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A POSTERIORI ANALYSIS OF A RICHARDS PROBLEM WITH A FREE BOUNDARY

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Abstract: Richards equation models the water flow in a partially saturated underground porous medium. When it rains on the surface, boundary conditions of Signorini type may be considered on this part of the boundary. We recall the well-posedness of this problem together with its discretization by an implicit Euler's scheme in time and finite elements in space. We establish quasi-optimal a posteriori estimates for the solution of the discrete problem. Some numerical experiments confirm the interest of this discretization.

1. Introduction

The following

$$\partial_t \Theta(\psi) - \nabla \cdot K_w(\Theta(\psi)) \nabla(\psi + z) = 0, \quad (1.1)$$

models the flow of a wetting fluid, mainly water, in the underground medium located just under the surface, hence in an unsaturated medium, see L.A. Richards [16] for the introduction of this type of models. In opposite to Darcy's or Brinkman's systems (see [15] for all these models), this equation which is derived by combining Darcy's generalized equation with the mass conservation law is highly nonlinear: This follows from the fact that, due to the presence of air above the surface, the porous medium is only partially saturated with water. The unknown ψ is the difference between the pressure of water and the atmospheric pressure.

This equation is usually provided with Dirichlet or Neuman type boundary conditions. Indeed, Neumann boundary conditions on the underground part of the boundary are linked to the draining of water outside of the domain and Dirichlet boundary conditions on the surface are introduced to take into account the rain. However, when it is raining cats and dogs, the part of the domain saturated with water reaches the boundary. Then the Dirichlet boundary conditions must be replaced by variational inequalities of the following type:

$$-\psi \geq 0, \quad \mathbf{v}(\psi) \cdot \mathbf{n} \geq \mathbf{v}_r \cdot \mathbf{n}, \quad \psi(\mathbf{v}(\psi) \cdot \mathbf{n} - \mathbf{v}_r \cdot \mathbf{n}) = 0, \quad (1.2)$$

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where $\mathbf{v}(\psi)$ is the flux

$$\mathbf{v}(\psi) = -K_w(\Theta(\psi))\nabla(\psi + z). \quad (1.3)$$

Here, \mathbf{n} stands for the unit normal vector to the surface and \mathbf{v}_r is a given rain fall rate. We refer to the thesis of H. Berninger [7] for the full derivation of this model from hydrology laws.

It is not so easy to give a mathematical sense to the system (1.1) – (1.2). As standard, the key argument for the analysis of problem (1.1) is to use Kirchhoff’s change of unknowns. Indeed, after this transformation, the new equation fits the general framework proposed in [2]. On the other hand, the inequality in (1.2) can be handled by a variational formulation. In [4], we prove that the corresponding variational problem is well-posed when the coefficients and the data are smooth enough, but without any restriction on their maximal value.

The discretization of problem (1.1) has been studied in many papers when provided with standard boundary conditions, see [1], [9], [14], [17], [18] and [20], however it seems that it has not been treated when provided with the boundary inequality (1.2). As first proposed in [4]Section 3, we write here a discretization of system (1.1) – (1.2), in two steps:

- (i) We first use the Euler’s implicit scheme to build a time semi-discrete problem, where one of the nonlinear terms is treated in an explicit way for simplicity;
- (ii) We then construct a fully discrete problem that relies on the Galerkin method and finite elements in the spatial domain.

We also recall the convergence properties of this discretization. We observe that the discrete problem is unconditionally stable and that its convergence does not require any limitation on the time step and mesh size. Indeed, these two properties are very important for performing adaptivity.

The aim of this paper is to prove a posteriori estimates for the discrete problem. We refer to [11] for the first a posteriori analysis of a problem with Signorini type boundary conditions but in the stationary case. Following the approach in [3], in this a posteriori analysis, we try to uncouple the errors issued from the time and space discretizations as much as possible. This leads to a simple adaptivity strategy, as first described in [6]. Some numerical experiments confirm the interest of the discretization.

The outline of the paper is as follows.

- In Section 2, we present the variational formulation of the full system and recall its well-posedness.
- Section 3 is devoted to the description of the time semi-discrete problem and of the fully discrete problem. There also, we recall their well-posedness and their convergence.
- The a posteriori analyses of the time semi-discretization and space discretization are performed in Sections 4 and 5, respectively.
- In Section 6, we present a few numerical experiments.

2. THE CONTINUOUS PROBLEM AND ITS WELL-POSEDNESS

Let Ω be a bounded connected open set in \mathbb{R}^d , $d = 2$ or 3 , with a Lipschitz-continuous boundary $\partial\Omega$. From now on and for simplicity, we assume that Ω is a polygon ($d = 2$) or a polyhedron ($d = 3$), and we denote by \mathbf{n} the unit outward normal vector to Ω on $\partial\Omega$. We assume that $\partial\Omega$ admits a partition without overlap into three parts Γ_B , Γ_F , and Γ_G (these indices mean “bottom”, “flux” and “ground”, respectively) and that Γ_B has a positive measure. We also assume that this partition is smooth enough, i.e. $\partial\Gamma_B$, $\partial\Gamma_F$ and $\partial\Gamma_G$ are Lipschitz submanifolds of $\partial\Omega$.

In order to perform the Kirchhoff’s change of unknowns in problem (1.1), we observe that, since the conductivity coefficient K_w is positive, the mapping:

$$x \mapsto \mathcal{K}(x) = \int_0^x K_w(\Theta(\xi)) d\xi,$$

is one-to-one from \mathbb{R} into itself. Thus, by setting $u = \mathcal{K}(\psi)$, we obtain an equivalent equation to (1.1). Moreover, the Kirchhoff's change of unknowns has the further property of preserving the positivity, so that (1.2) can be written in terms of u . All this leads to the following system

$$\alpha \partial_t u + \partial_t b(u) - \nabla \cdot (\nabla u + k \circ b(u) \mathbf{e}_z) = 0 \quad \text{in } \Omega \times]0, T[\quad (2.1)$$

$$u = u_B \quad \text{on } \Gamma_B \times]0, T[\quad (2.2)$$

$$-\left(\nabla u + k \circ b(u) \mathbf{e}_z\right) \cdot \mathbf{n} = f_F \quad \text{on } \Gamma_F \times]0, T[\quad (2.3)$$

$$u \leq 0, \quad -(\nabla u + k \circ b(u) \mathbf{e}_z) \cdot \mathbf{n} \geq \mathbf{q}_r \cdot \mathbf{n}, u (\nabla u + k \circ b(u) \mathbf{e}_z + \mathbf{q}_r) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_G \times]0, T[, \quad (2.4)$$

$$u|_{t=0} = u_0 \text{ in } \Omega, \quad (2.5)$$

where $-\mathbf{e}_z$ stands for the unit vector in the direction of gravity. The unknown is now the quantity u . The data are the Dirichlet boundary condition u_B on Γ_B and the initial condition u_0 on Ω , together with the boundary conditions f_F and \mathbf{q}_r on the normal component of the flux, with f_F corresponding to the draining of water and \mathbf{q}_r corresponding to the rain. Finally, the coefficients b and k are supposed to be nonnegative, while α is a positive constant. From now on, we assume that the function b is continuously differentiable on \mathbb{R} , with a nonnegative, bounded and Lipschitz-continuous first derivative b' , and that the function $k \circ b$ is continuous, bounded, and uniformly Lipschitz-continuous on \mathbb{R} .

Remark It must be observed that the unknown u has no physical meaning. However, the pressure ψ can easily be recovered from u , for instance by a dichotomy algorithm, see the definition of the operator \mathcal{K} .

In what follows, we use the whole scale of Sobolev spaces $W^{m,p}(\Omega)$, with $m \geq 0$ and $1 \leq p \leq +\infty$, equipped with the norm $\|\cdot\|_{W^{m,p}(\Omega)}$ and seminorm $|\cdot|_{W^{m,p}(\Omega)}$, with the usual notation $H^m(\Omega)$ when $p = 2$. For any separable Banach space E equipped with the norm $\|\cdot\|_E$, we denote by $\mathcal{C}^0(0, T; E)$ the space of continuous functions on $[0, T]$ with values in E . For each integer $m \geq 0$, we also introduce the space $H^m(0, T; E)$ as the space of measurable functions on $]0, T[$ with values in E such that the mappings: $v \mapsto \|\partial_t^\ell v\|_E$, $0 \leq \ell \leq m$, are square-integrable on $]0, T[$.

To write a variational formulation for the problem, we introduce the time-dependent subset

$$\mathbb{V}(t) = \{v \in H^1(\Omega); v|_{\Gamma_B} = u_B(\cdot, t) \text{ and } v|_{\Gamma_G} \leq 0\}.$$

It is readily checked that each $\mathbb{V}(t)$ is closed and convex, see [7, Prop. 1.5.5], when u_B belongs to $\mathcal{C}^0(0, T; H^{\frac{1}{2}}(\Gamma_B))$. Thus, we are led to consider the following variational problem (with obvious notation for $L^2(0, T; \mathbb{V})$)

Find u in $L^2(0, T; \mathbb{V})$ such that

$$u|_{t=0} = u_0,$$

and that, for a.e. t in $]0, T[$,

$$\forall v \in \mathbb{V}(t),$$

$$\begin{aligned} & \alpha \int_{\Omega} (\partial_t u)(\mathbf{x}, t)(v - u)(\mathbf{x}, t) \, d\mathbf{x} + \int_{\Omega} (\partial_t b(u))(\mathbf{x}, t)(v - u)(\mathbf{x}, t) \, d\mathbf{x} \\ & + \int_{\Omega} \left(\nabla u + k \circ b(u) \mathbf{e}_z\right)(\mathbf{x}, t) \cdot (\nabla(v - u))(\mathbf{x}, t) \, d\mathbf{x} \\ & \geq - \int_{\Gamma_F} f_F(\tau, t)(v - u)(\tau, t) \, d\tau - \int_{\Gamma_G} (\mathbf{q}_r \cdot \mathbf{n})(\tau, t)(v - u)(\tau, t) \, d\tau. \end{aligned}$$

We refer to [4, Prop. 2.2] for the proof of the following statement.

