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# Asymptotic Expansions of Vibrations with Small Unilateral Contact

Stéphane Junca and Bernard Rousselet

**Abstract** We study some spring mass models for a structure having a small unilateral contact with a small parameter  $\varepsilon$ . We valid an asymptotic expansion with the method of strained coordinates with new tools to handle such defects, including a non negligible cumulative effect over a long time:  $T_\varepsilon \sim \varepsilon^{-1}$  as usual; or, for a new critical case, we can only expect:  $T_\varepsilon \sim \varepsilon^{-1/2}$ .

**Key words:** : nonlinear vibrations, method of strained coordinates, piecewise linear, approximate nonlinear normal mode.

## 1 Introduction

For spring mass models, the presence of a small piecewise linear rigidity can model a small defect which implies unilateral reactions of the structure. For nondestructive testing we study a such singular nonlinear effect for large time by asymptotic expansion of the vibrations. New features and comparisons with classical cases of smooth perturbations are given, for instance for the Duffing equations:  $\ddot{u} + u + \varepsilon u^3 = 0$ . Indeed, piecewise non linearity is singular, lipschitz but not differentiable. We give some new results to validate such asymptotic expansions. Furthermore, these tools are also valid for a more general piecewise non linearity.

For short time, a linearization procedure is enough to compute a good approximation. But for large time, nonlinear cumulative effects drastically alter

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the nature of the solution. We will consider the classical method of strained coordinates to compute asymptotic expansions. The idea goes further back to Stokes, who in 1847 calculated periodic solutions for a weakly nonlinear wave propagation problem. Subsequent authors have generally referred to this as the method of Poincaré or the Lindstedt's method. It is a simple and efficient method which gives us approximate nonlinear normal modes with 1 or more degrees of freedom.

In section 2 we present the method on an explicit case with lipschitz force. We focus on an equation with one degree of freedom with expansions valid for time of order  $\varepsilon^{-1}$  or, more surprisingly,  $\varepsilon^{-1/2}$ . Section 3 contains a tool to expand  $(u + \varepsilon v)_+$  and some accurate estimate for the remainder. This is a new key point to validate the method of strained coordinates with unilateral contact. In Section 4, we extend previous results for systems with  $N$  degrees of freedom, first, with the same accuracy for approximate nonlinear normal modes, second, with less accuracy with all modes. Section 5 is an appendix containing some technical proofs and results.

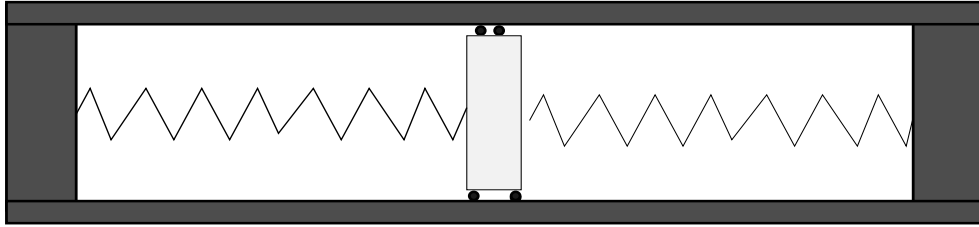
## 2 One degree of freedom

### 2.1 *Explicit pulsation*

We replace in the Duffing equation  $u^3$  by the piecewise linear term  $u_+ = \max(0, u)$ .

$$\ddot{u} + \omega_0^2 u + \varepsilon u_+ = 0, \quad (1)$$

where  $\omega_0$  a positive constant. This case has got a conserved energy  $E$ :  $\dot{E} = 0$ ,



**Fig. 1** Two springs, one on the right with an unilateral contact.

where  $2E = \dot{u}^2 + \omega_0^2 u^2 + \varepsilon(u_+)^2$ . Therefore, the level sets of  $E(u, \dot{u})$  will be made of two half ellipses. Indeed, for  $u < 0$  the level set is an half ellipse, and for  $u > 0$  is another half ellipse. Any solution  $u(t)$  is confined to a closed level curve of  $E(u, \dot{u})$  and is necessarily a periodic functions of  $t$ .

