \leq 3q-2} \sum_{\substack{1 \leq k_1 < \dots < k_q \leq 3q-2 \\ k_{j'} \neq i_j}} \prod_{j=1}^{q-1} t_{i_j} \prod_{j'=1}^q t_{k_{j'}}^2. \end{aligned}$$

By [4], Lemma 3.4, we have

- 1)  $\mathbb{B}_q = 1$ .
- 2)  $\mathbb{B}_{2q-1} = \theta - \sigma_{2q-1}(q)$ .
- 3)  $\mathbb{B}_{3q-2} = \theta^2 - \theta[\sigma_{3q-2}(q) + \sigma_{3q-2}(2q-1)] + [\sigma_{3q-2}(q-1, q) + \sigma_{3q-2}(2q)]$ .

## 2.2. A conjecture equivalent to Conjecture 1.2.

In this section, we use the several variable polynomials  $\mathbb{B}_s$  to formulate a conjecture equivalent to Conjecture 1.2, see Conjecture 2.3.

By Equation (2.1), we know that

$$\zeta_A(1, t) = \frac{\tilde{\pi}}{(\theta - t)\omega(t)}.$$



Since  $\tau\omega(t) = (\theta - t)\omega(t)$ , it follows that for  $m \geq 1$ , we have

$$\omega(t)\tau^{-m} \left( \frac{\zeta_A(1, t)}{\tilde{\pi}} \right) = (t - \theta^{\frac{1}{q^{m-1}}}) \cdots (t - \theta^{\frac{1}{q}}).$$

For  $m \geq 1$ , we set

$$b_m^*(t) := (t - \theta^{\frac{1}{q^{m-1}}}) \cdots (t - \theta^{\frac{1}{q}})$$

and

$$B_m^*(\underline{t}_s) := \prod_{i \in \Sigma} b_m^*(t_i).$$

By the previous discussion, we deduce that that Conjecture 1.2 is equivalent to the following Conjecture.

**Conjecture 2.3** ([25], Conjecture 9.7). *Suppose that  $q$  is large enough depending on  $m$ . Then the following formula holds:*

$$(2.3) \quad \mathbb{B}_s = (-1)^{m-1} \sum B_1^*(\underline{t}_{U_1}) \cdots \sum B_m^*(\underline{t}_{U_m})$$

where the sum runs through the set of ordered set partitions of  $\Sigma$

$$U_1 \sqcup \cdots \sqcup U_m = \Sigma$$

such that

$$\frac{|U_1|}{q} + \cdots + \frac{|U_m|}{q^m} = 1.$$

We now present the cases  $m = 1$  and  $m = 2$  to illustrate combinatorial computations which we may encounter. We follow the presentation of Pellarin given in [25], Section 9.1.1 and see that by direct calculations, Conjecture 2.3 holds in these cases.

- (1) For  $m = 1$ , Conjecture 2.3 holds since both sides of Equation (2.3) are equal to 1.
- (2) For  $m = 2$ , we have seen in the previous section that the left-hand side of Equation (2.3) is

$$\mathbb{B}_{2q-1} = \theta - \sigma_{2q-1}(q).$$

Since  $|\Sigma| = 2q - 1$ , we see that the only ordered set partitions appearing on the right-hand side of Equation (2.3) are  $U_1 \sqcup U_2 = \Sigma$  with  $|U_1| = q - 1$  and  $|U_2| = q$ . It follows that the right-hand side of Equation (2.3) is equal to

$$- \sum_{U_2 \subset \Sigma, |U_2|=q} \prod_{j \in U_2} (t_j - \theta^{1/q}).$$

We claim that this expression is equal to  $\theta - \sigma_{2q-1}(q)$ , which confirms Conjecture 2.3 for  $m = 2$ . In fact, it is easy to see that all the terms defined over  $\mathbb{F}_q[\theta^{1/q}]$  but not over  $\mathbb{F}_q[\theta]$  cancel. Further, the terms over  $\mathbb{F}_q[\theta]$  give exactly the polynomial  $\theta - \sigma_{2q-1}(q)$  as desired.

### 2.3. An expansion for the polynomials $\mathbb{B}_s$ .

For a monomial  $\underline{t}_s^m = \prod_{j=1}^s t_j^{m_j}$  with  $m_j \in \mathbb{N}$ , we define its  $q$ -weight by

$$w(\underline{t}_s^m) = \sum_{j=1}^s \frac{1}{q^{m_j+1}}.$$

For a polynomial  $P(\underline{t}_s) \in \mathbb{F}_q[\theta][\underline{t}_s]$ , its weight is defined to be the minimal weight of its monomials.

For an ordered set partition  $\underline{U} = U_1 \sqcup \cdots \sqcup U_m$  of  $\Sigma$ , we define its  $q$ -weight by

$$w(\underline{U}) = \sum_{j=1}^m \frac{|U_j|}{q^j}.$$

We set

$$K^{\text{rad}} = \mathbb{F}_q \left[ \theta^{\frac{1}{q^m}}, m \in \mathbb{N} \right].$$

**Lemma 2.4.** *For  $m \geq 1$ , we can write*

$$(2.4) \quad \zeta_A(1, \underline{t}_s) = \sum a_{\underline{U}} \mathcal{L}_{U_1}^{(-1)} \cdots \mathcal{L}_{U_m}^{(-m)}, \quad \text{with } a_{\underline{U}} \in K^{\text{rad}},$$

where the sum runs through a finite set of ordered set partitions  $\underline{U}$  of  $\Sigma$

$$U_1 \sqcup \cdots \sqcup U_m = \Sigma$$

such that  $w(\underline{U}) \geq w(\mathbb{B}_s)$ .

*Proof.* Recall that for  $m \geq 1$ , we have set

$$b_m^*(t) = (t - \theta^{\frac{1}{q^{m-1}}}) \cdots (t - \theta^{\frac{1}{q}}).$$

In particular, for  $i \in \mathbb{N}$ , we can write

$$t^i = \sum_{m=1}^{i+1} a_m b_m^*(t), \quad \text{with } a_m \in K^{\text{rad}}.$$

It follows that

$$\mathbb{B}_s = \sum_{\underline{m}=(m_1, \dots, m_s) \in (\mathbb{N}^*)^s} a_{\underline{m}} b_{m_1}^*(t_1) \cdots b_{m_s}^*(t_s), \quad \text{with } a_{\underline{m}} \in K^{\text{rad}}.$$

We claim that if  $a_{\underline{m}} \neq 0$ , then  $w(\mathbb{B}_s) \leq \sum_{j=1}^s \frac{1}{q^{m_j}}$ . In fact, if  $a_{\underline{m}} \neq 0$ , then there exists a monomial  $t_1^{i_1} \cdots t_s^{i_s}$  with nonzero coefficient in  $\mathbb{B}_s$  such that  $i_j + 1 \geq m_j$  for all  $1 \leq j \leq s$ . It follows that

$$w(\mathbb{B}_s) \leq w(t_1^{i_1} \cdots t_s^{i_s}) = \sum_{j=1}^s \frac{1}{q^{i_j+1}} \leq \sum_{j=1}^s \frac{1}{q^{m_j}}$$

as required.