Proposition 2.1. *Problems (2.1) and (2.3) – (2.4) are equivalent, in the following sense:*

- (i) *Any solution of problem (2.1) in $L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ is a solution of (2.3) – (2.4);*
- (ii) *Any solution of problem (2.3) – (2.4) is a solution of problem (2.1) in the distribution sense.*

Proving that problem (2.3) – (2.4) is well-posed is not at all obvious. Indeed, standard arguments yield the uniqueness of the solution, and its existence only when a rather restrictive condition on the function b is satisfied. In any case, we refer to [4, Prop. 2.3 and Thm 4.3] for the following result. For brevity, we set:

$$\mathbb{X} = L^2(0, T; \mathbb{V}) \cap H^1(0, T; L^2(\Omega)).$$

Theorem 2.1. *For any data u_B , f_F , \mathbf{q}_r and u_0 satisfying*

$$u_B \in H^1(0, T; H_{00}^{\frac{1}{2}}(\Gamma_B)), \quad f_F \in \mathcal{C}^0(0, T; H^{\frac{1}{2}}(\Gamma_F)),$$

$$\mathbf{q}_r \in \mathcal{C}^0(0, T; H^{\frac{1}{2}}(\Gamma_G)^d), \quad u_0 \in H^1(\Omega),$$

together with the compatibility conditions

$$u_0(\mathbf{x}) = u_B(\mathbf{x}, 0) \quad \text{for a.e. } \mathbf{x} \in \Gamma_B \quad \text{and} \quad u_0(\mathbf{x}) \leq 0 \quad \text{for a.e. } \mathbf{x} \in \Gamma_G, \quad (2.6)$$

and for any positive parameter α , problem (2.3) – (2.4) has a unique solution in \mathbb{X} .

Remark It is readily checked, see [4, Proof of Thm 4.3], that the norm in \mathbb{X} of the solution exhibited in Theorem 2.3 is bounded as a function of the norms of the data (see (2.6) for these norms).

3. THE DISCRETE PROBLEMS

We present successively the time semi-discrete problem constructed from the backward Euler's scheme, next a finite element discretization of this problem relying on standard, conforming, finite element spaces.

3.1. The time semi-discrete problem. Since we intend to work with non uniform time steps, we introduce a partition of the interval $[0, T]$ into subintervals $[t_{n-1}, t_n]$, $1 \leq n \leq N$, such that $0 = t_0 < t_1 < \dots < t_N = T$. We denote by τ_n the time step $t_n - t_{n-1}$, by τ the N -tuple (τ_1, \dots, τ_N) and by $|\tau|$ the maximum of the τ_n , $1 \leq n \leq N$.

As already hinted in Section 1, the time discretization mainly relies on a backward Euler's scheme, where the nonlinear term $k \circ b(u)$ is treated in an explicit way for simplicity. Thus, the semi-discrete problem reads

$$\text{Find } (u^n)_{0 \leq n \leq N} \text{ in } \prod_{n=0}^N \mathbb{V}(t_n) \text{ such that}$$

$$u^0 = u_0 \quad \text{in } \Omega, \quad (3.1)$$

and, for $1 \leq n \leq N$, $\forall v \in \mathbb{V}(t_n)$,

$$\alpha \int_{\Omega} \left(\frac{u^n - u^{n-1}}{\tau_n} \right) (\mathbf{x}) (v - u^n) (\mathbf{x}) \, d\mathbf{x} + \int_{\Omega} \left(\frac{b(u^n) - b(u^{n-1})}{\tau_n} \right) (\mathbf{x}) (v - u^n) (\mathbf{x}) \, d\mathbf{x}$$

$$+ \int_{\Omega} (\nabla u^n + k \circ b(u^{n-1}) \mathbf{e}_z) (\mathbf{x}) \cdot \nabla (v - u^n) (\mathbf{x}) \, d\mathbf{x} \quad (3.2)$$

$$\geq - \int_{\Gamma_F} f_F(\tau, t_n) (v - u^n) (\tau) \, d\tau - \int_{\Gamma_G} (\mathbf{q}_r \cdot \mathbf{n})(\tau, t_n) (v - u^n) (\tau) \, d\tau.$$

It can be noted that this problem makes sense when both f_F and \mathbf{q}_r are continuous in time. On the other hand, it is readily checked that the problem for each n is equivalent to the minimization of a convex functional on the convex set $\mathbb{V}(t_n)$, whence the following result.

Proposition 3.1. *For any data u_B , f_F , \mathbf{q}_r and u_0 satisfying*

$$u_B \in H^1(0, T; H_{00}^{\frac{1}{2}}(\Gamma_B)), \quad f_F \in \mathcal{C}^0(0, T; L^2(\Gamma_F)), \quad \mathbf{q}_r \in \mathcal{C}^0(0, T; L^2(\Gamma_G)^d), \quad u_0 \in H^1(\Omega), \quad (3.3)$$

and (2.7), for any nonnegative coefficient α , problem (3.1)–(3.2) has a unique solution in $\prod_{n=0}^N \mathbb{V}(t_n)$.

3.2. The fully discrete problem. We recall that Ω is a polygon ($d = 2$) or a polyhedron ($d = 3$). For $0 \leq n \leq N$, let $(\mathcal{T}_h^n)_h$ be a regular family of triangulations of Ω (by triangles or tetrahedra), in the sense that, for each h :

- $\bar{\Omega}$ is the union of all elements of \mathcal{T}_h^n ;
- The intersection of two different elements of \mathcal{T}_h^n , if not empty, is a vertex or a whole edge or a whole face of both of them;
- The ratio of the diameter h_K of any element K of \mathcal{T}_h^n to the diameter of its inscribed circle or sphere is smaller than a constant σ independent of h and n .

As usual, h_n stands for the maximum of the diameters h_K , $K \in \mathcal{T}_h^n$. We make the further and non restrictive assumption that, for all n , $\bar{\Gamma}_B$, $\bar{\Gamma}_F$ and $\bar{\Gamma}_G$ are the union of whole edges ($d = 2$) or whole faces ($d = 3$) of elements of \mathcal{T}_h^n .

We now introduce the finite element space

$$\bar{\mathbb{V}}_h^n = \{v_h \in H^1(\Omega); \forall K \in \mathcal{T}_h^n, v_h|_K \in \mathcal{P}_1(K)\},$$

where $\mathcal{P}_1(K)$ is the space of restrictions to K of affine functions on \mathbb{R}^d . Let \mathcal{I}_h^n denote the Lagrange interpolation operator at all the vertices of elements of \mathcal{T}_h^n with values in $\bar{\mathbb{V}}_h^n$ and i_h^{nB} the corresponding interpolation operator on Γ_B . Assuming that u_B is continuous where needed, we then define for each n , $0 \leq n \leq N$, the subset of $\bar{\mathbb{V}}_h^n$:

$$\mathbb{V}_h^n = \{v_h \in \bar{\mathbb{V}}_h^n; v_h|_{\Gamma_B} = i_h^{nB} u_B(\cdot, t_n) \text{ and } v_h|_{\Gamma_G} \leq 0\}.$$

We are thus in a position to write the discrete problem, constructed from problem (3.1)-(3.2) by the Galerkin method,

Find $(u_h^n)_{0 \leq n \leq N}$ in $\prod_{n=0}^N \mathbb{V}_h^n$ such that

$$u_h^0 = \mathcal{I}_h^0 u_0 \quad \text{in } \Omega, \tag{3.4}$$

and, for $1 \leq n \leq N$, $\forall v_h \in \mathbb{V}_h^n$

$$\alpha \int_{\Omega} \left(\frac{u_h^n - \mathcal{I}_h^n u_h^{n-1}}{\tau_n} \right) (\mathbf{x}) (v_h - u_h^n) (\mathbf{x}) \, d\mathbf{x} + \int_{\Omega} \left(\frac{b(u_h^n) - \mathcal{I}_h^n b(u_h^{n-1})}{\tau_n} \right) (\mathbf{x}) (v_h - u_h^n) (\mathbf{x}) \, d\mathbf{x} \tag{3.5}$$

$$+ \int_{\Omega} (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) e_z) (\mathbf{x}) \cdot \nabla (v_h - u_h^n) (\mathbf{x}) \, d\mathbf{x} \geq - \int_{\Gamma_F} f_F(\tau, t_n) (v_h - u_h^n) (\tau) \, d\tau \tag{3.6}$$

$$- \int_{\Gamma_G} (\mathbf{q}_r \cdot \mathbf{n})(\tau, t_n) (v_h - u_h^n) (\tau) \, d\tau. \tag{3.7}$$

We also recall the theorem below which is proved in [4].

Theorem 3.1. *For any data u_B , f_F , \mathbf{q}_r and u_0 satisfying 3.3, 2.6 and*

$$u_B \in \mathcal{C}^0(\bar{\Gamma}_B \times [0, T]), \quad u_0 \in \mathcal{C}^0(\bar{\Omega}),$$

for any nonnegative coefficient α , problem (3.4)-(3.5) has a unique solution.

3.3. Convergence of the discretization. We introduce a lifting u_B^* of an extension of u_B to $\partial\Omega$ which belongs to $H^1(0, T; \mathbb{V})$ and satisfies

$$\text{for a.e. } \mathbf{x} \in \Omega, \quad u_B^*(\mathbf{x}, 0) = u_0(\mathbf{x}),$$

(this requires assumption ??) together with the stability property

$$\|u_B^*\|_{H^1(0, T; H^1(\Omega))} \leq c \|u_B\|_{H^1(0, T; H_{00}^{\frac{1}{2}}(\Gamma_B))}.$$

We thus write the problem satisfied by the function $u^* = u - u_B^*$. Similarly, when setting $u_h^{*n} = u_h^n - \mathcal{I}_h^n u_B^*(t_n)$ (this requires a little more regularity on u_B^* , hence on u_B), we can write the discrete problem satisfied by the u_h^{*n} . Thus technical arguments, see [4, Lemma 4.2], lead to the following result.

Lemma 3.1. For any data u_B , f_F , \mathbf{q}_r and u_0 satisfying 2.6 and

$$u_B \in H^1(0, T; H^s(\Gamma_B)), \quad f_F \in C^0(0, T; H^{\frac{1}{2}}(\Gamma_F)), \quad \mathbf{q}_r \in C^0(0, T; H^{\frac{1}{2}}(\Gamma_G)^d), \quad u_0 \in H^{s+\frac{1}{2}}(\Omega), \quad (3.8)$$

for some $s > \frac{d-1}{2}$, the sequence $(u_h^{*n})_{0 \leq n \leq N}$ satisfies the following bound, for $1 \leq n \leq N$,

$$\alpha \sum_{m=1}^n \tau_m \left\| \frac{u_h^{*m} - u_h^{*m-1}}{\tau_m} \right\|_{L^2(\Omega)}^2 + |u_h^{*n}|_{H^1(\Omega)}^2 \leq c \left(1 + \|\mathcal{I}_h^n u_B^*\|_{H^1(0, T; H^1(\Omega))}^2 + \|f_F\|_{C^0(0, T; H^{\frac{1}{2}}(\Gamma_F))}^2 + \|\mathbf{q}_r\|_{C^0(0, T; H^{\frac{1}{2}}(\Gamma_G)^d)}^2 \right).$$

The following result is now easily derived from this boundedness. We denote by u_τ and $u_{h\tau}$ the functions which are affine on each interval $[t_{n-1}, t_n]$, $1 \leq n \leq N$, and equal to the solutions u^n and u_h^n of problems (3.1)-(3.2) and (3.4)-(3.5), respectively, in t_n , $0 \leq n \leq N$.