More precisely, a non trivial solution ( $E > 0$ ) is on the half ellipse:  $\dot{u}^2 + \omega_0^2 u = 2E$ , in the phase plane during the time  $T_C = \pi/\omega_0$ , and on the half ellipse  $\dot{u}^2 + (\omega_0^2 + \varepsilon)u = 2E$  during the time  $T_E = \pi/\sqrt{\omega_0^2 + \varepsilon}$ . The period  $P(\varepsilon)$  is then

$P(\varepsilon) = (1 + (1 + \varepsilon/\omega_0^2)^{-1/2})\pi/\omega_0$ , and the exact pulsation is:

$$\omega(\varepsilon) = 2\omega_0(1 + (1 + \varepsilon/\omega_0^2)^{-1/2})^{-1} = \omega_0 + \frac{\varepsilon}{(4\omega_0)} - \frac{\varepsilon^2}{(8\omega_0^3)} + \mathcal{O}(\varepsilon^3). \quad (2)$$

Let us compare with the pulsation for Duffing equation which depends on the amplitude  $a_0$  of the solution:  $\omega_D(\varepsilon) = \omega_0 + \frac{3}{8\omega_0^2}a_0^2\varepsilon - \frac{15}{256\omega_0^4}a_0^4\varepsilon^2 + \mathcal{O}(\varepsilon^3)$ .

## 2.2 The method of strained coordinates

Now, we compute, with the method of strained coordinates,  $\omega_\varepsilon$ , an *approximation* of the exact pulsation  $\omega(\varepsilon)$ . We expose completely this case to use the same method further when we will not have such explicit pulsation.

Let us define the new time  $s = \omega_\varepsilon t$  and the following notations:

$$\begin{aligned} \omega_\varepsilon &= \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2, & \omega_\varepsilon^2 &= \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \mathcal{O}(\varepsilon^3) & (3) \\ \alpha_0 &= \omega_0^2, \quad \alpha_1 = 2\omega_0\omega_1, & \alpha_2 &= \omega_1^2 + 2\omega_0\omega_2. & (4) \end{aligned}$$

The unknowns are  $\omega_1, \omega_2$  or  $\alpha_1, \alpha_2$ . Replacing the solution of (1) by the following ansatz with the following initial data to simplify the exposition:

$$\begin{aligned} u_\varepsilon(t) &= v_\varepsilon(\omega_\varepsilon t) + \varepsilon^2 r_\varepsilon(\omega_\varepsilon t), \\ v_\varepsilon(s) &= v_0(s) + \varepsilon v_1(s), \quad \text{where } s = \omega_\varepsilon t, \\ u_\varepsilon(0) &= a_0 > 0, \quad \dot{u}_\varepsilon(0) = 0, \end{aligned}$$

then, we obtain initial data and next differential equations for  $v_0, v_1, r_\varepsilon$ :

$$v_0(0) = a_0 \dot{v}_0(0) = 0, \quad 0 = v_1(0) = \dot{v}_1(0), \quad 0 = r_\varepsilon(0) = \dot{r}_\varepsilon(0).$$

We use the natural expansion:  $(u + \varepsilon v)_+ = u_+ + \varepsilon H(u)v + \dots$ , where  $H$  is the Heaviside function, equal to 1 if  $u > 0$  and else 0, (see Lemma 3.1 below).

$$\ddot{v}_0 + v_0 = 0, \quad (5)$$

$$-\alpha_0(\ddot{v}_1 + v_1) = (v_0)_+ + \alpha_1 \ddot{v}_0, \quad (6)$$

$$-\alpha_0(\ddot{r}_\varepsilon + r_\varepsilon) = H(v_0)v_1 + \alpha_2 \ddot{v}_0 + \alpha_1 \dot{v}_1 + R_\varepsilon(s). \quad (7)$$

We now compute,  $\alpha_1, v_1$  and then  $\alpha_2$ . We have  $v_0(s) = a_0 \cos(s)$ . A key point in the method of strained coordinates is to keep bounded  $v_1$  and  $r_\varepsilon$  for large time by a choice of  $\alpha_1$  for  $u_1$  and  $\alpha_2$  for  $r_\varepsilon$ . For this purpose,

we avoid resonant or **secular** term in the right-hand-side of equations (6), (7). Let us first focus on  $\alpha_1$ . Notice that,  $u_+ = \frac{u}{2} + \frac{|u|}{2} \cdot |\cos(s)|$  has no term with frequencies  $\pm 1$ , since there are only even frequencies. Thus  $-\alpha_0((v_0)_+ - \alpha_1 v_0) = a_0 \cos(s)(1/2 - \alpha_1) + a_0 |\cos(s)|/2$  has no secular term if and only if  $\alpha_1 = 1/2$ ,  $\omega_1 = 1/(4\omega_0)$ . Now,  $v_1$  satisfies:

$$-\omega_0^2(\ddot{v}_1 + v_1) = |v_0|/2, \quad v_1(0) = 0, \quad \dot{v}_1(0) = 0.$$