Thus we get

$$\begin{aligned} \zeta_A(1, \underline{t}_s) &= \tilde{\pi} \sum_{\underline{m}=(m_1, \dots, m_s) \in (\mathbb{N}^*)^s} a_{\underline{m}} \prod_{i=1}^s \tau^{-m_i} \left( \frac{\zeta_A(1, t_i)}{\tilde{\pi}} \right), \quad \text{with } a_{\underline{m}} \in K^{\text{rad}} \\ &= \tilde{\pi}^{1 - \sum_{i=1}^s 1/q^{m_i}} \sum_{\underline{m}=(m_1, \dots, m_s) \in (\mathbb{N}^*)^s} a_{\underline{m}} \prod_{i=1}^s \tau^{-m_i} (\zeta_A(1, t_i)) \end{aligned}$$

Further, if  $a_{\underline{m}} \neq 0$ , then  $w(\mathbb{B}_s) \leq \sum_{j=1}^s \frac{1}{q^{m_j}}$ . The Proposition follows immediately.  $\square$

We now calculate some coefficients of the above expression using an argument of specialization. Let  $\underline{k} = (k_1, \dots, k_s) \in \mathbb{N}^s$ . We study the specialization of

$$t_i = \theta^{-q^{k_i}}, \quad i = 1, \dots, s.$$

Recall that  $\zeta_A(n) = 0$  if  $n < 0$  and  $n \equiv 0 \pmod{q-1}$ ,  $\zeta_A(0) = 1$  and  $\zeta_A(n) \neq 0$  if  $n > 0$  (see for example [19], chapter 8). Thus we get

$$(2.5) \quad \tau^{-m_i} (\zeta_A(1, t_i)) \Big|_{t_i = \theta^{-q^{k_i}}} = \begin{cases} 0 & \text{if } m_i > k_i, \\ 1 & \text{if } m_i = k_i, \\ \neq 0 & \text{otherwise.} \end{cases}$$

Further, since  $s \equiv 1 \pmod{q-1}$ , we see that

$$(2.6) \quad \zeta_A(1, \underline{t}_s) \Big|_{t_i = \theta^{-q^{k_i}}} = \begin{cases} 0 & \text{if } \sum_{i=1}^s \theta^{-q^{k_i}} > 1, \\ 1 & \text{if } \sum_{i=1}^s \theta^{-q^{k_i}} = 1, \\ \neq 0 & \text{otherwise.} \end{cases}$$

**Lemma 2.5.** *We keep the notation of Lemma 2.4. Then for an ordered set partition  $\underline{U}$  of  $\Sigma$ , we have*

$$a_{\underline{U}} = \begin{cases} 0 & \text{if } w(\underline{U}) > 1, \\ 1 & \text{if } w(\underline{U}) = 1. \end{cases}$$

*Proof.* We divide the proof into two steps.

1) Suppose that there exists a ordered set partition  $\underline{U}$  of  $\Sigma$  such that  $w(\underline{U}) > 1$  and  $a_{\underline{U}} \neq 0$ . We choose a such partition of maximal weight, says  $\underline{V}$ . Thus  $w(\underline{V}) > 1$ . We will specialize to

$$t_i = \theta^{-q^{k_i}}, \quad i = 1, \dots, s$$

where  $k_i$  is the unique integer such that  $i \in V_{k_i}$ . By Equation (2.6), since  $w(\underline{V}) > 1$ , we have

$$\zeta_A(1, \underline{t}_s) \Big|_{t_i = \theta^{-q^{k_i}}} = 0.$$

Thus the specialization value of the left-hand side of Equation (2.4) is 0.

Next, by Equation (2.5), we see that

$$\mathcal{L}_{V_1}^{(-1)} \dots \mathcal{L}_{V_m}^{(-m)} \Big|_{t_i = \theta^{-q^{k_i}}} = 1.$$

For any ordered set partition  $\underline{U}$  of  $\Sigma$  which correspond to  $\underline{m} = (m_1, \dots, m_s)$ , there exists  $1 \leq i \leq s$  such that  $m_i > k_i$  (since otherwise we would have  $w(\underline{V}) < w(\underline{U})$ , which contradicts with the fact that the weight of  $\underline{V}$  is maximal). It follows that

$$\mathcal{L}_{U_1}^{(-1)} \dots \mathcal{L}_{U_m}^{(-m)} \Big|_{t_i = \theta^{-q^{k_i}}} = 0.$$

We obtain that the specialization value of the right-hand side of Equation (2.4) is equal to  $a_{\underline{V}}$ .

Putting altogether, we get  $a_{\underline{V}} = 0$ , which is a contradiction. We conclude that if  $w(\underline{U}) > 1$ , then  $a_{\underline{U}} = 0$ .

2) We have already known that if  $w(\underline{U}) > 1$ , then  $a_{\underline{U}} = 0$ . By a similar argument, we can prove that if  $w(\underline{U}) = 1$ , then  $a_{\underline{U}} = 1$ .  $\square$

As a direct consequence of Lemma 2.4 and Lemma 2.5, we have proved:

**Proposition 2.6.** *Suppose that  $w(\mathbb{B}_s) \geq 1$ . Then Conjecture 1.2 holds, i.e. we have*

$$\zeta_A(1, t_s) = \sum \mathcal{L}_{U_1}^{(-1)} \cdots \mathcal{L}_{U_m}^{(-m)}$$

where the sum runs through the set of ordered set partitions of  $\Sigma$

$$U_1 \sqcup \cdots \sqcup U_m = \Sigma$$

such that

$$\frac{|U_1|}{q} + \cdots + \frac{|U_m|}{q^m} = 1.$$

#### 2.4. An explicit formula for $\mathbb{B}_s$ when $m < q$ .

We recall that for any sequence  $\underline{u} = (u_1, \dots, u_d) \in \mathbb{N}^d$  of non-negative integers, we have set

$$u_0 = s - (u_1 + \cdots + u_d) \in \mathbb{Z}$$

and

$$\sigma_s(\underline{u}) = \sigma_s(u_1, \dots, u_d) = \sum \prod_{j=1}^d \prod_{i \in U_j} t_i^j,$$

where the sum runs through the ordered set partitions of  $\Sigma$  of type  $\underline{u}$

$$U_0 \sqcup \cdots \sqcup U_d = \Sigma \quad \text{with } |U_j| = u_j, \quad j = 0, \dots, d.$$

We are ready to state the second theorem of this paper.