Theorem 3.2. For any data u_B , f_F , \mathbf{q}_r and u_0 satisfying (2.6) and (3.8), there exists a subsequence of the $(u_{h\tau})_{\tau, h}$ which converges to the solution u of problem (2.1) weakly in \mathbb{X} .

4. A POSTERIORI ANALYSIS OF THE TIME SEMI-DISCRETIZATION

As already hinted, we wish to uncouple as much as possible the time and space errors. Equivalently, we use the triangle inequality

$$\|u - u_{h\tau}\|_{\mathbb{X}} \leq \|u - u_\tau\|_{\mathbb{X}} + \|u_\tau - u_{h\tau}\|_{\mathbb{X}}, \quad (4.1)$$

and we evaluate successively the two terms in the right-hand side. In this section, we are interested in the evaluation of the first term: We propose error indicators and establish their reliability (i.e., that they provide an upper bound for the error). Their efficiency (i.e. that they are bounded from above by the error) is only hinted.

To go further, we introduce a lifting u_B^* of an extension of u_B to $\partial\Omega$ which belongs to $H^1(0, T; \mathbb{V})$ and satisfies

$$\|u_B^*\|_{H^1(0, T; H^1(\Omega))} \leq c \|u_B\|_{H^1(0, T; H_{00}^{\frac{1}{2}}(\Gamma_B))}.$$

Then it is readily checked that the function $u^* = u - u_B^*$ is a solution of a problem very similar to 2.1, with a modified function b and an additional term in the right-hand side, namely the quantity $-\int_\Omega F_B(\mathbf{x}, t)(v - u)(\mathbf{x}) d\mathbf{x}$ defined by

$$\int_\Omega F_B(\mathbf{x}, t)v(\mathbf{x}) d\mathbf{x} = \alpha \int_\Omega (\partial_t u_B^*)(\mathbf{x}, t)v(\mathbf{x}) d\mathbf{x} + \int_\Omega (\nabla u_B^*)(\mathbf{x}, t) \cdot (\nabla v)(\mathbf{x}) d\mathbf{x}.$$

An analogous property holds for the semi-discrete problem (3.1)-(3.2).

So, from now on and for simplicity, we assume that u_B is equal to zero. In a first step, in order to evaluate the norms of $u - u_\tau$, we compute the quantity, for a.e. t in $[0, T]$

$$\begin{aligned} R(t) &= \alpha \int_\Omega (\partial_t(u - u_\tau))(\mathbf{x}, t)(u - u_\tau)(\mathbf{x}, t) d\mathbf{x} + \int_\Omega (\partial_t b(u) - \partial_t b(u_\tau))(\mathbf{x}, t)(u - u_\tau)(\mathbf{x}, t) d\mathbf{x} \\ &\quad + \int_\Omega (\nabla(u - u_\tau))(\mathbf{x}, t) \cdot (\nabla(u - u_\tau))(\mathbf{x}, t) d\mathbf{x} + \int_\Omega (k \circ b(u) - k \circ b(u_\tau))(\mathbf{x}, t) \mathbf{e}_z \cdot (\nabla(u - u_\tau))(\mathbf{x}, t) d\mathbf{x}. \end{aligned}$$

The interest of this quantity appears in the following lemma. We refer to [4, Prop. 2.3] for the arguments for proving it.

Lemma 4.1. Assume that the function b belongs to $W^{2, \infty}(\mathbb{R})$. The quantity R satisfies for all t in $[0, T]$

$$\int_0^t R(s) ds \geq \frac{\alpha}{2} \|(u - u_\tau)(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_0^t |(u - u_\tau)(s)|_{H^1(\Omega)}^2 ds - c_D \int_0^t \|(u - u_\tau)(s)\|_{L^2(\Omega)}^2 ds,$$

where the constant c_D only depends on the data.

Proof. Obviously, the first and third terms in $R(t)$ lead to (note that $u - u_\tau$ vanishes at time $t = 0$)

$$\begin{aligned} \int_0^t \left(\alpha \int_\Omega (\partial_t(u - u_\tau))(\mathbf{x}, s)(u - u_\tau)(\mathbf{x}, s) \, d\mathbf{x} + \int_\Omega (\nabla(u - u_\tau))(\mathbf{x}, s) \cdot (\nabla(u - u_\tau))(\mathbf{x}, s) \, d\mathbf{x} \right) ds \\ = \frac{\alpha}{2} \|(u - u_\tau)(t)\|_{L^2(\Omega)}^2 + \int_0^t \|(u - u_\tau)(s)\|_{H^1(\Omega)}^2 ds. \end{aligned}$$

So, we now investigate successively the two remaining terms.

1) We use the decomposition

$$\begin{aligned} \int_\Omega (\partial_t b(u) - \partial_t b(u_\tau))(\mathbf{x}, t)(u - u_\tau)(\mathbf{x}, t) \, d\mathbf{x} &= \int_\Omega b'(u)(\mathbf{x}, t)(\partial_t(u - u_\tau))(\mathbf{x}, t)(u - u_\tau)(\mathbf{x}, t) \, d\mathbf{x} \\ &+ \int_\Omega (b'(u) - b'(u_\tau))(\mathbf{x}, t)(\partial_t u_\tau)(\mathbf{x}, t)(u - u_\tau)(\mathbf{x}, t) \, d\mathbf{x}, \end{aligned}$$

and integrate the first term by parts with respect to t , which gives

$$\begin{aligned} \int_0^t \int_\Omega (\partial_t b(u) - \partial_t b(u_\tau))(\mathbf{x}, s)(u - u_\tau)(\mathbf{x}, s) \, d\mathbf{x} \, ds &= \int_\Omega \frac{b'(u)(\mathbf{x}, t)}{2} (u - u_\tau)^2(\mathbf{x}, t) \, d\mathbf{x} \\ &- \frac{1}{2} \int_0^t \int_\Omega b''(u)(\mathbf{x}, s)(\partial_t u)(\mathbf{x}, s)(u - u_\tau)^2(\mathbf{x}, s) \, d\mathbf{x} \, ds \\ &+ \int_0^t \int_\Omega (b'(u) - b'(u_\tau))(\mathbf{x}, s)(\partial_t u_\tau)(\mathbf{x}, s)(u - u_\tau)(\mathbf{x}, s) \, d\mathbf{x} \, ds, \end{aligned}$$

Next, the nonnegativity of b' , the boundedness of b'' and the Lipschitz-continuity of b' yields

$$\int_0^t \int_\Omega (\partial_t b(u) - \partial_t b(u_\tau))(\mathbf{x}, s)(u - u_\tau)(\mathbf{x}, s) \, d\mathbf{x} \, ds \geq -c(u, u_\tau) \int_0^t \|(u - u_\tau)(\cdot, s)\|_{L^4(\Omega)}^4 \, ds,$$

where $c(u, u_\tau) > 0$ depends on the norm of u and u_τ in \mathbb{X} . However, using a further decomposition allows us to replace it by a constant only depending on $\|u\|_{\mathbb{X}}$, and it follows from Remark ?? that this norm is bounded as a function of the data. Next, we use an interpolation inequality (see [13, Chap. 1, Prop. 2.3]) and the Poincaré–Friedrichs inequality: for any v in $H^1(\Omega)$ vanishing on Γ_B ,

$$\|v\|_{L^4(\Omega)} \leq \|v\|_{L^2(\Omega)}^{1-\frac{d}{4}} (c|v|_{H^1(\Omega)})^{\frac{d}{4}} \leq c'(1 - \frac{d}{4})\|v\|_{L^2(\Omega)} + c'\frac{d}{4}|v|_{H^1(\Omega)},$$

and conclude by applying it with $v = u - u_\tau$ and using a Young's inequality.

2) Finally, to bound the last term, we combine the Lipschitz-continuity of $k \circ b$ together with a Young's inequality

$$\begin{aligned} \int_0^t \int_\Omega \left(k \circ b(u) - k \circ b(u_\tau) \right)(\mathbf{x}, s) e_z \cdot (\nabla(u - u_\tau))(\mathbf{x}, s) \, d\mathbf{x} \, ds &\geq -\frac{1}{4} \left(\int_0^t \|(u - u_\tau)(\cdot, s)\|_{H^1(\Omega)}^2 \, ds \right) \\ &- c \left(\int_0^t \|(u - u_\tau)(\cdot, s)\|_{L^2(\Omega)}^2 \, ds \right). \end{aligned}$$

Combining all this yields the desired estimate.

We introduce the subset \mathbb{V}_0 :

$$\mathbb{V}_0 = \{v \in H^1(\Omega); v|_{\Gamma_B} = 0 \text{ and } v|_{\Gamma_G} \leq 0\}. \quad (4.2)$$

Thus, applying problem (2.1) with $v = u_\tau$ which now belongs to \mathbb{V}_0 , we derive

$$R(t) \leq -\alpha \int_{\Omega} (\partial_t u_\tau)(\mathbf{x}, t)(u - u_\tau)(\mathbf{x}, t) d\mathbf{x} - \int_{\Omega} (\partial_t b(\mathbf{u}_\tau))(\mathbf{x}, t)(u - u_\tau)(\mathbf{x}, t) d\mathbf{x} \int_{\Omega} (\nabla u_\tau)(\mathbf{x}, t) \cdot (\nabla(u - u_\tau))(\mathbf{x}, t) d\mathbf{x} \quad (4.3)$$

$$- \int_{\Omega} ((k \circ b(\mathbf{u}_\tau))\mathbf{e}_z)(\mathbf{x}, t) \cdot (\nabla(u - u_\tau))(\mathbf{x}, t) d\mathbf{x} - \int_{\Gamma_F} f_F(\tau, t)(u - u_\tau)(\tau, t) d\tau \quad (4.4)$$

$$- \int_{\Gamma_G} (\mathbf{q}_r \cdot \mathbf{n})(\tau, t)(u - u_\tau)(\tau, t) d\tau. \quad (4.5)$$

On the other hand, we define the space

$$\mathbb{W} = \{v \in H^1(\Omega); v|_{\Gamma_B} = 0 \text{ and } v|_{\Gamma_G} = 0\}. \quad (4.6)$$