To remove secular term in the equation (7) we have to obtain the Fourier expansion for  $H(v_0)$  and  $v_1$ . Some computations give us:

$$\begin{aligned} |\cos(s)| &= \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{+\infty} \frac{(-1)^k}{4k^2 - 1} \cos(2ks), \\ v_1(s) &= \frac{-a_0}{\omega_0^2} \left( \frac{1}{\pi} - \frac{2}{\pi} \sum_{k=1}^{+\infty} \frac{(-1)^k}{(4k^2 - 1)^2} \cos(2ks) \right), \\ H(v_0) &= \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{+\infty} \frac{(-1)^j}{2j + 1} \cos((2j + 1)s) \end{aligned}$$

To remove secular term of order one in (7), it suffices to take  $\alpha_2$  such that:

$$0 = \int_0^{2\pi} [H(v_0(s))v_1(s) + \alpha_2 \ddot{v}_0(s) + \alpha_1 \dot{v}_1(s)] \cdot v_0 ds. \quad (8)$$

For Duffing equation, see [6, 7, 8], the source term involve only few complex exponentials and the calculus of  $\alpha_2$  is explicit. For general smooth source term, Fourier coefficients decay very fast. Here, we have an infinite set of frequencies for  $v_1$  and  $H(v_0)$ , with only a small algebraic rate of decay for Fourier coefficients. So, numerical computations need to compute more Fourier coefficients. For our first simple example, we can compute explicitly  $\alpha_2$ . After lengthy and tedious computations involving numerical series, we obtain  $\alpha_2 = -3(4\omega_0)^{-2}$ , thus  $\omega_2 = -(2\omega_0)^{-3}$  as we have yet obtained in (2). More generally, we have:

**Proposition 2.1.** *Let  $u_\varepsilon$  be the solution of (1) with  $u_\varepsilon(0) = a_0 + \varepsilon a_1$ ,  $\dot{u}_\varepsilon(0) = 0$ , then, there exists  $\gamma > 0$ , such that, for all  $t < T_\varepsilon = \gamma\varepsilon^{-1}$ :*

$$u_\varepsilon(t) = v_0(\omega_\varepsilon t) + \varepsilon v_1(\omega_\varepsilon t) + \mathcal{O}(\varepsilon^2), \quad \omega_\varepsilon = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2,$$

where  $v_0(s) = a_0 \cos(s)$ ,  $\ddot{v}_1 + v_1 = -\frac{|v_0|}{2\omega_0^2}$ ,  $v_1(0) = a_1$ ,  $\dot{v}_1(0) = 0$ .  
 $\omega_1 = 1/4\omega_0$  and  $\omega_2$  is given by  $\alpha_2$  thanks equations (8), (3).

**Remarks:**

a new critical case: we give another simple example, with an asymptotic expansion only valid for time of order  $\frac{1}{\sqrt{\varepsilon}}$ . Consider, the solution  $u_\varepsilon$  of:

$$\ddot{u} + u + \varepsilon(u - 1)_+ = 0, \quad u_\varepsilon(0) = 1 + \varepsilon, \quad \dot{u}_\varepsilon(0) = 0. \quad (9)$$

The method of strained coordinates gives us the following approximation for  $u_\varepsilon(t)$ :  $v_\varepsilon(t) = (1 + \varepsilon) \cos(t)$  for  $t \leq T_\varepsilon$ . This system has got an energy:  $2E_\varepsilon = \dot{u}^2 + u^2 + \varepsilon[(u - 1)_+]^2$ . Since, 1 is the maximum of  $v_0(t) = \cos(t)$ , a new phenomenon appears, during each period,  $u_\varepsilon > 1$  for interval of time of order  $\sqrt{\varepsilon}$  instead of  $\varepsilon$ , and then  $T_\varepsilon$  is smaller and of order  $\frac{1}{\sqrt{\varepsilon}}$ . To explain this new phenomenon, we give precise estimates of the remainder when we expand  $(v_0 + \varepsilon v_1 + \varepsilon^2 r_\varepsilon)_+$  in the next section.