**Theorem 2.7.** *Recall that  $s = m(q-1) + 1$  and*

$$\mathbb{B}_s = \theta^{m-1} + B_1 \theta^{m-2} + \cdots + B_{m-1}, \quad B_i \in \mathbb{F}_q[t_s].$$

Let  $i \in \mathbb{N}$  such that  $1 \leq i \leq q-2$ . Then we have,

(2.7)

$$B_i = \sum_{\underline{n}} \sum_{\underline{x}} B(\underline{n}, \underline{x}) \sigma_s(x_1(q-1) + n_1 - n_2, \dots, x_{d-1}(q-1) + n_{d-1} - n_d, x_d(q-1) + n_d)$$

where

i) the first sum runs through the set of sequences  $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d$  such that  $\sum_{j=1}^d n_j = i$ ,

ii) the second sum runs through the set of sequences  $\underline{x} = (x_0, \dots, x_d) \in (\mathbb{N}^*)^{d+1}$  such that  $\sum_{j=0}^d x_j = m$ ,

iii) the coefficients  $B(\underline{n}, \underline{x})$  are given by

$$B(\underline{n}, \underline{x}) = (-1)^i \prod_{j=1}^d \binom{x_j - 1}{n_j - 1}.$$

In Section 3, we will give a proof of Theorem 2.7 (see Section 3.3) and show that Theorem 2.7 implies Theorem 1.3 (see Section 3.4).

## 3. PROOFS OF MAIN RESULTS

This section is devoted to the proofs of Theorems 1.3 and 2.7.

## 3.1. Preparatory lemmas.

We first collect several combinatorial lemmas which will be used in the sequel.

**Lemma 3.1.** *For any integer  $j \geq 0$  and any sequence  $(m_1, \dots, m_d) \in \mathbb{N}^d$ , we have*

$$\begin{aligned} \sigma_s(j)\sigma_s(m_1, \dots, m_d) &= \sum_{\underline{j}=(j_1, \dots, j_{d+1})} \binom{m_1 + j_1 - j_2}{j_1} \cdots \binom{m_d + j_d - j_{d+1}}{j_d} \times \\ &\quad \times \sigma_s(m_1 + j_1 - j_2, \dots, m_d + j_d - j_{d+1}, j_{d+1}) \end{aligned}$$

where the sum runs through the set of sequences  $\underline{j} = (j_1, \dots, j_{d+1}) \in \mathbb{N}^{d+1}$  such that  $j_1 + \dots + j_{d+1} = j$ .

*Proof.* Left to the reader. □

We recall that for any  $m \in \mathbb{N}$ , the expression

$$\binom{x}{m} := \frac{x(x-1)\cdots(x-m+1)}{m!}$$

represents a polynomial in variable  $x$  with rational coefficients.

**Lemma 3.2.** *Let  $n_1, n_2 \in \mathbb{N}$  be two integers. Then we have the following equality in  $\mathbb{Q}[x]$ :*

$$\sum_{m=0}^{n_1} (-1)^m \binom{x+n_2-m}{n_1-m} \binom{x}{m} = \frac{n_2 \cdots (n_2 - n_1 + 1)}{n_1!}.$$

*Proof.* We set

$$P_{n_1, n_2}(x) = \sum_{m=0}^{n_1} (-1)^m \binom{x+n_2-m}{n_1-m} \binom{x}{m} \in \mathbb{Q}[x].$$

For  $n_1 \in \mathbb{N}^*$  and  $n_2 \in \mathbb{N}$ , we have the following equality of polynomials:

$$P_{n_1, n_2+1}(x) - P_{n_1, n_2}(x) = P_{n_1-1, n_2}(x).$$

Thus we get

$$\begin{aligned} (3.1) \quad &\left( P_{n_1, n_2+1}(x) - \frac{(n_2+1)\cdots(n_2-n_1+2)}{n_1!} \right) - \left( P_{n_1, n_2}(x) - \frac{n_2 \cdots (n_2 - n_1 + 1)}{n_1!} \right) \\ &= P_{n_1-1, n_2}(x) - \frac{n_2 \cdots (n_2 - n_1 + 2)}{(n_1 - 1)!}. \end{aligned}$$

By induction on  $n_1$ , we will show that for all  $n_2 \in \mathbb{N}$ , we have

$$P_{n_1, n_2}(x) = \frac{n_2 \cdots (n_2 - n_1 + 1)}{n_1!}.$$

For  $n_1 = 0$ , the assertion is clear. Suppose that for all  $n_2 \in \mathbb{N}$ , we have

$$P_{n_1-1, n_2}(x) = \frac{n_2 \cdots (n_2 - n_1 + 2)}{(n_1 - 1)!}.$$

We have to show that for all  $n_2 \in \mathbb{N}$ , the following equality holds:

$$P_{n_1, n_2}(x) = \frac{n_2 \cdots (n_2 - n_1 + 1)}{n_1!}.$$

In fact, we will distinguish two cases.

