SUPPLEMENTARY MATERIAL: Spectral density estimation for nonstationary data with nonzero mean function

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In this companion document we give all the proofs of the results appearing in the paper.

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1 Proofs

At first we introduce a few auxiliary lemmas.

Lemma 1.1. *Assume that **A1-A3** holds. Then for any $\lambda \in [0, 2\pi)$, any sequences of integers $\{c_n\}$, $\{d_n\}$, where $d_n > 0$ we have*

$$E\left(\frac{1}{d_n}\sum_{t=c_n}^{c_n+d_n-1} X_t e^{-i\lambda t}\right) = m(\lambda) + R_n/d_n,$$

where $|R_n| \le C$ and C is a constant dependent on Ψ, λ and $m(\cdot)$.

Proof of the Lemma 1.1. For $\Psi = \emptyset$ the proof is obvious with $C = 0$. Assume that $\Psi \neq \emptyset$ and note that

$$\begin{aligned} E\left(\frac{1}{d_n}\sum_{t=c_n}^{c_n+d_n-1} X_t e^{-i\lambda t}\right) &= \frac{1}{d_n}\sum_{t=c_n}^{c_n+d_n-1}\sum_{\psi\in\Psi} m(\psi)e^{i\psi t}e^{-i\lambda t} = \sum_{\psi\in\Psi} m(\psi)\frac{1}{d_n}\left(\sum_{t=c_n}^{c_n+d_n-1} e^{i\psi t}e^{-i\lambda t}\right) \\ &= m(\lambda) + \frac{1}{d_n}\sum_{\psi\in\Psi\setminus\{\lambda\}} m(\psi)\left(\sum_{t=c_n}^{c_n+d_n-1} e^{i\psi t}e^{-i\lambda t}\right) \\ &= m(\lambda) + R_n/d_n, \end{aligned}$$

where

$$R_n = \sum_{\psi\in\Psi\setminus\{\lambda\}} m(\psi)\frac{e^{i\lambda}\left(e^{i\,c_n(\psi-\lambda)} - e^{i(\psi-\lambda)(c_n+d_n)}\right)}{e^{i\lambda} - e^{i\psi}}$$

and

$$|R_n| \le C = \sum_{\psi\in\Psi\setminus\{\lambda\}} \frac{2|m(\psi)|}{|e^{i\lambda} - e^{i\psi}|}.$$

□

Lemma 1.2. *Assume that **A1-A3** holds. Take $\lambda \in [0, 2\pi)$ and any sequences of integers $\{c_n\}$, $d_n = [C_1 n^{\kappa}]$ and $\lambda_n = C_2/n^{\rho}$, where $\kappa > 0$, $\rho > 0$ and C_1, C_2 are nonzero constants.*

Then, for n sufficiently large for $|\lambda_n| < C_3 = \min\{|\lambda - \psi| : \psi \in \Psi \setminus \{\lambda\}\}$ if $\Psi \neq \emptyset$ and for any n if $\Psi = \emptyset$, we have that

$$E\left(\frac{1}{d_n}\sum_{t=c_n}^{c_n+d_n-1} X_t e^{-i(\lambda+\lambda_n)t}\right) = W_n n^{-\kappa+2\rho},$$

where $|W_n| \leq C$ and C is a constant dependent on Ψ, λ, C_1, C_2, κ, ρ and $m(\cdot)$.

Proof of the Lemma 1.2. Note that For $\Psi = \emptyset$ the proof is obvious with $C = 0$. Assume that $\Psi \neq \emptyset$ and note that

$$\begin{aligned}
&E\left(\frac{1}{d_n}\sum_{t=c_n}^{c_n+d_n-1} X_t e^{-i(\lambda+\lambda_n)t}\right) = \frac{1}{d_n}\sum_{t=c_n}^{c_n+d_n-1}\sum_{\psi\in\Psi} m(\psi)e^{i\psi t}e^{-i(\lambda+\lambda_n)t} \\
&= \sum_{\psi\in\Psi} m(\psi)\frac{1}{d_n}\left(\sum_{t=c_n}^{c_n+d_n-1} e^{i\psi t}e^{-i(\lambda+\lambda_n)t}\right) \\
&= m(\lambda)\frac{1}{d_n}\sum_{t=c_n}^{c_n+d_n-1} e^{-i\lambda_n t} + \frac{1}{d_n}\sum_{\psi\in\Psi\setminus\{\lambda\}} m(\psi)\left(\sum_{t=c_n}^{c_n+d_n-1} e^{i\psi t}e^{-i(\lambda+\lambda_n)t}\right) \\
&= m(\lambda)\frac{1}{d_n}\underbrace{\frac{e^{i\lambda_n}\left(e^{-ic_n\lambda_n} - e^{-i\lambda_n(c_n+d_n)}\right)}{e^{i\lambda_n}-1}}_{Q_n} + \frac{1}{d_n}\underbrace{\sum_{\psi\in\Psi\setminus\{\lambda\}} m(\psi)\left(\sum_{t=c_n}^{c_n+d_n-1} e^{i\psi t}e^{-i(\lambda+\lambda_n)t}\right)}_{R_n} \\
&= m(\lambda)Q_n + \frac{1}{C_1 n^{\kappa}}R_n = \underbrace{\left(\frac{m(\lambda)Q_n}{n^{-\kappa+2\rho}} + \frac{R_n}{C_1 n^{2\rho}}\right)}_{W_n} n^{-\kappa+2\rho}.
\end{aligned}$$