Nonlinear dependence of pulsation with respect to the amplitude : Previous examples have pulsation independent of the amplitude. It is not always the case, as we can see on following case. Let  $b$  be a real number and consider, the solution  $u_\varepsilon$  of:

$$\ddot{u} + u + \varepsilon(u - b)_+ = 0, \quad u_\varepsilon(0) = a_0 > |b|, \quad \dot{u}_\varepsilon(0) = 0. \quad (10)$$

At the first order, the method of strained coordinates gives us following equations:

$$\ddot{v}_0 + v_0 = 0, \quad -\alpha_0(\ddot{v}_1 + v_1) = (v_0 - b)_+ + \alpha_1 \ddot{v}_0 + O(\varepsilon).$$

Then  $v_0(s) = a_0 \cos(s)$  and  $\alpha_1$  satisfies following equation:

$$\alpha_1 = \frac{1}{\pi} \int_0^{2\pi} (a_0 \cos(s) - b)_+ \cos(s) ds = \frac{a_0}{2\pi} (2\beta + \sin(2\beta) - 4b \sin(\beta)),$$

$$\beta = \beta(b, a_0) = \arccos\left(\frac{b}{a_0}\right) \in [0, \pi].$$

Notice the nonlinear dependence of  $\omega_1 = \alpha_1/2$  with respect to  $b$  and  $a_0$ . Furthermore, at the first order, and for time of the order  $\varepsilon^{-1}$ , we have:  $u_\varepsilon(t) = a_0 \cos((1 + \varepsilon\alpha_1/2)t) + O(\varepsilon)$ .

### 3 Expansion of $(u + \varepsilon v)_+$

We give some useful lemmas to make asymptotic expansions and to estimate precisely the remainder for the basic piecewise linear map  $u \rightarrow u_+ = \max(0, u)$ .

**Lemma 3.1.** [Asymptotic expansion for  $(u + \varepsilon v)_+$  ] *Let be  $T > 0$ ,  $M > 0$ ,  $u, v$  two real valued functions defined on  $I = [0, T]$ ,  $J_\varepsilon = \{t \in I, |u(t)| \leq \varepsilon M\}$ ,  $\mu_\varepsilon(T)$  the measure of the set  $J_\varepsilon$  and  $H$  is the Heaviside step function, then*

$$(u + \varepsilon v)_+ = (u)_+ + \varepsilon H(u)v + \varepsilon \chi_\varepsilon(u, v), \quad \text{with } H(u) = \begin{cases} 1 & \text{if } u > 0 \\ 0 & \text{else} \end{cases},$$

where  $\chi_\varepsilon(u, v)$  is a non negative piecewise linear function and 1-Lipschitz with respect to  $v$ , which satisfies for all  $\varepsilon$ , If  $|v(t)| \leq M$  for any  $t \in I$ ,

$$|\chi_\varepsilon(u, v)| \leq |v| \leq M, \quad \int_0^T |\chi_\varepsilon(u(t), v(t))| dt \leq M\mu_\varepsilon(T). \quad (11)$$

The point in inequality (11) is the remainder  $\varepsilon\chi_\varepsilon$  is only of order  $\varepsilon$  in  $L^\infty$  but of order  $\varepsilon\mu_\varepsilon$  in  $L^1$ . In general,  $\mu_\varepsilon$  is not better than a constant, take for instance  $u \equiv 0$ . Fortunately, it is proved below that  $\mu_\varepsilon$  is often of order  $\varepsilon$ , and for some critical cases of order  $\sqrt{\varepsilon}$ .

**Proof :** Equality (11) defines  $\chi_\varepsilon$  and can be rewritten as follow:

$$\chi_\varepsilon(u, v) = \frac{(u + \varepsilon v)_+ - u_+ - \varepsilon H(u)v}{\varepsilon}. \quad (12)$$

So,  $\chi_\varepsilon$  is non negative since  $u \rightarrow u_+$  is a convex function. We also easily see that the map  $(u, v) \rightarrow \chi_\varepsilon(u, v)$  is piecewise linear, continuous except on the line  $u = 0$  where  $\chi_\varepsilon$  has a jump  $-v$ . This jump comes from the Heaviside step function. An explicit computations gives us the simple and useful formula:  $0 \leq \varepsilon\chi_\varepsilon(u, v) = \begin{cases} |u + \varepsilon v| & \text{if } |u + \varepsilon v| < |\varepsilon v| \\ 0 & \text{else} \end{cases}$ . We then have immediately  $0 \leq \chi_\varepsilon(u, v) \leq |v|$ . Let  $u$  be fixed, then  $v \rightarrow \chi_\varepsilon(u, v)$  is one Lipschitz with respect to  $v$ . Furthermore, the support of  $\chi_\varepsilon$  is included in  $J_\varepsilon$ , which concludes the proof.  $\square$

Now, we investigate the size of  $\mu_\varepsilon(T)$  with notations of Lemma 3.1.