**Case 1:** we suppose first that  $n_2 < n_1$ . We write

$$\begin{aligned} P_{n_1, n_2}(x) &= \sum_{m=0}^{n_1} (-1)^m \binom{x + n_2 - m}{n_1 - m} \binom{x}{m} \\ &= (x + n_2 - n_1 + 1) \cdots x \sum_{m=0}^{n_1} (-1)^m \frac{(x - m + 1)(x - m + 2) \cdots (x - m + n_2)}{(n_1 - m)!m!}. \end{aligned}$$

We consider the following polynomial in variable  $t$ :

$$Q(t) = \frac{(-t + x + 1)(-t + x + 2) \cdots (-t + x + n_2)}{n_1!}.$$

It is a polynomial in  $t$  of degree  $n_2$ , which implies that  $\deg Q < n_1$ . Thus

$$\sum_{m=0}^{n_1} (-1)^m \binom{n_1}{m} Q(m) = 0.$$

and we obtain

$$\sum_{m=0}^{n_1} (-1)^m \frac{(x - m + 1)(x - m + 2) \cdots (x - m + n_2)}{(n_1 - m)!m!} = 0.$$

Hence we get

$$P_{n_1, n_2}(x) = 0 = \frac{n_2 \cdots (n_2 - n_1 + 1)}{n_1!}.$$

**Case 2:** we now work with  $n_2 \geq n_1$ . By Equation (3.1), Step 1 and the induction hypothesis, we deduce that for all  $n_2 \in \mathbb{N}$ ,

$$P_{n_1, n_2}(x) = \frac{n_2 \cdots (n_2 - n_1 + 1)}{n_1!}.$$

The proof is finished.  $\square$

**Lemma 3.3.** *Let  $n_1, n_2 \in \mathbb{N}$  be two integers such that  $n_1 \leq n_2$ . Then we have the following equality in  $\mathbb{Q}[x]$ :*

$$\sum_{m=0}^{n_1} (-1)^m \binom{x - n_2 + m - 1}{m} \binom{x}{n_1 - m} = \binom{n_2}{n_1}.$$

*Proof.* We consider the polynomial in  $\mathbb{Q}[x]$  given by

$$\begin{aligned} P(x) &= \sum_{m=0}^{n_1} (-1)^m \binom{x - n_2 + m - 1}{m} \binom{x}{n_1 - m} \\ &= \sum_{m=0}^{n_1} \frac{(n_2 - x) \cdots (n_2 - x - m + 1)}{m_1!} \frac{x \cdots (x - (n_1 - m) + 1)}{(n_1 - m)!}. \end{aligned}$$

Since  $n_1 \leq n_2$ , the degree of  $P(x)$  is bounded by  $n_2$ . We know that for all integer  $x$  with  $0 \leq x \leq n_2$ , we have the equality:

$$P(x) = \sum_{m=0}^{n_1} \binom{n_2-x}{m} \binom{x}{n_1-m} = \binom{n_2}{n_1}.$$

It follows that  $P(x)$  is the constant polynomial  $\binom{n_2}{n_1}$ .  $\square$

### 3.2. Basic sums.

Following [4], Section 5.2, we recall some facts of basic sums. For a sequence  $\underline{k} = (k_0, \dots, k_{d-1}) \in \mathbb{N}^d$ , we set

$$\begin{aligned} l(\underline{k}) &= d, \\ w(\underline{k}) &= dk_0 + (d-1)k_1 + \dots + k_{d-1}, \\ |\underline{k}| &= k_0 + \dots + k_{d-1}, \\ C_{\underline{k}} &= (-1)^{|\underline{k}|} \frac{|\underline{k}|!}{k_0! \dots k_{d-1}!} \in \mathbb{F}_p. \end{aligned}$$

Letting  $a \in A_{+,d}$ , we write  $a = a_0 + a_1\theta + \dots + a_{d-1}\theta^{d-1} + \theta^d$ . Thus we get

$$\frac{1}{a} = \frac{1}{\theta^d} \sum_{\underline{k} \in \mathbb{N}^d} C_{\underline{k}} a^{\underline{k}} \frac{1}{\theta^{w(\underline{k})}}$$

where we put  $a^{\underline{k}} = \prod_{i=0}^{d-1} a_i^{k_i}$ .

It follows that

$$\begin{aligned} l_{d,s} &= \sum_{a \in A_{+,d}} \frac{a(t_1) \dots a(t_s)}{a} \\ &= \frac{1}{\theta^d} \sum_{\underline{k} \in \mathbb{N}^d} C_{\underline{k}} \frac{1}{\theta^{w(\underline{k})}} \sum_{a \in A_{+,d}} a^{\underline{k}} a(t_1) \dots a(t_s) \\ &= \frac{1}{\theta^d} \sum_{\underline{k} \in \mathbb{N}^d} C_{\underline{k}} \frac{1}{\theta^{w(\underline{k})}} \sum_{a \in A_{+,d}} \sum_{\underline{m} \in \mathbb{N}^{d+1}, |\underline{m}|=s} a^{\underline{k}} a^{\underline{m}} \sigma_s(m_1, \dots, m_d) \\ &= \frac{1}{\theta^d} \sum_{\underline{k} \in \mathbb{N}^d} C_{\underline{k}} \frac{1}{\theta^{w(\underline{k})}} \sum_{\underline{m} \in \mathbb{N}^{d+1}, |\underline{m}|=s} \sigma_s(m_1, \dots, m_d) \sum_{a \in A_{+,d}} a^{\underline{k}+\underline{m}}. \end{aligned}$$

We say that  $\underline{m}$  is  $\underline{k}$ -admissible if  $(k_0 + m_0, \dots, k_{d-1} + m_{d-1}) \in ((q-1)\mathbb{N}^*)^d$ . We see that if  $\underline{m}$  is  $\underline{k}$ -admissible, then the sum  $\sum_{a \in A_{+,d}} a^{\underline{k}+\underline{m}}$  is equal to  $(-1)^d$ . Otherwise, this sum is equal to 0.

Given a sequence  $\underline{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$  as above, we define another sequence  $\underline{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$  as follows:

$$\begin{aligned} n_1 &= k_0 + 1, \\ n_2 &= k_0 + k_1 + 1, \\ &\dots \\ n_d &= k_0 + \dots + k_{d-1} + 1. \end{aligned}$$

The sequence  $\underline{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$  satisfies the following properties:

- i)  $1 \leq n_1 \leq \dots \leq n_d$ ,

$$ii) \sum_{j=1}^d n_j = i.$$

We observe that the sequence  $\underline{k}$  is completely determined by the associated sequence  $\underline{n}$ .