Using problem (3.2) with $v = u^n \pm w$ (which also belongs to \mathbb{V}_0 for any w in \mathbb{W}), we obtain the equation for all $w \in \mathbb{W}$,

$$\alpha \int_{\Omega} \left(\frac{u^n - u^{n-1}}{\tau_n}\right)(\mathbf{x})w(\mathbf{x}) d\mathbf{x} + \int_{\Omega} \left(\frac{b(u^n) - b(u^{n-1})}{\tau_n}\right)(\mathbf{x})w(\mathbf{x}) d\mathbf{x} \quad (4.7)$$

$$+ \int_{\Omega} (\nabla u^n + k \circ b(u^{n-1}))\mathbf{e}_z(\mathbf{x}) \cdot \nabla w(\mathbf{x}) d\mathbf{x} = - \int_{\Gamma_F} f_F(\tau, t_n)w(\tau) d\tau, \quad (4.8)$$

or, equivalently, by taking w successively in $\mathcal{D}(\Omega)$ and in $\mathcal{D}(\Omega \cup \Gamma_F)$,

$$\alpha \frac{u^n - u^{n-1}}{\tau_n} + \frac{b(u^n) - b(u^{n-1})}{\tau_n} - \nabla \cdot \left(\nabla u^n + k \circ b(u^{n-1})\mathbf{e}_z\right) = 0 \quad \text{in } \Omega, \quad (4.9)$$

$$-\left(\nabla u^n + k \circ b(u^{n-1})\mathbf{e}_z\right) \cdot \mathbf{n} = f_F(\cdot, t_n) \quad \text{on } \Gamma_F. \quad (4.10)$$

Inserting these equations into (3.2) also yields

$$\forall v \in \mathbb{V}_0, \quad \int_{\Gamma_G} \left(\nabla u^n + k \circ b(u^{n-1})\mathbf{e}_z\right) \cdot \mathbf{n}(\tau)(v - u^n)(\tau) d\tau \leq - \int_{\Gamma_G} (\mathbf{q}_r \cdot \mathbf{n})(\tau, t_n)(v - u^n)(\tau) d\tau. \quad (4.11)$$

For any function f continuous on $[0, T]$, we agree to denote by $\pi_\tau f$ the piecewise constant function equal to $f(t_n)$ on each interval $]t_{n-1}, t_n]$, $1 \leq n \leq N$. Multiplying both equations in (4.9) by $u - u_\tau$ and adding them to (4.3) (note that $\partial_t u_\tau$ is equal to $\frac{u^n - u^{n-1}}{\tau_n}$) and also using (4.11) with $v = u$ gives on the interval $]t_{n-1}, t_n[$

$$R(t) \leq \sum_{j=1}^4 R_j(t) + D(t),$$

with

$$R_1(t) = \int_{\Omega} \left(\frac{b(u^n) - b(u^{n-1})}{\tau_n} - \partial_t b(\mathbf{u}_\tau)\right)(\mathbf{x}, t)(u - u_\tau)(\mathbf{x}, t) d\mathbf{x},$$

$$R_2(t) = \int_{\Omega} \nabla(u^n - u_\tau)(\mathbf{x}, t) \cdot \nabla(u - u_\tau)(\mathbf{x}, t) d\mathbf{x},$$

$$R_3(t) = \int_{\Omega} (k \circ b(u^{n-1}) - k \circ b(\mathbf{u}_\tau))(\mathbf{x}, t)\mathbf{e}_z \cdot \nabla(u - u_\tau)(\mathbf{x}, t) d\mathbf{x},$$

$$R_4(t) = - \int_{\Gamma_G} (\nabla u^n + k \circ b(u^{n-1})\mathbf{e}_z + \pi_\tau \mathbf{q}_r) \cdot \mathbf{n}(\tau, t)(u^n - u_\tau)(\tau, t) d\tau,$$

while the data depending term D is given by

$$D(t) = - \int_{\Gamma_F} (f_F - \pi_\tau f_F)(\tau, t)(u - u_\tau)(\tau, t) d\tau - \int_{\Gamma_G} (\mathbf{q}_r - \pi_\tau \mathbf{q}_r) \cdot \mathbf{n}(\tau, t)(u - u_\tau)(\tau, t) d\tau.$$

It also follows from (4.11) that the quantity $R_4(t)$ is nonpositive.

Despite its complexity, the first a posteriori bound for the error can easily be derived from the last inequality. Indeed, we observe that, for all t in $[t_{n-1}, t_n]$,

$$u_\tau = u^n - \frac{t_n - t}{\tau_n}(u^n - u^{n-1}).$$

This leads to the following definition of the time error indicators:

$$\eta_\tau^n = \tau_n^{\frac{1}{2}} \|u_h^n - u_h^{n-1}\|_{H^1(\Omega)} + \tau_n^{\frac{1}{2}} \left\| \frac{b(u_h^n) - b(u_h^{n-1})}{\tau_n} - B_h^n \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_{L^2(\Omega)},$$

where the function B_h^n is defined as, for $1 \leq n \leq N$,

$$B_h^n = \frac{b'(u_h^{n-1}) + b'(u_h^n)}{2}.$$

Note that the first term in the indicator is fully standard for a parabolic equation, see [3, equation (3.1)] or [12], while the second one comes from the nonlinear term $\partial_t b(u)$ and is the same as introduced in [5, equation (4.1)].

Theorem 4.1. *Assume that the function b belongs to $W^{2,\infty}(\mathbb{R})$. The following a posteriori estimate holds between the solution u of problem (2.1) and the function u_τ associated with the solution $(u^n)_{0 \leq n \leq N}$ of problem (3.1)-(3.2).*

$$\sup_{0 \leq t \leq T} \|(u - u_\tau)(t)\|_{L^2(\Omega)} + \|u - u_\tau\|_{L^2(0,T;H^1(\Omega))} \leq c \left(\left(\sum_{n=1}^N (\eta_\tau^n)^2 \right)^{\frac{1}{2}} + \|u_\tau - u_{h\tau}\|_{\mathbb{X}} + \varepsilon^{(D)} + \varepsilon^{(b)} \right),$$

where the quantities $\varepsilon^{(D)}$ and $\varepsilon^{(b)}$ are given by (with obvious definition of $u_{B\tau}^*$)

$$\begin{aligned} \varepsilon^{(D)} &= \|f_F - \pi_\tau f_F\|_{L^2(\Gamma_F \times]0,T])} + \|\mathbf{q}_r - \pi_\tau \mathbf{q}_r\|_{L^2(\Gamma_G \times]0,T])^d} + \|u_B^* - u_{B\tau}^*\|_{\mathbb{X}} \\ \varepsilon^{(b)} &= \left(\sum_{n=1}^N (\varepsilon^{(b)n})^2 \right)^{\frac{1}{2}}, \quad \text{with} \quad \varepsilon^{(b)n} = \left\| (b'(u_{h\tau}) - B_h^n) \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_{L^2(t_{n-1}, t_n; L^2(\Omega))}. \end{aligned}$$

Proof. Using Cauchy–Schwarz and Young inequalities on each $R_j(t)$ and $D(t)$ together with a triangle inequality to replace u^n by u_h^n and u^{n-1} by u_h^{n-1} leads to

$$\int_0^t R(s) ds \leq c \left(\left(\sum_{n=1}^N (\eta_\tau^n)^2 \right) + \|u_\tau - u_{h\tau}\|_{\mathbb{X}}^2 + (\varepsilon^{(D)})^2 + \left(\sum_{n=1}^N (\varepsilon^{(b)n})^2 \right) \right) + c' \int_0^t \|(u - u_\tau)(s)\|_{L^2(\Omega)}^2 ds.$$

We then apply Lemma 4.1 and use the Gronwall's lemma, see [10, chap. V, lemma 1.8] for instance, to derive the desired bound (the further term including u_B^* comes from standard arguments).

Remark More complex arguments would also allow us to derive a bound for the term $\|u - u_\tau\|_{H^1(0,T;L^2(\Omega))}$, we prefer to skip them for brevity.

Proving the efficiency of these indicators, i.e. deriving an upper bound of them as a function of the error apparently requires much more technical arguments than previously, so that we prefer to skip them. Since this estimate holds (and is not simple to establish) for the heat equation [3, Prop. 3.6] and also for the Richards's equations when $\Gamma_G = \emptyset$, see [5, Prop. 4.10], both of them being provided with the same time discretization, we only say that it is likely.

5. A POSTERIORI ANALYSIS OF THE SPACE DISCRETIZATION

We now evaluate the second term of (4.1), following the same steps as in Section 4: We exhibit error indicators and prove first their reliability, second their efficiency.

5.1. The indicators and their reliability. In order to write the residual equation applied to $u_\tau - u_{h\tau}$, we compute the quantity, for $1 \leq n \leq N$,

$$\begin{aligned} R_h^n = & \alpha \int_{\Omega} \left(\frac{(u^n - u_h^n) - (u^{n-1} - \mathcal{I}_h^n u_h^{n-1})}{\tau_n} \right) (\mathbf{x}) (u^n - u_h^n) (\mathbf{x}) \, d\mathbf{x} \\ & + \int_{\Omega} \left(\frac{(b(u^n) - b(u_h^n)) - (b(u^{n-1}) - \mathcal{I}_h^n b(u_h^{n-1}))}{\tau_n} \right) (\mathbf{x}) (u^n - u_h^n) (\mathbf{x}) \, d\mathbf{x} \\ & + \int_{\Omega} (\nabla(u^n - u_h^n)) (\mathbf{x}) \cdot (\nabla(u^n - u_h^n)) (\mathbf{x}) \, d\mathbf{x} + \int_{\Omega} (k \circ b(u^{n-1}) - \mathcal{I}_h^n k \circ b(u_h^{n-1})) (\mathbf{x}) \mathbf{e}_z \cdot (\nabla(u^n - u_h^n)) (\mathbf{x}) \, d\mathbf{x}. \end{aligned}$$

We now prove the analogue of Lemma 4.1 for this quantity.