**Lemma 3.2 (Order of  $\mu_\varepsilon(T)$ ).** *Let  $u$  be a smooth periodic function. If  $u$  has only simple roots on  $I = [0, T]$ , then , for some positive  $C$ :  $\mu_\varepsilon(T) \leq C\varepsilon T$ . More generally, if  $u$  has also double roots then  $\mu_\varepsilon(T) \leq C\sqrt{\varepsilon}T$ .*

Notice that any non zero solution of any linear homogeneous second order ordinary differential equation has always simple zeros.

**Proof :** First assume  $u$  only has simple roots on a period  $[0, P]$ , and let  $Z = \{t_0 \in [0, P], u(t_0) = 0\}$ . A well known result state that  $Z$  is a discret set since  $u$  has only simple roots. Thus  $Z$  is a finite subset of  $[0, P]$ :  $Z = \{t_1, t_2, \dots, t_N\}$ . We can choose an open neighborhood  $V_j$  of each  $t_j$  such that  $u$  is a diffeomorphism on  $V_j$  with derivative  $|\dot{u}| > |\dot{u}(t_j)|/2$ . On the compact set  $K = [0, P] - \cup V_j$ ,  $u$  never vanishes, then  $\min_{t \in K} |u(t)| = \varepsilon_0 > 0$ . Thus, we

have for all  $\varepsilon M < \varepsilon_0$ , the length of  $J_\varepsilon$  in  $V_j$  is  $|V_j \cap J_\varepsilon| \leq \frac{4\varepsilon M}{|\dot{u}(t_j)|}$ .  $\mu_\varepsilon$  is additive:

$\mu_\varepsilon(P + t) = \mu_\varepsilon(P) + \mu_\varepsilon(t)$  which give the linear growth of  $\mu_\varepsilon(T) = \mathcal{O}(\varepsilon T)$  for the case with simple roots.

For the general case, on each small neighborhood of  $t_j$ :  $V_j$ , we have with a Taylor expansion,  $|u(t_j + s)| \geq d_j |s|^l$ , with  $1 \leq l \leq 2$ ,  $d_j > 0$ , so,  $|V_j \cap J_\varepsilon| \leq$

$2(\varepsilon M/d_j)^{1/l}$ , then  $\mu_\varepsilon(P) = \mathcal{O}(\sqrt{\varepsilon})$ , which is enough to conclude the proof.  $\square$

## 4 N degrees of freedom

$$M\ddot{U} + KU + \varepsilon(AU - B)_+ = 0, \text{ where } [(AU - B)_k]_+ = \left( \sum_{j=1}^N a_{kj}u_j - b_k \right)_+,$$

$M$  is the diagonal mass matrix with positive term on the diagonal,  $K$  is the stiffness matrix which is symmetric definite positive. For the term  $\varepsilon(AU - B)_+$ , modeling small defect, it is possible to add many of such terms. For a such system, endowed with a natural energy for the linearized part, we control the  $\varepsilon$ -Lipschitz last term, and the solutions remain bounded for all time. Without losing generality, with a change of variables, we deal with following diagonalized system for the linear part, keeping the same notation, except for the positive diagonal matrix  $\Lambda$ :

$$\ddot{U} + \Lambda^2 U + \varepsilon(AU - B)_+ = 0, \quad (13)$$

### 4.1 Nonlinear normal mode, second order approximation

For the system (13) with an initial condition on an eigenmode of the linearized system:  $u_1^\varepsilon(0) = a_0 + \varepsilon a_1$ ,  $\dot{u}_1^\varepsilon(0) = 0$  and, for  $k \neq 1$ :  $u_k^\varepsilon(0) = 0$ ,  $\dot{u}_k^\varepsilon(0) = 0$ . Using the same time  $s = \omega_\varepsilon t$  for each component and following notations:

$$\begin{aligned} \omega_\varepsilon &= \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2, & \omega_0 &= \lambda_1, & (\omega_\varepsilon)^2 &= \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \mathcal{O}(\varepsilon^3), \\ u_j^\varepsilon(t) &= v_j^\varepsilon(s), & v_j^\varepsilon(s) &= v_j^0 + \varepsilon v_j^1 + \varepsilon^2 r_j^\varepsilon, & j &= 1, \dots, N. \end{aligned}$$