Note that for sufficiently large n such that $|\lambda_n| < C_3 = \min\{|\lambda - \psi| : \psi \in \Psi \setminus \{\lambda\}\}$, R_n can be bounded from above as follows

$$|R_n| \leq \left|\sum_{\psi\in\Psi\setminus\{\lambda\}} m(\psi)\frac{e^{i(\lambda+\lambda_n)}\left(e^{ic_n(\psi-(\lambda+\lambda_n))} - e^{i(\psi-(\lambda+\lambda_n))(c_n+d_n)}\right)}{e^{i(\lambda+\lambda_n)} - e^{i\psi}}\right| \leq \sum_{\psi\in\Psi\setminus\{\lambda\}} \frac{2|m(\psi)|}{|e^{i(\lambda+\lambda_n)} - e^{i\psi}|}.$$

Moreover

$$|Q_n| \leq \left|\frac{2}{d_n\left(-1+e^{i\lambda_n}\right)}\right| = \frac{1}{d_n(1-\cos\lambda_n)} \leq C_4 n^{-\kappa+2\rho},$$

where $C_4 = C_4(\rho, \kappa, C_1, C_2)$ is some positive constant. Its existence is a consequence of the following convergence $\lim_{n\to\infty} \frac{\lambda_n^2}{1-\cos(\lambda_n)} = 2$. This fact ends the proof. □

The following lemma is a generalization of Lemma A.5 from Lenart (2011).

Lemma 1.3. *Assume that the conditions* ***A1-A3*** *hold and that there exists a positive constant C such that for any $\lambda \in \mathbb{R}$ we have*

$$\sum_{\tau=-\infty}^{\infty} |\tau||a(\lambda, \tau)| < C. \tag{1}$$

Then for any pair of frequencies $(\nu, \omega) \in [0, 2\pi)^2$, for any real valued sequence $\{v_n\}_{n\in\mathbb{N}}$ convergent to zero and for any sequence of integers $\{c_n\}$ and any sequence of positive integers $\{d_n\}$ tending to infinity with $n \to \infty$ we have

$$\lim_{n\to\infty} E\big(\widehat{P}_{d,c}(\nu + v_n, \omega + v_n) - P(\nu, \omega)\big) = 0,$$

where for any $(x, y) \in \mathbb{R}^2$

$$\widehat{P}_{d,c}(x, y) = \frac{1}{2\pi d_n} \sum_{s=c_n}^{c_n+d_n-1} \sum_{t=c_n}^{c_n+d_n-1} (X_s - \mu(s))(X_t - \mu(t)) e^{-ixs} e^{iyt}.$$

Proof of the Lemma 1.3. The proof is similar to the proof of Lemma A.5 from Lenart (2011). Therefore we show only main steps. Take any $(\nu, \omega) \in [0, 2\pi)^2$. Note that

$$E(\widehat{P}_{d,c}(\nu + v_n, \omega + v_n)) = \frac{1}{2\pi d_n} \sum_{s=c_n}^{c_n+d_n-1} \sum_{t=c_n}^{c_n+d_n-1} \operatorname{cov}(X_s, X_t) e^{-i(\nu-\omega)t} e^{-i(\nu+v_n)(s-t)}.$$

Setting $j = t$ and $\tau = s - t$ we get

$$
\begin{aligned}
E(\widehat{P}_{d,c}(\nu + v_n, \omega + v_n)) &= \frac{1}{2\pi d_n} \sum_{j=c_n}^{c_n+d_n-1} \sum_{\tau=c_n-j}^{c_n+d_n-1-j} B(j,\tau) e^{-i(\nu-\omega)j} e^{-i(\nu+v_n)\tau} \\
&= \frac{1}{2\pi d_n} \sum_{j=c_n}^{c_n+d_n-1} \sum_{\tau=c_n-j}^{c_n+d_n-1-j} \sum_{\lambda\in\Lambda_\tau} a(\lambda,\tau) e^{i(\lambda-(\nu-\omega))j} e^{-i(\nu+v_n)\tau} \\
&= \frac{1}{2\pi d_n} \sum_{j=c_n}^{c_n+d_n-1} \sum_{\tau=c_n-j}^{c_n+d_n-1-j} a(\nu-\omega,\tau) e^{-i(\nu+v_n)\tau} \\
&+ \frac{1}{2\pi d_n} \sum_{j=c_n}^{c_n+d_n-1} \sum_{\tau=c_n-j}^{c_n+d_n-1-j} \sum_{\lambda\in\Lambda_\tau\setminus\{\nu-\omega\}} a(\lambda,\tau) e^{i(\lambda-(\nu-\omega))j} e^{-i(\nu+v_n)\tau} \\
&= \frac{1}{2\pi} \sum_{|\tau|<d_n} \left(1 - \frac{|\tau|}{d_n}\right) a(\nu-\omega,\tau) e^{-i(\nu+v_n)\tau} \\
&+ \frac{1}{2\pi d_n} \sum_{j=c_n}^{c_n+d_n-1} \sum_{\tau=c_n-j}^{c_n+d_n-1-j} \sum_{\lambda\in\Lambda_\tau\setminus\{\nu-\omega\}} a(\lambda,\tau) e^{i(\lambda-(\nu-\omega))j} e^{-i(\nu+v_n)\tau} \\
&= \frac{1}{2\pi} \sum_{|\tau|<d_n} a(\nu-\omega,\tau) e^{-i\nu\tau} \\
&+ \frac{1}{2\pi} \sum_{|\tau|<d_n} a(\nu-\omega,\tau)\big(e^{-i(\nu+v_n)\tau} - e^{-i\nu\tau}\big) - \frac{1}{2\pi} \sum_{|\tau|<d_n} \frac{|\tau|}{d_n} a(\nu-\omega,\tau) e^{-i(\nu+v_n)\tau} \\
&+ \frac{1}{2\pi d_n} \sum_{j=c_n}^{c_n+d_n-1} \sum_{\tau=c_n-j}^{c_n+d_n-1-j} \sum_{\lambda\in\Lambda_\tau\setminus\{\nu-\omega\}} a(\lambda,\tau) e^{i(\lambda-(\nu-\omega))j} e^{-i(\nu+v_n)\tau}.
\end{aligned}
$$

We denote the summands on the right-hand side by $\epsilon_{1,n}$, $\epsilon_{2,n}$, $\epsilon_{3,n}$ and $\epsilon_{4,n}$, respectively. Note that $\epsilon_{1,n} \to P(\nu,\omega)$ as $n \to \infty$. Thus, it is sufficient to show that terms $\epsilon_{2,n}$, $\epsilon_{3,n}$ and $\epsilon_{4,n}$ tend to 0 as $n \to \infty$.