Lemma 5.1. *Assume that the function b is of class \mathcal{C}^2 with bounded derivatives. The quantities R_h^n satisfy for all n , $1 \leq n \leq N$,*

$$\sum_{m=1}^n \tau_m R_h^m \geq \frac{\alpha}{2} \|u^n - u_h^n\|_{L^2(\Omega)}^2 + \frac{1}{2} \sum_{m=1}^n \tau_m |u^m - u_h^m|_{H^1(\Omega)}^2 - c \sum_{m=1}^{n-1} \tau_m \|u^m - u_h^m\|_{L^2(\Omega)}^2 - \sum_{m=1}^n (\varepsilon^{(I)m})^2,$$

where the quantity $\varepsilon^{(I)n}$ is defined by

$$\varepsilon^{(I)n} = \tau_n^{-\frac{1}{2}} \|(Id - \mathcal{I}_h^n)(u_h^n - u_h^{n-1})\|_{L^2(\Omega)} + \tau_n^{-\frac{1}{2}} \|(Id - \mathcal{I}_h^n)b(u_h^{n-1})\|_{L^2(\Omega)} + \tau_n \|(Id - \mathcal{I}_h^n)k \circ b(u_h^{n-1})\|_{L^2(\Omega)}. \quad (5.1)$$

Proof. For brevity, we denote by R^1, \dots and R^4 the four lines in R_h^n . As for Lemma 4.1, the first and third terms in R_h^n can be handled obviously

$$\begin{aligned} \tau_n R^1 & \geq \frac{\alpha}{2} (\|u^n - u_h^n\|_{L^2(\Omega)}^2 - \|u^{n-1} - u_h^{n-1}\|_{L^2(\Omega)}^2) + \|(u^n - u_h^n) - (u^{n-1} - u_h^{n-1})\|_{L^2(\Omega)}^2 \\ & \quad - \alpha \|(Id - \mathcal{I}_h^n)(u_h^n - u_h^{n-1})\|_{L^2(\Omega)} \|u^n - u_h^n\|_{L^2(\Omega)}, \\ \tau_n R^3 & = \tau_n |u^n - u_h^n|_{H^1(\Omega)}^2. \end{aligned}$$

To estimate the second term in $R_h(n)$, we use the Lipschitz-continuity of b and the fact that it is not decreasing: if κ denotes its Lipschitz constant,

$$\tau_n R_2 \geq \frac{\kappa}{2} (\|u^n - u_h^n\|_{L^2(\Omega)}^2 - \|u^{n-1} - u_h^{n-1}\|_{L^2(\Omega)}^2) - c \|(Id - \mathcal{I}_h^n)b(u_h^{n-1})\|_{L^2(\Omega)}^2.$$

Finally evaluating the last term relies on the Lipschitz property of $k \circ b$:

$$\tau_n R_4 \geq -\frac{\tau_n}{2} |u^n - u_h^n|_{H^1(\Omega)}^2 - c \tau_n \|u^{n-1} - u_h^{n-1}\|_{L^2(\Omega)}^2.$$

We conclude by summing these 4 inequalities and then summing up on n .

As in Section 4, in order to avoid the technicality linked to the lifting of u_B , we assume that u_B and thus its interpolate are zero. It follows from problem (3.1)-(3.2) that each R_h^n satisfies

$$\begin{aligned} R_h^n \leq & -\alpha \int_{\Omega} \left(\frac{u_h^n - \mathcal{I}_h^n u_h^{n-1}}{\tau_n} \right) (\mathbf{x}) (u^n - u_h^n) (\mathbf{x}) \, d\mathbf{x} \int_{\Omega} \left(\frac{b(u_h^n) - \mathcal{I}_h^n b(u_h^{n-1})}{\tau_n} \right) (\mathbf{x}) (u^n - u_h^n) (\mathbf{x}) \, d\mathbf{x} \\ & - \int_{\Omega} (\nabla u_h^n) (\mathbf{x}) \cdot (\nabla(u^n - u_h^n)) (\mathbf{x}) \, d\mathbf{x} - \int_{\Omega} (\mathcal{I}_h^n k \circ b(u_h^{n-1})) (\mathbf{x}) \mathbf{e}_z \cdot (\nabla(u^n - u_h^n)) (\mathbf{x}) \, d\mathbf{x} \\ & - \int_{\Gamma_F} f_F(\tau, t_n) (u^n - u_h^n) (\tau) \, d\tau - \int_{\Gamma_G} (\mathbf{q}_r \cdot \mathbf{n})(\tau, t_n) (u^n - u_h^n) (\tau) \, d\tau. \end{aligned}$$

For the space \mathbb{W} defined in (4.6), we set

$$\mathbb{W}_h = \overline{\mathbb{V}}_h^n \cap \mathbb{W}.$$

Indeed, taking $v_h = u_h^n \pm w_h$ in problem (3.5) yields $\forall w_h \in \mathbb{W}_h$,

$$\begin{aligned} \alpha \int_{\Omega} \left(\frac{u_h^n - \mathcal{I}_h^n u_h^{n-1}}{\tau_n} \right) (\mathbf{x}) w_h(\mathbf{x}) d\mathbf{x} + \int_{\Omega} \left(\frac{b(u_h^n) - \mathcal{I}_h^n b(u_h^{n-1})}{\tau_n} \right) (\mathbf{x}) w_h(\mathbf{x}) d\mathbf{x} \\ + \int_{\Omega} (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z) (\mathbf{x}) \cdot \nabla w_h(\mathbf{x}) d\mathbf{x} = - \int_{\Gamma_F} f_F(\tau, t_n) w_h(\tau) d\tau. \end{aligned}$$

We use this equation by taking w_h equal to $v_h - u_h^n$, where v_h is any element in \mathbb{V}_h^n which coincides with u_h^n on Γ_G . This yields

$$\begin{aligned} R_h^n \leq & -\alpha \int_{\Omega} \left(\frac{u_h^n - \mathcal{I}_h^n u_h^{n-1}}{\tau_n} \right) (\mathbf{x}) (u^n - v_h)(\mathbf{x}) d\mathbf{x} - \int_{\Omega} \left(\frac{b(u_h^n) - \mathcal{I}_h^n b(u_h^{n-1})}{\tau_n} \right) (\mathbf{x}) (u^n - v_h)(\mathbf{x}) d\mathbf{x} \\ & - \int_{\Omega} (\nabla u_h^n) (\mathbf{x}) \cdot (\nabla (u^n - v_h)) (\mathbf{x}) d\mathbf{x} - \int_{\Omega} (\mathcal{I}_h^n k \circ b(u_h^{n-1})) (\mathbf{x}) \mathbf{e}_z \cdot (\nabla (u^n - v_h)) (\mathbf{x}) d\mathbf{x} \\ & - \int_{\Gamma_F} f_F(\tau, t_n) (u^n - v_h)(\tau) d\tau - \int_{\Gamma_G} (\mathbf{q}_r \cdot \mathbf{n})(\tau, t_n) (u^n - v_h)(\tau) d\tau. \end{aligned}$$

We need some further notation: For each K in \mathcal{T}_h^n , let \mathcal{E}_K^n , \mathcal{E}_{FK}^n and \mathcal{E}_{GK}^n denote the sets of edges ($d = 2$) or faces ($d = 3$) of K that are not contained in $\partial\Omega$ or lie in $\bar{\Gamma}_F$ or lie in $\bar{\Gamma}_G$, respectively. For any edge or face e in any of these sets, h_e stands for the diameter of e . We also need the notation, for any quantity a ,

$$[a]_+ = \max\{a, 0\}, \quad [a]_- = \min\{a, 0\}.$$

As standard, we introduce an approximation f_{Fh}^n of $f_F^n = f_F(\cdot, t_n)$ in the space of piecewise constant functions relying on the triangulation of Γ_F by edges or faces of elements of \mathcal{T}_h^n and an approximation \mathbf{q}_{rh}^n of $\mathbf{q}_r(\cdot, t_n)$ in the space of piecewise constant functions on Γ_G .

We are thus in a position to define the error indicators. Since we prefer to treat separately the boundary term linked to the inequality, we define:

(i) For each K in \mathcal{T}_h^n ,

$$\eta_K^n = h_K \left\| \alpha \frac{u_h^n - \mathcal{I}_h^n u_h^{n-1}}{\tau_n} + \frac{b(u_h^n) - \mathcal{I}_h^n b(u_h^{n-1})}{\tau_n} - \nabla \cdot (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z) \right\|_{L^2(K)} \quad (5.2)$$

$$+ \sum_{e \in \mathcal{E}_K^n} h_e^{\frac{1}{2}} \left\| [(\nabla u_h^n + (\mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z) \cdot \mathbf{n})]_e \right\|_{L^2(e)} + \sum_{e \in \mathcal{E}_{FK}^n} h_e^{\frac{1}{2}} \left\| (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z) \cdot \mathbf{n} + f_{Fh}^n \right\|_{L^2(e)}, \quad (5.3)$$

where $[\cdot]_e$ stands for the jump through e (making its sign precise is not necessary).

(ii) For each e in \mathcal{E}_{GK}^n ,

$$\eta_{Ge}^n = \eta_{Ge1}^n + \eta_{Ge2}^n, \quad \text{with } \eta_{Ge1}^n = \left\| [\mathbf{q}_{rh}^n + (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z) \cdot \mathbf{n}]_+ \right\|_{L^2(e)}, \quad (5.4)$$

$$\eta_{Ge2}^n = \int_e \left[(\mathbf{q}_{rh}^n + (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z)(\tau) \cdot \mathbf{n}) \right]_- u_h^n(\tau) d\tau. \quad (5.5)$$

Despite their apparent complexity, these indicators only involve polynomials of low degree, so they are easy to compute. Moreover, the indicators η_{Ge}^n are not zero only for the e contained in $\bar{\Gamma}_G$, hence are not so many, and are the analogues of those proposed in [11] in a simpler case. With each element K in \mathcal{T}_h^n , we associate the real number s_K , which is positive in dimension $d = 2$ if K intersects $\bar{\Gamma}_G$ and equal to zero otherwise. We are in a position to prove the next result.

Theorem 5.1. *The following a posteriori estimate holds between the solution $(u^n)_{0 \leq n \leq N}$ of problem (3.1)-(3.2) and the solution $(u_h^n)_{0 \leq n \leq N}$ of problem (3.4)-(3.5)*

$$\max_{0 \leq n \leq N} \|u^n - u_h^n\|_{L^2(\Omega)}^2 + \frac{1}{2} \sum_{m=1}^N \tau_m \|u^m - u_h^m\|_{H^1(\Omega)}^2 \leq c \sum_{m=1}^N \tau_m \sum_{K \in \mathcal{T}_h^m} (h_K^{-2s_K} (\eta_K^m)^2 + \sum_{e \in \mathcal{E}_{GK}^m} \eta_{Ge}^m) \quad (5.6)$$

$$+ \sum_{m=1}^N (\varepsilon^{(I)m})^2 + \varepsilon_h^{(D)}, \quad (5.7)$$

where the quantity $\varepsilon_h^{(D)}$ is given by

$$\varepsilon_h^{(D)} = \left(\sum_{m=1}^n \sum_{e \in \mathcal{E}_{KF}^m} \tau_m h_e^{1-2s_K} \|f_F^m - f_{Fh}^m\|_{L^2(e)}^2 \right)^{\frac{1}{2}} + \left(\sum_{m=1}^n \sum_{e \in \mathcal{E}_{GK}^m} \tau_m h_e^{-2} \|\mathbf{q}_r^m - \mathbf{q}_{rh}^m\|_{L^2(e)}^2 \right)^{\frac{1}{2}},$$

and the $\varepsilon^{(I)m}$ are defined in (5.1).