Let  $\underline{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$  be such a sequence and let  $\underline{k}$  be the associated sequence. We set

$$\mathfrak{B}_{\underline{n},s} := \{\underline{m} = (m_0, \dots, m_d) \in \mathbb{N}^{d+1} : \underline{m} \text{ is } \underline{k}\text{-admissible et } |\underline{m}| = s\}.$$

Then

$$l_{d,s} = \frac{(-1)^d}{\theta^d} \sum_{\underline{k} \in \mathbb{N}^d} C_{\underline{k}} \frac{1}{\theta^{w(\underline{k})}} \sum_{\underline{m} \in \mathfrak{B}_{\underline{n},s}} \sigma_s(m_1, \dots, m_d).$$

Let  $\underline{m} = (m_0, \dots, m_d) \in \mathfrak{B}_{\underline{n},s}$ . By definition, there exist strictly positive integers  $x_0, \dots, x_{d-1} \in \mathbb{N}^*$  such that

$$\begin{aligned} m_0 &= x_0(q-1) - n_1 + 1, \\ m_1 &= x_1(q-1) + n_1 - n_2, \\ &\dots \\ m_{d-1} &= x_{d-1}(q-1) + n_{d-1} - n_d. \end{aligned}$$

Thus we get

$$m_d = s - \sum_{j=0}^{d-1} m_j = x_d(q-1) + n_d$$

with  $x_d = m - \sum_{j=0}^{d-1} x_j$ . Since  $\underline{m} \in \mathbb{N}^{d+1}$ , we deduce the following lemma:

**Lemma 3.4.** *The set  $\mathfrak{B}_{\underline{n},s}$  consists of the elements  $\underline{m} = (m_0, \dots, m_d) \in \mathbb{N}^{d+1}$  of the form*

$$\begin{aligned} m_0 &= x_0(q-1) - n_1 + 1, \\ m_1 &= x_1(q-1) + n_1 - n_2, \\ &\dots \\ m_{d-1} &= x_{d-1}(q-1) + n_{d-1} - n_d, \\ m_d &= x_d(q-1) + n_d \end{aligned}$$

where  $x_0, \dots, x_d$  are integers such that

$$\begin{aligned} i) \quad &x_0 > 0, \dots, x_{d-1} > 0, \\ ii) \quad &\sum_{j=0}^d x_j = m. \end{aligned}$$

To summarize, we have proved the following proposition.

**Proposition 3.5.** *We have*

$$\zeta_A(1, t_s) = \sum_{i \geq 0} \alpha_{i,s} \theta^{-i}$$

with

$$\alpha_{i,s} = \sum_{\underline{n}} \tilde{C}(\underline{n}) \sum_{\underline{m} \in \mathfrak{B}_{\underline{n},s}} \sigma_s(\underline{m})$$

where



i) the first sum runs through the set of sequences of integers  $\underline{n} = (n_1, \dots, n_d)$  verifying  $1 \leq n_1 \leq \dots \leq n_d$  and  $\sum_{j=1}^d n_j = i$ ,

ii) the coefficient  $\tilde{C}(\underline{n})$  equals to

$$(-1)^{d+n_d-1} \frac{(n_d - 1)!}{(n_1 - 1)!(n_2 - n_1)! \dots (n_d - n_{d-1})!}.$$

### 3.3. Proof of Theorem 2.7.

In this section, we give a proof of Theorem 2.7 which gives an explicit formula for the polynomial  $\mathbb{B}_s$  under the hypothesis  $m < q$ .

We now suppose that  $m < q$ . Under this hypothesis, Lemma 3.4 implies:

**Lemma 3.6.** *Let  $i$  be an integer such that  $1 \leq i \leq m-1$  and  $\underline{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$  with  $1 \leq n_1 \leq \dots \leq n_d$  and  $\sum_{j=1}^d n_j = i$ . Then the set  $\mathfrak{B}_{\underline{n}, s}$  consists of the sequences  $\underline{m} = (m_0, \dots, m_d) \in \mathbb{N}^{d+1}$  of the form*

$$\begin{aligned} m_0 &= x_0(q-1) - n_1 + 1, \\ m_1 &= x_1(q-1) + n_1 - n_2, \\ &\dots \\ m_{d-1} &= x_{d-1}(q-1) + n_{d-1} - n_d, \\ m_d &= x_d(q-1) + n_d \end{aligned}$$

where  $x_0, \dots, x_d$  are integers such that

- i)  $x_0 > 0, \dots, x_{d-1} > 0$  et  $x_d \geq 0$ ,
- ii)  $\sum_{j=0}^d x_j = m$ .

To continue, we introduce the following sets:

$$\begin{aligned} \mathfrak{N}_i &:= \{\underline{n} = (n_1, \dots, n_d) \in \mathbb{N}^d : 1 \leq n_1 \leq \dots \leq n_d, \sum_{j=1}^d n_j = i\}, \\ \mathfrak{X} &:= \{\underline{x} = (x_0, \dots, x_d) \in (\mathbb{N}^*)^d \times \mathbb{N} : \sum_{j=0}^d x_j = m\}. \end{aligned}$$

As an immediate consequence of Lemma 3.6, we obtain:

**Corollary 3.7.** *We continue with the notation of Proposition 3.5. Let  $i$  be an integer such that  $1 \leq i \leq m-1$ . Then*

$$(3.2) \quad \alpha_{i, s} = \sum_{\underline{n} \in \mathfrak{N}_i} \sum_{\underline{x} \in \mathfrak{X}} \tilde{C}(\underline{n}) \sigma_s(x_1(q-1) + n_1 - n_2, \dots, x_d(q-1) + n_d)$$

where the coefficient  $\tilde{C}(\underline{n})$  equals to

$$\tilde{C}(\underline{n}) = (-1)^{d+n_d-1} \frac{(n_d - 1)!}{(n_1 - 1)!(n_2 - n_1)! \dots (n_d - n_{d-1})!}.$$

We recall that  $\mathbb{B}_s$  is a monic polynomial in  $\theta$  of degree  $m-1$ . To calculate  $\mathbb{B}_s$ , it suffices to look at both sides of Equation (2.2) modulo  $\theta^{-m}$ . Since  $q > m$ , we have

$$(3.3) \quad \prod_{i \geq 1} \left(1 - \frac{\theta}{\theta q^i}\right) \equiv 1 \pmod{\theta^{-m}}.$$

and

$$\begin{aligned}
\prod_{j=1}^s \prod_{i \geq 0} \left(1 - \frac{t_j}{\theta^{q^i}}\right)^{-1} &\equiv \prod_{j=1}^s \left(1 - \frac{t_j}{\theta}\right)^{-1} && (\text{mod } \theta^{-m}) \\
(3.4) \qquad \qquad \qquad &\equiv (1 - \sigma_s(1)\theta^{-1} + \dots + (-1)^r \sigma_s(r)\theta^{-r})^{-1} && (\text{mod } \theta^{-m}).
\end{aligned}$$