Using inequality $|e^{ix} - e^{iy}| \le |x - y|$ (which holds for any $x, y \in \mathbb{R}$) and (1) we get

$$
|\epsilon_{2,n}| \le \frac{1}{2\pi} \sum_{|\tau|<d_n} |a(\nu-\omega,\tau)|\big|e^{-i(\nu+v_n)\tau} - e^{-i\nu\tau}\big| \le \frac{1}{2\pi} \sum_{|\tau|<d_n} |v_n||\tau||a(\nu-\omega,\tau)| \longrightarrow 0,
$$

as $n \longrightarrow \infty$. Moreover,

$$|\epsilon_{3,n}| \leq \frac{1}{2\pi} \sum_{|\tau|<d_n} \frac{|\tau|}{d_n} |a(\nu-\omega,\tau)| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

After changing the order of summation in term $\epsilon_{4,n}$, calculating the sum over j and using the equality $1/|1-e^{ix}| = |\text{cosec}(x/2)|$ we get that

$$\begin{aligned}
|\epsilon_{4,n}| &= \frac{1}{2\pi d_n} \left| \left(\sum_{\tau=-d_n+1}^{0} \sum_{j=c_n-\tau}^{c_n+d_n-1} + \sum_{\tau=1}^{d_n-1} \sum_{j=c_n}^{c_n+d_n-1-\tau} \right) \sum_{\lambda \in \Lambda_\tau \setminus \{\nu-\omega\}} a(\lambda,\tau) e^{i(\lambda-(\nu-\omega))j} e^{-i(\nu+v_n)\tau} \right| \\
&\leq \frac{1}{2\pi d_n} \sum_{\tau=-d_n+1}^{d_n-1} \sum_{\lambda \in \Lambda_\tau \setminus \{\nu-\omega\}} |a(\lambda,\tau) \text{cosec}((\lambda-(\nu-\omega))/2)| \\
&\leq \frac{1}{2\pi d_n} \sum_{\lambda \in \Lambda \setminus \{\nu-\omega\}} |\text{cosec}((\lambda-(\nu-\omega))/2)| \sum_{\tau=-d_n+1}^{d_n-1} |a(\lambda,\tau)|.
\end{aligned}$$

By the assumption that the set Λ is finite and by (1) we have that $\epsilon_{4,n} \to 0$ as $n \to \infty$, which ends the proof. □

The following lemma is a generalization of Theorem 1(a) from Fukuchi (1999).

Lemma 1.4. *Let $(X_1, X_2, \ldots, X_n)$ be sample from an α-mixing time series $\{X_t, t \in \mathbb{Z}\}$ and let $f_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a measurable function. Take two sequences of integers $b_1 = b_1(n)$, $b_2 = b_2(n) \longrightarrow \infty$ as $n \longrightarrow \infty$ such that $b_2/n \longrightarrow 0$ as $n \longrightarrow \infty$, $b_1 \leq b_2$. Define the subsampling estimator*

$$\bar{f}_{n,b_1,b_2} = w \sum_{b_1 \leq b \leq b_2} \frac{1}{q} \sum_{1 \leq t \leq q} f_{b,t}, \tag{2}$$

based on whole sample $(X_1, X_2, \ldots, X_n)$, where $f_{b,t} = f_b(X_t, \ldots, X_{t+b-1})$ is based on subsample $(X_t, \ldots, X_{t+b-1})$, $b_1 = b_1(n)$, $b_2 = b_2(n)$, $w = w(b_1, b_2) = 1/(b_2 - b_1 + 1)$, $q = q(n,b) = n - b + 1$, $b_1 \leq b \leq b_2$. Assume that

(i) $\mathrm{E}(\bar{f}_{n,b_1,b_2}) \longrightarrow \Upsilon$ *as* $n \longrightarrow \infty$, *where* Υ *is some constant;*

(ii) there exist constants $\tilde{\xi} > 0$ *and* $\tilde{\Delta} > 0$ *such that*

$$\sup_{b_1 \le b \le b_2,\ 1 \le t \le q,\ n \in \mathbb{Z}} \|f_{b,t}\|_{2+\tilde{\xi}} < \tilde{\Delta} \quad \textit{and} \quad \sum_{k=1}^{\infty} \alpha^{\frac{\tilde{\xi}}{2+\tilde{\xi}}}(k) < \infty;$$

