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Abstract

In this paper, an optimal input design framework for continuous-time system identification is proposed. The problem is formulated in a convex finite dimensional form. For that purpose, input spectrum is decomposed onto a Laguerre basis and an LMI-problem is solved. The proposed solution is used to compute the optimal spectrum for a fractional system identification.

Key words: Optimal input design, Experiment planning, Continuous-time system identification, Fractional systems, Basis functions decomposition, Laguerre functions.

1. Introduction

Nowadays, most identification experiments and models are treated in discrete-time. However, some systems are naturally described by C-T differential equations in order to keep the physical meaning of their parameters. Several identification methods are developed for C-T system identification using either rational or fractional models [1, 2, 3, 4, 5, 6]. It can be useful to optimize their experiment in order to improve the estimation quality or to reduce the experiment cost. For these reasons, it is necessary to obtain a framework permitting to apply optimal input design when C-T models are involved.

The question of optimal experiment design appeared in the early 1970's. In [7, 8] first interesting results were presented, based on maximization of the information obtained from the system when the input signal is a multi-sine function. These results set up a basis for more complex approaches. The most well-known ones are subjected to Ljung's asymptotic variance formula [9, 10]. Many efforts were made for closed-loop experiment design analysis [11, 12]. Meanwhile, different optimization methods were applied to solve the problems arising in this context [13]. There exists other approaches based on projection in noise-free space, time-domain optimization or moving horizon, rather than variance estimation [14, 15]. Experiment design for system identification using fractional models has recently been treated in [16, 17]. The proposed method is based on finding the optimal sine input signal in the case where one, two or three parameters are unknown by minimizing the D-optimality criterion of the covariance matrix. Up to authors knowledge, experiment design in the C-T case was not much investigated. This paper attempts to fill this gap and provides an extension for authors previous work [18] by formalizing the main result as theorem 4, providing a rigorous proof, and extending the examples to include a comparison with a multi-sine solution.

The paper is organized as follows: at first, main definitions and problem formulation are presented in section 2. Input spectrum parametrization by C-T Laguerre functions is described in section 3. The problem is then solved via spectrum optimization in section 4. Finally, numerical illustration on a fractional system is presented in section 5, before concluding.

2. Problem formulation

Consider a single-input-single-output (SISO) C-T system operating in open-loop, corrupted by a white noise:

$$y(t) = G(p, \Theta)u(t) + e(t) \tag{1}$$

where $G(p, \Theta)$ is a real strictly proper stable transfer function describing the system dynamics, Θ is the vector of unknown parameters to identify corresponding

to the minimal parametrization, $y(t)$ is the output, $u(t)$ the input, $p = \frac{d}{dt}$ the differentiation operator and $e(t)$ a zero mean Gaussian noise with a constant spectrum equal to σ^2. The parameters covariance matrix depends on the input signal spectrum $\Phi_u(\omega)$ and is given by [19]:

$$P^{-1}(\Theta_0) = \frac{1}{\sigma^4}\int_{-\infty}^{\infty} F(j\omega,\Theta_0)\Phi_u(\omega)F^*(j\omega,\Theta_0)d\omega + R_0, \tag{2}$$

with $F(j\omega,\Theta) = \frac{dG(j\omega,\Theta)}{d\Theta}$, Θ_0 is the true parameter vector, $P(\Theta_0)$ is a covariance matrix, and R_0 is a constant, independent of $\Phi_u(\omega)$. The true system transfer function is denoted by $G_0(s) = G(s,\Theta_0)$.

It is clear from (2) that it is possible to change the parameters estimation variance by shaping the input spectrum. Different estimation quality criteria could be proposed, for example D-optimality ($\det(P)$), A-optimality ($\text{trace}(P)$), E-optimality ($\lambda_{max}(P)$) etc. It is shown in [8] that, for these criteria, the optimal solution can be expressed as a finite number of sine-functions, which depends on the number of parameters to estimate. However, for control system applications, it is useful to formulate the problem in terms of transfer function variance at each frequency. The relative error on frequency function estimation is given by [13]:

$$\varepsilon(j\omega,\Theta) = \frac{G_0(j\omega) - G(j\omega,\Theta)}{G_0(j\omega)}. \tag{3}$$

The first term of the Taylor series expansion of the variance of this error estimation is given by [13]:

$$\Delta(\omega) \;=\; \text{Var}(\varepsilon(j\omega,\Theta)) \;=\; \left|\frac{1}{G_0(j\omega)}\right|^2 \frac{1}{N}\frac{dG^*(j\omega,\Theta)}{d\Theta}P(\Theta)\frac{dG(j\omega,\Theta)}{d\Theta}. \tag{4}$$

The value $\max_\omega \Delta(\omega)$ will be referred to later as estimation quality.

