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for the computation of the probability of collision
between space objects

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Notations

Mathematics

Proofs are ended by the \square symbol.

Sets

- \mathbb{N} : set of natural numbers.
- \mathbb{Z} : set of integers.
- \mathbb{R} : set of real numbers.
- \mathbb{C} : set of complex numbers.
- $\mathbb{N}^* = \{n \in \mathbb{N} \text{ s.t. } n \geq 1\}$.
- ${}_A^c B := A \setminus B = \{x \in A \text{ s.t. } x \notin B\}$: complement of B in A . The prescript A may be omitted if the context makes it clear and obvious.
- ∂A : boundary of the topological space A .
- $\text{cl}(A) = A \cup \partial A$: closure of the topological space A .
- $\overline{B}(x, R) = \{y \in \mathbb{R}^n \text{ s.t. } \|x - y\|_2 \leq R\}$ ($n \in \mathbb{N}$, $x \in \mathbb{R}^n$, $R > 0$).
- $\text{supp}(f) = \text{cl}(\{x \in A \text{ s.t. } f(x) \neq 0\})$: support of $f : A \rightarrow \mathbb{R}$.
- $\text{supp}(\mu)$: support of the measure μ .

Functions and Measures

- id : identity function i.e. $id : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $x \mapsto x$.
- 1_B : indicator function of B .
- $|\cdot|$: absolute value.
- $\|x\|_2 = \left[x \in \mathbb{R}^n \mapsto \sqrt{\sum_{i=1}^n x_i^2} \right]$: Euclidean norm.
- $\mu_1 \otimes \mu_2$: product measure of the measures μ_1 and μ_2 .

Derivatives and vector analysis

Let $\mathcal{T} \subseteq \mathbb{R}$, $n \in \mathbb{N}$, $m \in \mathbb{N}$, $f : \mathcal{T} \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a scalar field, and $g : \mathcal{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a vector field.

- $\partial_{x_i} f = \frac{\partial f}{\partial x_i}$: partial derivative of f with respect to the variable x_i ($i = 1, \dots, n$).
- $\dot{g} = \partial_t g = (\partial_t g_1, \partial_t g_2, \dots, \partial_t g_m)$: element-wise partial time derivative of g .
- $\nabla f = (\partial_t f, \partial_{x_1} f, \partial_{x_2} f, \dots)^T$: gradient operator.
- $\text{div}(g) = \partial_{x_1} g_1 + \partial_{x_2} g_2 + \dots + \partial_{x_n} g_n$: divergence operator.
- $J(g) = (\partial_{x_1} g, \partial_{x_2} g, \dots, \partial_{x_n} g)$: Jacobian operator.

- $d_t f = \frac{df}{dt} = \partial_t f + (\nabla f) \cdot g$: total derivative of f under the vector field g .

Linear Algebra

- $\mathcal{M}_{m,n}(\mathbb{K})$: matrices with $m \in \mathbb{N}$ rows and $n \in \mathbb{N}$ columns, and which components are in the field \mathbb{K} (\mathbb{R} or \mathbb{C}).
- I_n : identity square matrix of dimension n .
- Let $(A, B) \in (\mathcal{M}_{n,n}(\mathbb{K}))^2$ ($n \in \mathbb{N}$). The notation " $A \succeq 0$ " means A is positive semidefinite. " $A \succeq B$ " means " $A - B \succeq 0$ ".
- $u \cdot v$: scalar product between the vectors u and v .
- $\langle A, B \rangle_F = \text{trace}(A^T B) = \sum_{i,j} A_{ij} B_{ij}$: Frobenius inner product of two matrices $(A, B) \in (\mathcal{M}_{n,m}(\mathbb{K}))^2$ ($n, m \in \mathbb{N}$).
- $0_{n,m}$ is a matrix of zeros of dimensions $n \times m$.

Probabilities and Statistics

- $\mathbb{P}(E)$: probability of the event E .
- $x \sim \mathcal{N}(\mu, \Sigma)$: x is a n -dimensional ($n \in \mathbb{N}$) random vector following a multivariate Gaussian distribution with mean vector $\mu \in \mathbb{R}^n$ and with covariance matrix $\Sigma \in \mathcal{M}_{n,n}(\mathbb{R})$ ($\Sigma \succeq 0$).

Orbital Model

Those notations describe what the adopted letters usually denote in this report, unless otherwise stated.

- $r_\star = (r_{\star 1}, r_{\star 2}, r_{\star 3})^T$: position vector of object \star in some reference frame, (primary: $\star = p$, secondary: $\star = s$, relative: $\star = r$).
- $v_\star = (v_{\star 1}, v_{\star 2}, v_{\star 3})^T$: velocity vector of object \star .
- $x_\star = (r_\star^T, v_\star^T)^T$: state vector of object \star .
- $X_{\star b} = (x_\star^T, x_b^T)^T$: state vector of combined object composed of \star and b .
- $f(.,.)$: system dynamics.
- $X(.|X^0)$: trajectory, solution of the equations of motion starting from the initial condition X^0 .

Abbreviations

- CSM: Conjunction Summary Report.
- CAR: Conjunction Assessment Report.
- TLE: Two-Line Elements.
- TCA: Time of Closest Approach.

1 State of the Art of Collision Risk Assessment

Since the collision between the Russian satellite COSMOS 1934 and one debris of COSMOS 926 in December 1991, no less than eight orbital collisions have been reported between operational satellites (e.g. IRIDIUM 33 and COSMOS 2251 collision in February 2009, the 10th), or between satellites and debris (e.g. the French satellite CERISE hit by a debris in July 1996, the 24th or the collision between THOR BURNER and a debris of Long March in January 2005, the 17th). Collision risk is particularly high in low orbits and the different space agencies (CNES, ESA, NASA) and the operators of the field (Airbus Defense and Space (ADS), GMV) have established alert procedures to assess the risks of collision for controlled satellites, and to authorize avoidance maneuvers if the predicted risk exceeds some tolerance threshold. These procedures have undergone many changes in recent years and the field of collision avoidance techniques is currently in full development. For the different evolutions of these procedures, according to the agencies and the operators, we refer to [19], [18] for ESA, [25, 24] for CNES and [9] for ADS.

At the origin of any procedure of collision avoidance between two objects controlled or not in orbit, is the information of conjunction between the two objects. Since 2009, a Conjunction Message is sent by the Combined Space Operations Center (CSpOC) (formerly the Joint Space Operations Center (JSpOC)) to all spacecraft owners and operators, concerning approximately 15000 objects listed in the Two-Line Elements (TLE) catalog provided by USSTRATCOM (US Strategic Command). The information provided by the JSpOC consists of a Conjunction Assessment Report (CAR) containing few information: the Time of Closest Approach (TCA), the miss distance between the two objects, statistical and geometrical information on the position and the velocities of each object. These messages are sent only three days before the date of the encounter. To obtain more accurate information on the possible encounter, it is necessary to subscribe to a service which will in return provide a Conjunction Summary Report (CSM) from which is extracted the information needed to calculate the risk of collision between both objects.

The most general methods to accurately compute the global collision probability, without any additional assumption, are based on Monte-Carlo simulations, see e.g. [5, 17] in the context of a simple encounter or [21] in the context of a multiple encounter. These methods use a random sampling of N vectors in the space of initial conditions. For each of them, the corresponding trajectories are propagated according to the dynamical model adopted on the discretized time interval $[0, T]$. We count 1 if there is a collision, 0 otherwise. At the end, the collision probability is given by the formula: $\mathcal{P}_c([0, T]) = \frac{1}{N} \sum_{i=1}^N \delta_i$. The number of trials to be made depends on the requested precision as well as the value of the probability: a low value requires a lot of samples to be correctly estimated, and simulations can be dramatically time-consuming. This is one of the major disadvantages of Monte Carlo methods which makes them unsuitable for detecting low probability events in high dimension such as multiple events [21]. Therefore alternative approaches had to be explored to assess the risk of collision between two or more objects.

In the particular context of encounters between two objects, encounters are usually classified into two families: the short-term encounters [1, 13, 14, 20] and the long-term encounters [13, 17] for which many simplifying assumptions, enabling the computation of the collision probability can be made. In the context of short-term encounters, conjunctions are assumed to be short and rare. The relative velocity between the two objects is assumed to be very high (higher than the km/s) and the relative motion is assumed rectilinear on the time interval of the encounter. Finally, it is also assumed that the cross-correlations between the estimated states of the two objects are very small and therefore negligible. Such encounters typically occur in low orbits where the orbital velocities are high. Long-term encounters are characterized by relative velocities of the order of m/s, and correspond to situations where both objects spend significant time in proximity to each other. The

motion equations of both objects are linearized around the reference orbit. This type of encounter is more common in the context of formation flying or proximity operations.

In the specific context of short-term encounters, several techniques for calculating the probability of collision have been developed. In historical order, the main methods are identified as those of Foster (1992) [20], Patera (2001) [27] and Alfano (2002) [2], based on numerical integration schemes and the method of Chan (1997) [12, 13] based on an analytic formula of the probability of collision in the form of a convergent series with positive terms. All these methods have already been the subject of several comparative studies [4, 13]. A new method for calculating the collision probability in short-term encounter and the instantaneous probability of collision under Gaussian uncertainty, has been developed during the PhD thesis of R. Serra [31] and published in [33, 34, 32]. The first gives a conservative estimate of the risk for a large part of actual encounters, those falling within the short-term encounter framework, while the second provides, in a more general context, an instantaneous information which then makes it possible to obtain lower bounds for the risk of collision over time intervals. This new method has the advantage of being based on an exact analytical formula, in the form of a convergent series with positive terms, and is the exact version of the approximate one by Chan [12, 13]. Numerical examples are given and demonstrate the efficiency of the proposed formulas compared to existing methods. In addition, analytical bounds are obtained for the probability of collision: they represent a significant tool for mission analysis since their efficient evaluation, allows in many real cases to rule on the importance of the risk.