Then

$$\bar{f}_{n,b_1,b_2} \xrightarrow{L_2} \Upsilon.$$

Proof of Lemma 1.4. By (i) it is sufficient to show that $\mathrm{var}(\bar{f}_{n,b_1,b_2}) \to 0$. We have

$$\begin{aligned}
\mathrm{var}(\bar{f}_{n,b_1,b_2}) &= w^2 \sum_{b_1 \le b \le b_2} \sum_{b_1 \le b' \le b_2} \frac{1}{q(n,b)q(n,b')} \sum_{1 \le t \le q(n,b)} \sum_{1 \le t' \le q(n,b')} \mathrm{cov}(f_{b,t}, f_{b',t'}) \\
&\le w^2 \sum_{b_1 \le b \le b_2} \sum_{b_1 \le b' \le b_2} \frac{1}{q(n,b)q(n,b')} \sum_{1 \le t \le q(n,b)} \sum_{1 \le t' \le q(n,b')} |\mathrm{cov}(f_{b,t}, f_{b',t'})| \\
&\le w^2 \sum_{b_1 \le b \le b_2} \sum_{b_1 \le b' \le b_2} \frac{1}{q(n,b)q(n,b')} \sum_{|t-t'| \le b_2} |\mathrm{cov}(f_{b,t}, f_{b',t'})| \qquad (3) \\
&+ w^2 \sum_{b_1 \le b \le b_2} \sum_{b_1 \le b' \le b_2} \sum_{s=b_2+1}^{n-b_1+1} \frac{1}{q(n,b)q(n,b')} \sum_{|t-t'|=s} |\mathrm{cov}(f_{b,t}, f_{b',t'})| \\
&= A_1 + A_2.
\end{aligned}$$

Note that for any b and b' ($b_1 \le b, b' \le b_2$) we get that

$$\begin{aligned}
&\sharp\{|t-t'| \le b_2 : 1 \le t \le q(n,b) \wedge 1 \le t' \le q(n,b')\} \le 2(n-b_1+1)b_2, \\
&\sharp\{|t-t'| = s : 1 \le t \le q(n,b) \wedge 1 \le t' \le q(n,b')\} \le 2[(n-b_1+1)-s],
\end{aligned}$$

for $s = b_2+1, b_2+2, \ldots, n-b_1+1$. Hence, by inequality $|\mathrm{cov}(f_{b,t}, f_{b',t'})| \le 8\tilde{\Delta}^2$ uniformly

at b, t, b', t' (which follows from Theorem 3 from Doukhan (1994)) (pp. 9) we have

$$\begin{aligned} A_1 &\leq w^2 \sum_{b_1 \leq b \leq b_2} \sum_{b_1 \leq b' \leq b_2} \frac{1}{q(n,b)q(n,b')} 2(n-b_1+1)b_2 8\tilde{\Delta}^2 \\ &\leq (n-b_1+1)b_2 \frac{(b_2-b_1+1)^2}{(n-b_2+1)^2(b_2-b_1+1)^2} 8\tilde{\Delta}^2 \\ &= \frac{(n-b_1+1)b_2 8\tilde{\Delta}^2}{(n-b_2+1)^2} = O(b_2/n) \end{aligned}$$

and hence $A_1 \to 0$ as $n \to \infty$. Using now Theorem 3 from Doukhan (1994) (pp. 9) for A_2 we have

$$\begin{aligned} A_2 &\leq w^2 \sum_{b_1 \leq b \leq b_2} \sum_{b_1 \leq b' \leq b_2} \frac{1}{q(n,b)q(n,b')} \sum_{s=b_2+1}^{n-b_1+1} \sum_{|t-t'|=s} 8\tilde{\Delta}^2 \alpha^{\frac{\tilde{\xi}}{2+\tilde{\xi}}}(|t-t'|-b_2) \\ &\leq w^2 \sum_{b_1 \leq b \leq b_2} \sum_{b_1 \leq b' \leq b_2} \frac{1}{q(n,b)q(n,b')} \sum_{s=b_2+1}^{n-b_1+1} 2[(n-b_1+1)-s] 8\tilde{\Delta}^2 \alpha^{\frac{\tilde{\xi}}{2+\tilde{\xi}}}(s-b_2) \\ &= w^2 \sum_{b_1 \leq b \leq b_2} \sum_{b_1 \leq b' \leq b_2} \frac{1}{q(n,b)q(n,b')} \sum_{h=1}^{n-b_1+1-b_2} 2[(n-b_1+1)-(h+b_2)] 8\tilde{\Delta}^2 \alpha^{\frac{\tilde{\xi}}{2+\tilde{\xi}}}(h) \\ &\leq \sum_{h=1}^{n-b_1+1-b_2} 2[(n-b_1+1)-(h+b_2)] \frac{8\tilde{\Delta}^2(b_2-b_1+1)^2}{(n-b_2+1)^2(b_2-b_1+1)^2} \alpha^{\frac{\tilde{\xi}}{2+\tilde{\xi}}}(h) \\ &\leq O(1/n) \sum_{h=1}^{n} \alpha^{\frac{\tilde{\xi}}{2+\tilde{\xi}}}(h) \end{aligned}$$

and hence $A_2 \to 0$ as $n \to \infty$. This completes the proof of this lemma. □

Proof of Theorem 3.1. We prove Theorem 3.1 using the same arguments as Carlstein (1986). In the first step we use the well known property: $\mathrm{cov}(X,Y) = \mathrm{cov}(X-x, Y-y) = \mathrm{E}(X-x)(Y-y) - \mathrm{E}(X-x)\mathrm{E}(Y-y)$ which is true for any second order random variables