Replacing, this ansatz in the System (13) we have in variable  $s$ :

$$\begin{aligned} (\omega_\varepsilon)^2 \ddot{v}_k^\varepsilon + \lambda_k^2 v_k^\varepsilon &= -\varepsilon \left( \sum_{j=1}^N a_{kj} v_j^\varepsilon(s) - b_k \right)_+, \\ L_k v_k^0 &= \alpha_0 \ddot{v}_k^0 + \lambda_k^2 v_k^0 = 0, & -L_k v_k^1 &= \left( \sum_{j=1}^N a_{kj} v_j^0 - b_k \right)_+ + \alpha_1 \ddot{v}_k^0 = S_k^1, \\ -L_k r_k^\varepsilon &= H \left( \sum_{j=1}^N a_{kj} v_j^0 - b_k \right) \left( \sum_{j=1}^N a_{kj} v_j^1 \right) + \alpha_2 \ddot{v}_k^0 + \alpha_1 \dot{v}_k^1 + \varepsilon \dots = S_k^2 + \varepsilon \dots \end{aligned}$$



Equations for  $v_k^0$ , for all  $k \neq 1$ , with zero initial data give us  $v_k^0 = 0$ . In equation for  $v_1^1$ , we remove the secular term for the right hand side. If  $b_1 = 0$  we have  $\omega_1 = \frac{a_{11}}{4\lambda_1}$ . Then, for  $k \neq 1$ , we can compute  $v_k^1$  since:  $\alpha_0 \ddot{v}_k^1 + \lambda_k^2 v_k^1 = - (a_{k1} v_1^0 - b_k)_+$ .  $v_k^1$  is a  $2\pi$ -periodic bounded function since  $\lambda_k \neq \lambda_1$ . Simplifying equation for  $r_1^\varepsilon$  we can compute numerically  $\alpha_2$  and then  $\omega_2$  as in the Propoposition 2.1. Then we check, for all  $k \neq 1$  that  $r_k^\varepsilon$  stay bounded for large time since there is no resonance of order one. We have obtained following results with previous notations:

**Theorem 4.1.** *The following expansion of the nonlinear normal mode is valid on  $(0, T_\varepsilon)$ , under assumption  $\lambda_1 \neq \lambda_k$  for all  $k \neq 1$ :*

$$u_1^\varepsilon(t) = v_1^0(\omega_\varepsilon t) + \varepsilon v_1^1(\omega_\varepsilon t) + \mathcal{O}(\varepsilon^2), \quad u_k^\varepsilon(t) = 0 + \varepsilon v_k^1(\omega_\varepsilon t) + \mathcal{O}(\varepsilon^2),$$

where  $v_1^0(s) = a_0 \cos(s)$ , and  $\omega_1, v_1^1, v_k^1, \omega_2$  are given by following equations, in the sense that we compute successively  $\alpha_1, \omega_1, v_1^1, v_k^1, \alpha_2, \omega_2$ :

$$\begin{aligned} 0 &= \int_0^{2\pi} S_1^1 \cdot v_1^0 ds, \quad \text{where } S_1^1 = (a_{11} v_1^0 - b_1)_+ + \alpha_1 \ddot{v}_1^0, \\ -L_1 v_1^1 &= (a_{11} v_1^0 - b_1)_+ + \alpha_1 \ddot{v}_1^0 = S_1^1, \quad v_1^1(0) = a_1, \quad \dot{v}_1^1(0) = 0, \\ -L_k v_k^1 &= (a_{k1} v_1^0 - b_k)_+ = S_k^1, \quad v_k^1(0) = 0, \quad \dot{v}_k^1(0) = 0, \quad \text{for } k \neq 1, \\ 0 &= \int_0^{2\pi} S_1^2 \cdot v_1^0 ds \quad \text{where } S_1^2 = H(a_{11} v_1^0 - b_1) \left( \sum_{j=1}^N a_{1j} v_j^1 \right) + \alpha_2 \ddot{v}_1^0 + \alpha_1 \ddot{v}_1^0. \end{aligned}$$

Furthermore, if  $(a_{j1} v_1^0 - b_j)$  has got only simple roots for all  $j = 1, \dots, N$ , then  $T^\varepsilon$  is of order  $\varepsilon^{-1}$ , else  $T^\varepsilon$  is of order  $\varepsilon^{-1/2}$ .