If the number of unknown parameters is big, asymptotic variance formula [20], can be applied. However, in parametric system identification, the unknown parameters are usually limited to a small number. So, it is necessary to use the Taylor expansion of non-asymptotic error variance (4).

It is evident from (2) and (4) that increasing the input signal power reduces the estimation variance. Consequently, evaluating the estimation quality due to the input signal must be done at a fixed input power:

$$\begin{cases} \min_{\Phi_u} \max_{\omega} \Delta(\omega), & \forall \omega \in \mathbb{R}^+; \\ \int_{-\infty}^{\infty} \Phi_u d\omega = \alpha, & \Phi_u(\omega) \geq 0. \end{cases} \tag{5}$$

Otherwise, a minimum total input power can be computed for a fixed estimation quality:

$$\begin{cases} \min_{\Phi_u} \alpha = \int_{-\infty}^{\infty} \Phi_u d\omega, & \Phi_u(\omega) \geq 0; \\ \Delta(\omega) \leq \gamma, \gamma > 0, & \forall \omega \in \mathbb{R}^+. \end{cases} \tag{6}$$

Whatever the formulation, the posed problem has two main disadvantages:

(i) it is of infinite dimension, as ω ranges from $-\infty$ to $+\infty$ and the input spectrum Φ_u is defined in the frequency space;

(ii) it is non-convex, due to the inverse of the covariance matrix P.

The way to overcome these disadvantages is described in the next section.

3. Spectrum decomposition in Laguerre polynomial basis

It is necessary to define a finite dimensional parametrization of the input spectrum. In [13], Jansson and Hjalmarsson propose to represent the spectrum using FIR filters which could be extended to any discrete-time orthogonal basis function. Here, a C-T version is considered:

$$\Phi_u(\omega) = \sum_{k=-\infty}^{\infty} r_{|k|} \mathcal{B}_k(j\omega) \tag{7}$$

where $r_{|k|}$ are the coefficients associated to the orthogonal basis functions $\{\mathcal{B}_k(j\omega)\}_{k=0}^{\infty}$ which are proper, stable and rational with $B_{-k}(j\omega) = B_k(-j\omega)$. The choice of basis functions is necessary for the realization of corresponding input signal by spectrum factorization. This aspect, as well as other possible choices of basis functions, will be discussed in the next sections. A limited parametrization of the positive part of $\Phi_u(\omega)$ is usually chosen:

$$\Psi_u(s) = \sum_{k=0}^{M-1} r_k \mathcal{B}_k(s) \tag{8}$$

$$\Phi_u(\omega) = \Psi_u(j\omega) + \Psi_u^*(j\omega) \tag{9}$$

where M is the desired number of basis functions and $(^*)$ stands for conjugate transpose. For $\Phi_u(\omega)$ to be a spectrum, it is necessary that:

$$\Phi_u(\omega) \geq 0, \qquad \forall \omega \in \mathbb{R}. \tag{10}$$

To satisfy (10) with parametrization (8), a basic idea is to define a frequency-by-frequency positivity constraint for a certain number of frequencies. However, this approach is not always consistent, as it cannot insure the positivity of the spectrum for intermediate frequencies. Another approach, coming from discrete-time filter design, discussed in [21] can be used. It is based on the real positive lemma which states in the C-T case as follows.