Proof. By integrating by parts on each K and using Cauchy-Schwarz inequalities, we derive

$$\begin{aligned} R_h^n &\leq \sum_{K \in \mathcal{T}_h^n} \left(\|\alpha \frac{u_h^n - \mathcal{I}_h^n u_h^{n-1}}{\tau_n} + \frac{b(u_h^n) - \mathcal{I}_h^n b(u_h^{n-1})}{\tau_n} - \nabla \cdot (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}))\|_{L^2(K)} \|u^n - v_h\|_{L^2(K)} \right. \\ &\quad + \sum_{e \in \mathcal{E}_K^n} \|[(\nabla u_h^n + (\mathcal{I}_h^n k \circ b(u_h^{n-1}))) \cdot \mathbf{n}]_e\|_{L^2(e)} \|u^n - v_h\|_{L^2(e)} \\ &\quad + \sum_{e \in \mathcal{E}_{FK}^n} \|(\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1})) \cdot \mathbf{n} + f^n\|_{L^2(e)} \|u^n - v_h\|_{L^2(e)} \\ &\quad \left. - \sum_{e \in \mathcal{E}_{GK}^n} \int_e (\mathbf{q}_r^n + (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1})) \cdot \mathbf{n})(\tau) \cdot \mathbf{n}(\tau) (u^n - u_h^n)(\tau) d\tau \right) \end{aligned}$$

We treat separately the first three lines of R_h^n that we denote by R_1^n and the fourth line that we denote by R_2^n .

1) We insert the approximation of the datum f^n , use the trace inequality (obtained by switching to the reference element and using [8, Thm 1.6.6])

$$\|u^n - v_h\|_{L^2(e)} \leq c(h_K^{\frac{1}{2}} |u^n - v_h|_{H^1(K)} + h_K^{-\frac{1}{2}} \|u^n - v_h\|_{L^2(K)}),$$

and also triangle and Young's inequalities. We also choose v_h equal to $u_h^n + w_h$, where w_h is the function defined in Lemma A.1 of the Appendix with u replaced by $u^n - u_h^n$. All this gives

$$\|u^n - v_h\|_{L^2(K)} \leq c h_K^{1-s_K} \|u^n - u_h^n\|_{H^1(K)}, \quad \|u^n - v_h\|_{H^1(K)} \leq c h_K^{-s_K} \|u^n - u_h^n\|_{H^1(K)}.$$

Thus, we derive, with obvious definition of $\varepsilon_h^{(D)n}$,

$$R_1^n \leq c \left(\sum_{K \in \mathcal{T}_h^n} (h_K^{-2s_K} (\eta_K^n)^2 + (\varepsilon_h^{(D)n})^2) \right)^{\frac{1}{2}} \|u^n - u_h^n\|_{H^1(\Omega)}.$$

2) To evaluate R_2^n , we observe that, since both u^n and u_h^n are nonpositive on Γ_G ,

$$\begin{aligned} R_2^n &\leq - \sum_{e \in \mathcal{E}_{GK}^n} \int_e [\mathbf{q}_r^n + (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1})) \cdot \mathbf{n}(\tau)]_+ u^n(\tau) d\tau \\ &\quad + \sum_{e \in \mathcal{E}_{GK}^n} \int_e [\mathbf{q}_r^n + (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1})) \cdot \mathbf{n}(\tau)]_- u_h^n(\tau) d\tau, \end{aligned}$$

whence, for a constant c that there also only depends on the data,

$$R_2^n \leq c \eta_{Ge1}^n + \eta_{Ge2}^n.$$

In a final step, we replace n by m , multiply this equation by τ_m and sum up on m . When combining the result with Lemma 5.1 and taking h_0 smaller than 1, we derive

$$\frac{\alpha}{2} \|u^n - u_h^n\|_{L^2(\Omega)}^2 + \frac{1}{2} \sum_{m=1}^n \tau_m \|u^m - u_h^m\|_{H^1(\Omega)}^2 \leq c \sum_{m=1}^n \tau_m \|u^m - u_h^m\|_{L^2(\Omega)}^2 + R_n,$$

where the quantity R_n is given by

$$R_n = \sum_{m=1}^n \left((\varepsilon^{(I)m})^2 + \frac{\tau_m}{2} \sum_{K \in \mathcal{T}_h^m} (h_K^{-2s_K} (\eta_K^m)^2 + (\varepsilon_h^{(D)m})^2) + \sum_{e \in \mathcal{E}_{GK}^m} \eta_{Ge}^m \right).$$

Thus, applying the discrete Gronwall's lemma (see [10, Chap. V, Lemma 2.4]) and noting that $\sum_{m=1}^n \tau_m$ is smaller than T , we obtain the desired result.

5.2. Efficiency of the indicators. We successively prove an upper bound first for the indicators η_K^n , next for the indicators η_{Ge}^n . For the set \mathbb{V}_0 defined in (4.2), we compute the residual term $R_h^n(w)$ for any w in \mathbb{V}_0

$$\begin{aligned} R_h^n(w) = & \alpha \int_{\Omega} \left(\frac{(u^n - u_h^n) - (u^{n-1} - \mathcal{I}_h^n u_h^{n-1})}{\tau_n} \right) (\mathbf{x}) w(\mathbf{x}) \, d\mathbf{x} \\ & + \int_{\Omega} \left(\frac{(b(u^n) - b(u_h^n)) - (b(u^{n-1}) - \mathcal{I}_h^n b(u_h^{n-1}))}{\tau_n} \right) (\mathbf{x}) w(\mathbf{x}) \, d\mathbf{x} \\ & + \int_{\Omega} (\nabla(u^n - u_h^n) + (k \circ b(u^{n-1}) - \mathcal{I}_h^n k \circ b(u_h^{n-1})) \mathbf{e}_z) (\mathbf{x}) \cdot \nabla w(\mathbf{x}) \, d\mathbf{x}. \end{aligned}$$

Indeed, the following result follows from Cauchy–Schwarz inequalities and the Lipschitz-continuity of b and $k \circ b$.

For brevity, we use the notation δu^n for the quantity $\frac{u^n - u^{n-1}}{\tau_n}$ and its analogues δu^n , $\delta(u - u_h)^n$.

Lemma 5.2. *For all n , $1 \leq n \leq N$, and for all w in \mathbb{W} , the quantity $R_h^n(w)$ satisfies*

$$|R_h^n(w)| \leq |(\|\delta(u - u_h)^n\|_{L^2(\omega_w)} + \varepsilon_1^{(I)n}) \|w\|_{L^2(\omega_w)} + (|u^n - u_h^n|_{H^1(\omega_w)} + c \|u^{n-1} - u_h^{n-1}\|_{L^2(\omega_w)} + \varepsilon_2^{(I)n}) |w|_{H^1(\omega_w)},$$

where ω_w stands for the support of w and the quantities $\varepsilon_1^{(I)n}$ and $\varepsilon_2^{(I)n}$ are defined by

$$\varepsilon_1^{(I)n} = \tau_n^{-1} \|(Id - \mathcal{I}_h^n)(u_h^n - u_h^{n-1})\|_{L^2(\omega_w)} + \tau_n^{-1} \|(Id - \mathcal{I}_h^n)b(u_h^{n-1})\|_{L^2(\omega_w)}, \quad \varepsilon_2^{(I)n} = \|(Id - \mathcal{I}_h^n)k \circ b(u_h^{n-1})\|_{L^2(\omega_w)}. \quad (5.8)$$

To bound the indicators η_K^n , we take w in the space \mathbb{W} defined in (4.6). The residual equation can easily be derived from (4.7):

$$\begin{aligned} R_h^n(w) = & -\alpha \int_{\Omega} \left(\frac{u_h^n - \mathcal{I}_h^n u_h^{n-1}}{\tau_n} \right) (\mathbf{x}) w(\mathbf{x}) \, d\mathbf{x} - \int_{\Omega} \left(\frac{b(u_h^n) - \mathcal{I}_h^n b(u_h^{n-1})}{\tau_n} \right) (\mathbf{x}) w(\mathbf{x}) \, d\mathbf{x} \\ & + \int_{\Omega} (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1})) \mathbf{e}_z (\mathbf{x}) \cdot \nabla w(\mathbf{x}) \, d\mathbf{x} + \int_{\Gamma_F} f_F(\tau, t_n) w(\tau) \, d\tau, \end{aligned}$$

or equivalently, by integrating by parts on each K ,

$$\begin{aligned} R_h^n(w) &= \sum_{K \in \mathcal{T}_h^n} \left(- \int_K \left(\alpha \frac{u_h^n - \mathcal{I}_h^n u_h^{n-1}}{\tau_n} + \frac{b(u_h^n) - \mathcal{I}_h^n b(u_h^{n-1})}{\tau_n} - \nabla \cdot (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z) \right) w(x) dx \right. \\ &\quad - \frac{1}{2} \sum_{e \in \mathcal{E}_K^n} \int_e [\nabla u_h^n + (\mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z) \cdot \mathbf{n}]_e(\tau) w(\tau) d\tau - \sum_{e \in \mathcal{E}_{FK}^n} \int_e ((\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z)(\tau) \cdot \mathbf{n}(\tau) \\ &\quad \left. + f_F^n(\tau)) w(\tau) d\tau \right). \end{aligned}$$

The bound for η_K^n can easily be derived from this equation thank to standard choices of the function w .

Proposition 5.1. *For $1 \leq n \leq N$ and for all K in \mathcal{T}_h^n , the following bound holds for the indicators η_K^n defined in (5.2)*

$$(\text{??}) \eta_K^n \leq c h_K (\|\delta(u - u_h)^n\|_{L^2(K)} + \varepsilon_1^{(I)n}) + (|u^n - u_h^n|_{H^1(K)} + \|u^{n-1} - u_h^{n-1}\|_{L^2(K)} + \varepsilon_2^{(I)n} + \varepsilon^{(D)n}), \quad (5.9)$$

where ω_K is the union of elements of \mathcal{T}_h^n that share at least an edge ($d = 2$) or a face ($d = 3$) with K , the quantities $\varepsilon_1^{(I)n}$ and $\varepsilon_2^{(I)n}$ are defined in (5.8), and the quantity $\varepsilon^{(D)n}$ is given by

$$\varepsilon^{(D)n} = \sum_{e \in \mathcal{E}_{KF}} h_e^{\frac{1}{2}} \|f_F^n - f_{Fh}^n\|_{L^2(e)}.$$

Proof. We bound successively the three terms in η_K^n .