## 4.2 First order asymptotic expansion

The method of strained coordinates is used for each normal component, with general initial data  $u_k^\varepsilon(0) = a_k, \dot{u}_k^\varepsilon(0) = 0$  and, with following anatz:

$$\lambda_k^\varepsilon = \lambda_k^0 + \varepsilon \lambda_k^1, \quad \lambda_k^0 = \lambda_k, \quad u_k^\varepsilon(t) = v_k^\varepsilon(s_k) \quad \text{where } s_k = \lambda_k^\varepsilon t, v_k^\varepsilon(s) = v_k^0 + \varepsilon r_k^\varepsilon.$$

Replacing, this anatz in the system (13) we have:

$$\begin{aligned} (\lambda_k^\varepsilon)^2 \ddot{v}_k^\varepsilon(s_k) + \lambda_k^2 v_k(s_k) &= -\varepsilon \left( \sum_{j=1}^N a_{kj} v_j^\varepsilon \left( \frac{\lambda_j^\varepsilon}{\lambda_k^\varepsilon} s_k \right) - b_k \right)_+, \\ L_k v_k^0 &= (\lambda_k^0)^2 \ddot{v}_k^0(s_k) + \lambda_k^2 v_k^0(s_k) = 0, \\ -L_k r_k^\varepsilon(s_k) &= \left( \sum_{j=1}^N a_{kj} v_j^0 \left( \frac{\lambda_j^0}{\lambda_k^0} s_k \right) - b_k \right)_+ + 2\lambda_k \lambda_k^1 \ddot{v}_k^0 + \varepsilon(\dots) \equiv S_k^1 + \varepsilon(\dots). \end{aligned}$$

If  $b_k = 0$ , we identify the secular term with the Lemma 5.4 since  $S_+ = S/2 + |S|/2$ . Then, we remove the resonant term in the source term for the remainder  $r_k^\varepsilon$ , which gives us  $\lambda_k^1 = \frac{a_{kk}}{4\lambda_k}$ . If  $b_k \neq 0$ , we compute  $\lambda_k^1$  numerically.

Noting that, replacing  $v_j^\varepsilon(s_j)$  by  $v_j^0\left(\frac{\lambda_j^0}{\lambda_k^0}s_k\right)$  implies a secular term of order  $\varepsilon t$ . Since the map  $S \rightarrow S_+$  is one-Lipschitz, the error goes to the right-hand side of equation (14). Furthermore,  $r_k^\varepsilon = r_k^\varepsilon(s_1, \dots, s_N)$ , and the method of strained coordinates is not valid to get  $u_k^1$  and  $\lambda_k^2$ . Nevertheless, we obtain:

**Theorem 4.2.** *If  $\lambda_1, \dots, \lambda_N$  are  $\mathbb{Z}$  independent, then, for all  $k$ , for  $t < T_\varepsilon \sim \varepsilon^{-1}$ :*

$$u_k^\varepsilon(t) = v_k^0(\lambda_k^\varepsilon t) + \mathcal{O}(\varepsilon) \quad \text{where } \lambda_k^\varepsilon = \lambda_k + \varepsilon\lambda_k^1,$$

where  $v_k^0(s) = a_k \cos(s)$ , and  $\lambda_k^1$  is defined by the equation:

$$0 = \int_0^{2\pi} \left[ \left( \sum_{j=1}^N a_{kj} v_j^0 \left( \frac{\lambda_j^0}{\lambda_k^0} s_k \right) - b_k \right) + 2\lambda_k \lambda_k^1 \ddot{v}_k^0 \right] \cdot v_k^0 ds$$

Furthermore, if  $b_k = 0$  we have:  $\lambda_k^1 = \frac{a_{kk}}{4\lambda_k}$ .

## 5 Appendix: technical proofs

We briefly give some results used before. Complete proofs are available in [4]. The following Lemma is useful to prove an expansion for large time. There is a similar version for system.