By Equations (3.2), (3.4) and (3.3), we deduce the following equality:

$$(1 + B_1\theta^{-1} + \dots + B_{m-1}\theta^{-(m-1)})(1 - \sigma_s(1)\theta^{-1} + \dots + (-1)^r \sigma_s(m-1)\theta^{-(m-1)}) = \sum_{i=0}^{m-1} \alpha_{i,s} \theta^{-i}.$$

In other words, if we set  $B_0 := 1$ , then for  $1 \leq i \leq m-1$ ,

$$(3.5) \qquad B_i - \sigma_s(1)B_{i-1} + \dots + (-1)^i \sigma_s(i)B_0 = \alpha_{i,s}.$$

**Proposition 3.8.** *Let  $i$  be an integer such that  $1 \leq i \leq m-1$ . Then*

$$(3.6) \qquad B_i = \sum_{\underline{n} \in \mathfrak{N}_i} \sum_{\underline{x} \in \mathfrak{X}} C(\underline{n}, \underline{x}) \sigma_s(x_1(q-1) + n_1 - n_2, \dots, x_d(q-1) + n_d)$$

where the coefficient  $C(\underline{n}, \underline{x})$  is equal to

$$C(\underline{n}, \underline{x}) = (-1)^d \prod_{j=1}^d \binom{x_j(q-1) + n_j - 1}{n_j - 1}.$$

*Proof.* Let  $i$  be an integer with  $1 \leq i \leq m-1$ . It suffices to prove the equality (3.5) where  $B_i$  are given by Equation (3.6) and  $\alpha_{i,s}$  given by Equation (3.2).

We set

$$S := B_i - \sigma_s(1)B_{i-1} + \dots + (-1)^i \sigma_s(i)B_0.$$

By Lemma 3.1, we have

$$\begin{aligned}
S &= B_i - \sigma_s(1)B_{i-1} + \dots + (-1)^i \sigma_s(i)B_0 \\
&= \sum_{j=0}^i (-1)^j \sigma_s(j) \sum_{\underline{m}=(m_1, \dots, m_d) \in \mathfrak{N}_{i-j}} \sum_{\underline{x}=(x_0, \dots, x_d) \in \mathfrak{X}} C(\underline{m}, \underline{x}) \times \\
&\quad \times \sigma_s(x_1(q-1) + m_1 - m_2, \dots, x_d(q-1) + m_d) \\
&= \sum_{j=0}^i (-1)^j \sum_{\underline{m} \in \mathfrak{N}_{i-j}} \sum_{\underline{x} \in \mathfrak{X}} C(\underline{m}, \underline{x}) \sigma_s(j) \sigma_s(x_1(q-1) + m_1 - m_2, \dots, x_d(q-1) + m_d) \\
&= \sum_{j=0}^i (-1)^j \sum_{\underline{m} \in \mathfrak{N}_{i-j}} \sum_{\underline{x} \in \mathfrak{X}} C(\underline{m}, \underline{x}) \times \\
&\quad \times \sum_{j_1 + \dots + j_{d+1} = j} \binom{x_1(q-1) + m_1 - m_2 + j_1 - j_2}{j_1} \dots \binom{x_d(q-1) + m_d + j_d - j_{d+1}}{j_d} \\
&\quad \times \sigma_s(x_1(q-1) + m_1 - m_2 + j_1 - j_2, \dots, x_d(q-1) + m_d + j_d - j_{d+1}, j_{d+1}) \\
&= \sum_{\underline{n}=(n_1, \dots, n_{d+1})} \sum_{\underline{x}=(x_1, \dots, x_d)} \sigma_s(x_1(q-1) + n_1 - n_2, \dots, x_d(q-1) + n_d, n_{d+1}) \\
&\quad \times \sum_{\underline{m}=(m_1, \dots, m_d) \leq (n_1, \dots, n_d)} (-1)^{d+(n_1-m_1)+\dots+(n_d-m_d)+n_{d+1}} \\
&\quad \times \binom{x_1(q-1) + n_1 - n_2}{n_1 - m_1} \dots \binom{x_d(q-1) + n_d - n_{d+1}}{n_d - m_d} \\
&\quad \times \binom{x_1(q-1) + m_1 - 1}{m_1 - 1} \dots \binom{x_d(q-1) + m_d - 1}{m_d - 1}.
\end{aligned}$$

Here in the last line,

i) the first sum runs through the set of sequences  $\underline{n} = (n_1, \dots, n_{d+1}) \in (\mathbb{N}^*)^d \times \mathbb{N}$  such that  $n_1 + \dots + n_{d+1} = i$ ,

ii) the second sum runs through the set of sequences  $\underline{x} = (x_0, \dots, x_d) \in (\mathbb{N}^*)^{d+1}$  such that  $\sum_{j=0}^d x_j = m$ ,

iii) the third sum runs through the set of sequences  $\underline{m} = (m_1, \dots, m_d) \in (\mathbb{N}^*)^d$  such that  $m_j \leq n_j$  for all  $1 \leq j \leq d$ .

To finish the proof of Proposition 3.8, it suffices to prove the following proposition:

**Proposition 3.9.** *Let  $1 \leq i \leq m-1$  be an integer,  $\underline{n} = (n_1, \dots, n_{d+1}) \in (\mathbb{N}^*)^d \times \mathbb{N}$  be a sequence such that  $n_1 + \dots + n_{d+1} = i$  and  $\underline{x} = (x_1, \dots, x_d) \in \mathbb{Z}^d$  be a sequence of integers. We set*

$$\begin{aligned}
S_0 := & \sum_{\underline{m}=(m_1, \dots, m_d)} (-1)^{d+(n_1-m_1)+\dots+(n_d-m_d)+n_{d+1}} \binom{x_1(q-1) + n_1 - n_2}{n_1 - m_1} \dots \binom{x_d(q-1) + n_d - n_{d+1}}{n_d - m_d} \times \\
& \times \binom{x_1(q-1) + m_1 - 1}{m_1 - 1} \dots \binom{x_d(q-1) + m_d - 1}{m_d - 1}
\end{aligned}$$

where the sum runs through the set of sequences  $\underline{m} = (m_1, \dots, m_d) \in (\mathbb{N}^*)^d$  such that  $m_j \leq n_j$  for all  $1 \leq j \leq d$ .