In the general context of satellite flying formation or proximity operations, the hypothesis of short-term encounters can no longer be considered valid for the calculation of the overall risk of collision [21, 11]. The assumption of short-time spent in the encounter area is no longer verified when considering a cluster of satellites and, even in the case of only two space objects, the encounter could sometimes occur several times per orbit, for several consecutive orbits [21]. For such multiple encounters, the cross-correlations between the estimated states are not negligible anymore, and assessing the risk separately for each individual encounter as if they were independent events, is not sufficient to characterize the real risk of such a conjunction.

The extension of the probability calculation in the case of long-term encounters, also called nonlinear framework, represents a first generalization attempt for specific cases of configurations [28, 26, 13, 17]. Unfortunately, these approaches are relatively limited because of their characterization for particular relative trajectories and are only imperfectly generalized to other cases [29]. In this last reference or more recently in [16], different metrics are proposed: distribution sampling, the Mahalanobis minimum distance and its upper bound on the collision probability, the Maximum Instantaneous probability and its lower bound on the collision probability, a hybrid probability combining the last two metrics for which the calculations can be shared or the symmetric Kullback-Leibler (KL) divergence. Roughly, the Mahalanobis distance or the symmetric KL divergence can be seen as a measure of similarity between two pdfs. The idea is to reduce the computational complexity of the distribution sampling approach by improving the determination of an interval over which two objects are in close proximity while accounting for the uncertainties of these objects. A similar approach based on computational tools of increasing complexity and precision (approximate ellipsoids for the probability density, probability density level curves, etc.) is also proposed in [10]. In [14], V. Coppola proposes a mathematical formalization and generalization of the short-term encounter formula: the two objects are modeled with uncertainty in both position and velocity, and their trajectories are not assumed to be rectilinear anymore. More recently, the Coppola's formula has been revisited and generalized to non-spherical objects [23]: instead of directly computing the collision probability, G. Krier first computes the collision probability per time unit, called the hazard function. The total probability of collision is the time integral of this hazard function, and this integration is performed numerically. This method has been successfully validated by comparing

the obtained results with Monte Carlo simulations.

The objective of this report is to give an exact and rigorous mathematical modelling of the problem of the computation of the probability of collision as derived by V. Coppola in the reference [14].

2 Preliminaries

2.1 Encounter Modeling and Problem Statement

Consider an operational spacecraft (called the primary) in orbit around the Earth and a space debris (called the secondary). The state of each orbiting object is described by their position and velocity in a reference frame \mathcal{R} , and gathered in a global state vector X_{ps} :

$$X_{ps} = (r_p, v_p, r_s, v_s)^T \in \mathbb{R}^n,$$

where $n = 12$.

Let $[0, T]$ be the time interval of the encounter. Consider now the dynamics of the 2 objects:

$$\begin{cases} \dot{X}_{ps}(t) &= f(t, X_{ps}(t)), \quad t \in [0, T], \\ X_{ps}(0) &= X_{ps}^0. \end{cases} \quad (1)$$

where f is a real vector field, which is supposed to be at least Lipschitz continuous, and T is a given terminal time defining the time interval of the encounter. The initial conditions $X_{ps}^0 \in \mathbb{R}^n$ (e.g. position and velocity) are usually subject to uncertainties, and so, they are supposed to be distributed according to a given probability measure μ_I with its density function $\rho_I = \frac{d\mu_I}{d\lambda}$ (cf. the notation defined in Appendix B). These equations include the Newtonian gravitational central field and possible orbital perturbations (non spherical Earth, atmospheric drag, e.g.). Whatever model is adopted, it is assumed that, for given initial conditions, the solutions of the system (1) exist and are unique. For each fixed initial condition $X_{ps}^0 \in \mathbb{R}^n$, a trajectory, or sample path, is then defined as follows:

Definition 1 (Trajectory/Sample path). *Given an initial condition $X_{ps}^0 \in \mathbb{R}^n$, a trajectory, or sample path, starting from X_{ps}^0 is the unique solution of:*

$$X_{ps}(t|X_{ps}^0) = X_{ps}^0 + \int_0^t f(\tau, X_{ps}(\tau|X_{ps}^0))d\tau. \quad (2)$$

Classically the objects are assumed to be spherical [1, 28, 3, 13, 14]: this assumption enables one to ignore the orientation of the objects, and to model conservatively the secondary object whose geometry is often poorly known (see Figure 1).

Assumption 1.

The objects involved in the conjunction have a spherical shape.

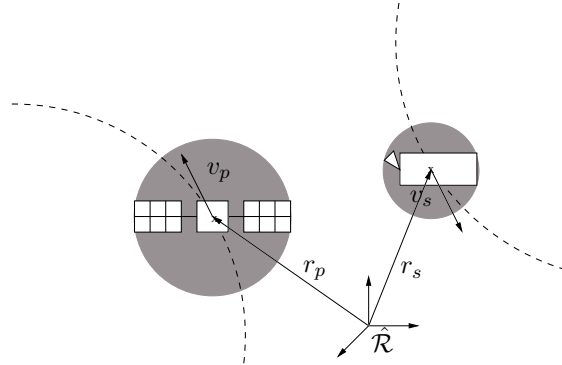


Figure 1: Encounter between two spherical objects.

Let us now define the notion of a collision. Suppose that for any initial condition $X_{ps}^0 \in \mathbb{R}^n$ the corresponding trajectory exists on the time interval $[0, T]$, then roughly speaking, a collision occurs when a trajectory enters a certain *forbidden region* \mathcal{X}_R . For instance, in practice, this describes the fact that the relative distance between two objects is less than a certain given radius threshold $R > 0$.

If the state x is given by the positions and velocities of 2 objects, then the *forbidden region* is expressed as

$$\mathcal{X}_R = \{X_{ps} = (r_p, v_p, r_s, v_s) \in \mathbb{R}^{12} \mid \|r_s - r_p\|^2 - R^2 < 0\}. \quad (3)$$

Note that, in case of two objects, the forbidden region (which depends on R) \mathcal{X}_R is also called the combined spherical object and let its complement, so-called *safe/admissible region*, be ${}^c\mathcal{X}_R := \mathbb{R}^n \setminus \mathcal{X}_R$.

Definition 2 (Collision). *Given an initial condition $X_{ps}^0 \in \mathbb{R}^n$, a maximum time of interest $T > 0$, and a forbidden region \mathcal{X}_R , a collision occurs if there exists $t \in [0, T]$ such that $X_{ps}(t|X_{ps}^0) \in \mathcal{X}_R$.*

Definition 3 (Collision domain). *The domain of collision \mathcal{X}_T^0 over the time interval $[0, T]$ is the set of initial conditions leading to a collision between any pair of objects during $[0, T]$, namely:*

$$\mathcal{X}_T^0 = \{X_{ps}^0 \in \mathbb{R}^n \mid \exists t \in [0, T], X_{ps}(t|X_{ps}^0) \in \mathcal{X}_R\}.$$

Remark 1. In Equation (3) and Definition 3, the sets \mathcal{X}_R and \mathcal{X}_T^0 have been defined based on the combined object state vector X_{ps} . Equivalent definitions could be given by using any combined state vector such as X_{rp} for instance, as it is the case in the rest of this report.

Historically, the first criterion for collision risk assessment between two space objects was geometric: it relies on checking whether or not the nominal trajectory enters a *no-go zone* [6, 15] (including the forbidden region) whose size is either arbitrarily set or calculated with respect to uncertainties' standard deviations. A more natural formulation of the problem of collision risk assessment consists in *computing the probability that a collision occurs*:

Problem 1. *Let the dynamics in (1), a maximum time of interest $T > 0$ and a safe region ${}^c\mathcal{X}_R$. Provided that the initial conditions $X_{ps}^0 \in \mathbb{R}^n$ are distributed according to a given probability measure μ_I with its density function ρ_I , the probability that a collision occurs is computed simply by:*

$$\mathcal{P}_c([0, T]) = \mathcal{P}_c = \mathbb{P}(X_{ps}^0 \in \mathcal{X}_T^0) = \mu_I(\mathcal{X}_T^0) = \int_{\mathcal{X}_T^0} d\mu_I. \quad (4)$$

The analytical calculation of the collision probability as defined in (4) is a very difficult problem: the first issue is to determine the domain of integration, which strongly depends on the chosen model for the dynamics when propagating the distribution of probability of the initial state. In addition, the integration of the density of probability on this set may be very complex, even for a Gaussian distribution [13].

2.2 Measures for the collision probability

As in [14], we define the set \mathcal{X}_T^{00} of initial states for which a collision occurs at $t = 0$ and the set $\mathcal{X}_T^{0tc} := \mathcal{X}_T^0 \setminus \mathcal{X}_T^{00}$ of the remaining initial states for which a collision occurs later. The collision probability $\mathcal{P}_c := \mu_I(\mathcal{X}_T^0)$ is thus, $\mathcal{P}_c = \mu_I(\mathcal{X}_T^{00}) + \mu_I(\mathcal{X}_T^{0tc})$. Often, in practice, the probability that a collision occurs at $t = 0$ is very small. However, the general computation of an *instantaneous* collision probability for a fixed $t = t_0$ (when the distribution of states at t_0 is Gaussian) is of interest and may be analyzed independently.