Lemma 3.1 ([22]). *Let $\{A, B, C, D\}$ be a minimal controllable state space realization of $\Psi_u(s) = \sum_{k=0}^{M-1} r_k \mathcal{B}_k(s)$. Then $\Phi_u(\omega) = \Psi_u(j\omega) + \Psi_u^*(j\omega) > 0$, if and only if there exists a symmetric definite positive matrix $Q = Q^T \succ 0$ such that:*

$$\begin{bmatrix} A^TQ + QA & QB - C^T \\ (QB - C^T)^T & -(D + D^T) \end{bmatrix} \preceq 0. \,\square \tag{11}$$

The Lemma 3.1 is not restricted to Laguerre basis, but to any fixed realisation of generalized orthonormal basis functions.

To check the positivity, it is only necessary to choose suitable basis functions $\mathcal{B}_k$ and to obtain their state-space representation. The C-T Laguerre basis functions are used in the remainder of the paper:

$$\mathcal{B}_k(s) = \sqrt{2\lambda}\frac{(s-\lambda)^k}{(s+\lambda)^{k+1}}. \tag{12}$$

The properties of Laguerre basis are described more in details in [23] and the choice of M and λ is discussed in the illustration section.

Substituting (12) in (8) yields:

$$\Psi_u(s) = \sqrt{2\lambda}\frac{1}{(s+\lambda)} \times \sum_{k=0}^{M-1} r_k \left(\frac{s-\lambda}{s+\lambda}\right)^k. \tag{13}$$

Using such a decomposition of the spectrum, a low-pass filter appears explicitly as a factorized term. The second term of the product corresponds to the bilinear transform. It can be noticed, that λ is a cut-off frequency. In a zone much higher than λ the low-pass behavior is dominating. In a zone much lower than λ a decomposition corresponds to a static gain value. Consequently, only for frequencies around λ the spectrum varies significantly, so the choice of λ is very important. It must be close enough to system highest proper frequency.

This parametrization is linear in parameters r_k. To satisfy the positivity constraint (10), it is necessary to obtain a state-space model of (13) with known A and B matrices.

State-space parametrization of $\Psi_u(s)$

Eq. (13) can be reformulated in the following expanded form:

$$\Psi_u(s) = \sqrt{2\lambda}\frac{\sum_{i=0}^{M-1} n_i s^i}{\sum_{j=0}^{M} d_j s^j}. \tag{14}$$

To obtain the state-space representation it is necessary to determine d_j and n_i terms which is done in [18]:

$$n_i = \lambda^{M-1-i} \sum_{k=0}^{M-1} \left(r_k \sum_{j=\max(0,i+k-(M-1))}^{\min(i,k)} (-1)^j C_j^k C_{i-j}^{M-1-k} \right), \tag{15}$$

$$d_j = \lambda^{M-j} C_j^M, \tag{16}$$

where $C_i^k = \frac{k!}{i!(k-i)!}$ is Newton binomial coefficient.

Using (14)-(16) the minimal state-space representation $\{A, B, C, D\}$ of $\Psi_u(s)$ can be formulated:

$$A = \begin{bmatrix} -[d_{M-1}, d_{M-2}, \ldots, d_0] \\ [I_{M-1\times M-1}\ 0_{M-1\times 1}] \end{bmatrix}, \tag{17}$$

$$B = [1, 0_{1\times M-1}]^T, \tag{18}$$

$$C = \sqrt{2\lambda}[n_{M-1}, n_{M-2}, \ldots, n_0], \tag{19}$$

$$D = 0. \tag{20}$$

where $I_{m\times m}$ is the identity matrix of size m and $0_{m\times n}$ a matrix of zeros of size $m \times n$.

Now, using (17-20) in lemma 3.1, spectrum positivity constraint can be formulated.

4. Input spectrum optimization

The two problems described in (5) and (6) are quite similar. Hence, only the problem described in (6) is treated in the remainder of the paper. The following theorem allows to overcome the disadvantage (ii) p.2 and to provide a convex formulation of the problem.