X, Y and any $x, y \in \mathbb{R}$. Hence,

$$\begin{aligned}
\widehat{\sigma}_n &= \sum_{i=1}^{r}\sum_{j=1}^{r} a_{i\,j}\left[\sum_{b_1\le b\le b_2}\frac{\tau_b^2 w}{q(n,b)}\sum_{1\le t\le q(n,b)}\left(\widehat{\theta}^{(i)}_{n,b,t}-\widehat{\theta}^{(i)}_{n,b,\cdot}\right)\left(\widehat{\theta}^{(j)}_{n,b,t}-\widehat{\theta}^{(j)}_{n,b,\cdot}\right)\right]\\
&= \sum_{i=1}^{r}\sum_{j=1}^{r} a_{i\,j}\left[\sum_{b_1\le b\le b_2}\frac{\tau_b^2 w}{q(n,b)}\sum_{1\le t\le q(n,b)}\left(\widehat{\theta}^{(i)}_{n,b,t}-E(\widehat{\theta}^{(i)}_{n,b,\cdot})\right)\left(\widehat{\theta}^{(j)}_{n,b,t}-E(\widehat{\theta}^{(j)}_{n,b,\cdot})\right)\right]\\
&- \sum_{i=1}^{r}\sum_{j=1}^{r} a_{i\,j}\left[\sum_{b_1\le b\le b_2}\frac{\tau_b^2 w}{q(n,b)^2}\sum_{1\le t\le q(n,b)}\left(\widehat{\theta}^{(i)}_{n,b,t}-E(\widehat{\theta}^{(i)}_{n,b,\cdot})\right)\sum_{1\le t\le q}\left(\widehat{\theta}^{(j)}_{n,b,t}-E(\widehat{\theta}^{(j)}_{n,b,\cdot})\right)\right]\\
&= \sum_{b_1\le b\le b_2}\frac{w}{q(n,b)}\sum_{1\le t\le q(n,b)}\left[\sum_{i=1}^{r}\sum_{j=1}^{r} a_{i\,j}\tau_b^2\left(\widehat{\theta}^{(i)}_{n,b,t}-E(\widehat{\theta}^{(i)}_{n,b,\cdot})\right)\left(\widehat{\theta}^{(j)}_{n,b,t}-E(\widehat{\theta}^{(j)}_{n,b,\cdot})\right)\right]\\
&- \sum_{i=1}^{r}\sum_{j=1}^{r} a_{i\,j}\left[\sum_{b_1\le b\le b_2}\frac{\tau_b^2 w}{q(n,b)^2}\sum_{1\le t\le q(n,b)}\left(\widehat{\theta}^{(i)}_{n,b,t}-E(\widehat{\theta}^{(i)}_{n,b,\cdot})\right)\sum_{1\le t\le q(n,b)}\left(\widehat{\theta}^{(j)}_{n,b,t}-E(\widehat{\theta}^{(j)}_{n,b,\cdot})\right)\right]\\
&= I_n + II_n.
\end{aligned}$$

Below at first we show that $I_n \xrightarrow{L_2} \sigma$ (Step 1) and then that $II_n \xrightarrow{L_2} 0$ (Step 2).

Step 1. To obtain the desired convergence we use Lemma 1.4. Note that under the assumption (i) of Theorem 3.1, the condition (i) of Lemma 1.4 holds with

$$f_{b,t} = \sum_{i=1}^{r}\sum_{j=1}^{r} a_{i\,j}\tau_b^2\left(\widehat{\theta}^{(i)}_{n,b,t}-E(\widehat{\theta}^{(i)}_{n,b,\cdot})\right)\left(\widehat{\theta}^{(j)}_{n,b,t}-E(\widehat{\theta}^{(j)}_{n,b,\cdot})\right).$$

Moreover, notice that under the assumption (ii) of Theorem 3.1 and Hölder inequality we have that

$$\sup_{b_1\le b\le b_2,\ 1\le t\le q,\ n\in\mathbb{Z}} \|f_{b,t}\|_{2+\xi} \le \Delta^2 \sum_{i=1}^{r}\sum_{j=1}^{r} |a_{i\,j}| < \infty,$$

which means that the assumption (ii) of Lemma 1.4 is fulfilled with $\tilde{\xi} = \xi$ and $\tilde{\Delta} = \Delta^2 \sum_{i=1}^{r}\sum_{j=1}^{r} |a_{i\,j}|$. Thus, $I_n \xrightarrow{L_2} \sigma$.

Step 2. It is sufficient to show that $E|II_n| \to 0$ as $n \to \infty$. By Hölder inequality we have

$$\begin{aligned} E|II_n| &\leq \sum_{i=1}^{r}\sum_{j=1}^{r}|a_{i\,j}|\left[\sum_{b_1\leq b\leq b_2}\frac{\tau_b^2 w}{q(n,b)^2}E\left|\sum_{1\leq t\leq q(n,b)}\left(\widehat{\theta}^{(i)}_{n,b,t}-E(\widehat{\theta}^{(i)}_{n,b,\cdot})\right)\sum_{1\leq t\leq q(n,b)}\left(\widehat{\theta}^{(j)}_{n,b,t}-E(\widehat{\theta}^{(j)}_{n,b,\cdot})\right)\right|\right] \\ &\leq \sum_{i=1}^{r}\sum_{j=1}^{r}|a_{i\,j}|\sum_{b_1\leq b\leq b_2}w\left\|\frac{\tau_b}{q}\sum_{1\leq t\leq q(n,b)}\left(\widehat{\theta}^{(i)}_{n,b,t}-E(\widehat{\theta}^{(i)}_{n,b,\cdot})\right)\right\|_2\left\|\frac{\tau_b}{q}\sum_{1\leq t\leq q(n,b)}\left(\widehat{\theta}^{(j)}_{n,b,t}-E(\widehat{\theta}^{(j)}_{n,b,\cdot})\right)\right\|_2. \end{aligned}$$

Hence under the assumption (iii) of Theorem 3.1 and using the Toeplitz lemma we get that $II_n \xrightarrow{L_2} 0$. This fact ends proof of theorem. □

Proof of Theorem 3.2. Theorem 3.2 is a simple generalization of Theorem 2.1 form Lenart (2013). Therefore we omit the proof. □

Proof of Theorem 3.3. It is sufficient to show that the assumptions $(i)-(iii)$ of Theorem 3.1 hold. We split the proof into three steps.