We now focus on computing $\mu_I(\mathcal{X}_T^{0t_c})$. More precisely, we are given the distribution (measure) of all initial states μ_I , the dynamics, as well as the forbidden region \mathcal{X}_R . The computation of $\mu_I(\mathcal{X}_T^{0t_c})$ amounts in *finding* (in some sense) an unknown initial measure μ_0 which can be seen as the restriction of μ_I to initial points leading to a collision on the time interval $(0, T]$, denoted by $1_{\mathcal{X}_T^{0t_c}}\mu_I$. Assume that trajectories $X_{ps}(\cdot|X_{ps}^0)$ starting at $X_{ps}^0 \in {}^c\mathcal{X}_R$ are continuous functions and let $({}^c\mathcal{X}_R)^\circ$ be the interior of ${}^c\mathcal{X}_R$. Then, over the fixed period $[0, T]$, these trajectories either are in $({}^c\mathcal{X}_R)^\circ$ or will touch its topological boundary $\partial {}^c\mathcal{X}_R := {}^c\mathcal{X}_R \setminus ({}^c\mathcal{X}_R)^\circ$ at a so-called first hitting time $\tau(X_{ps}^0) \in (0, T]$,

$$\tau(X_{ps}^0) := \min\{T, \inf\{t \geq 0 \text{ s.t. } X_{ps}(t|X_{ps}^0) \in \partial {}^c\mathcal{X}_R\}\}.$$

Finally, one defines the final measure $\mu_F \in M([0, T] \times \mathbb{R}^n)_+$ which captures the distribution of the first hitting times $\tau(X_{ps}^0)$ and the corresponding state $X_{ps}(t|X_{ps}^0)$ after it has been propagated by the dynamics starting at $t = 0$ from the initial measure μ_0 :

$$\mu_F(A \times B) := \int_{[0, T] \times \mathbb{R}^n} 1_{A \times B}(\tau(X_{ps}^0), X_{ps}(\tau(X_{ps}^0)|X_{ps}^0)) d\mu_0(X_{ps}^0), \quad (5)$$

for all Borel measurable sets $A \times B \subseteq (0, T] \times \mathbb{R}^n$. The final measure μ_F is the pushforward measure $h_{\tau\star}\mu_0$ of μ_0 , via the mapping:

$$\begin{aligned} h_\tau : \quad \mathbb{R}^n &\rightarrow [0, T] \times \mathbb{R}^n, \\ X_{ps}^0 &\mapsto (\tau(X_{ps}^0), X_{ps}(\tau(X_{ps}^0)|X_{ps}^0)), \end{aligned} \quad (6)$$

that is:

$$\mu_F(A \times B) = \mu_0(h_\tau^{-1}(A \times B)), \quad (7)$$

for any Borel measurable sets $A \times B \subseteq (0, T] \times \mathbb{R}^n$ (cf. Figure 2).

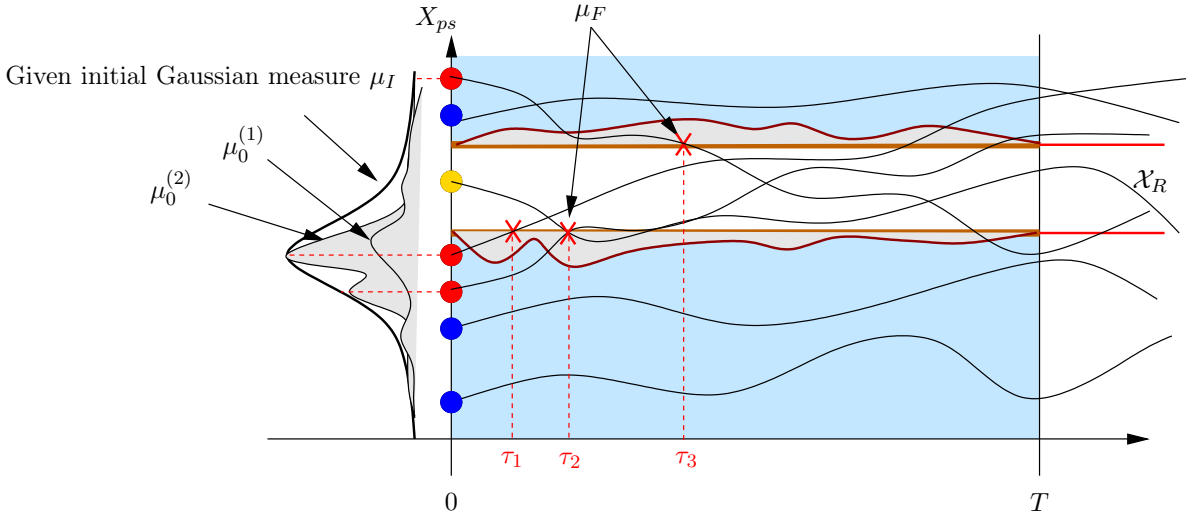


Figure 2: Yellow points: $X_{ps}^0 \in \mathcal{X}_T^{00}$, red points: $X_{ps}^0 \in \mathcal{X}_T^{0\tau_i}$, blue points: $X_{ps}^0 \in {}^c\mathcal{X}_T^0$.

3 Coppola's formula revisited

The objective of this section is to derive the formula for the computation of the probability of collision given in [14] and obtained in a general context. Our intention is to give a thorough and rigorous treatment to the solution proposed in [14].

We assume that the dynamical model under consideration (regardless of how the state vector X_{**} of the combined object is defined) $\dot{X}(t) = f(t, X(t))$ gives rise to an invertible flow φ_0^t , behaving as a C^1 -diffeomorphism for $t \in [0, T]$:

$$\begin{aligned} \varphi_0^t &: \mathbb{R}^{12} \rightarrow \mathbb{R}^{12} \\ X^0 &\mapsto X(t|X^0), \end{aligned}$$

where $\varphi_0^t(\cdot)$ is the map $\varphi(t, t_0, \cdot)$ for a given $(t, t_0) \in \mathbb{R} \times \mathbb{R}$. Assuming that the function $\varphi(\cdot, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{12} \rightarrow \mathbb{R}^{12}$ remains unchanged under time translation, we can set $t_0 = 0$ without loss of generality [22]. The two parameter family of mappings satisfies:

$$\frac{\partial}{\partial t} \varphi_0^t(X^0) = f(t, \varphi_0^t(X^0)).$$

Its inverse is given by:

$$(\varphi_0^t)^{-1} = \varphi_t^0 : X(t|X^0) \mapsto X^0.$$

When manipulating density functions and the change of variable formula in the calculations below, we will also need the *local linearization of the flow* with respect to the initial conditions, denoted by $D\varphi_0^t(X^0) \in \mathbb{R}^{12 \times 12}$. Indeed, if \tilde{X}^0 is close to X^0 , then:

$$\varphi_0^t(\tilde{X}^0) \approx \varphi_0^t(X^0) + D\varphi_0^t(X^0)(\tilde{X}^0 - X^0).$$

Note that the local linearization of the inverse flow φ_t^0 satisfies the relation:

$$D\varphi_t^0(\varphi_0^t(X^0)) = (D\varphi_0^t(X^0))^{-1},$$

since by differentiating with respect to X^0 the equation $\varphi_t^0 \circ \varphi_0^t = \text{Id}_{\mathbb{R}^{12}}$, we get $D\varphi_t^0(\varphi_0^t(X^0)) \circ D\varphi_0^t(X^0) = \text{Id}_{\mathbb{R}^{12}}$.

Example 1.

Under the assumption that the flow is linear, $\dot{X}(t) = A(t)X(t)$, the solution of the dynamics equation via the state transition matrix $\Phi(t, 0)$, is such that

$$X(t|X^0) = \varphi_0^t(X^0) = \Phi(t, 0)X^0,$$

that is, the initial conditions are propagated from time 0 to time t via the matrix $\Phi(t, 0)$. One has also $X^0 = \Phi(t, 0)^{-1}X(t) = \Phi(0, t)X(t)$.

Also, suppose that there is a suitable surface parametrization of the whole or part of $\partial^c \mathcal{X}_R$ given by $P : S \rightarrow \mathbb{R}^n$, with $S \subseteq \mathbb{R}^{n-1}$ and with $P(s) = (p_1(s), \dots, p_n(s)) \in \partial^c \mathcal{X}_R$.

Example 2.

(a) (Circle) The coordinates $(x, y) \in \mathbb{R}^2$ of a point on the circle of radius R are parameterized

on the circle via the change of coordinates:

$$(x, y)^T := P(\theta) = (R \cos(\theta), R \sin(\theta))^T,$$

with $\theta \in S = [0, 2\pi]$.

(b) (Sphere) The coordinates $(x, y, z) \in \mathbb{R}^3$ of a point on the sphere of radius R are parameterized on the sphere via the change of coordinates:

$$(x, y, z)^T := P(\theta, \phi) = (R \cos(\theta) \sin(\phi), R \sin(\theta) \sin(\phi), R \cos(\phi))^T,$$

with $\theta \in [0, 2\pi), \phi \in [0, \pi]$ and $S = [0, 2\pi) \times [0, \pi]$.