Theorem 4.1. *Consider the C-T linear system defined in* (1)*. Then, the problem formulated in* (6) *can be solved by minimizing:*

$$\min_{r_k} \alpha = \sum_{k=0}^{M-1} r_k \int_{-\infty}^{\infty} [\mathcal{B}_k(j\omega) + \mathcal{B}_k^*(j\omega)] d\omega, \tag{21}$$

under the following linear matrix inequality (LMI) constraints:

$$\begin{bmatrix} A^TQ + QA & QB - C^T \\ (QB - C^T)^T & -(D + D^T) \end{bmatrix} \preceq 0, \tag{22}$$

$$\begin{bmatrix} \gamma & V^*(\omega_i) \\ V(\omega_i) & P(\Theta)^{-1} \end{bmatrix} \succeq 0, i = 0 \dots N-1. \tag{23}$$

where $\mathcal{B}_k(s)$ are the Laguerre functions defined in (12), $\{A, B, C, D\}$ is the state-space representation of basis decomposition defined in (17)-(20), Θ is the vector of unknown parameters, $P(\Theta)$ is the parameter covariance matrix and $V(\omega) = \frac{1}{G_0(j\omega)\sqrt{N}} \frac{dG(j\omega)}{d\Theta}$. □

Proof First, the total input signal power that appears in the objective function of problem (6) is obtained using (8):

$$\alpha = \int_{-\infty}^{\infty} \Phi_u d\omega = \sum_{k=0}^{M-1} r_k \int_{-\infty}^{\infty} [\mathcal{B}_k(j\omega) + \mathcal{B}_k^*(j\omega)] d\omega. \tag{24}$$

The next step is to obtain a convex formulation of the problem (6). From (4), it is possible to rewrite (6) as:

$$\gamma - V^*(\omega)P(\Theta)V(\omega) \geq 0, \quad \forall\omega \in \mathbb{R}^+. \tag{25}$$

Inequality (25) can be expressed as an LMI on $P^{-1}(\Theta_0)$ using Schur's complement:

$$\begin{bmatrix} \gamma & V^*(\omega) \\ V(\omega) & P^{-1}(\Theta_0) \end{bmatrix} \succeq 0, \forall\omega \in \mathbb{R}^+. \tag{26}$$

The only unknown term in the LMI (26) is the inverse covariance matrix $P^{-1}(\Theta_0)$. It can be expressed as a linear equation, using the input spectrum parametrization (13) in the covariance matrix expression (2):

$$P^{-1}(\Theta_0) = \frac{1}{\sigma^4}\sum_{k=0}^{M-1} r_k \int_{-\infty}^{\infty} F(j\omega,\Theta_0)[B_k(j\omega)+B_k^*(j\omega)]F^*(j\omega,\Theta_0)d\omega + R_0. \tag{27}$$

Using (27), the problem becomes convex and linear with only parameters $r_k, k = 0 \dots M-1$ being unknown. However, it is still infinite-dimensional with respect to ω. The most simple solution consists in replacing condition (26) with a finite number of identical conditions for some prescribed frequencies $\omega_i, i = 0...N-1$ which yields (23) and completes the proof □

The problem formulated in theorem 4.1 can be solved for any C-T system. The only unknown parameters are the coefficients r_k. The obtained optimal spectrum can be realized in time-domain using spectral factorization method, described in [24].

5. Exemple: experiment planning for a C-T fractionnal system

Consider a fractional system of the first kind [25, 26]

$$G(s) = \frac{K}{(\tau s)^\nu + 1}, \tag{28}$$

with the true system parameters $K = 10, \tau = 1$ and $\nu = 1.4$. The system output is corrupted by an additive white noise as in (1). The objective is to design an