**Lemma 5.3. [Bounds for large time]** *Let  $w_\varepsilon$  be a solution of*

$$w_\varepsilon'' + w_\varepsilon = S_\varepsilon(s) + f_\varepsilon(s) + \varepsilon g_\varepsilon(s, w_\varepsilon), \quad w_\varepsilon'(0) = 0, w_\varepsilon''(0) = 0. \quad (14)$$

*If source terms satisfy the following conditions with  $M > 0$ :*

1.  $S_\varepsilon$  are periodic functions orthogonal to  $e^{\pm it}$ , and  $|S_\varepsilon(t)| \leq M$
2.  $|f_\varepsilon| \leq M$  and for all  $T$ ,  $\int_0^T |f_\varepsilon(s)| ds \leq C\varepsilon T$  or  $C\sqrt{\varepsilon}T$ ,
3. there exists  $R > 0$  such that:  $M_R = \sup_{\varepsilon \in (0,1), s > 0, R > u^2} |g_\varepsilon(s, u)| < \infty$ ,

*then,  $w_\varepsilon$  is uniformly bounded in  $L^\infty(0, T_\varepsilon)$ , where  $T_\varepsilon = \frac{\gamma}{\varepsilon}$  or  $\frac{\gamma}{\sqrt{\varepsilon}}$  and  $\gamma > 0$ .*

For system we have to work with linear combination of periodic functions with different periods and nonlinear function of such sum. So we work with the adherence in  $L^\infty(\mathbb{R}, \mathbb{R})$  of  $\text{span}\{e^{i\lambda t}, \lambda \in \mathbb{R}\}$ , namely the set of almost

periodic functions  $C_{ap}^0(\mathbb{R}, \mathbb{R})$ , see [1]. We first give an useful Lemma about the spectrum of  $|w|$  for  $u \in C_{ap}^0(\mathbb{R}, \mathbb{R})$ . Let us recall definitions for the Fourier coefficient of  $u$  associated to frequency  $\lambda$ :  $c_\lambda[u]$  and its spectrum:  $Sp[u]$ ,

$$c_\lambda[u] = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T u(t) e^{-i\lambda t} dt, \quad Sp[u] = \{\lambda \in \mathbb{R}, c_\lambda[u] \neq 0\}.$$

**Lemma 5.4. [About spectrum of  $|u|$  ]** *If  $u \in C_{ap}^0(\mathbb{R}, \mathbb{R})$ ,  $u$  has got a finite spectrum:  $Sp[u] \subset \{\pm\lambda_1, \dots, \pm\lambda_N\}$ ,  $(\lambda_1, \dots, \lambda_N)$  are  $\mathbb{Z}$ -independent,  $0 \notin Sp[u]$ , then  $\lambda_k \notin Sp[|u|]$  for all  $k$ .*

## References

1. Corduneanu, *Almost periodic functions*, Interscience, New York, 1968.
2. H. Hazim, B. Rousselet, *Finite Elements for a Beam System with Nonlinear Contact Under Periodic Excitation*, preprint 2008.
3. D. Jiang, C. Pierre, S.W. Shaw, *Large-amplitude non-linear normal mode of piecewise linear systems*. Journal of Sound and Vibration, 272, p. 869-891, 2004.
4. S. Junca, B. Rousselet, *The Method of Strained Coordinates for Discrete Models with Small Unilateral Contacts*, preprint 2008.
5. J.B. Keller & S. Kogelman, *Asymptotic solutions of initial value problems for nonlinear parabolic differential equations*, S.I.A.M. J. Appl. Math. 18, p. 748-758, 1970.
6. Kevorkian, J.; Cole, Julian D., *Perturbation methods in applied mathematics*. Applied Mathematical Sciences, 34. Springer-Verlag, New York-Berlin, 1981.
7. J. Kevorkian and J. Cole, *Multiple Scale and Singular Perturbations Problems*, Applied Mathematical Sciences, volume 114, Springer, Berlin, 1996.
8. Peter D. Miller, *Applied Asymptotic Analysis*, American Mathematical Society, Providence, Rhode Island, volume 75, ch 9-10, 2006.
9. Nayfeh, Ali Hasan, *Introduction to perturbation techniques*. Wiley-Interscience [John Wiley & Sons], New York, 1981.
10. M. Roseau, *Vibrations des systèmes mécaniques. Méthodes analytiques et applications*. Masson, Paris, 489 pp, 1984.
11. B. Rousselet, G. Vanderborck, *Non destructive control of cables: O.D.E. models of non linear vibrations*. Variational Formulations in Mechanics : Theory and Applications - A Workshop dedicated to the 60th Birthday of Professor Raül A. Feijoo; 3-5/9/2006, 2006.
12. G. Vanderborck, B. Rousselet, *Structural damage detection and localization by non-linear acoustic spectroscopy*, Saviac, 76th Shock and Vibration Symposium, October 31 - November 3, 2005, Destin (Florida / USA), 2005.