(c) (Practical case in Coppola's article)

For $X_{rp} = (r_{r1}, r_{r2}, r_{r3}, v_{r1}, v_{r2}, v_{r3}, r_{p1}, r_{p2}, r_{p3}, v_{p1}, v_{p2}, v_{p3})^T \in \mathbb{R}^{12}$ the surface $r_{r1}^2 + r_{r2}^2 + r_{r3}^2 = R^2$, is parameterized on \mathbb{R}^{11} via the change of coordinates:

$$X_{rp} = (r_{r1}, r_{r2}, r_{r3}, v_{r1}, v_{r2}, v_{r3}, r_{p1}, r_{p2}, r_{p3}, v_{p1}, v_{p2}, v_{p3})^T := P(\theta, \phi, v_{r1}, \dots, v_{p3}), \quad (8)$$

$$= (R \cos(\theta) \sin(\phi), R \sin(\theta) \sin(\phi), R \cos(\phi), v_{r1}, v_{r2}, v_{r3}, r_{p1}, r_{p2}, r_{p3}, v_{p1}, v_{p2}, v_{p3})^T, \quad (9)$$

with $\theta \in [0, 2\pi), \phi \in [0, \pi]$ and $S = [0, 2\pi) \times [0, \pi] \times \mathbb{R}^9$. Note that this parametrization is based on spherical coordinates used in Mathematics and which convention is different (θ and ϕ are swapped) from the one preferentially used in Physics and in [14].

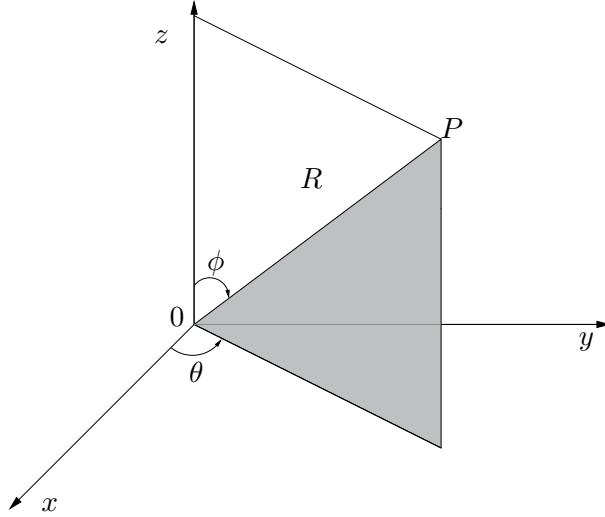


Figure 3: Spherical coordinates for the parametrization of a sphere.

The *a priori* unknown and possibly complicated shape of $\mathcal{X}_T^{0t_c}$ makes it difficult to get the probability of collision by directly computing an integral over this set. The principle of Coppola's formula is to compute an equivalent integral over the set of collision states instead, via a change of variables given by the function $\psi : [0, T] \times S \rightarrow \mathcal{X}_T^{0t_c}$, $\psi(t, s) = \varphi_t^0(P(s))$ (note that this definition is a partial composition only, since φ_t^0 depends on the first argument t of ψ). For this substitution to be licit, as stated by the following lemma, we need ψ to be 1-1 (more precisely, a C^1 -diffeomorphism). The following assumption is sufficient for the bijectivity of ψ .

Assumption 2.

For each trajectory leading to a collision, there is only one entering crossing in (or only one tangent to) the forbidden region \mathcal{X}_R for all $t \in (0, T]$. Therefore, the collision domain $\mathcal{X}_T^{0t_c}$ and the surface S are respectively defined by:

$$\mathcal{X}_T^{0t_c} = \{X^0 \in \mathbb{R}^n \mid \exists! t \in (0, T], X(t|X^0) \in \mathcal{X}_R \text{ and } v_r(t)^T r_r(t) \leq 0\}, \quad (10)$$

and

$$S = \{(\theta, \phi, v_r, r_p, v_p) \in [0, 2\pi) \times [0, \pi] \times \mathbb{R}^9 : v_{r1} \cos(\theta) + v_{r2} \sin(\theta) \sin(\phi) + v_{r3} \cos(\phi) \leq 0\}. \quad (11)$$

In order to compute the part of the collision probability \mathcal{P}_c given by $\mu_I(\mathcal{X}_T^{0t_c}) := \mu_0(\mathbb{R}^n)$, the next derivations rely on the following lemma:

Lemma 1.

$$(i) \quad \mu_F = ((\text{Id}_{\mathbb{R}} \times P) \circ \psi^{-1})_* \mu_0;$$

(ii)

$$\int_{\mathbb{R}^n} d\mu_0 = \int_{\mathcal{X}_T^{0t_c}} \rho_I(X^0) d\lambda(X^0) = \int_{[0, T] \times S} (\rho_I \circ \psi)(t, s) |\det(J(\psi)(t, s))| d\lambda(t, s), \quad (12)$$

where $J(\psi)$ is the Jacobian matrix of the mapping ψ .

Proof.

(i) One notices that since ψ is 1-1, each point which achieves a collision, $X^0 \in \mathcal{X}_T^{0t_c} \subset \mathbb{R}^n$ is transported via $(\text{Id}_{\mathbb{R}} \times P) \circ \psi^{-1}$ on the surface $\partial^c \mathcal{X}_R$ at a certain time $\tau(X^0) \in (0, T]$ and reciprocally since:

$$\begin{array}{ccc} \mathcal{X}_T^{0t_c} \subset \mathbb{R}^n & \xrightarrow{\psi^{-1}} & [0, T] \times S \\ X^0 & \mapsto & (\tau(X^0), s(X^0)) \end{array} \quad \xrightarrow{\text{Id}_{\mathbb{R}} \times P} \quad \begin{array}{c} [0, T] \times \mathbb{R}^n, \\ (\tau(X^0), X(\tau(X^0)|X^0)). \end{array}$$

Therefore, we have that $h_\tau = ((\text{Id}_{\mathbb{R}} \times P) \circ \psi^{-1})$ and by 2 in Appendix C,

$$\int_{[0, T] \times \mathbb{R}^n} 1_{A \times B}(t, X) d\mu_F = \int_{\mathbb{R}^n} 1_{A \times B} \circ (\text{Id}_{\mathbb{R}} \times P) \circ \psi^{-1}(X^0) d\mu_0.$$

(ii) One uses the Jacobi's change of variables formula [30, Chapter 15], since ψ is a C^1 -diffeomorphism. Note that this formula is obtained by applying (42) with $X = [0, T] \times S$, $Y = \mathcal{X}_T^{0t}$, $g(\psi) = \rho_I(\psi) |\det(J(\psi))|$, $\mu = \lambda$ and noting that $d(\psi_* \mu) = d(\psi_* \lambda) = |\det(J(\psi^{-1}))| d\lambda$ [30, Remark 15.10]. Indeed,

$$\begin{aligned} & \int_{[0, T] \times S} (\rho_I \circ \psi)(t, s) |\det(J(\psi)(t, s))| d\lambda(t, s) = \int_{\mathcal{X}_T^{0t}} \rho_I(X^0) |\det(J(\psi)(\psi^{-1}(X^0)))| d\psi_* \lambda(X^0) = \\ & \int_{\mathcal{X}_T^{0t}} \rho_I(X^0) |\det(J(\psi)(\psi^{-1}(X^0)))| |\det(J(\psi^{-1})(X^0))| d\lambda(X^0) = \\ & \int_{\mathcal{X}_T^{0t}} \rho_I(X^0) |\det(J(\psi)(\psi^{-1}(X^0)))| |\det(J(\psi)(\psi^{-1}(X^0)))|^{-1} d\lambda(X^0) = \\ & \int_{\mathcal{X}_T^{0t}} \rho_I(X^0) d\lambda(X^0) = \int_{\mathcal{X}_T^{0t}} d\mu(X^0). \end{aligned}$$

Note that $J(\psi)(\psi^{-1}(X^0))J(\psi^{-1})(X^0) = I_n$. □

Example 3.

If the flow is linear, the function ψ is now defined as $\psi : [0, T] \times S \rightarrow \mathcal{X}_T^{0t_c}$, $\psi(t, s) = \Phi(t, 0)^{-1}P(s)$.

Remark 2.

Assumption 2 is equivalent to the Assumptions (A1) and (A2) proposed in Coppola's article and reminded below.

(A1) Only one crossing.

(A2) Trajectories must cross.

For computing the Jacobian of ψ , one has:

$$\frac{\partial \psi}{\partial t}(t, s) = -D\varphi_t^0(P(s))f(t, P(s)) \in \mathbb{R}^{12}.$$

This indeed follows from differentiating with respect to t the equality $P(s) = \varphi_0^t(\psi(t, s))$ by applying the chain rule:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} (\varphi_0^t(\psi(t, s))) \\ &= \frac{\partial \varphi_0^t}{\partial t}(\psi(t, s)) + D\varphi_0^t(\psi(t, s)) \frac{\partial \psi}{\partial t}(t, s) \\ &= f(t, \varphi_0^t(\psi(t, s))) + D\varphi_0^t(\psi(t, s)) \frac{\partial \psi}{\partial t}(t, s) \\ &= f(t, P(s)) + (D\varphi_t^0(P(s)))^{-1} \frac{\partial \psi}{\partial t}(t, s). \end{aligned}$$

In addition, we have that:

$$\frac{\partial \psi}{\partial s}(t, s) = D\varphi_t^0(P(s)) \frac{\partial P(s)}{\partial s} \in \mathbb{R}^{12 \times 11}.$$

Hence

$$J(\psi)(t, s) = D\varphi_t^0(P(s)) \left(-f(t, P(s)) \mid \frac{\partial P(s)}{\partial s} \right) \in \mathbb{R}^{12 \times 12}. \quad (13)$$

Example 4.