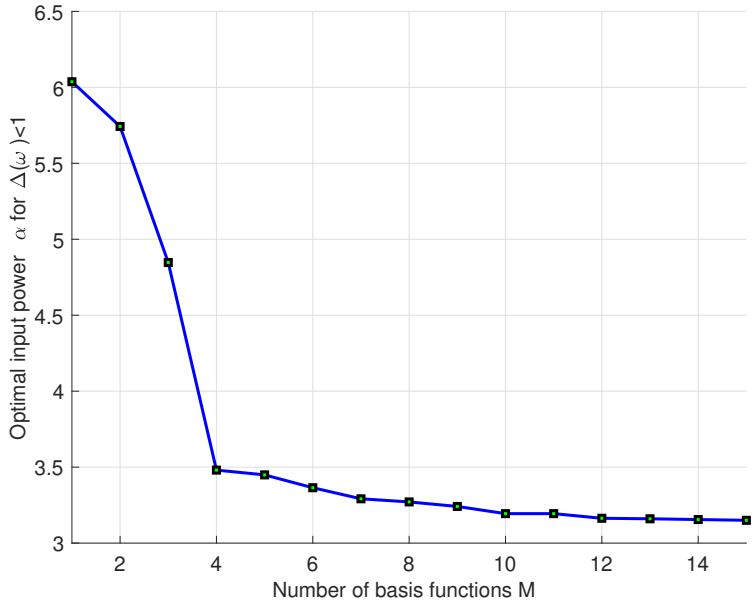

Figure 1: Optimal input power versus the number of basis functions M for $\lambda = 1.5$

experiment with a minimum total power of the input signal, so that parameters τ, K and ν are estimated and the precision on the frequency error estimation (4) is $\Delta(\omega) \leq 1, \forall \omega \in \mathbb{R}^+$. The estimation quality is assessed through (4) by the maximum of criterion $\Delta(\omega)$ and the complete problem is formulated in the form (6). Parameters are estimated using "Output Error" fractional model [5, 6]. As a consequence, the problem can be formulated in the LMI form (21)-(23) and solved numerically.

To apply the proposed method, it is necessary to choose the basis functions parameter λ and the number of basis functions M. Fig.1 shows optimal input spectrum power, required to have $\Delta(\omega) \leq 1$, for different number of basis functions and for $\lambda = 1.5$. Increasing the number of basis functions increases the accuracy of the computed spectrum. Indeed, the higher the decomposition order, the better the precision. However, beyond $M = 15$ the numerical burden is significantly increased and causes numerical problems while solving the LMI.

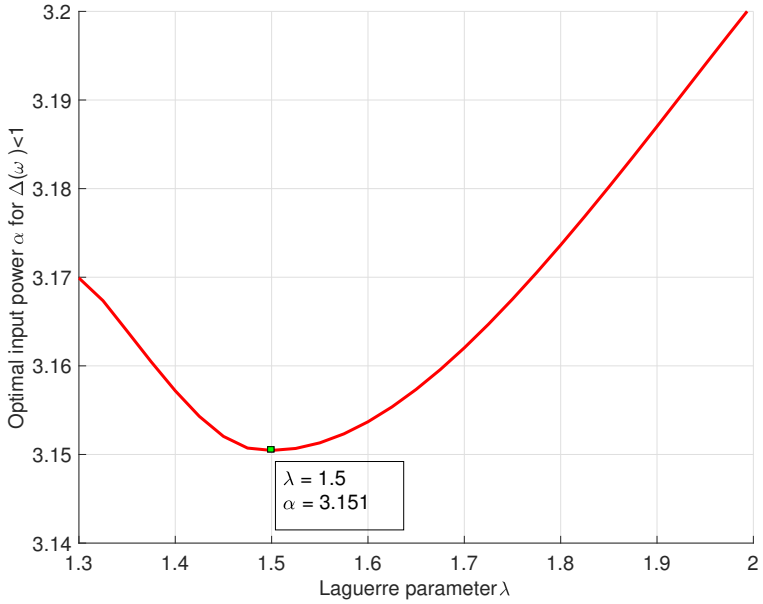

Figure 2: Optimal input power versus λ for $M = 15$

The choice of λ seems to be more complicated. System cut-off frequency equals $\frac{1}{\tau}$. It is rather reasonable to choose λ in some range close to $\frac{1}{\tau}$. Fig. 2 shows the optimal input spectrum power α versus λ for $M = 15$.

It is evident that λ influences the precision of the estimation. The parameter λ can also be estimated by any non-linear optimization method.

Now the influence of error bound is studied. Fig. 3 shows the optimal input power α versus γ for $\lambda = 1.5$ and $M = 15$. The total power decreases if the error bound is relaxed, since more power is necessary to obtain a better evaluation quality.