1) The idea consists in taking in (??) w equal to the function w_K defined by

$$w_K = - \left(\alpha \frac{u_h^n - \mathcal{I}_h^n u_h^{n-1}}{\tau_n} + \frac{b(u_h^n) - \mathcal{I}_h^n b(u_h^{n-1})}{\tau_n} - \nabla \cdot (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z) \right) \psi_K \text{ on } K,$$

where ψ_K is the bubble function on K (equal to the product of the barycentric coordinates associated with the vertices of K). Indeed, this yields

$$R_h^n(w_K) = \left\| \left(\alpha \frac{u_h^n - \mathcal{I}_h^n u_h^{n-1}}{\tau_n} + \frac{b(u_h^n) - \mathcal{I}_h^n b(u_h^{n-1})}{\tau_n} - \nabla \cdot (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z) \right) \psi_K^{\frac{1}{2}} \right\|_{L^2(K)}^2.$$

By combining this with the inverse inequalities, valid for any polynomial φ of fixed degree [19, Prop. 3.37]

$$\|\varphi\|_{L^2(K)} \leq c \|\varphi \psi_K^{\frac{1}{2}}\|_{L^2(K)}, \quad \|\varphi \psi^K\|_{L^2(K)} \leq \|\varphi\|_{L^2(K)} \quad |\varphi \psi^K|_{H^1(K)} \leq c h_K^{-1} \|\varphi\|_{L^2(K)},$$

and using Lemma 5.3, we derive

$$h_K \left\| \alpha \frac{u_h^n - \mathcal{I}_h^n u_h^{n-1}}{\tau_n} + \frac{b(u_h^n) - \mathcal{I}_h^n b(u_h^{n-1})}{\tau_n} - \nabla \cdot (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z) \right\|_{L^2(K)} \quad (5.10)$$

$$\leq c h_K (\|\delta(u - u_h)^n\|_{L^2(K)} + \varepsilon_1^{(I)n}) + (|u^n - u_h^n|_{H^1(K)} + \|u^{n-1} - u_h^{n-1}\|_{L^2(K)} + \varepsilon_2^{(I)n}). \quad (5.11)$$

2) For each edge or face e of an element K , we introduce a lifting operator $\mathcal{L}_{e,K}$ which \bullet maps polynomials on e vanishing on ∂e into polynomials on K vanishing on $\partial K \setminus e$, \bullet and is constructed from a fixed operator on the reference element. Next, for each edge or face e shared by two elements K and K' , we set

$$\begin{aligned} w_e &= \mathcal{L}_{e,\kappa} \left(- [(\nabla u_h^n + (\mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z) \cdot \mathbf{n}]_e \psi_e \right) && \text{on } \kappa \in \{K, K'\}, \\ &= 0 && \text{on } \Omega \setminus (K \cup K'), \end{aligned}$$

where ψ_e is now the bubble function on e . By using the property of the operator $\mathcal{L}_{e,\kappa}$ which is easily derived by switching to the reference element (recall that φ is any polynomial of fixed degree)

$$\|\mathcal{L}_{e,\kappa}\varphi\|_{L^2(\kappa)} + h_\kappa \|\mathcal{L}_{e,\kappa}\varphi\|_{H^1(\kappa)} \leq h_e^{\frac{1}{2}} \|\varphi\|_{L^2(e)},$$

the new inverse inequalities [19, Prop. 3.37]

$$\|\varphi\|_{L^2(e)} \leq c \|\varphi\|_{L^2(e)}^{\frac{1}{2}} \|\varphi\|_{L^2(K)}, \quad \|\varphi\|_{L^2(e)} \leq \|\varphi\|_{L^2(K)},$$

and (5.10), we derive

$$h_e^{\frac{1}{2}} \left\| \left[(\nabla u_h^n + (\mathcal{I}_h^n k \circ b(u_h^{n-1}))) \cdot \mathbf{n} \right]_e \right\|_{L^2(e)} \leq c \left(\sum_{\kappa \in \{K, K'\}} h_\kappa (\|\delta(u - u_h)^n\|_{L^2(\kappa)} + \varepsilon_1^{(I)n}) + c' (|u^n - u_h^n|_{H^1(K \cup K')}) \right. \\ \left. + \|u^{n-1} - u_h^{n-1}\|_{L^2(K \cup K')} + \varepsilon_2^{(I)n} \right).$$

3) To bound the terms $h_e^{\frac{1}{2}} \|(\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z) \cdot \mathbf{n} + f_{Fh}^n\|_{L^2(e)}$, if e is contained in K , we take w equal to \tilde{w}_e , with

$$\tilde{w}_e = \begin{cases} \mathcal{L}_{e,\kappa} \left(- \left((\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z) \cdot \mathbf{n} + f_{Fh}^n \right) \psi_e \right) & \text{on } K, \\ 0 & \text{on } \Omega \setminus K. \end{cases} \quad (5.12)$$

We use exactly the same arguments as in the previous step and obtain the same estimate only with the further term $h_e^{\frac{1}{2}} \|f_F^n - f_{Fh}^n\|_{L^2(e)}$ in the right-hand side. All these bounds give estimate (??).

Finally, to bound the indicators η_{Ge}^n , we use the quantity $R_h^n(w)$ for w in \mathbb{V}_0 . For simplicity, we define the set Ω_{Gh}^n as the domain made of Γ_G and the interior of all triangles K in \mathcal{T}_h^n having at least an edge ($d=2$) or a face ($d=3$) lying on Γ_G . We denote \mathcal{T}_{Gh}^n the set of triangles contained in $\overline{\Omega}_{Gh}^n$ and we assume (this is not restrictive) that two elements in \mathcal{T}_{Gh}^n share at most a vertex ($d=2$) or an edge ($d=3$). Thus, using problem (3.2) and integrating by parts on each K yields for any w having its support in Ω_{Gh}^n

$$R_h^n(w) \geq \sum_{K \in \mathcal{T}_{Gh}^n} \left(- \int_K \left(\alpha \frac{u_h^n - \mathcal{I}_h^n u_h^{n-1}}{\tau_n} + \frac{b(u_h^n) - \mathcal{I}_h^n b(u_h^{n-1})}{\tau_n} - \nabla \cdot (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z) \right) (\mathbf{x}) w(\mathbf{x}) dx \right. \\ \left. - \sum_{e \in \mathcal{E}_{GK}^n} \int_e \left((\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z + \mathbf{q}_r^n) (\tau) \cdot \mathbf{n}(\tau) w(\tau) d\tau \right) \right).$$

Proposition 5.2. *For $1 \leq n \leq N$ and for all K in \mathcal{T}_h^n and each e in \mathcal{E}_{GK}^n , the following bound holds for the indicators η_{Ge1}^n defined in (5.4)*

$$\eta_{Ge1}^n \leq c h_K \left(\|\delta(u - u_h)^n\|_{L^2(K)} + \varepsilon_1^{(I)n} \right) + (|u^n - u_h^n|_{H^1(K)} + \|u^{n-1} - u_h^{n-1}\|_{L^2(K)} + \varepsilon_2^{(I)n} + \varepsilon^{(D)n}), \quad (5.13)$$

where the quantities $\varepsilon_1^{(I)n}$ and $\varepsilon_2^{(I)n}$ are defined in (5.1), and the quantity $\varepsilon^{(D)n}$ is given by

$$\varepsilon^{(D)n} = \sum_{e \in \mathcal{E}_{GK}} h_e^{\frac{1}{2}} \|(\mathbf{q}_r^n - \mathbf{q}_{rh}^n) \cdot \mathbf{n}\|_{L^2(e)}. \quad (5.14)$$

Proof. With the same notation as in the previous proof, we take w equal to \bar{w}_e , with

$$\bar{w}_e = \begin{cases} \mathcal{L}_{e,K} \left(- \left[(\mathbf{q}_{rh}^n + \nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z) \cdot \mathbf{n} \right]_+ \psi_e \right) & \text{on } K, \\ 0 & \text{on } \Omega \setminus K. \end{cases} \quad (5.15)$$

It can be noted that w is nonpositive on e , so that it belongs to \mathbb{V}_0 . By combining the same inverse inequalities as previously together with a Cauchy-Schwarz inequality and Proposition 5.4, we obtain

the estimate for η_{Ge1}^n .

A slightly different argument is needed to bound η_{Ge2}^n .

Proposition 5.3. *For $1 \leq n \leq N$ and for all K in \mathcal{T}_h^n and each e in \mathcal{E}_{GK}^n , the following bound holds for the indicators η_{Ge2}^n defined in (5.4)*

$$\eta_{Ge2}^n \leq c h_K (\|\delta(u - u_h)^n\|_{L^2(K)} + \varepsilon_1^{(I)n}) + c (\|u^n - u_h^n\|_{H^1(K)} + \|u^{n-1} - u_h^{n-1}\|_{H^1(K)} + \|\partial_n(u^n - u_h^n)\|_{H^{-\frac{1}{2}}(e)}) \quad (5.16)$$

$$+ \varepsilon_2^{(I)n} + \varepsilon^{(D)n}, \quad (5.17)$$

where the quantity $\varepsilon_2^{(I)n}$ is defined in (5.1), and the quantity $\varepsilon^{(D)n}$ is defined in (4.11).