In the case of linear dynamics, the computations above become

$$\frac{\partial \psi}{\partial t} = -\Phi^{-1}(t, 0) \frac{\partial \Phi(t, 0)}{\partial t} \Phi^{-1}(t, 0) P(s) = -\Phi^{-1}(t, 0) A(t) P(s),$$

since $\frac{\partial \Phi(t, 0)}{\partial t} = A(t) \Phi(t, 0)$. Moreover, we have

$$\frac{\partial \psi}{\partial s} = \Phi(t, 0)^{-1} \frac{\partial P(s)}{\partial s},$$

and (13) becomes

$$J(\psi) = \Phi^{-1}(t, 0) \left(-A(t)P(s) \mid \frac{\partial P(s)}{\partial s} \right).$$

Practical computations for the Jacobian matrix in Coppola's case (Example 2(c)), are as follows:

$$\frac{\partial P(s)}{\partial s} = \left(\begin{array}{cc|c} -R \sin(\theta) \sin(\phi) & R \cos(\theta) \cos(\phi) & 0_{3,9} \\ R \cos(\theta) \sin(\phi) & R \sin(\theta) \cos(\phi) & \\ 0 & -R \sin(\phi) & \\ \hline & 0_{9,2} & I_9 \end{array} \right).$$

Due to 0 blocks in the above matrix, to compute the determinant of $J(\psi)$, one only needs the first three coordinates of the $f(t, P(s))$ vector, and since $f(t, \cdot)$ is the dynamics, one has

$$f(t, P(s)) = (v_{r1}, v_{r2}, v_{r3}, \dot{v}_{r1}, \dot{v}_{r2}, \dot{v}_{r3}, v_{p1}, v_{p2}, \dot{v}_{p3}, \dot{v}_{p1}, \dot{v}_{p2}, \dot{v}_{p3})^T \in \mathbb{R}^{12}.$$

This gives:

$$\det(J(\psi)(t, s)) = \begin{vmatrix} -v_{r1} & -R \sin(\theta) \sin(\phi) & R \cos(\theta) \cos(\phi) \\ -v_{r2} & R \cos(\theta) \sin(\phi) & R \sin(\theta) \cos(\phi) \\ -v_{r3} & 0 & -R \sin(\phi) \end{vmatrix} \det(D\varphi_t^0(P(s))),$$

which amounts to:

$$\begin{aligned} \det(J(\psi)) &= R^2 \sin(\phi) (\cos(\theta) \sin(\phi) v_{r1} + \sin(\phi) \sin(\theta) v_{r2} + \cos(\phi) v_{r3}) \det(D\varphi_t^0(P(s))) \\ &= R^2 \sin(\phi) v_r \cdot \hat{n} \det(D\varphi_t^0(P(s))), \end{aligned} \tag{14}$$

where $\hat{n} = [\cos(\theta) \sin(\phi) \quad \sin(\phi) \sin(\theta) \quad \cos(\phi)]^T$ is the unit vector normal to the sphere.

Example 5.

For linear dynamics, one may easily deduce that

$$\det(J(\psi)) = \begin{vmatrix} -v_{r1} & -R \sin(\theta) \sin(\phi) & R \cos(\theta) \cos(\phi) \\ -v_{r2} & R \cos(\theta) \sin(\phi) & R \sin(\theta) \cos(\phi) \\ -v_{r3} & 0 & -R \sin(\phi) \end{vmatrix} \det(\Phi^{-1}(t, 0)),$$

and finally

$$\begin{aligned} \det(J(\psi)) &= R^2 \sin(\phi) (\cos(\theta) \sin(\phi) v_{r1} + \sin(\phi) \sin(\theta) v_{r2} + \cos(\phi) v_{r3}) \det(\Phi^{-1}(t, 0)) \\ &= R^2 \sin(\phi) v_r \cdot \hat{n} \det(\Phi^{-1}(t, 0)). \end{aligned} \tag{15}$$

Now, since ψ is 1-1, one can apply Equation (12) and get:

$$\int_{\mathbb{R}^n} d\mu_0 = \int_0^T \int_0^{2\pi} \int_0^\pi \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \rho_I(\psi(t, \theta, \phi, v_r, r_p, v_p)) |\det(D\varphi_t^0(P(t, \theta, \phi, v_r, r_p, v_p)))| \\ 1_{v_r \cdot \hat{n} \leq 0}(v_r) |R^2 \sin(\phi) v_r \cdot \hat{n}| dv_p dr_p dv_r d\phi d\theta dt \tag{16}$$

Example 6.

For linear dynamics, we have

$$\int_{\mathbb{R}^n} d\mu_0 = \int_{t=0}^T \int_0^{2\pi} \int_0^\pi \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \rho_I(\psi(t, \theta, \phi, v_r, r_p, v_p)) |\det(\Phi^{-1}(t, 0))| 1_{v_r \cdot \hat{n} \leq 0}(v_r) \\ |R^2 \sin(\phi) v_r \cdot \hat{n}| dv_p dr_p dv_r d\phi d\theta dt \quad (17)$$

Lemma 2. Let a given random vector X with probability density ρ_X and another random vector Y related to X by the equation $y = g(x)$, and g bijective. The probability density ρ_Y for Y is:

$$\rho_Y(y) = \rho_X(g^{-1}(y)) |\det(J(g^{-1})(y))|. \quad (18)$$

Proof. Let $\mathbb{P}(Y \in S)$ be the probability that Y takes a value in some particular subset S , so that we get:

$$\mathbb{P}(Y \in S) = \int_S \rho_Y(y) dy.$$

Since Y takes a value in S whenever X takes a value in $g^{-1}(S)$, one has:

$$\mathbb{P}(Y \in S) = \int_{g^{-1}(S)} \rho_X(x) dx.$$

Now, changing from variable x to y gives:

$$\mathbb{P}(Y \in S) = \int_{g^{-1}(S)} \rho_X(x) dx = \int_S \rho_X(g^{-1}(y)) |\det(J(g^{-1})(y))| dy.$$

□

In our case, let g be the flow φ_0^t , and ρ_I be the probability density of the initial random state X^0 , one obtains the probability density $\rho_t(X)$ of the random state X at a given fixed time t from Equation (18):

$$\rho_t(X) = \rho_I(\varphi_0^t(X)) |\det(D\varphi_0^t(X))|. \quad (19)$$

Example 7.

In the linear case, the flow is given by $X^0 \mapsto \Phi(t, 0)X^0$ and then, we have that the probability density $\rho_t(X)$ of the random state X at a given fixed time t is given by

$$\rho_t(X) = \rho_I(\Phi^{-1}(t, 0)X) |\det(\Phi^{-1}(t, 0))|. \quad (20)$$

From Equations (19) and (16), one has:

$$P_I = \int_{\mathbb{R}^n} d\mu_0 = \int_{t=0}^T \int_{\theta=0}^{2\pi} \int_{\phi=0}^\pi \int_{v_r=-\infty}^\infty \int_{r_p=-\infty}^\infty \int_{v_p=-\infty}^\infty \rho_t(P(\theta, \phi, v_r, r_p, v_p)) 1_{v_r \cdot \hat{n} \leq 0}(v_r) \\ |R^2 \sin(\phi) v_r \cdot \hat{n}| dv_p dr_p dv_r d\phi d\theta dt. \quad (21)$$

Remark 3. Equation (21) is similar to formula (15) in Coppola's article, with two differences. Firstly, the parametrization in spherical coordinates is different, but this does not affect the result in any way. Secondly, there is a slight abuse of notation in formula (15) of Coppola's article, concerning the probability density function $\rho_t(X)$ which appears instead of ours $\rho_t(P(s))$. More precisely, this means that ρ_t is the probability density of the state random variable X at each fixed time t , but this function should be applied to $P(s)$ (after the reparametrization). There is also a slight abuse of notations in this document regarding the definition of a probability density function $\rho_t(P(s))$ as shown in Lemma 2 since the random vector and the variable of the density function coincide.

We move on to the next assumption made in [14]:

Assumption 3. Independence of the two random vectors x_p and x_s .

The dynamic model and probability distribution function for one object are independent from the dynamic model and probability distribution function of the other object.

Keeping in mind the slight abuse of notations mentioned in Remark 3 and if $\rho_{tps}(x_p, x_s)$, $\rho_{tp}(x_p)$, $\rho_{ts}(x_s)$ denote respectively the joint density functions of the random vectors $X^T = [x_p^T \ x_s^T]$, x_p and x_s then Assumption 3 means that:

$$\rho_{tps}(x_p, x_s) = \rho_{tp}(x_p)\rho_{ts}(x_s). \quad (22)$$

This implies that the density $\rho_t(P(s)) = \rho_t(x_r(\theta, \phi, v_r), x_p)$ satisfies:

$$\rho_t(x_r, x_p) = \rho_{tps}(x_p, x_s) = \rho_{tp}(x_p)\rho_{ts}(x_p + x_r), \quad (23)$$

where $x_r = x_p - x_s$ is the relative state on the sphere of radius R . This may be proved by noting that $X_{rp} = \begin{bmatrix} x_r \\ x_p \end{bmatrix} = \begin{bmatrix} -I_6 & I_6 \\ I_6 & 0_{6,6} \end{bmatrix} \begin{bmatrix} x_p \\ x_s \end{bmatrix}$ and by applying Lemma 2. Therefore, Equation (16) becomes:

$$P_I = \int_0^T \int_0^{2\pi} \int_0^\pi |\sin(\phi)| \int_{-\infty}^\infty R^2 |v_r \cdot \hat{n}| 1_{v_r \cdot \hat{n} \leq 0}(v_r) \int_{-\infty}^\infty \int_{-\infty}^\infty \rho_{tp}(x_p)\rho_{ts}(x_p + x_r) dv_p dr_p dv_r d\phi d\theta dt. \quad (24)$$

The next assumption from [14] is related to the Gaussian nature of the two density functions of the random vectors x_p and x_s .