In Fig. 4, optimal spectrum is plotted for $\lambda = 1.5$ and $M = 15$. It is worth noting that a similar problem is treated in [16] with an input signal composed of a sum of two sine functions having a unit amplitude. The D-optimality criterion of the inverse covariance matrix is optimized and the obtained optimal frequencies, at 0.54 rad/s and 1.1 rad/s, are very close to the ones plotted in

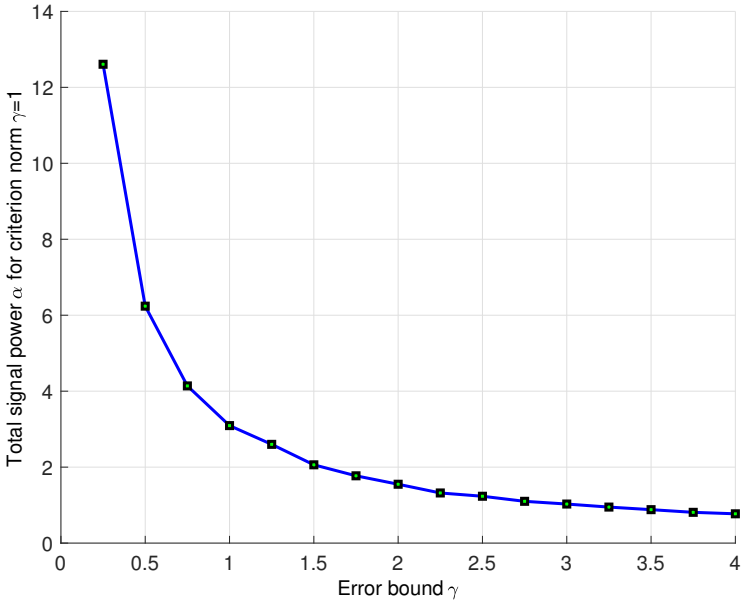

Figure 3: Optimal input power versus γ for $M = 15$ and $\lambda = 1.5$

Fig.4.

Monte Carlo analysis is now used to confirm the results of the paper. The criterion (5) is evaluated for three different input signals, having the same power density:

- pseudo-random binary sequence (PRBS), generated using 8 registers, such that the signal remains constant at least over intervals of 10 samples;
- optimal multi-sine signal presented in [16] with two components of the same amplitude with frequencies 0.54 rad/s and 1.1 rad/s;
- the signal with the optimal spectrum plotted in Fig.4.

It is worth mentionning that solutions to problems (5) and (6) give spectra of the same form. Hence, it is possible to use the results of (6) by a simple scaling of the total power . The input signals have the same power and the sampling period equals 0.02 s. The sampling period is chosen to be sufficiently low for the simulation error to be negligible. The output signal is corrupted by a centered

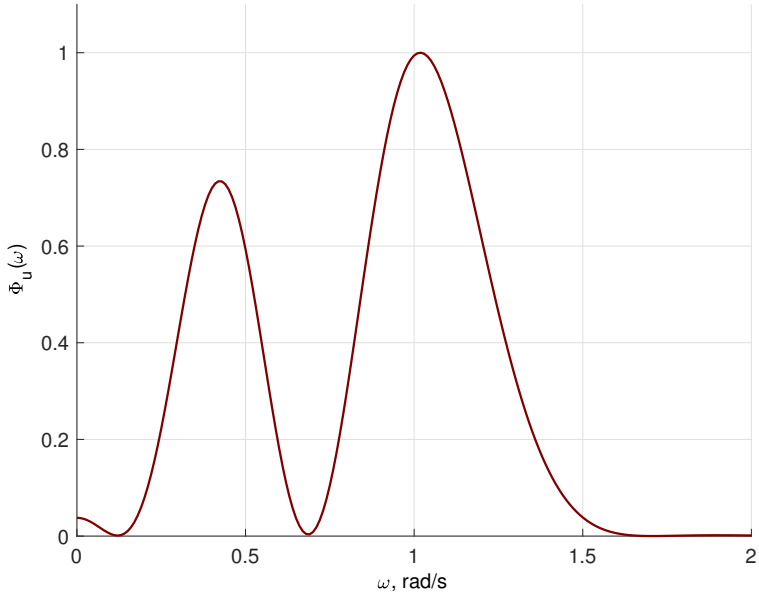

Figure 4: Normalized optimal spectrum for Laguerre basis decomposition

white noise with a gaussian distribution and a unity variance. For each input signal, 50 runs of Monte-Carlo simulations with different noise realizations are carried out. Spectral factorization is used for generating the signal with optimal form of spectrum. For each realization the system is identified using an output error fractional model [27, 28]. Finally, "frequency-by-frequency" variance is computed using (3) and (4). Results are presented in Fig. 6, which shows that using the PRBS leads to a higher frequency-by-frequency variance than the optimal spectrum. The presented result compares the criterion of problem (5) for three input signals.