Step 1. To show (i) note that for any $i,j\in\{1,2,3,4\}$ we have

$$\begin{aligned} &\tilde{\tau}_b^2\left(\widehat{\theta}^{(i)}_{n,\tilde{b},t_n}-E\left(\widehat{\theta}^{(i)}_{n,\tilde{b},\cdot}\right)\right)\left(\widehat{\theta}^{(j)}_{n,\tilde{b},t_n}-E\left(\widehat{\theta}^{(j)}_{n,\tilde{b},\cdot}\right)\right) \\ &\qquad=\tau_{\tilde{b}}^2\left(\widehat{\theta}^{(i)}_{n,\tilde{b},t_n}-E\left(\widehat{\theta}^{(i)}_{n,\tilde{b},t_n}\right)\right)\left(\widehat{\theta}^{(j)}_{n,\tilde{b},t_n}-E\left(\widehat{\theta}^{(j)}_{n,\tilde{b},t_n}\right)\right) \\ &\qquad+\tau_{\tilde{b}}^2\left(\widehat{\theta}^{(i)}_{n,\tilde{b},t_n}-E\left(\widehat{\theta}^{(i)}_{n,\tilde{b},t_n}\right)\right)\left(E\left(\widehat{\theta}^{(j)}_{n,\tilde{b},t_n}\right)-E\left(\widehat{\theta}^{(j)}_{n,\tilde{b},\cdot}\right)\right) \\ &\qquad+\tau_{\tilde{b}}^2\left(E\left(\widehat{\theta}^{(i)}_{n,\tilde{b},t_n}\right)-E\left(\widehat{\theta}^{(i)}_{n,\tilde{b},\cdot}\right)\right)\left(\widehat{\theta}^{(j)}_{n,\tilde{b},t_n}-E\left(\widehat{\theta}^{(j)}_{n,\tilde{b},t_n}\right)\right) \\ &\qquad+\tau_{\tilde{b}}^2\left(E\left(\widehat{\theta}^{(i)}_{n,\tilde{b},t_n}\right)-E\left(\widehat{\theta}^{(i)}_{n,\tilde{b},\cdot}\right)\right)\left(E\left(\widehat{\theta}^{(j)}_{n,\tilde{b},t_n}\right)-E\left(\widehat{\theta}^{(j)}_{n,\tilde{b},\cdot}\right)\right). \end{aligned}$$

Hence,

$$
\begin{aligned}
E\left[\tau_{\tilde{b}}^2\left(\widehat{\theta}_{n,\tilde{b},t_n}^{(i)}-E\left(\widehat{\theta}_{n,\tilde{b},\cdot}^{(i)}\right)\right)\left(\widehat{\theta}_{n,\tilde{b},t_n}^{(j)}-E\left(\widehat{\theta}_{n,\tilde{b},\cdot}^{(j)}\right)\right)\right] &= \tau_{\tilde{b}}^2\mathrm{cov}\left(\widehat{\theta}_{n,\tilde{b},t_n}^{(i)},\widehat{\theta}_{n,\tilde{b},t_n}^{(j)}\right) \\
&+3\tau_{\tilde{b}}^2\left(E\left(\widehat{\theta}_{n,\tilde{b},t_n}^{(i)}\right)-E\left(\widehat{\theta}_{n,\tilde{b},\cdot}^{(i)}\right)\right)\left(E\left(\widehat{\theta}_{n,\tilde{b},t_n}^{(j)}\right)-E\left(\widehat{\theta}_{n,\tilde{b},\cdot}^{(j)}\right)\right) \\
&= III_n + IV_n \qquad (4)
\end{aligned}
$$

Note that by Lemma 1.1 we have that $IV_n \to 0$ as $n \to \infty$. Moreover, from Theorem 3.2 we get that $III_n = \tau_{\tilde{b}}^2\mathrm{cov}\left(\widehat{\theta}_{n,\tilde{b},t_n}^{(i)},\widehat{\theta}_{n,\tilde{b},t_n}^{(j)}\right) \to \gamma_{ij}$ as $n \to \infty$, where γ_{ij} is element of matrix $\mathbf{\Gamma}(\nu,\omega)$ (see Theorem 3.2). Hence, the condition (i) of Theorem 3.1 holds.

Step 2. To get (ii) it is sufficient to show that there exists $\Delta > 0$ such that

$$
\sup_{b_1\le b\le b_2,\, 1\le t\le n-b+1,\, n\in\mathbb{Z}} \tau_b\left\|\widehat{\theta}_{n,b,t}^{(i)}-E\left(\widehat{\theta}_{n,b,\cdot}^{(i)}\right)\right\|_{4+2\xi} < \Delta,
$$

for $1 \le i \le 4$. Without loss of generality we take $i = 1$. In such a case $\widehat{\theta}_{n,b,t}^{(1)} = \frac{1}{b}\sum_{k=t}^{t+b-1} X_k\cos(\nu k)$. For any b such that $b_1 \le b \le b_2$ and any t such that $1 \le t \le n-b+1$ we have

$$
\begin{aligned}
\tau_b\left\|\widehat{\theta}_{n,b,t}^{(i)}-E\left(\widehat{\theta}_{n,b,\cdot}^{(i)}\right)\right\|_{4+2\xi} &\le \tau_b\left\|\widehat{\theta}_{n,b,t}^{(i)}-E\left(\widehat{\theta}_{n,b,t}^{(i)}\right)\right\|_{4+2\xi} + \tau_b\left|E\left(\widehat{\theta}_{n,b,t}^{(i)}\right)-E\left(\widehat{\theta}_{n,b,\cdot}^{(i)}\right)\right| \\
&= V_n + VI_n,
\end{aligned}
$$

where

$$
V_n = \tau_b\left\|\widehat{\theta}_{n,b,t}^{(1)}-E\left(\widehat{\theta}_{n,b,t}^{(1)}\right)\right\|_{4+2\xi} = \sqrt{b}\left\|\frac{1}{b}\sum_{k=t}^{t+b-1}(X_k-\mu(k))\cos(\nu k)\right\|_{4+2\xi}.
$$