Then we have

i) If  $n_{d+1} > 0$ , then

$$S_0 = \tilde{C}(n_1, \dots, n_{d+1}) = (-1)^{d+n_{d+1}} \frac{(n_{d+1} - 1)!}{(n_1 - 1)!(n_2 - n_1)! \dots (n_{d+1} - n_d)!}.$$

ii) If  $n_{d+1} = 0$ , then

$$S_0 = \tilde{C}(n_1, \dots, n_d) = (-1)^{d+n_d-1} \frac{(n_d - 1)!}{(n_1 - 1)!(n_2 - n_1)! \dots (n_d - n_{d-1})!}.$$

*Proof.* In fact, we write

$$S_0 = (-1)^{d+n_{d+1}} \prod_{j=1}^d \sum_{1 \leq m_j \leq n_j} (-1)^{n_j - m_j} \binom{x_j(q-1) + n_j - n_{j+1}}{n_j - m_j} \binom{x_j(q-1) + m_j - 1}{m_j - 1}.$$

For  $1 \leq j \leq d-1$ , Lemma 3.2 implies

$$\sum_{1 \leq m_j \leq n_j} (-1)^{n_j - m_j} \binom{x_j(q-1) + n_j - n_{j+1}}{n_j - m_j} \binom{x_j(q-1) + m_j - 1}{m_j - 1} = \binom{n_{j+1} - 1}{n_j - 1}.$$

We now study the last factor (e.g. when  $j = d$ ).

• If  $n_{d+1} > 0$ , then Lemma 3.2 gives

$$\sum_{1 \leq m_d \leq n_d} (-1)^{n_d - m_d} \binom{x_d(q-1) + n_d - n_{d+1}}{n_d - m_d} \binom{x_d(q-1) + m_d - 1}{m_d - 1} = \binom{n_{d+1} - 1}{n_d - 1}.$$

• If  $n_{d+1} = 0$ , then Lemma 3.3 below implies

$$\sum_{1 \leq m_d \leq n_d} (-1)^{n_d - m_d} \binom{x_d(q-1) + n_d - n_{d+1}}{n_d - m_d} \binom{x_d(q-1) + m_d - 1}{m_d - 1} = (-1)^{n_d - 1}.$$

To summarize, we obtain

i) If  $n_{d+1} > 0$ , then

$$S_0 = (-1)^{d+n_{d+1}} \frac{(n_{d+1} - 1)!}{(n_1 - 1)!(n_2 - n_1)! \dots (n_{d+1} - n_d)!} = \tilde{C}(n_1, \dots, n_{d+1}).$$

ii) If  $n_{d+1} = 0$ , then

$$S_0 = (-1)^{d+n_d-1} \frac{(n_d - 1)!}{(n_1 - 1)!(n_2 - n_1)! \dots (n_d - n_{d-1})!} = \tilde{C}(n_1, \dots, n_d).$$

The proof is complete.  $\square$

The proof of Proposition 3.8 is finished.  $\square$

**Lemma 3.10.** *We keep the notation of Proposition 3.8. Then we have the following equality:*

$$C(\underline{n}, \underline{x}) = (-1)^i \prod_{j=1}^d \binom{x_j - 1}{n_j - 1}.$$

In particular, if  $x_j < n_j$  for some  $1 \leq j \leq d$ , then  $C(\underline{n}, \underline{x}) = 0$ .

*Proof.* We observe that for  $1 \leq n < q$ ,

$$\frac{(x+q)\dots(x+q-n+1)}{n!} = \frac{x\dots(x-n+1)}{n!}$$

It follows that

$$\begin{aligned} C(\underline{n}, \underline{x}) &= (-1)^d \prod_{j=1}^d \binom{x_j(q-1) + n_j - 1}{n_j - 1} \\ &= (-1)^d \prod_{j=1}^d \frac{(-x_j + n_j - 1) \dots (x_j - 1)}{(n_j - 1)!} \\ &= (-1)^d \prod_{j=1}^d (-1)^{n_j - 1} \binom{x_j - 1}{n_j - 1} \\ &= (-1)^{n_1 + \dots + n_d} \prod_{j=1}^d \binom{x_j - 1}{n_j - 1} \\ &= (-1)^i \prod_{j=1}^d \binom{x_j - 1}{n_j - 1}. \end{aligned}$$

The proof is finished.  $\square$

As an immediate consequence of Proposition 3.8 and the previous lemma, we obtain Theorem 2.7.

### 3.4. Proof of Theorem 1.3.

In this section, we will deduce Theorem 1.3 from Theorem 2.7.

We see that Theorem 2.7 implies immediately the following corollary:

**Corollary 3.11.** *Suppose that  $m < q$ . Then  $w(\mathbb{B}_s) \geq 1$ .*

*Proof.* Since  $m < q$ , the polynomial  $\mathbb{B}_s$  is completely determined by Theorem 2.7. In particular, the monomials appearing in  $\mathbb{B}_s$  are of the form

$$\sigma_s(x_1(q-1) + n_1 - n_2, \dots, x_d(q-1) + n_d)$$

with

- i)  $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{N}^*)^d$ ,
- ii)  $\underline{x} = (x_0, \dots, x_d) \in (\mathbb{N}^*)^{d+1}$  such that  $\sum_{j=0}^d x_j = m$ ,
- iii)  $x_j \geq n_j$  for  $j = 1, \dots, d$ .

It implies that the weights of monomials in  $\sigma_s(x_1(q-1) + n_1 - n_2, \dots, x_d(q-1) + n_d)$  are greater or equal to

$$\frac{q - n_1}{q} + \frac{n_1 q - n_2}{q^2} + \dots + \frac{n_d q}{q^{d+1}} = 1.$$

Since the weight of  $\mathbb{B}_s$  is defined to be the minimal weight of its monomials, we deduce that  $w(\mathbb{B}_s) \geq 1$  as required.  $\square$

Thus Theorem 1.3 follows immediately from Corollary 3.11 and Proposition 2.6.

## 4. CONCLUDING REMARKS

We have succeeded in proving Conjecture 1.2 and thus get a partial answer to Conjecture 1.1. We expect that Conjecture 1.1 always holds. Thus it is tempting to ask whether Theorem 2.7 holds in full generality, that means we can remove the restriction  $1 \leq i \leq q - 2$ .