The multi-sine input and the optimal signal have similar performances with maximum error of the same order. However, for multi-sine signal the error slightly increases for higher frequencies and for optimal signal it remains constant. The advantage of the proposed approach is that it allows, by solving an LMI and using linear programming, to determine the optimal spectrum.

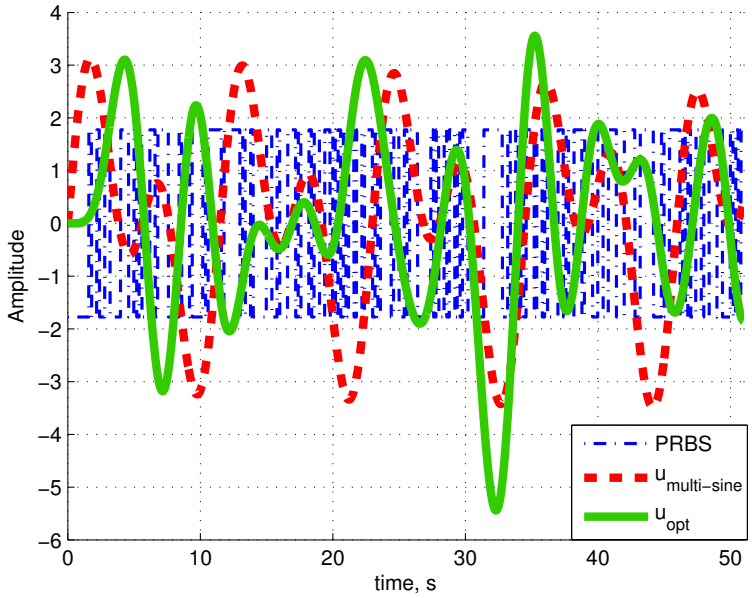

Figure 5: Input signals for Monte-Carlo simulations

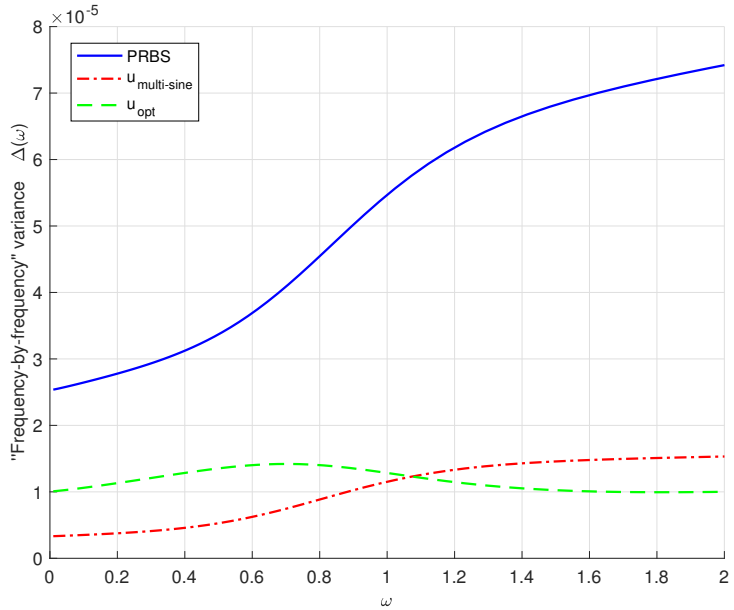

Figure 6: "Frequency-by-frequency" variance $\Delta(\omega)$ for three kind of inputs

6. Conclusion

The paper describes a framework for optimal input design for C-T system identification. The problem is formulated in convex form and can be solved numerically. The input spectrum is parametrized by Laguerre functions, which require to choose a tuning parameter λ. The provided example shows the efficacy of the computed optimal spectrum as compared to the classical PRBS and optimal multi-sine signal.

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