Assumption 4. Gaussian distributions.

The probability distribution functions $\rho_{tp} \sim \mathcal{N}(m_p(t), P_p(t))$ and $\rho_{ts} \sim \mathcal{N}(m_s(t), P_s(t))$ remain Gaussian at each time t .

It may be deduced that $x_r = x_s - x_p \sim \mathcal{N}(m_s(t) - m_p(t), \underbrace{P_p(t) + P_s(t)}_{P_r(t)})$.

Let us define the following matrices:

$$G^{-1} = P_p^{-1} + P_s^{-1}, \quad T = GP_s^{-1}, \quad (25)$$

where G is a symmetric positive semidefinite matrix. Keeping in mind that:

$$\begin{aligned} \rho_{tp}(x_p) &= \frac{1}{\sqrt{(2\pi)^6} \sqrt{\det(P_p)}} e^{-\frac{1}{2}(x_p - m_p)^T P_p^{-1}(x_p - m_p)}, \\ \rho_{ts}(x_s) &= \frac{1}{\sqrt{(2\pi)^6} \sqrt{\det(P_s)}} e^{-\frac{1}{2}(x_s - m_s)^T P_s^{-1}(x_s - m_s)}, \end{aligned} \quad (26)$$

then

$$\rho_t(x_r, x_p) = \rho_{tp}(x_p)\rho_{ts}(x_p + x_r) = \rho_t(w, x_r) = \rho_{tw}(w)\rho_{tr}(x_r), \quad (27)$$

is obtained after tedious algebraic manipulations for which the following identities are used:

$$\begin{aligned} TP_s T^T + (I_6 - T)P_p(I_6 - T)^T &= GT^T + G(I_6 - T)^T = G, \\ \det(P_p)\det(P_s) &= \det(P_p P_s) = \det(GP) = \det(G)\det(P), \end{aligned} \quad (28)$$

and:

$$\begin{aligned} \rho_{tw} &\sim \mathcal{N}(m_p(t) + T \underbrace{(m_s(t) - m_p(t))}_{m_r(t)}, G(t)), \\ \rho_{tr} &\sim \mathcal{N}(m_r(t), P_r(t)). \end{aligned} \quad (29)$$

Equation (27) shows that the random vectors $w = x_p + Tx_r$ and x_r are independent. Therefore, we get that:

$$\begin{aligned} P_I &= \int_0^T \int_0^{2\pi} \int_0^\pi |\sin(\phi)| \int_{-\infty}^\infty R^2 |v_r \cdot \hat{n}| 1_{v_r \cdot \hat{n} \leq 0}(v_r) \rho_{tr}(x_r) \underbrace{\int_{-\infty}^\infty \rho_{tw}(w) dw}_{=1} dv_r d\phi d\theta dt \\ &= \int_0^T \int_0^{2\pi} \int_0^\pi |\sin(\phi)| \int_{-\infty}^\infty R^2 |v_r \cdot \hat{n}| 1_{v_r \cdot \hat{n} \leq 0}(v_r) \rho_{tr}(x_r(\theta, \phi, v_r)) dv_r d\phi d\theta dt. \end{aligned} \quad (30)$$

The next step consists in partitioning the mean vector $m_r(t) = \begin{bmatrix} m_{r_r}(t) \\ m_{v_r}(t) \end{bmatrix}$ and the covariance matrix $P_r(t)$ as $P_r(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}^T(t) & P_{22}(t) \end{bmatrix}$ leading to:

$$\begin{aligned} P_r^{-1}(t) &= \begin{bmatrix} P_{11}^{-1}(t) + P_{11}^{-1}(t)P_{12}(t)\Delta^{-1}(t)P_{12}^T(t)P_{11}^{-1}(t) & -P_{11}^{-1}(t)P_{12}(t)\Delta^{-1}(t) \\ -\Delta^{-1}(t)P_{12}^T(t)P_{11}^{-1}(t) & \Delta^{-1}(t) \end{bmatrix}, \\ \Delta(t) &= P_{22}(t) - P_{12}^T(t)P_{11}^{-1}(t)P_{12}(t). \end{aligned}$$

Consequently, we get that:

$$\begin{aligned} -\frac{1}{2}(x_r - m_r)^T P_r^{-1}(x_r - m_r) &= -\frac{1}{2}(r_r - m_{r_r})^T P_{11}^{-1}(r_r - m_{r_r}) \dots \\ &\quad -\frac{1}{2}(\cdot)^T \Delta^{-1} \underbrace{\left(\overbrace{(v_r - P_{12}^T P_{11}^{-1} r_r)}^{v'} - \overbrace{(m_{v_r} - P_{12}^T P_{11}^{-1} m_{r_r})}^{m'_v} \right)}_{v' - m'_v}, \\ &= -\frac{1}{2}(r_r - m_{r_r})^T P_{11}^{-1}(r_r - m_{r_r}) - \frac{1}{2}(v' - m'_v)^T \Delta^{-1}(v' - m'_v), \end{aligned}$$

and $\rho_{tr}(x_r) = \rho_{tr}(r_r)\rho_{tv'}(v')$ with:

$$\begin{aligned} \rho_{tr} &\sim \mathcal{N}(m_{r_r}(t), P_{11}(t)), \\ \rho_{tv'} &\sim \mathcal{N}(m'_v(t), \Delta(t)). \end{aligned} \quad (31)$$

Moreover, $|v_r \cdot \hat{n}| = |v'(t) \cdot \hat{n} + R\hat{n}^T P_{11}^{-1}(t)P_{12}(t)\hat{n}| = |v'(t) \cdot \hat{n} + \epsilon_0(\hat{n}, t)|$ and we get:

$$P_I = \int_0^T \int_0^{2\pi} \int_0^\pi R^2 |\sin(\phi)| \rho_{tr}(r_r) \cdot \underbrace{\int_{-\infty}^\infty |v'(t) \cdot \hat{n} + \epsilon_0(\hat{n}, t)| 1_{v'(t) \cdot \hat{n} + \epsilon_0(\hat{n}, t) \leq 0}(v') \rho_{tv'}(v') dv'}_{I(\hat{n}, t)} d\phi d\theta dt. \quad (32)$$

The unit vector normal to the sphere \hat{n} is defined by $\hat{n} = [\cos(\theta) \sin(\phi) \quad \sin(\theta) \sin(\phi) \quad \cos(\phi)]^T$ and the orthogonal matrix \bar{T} defined by:

$$\bar{T} = \begin{bmatrix} \cos(\theta) \sin(\phi) & \sin(\theta) \sin(\phi) & \cos(\phi) \\ -\sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) \cos(\phi) & -\sin(\theta) \cos(\phi) & \sin(\phi) \end{bmatrix}, \quad (33)$$

is such that $\vec{i} = [1 \ 0 \ 0]^T = \bar{T}\hat{n}$. With the notations:

$$\begin{aligned} v''(t) &= \bar{T}v'(t) = \begin{bmatrix} \epsilon(t) \\ \zeta(t) \end{bmatrix}, \\ m_v''(t) &= \bar{T}m'_v(t) = \begin{bmatrix} m_\epsilon(t) \\ m_\zeta(t) \end{bmatrix}, \\ \bar{\Delta}(\theta, \phi, t) &= \bar{T}\bar{\Delta}(t)\bar{T}^T = \begin{bmatrix} \sigma^2(\theta, \phi, t) & \bar{\Delta}_{12}(t) \\ \bar{\Delta}_{12}^T(t) & \bar{\Delta}_{22}(t) \end{bmatrix}, \\ \sigma^2(\theta, \phi, t) &= \hat{n}^T (P_{22}(t) - P_{12}^T(t)P_{11}^{-1}(t)P_{12}(t))\hat{n}, \end{aligned} \quad (34)$$

we have that $\rho_{tv'}(v') = \rho_{tv''}(v'')$ with $v'' \sim \mathcal{N}(m_v''(t), \bar{\Delta}(t))$ and simple algebraic computations lead to $\rho_{tv''}(v'') = \rho_{t\epsilon}(\epsilon)\rho_{t\zeta}(\zeta)$ with:

$$\begin{aligned} \epsilon &\sim \mathcal{N}(m_\epsilon, \sigma^2), \\ \bar{\zeta} &= \zeta - \frac{\epsilon \bar{\Delta}_{12}^T}{\sigma^2} \sim \mathcal{N}\left(m_\zeta - \frac{m_\epsilon \bar{\Delta}_{12}^T}{\sigma^2}, \bar{\Delta}_{22} - \frac{\bar{\Delta}_{12}^T \bar{\Delta}_{12}}{\sigma^2}\right). \end{aligned}$$