**Conjecture 4.1.** *Recall that*

$$\mathbb{B}_s = \theta^{m-1} + B_1\theta^{m-2} + \dots + B_{m-1}, \quad B_i \in \mathbb{F}_q[t_s].$$

Then for  $1 \leq i \leq m - 1$ , we have

$$B_i = \sum_{\underline{n}} \sum_{\underline{x}} B(\underline{n}, \underline{x}) \sigma_s(x_1(q-1) + n_1 - n_2, \dots, x_d(q-1) + n_d)$$

where

i) the first sum runs through the set of sequences  $\underline{n} = (n_1, \dots, n_d)$  such that  $n_j \in \mathbb{N}^*$  and  $\sum_{j=1}^d n_j = i$ ,

ii) the second sum runs through the set of sequences  $\underline{x} = (x_0, \dots, x_d) \in (\mathbb{N}^*)^{d+1}$  such that  $\sum_{j=0}^d x_j = m$ ,

iii) the coefficients  $B(\underline{n}, \underline{x})$  are given by

$$B(\underline{n}, \underline{x}) = (-1)^i \prod_{j=1}^d \binom{x_j - 1}{n_j - 1}.$$

As expected, we prove:

**Proposition 4.2.** *Conjecture 4.1 implies Conjecture 1.1.*

*Proof.* Suppose that Conjecture 4.1 holds. From the explicit formula for  $\mathbb{B}_s$ , by similar arguments to the proof of Corollary 3.11, we see that  $w(\mathbb{B}_s) \geq 1$ . Combined with Proposition 2.6, it implies immediately Conjecture 1.1.  $\square$

**Remark 4.3.**

- 1) For  $m = 1, 2, 3$ , we have explicit formulas for  $\mathbb{B}_s$  (see Section 2.1) and see easily that Conjecture 4.1 holds for these small values. They provide the first evidence to support our conjecture.
- 2) We should mention that the first author is currently investigating the above Conjecture in his PhD thesis.

## REFERENCES

- [1] G. Anderson and D. Thakur. Tensor powers of the Carlitz module and zeta values. *Ann. of Math. (2)*, 132(1):159–191, 1990.
- [2] B. Anglès, T. Ngo Dac, and F. Tavares Ribeiro. Special functions and twisted  $L$ -series. *J. Théor. Nombres Bordeaux*, 29:931–961, 2017.
- [3] B. Anglès, T. Ngo Dac, and F. Tavares Ribeiro. Stark units in positive characteristic. *Proc. Lond. Math. Soc. (3)*, 115(4):763–812, 2017.
- [4] B. Anglès, T. Ngo Dac, and F. Tavares Ribeiro. Exceptional zeros of  $L$ -series and Bernoulli-Carlitz numbers. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 19(3):981–1024, 2019.
- [5] B. Anglès and F. Pellarin. Functional identities for  $L$ -series values in positive characteristic. *J. Number Theory*, 142:223–251, 2014.

- [6] B. Anglès and F. Pellarin. Universal Gauss-Thakur sums and  $L$ -series. *Invent. Math.*, 200(2):653–669, 2015.
- [7] B. Anglès, F. Pellarin, and F. Tavares Ribeiro. Arithmetic of positive characteristic  $L$ -series values in Tate algebras. With an appendix by F. Demeslay. *Compos. Math.*, 152(1):1–61, 2016.
- [8] B. Anglès, F. Pellarin, and F. Tavares Ribeiro. Anderson-Stark units for  $\mathbb{F}_q[\theta]$ . *Trans. Amer. Math. Soc.*, 370(3):1603–1627, 2018.
- [9] B. Anglès and F. Tavares Ribeiro. Arithmetic of function fields units. *Math. Ann.*, 367(1-2):501–579, 2017.
- [10] L. Carlitz. On certain functions connected with polynomials in Galois field. *Duke Math. J.*, 1(2):137–168, 1935.
- [11] F. Demeslay. A class formula for  $L$ -series in positive characteristic. *arXiv:1412.3704*, 2014.
- [12] F. Demeslay. *Formules de classes en caractéristique positive*. PhD thesis, Université de Caen Normandie, October 2015.
- [13] E.-U. Gekeler. On the coefficients of Drinfeld modular forms. *Invent. Math.*, 93:667–700, 1988.
- [14] O. Gezmis. Taelman  $L$ -values for Drinfeld modules over Tate algebras. *Res. Math. Sci.*, 6(1):Paper No. 18, 25pp, 2019.
- [15] O. Gezmis and M. Papanikolas. The de Rham isomorphism for Drinfeld modules over Tate algebras. *J. Algebra*, 525:454–496, 2019.
- [16] D. Goss. Modular forms for  $\mathbb{F}_r[T]$ . *J. Reine Angew. Math.*, 317:16–39, 1980.
- [17] D. Goss.  $\pi$ -adic Eisenstein series for function fields. *Compositio Math.*, 41:3–38, 1980.
- [18] D. Goss. The algebraist’s upper half-plane. *Bull. Amer. Math. Soc. (N.S.)*, 2:391–415, 1980.
- [19] D. Goss. *Basic Structures of function field arithmetic*, volume 35 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1996.
- [20] D. Goss, B. Anglès, T. Ngo Dac, F. Pellarin, and F. Tavares Ribeiro. The digit principle and derivatives of certain  $L$ -series. *Publ. Math. Besançon Algèbre Théorie Nr.*, 2019(1):81–102, 2019.
- [21] N. Green. Special zeta values using tensor powers of Drinfeld modules. *to appear, Math. Res. Letters*, *arXiv:1706.06048*, 2017.
- [22] N. Green. Tensor powers of rank 1 Drinfeld modules and periods. *to appear, J. Number Theory*, *arXiv:1706.03854*, 2017.
- [23] N. Green and M. Papanikolas. Special  $L$ -values and shtuka functions for Drinfeld modules on elliptic curves. *Res. Math. Sci.*, 5(1):Paper No. 4, 47pp, 2018.
- [24] F. Pellarin. Values of certain  $L$ -series in positive characteristic. *Ann. of Math.*, 176(3):2055–2093, 2012.
- [25] F. Pellarin. On the behaviour at the cusps of Drinfeld modular forms. *arXiv:1910.12743*, 2019.
- [26] F. Pellarin and R. Perkins. On twisted  $A$ -harmonic sums and Carlitz finite zeta values. *J. Number Theory*, *forthcoming*, 2015.
- [27] F. Pellarin and R. Perkins. Vectorial Drinfeld modular forms over Tate algebras. *Int. J. Number Theory*, 14(6):1729–1783, 2018.
- [28] L. Taelman. A Dirichlet unit theorem for Drinfeld modules. *Math. Ann.*, 348(4):899–907, 2010.
- [29] L. Taelman. Special  $L$ -values of Drinfeld modules. *Ann. of Math.*, 175(1):369–391, 2012.
- [30] D. Thakur. Multizeta values for function fields: a survey. *J. Théor. Nombres Bordeaux*, 29(3):997–1023, 2017.

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