Proof. We first bound the quantity

$$E = \int_e (\mathbf{q}_{rh}^n + (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z)(\tau) \cdot \mathbf{n}(\tau)) u_h^n(\tau) d\tau.$$

We deduce from (4.11) (with $v = 2u^n$) that

$$\begin{aligned} E &\leq \int_e (\mathbf{q}_{rh}^n + (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z)(\tau) \cdot \mathbf{n}(\tau)) u_h^n(\tau) d\tau - \int_e (\mathbf{q}_r^n + (\nabla u^n + k \circ b(u^{n-1}) \mathbf{e}_z)(\tau) \cdot \mathbf{n}(\tau)) u^n(\tau) d\tau \\ &\leq \int_e (\mathbf{q}_{rh}^n + (\nabla u_h^n + \mathcal{I}_h^n k \circ b(u_h^{n-1}) \mathbf{e}_z)(\tau) \cdot \mathbf{n}(\tau)) (u_h^n - u^n)(\tau) d\tau \\ &\quad + \int_e (\mathbf{q}_{rh}^n - \mathbf{q}_{rn} + \nabla(u_h^n - u^n) - \mathcal{I}_h^n k \circ b(u_h^{n-1}) + k \circ b(u^{n-1}) \mathbf{e}_z)(\tau) \cdot \mathbf{n}(\tau) u^n(\tau) d\tau \end{aligned}$$

Cauchy–Schwarz inequalities combined with the Lipschitz–continuity of $k \circ b$ yield

$$E \leq c (\|u^n - u_h^n\|_{H^1(K)} + \|u^{n-1} - u_h^{n-1}\|_{H^1(K)} + \|\partial_n(u^{n-1} - u_h^{n-1})\|_{H^{-\frac{1}{2}}(e)} + \varepsilon_2^{(I)n} + \varepsilon^{(D)n}),$$

(it can be noted from problem (3.2) that Δu^n belongs to $L^2(\Omega)$, so that the quantity $\|\partial_n(u^{n-1} - u_h^{n-1})\|_{H^{-\frac{1}{2}}(e)}$ is well-defined).

To conclude, we deduce from the formula $a = [a]_+ + [a]_-$ that

$$\eta_{Ge2}^n \leq E + c \eta_{Ge1}^n,$$

and we use estimate (5.13).

Due to the presence of s_K , estimate (5.6) is not fully optimal, but the lack of optimality seems negligible (s_K is equal to zero in dimension $d = 3$ and also in nearly all the triangles in dimension $d = 2$ and is any positive number elsewhere). Estimate (5.16) is also not optimal because of the term involving the normal derivative. However, estimates (??) (5.13) and (5.16) are local (i.e. only involve the error in a neighborhood of K), so that they are an efficient tool for mesh adaptivity.

6. NUMERICAL RESULTS

In this section we consider the well-known test case of Abdull and Gilham ([]). The experiment consists in a Plexiglas trapezoidal sandbox filled with medium-fine sand. The domain is represented at figure 6 where the dimension are indicated. Note that the surface slope is uniform equal to 14° . The initial water table is assigned at the height of 0.78 m.

The volumetric water content given by the Van Genuchten formula

$$\theta(\psi) = \begin{cases} \theta_s (1 + (\varepsilon|h|)^n)^{-m} & \text{if } h < 0, \\ \theta_s & \text{if } h \geq 0, \end{cases}$$

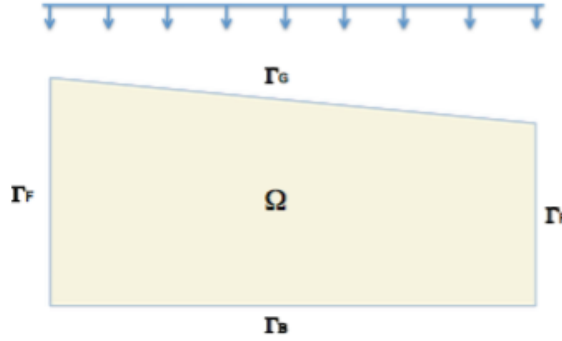


FIGURE 1. The computational domain

$\theta_s = 0.55, n = 1.9, m = 1 - 1/n, \varepsilon = 0.036\text{cm}^{-1}$.

The conductivity is given by

$$K(h) = \begin{cases} \frac{K_s(1-(\varepsilon|h|)^{n-1})(1+(\varepsilon|h|)^n)^{-m}}{(1+(\varepsilon|h|)^n)^{m/2}} & \text{if } h < 0, \\ K_s & \text{if } h \geq 0, \end{cases}$$

with $K_s = 5 \cdot 10^{-4} \text{cm s}^{-1}$. A constant rainfall rate of $0.1 K_s$ is applied on the whole surface domain, and no flow boundary is enforced on the bottom and on the side walls.

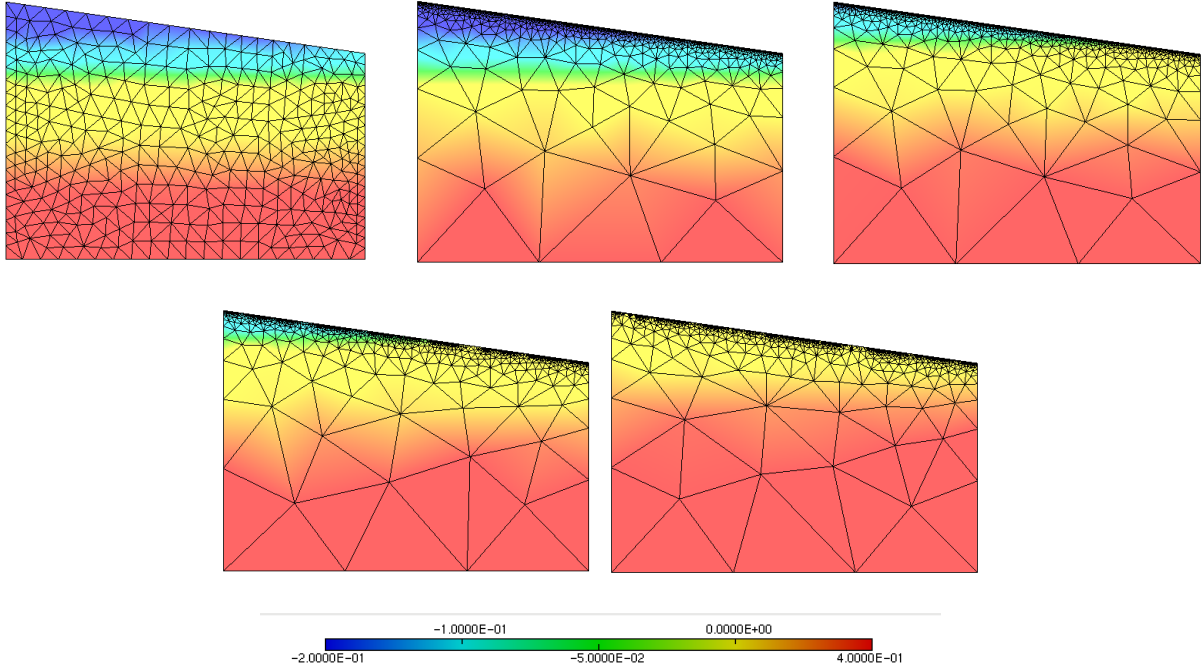
We denote by p the percentage of the underground porous media which is unsaturated and we set $\mathcal{P} = \{25\%, 15\%, 10\%, 5\%, 1\%\}$.

The time-space adaptive algorithm we propose is the following.

Set $\mathcal{T}_h^0, \tau_0, u_h^0, n = 1$. While $t \leq T$:

- i) Compute $u_h^n, \eta_\tau^n, \eta^n, \eta_K^n, \eta_{Ge}^n, \eta_{Ge2}^n$ on \mathcal{T}_h^n .
- ii) if $\eta_\tau^n \geq \sum_{K \in \mathcal{T}_h^n} \eta_K^n + \sum_{e \in \mathcal{E}_{GK}^n} \eta_{Ge}^n + \eta_{Ge2}^n$, then $\tau_{n+1} = 0.5\tau_n$ else $\tau_{n+1} = \tau_n$.
- iii) if $p \in \mathcal{P}$, \mathcal{T}_h^{n+1} is derived from \mathcal{T}_h^n by 2 iterations of the standard space adaptive algorithm based on an equirepartition of the error indicators. Else $\mathcal{T}_h^{n+1} = \mathcal{T}_h^n$.
- iv) $t := t + \tau_n, n := n + 1$.

We present below the refined meshes obtained thanks to our time-mesh adaptive algorithm.

FIGURE 2. Mesh at time $t = 0$, $t = 200$ and $t = 500$

Appendix

We prove here a technical lemma. Recall that the quantity s_K is positive in dimension $d = 2$ if K intersects $\bar{\Gamma}_G$ and equal to zero otherwise.

Lemma A.1. *For any function u in $H^1(\Omega)$ vanishing on Γ_B , there exists a function w_h in \mathbb{W}_h such that, for any K in \mathcal{T}_h^n ,*

$$\|u - w_h\|_{L^2(K)} \leq c h_K^{1-s_K} \|u\|_{H^1(\Delta_K)}, \quad \|u - w_h\|_{H^1(K)} \leq c h_K^{-s_K} \|u\|_{H^1(\Delta_K)}, \quad (\text{A.1})$$

where Δ_K is the union of all elements of \mathcal{T}_h^n that intersect K .

Proof. We choose w_h equal to $w_h^1 + w_h^2$, where w_h^1 is the image of u by any type of Clément regularization operator preserving the nullity on Γ_B and w_h^2 has its support contained in the set $\tilde{\Omega}_{Gh}^n$ (made of Γ_G and all triangles K in \mathcal{T}_h^n having at least a corner in $\bar{\Gamma}_G$). Thus estimate (A.1) holds (with $s_K = 0$) on all triangles K that does not intersect Γ_G and also on the other triangles with w_h replaced by w_h^1 .

On the other hand, let \mathcal{C}^n be the set of the corners of elements of \mathcal{T}_h^n which belong to $\bar{\Gamma}_G$. For each c in \mathcal{C}^n , we denote by K_c an element of \mathcal{T}_h^n containing c and by φ_c the Lagrange function associated with c (with values in $\bar{\mathbb{V}}_h^n$). We assume without restriction that

$$w_h^1|_{\Gamma_G} = \sum_{c \in \mathcal{C}^n} \alpha_c \varphi_c, \quad \text{with} \quad \alpha_c = \frac{1}{\text{meas}(K_c)} \int_{K_c} u(x) dx,$$

and we take

$$w_h^2 = - \sum_{c \in \mathcal{C}^n} \alpha_c \varphi_c.$$

Thus, w_h belongs to \mathbb{W}_h . To evaluate the norms of w_h^2 we first observe by switching to the reference element that

$$\|\varphi_c\|_{L^2(\Omega)} \leq c h_{K_c}^{\frac{d}{2}}, \quad |\varphi_c|_{H^1(\Omega)} \leq c h_{K_c}^{\frac{d}{2}-1}.$$

It also follows from a Hölder's inequality that, with $\frac{1}{q} + \frac{1}{q'} = 1$,

$$|\alpha_c| \leq c h_{K_c}^{-d} h_{K_c}^{\frac{d}{q'}} \|u\|_{L^q(K_c)} = c h_{K_c}^{-\frac{d}{q}} \|u\|_{L^q(K_c)}.$$

Using the imbedding of $H^1(K_c)$ into $L^q(K_c)$ for all $q < \infty$ in dimension $d = 2$ and for $q = 6$ in dimension 3 gives the desired result.

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