Note that $m_\epsilon(\theta, \phi, t) = \hat{n}^T (m_{v_r} - P_{12}^T(t)P_{11}^{-1}(t)m_{r_r}) = \vec{i}^T m_v''(t)$. Therefore,

$$\begin{aligned} I(\hat{n}, t) &= \int_{-\infty}^\infty |v'(t) \cdot \hat{n} + \epsilon_0(\hat{n}, t)| 1_{v'(t) \cdot \hat{n} + \epsilon_0(\hat{n}, t) \leq 0}(v') \rho_{tv'}(v') dv', \\ &= \int_{-\infty}^\infty |\epsilon(t) \cdot \hat{n} + \epsilon_0(\hat{n}, t)| 1_{\epsilon + \epsilon_0(\hat{n}, t) \leq 0}(\epsilon) \rho_{tv''}(v'') d\bar{\zeta} d\epsilon, \\ &= \int_{-\infty}^\infty |\epsilon(t) \cdot \hat{n} + \epsilon_0(\hat{n}, t)| 1_{\epsilon + \epsilon_0(\hat{n}, t) \leq 0}(\epsilon) \rho_{t\epsilon}(\epsilon) \underbrace{\int_{-\infty}^\infty \rho_{t\bar{\zeta}}(\bar{\zeta}) d\bar{\zeta}}_1 d\epsilon, \\ &= -\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{-\epsilon_0(\hat{n}, t)} (\epsilon(t) \cdot \hat{n} + \epsilon_0(\hat{n}, t)) e^{-\frac{(\epsilon - m_\epsilon)^2}{2\sigma^2}} d\epsilon, \end{aligned} \quad (35)$$

since $v'(t) \cdot \hat{n} = v''(t) \cdot \bar{T}\hat{n} = v''(t) \cdot \vec{i}$. Finally, remembering that the error function is given by:

$$\begin{aligned} \text{erf} : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto \text{erf}(x) = \frac{1}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \end{aligned} \quad (36)$$

the integral $I(\hat{n}, t)$ may be readily computed as:

$$I(\hat{n}, t) = \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{(\epsilon_0(\hat{n}, t) + m_\epsilon)^2}{2\sigma^2}} - \frac{(m_\epsilon + \epsilon_0(\hat{n}, t))}{2} \left[1 - \operatorname{erf} \left(\frac{m_\epsilon + \epsilon_0(\hat{n}, t)}{\sqrt{2} \sigma} \right) \right], \quad (37)$$

and the probability P_I is finally obtained as:

$$P_I = \int_0^T \int_0^{2\pi} \int_0^\pi R^2 |\sin(\phi)| \rho_{tr}(r_r(\theta, \phi)) I(\hat{n}, t) d\phi d\theta dt. \quad (38)$$

Appendix

This appendix quickly summarizes the main definitions from measure theory and integration that are needed for the developments presented in this report. The material here in is mainly borrowed from the references [8], [7] and [30].

A Basic definitions, results and facts from measure theory

Definition 4 (Indicator Function). *Let $B \subset A$. The indicator function 1_B of B is defined as:*

$$\begin{aligned} 1_B : A &\rightarrow \{0, 1\}, \\ x &\mapsto \begin{cases} 0 & \text{if } x \notin B, \\ 1 & \text{if } x \in B. \end{cases} \end{aligned}$$

Definition 5 (σ -algebra). *Let A be a set. A σ -algebra of subsets of A (sometimes called a σ -field) is a family \mathcal{A} of subsets of A such that:*

1. $\emptyset \in \mathcal{A}$,
2. $\forall E \in \mathcal{A}, A \setminus E \in \mathcal{A}$,
3. for every sequence $(E_n)_{n \in \mathbb{N}} \in \mathcal{A}$, $\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{A}$.

Example 8.

- $\mathcal{P}(A)$ is a (the maximal) σ -algebra on A .
- $\{\emptyset, A\}$ is a (the minimal) σ -algebra on A .
- $\{\emptyset, B, A \setminus B, A\}$ is a σ -algebra on A .
- If $B \neq \emptyset$ and $B \neq A$ then $\{\emptyset, B, A \setminus B, A\}$ is not a σ -algebra on A .

Definition 6. *For every system of sets $\mathcal{G} \in \mathcal{P}(A)$, there exists a smallest σ -algebra containing \mathcal{G} . This σ -algebra, denoted by $\sigma(\mathcal{G})$ is the σ -algebra generated by \mathcal{G} .*

Definition 7 (Borel σ -algebra or topological σ -algebra). *The σ -algebra $\sigma(\mathcal{O}^n)$, denoted $\mathcal{B}(\mathbb{R}^n)$, generated by the system \mathcal{O}^n of all open sets of \mathbb{R}^n is called the Borel σ -algebra on \mathbb{R}^n and its members are the Borel sets or Borel measurable sets.*

Note that the Borel σ -algebra $\mathcal{B}(\mathbb{R}^n)$ may be generated by the family of half-open rectangles in \mathbb{R}^n , i.e. $\mathcal{B}(\mathbb{R}^n) = \mathcal{I}$ where $\mathcal{I} = \mathcal{I}(\mathbb{R}^n) = \{[a_1, b_1] \times \cdots \times [a_n, b_n] : a_j, b_j \in \mathbb{R}\}$.

Definition 8 (Measurable Space). *A measurable space is a pair (A, \mathcal{A}) where A is a set and \mathcal{A} is a σ -algebra of subsets of A .*

Definition 9 (Measure). *Let (A, \mathcal{A}) be a measurable space (see Definition 8). A positive measure on A (or, more precisely, on (A, \mathcal{A})) is a mapping $\mu : \mathcal{A} \rightarrow [0, +\infty]$ satisfying:*

1. $\mu(\emptyset) = 0$,

2. for every sequence $(E_n)_{n \in \mathbb{N}}$ of pairwise disjoint sets in \mathcal{A} , $\mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \sum_{n \in \mathbb{N}} \mu(E_n)$.

Example 9.

- *Borel measure* : A measure defined on the Borel σ -algebra $\mathcal{B}(\mathbb{R}^n)$ is called a Borel measure.
- *Dirac Measure at a* : Let (A, \mathcal{A}) be any measurable space and let $a \in A$. Then $\delta_a : \mathcal{A} \rightarrow \{0, 1\}$, defined for $A \in \mathcal{A}$ by:

$$\delta_a(A) = \begin{cases} 0 & \text{if } a \notin A, \\ 1 & \text{if } a \in A, \end{cases}$$

is called the Dirac measure at the point a .

- *Lebesgue measure*: The set-function λ^n on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ that assigns the value :

$$\lambda^n([a_1, b_1) \times \cdots \times [a_n, b_n)) = \prod_{j=1}^n (b_j - a_j)$$

to every half-open rectangle $[a_1, b_1) \times \cdots \times [a_n, b_n) \in \mathcal{I}$, is called the n -dimensional Lebesgue measure. If $A \subset \mathbb{R}^n$ then

$$\lambda^n(A) = \inf_{A \subset \bigcup_{i=1}^{\infty} R_i} \sum (b_1^i - a_1^i) \cdots (b_n^i - a_n^i),$$

where the R_i 's are half-open rectangles, i.e. $R_i = [a_1^i, b_1^i) \times \cdots \times [a_n^i, b_n^i)$.

The 1-dimensional-Lebesgue measure represents the conventional "length" of a segment:

$$\forall (a, b) \in \mathbb{R}^2 \text{ s.t. } a < b, \quad \lambda([a, b]) = b - a.$$

Remark that, for $a \in \mathbb{R}$, $\lambda(\{a\}) = 0$. From the second point of Definition 9:

$$\forall (a, b) \in \mathbb{R}^2 \text{ s.t. } a < b, \quad \lambda([a, b]) = \lambda(\{a\}) + \lambda(]a, b[) + \lambda(\{b\}) = \lambda(]a, b[).$$

Definition 10 (Measure Space). A measure space is a triplet (A, \mathcal{A}, μ) where A is a set, \mathcal{A} is a σ -algebra of subsets of A , and μ is a positive measure on (A, \mathcal{A}) .

Property 1. Let (A, \mathcal{A}, μ) be a measure space and $B, C \in \mathcal{A}$. Then, $C \subset B \Rightarrow \mu(C) \leq \mu(B)$.

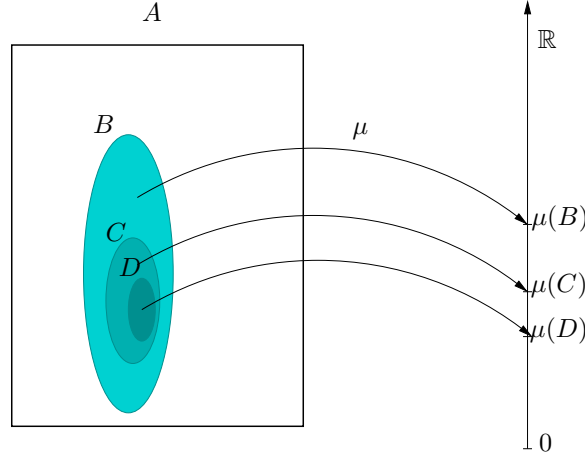


Figure 4: Monotonicity of positive measures.

Definition 11 (σ -finite measure). A measure μ defined on (A, \mathcal{A}) is said to be σ -finite and (A, \mathcal{A}, μ) is called a σ -finite measure space if \mathcal{A} contains an increasing sequence $(A_j)_{j \in \mathbb{N}} \subset \mathcal{A}$ of sets $A_1 \subset A_2 \subset \dots$ such that $\cup_{j \in \mathbb{N}} A_j = A$ (exhausting sequence) verifying $\mu(A_j) < \infty$ for all $j \in \mathbb{N}$.

Definition 12 (Support of a Measure). Let (A, \mathcal{A}, μ) be a measure space. The support of the positive measure μ is the smallest closed set defined as:

$$\text{supp}(\mu) = \{x \in A \text{ s.t. } \mu(U) > 0 \text{ for every open neighborhood } U \text{ of } x\}.$$

Example 10.

- The support of the Dirac measure at 0 is the singleton $\{0\}$.
- Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The support of the Lebesgue measure λ is \mathbb{R} . Note that every single point has Lebesgue measure zero and so has every countable union of points.