Note that $\{X_t : t \in \mathbb{Z}\}$ and $\{(X_t-\mu(t))\cos(\nu t) : t \in \mathbb{Z}\}$ have the same mixing coefficients. Under moment and mixing assumptions of Theorem 3.3 and using Theorem 2 from Doukhan (1994) (see p. 26) with $\tau = 4+2\xi$, we get that V_n is uniformly bounded in t, b, n.

Moreover, from Lemma 1.1 we have that $VI_n = O(\tau_b/b)$. Hence VI_n is also uniformly bounded, since $b_1 < b$ and $b_1 \to \infty$ as $n \to \infty$.

Step 3. Without loss of generality we take $i = 1$. Let sequence $\tilde{b}$ be such that $b_1 \leq \tilde{b} \leq b_2$. To get (iii) it is sufficient to show that

$$E\left|\frac{\tau_{\tilde{b}}}{q(n,\tilde{b})}\sum_{1\leq t\leq q(n,\tilde{b})}\left(\widehat{\theta}^{(1)}_{n,\tilde{b},t} - E\left(\widehat{\theta}^{(1)}_{n,\tilde{b},\cdot}\right)\right)\right|^2 \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty,$$

where $\widehat{\theta}^{(1)}_{n,b,t} = \frac{1}{\tilde{b}}\sum_{k=t}^{t+\tilde{b}-1} X_k \cos(\nu k)$, $\widehat{\theta}^{(1)}_{n,\tilde{b},\cdot} = \frac{1}{q(n,\tilde{b})}\sum_{s=1}^{q}\widehat{\theta}^{(1)}_{n,\tilde{b},s}$. Note that from inequality $|z_1+z_2|^2 \leq 2|z_1|^2 + 2|z_2|^2$ (which is true for any $z_1, z_2 \in \mathbb{C}$) we have

$$\begin{aligned}
&\left|\frac{\tau_{\tilde{b}}}{q(n,\tilde{b})}\sum_{1\leq t\leq q(n,\tilde{b})}\left(\widehat{\theta}^{(1)}_{n,\tilde{b},t} - E\left(\widehat{\theta}^{(1)}_{n,\tilde{b},\cdot}\right)\right)\right|^2 \\
&= \left|\frac{\tau_{\tilde{b}}}{q(n,\tilde{b})}\sum_{1\leq t\leq q(n,\tilde{b})}\left(\widehat{\theta}^{(1)}_{n,\tilde{b},t} - E\left(\widehat{\theta}^{(1)}_{n,\tilde{b},t}\right)\right) + \frac{\tau_{\tilde{b}}}{q(n,\tilde{b})}\sum_{1\leq t\leq q(n,\tilde{b})}\left(E(\widehat{\theta}^{(1)}_{n,\tilde{b},t}) - E\left(\widehat{\theta}^{(1)}_{n,\tilde{b},\cdot}\right)\right)\right|^2 \\
&\leq 2\left|\frac{\tau_{\tilde{b}}}{q(n,\tilde{b})}\sum_{1\leq t\leq q(n,\tilde{b})}\left(\widehat{\theta}^{(1)}_{n,\tilde{b},t} - E\left(\widehat{\theta}^{(1)}_{n,\tilde{b},t}\right)\right)\right|^2 + 2\left|\frac{\tau_{\tilde{b}}}{q(n,\tilde{b})}\sum_{1\leq t\leq q(n,\tilde{b})}\left(E(\widehat{\theta}^{(1)}_{n,\tilde{b},t}) - E\left(\widehat{\theta}^{(1)}_{n,\tilde{b},\cdot}\right)\right)\right|^2 \\
&= VII_n + VIII_n.
\end{aligned}$$

Additionally, using inequality $\max\{|Re(z)|^2, |Im(z)|^2\} \leq |z|^2$ (which is true for any $z \in \mathbb{C}$) and Lemma 1.1 we get that $VIII_n \to 0$ as $n \to \infty$. Moreover,

$$VII_n = 2\left|\frac{\tau_{\tilde{b}}}{q(n,\tilde{b})}\sum_{1\leq t\leq q(n,\tilde{b})}\frac{1}{\tilde{b}}\sum_{k=t}^{t+\tilde{b}-1}(X_k - \mu(k))\cos(\nu k)\right|^2 = 2\frac{n\tau^2_{\tilde{b}}}{q(n,\tilde{b})^2}\left|\frac{1}{\sqrt{n}}\sum_{j=1}^{n} Y_{n,j}\right|^2,$$

where $Y_{n,t} = s_{n,t}(X_t - \mu(t))\cos(\nu t)$ and

$$s_{n,j} = \begin{cases} j/\tilde{b} & \text{for } j = 1, 2, \ldots, \tilde{b} - 1 \\ 1 & \text{for } j = \tilde{b}, \tilde{b}+1, \ldots, n - \tilde{b} \\ (n - j + 1)/\tilde{b} & \text{for } j = n - \tilde{b} + 1, \ldots, n \end{cases}$$

Note that $|s_{n,j}| \leq 1$. Moreover, $Y_{n,j}$, is the zero mean triangular array with the α-mixing coefficients of the nth row $\alpha_n(k) = \alpha_X(k)$, $k = 1, 2, \ldots, n$. Thus, using Lemma 7.5 from Lenart (2016) we get that

$$E\left|\frac{1}{\sqrt{n}}\sum_{j=1}^{n} Y_{n,j}\right|^2 \leq C_1,$$

where C_1 is some positive constants. Additionally, since $n\tau_{\tilde{b}}^2/q(n,\tilde{b})^2 \leq nb_2/(n-b_2+1)^2 \to 0$ as $n \to \infty$, we get that $E(VII_n) \to 0$ as $n \to \infty$. □

Proof of the Theorem 3.4. The proof of Theorem 3.4 follows the same reasoning as was used in the proof of Theorem 3.3. Therefore, below we only discuss shortly the main steps. To prove theorem it is sufficient to show that the conditions $(i) - (iii)$ of Theorem 3.1 hold.

To get (i) we use decomposition (4). Thus, we have two summands III_n and IV_n. Using Lemma 1.2 we get that $IV_n \to 0$ as $n \to \infty$. Now we consider the term III_n. Following the same reasoning as was used in the proof of Theorem 2.1 from Lenart (2013), we get

$$\begin{aligned} p_1 &= d_n \mathrm{cov}(\mathrm{Re}[\widehat{m}_{d,c}(\nu + v_n)], \mathrm{Re}[\widehat{m}_{d,c}(\omega + v_n)]) \\ &= \pi E\left(\mathrm{Re}[\widehat{P}_{d,c}(\nu + v_n, \omega + v_n)] + \mathrm{Re}[\widehat{P}_{d,c}(\nu + v_n, -(\omega + v_n))]\right), \\ p_2 &= d_n \mathrm{cov}(\mathrm{Im}[\widehat{m}_{d,c}(\nu + v_n)], \mathrm{Im}[\widehat{m}_{d,c}(\omega + v_n)]) \\ &= \pi E\left(\mathrm{Re}[\widehat{P}_{d,c}(\nu + v_n, \omega + v_n)] - \mathrm{Re}[\widehat{P}_{d,c}(\nu + v_n, -(\omega + v_n))]\right), \end{aligned}$$

$$
\begin{aligned}
q_1 &= d_n \mathrm{cov}(\mathrm{Re}[\widehat{m}_{d,c}(\nu+v_n)], \mathrm{Im}[\widehat{m}_{d,c}(\omega+v_n)]) \\
&= \pi E\left(\mathrm{Im}[\widehat{P}_{d,c}(\nu+v_n, -(\omega+v_n))] - \mathrm{Im}[\widehat{P}_{d,c}(\nu+v_n, \omega+v_n)]\right),
\end{aligned}
$$

$$
\begin{aligned}
q_2 &= d_n \mathrm{cov}(\mathrm{Im}[\widehat{m}_{d,c}(\nu+v_n)], \mathrm{Re}[\widehat{m}_{d,c}(\omega+v_n)]) \\
&= \pi E\left(\mathrm{Im}[\widehat{P}_{d,c}(\nu+v_n, \omega+v_n)] + \mathrm{Im}[\widehat{P}_{d,c}(\nu+v_n, -(\omega+v_n))]\right).
\end{aligned}
$$

Notice that for any $(x, y) \in \mathbb{R}^2$

$$
\widehat{P}_{d,c}(x,y) + \widehat{P}_{d,c}(-x,-y) = 2\mathrm{Re}[\widehat{P}_{d,c}(x,y)], \tag{5}
$$

$$
\widehat{P}_{d,c}(x,y) - \widehat{P}_{d,c}(-x,-y) = 2i\ \mathrm{Im}[\widehat{P}_{d,c}(x,y)]. \tag{6}
$$

Thus, using Lemma 1.3 we obtain that $p_1 + p_2 \to 2\pi \mathrm{Re}[(\nu, \omega)]$ and $q_2 - q_1 \to 2\pi \mathrm{Im}[P(\nu, \omega)]$ as $n \to \infty$. Note that the condition (1) follows from the inequality $|a(\lambda, \tau)| \leq C_4 \alpha^{\frac{\delta}{2+\delta}}(|\tau|)$ (which is true for any $\tau \in \mathbb{Z}$ and follows from Theorem 3, pp. 9 Doukhan (1994)), where C_4 is some positive constant. This completes the proof of (i).

To show (ii) and (iii) it is sufficient to use the same steps as in the proof of Theorem 3.3. The only difference is that we consider $\nu + v_n$ and $\omega + v_n$ instead of ν and ω. □

Proof of the Theorem 3.5. Proof of Theorem 3.5 is a direct consequence of the following lemma.

Lemma 1.5. *Let $k > 0$ be a fixed integer and χ be an unknown real valued parameter. Let $\widehat{\chi}_{n,1}, \ldots, \widehat{\chi}_{n,k}$ be estimators of χ such that $\widehat{\chi}_{n,i} \xrightarrow{L^2} \chi, i = 1, \ldots, k$ as $n \to \infty$. Moreover, let $\widehat{\zeta}_n$ be such that*

$$
\min\{\widehat{\chi}_{n,1}, \ldots, \widehat{\chi}_{n,k}\} \leq \widehat{\zeta}_n \leq \max\{\widehat{\chi}_{n,1}, \ldots, \widehat{\chi}_{n,k}\}.
$$

Then

$$\widehat{\zeta}_n \xrightarrow{L^2} \chi \qquad as\ n \longrightarrow \infty.$$

Proof: Note that

$$E(\widehat{\zeta}_n - \chi)^2 \leq \sum_{i=1}^{k} E(\widehat{\chi}_{n,i} - \chi)^2 \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

This completes the proof. □

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