Definition 13 (Product Measure). Let $(A_1, \mathcal{A}_1, \mu_1)$ and $(A_2, \mathcal{A}_2, \mu_2)$ be two σ -finite measure spaces. Then the set-function

$$\begin{aligned} \rho : \mathcal{A}_1 \times \mathcal{A}_2 &\rightarrow [0, \infty], \\ A_1 \times A_2 &\mapsto \mu_1(A_1)\mu_2(A_2) \end{aligned}$$

is the unique measure called product measure, denoted by $\rho = \mu_1 \otimes \mu_2$ and defined on the measurable space $(A_1 \times A_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ where $\mathcal{A}_1 \otimes \mathcal{A}_2 = \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ is a product σ -algebra. $(A_1 \times A_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu_1 \otimes \mu_2)$ is called the product measure space.

The product measure ρ is a σ -finite measure on $(A_1 \times A_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ such that

$$\rho(E) = \int \int 1_E(x, y) \mu_1(dx) \mu_2(dy) = \int \int 1_E(x, y) \mu_2(dy) \mu_1(dx),$$

holds for all $E \in \mathcal{A}_1 \otimes \mathcal{A}_2$.

Example 11.

The n -dimensional Lebesgue measure λ^n on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ may be defined as a product measure, i.e.

$$(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \lambda^n) = (\mathbb{R}^d \times \mathbb{R}^{n-d}, \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^{n-d}), \lambda^d \otimes \lambda^{n-d}), \quad \forall n > d \geq 1.$$

B Measurable functions and integration

Definition 14 ($(\mathcal{A}_1, \mathcal{A}_2)$ -measurable mapping). Let (A_1, \mathcal{A}_1) and (A_2, \mathcal{A}_2) be two measurable spaces. A mapping $f : A_1 \rightarrow A_2$ is $(\mathcal{A}_1, \mathcal{A}_2)$ -measurable if

$$f^{-1}(B) \in \mathcal{A}_1, \quad \forall B \in \mathcal{A}_2.$$

Definition 15 (Measurable function). Let (A, \mathcal{A}) be a measurable space. A function $f : A \rightarrow \mathbb{R}$ is \mathcal{A} -measurable (or measurable in short) if $\{x : f(x) < c\} \in \mathcal{A}$ for every $c \in \mathbb{R}$ or equivalently

$$f^{-1}(B) \in \mathcal{A}, \quad \forall B \in \mathcal{B}(\mathbb{R}^n).$$

A measurable function $f : A \rightarrow \mathbb{R}$ is a $(\mathcal{A}_1, \mathcal{B}(\mathbb{R}))$ -measurable mapping.

Example 12.

Let (X, \mathcal{A}) be a measurable space. The indicator function $f(x) = 1_A(x)$ is measurable if and only if $A \in \mathcal{A}$. Indeed, for a set $A \in \mathcal{A}$

$$\begin{aligned} \{x : 1_A(x) < c\} &= \emptyset \in \mathcal{A} && \text{if } c \leq 0, \\ \{x : 1_A(x) < c\} &= X \setminus A \in \mathcal{A} && \text{if } c \in (0, 1], \\ \{x : 1_A(x) < c\} &= X \in \mathcal{A} && \text{if } c > 1. \end{aligned}$$

Definition 16 (Simple function). A simple function $g : A \rightarrow \mathbb{R}$ on a measurable space (A, \mathcal{A}) is a function of the form:

$$g(x) = \sum_{j=1}^M y_j 1_{A_j}(x),$$

with finitely many disjoint sets $A_1, \dots, A_M \in \mathcal{A}$ and $y_1, \dots, y_M \in \mathbb{R}$.

If $y_i \geq 0$ for all i , the function g is called a positive simple function.

Definition 17. Suppose μ is a positive measure on the measurable space (A, \mathcal{A}) and f is measurable

- If $f = \sum_{j=1}^M y_j 1_{A_j}$ is a positive simple function then

$$\int f d\mu = \sum_{i=1}^M y_i \mu(A_i).$$

- If $f \geq 0$ then

$$\int f d\mu = \sup \left\{ \int g d\mu : g \leq f, g \text{ is positive simple} \right\}.$$

- For arbitrary f ,

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu,$$

where $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$.

If (B, \mathcal{B}) is measurable, the integral of f over B is defined by:

$$\int_B f d\mu = \int f 1_B d\mu.$$

Theorem 1 (Fubini). *Let $(A_1, \mathcal{A}_1, \mu_1)$ and $(A_2, \mathcal{A}_2, \mu_2)$ be σ -finite measure spaces and let $f : A_1 \times A_2 \rightarrow \mathbb{R}$ be $\mathcal{A}_1 \otimes \mathcal{A}_2$ -measurable. If at least one of the following is finite*

$$\int_{A_1 \times A_2} |f| d(\mu_1 \otimes \mu_2), \quad \int_{A_2} \int_{A_1} |f(x, y)| \mu_1(dx) \mu_2(dy), \quad \int_{A_1} \int_{A_2} |f(x, y)| \mu_2(dy) \mu_1(dx)$$

then

$$\int_{A_1 \times A_2} f d(\mu_1 \otimes \mu_2) = \int_{A_2} \int_{A_1} f(x, y) \mu_1(dx) \mu_2(dy) = \int_{A_1} \int_{A_2} f(x, y) \mu_2(dy) \mu_1(dx).$$

Definition 18 (Density). *Let (A, \mathcal{A}, μ) be a measure space and ρ a positive real \mathcal{A} -measurable function, the set-function*

$$\nu : A \mapsto \int_A \rho d\mu = \int 1_A \rho d\mu = \int 1_A(x) \rho(x) \mu(dx),$$

is a measure on (A, \mathcal{A}) called the measure with density function ρ with respect to μ and denoted $\nu = \rho\mu$. Traditionally, the density is denoted $\rho = \frac{d\nu}{d\mu}$.

For instance, if the measure μ_I is Gaussian ($\mu_I = \mu_g$), with mean $m \in \mathbb{R}^n$, and covariance matrix Σ , given by:

$$\mu_I(A) := \frac{1}{\sqrt{(2\pi \det(\Sigma))^n}} \int_A \exp\left(-\frac{(X - m)^T \Sigma^{-1} (X - m)}{2}\right) dX, \quad (39)$$

where $A \in \mathcal{B}(\mathbb{R}^n)$, and $\mathcal{B}(\mathbb{R}^n)$ denotes the completion of the Borel σ -algebra on \mathbb{R}^n and the integral in (39) is with respect to the standard n -dimensional Lebesgue measure. The density of the measure μ_I is a function $\rho_I : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by:

$$\rho_I(X) = \frac{1}{\sqrt{(2\pi \det(\Sigma))^n}} \exp\left(-\frac{(X - m)^T \Sigma^{-1} (X - m)}{2}\right). \quad (40)$$

Using the notation of Definition 18, one has $\rho_I = \frac{d\mu_I}{d\lambda}$.

C Image of a measure

Let us denote by $M(S)_+$ the cone of all nonnegative Borel measures on the subset S of a measurable space. For completeness, let us recall the classical notions related to the *pushforward measure*, which is roughly speaking *the image of a given measure under a given mapping*:

Definition 19. [35, Theorem 1.44] (*Pushforward measure*) Let two measurable spaces (X, \mathcal{A}) and (Y, \mathcal{B}) , a $(\mathcal{A}, \mathcal{B})$ -measurable mapping $h : X \rightarrow Y$ and a measure $\mu \in M(X)_+$. The pushforward measure (or image measure under the mapping h) $\nu = h_*\mu \in M(Y)_+$ defined on \mathcal{B} is given by:

$$\nu(B) = h_*\mu(B) = \mu(h^{-1}(B)), \quad (41)$$

for all Borel measurable sets $B \subseteq Y$.

For an arbitrary set $B \subseteq Y$, $h^{-1}(B)$ is the preimage of B under the mapping h , i.e.

$$h^{-1}(B) = \{x \in X : h(x) \in B\}.$$

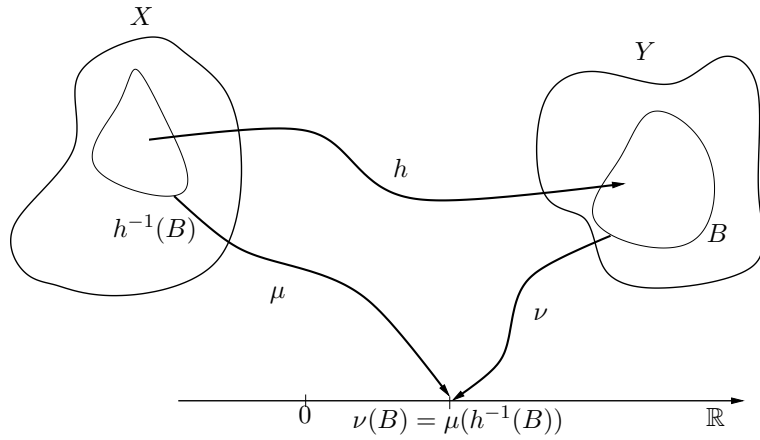


Figure 5: Pushforward measure or image measure ν .

Theorem 2. [8, Theorem 3.6.1] (*Change of variables*) Let $\mu \in M(X)_+$. A measurable function g on Y is integrable with respect to the pushforward measure $h_*\mu \in M(Y)_+$ if and only if the composition $g \circ h$ is integrable with respect to the measure $\mu \in M(X)_+$. In that case:

$$\int_Y g \, d(h_*\mu) = \int_X g \circ h \, d\mu. \quad (42)$$

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